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Hamilton–Jacobi method in non-minimal coupling inflation: metric vs. Palatini

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Abstract The Hamilton–Jacobi approach offers a natural framework for analyzing inflationary dynamics, relying on the specified Hubble parameter rather than the potential, particularly in extended gravity theories. In this study, we apply this method to investigate inflation with non-minimal coupling, comparing the metric and Palatini formulations. Using a power-law Hubble parameter in the Jordan frame, we find that both formalisms satisfy the attractor condition, with a slight suppression in the Palatini case. Our results align closely with the latest observational data, demonstrating the Palatini formalism's heightened sensitivity to coupling constants and model parameters. Furthermore, we show that within the model's viable parameter space, the reheating process can achieve sufficiently high temperatures to support successful leptogenesis.

1 Introduction

The standard cosmological model provides several predictions that have been successfully verified experimentally. However, it encounters three significant issues: the horizon dilemma, the flatness issue, and the magnetic monopole challenge. These challenges are addressed by inflation theory, which suggests a period of rapid exponential expansion in the early universe [1–4]. This inflationary phase leads to a nearly scale-invariant spectrum, which is crucial for the development of large-scale structures in the universe and aligns well with the cosmic microwave background (CMB) observations [5–11]. CMB observations place constraints on two key inflationary parameters: the spectral index of the primordial curvature perturbations, n_s, and the tensor-to-scalar ratio, r. According to the latest Planck 2018 data, the spectral index is constrained to $n_s = 0.965 \pm 0.004$ at the 68% confidence level (CL) [12]. Meanwhile, the BICEP/Keck survey sets an upper limit on r as $r < 0.036$ at the 95% CL [13].

Observational constraints on n_s and r exclude several inflationary models, including those with simple quadratic and quartic potentials [12]. However, when considering extensions to gravity, such as non-minimal coupling of the inflaton to gravity, these models can align well with the observational data [14–16]. The discussion of gravitational degrees of freedom is particularly prominent in non-minimal coupling theories, especially in the metric and Palatini formalisms [17–36]. In the metric formalism, the independent variables are the metric and its first-order derivatives, whereas in the Palatini formalism, the independent variables are the metric and the connection. Although these two formalisms yield identical field equations in general relativity, they lead to different field equations and distinct physical outcomes when non-minimal coupling is introduced [17,19]. Higgs inflation with non-minimal coupling encounters unitarity violations in the metric approach, relying on unknown ultraviolet (UV) physics [37–41], while the Palatini formalism ensures UV safety and avoids these issues [18]. Additionally, in Palatini formalism, the tensor scale ratio is consistently suppressed compared to metric formalism [42].

Recently, the Hamilton–Jacobi method has been widely employed for analyzing various inflationary models [43–66]. Unlike the traditional slow-roll approximation, the Hamilton–Jacobi formalism allows the scalar field to be treated as the time variable in cosmological background equations, provided the field evolves monotonically. This approach shifts the focus from specifying the inflaton potential $V(\phi)$ to specifying the Hubble parameter $H(\phi)$, a geometric quantity that directly characterizes the expansion rate

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of the universe. Furthermore, this method simplifies the analysis of inflation by bypassing some complex perturbation problems, especially in extended gravity theories, such as those involving non-minimal derivative coupling [58]. The Hamilton–Jacobi formalism enables the reconstruction of inflationary potentials beyond the slow-roll approximation. Models such as tachyonic inflation [55], quasi-exponential inflation [60], anisotropic inflation [66], and k-inflation [65], among others, consistently produce predictions that align well with the observational data.

The Hamilton–Jacobi method has been applied in non-minimal coupling inflation to explore inflationary solutions by specifying different scalar potentials [67]. However, this method has not yet been used to investigate inflationary predictions in the scalar field theories with non-minimal coupling. To address this gap, we analyze inflationary predictions for non-minimal coupling in two formulations: metric and Palatini, by giving the specific form of the Hubble parameter, and compare these predictions with the latest observations. This paper is structured in the following manner: Sect. 2 presents the cosmological background equations for the non-minimal coupling theories. Section 3 applies the Hamilton–Jacobi method to inflationary scenarios. Section 4 examines the inflationary results using the specific form of the Hubble parameter. Finally, Sect. 6 presents our conclusions. This paper uses the metric signature $(-,+,+,+)$, adopts natural units with $c=\hbar=1$, and establishes the reduced Planck mass as $M_{pl}=1/\sqrt{8\pi G}=1$.

2 Background equations in the scalar field theory with non-minimal coupling

In this section, we provide a brief review of the cosmological background equations for the scalar field theory with non-minimal coupling in both the metric and Palatini formalisms, considering the Jordan and Einstein frames.

