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A Mass Term for Three-Dimensional Gauge Fields

JONATHAN F. SCHONFELD
Laboratoire de Physique Theorique de l'École Normale Supérieure, Paris, France*

and

Fermi National Accelerator Laboratory, Batavia, Illinois 60510†

ABSTRACT

We propose the interaction

$$\mathscr{L}_\xi \equiv (\xi/2)\,\varepsilon_{\mu\nu\lambda}\,\mathrm{Tr}\,\underset{\sim}{A}^\mu\,[\,\partial^\nu A^\lambda - (2ig/3)A^\nu A^\lambda\,]$$

as a mass term for gauge fields in three-dimensional spacetime. The A^μ belong to a Lie algebra (represented here in terms of matrices), $\varepsilon_{\mu\nu\lambda}$ is the completely antisymmetric symbol, the coupling g has units $[\mathrm{mass}]^{1/2}$, and the parameter ξ has units [mass], $\mathscr{L}_\xi$, related to the instanton current of four dimensions, is gauge invariant up to a total divergence and a topological density. (There is a supersymmetric extension with the same property.) When technical complications can be ignored, $\mathscr{L}_\xi$ provides gauge particles with mass without breaking local symmetry and without introducing auxiliary fields. Perturbative analysis of models involving $\mathscr{L}_\xi$ (collectively called "ξ theories") is complicated by gauge-noninvariant infrared singularities in gauge-field propagators. Nevertheless, quantized abelian ξ-theories (collectively called "ξ QED") do define gauge invariant and infrared finite

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*Address for 1979-80 while on leave from School of Physics and Astronomy, University of Minnesota.

†Current address.

scattering in perturbation theory. The consistency of nonabelian ξ theories is not yet established. The physics of nonrelativistic charges in ξ QED is, in its gross features, the same as that of the Aharanov-Bohm effect--the static field of a point charge is a nontrivial pure gauge at large distances. (We argue that in spite of the long range fields, propagation of charges at large times is free; so that in ξ QED there should be no unexpected subtleties in the axiomatic definition of scattering amplitudes.) Compatibility of gauge invariance and mass in three dimensions is related to the existence of massive spinning representations of the Poincaré algebra with only one polarization per momentum. The massive spin-one photon of ξ QED is such a particle. (There is in fact a massive unitary representation of the three-dimensional Poincaré algebra with only one polarization for spin equal to <u>any</u> real number, integral multiple of one-half or otherwise. It is possible that particles with such an anomalous spin are present in some field theories.) $\mathscr{L}_{\xi}$ is invariant under the discrete transformations C and P but not under T, nor under any reflection P' in a single spatial axis. In 2 + 1 dimensions the analogue of the PCT theorem refers to P'CT.

The notion that model systems in lower-dimensional spacetimes could be useful in developing ideas for four-dimensional particle physics has been reinforced in the last few years, most recently by studies of the U(1) problem in two-dimensional CP^{n-1} models [1] and, earlier, by Polyakov's demonstration [2] that instantons give rise to confinement in the three-dimensional Georgi-Glashow model.[F1]

These analyses were essentially nonperturbative. Perturbative studies of gauge theories and sigma models in low-dimensional spacetimes are limited by infrared problems that are more severe than those encountered in four dimensions. With this in mind, we discuss in this paper a vector field self-coupling in three-dimensional spacetime that might be useful as a mass term for gauge theories. In the case of abelian gauge fields, we are able to show, in perturbation theory, that it is free of inconsistency. When such a coupling can be defined without inconsistency, it provides gauge particles with mass without (at least in perturbation theory) breaking local symmetry, and without introduction of auxiliary fields (in contrast with the Higgs [4] and Schwinger [5] mechanisms).