2.1 Solutions in the Jordan frame

The action in the Jordan frame includes an inflaton field, ϕ, which is non-minimally coupled to the Ricci scalar R, a function of the connection Γ, and is given by the following expression:

$$S_J=\int d^4x\sqrt{-g}\left[\frac{1}{2}f(\phi)R(\Gamma)-\frac{1}{2}\partial_\mu\phi\partial^\mu\phi-V(\phi)\right], \tag{1}$$

where g denotes the determinant of the metric tensor $g_{\mu\nu}$, and $f(\phi)\equiv 1-\xi\phi^2$ is the coupling function [68]. Here, ξ represents the non-minimal coupling constant, which is limited to $\xi\leq 10^{-3}$ in chaotic models [69], and $V(\phi)$ is the potential of the inflaton field.

In understanding this framework, it is essential to distinguish between the metric and Palatini formalisms, as they treat the connection Γ differently. In the metric formalism, the connection is the Levi-Civita connection, expressed as $\Gamma=\tilde{\Gamma}(g_{\mu\nu})$, which depends on the metric tensor. However, in the Palatini case, $g_{\mu\nu}$ and Γ are treated as independent variables, and Γ is determined by solving the field equations derived from the variation of the action with respect to the connection [17]:

$$\begin{aligned}\Gamma^\gamma_{\mu\nu}=&\tilde{\Gamma}^\gamma_{\mu\nu}+\delta^\gamma_\mu\partial_\nu\ln\sqrt{f(\phi)}\\&+\delta^\gamma_\nu\partial_\mu\ln\sqrt{f(\phi)}-g_{\mu\nu}\partial^\gamma\ln\sqrt{f(\phi)}.\end{aligned} \tag{2}$$

When $f(\phi)=1$ (i.e., in the minimal case where $\xi=0$), the scalar field theory with non-minimal coupling reduces to general relativity, and the metric and Palatini formalisms yield identical field equations. However, when non-minimal coupling is introduced, the gravitational theories in two formalisms differ, as shown in Eq. (2).

To explore these dynamics, we use the spatially flat Friedmann-Robertson-Walker (FRW) metric:

$$ds^2=-dt^2+a^2(t)\delta_{ij}dx^idx^j, \tag{3}$$

to derive the background equations as follows:

$$3H^2=\frac{1}{(1-\alpha)}\left[\frac{1}{2}\dot{\phi}^2+V(\phi)+6\xi H\phi\dot{\phi}-\frac{3\sigma\xi\alpha\dot{\phi}^2}{1-\alpha}\right], \tag{4}$$

$$\begin{aligned}&\ddot{\phi}+3H\dot{\phi}+\frac{(1-\alpha)V_{,\phi}(\phi)}{\beta}\\&\quad+\frac{\xi\phi\left\{4V(\phi)-[1-(1-\sigma)6\xi]\dot{\phi}^2\right\}}{\beta}=0,\end{aligned} \tag{5}$$

where $a(t)$ is the scale factor, $H\equiv\dot{a}/a$ is the Hubble parameter, $\alpha\equiv\xi\phi^2$, and $\beta\equiv 1-\alpha[1-(1-\sigma)6\xi]$. The dot notation indicates a derivative with respect to cosmic time t, while the subscript "$,\phi$" signifies a derivative with respect to the scalar field ϕ. The parameter σ distinguishes between the metric ($\sigma=0$) and Palatini ($\sigma=1$) formulations.

2.2 Solutions in the Einstein frame

For convenience, we apply a conformal transformation to convert the action (1) in the Jordan frame into the Einstein frame:

$$\hat{g}_{\mu\nu}=f(\phi)g_{\mu\nu}. \tag{6}$$

Therefore, the action in the Einstein frame is given by

$$S_E = \int d^4x\sqrt{-\hat{g}}\left\{\frac{1}{2}\hat{R} - \frac{1}{2}\hat{\partial}_\mu\hat{\phi}\hat{\partial}^\mu\hat{\phi} - \hat{V}(\hat{\phi})\right\}, \quad (7)$$

where a hat denotes quantities in the Einstein frame. The potential $\hat{V}$ is connected to the potential V in the Jordan frame by:

$$\hat{V}\left(\hat{\phi}(\phi)\right) = \frac{V(\phi)}{f^2(\phi)}, \quad (8)$$

and the new scalar field $\hat{\phi}$ is defined as:

$$\frac{d\hat{\phi}}{d\phi} = \frac{\sqrt{\beta}}{1-\alpha}. \quad (9)$$

From Eq. (6) we obtain the line element relation between the Einstein and Jordan frames:

$$d\hat{s}^2 = f(\phi)ds^2 = -d\hat{t}^2 + \hat{a}^2(t)\delta_{ij}dx^i dx^j, \quad (10)$$

which leads to

$$\hat{a}(\hat{t}) = \sqrt{f(\phi)}a(t), \quad \hat{t} = \sqrt{f(\phi)}t. \quad (11)$$

Thus, the Friedmann background equation and the scalar field equation in the Einstein frame can be written as

$$3\hat{H}^2 = \frac{1}{2}\hat{\phi}'^2 + \hat{V}(\hat{\phi}) \quad (12)$$

and

$$\hat{\phi}'' + 3\hat{H}\hat{\phi}' + \hat{V}_{,\hat{\phi}}(\hat{\phi}) = 0, \quad (13)$$

respectively, the prime indicates differentiation with respect to the new time $\hat{t}$. The quantities $\hat{\phi}'$ and $\hat{H}$ can be represented in terms of the corresponding quantities in the Jordan frame:

$$\hat{\phi}' = \frac{d\hat{\phi}}{d\phi}\frac{d\phi}{dt}\frac{dt}{d\hat{t}} = \frac{\sqrt{\beta}}{(1-\alpha)^{\frac{3}{2}}}\dot{\phi}, \quad (14)$$

$$\hat{H} \equiv \frac{\hat{a}'}{\hat{a}} = \frac{1}{\sqrt{f}}\left(H + \frac{1}{2}\frac{\dot{f}}{f}\right). \quad (15)$$

3 Hamilton–Jacobi approach to inflationary processes

We will explore the dynamics of inflation with non-minimal coupling by employing the Hamilton–Jacobi method [43] in this part. This analysis will be conducted within both the metric and Palatini formalisms.

3.1 Inflationary dynamics

To achieve approximately exponential expansion, the inflaton field must satisfy the slow-roll approximation: $|\dot{\phi}/\phi| \ll H$, $|\ddot{\phi}/\dot{\phi}| \ll H$, and $\dot{\phi}^2 \ll V(\phi)$. Under these conditions, Eqs. (4) and (5) can be approximated as follows:

$$3(1-\alpha)H^2 \simeq V(\phi), \quad (16)$$

$$3H\dot{\phi} \simeq -\frac{4\xi\phi V(\phi) + (1-\alpha)V_{,\phi}(\phi)}{\beta}. \quad (17)$$

By differentiating Eq. (16) with respect to ϕ and combining it with Eq. (17), we derive:

$$\dot{\phi} \simeq -\frac{2(1-\alpha)\mathcal{H}(\phi)}{\beta}, \quad (18)$$

where $\mathcal{H}(\phi) \equiv \xi\phi H(\phi) + (1-\alpha)H_{,\phi}(\phi)$. Substituting Eq. (18) into Eq. (4), we obtain the Hamilton–Jacobi equation:

$$3(1-\alpha)H^2(\phi) = V(\phi) + \frac{2(1-\alpha)^2\mathcal{H}^2(\phi)}{\beta^2} - \frac{12\xi\phi(1-\alpha)H(\phi)\mathcal{H}(\phi)}{\beta} - \frac{12\xi\sigma\alpha(1-\alpha)\mathcal{H}^2(\phi)}{\beta^2}. \quad (19)$$

From this equation, we can express the potential $V(\phi)$ as:

$$V(\phi) = 3(1-\alpha)H^2(\phi) - \frac{2(1-\alpha)^2\mathcal{H}^2(\phi)}{\beta^2} + \frac{12\xi\phi(1-\alpha)H(\phi)\mathcal{H}(\phi)}{\beta} + \frac{12\xi\sigma\alpha(1-\alpha)\mathcal{H}^2(\phi)}{\beta^2}. \quad (20)$$

To effectively characterize the inflationary process, we introduce two slow-roll parameters as: [44]

$$\epsilon \equiv -\frac{\dot{H}}{H^2} = \frac{2(1-\alpha)H_{,\phi}(\phi)\mathcal{H}(\phi)}{\beta^2 H^2(\phi)}, \quad (21)$$

$$\eta \equiv -\frac{d\ln H_{,\phi}}{d\ln a} = -\frac{H_{,\phi\phi}\dot{\phi}}{HH_{,\phi}} = \frac{2(1-\alpha)H_{,\phi\phi}\mathcal{H}(\phi)}{\beta H H_{,\phi}} \quad (22)$$

In deriving Eq. (21), we used the relations $\dot{H} = \dot{\phi}H_{,\phi}$ and Eq. (18). During slow-roll inflation, the slow-roll parameter ϵ satisfies the condition $\epsilon, \eta \ll 1$, and inflation ends when $\epsilon(\phi_{\rm end}) = 1$ or $\eta(\phi_{\rm end}) = 1$. By utilizing the relation $\dot{a} = \dot{\phi}a_{,\phi}$ and integrating, we obtain the scale factor:

$$a(t) = a_i \exp\left\{-\int\frac{\beta H(\phi)}{2(1-\alpha)\mathcal{H}(\phi)}d\phi\right\}, \quad (23)$$

where a_i is the constant of integration. The e-folding number during inflation is given by:

$$N_* \equiv \int_{t_*}^{t_{\rm end}} H dt = \int_{\phi_*}^{\phi_{\rm end}} \frac{H(\phi)}{\dot{\phi}} d\phi, \tag{24}$$

where the subscripts "$_*$" and "$_{\rm end}$" indicate two specific moments: the exit of the pivot scale from the Hubble horizon and the end of inflation.