In Lagrangian form, the coupling is[F2]

$$\mathscr{L}_\xi \equiv (\xi/2)\,\epsilon_{\mu\nu\lambda}\,\mathrm{Tr}\,A^\mu[\,\partial^\nu A^\lambda - (2ig/3)A^\nu A^\lambda\,] \quad , \tag{1}$$

where the components of the gauge field A_μ are elements of a Lie algebra, represented here in terms of matrices. (We take elements of the gauge group to be exponentials of i times elements of this algebra.) The symbol $\epsilon_{\mu\nu\lambda}$ is completely antisymmetric, with $\epsilon_{012} \equiv 1$. The parameter ξ has units [mass]; the gauge coupling g has units $[\mathrm{mass}]^{1/2}$.

In what follows, we shall only consider total Lagrangians that can be expressed in terms of A_μ and the field strength $F_{\mu\nu} \equiv \partial_\mu A_\nu - \partial_\nu A_\mu - ig[A_\mu, A_\nu]$ as

$$\mathscr{L} = -\tfrac{1}{4}\mathrm{Tr}\, F_{\mu\nu}F^{\mu\nu} + \mathscr{L}_\xi + \begin{array}{l}\text{gauge-invariant matter terms}\\ \text{minimally coupled to } A_\mu\end{array} \quad . \tag{2}$$

We shall refer to any such model as a "ξ theory." A class of supersymmetric ξ theories is described in detail in appendix A. The quantized ξ theories with gauge group U(1) will be referred to generically as "ξ QED."

At the classical level, $\mathscr{L}_\xi$ displays local symmetry in the sense that when the vector field undergoes a gauge transformation, the change in (1) consists of a total divergence and a field-independent remainder. Specifically, when A_μ changes according to

$$A_\mu \to UA_\mu U^{-1} - \frac{i}{g}(\partial_\mu U)U^{-1} \quad , \tag{3}$$

then the change in $\mathscr{L}_\xi$ is

$$\Delta\mathscr{L}_\xi = \partial^\mu\Big[(-i\xi/2g)\,\varepsilon_{\mu\nu\lambda}\mathrm{Tr}\,(A^\nu U^{-1}\partial^\lambda U)\Big]$$

$$+ (\xi/6g^2)\,\varepsilon_{\mu\nu\lambda}\,\mathrm{Tr}\,(U^{-1}\partial^\mu U)(U^{-1}\partial^\nu U)(U^{-1}\partial^\lambda U) \quad . \tag{4}$$

The function U of spacetime takes values in the gauge group. The second term in $\Delta\mathscr{L}_\xi$ is proportional to the topological charge density [7] of U. The form (1) was suggested by the identity [7]

$$\mathrm{Tr}\, F_{\mu\nu}\tilde{F}^{\mu\nu} = 2\partial^\sigma \varepsilon_{\sigma\mu\nu\lambda}\ \mathrm{Tr}\, A^\mu [\partial^\nu A^\lambda - (2ig/3)A^\nu A^\lambda] \tag{5}$$

for the gauge-invariant instanton density in four dimensions. In a purely formal sense, ξ is a three-dimensional analogue of the four-dimensional vacuum angle θ.

The parameter ξ is to be interpreted as the mass of the gauge-invariant excitation of the vector field, to lowest order in g. To show this, we first define $B_\mu \equiv \varepsilon_{\mu\nu\lambda}F^{\nu\lambda}$. This definition is invertible for F: $F^{\mu\nu} = \tfrac{1}{2}\varepsilon^{\mu\nu\lambda}B_\lambda$. To lowest order in g, B_μ satisfies the identity

$$\partial_\mu B^\mu = 0 \quad . \tag{6}$$

Again to lowest order in g, the gauge field equations of motion corresponding to (2) can be written in terms of B as

$$0 = (\xi/2)\,\varepsilon^{\mu\nu\lambda}F_{\nu\lambda} + \partial_\nu F^{\nu\mu} = \xi B^\mu - \varepsilon^{\nu\lambda\mu}\,\partial_\nu B_\lambda \quad . \tag{7}$$