3.2 Attractor behavior

Inflationary models must exhibit attractor behavior, meaning that solutions with different initial conditions converge to a unique solution [43]. This property ensures that these models possess genuine predictive power. The Hamilton–Jacobi method is particularly useful for investigating whether an inflationary model has such an attractor solution. To analyze this, we decompose the Hubble parameter $H(\phi)$ into a background component $H_0(\phi)$ and a linear perturbation component $\delta H(\phi)$. A model is considered to exhibit attractor behavior if the linear perturbation $\delta H(\phi)$ diminishes over time [49].

By replacing $H(\phi) = H_0(\phi) + \delta H(\phi)$ into Eq. (19) and linearizing, we get:

$$\delta H(\phi) \simeq \frac{2(1-\alpha)\left[(\beta - 6\alpha\xi)\mathcal{H}_0(\phi) - 3\xi\phi\beta H_0(\phi)\right]}{4\xi\phi\mathcal{H}_0(\phi)(\beta + 3\alpha\xi) + 3\beta H_0(\phi)(\beta + 2\alpha\xi)} \delta H_{,\phi}, \tag{25}$$

where $\mathcal{H}_0(\phi) = \xi\phi H_0(\phi) + (1-\alpha)H_{0,\phi}(\phi)$. Integrating this differential equation yields:

$$\frac{\delta H}{\delta H_*} = \exp\left\{\int_{\phi_*}^{\phi} \frac{4\xi\phi\mathcal{H}_0(\beta + 3\alpha\xi) + 3\beta H_0(\beta + 2\alpha\xi)}{2(1-\alpha)\left[(\beta - 6\alpha\xi)\mathcal{H}_0 - 3\xi\phi\beta H_0\right]} d\phi\right\}, \tag{26}$$

where δH_* represents the initial value of the perturbation at $\phi = \phi_*$. If a specific form of $H(\phi)$ is given, Eq.(26) can be used to analyze the behavior of the perturbation $\delta H(\phi)$.

3.3 Cosmological perturbations and observational quantities

Inflation predicts a power spectrum for curvature perturbations that is approximately scale-invariant. In the case of a scalar field with non-minimal coupling to gravity, the scalar and tensor power spectra are invariant under the conformal transformation from the Jordan frame to the Einstein frame [70–75]. Therefore, we can express the scalar and tensor power spectra as:

$$P_{\rm s} = \hat{P}_{\rm s} = \frac{\hat{H}^4}{4\pi^2\hat{\dot{\phi}}'^2} = \frac{\left[(1-\alpha)H - \xi\phi\dot{\phi}\right]^4}{4\pi^2\dot{\phi}^2(1-\alpha)^3\beta}, \tag{27}$$

and

$$P_{\rm T} = \hat{P}_{\rm T} = \frac{2\hat{H}^2}{\pi^2} = \frac{2\left[(1-\alpha)H - \xi\phi\dot{\phi}\right]^2}{\pi^2(1-\alpha)^3}, \tag{28}$$

respectively. For the last step of the two equations above, we used Eqs. (14) and (15).

The scalar spectral index is defined as

$$n_{\rm s} \equiv 1 + \left(\frac{d\ln P_{\rm s}}{d\ln k}\right)_{k=aH} \approx 1 + \frac{\dot{\phi}P_{{\rm s},\phi}(\phi)}{HP_{\rm s}(\phi)}, \tag{29}$$

where k is the wave number. In deriving Eq. (29), we used the relations $d\ln k/(dt) = d\ln(aH)/(dt) = (1-\epsilon)H \approx H$ and $dP_{\rm s}/(dt) = \dot{\phi}P_{{\rm s},\phi}(\phi)$.

Combining Eqs. (28) and (27), we obtain the tensor-to-scalar ratio as

$$r \equiv \frac{P_{\rm T}}{P_{\rm s}} = \frac{8\dot{\phi}^2\beta}{\left[(1-\alpha)H - \xi\phi\dot{\phi}\right]^2}, \tag{30}$$

4 Concrete applications

In this section, we apply a specific form of $H(\phi)$ to derive concrete results for an inflaton field with non-minimal coupling in both formulations.

4.1 Inflationary dynamics

We begin by examining the Hubble parameter expressed as a classical power-law function of the scalar field, represented by:

$$H(\phi) = \lambda\phi^n, \tag{31}$$

where λ and n are constants. Using the Hubble parameter (31), we can rewrite Eq. (18) as

$$\dot{\phi} = -\frac{2\lambda\phi^{n-1}(1-\alpha)(\alpha + n - n\alpha)}{\beta}. \tag{32}$$

Similarly, using Eq. (20), the potential can be derived as:

$$V(\phi) = \lambda^2(1-\alpha)\left[3\phi^{2n} - \frac{2\phi^{2n-2}(1-\alpha)(\alpha + n - n\alpha)^2}{\beta^2}\right.$$

$$+\frac{12\xi\phi^{2n}(\alpha+n-n\alpha)}{\beta}+\frac{12\sigma\xi^2\phi^{2n}(\alpha+n-n\alpha)^2}{\beta^2}\Bigg]. \tag{33}$$

Figure 1 displays the potential variations with the scalar field during inflation for various values of ξ and n in two formalisms. The results indicate that the potential steepens as $|\xi|$ or n increases.

The slow-roll parameters ϵ and η can be written as

$$\epsilon=\frac{2n(1-\alpha)(\alpha+n-n\alpha)}{\phi^2\beta} \tag{34}$$

and

$$\eta=\frac{2(n-1)(1-\alpha)(\alpha+n-n\alpha)}{\phi^2\beta}, \tag{35}$$

respectively. From Eqs. (34) and (35), it can be seen that ϵ will reach 1 before η. Therefore, the inflation process will end when $\epsilon=1$, at which point the scalar field takes the following value:

$$\phi_{\text{end}}=2\sqrt{\frac{n^2}{1+2n\xi(2n-1)+\sqrt{1-4n\xi\left[1+n\xi(12\sigma-13)\right]}}}. \tag{36}$$

These results are consistent with the minimal coupling case in Ref. [58] when $\xi=0$. Note that if we consider the strong coupling case $|\alpha|\gg 1$, the slow-roll parameter will be approximated as:

$$\epsilon\approx\frac{2n\xi\,(1-n)}{1-(1-\sigma)6\xi}. \tag{37}$$

In this scenario, the slow-roll parameter becomes constant, dependent only on ξ and n, preventing inflation from either occurring or ending the analysis is similar for η. Therefore, we exclude the strong coupling case from further consideration. The e-folding number, N_*, is given by:

$$N_*=\int_{\phi_{\text{end}}}^{\phi_*}\frac{\phi\beta}{2(1-\alpha)(\alpha+n-n\alpha)}d\phi. \tag{38}$$

4.2 Attractor behavior

To investigate the attractor behavior, we substitute the Hubble function (31) into the perturbative expression (26). Taking $\xi=-0.1$ and $n=1$ as an example, we can get the representation of perturbation in the metric formulation as:

$$\frac{\delta H}{\delta H_*}=\exp\left\{\left[\frac{5}{2}\ln\left(10+\phi^2\right)+\ln\left(125+50\phi^2+6\phi^4\right)\right]\Bigg|_{\phi_*}^{\phi}\right\} \tag{39}$$

From Eq. (32), we see that the inflaton field reduces as time progresses. The value of ϕ_*, determined from Eq. (38), is based on the chosen value of N_*. Consequently, the exponential term of Eq. (39) decreases with time and approaches zero rapidly. As a result, the perturbed part of the Hubble parameter vanishes, indicating attractor behavior of the model.

Figure 2 displays the results of the perturbation for various parameters in both the metric (Fig. 2a) and Palatini (Fig. 2b) formalisms. It is evident from the figure that the perturbation converges to a small value over time, demonstrating attractor behavior. However, larger values of $|\xi|$ lead to a slower convergence to zero. Additionally, in the Palatini formalism, the perturbation decreases more slowly with increasing $|\xi|$ compared to the metric formalism.

4.3 Inflationary predictions

In this subsection, we will conduct a numerical analysis of the inflationary predictions related to the inflaton field with non-minimal coupling. Based on the expressions in Eqs. (31) and (32), we can eliminate the parameter λ from Eqs. (29) and (30). Consequently, by providing the values of ϕ_* (or N_*), we can calculate n_s and r, given the values of ξ and n.