Equations (6) and (7) imply

$$\begin{aligned} 0 &= (\xi g_{\sigma\mu} + \varepsilon_{\sigma\rho\mu}\,\partial^\rho)(\xi B^\mu - \varepsilon^{\nu\lambda\mu}\,\partial_\nu B_\lambda) \\ &= (\xi^2 + \partial^\nu\partial_\nu)B_\sigma - \partial_\sigma(\partial^\lambda B_\lambda) = (\xi^2 + \Box)B_\sigma \quad . \end{aligned} \tag{8}$$

In other words, to lowest order in g the components of the field strength tensor propagate as free fields of mass ξ, which is what we wanted to show.

At the quantum-mechanical level, the interaction $\mathscr{L}_\xi$ presents two technical problems that we have not solved in complete generality:

First, there can, because of (4), be boundary and topological terms in the gauge transformation of the action. This could invalidate the usual functional proof [8] of the gauge invariance of physical cross sections. To be safe, one should prove gauge invariance in some other way.

Second, when the total Lagrangian has the form (2), the free gauge field propagator has gauge-invariant poles at mass ξ, but it also has other noninvariant singularities in the infrared. For example, the free photon propagator for ξ QED is

$$D^C_{\mu\nu} \equiv \frac{i}{P^2 - \xi^2}\left[-g_{\mu\nu} + ||\vec{P}||^{-2}\left(-P_\mu P_\nu + P_\mu P^o g_{\nu o} + P_\nu P^o g_{\mu\nu} - (P^2 - \xi^2) g_{\mu o} g_{\nu o} \right) - \left(\frac{i\xi}{||\vec{P}||^2} \right) \varepsilon_{\mu\nu j} P^j \right] \tag{9}$$

in Coulomb gauge, and

$$D^L_{\mu\nu} \equiv \frac{i}{P^2 - \xi^2}\left[-\left(g_{\mu\nu} - \frac{P_\mu P_\nu}{P^2} \right) + \left(\frac{i\xi}{P^2} \right) \varepsilon_{\mu\nu\lambda} P^\lambda \right] \tag{10}$$

in Landau gauge. To be sure that such models are self-consistent theories of massive particles, one should show that these infrared singularities do not lead to divergences in perturbative expressions for physical transition amplitudes.

In response to these problems, we can show that in ξ QED physical cross-sections in Landau and Coulomb gauges are identically equal and free of infrared divergence in every order of the perturbative expansion in powers of the electric charges. The arguments that we use in the abelian case do not apply to the general nonabelian theory.

Here is the proof of gauge invariance in ξ QED: Following arguments well-known in the context of four-dimensional QED [9], we assume that in Feynman graphs with external lines on shell, the photon propagator in any gauge is emitted and absorbed by conserved currents consisting exclusively of charged-particle lines.[F3] Thus, as usual, all terms in (9) and (10) proportional to P_μ or P_ν make no contribution to a scattering amplitude. The term $g_{\mu o}g_{\nu o}$ in (9) is cancelled as usual by the instantaneous Coulomb interaction. As for the antisymmetric terms: Call the emitting and absorbing currents $J_e{}^\mu$ and $J_a{}^\mu$. Because spacetime has only three dimensions, we can say that since the exchanged momentum P^μ is orthogonal to both of these currents, it is parallel to their cross-product. In symbols,

$$\varepsilon_{\mu\nu\lambda} J_e{}^\mu J_a{}^\nu = \beta P_\lambda \qquad , \qquad (11)$$

where β is some gauge-invariant function of the external and internal momenta characterizing the process in question. The equivalence of the contributions of the antisymmetric terms in (9) and (10) to cross sections follows directly from (11):

$$\begin{aligned} - \varepsilon_{\mu\nu j} P^j J_e{}^\mu J_a{}^\nu / \|\vec{P}\|^2 &= - \beta P^j P_j / \|\vec{P}\|^2 \\ &= \beta \\ &= \beta P_\lambda P^\lambda / P^2 = \varepsilon_{\mu\nu\lambda} P^\lambda J_e{}^\mu J_a{}^\nu / P^2 \qquad . \qquad (12) \end{aligned}$$

This completes the proof.

As for infrared finiteness: The familiar analysis of on-shell infrared divergences developed in the context of four-dimensional QED [11] is applicable to ξ QED. In physical amplitudes, infrared divergences (if they exist at all) show up only in Feynman graphs in which the momenta through two internal charged-particle line segments go on shell when the momentum through some internal photon line vanishes. Such a graph is actually singular only if at least one of the integrals

$$\int d^3k \frac{p_\mu q_\nu S^{\mu\nu}(k)}{(p\cdot k - i\eta_p\varepsilon)(q\cdot k - i\eta_q\varepsilon)} \tag{13}$$

receives a divergent contribution from the region near $k = 0$. The momenta p and q in (13) refer to any such pair of charged lines and are on the appropriate mass shells; the tensor $S^{\mu\nu}(k)$ is the infrared singular part of the free photon propagator. As usual, the positive number ε is to approach zero after one evaluates the integral; the signs η_p and η_q are determined by the directions--incoming or outgoing--of the external charged lines that feed p and q. Expression (13) is adapted from expression (2.10) of reference [11].

In ξ QED, using Landau gauge for convenience, the singular function $S^{\mu\nu}(k)$ is $\varepsilon^{\mu\nu\lambda} k_\lambda/(\xi k^2)$. Upon substitution into (13), naive power counting would have the integral diverge logarithmically in the infrared as $\varepsilon \to 0$; but because the numerator in $S^{\mu\nu}$ changes sign when k does, the divergence is washed out. This establishes infrared finiteness.

It should be emphasized that we have not shown that the infrared singular antisymmetric terms in $D^C_{\mu\nu}$ and $D^L_{\mu\nu}$ make no contribution to physical processes. Indeed, explicit calculation to lowest order in the fine structure constant shows that because of these terms the amplitude for elastic scattering of two charges is singular in the forward direction--in the center-of-mass frame, as the scattering angle ϕ approaches zero the amplitude approaches a constant times $(1/\phi)$. In terms of the Lorentz-invariant Mandelstam variable t, this is proportional to $1/\sqrt{-t}$, the sign of the square root changing with the sense--clockwise or counterclockwise--of deflection.[F4]

We remark in passing that in general the ultraviolet divergences of ξ theories cannot be regularized by the dimensional method [13] because the three-index antisymmetric symbol is specific to three-dimensional space-time. We do not know whether there are analogues of $\mathscr{L}_\xi$ in the form of lattice actions [14]. The Pauli-Villars [15] technique should be sufficient to regularize ultraviolet divergences of the abelian ξ-theories without breaking gauge invariance.

In any case the general ξ theory with only spinor matter, with no spinor self-couplings beyond mass terms, if it can be defined as a gauge-invariant system at all, should in fact be ultraviolet finite. According to naive power counting, the only primitively divergent graphs are the one-loop contributions to the spinor self-energies and the vector field three-point function (logarithmic divergences), and the one- and two-loop contributions to the vector self-energy (linear and logarithmic divergences). The logarithmic divergences in the spinor self-energies and the vector three-point function should be washed out by symmetric integration. The linear divergence in the vector self-energy is symmetric in the Lorentz indices and should go away when the symmetric transverse polynomial $P_\mu P_\nu - P^2 g_{\mu\nu}$ is extracted. The symmetric part of the logarithmic divergence should also go away upon extraction of $P_\mu P_\nu - P^2 g_{\mu\nu}$; the antisymmetric part should go away upon extraction of $\varepsilon_{\mu\nu\lambda} P^\lambda$.