In Fig. 3, we present the predicted values of r and n_s for different parameter settings in the metric (Fig. 3a) and Palatini (Fig. 3b) formalisms. From Fig. 3a, it is clear that as $|\xi|$ increases, the corresponding value of r decreases for a fixed n. The predicted results converge when $|\xi|>1$, with this effect becoming more pronounced as n increases. However, as n increases, both r and n_s increase for a fixed ξ. All cases with $n=1$ or $\xi=-0.1$ are consistent with the BICEP/Keck data at the 68% CL. For the case of $n=0.98$ with $\xi=-1$, which is consistent with the BICEP/Keck data for large values of N_*, specifically when N_* approaches 70. Conversely, the cases of $n=1.02$ with $\xi=-1$ and $\xi=-10$ are in good agreement with the BICEP/Keck data when N_* is closer to 50. In contrast, the minimal coupling case[1] ($\xi=0$) and the case of $n=0.98$ with $\xi=-10$ are ruled out by the observations. These results suggest that the Hamilton–Jacobi method applied to a non-minimally coupled inflaton field model better fits observational data compared to the minimally coupled case.

However, for the Palatini formalism presented in Fig. 3b, the predicted values of r and n_s exhibit greater sensitivity to changes in $|\xi|$ and n compared to the metric case. In contrast to the metric formalism, the value of r tends to decline with increasing ξ in the Palatini formalism. All the n values we consider for $\xi=-0.01$ and $\xi=-0.1$ are consistent with the BICEP/Keck data at the 68% CL. While for $\xi=-0.5$, the cases of $n=0.99$ and $n=1.01$ are excluded by the

[1] Note that for $\xi=0$ we only choose the case of $n=1$, due to the fact that the deviation of 0.02 is negligible in the minimal case.

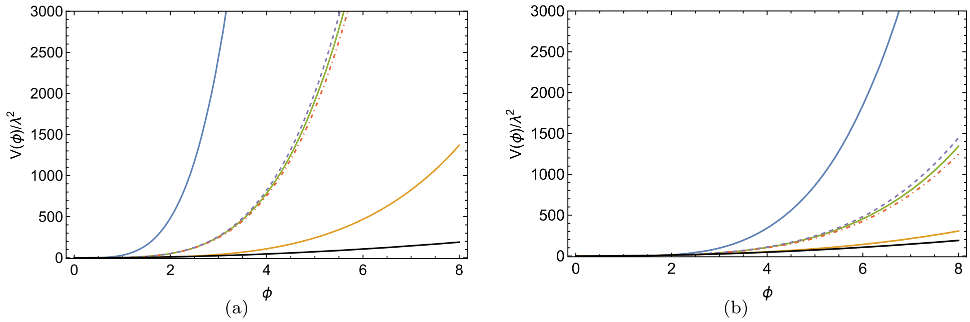

Fig. 1 Plot of the variation of $V(\phi)/\lambda^2$ with ϕ from Eq. (33) for different parameter values. **a** Metric case: $\xi = 0$, $n = 1$: (black, solid); $\xi = -0.1, n = 1$: (orange, solid); $\xi = -1, n = 0.98$: (red, dot-dashed); $\xi = -1$, $n = 1$: (green, solid); $\xi = -1$, $n = 1.02$: (purple, dashed); $\xi = -10$, $n = 1$: (blue, solid). **b** Palatini case: $\xi = 0$, $n = 1$: (black, solid); $\xi = -0.05$, $n = 1$: (orange, solid); $\xi = -0.1$, $n = 0.99$: (red, dot-dashed); $\xi = -0.1$, $n = 1$: (green, solid); $\xi = -0.1$, $n = 1.01$: (purple, dashed); $\xi = -0.5$, $n = 1$: (blue, solid)

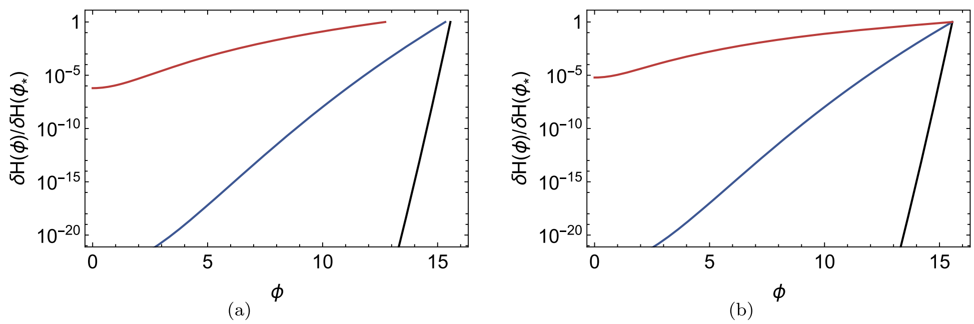

Fig. 2 Plot of the variation of $\delta H(\phi)/\delta H(\phi_*)$ with ϕ in the metric (**a**) and Palatini (**b**) cases for different parameter values: $\xi = 0$, $n = 1$ (black); $\xi = -0.01$, $n = 1$ (blue); $\xi = -0.1$, $n = 1$ (red). The value of ϕ_* is fixed by setting $N_* = 60$