To get a qualitative picture of the kind of physics described by the ξ theories, consider the stationary solution to the free abelian equations of motion in the presence of a static point source of electric charge e,

$$\partial_\nu F^{\nu\mu} + \frac{\xi}{2} \varepsilon^{\mu\nu\lambda} F_{\nu\lambda} + e g^{o\mu} \delta^2(\vec{x}) = 0 \qquad . \tag{14}$$

One could use (9) to solve (14) explicitly by evaluating the Fourier transform

$$A_\mu(x) = \frac{ie}{4\pi^2}\int (d^2P)\delta(P^o)D^c_{\mu o}(P) \quad , \tag{15}$$

but we can proceed to the main points more efficiently by looking directly at the $\mu = 0$ component of (14), integrated over all of two-dimensional space,

$$\int d^2x(-\vec{\nabla}\cdot\vec{E} + \xi B + e\delta^2(\vec{x})) = 0 \quad . \tag{16}$$

In (16), E_j is F^{oj} and B is F^{12}.

To simplify equation (16), observe that because of equation (8), the components of $\vec{E}$ decrease with distance r from the origin as some power of r times $e^{-\xi r}$ (for large r), so that $\int d^2x\vec{\nabla}\cdot\vec{E}$ vanishes. This reduces (16) to

$$\int Bd^2x = -e/\xi \quad . \tag{17}$$

So ξ QED is a theory of massive photons, and of charged particles whose static fields have finite range and contain magnetic as well as electric components. The magnetic field surrounding a charged particle has nonzero total flux $-e/\xi$; so that at large distances, by virtue of Stokes' theorem, the vector potential in Coulomb gauge is given in terms of the polar angle θ by

$$\tilde{A} \underset{r\to\infty}{\sim} \left(\frac{-e}{2\pi\xi}\right)\vec{\nabla}\theta \quad . \tag{18}$$

The right-hand side of (18) falls off like 1/r and is locally but not globally a pure gauge. Thus, on the largest scale, the physics of heavy nonrelativistic charges in ξ QED is the same as that of the Aharonov-Bohm effect [16].[F5]

One might wonder whether the presence of both magnetic and electric components signals nonzero angular momentum residing in the electromagnetic field of a point charge in ξ QED. The electromagnetic contribution to the Noether generator [9] of rotations in ξ QED, in a rotationally covariant gauge (so that rotations do not involve gauge transformations), is

$$M^{12}_{em} \equiv \int d^2x \left[(x^1 T^{02} - x^2 T^{01}) + \partial_\sigma (x^1 T^{\sigma 02} - x^2 T^{\sigma 01}) \right] \quad , \tag{19}$$

where[F6]

$$T^{\mu\nu} \equiv F^{\mu\sigma} F_\sigma{}^\nu + \frac{1}{4} g^{\mu\nu} F_{\rho\sigma} F^{\rho\sigma} \tag{20}$$

$$T^{\sigma\mu\nu} \equiv \left(\frac{\xi}{2} \varepsilon^{\mu\sigma\lambda} A_\lambda - F^{\mu\sigma} \right) A^\nu \quad . \tag{21}$$

The sum of the $T^{\mu\nu}$ terms in (19) reduces to $\int d^2x\, [-B(\vec{x}\cdot\vec{E})]$, the 2 + 1-dimensional analogue of the familiar $\int d^3x\, [\vec{x} \times (\vec{E} \times \vec{B})]$. This takes the value $e^2/4\pi\xi$ for the exact static solution $\{ B = -e\xi/2\pi\, K_0(\xi r),\ \vec{E} = -e/2\pi\, \vec{\nabla} K_0(\xi r) \}$ to equation (14), where K_0 is a Bessel function. However, in this case the total divergence term also contributes to M^{12}_{em}. Using the asymptotic behavior (18) we find the value $(-e^2/4\pi\xi)$ for the surface contribution to M^{12}.

Thus, in toto, the electromagnetic contribution to M^{12} for a point source is zero. This is in accord with general requirements of quantum field theory. Had the result for M^{12}_{em} been nonzero, we would have concluded that in ξ QED the spin of a particle depends continuously on its charge. But this is certainly not possible for particles created by applying fundamental matter fields to the vacuum, because in a rotationally covariant gauge the rotational behavior of matter fields is not affected by the coupling of gauge fields.

We stress, however, that outside the restrictions of perturbative quantum field theory there is nothing in the geometry of 2 + 1-dimensional spacetime that constrains the spins of particles to be integral multiples of ½. In appendix C we describe an explicit construction of an irreducible unitary representation of the Poincaré algebra in 2 + 1 dimensions in which the mass m and the rest-frame angular momentum $(1/m)\varepsilon_{\mu\nu\lambda} M^{\mu\nu}p^{\lambda}$ both take arbitrary values. Perhaps (with or without $\mathscr{L}_{\xi}$) some solitons provide examples of anomalous spin,[F7] either on account of quantum-mechanical effects, or already at the classical level.

We learn from the construction in appendix C that (among other things) in 2 + 1 dimensions there is only one polarization per momentum in a massive spinning irreducible representation of the Poincaré algebra. Since the number of transverse polarizations of a massless vector field in 2 + 1 dimensions is also one, this explains, roughly, why auxiliary degrees of freedom (as in the Higgs mechanism) are not necessary in giving mass to three-dimensional gauge mesons.

Indeed, per momentum there is precisely one real gauge-invariant normal mode of the equations of motion (7) of free ξ QED. For example, the unique real gauge-invariant mode with frequency $|\xi|$ and zero wavenumber is

$$B^{\mu} = A(0, \cos(|\xi|\, t - \delta), (\xi/|\xi|) \sin(|\xi|\, t - \delta)) \qquad , \qquad (22)$$

where t is time, and the amplitude A and phase δ are time-independent and arbitrary. Upon quantization, this mode corresponds to a state of mass $|\xi|$ at rest, with angular momentum $\xi/|\xi|$. There is also, by the way, precisely one real normal mode of the Dirac equation

$$\left(i\gamma^{\mu}\partial_{\mu} - m \right)\psi = 0 \qquad , \qquad (23)$$

when the Clifford algebra is represented in Majorana form by[F8]

$$\gamma_0 \equiv \sigma_2 \quad , \quad \gamma_1 \equiv i\sigma_1 \quad , \quad \gamma_2 \equiv i\sigma_3 \quad . \tag{24}$$

The unique mode with frequency $|m|$ and wavenumber zero corresponds, upon quantization of ψ as a Majorana field, to a state of mass $|m|$ at rest, with angular momentum $(-m/|m|)\cdot\frac{1}{2}$.

To see this in perspective, recall that in 3 + 1 dimensions a massive spinning particle must have more than one polarization because the spatial rotation group has more than one generator. In 2 + 1 dimensions the spatial group has only one generator; so once the rest-frame polarization has been chosen to be an eigenstate of this operator, there are no generators left with which to form other polarizations.

A related fact is that $\mathscr{L}_\xi$ is not invariant under time-reversal, T, nor under any reflection, P', in a single spatial axis. Either T or P' would change the angular momentum of the stationary massive photon in ξ QED from $\xi/|\xi|$ to $(-\xi/|\xi|)$, but we have seen that there is no gauge-invariant mode with zero wavenumber corresponding to spin $(-\xi/|\xi|)$. The same statements, mutatis mutandis, hold for the Majorana system (23)-(24).

We note in passing that both $\mathscr{L}_\xi$ and system (23)-(24) are invariant under C and P (parity inversion is equivalent to a proper rotation in two-dimensional space). The PCT theorem does not apply to 2 + 1 dimensions because the geometrical transformation PT is not a proper three-dimensional Euclidean rotation (its determinant is -1) [18]. The geometrical transformation P'T, by contrast, is proper, so there should be a P'CT theorem.[F9] Both $\mathscr{L}_\xi$ and the two-component Dirac system are invariant under P'CT.