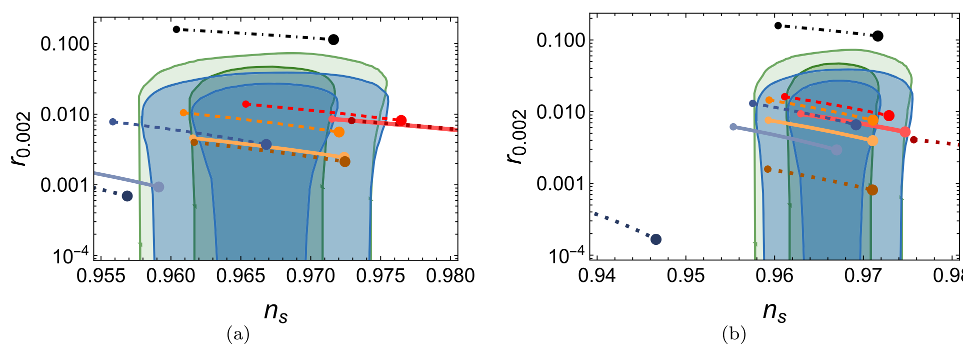

Fig. 3 The predicted values of r and n_s for the non-minimally coupled inflaton field with various N_* are shown. (a) Metric case: The dashed, solid, and dotted lines correspond to $\xi = -0.1$, -1, and -10, respectively. The blue, orange, and red lines correspond to $n = 0.98$, 1, and 1.02, respectively. (b) Palatini case: The dashed, solid, and dotted lines correspond to $\xi = -0.05$, -0.1, and -0.5, respectively. The blue, orange, and red lines correspond to $n = 0.99$, 1, and 1.01, respectively. The black dot-dashed line represents $n = 1$ with $\xi = 0$, and the small and large dots indicate $N_* = 50$ and 70, respectively. The green and blue shaded regions depict the constraints on n_s and r at the pivot scale $k_* = 0.002\ \mathrm{Mpc}^{-1}$ from the Planck 2018 CMB observations [12] and the BICEP/Keck survey [13], respectively. These regions represent the 68% and 95% CL contours, shown with dark and light shading, respectively

observations, with the exception of the case when $n = 1$. This indicates that values of n closer to 1 and smaller $|\xi|$ are required within the Palatini case as opposed to the metric case for better agreement with observational data.

5 Reheating

At the end of inflation, the universe transitions into the reheating phase, during which the oscillating inflaton field transfers its energy to other particles through decay and scattering processes [76–78]. During reheating, the decay of the inflaton and subsequent interactions are the pivotal steps in setting the stage for scenarios such as leptogenesis. In this section, we briefly analyze the perturbative decay of the inflaton into fermion within the non-minimal coupling inflation model.

The Lagrangian of the inflaton field couples to the fermion one in the Einstein frame is given by [79–81]:

$$\mathcal{L}_{\rm int}^E = -\frac{y}{\sqrt{f}}\phi(\hat{\phi})\bar{\hat{\psi}}\hat{\psi}, \quad (40)$$

where y is the Yukawa coupling constant in the Jordan frame and $\hat{\psi} \equiv \psi f^{-3/4}$ is the fermion field.

5.1 Ihe metric case

During reheating, we consider that the value of the inflaton field is much smaller than $1/\sqrt{|\xi|}$. Therefore, the interaction Lagrangian in the Einstein frame is given by [82]

$$\mathcal{L}_{\rm int}^E \approx -y_{\rm eff}\hat{\phi}\bar{\hat{\psi}}\hat{\psi}, \quad (41)$$

where $y_{\rm eff} = y\sqrt{\frac{\sqrt{2}}{\sqrt{3|\xi|\Phi}}}$ and Φ denotes the amplitude of the background inflaton field oscillations.

For perturbation theory to remain valid, $y_{\rm eff} < 1$ must hold. As $|\xi|$ increases, $y_{\rm eff}$ decreases, ensuring perturbative validity for $y < 1$. When $|\xi|$ is small, perturbation theory remains valid with a smaller y.