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APPENDIX A: SUPERSYMMETRIC ξ THEORIES

Let the two functions ψ_a (a = 1,2) be objects of the form $\Sigma_j \lambda_j G_j$, where the functions λ_j of spacetime are real anticommuting numbers and the G_j are generators of the matrix Lie algebra to which the A_μ belong. (Assume the numbers λ_j commute with the entries of the G_j.) In this appendix we show that the Lagrangian density

$$\mathscr{L}_s = -\frac{1}{4}\mathrm{Tr}F_{\mu\nu}F^{\mu\nu} + \mathscr{L}_\xi + \frac{i}{2}\mathrm{Tr}\left[\overline{\psi}_a\gamma^\mu_{ab}(D_\mu\psi_b) - (D_\mu\overline{\psi}_a)\gamma^\mu_{ab}\psi_b\right] + \xi\mathrm{Tr}\,\overline{\psi}_a\psi_a \quad (A.1)$$

possesses a supersymmetry at the classical level. The gamma matrices in (A.1) are taken from (24); the notation $\overline{\psi}_a$ means $\psi_b\gamma^\nu_{ba}$; the action of the gauge-covariant derivative D_μ is given by

$$D_\mu\psi_a \equiv \partial_\mu\psi_a - ig\,[\,A_\mu,\,\psi_a\,] \quad . \qquad (A.2)$$

We begin the proof by cataloguing the basic definitions that we shall need:

The smallest supersymmetry algebra in 2 + 1 dimensions has two generators beyond those of the translation group, and can be represented as simple operators on functions ("superfields") of three real variables x^μ and two real anticommuting parameters θ_a. The explicit form of this representation is

$$Q_a \equiv \left(\frac{\partial}{\partial\theta_b} + i\bar{\theta}_c \partial_\mu \gamma^\mu{}_{cb}\right)\gamma^o_{ba} \quad , \tag{A.3}$$

$$P_\mu \equiv -i\partial_\mu \quad , \tag{A.4}$$

and the algebra is

$$\{Q_a, \bar{Q}_b\} = 2P_\mu \gamma^\mu \tag{A.5}$$

$$[Q_a, P_\mu] = [P_\nu, P_\mu] = 0 \quad . \tag{A.6}$$

The general infinitesimal supersymmetry transformation is

$$f \to f - i\alpha^\mu \partial_\mu f + (\bar{\beta}_a Q_a) f \quad , \tag{A.7}$$

where the α^μ and the β_a are respectively commuting and anticommuting real parameters.

The supercovariant derivatives, defined by

$$P_a \equiv \left(\frac{\partial}{\partial\bar{\theta}_b} - i\bar{\theta}_c \partial_\mu \gamma^\mu{}_{cb}\right)\gamma^o_{ba} \quad , \tag{A.8}$$

satisfy

$$\{P_a, Q_b\} = [P_a, P_\mu] = 0 \quad . \tag{A.9}$$

Thus, like f itself, every function of f, the $P_a f$ and the $P_\mu f$ transforms under supersymmetry according to (A.7).

The expansion of any function in powers of the θ_a always terminates at the quadratic term because in monomials of order three or higher at least one of the two θ_a will appear more than once, and the anticommutation rules force such a combination to vanish. For the same reason there is only one quadratic monomial, $\theta_1\theta_2 = i/2(\bar{\theta}\theta)$. Thus we may write a general function f as

$$f(x,\theta) = a(x) + i\bar{\theta}_a b_a(x) + \tfrac{1}{2}(\bar{\theta}\theta)c(x) \quad . \tag{A.10}$$

Under an infinitesimal transformation parametrized as in (A.7), the change in c is a total divergence: This is obvious in the case of the term $-i\alpha^\mu\partial_\mu f$. The c component of $(\bar{\beta}Q)f$ is also a divergence because of the two terms in the definition (A.3) of the Q_a, only the second (involving the spacetime derivatives) can give a term quadratic in the θ_a when acting on something truncated as in (A.10). (Similarly, the last component of $P_a f$ is also a total divergence for any a.)