The decay width for the process $\hat{\phi} \to \bar{\hat{\psi}}\hat{\psi}$ in the Einstein frame is then given by:

$$\Gamma_{\hat{\phi}\to\bar{\hat{\psi}}\hat{\psi}} \simeq \frac{y_{\rm eff}^2}{8\pi}m_{\hat{\phi}}, \quad (42)$$

where $m_{\hat{\phi}} \equiv \hat{V}_{,\hat{\phi}\hat{\phi}}$ is the inflaton mass. Using the Friedmann equation and radiation energy density $\rho = (\pi^2 g_*/30)T_{\rm reh}^4$, we obtain the reheating temperature:

$$T_{\rm reh} \simeq 1.41\left(\frac{\Gamma_{\hat{\phi}\to\bar{\hat{\psi}}\hat{\psi}}}{g_*}\right)^{1/4}, \quad (43)$$

where $g_* = 106.5$ is the effective number of relativistic degrees of freedom at reheating. Taking the case of $n = 1$ as an example, we adopt $y = 10^{-3}$ and $\xi > -25$ (derived from relation (**??**)), and substitute these into Eq. (43) to obtain the range of reheating temperatures, $T_{\rm reh} > 3.4 \times 10^{10}$GeV.[2] Similarly, for $n = 1.02$, the reheating temperature is found to be $T_{\rm reh} > 6.3 \times 10^{10}$GeV. However, for $n = 0.98$, observational consistency (see Fig. 3a) requires $\xi > -1$ satisfying the observation (see Fig. 3a), leading to a corresponding reheating temperature range of $T_{\rm reh} > 3.2 \times 10^{11}$GeV. The derived reheating temperature satisfies both the Big Bang Nucleosynthesis (BBN) limit, $T_{\rm reh} > 4$MeV [83], and the electroweak scale limit, $T_{\rm reh} > 100$GeV.

5.2 The Palatini case

Similarly, for the Palatini case, we also consider the region: $\phi < 1/\sqrt{|\xi|}$. Consequently, we have $\hat{\phi} \approx \phi$, as derived from Eq. (9). The interaction Lagrangian in this case reduces to the standard form in the general relativity frame, i.e., $\mathcal{L}_{\rm int} = -y\phi\bar{\psi}\psi$. For perturbation theory to remain valid, it is sufficient to have $y < 1$. In our calculations, we choose $y = 10^{-3}$ for the cases $n = 0.99$, 1 and 1.01, which gives the corresponding reheating temperatures of $T_{\rm reh} = 5.19 \times 10^{11}$GeV, 5.17×10^{11}GeV and 5.15×10^{11}GeV, respectively.

In both the metric and Palatini formalisms, the calculated reheating temperatures are sufficiently high to support successful leptogenesis scenarios. These constraints on ξ are consistent with the inflationary predictions in our model.

6 Results

In this work, we investigated inflationary scenarios involving a non-minimally coupled inflaton field, characterized by a coupling function, $f(\phi) = 1 - \xi\phi^2$, using the Hamilton–Jacobi approach. We compared the behavior of this non-minimally coupled field under two different gravitational formalisms: metric and Palatini. By applying the Hamilton–Jacobi method, we reformulated the background equations in terms of the scalar field as a function of time, allowing for a natural analysis of the inflationary process based on a given form of the Hubble parameter. Using a power-law form of the Hubble parameter, $H(\phi) = \lambda\phi^n$, as an example, we found that strong coupling scenarios are unsuitable for inflation with non-minimal coupling, as inflation either fails to occur or ceases prematurely. In terms of attractor behavior, we observed that the linear perturbation of the Hubble parameter diminishes over time in both formalisms, satisfying the attractor condition. However, as the coupling constant $|\xi|$ increases, the perturbation decays more slowly over time.

[2] The model parameter λ is determined by the power spectrum amplitude of curvature perturbations, $\ln(10^{10}A_s) = 3.044 \pm 0.014$ [12], and by taking the e-folding number $N_* = 60$, we can approximately obtain the relation between λ and ξ.

Using the latest Planck 2018 and BICEP/Keck data, we compared the inflationary predictions for both the metric and Palatini cases. Our analysis shows that the inflationary predictions based on the Hamilton–Jacobi method are inconsistent with the observations for the minimal coupling case. However, when non-minimal coupling is introduced, the predictions fit well with the observations in both formalisms. Notably, the tensor to scalar ratio r is suppressed as $|\xi|$ increases in both formalisms, with convergence occurring for large $|\xi|$ ($|\xi| > 1$) in the metric formalism. Additionally, for fixed ξ, the tensor to scalar ratio r and the spectral index n_s increase as n increase in both formalisms. Furthermore, the predicted r and n_s are more sensitive to variations in ξ and n, leading to the requirement of values of n closer to 1 and smaller $|\xi|$ in the context of the Palatini formulation relative to the metric formulation, for better agreement with observational data.

Finally, we analyzed the decay of inflaton perturbations into fermion following the end of inflation. Within the parameter space allowed by our model, we demonstrated that this process could yield a sufficiently high reheating temperature to support successful leptogenesis. In summary, the Hamilton–Jacobi method effectively analyzes inflationary models involving inflaton fields with non-minimal coupling. Both the metric and Palatini formulations yield predictions that are consistent with recent observations, though notable differences in inflationary behavior exist between the two formalisms.

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