It follows that the integral $\int c(x)d^3x$ is a supersymmetric invariant, at least for sufficiently localized configurations. In particular, c-components of functions ("super Lagrangians") of superfields and their supercovariant and ordinary derivatives make supersymmetric Lagrangian densities.

Now define the three superfields V_μ to belong to the Lie algebra under study and the two superfields V_a each to be of the form $\Sigma_j\lambda_j G_j$, just like the ψ_a. Define the action of a (super-) local gauge transformation $U(x,\theta)$ on the V_a and the V_μ by

$$V_a \rightarrow UV_aU^{-1} - \frac{i}{g}(P_aU)U^{-1} \quad , \tag{A.11}$$

$$V_\mu \rightarrow UV_\mu U^{-1} - \frac{i}{g}(\partial_\mu U)U^{-1} \quad . \tag{A.12}$$

From the point of view of local symmetry the vector superfield V_μ is to some extent redundant. To see this, observe that if a gauge transformation changes some multicomponent superfield f to Uf, then by definition of V_a it does the same thing to $(P_a - igV_a)f$ for each a, and therefore also to

$$(\bar{P}_a - ig\,\bar{V}_a)\gamma^\mu_{ab}(P_b - igV_b)f \;=\; (\bar{P}\gamma^\mu P - g^2\bar{V}\gamma^\mu V - ig\bar{P}\gamma^\mu V)f$$

$$= 2i\left(\partial^\mu + \frac{ig}{2}\,[g\bar{V}\gamma^\mu V + i\bar{P}\gamma^\mu V\,]\right)f \qquad , \qquad \text{(A.13)}$$

where $\bar{P}\gamma^\mu P$, $\bar{V}\gamma^\mu V$, $\bar{P}\gamma^\mu V$ mean $\bar{P}_a\gamma^\mu_{ab}P_b$, etc., and we have used the identity $\bar{P}\gamma^\mu P \equiv 2i\,\partial^\mu$. It follows that the combination $[\,-\tfrac{1}{2}(g\bar{V}\gamma^\mu V + i\bar{P}\gamma^\mu V)\,]$ gauge transforms exactly as V^μ; and in particular that the constraint

$$V^\mu + \tfrac{1}{2}(g\bar{V}\gamma^\mu V + i\bar{P}\gamma^\mu V) \;=\; 0 \qquad \text{(A.14)}$$

entails no conflict with local symmetry.

We are finally in a position to prove that (A.1) possesses a supersymmetry: The Lagrange function (A.1) turns out to be the c-component of a superLagrangian constructed from such V_a and V_μ, constrained by (A.14), and their ordinary and supercovariant derivatives. Specifically:

Define the field strength $F_{\mu a}$ by

$$F_{\mu a} \;\equiv\; \partial_\mu V_a - P_a V_\mu - ig[\,V_\mu, V_a] \qquad . \qquad \text{(A.15)}$$

A gauge transformation U changes $F_{\mu a}$ to $UF_{\mu a}U^{-1}$. The superLagrangian is then

$$L = \frac{1}{12}\left\{ - \mathrm{Tr}\, \bar{F}_{\mu a} F^{\mu}{}_{a} + i\xi \mathrm{Tr}\, \bar{V}_a \gamma^{\mu}_{ab}\left(F_{\mu b} + \frac{ig}{2}\left[V_{\mu}, V_b \right] \right) \right\} \quad . \tag{A.16}$$

The precise connection between (A.16) and (A.1) follows from the decomposition

$$V_a \equiv \chi_a(x) + S(x)\theta_a + (A_{\mu}(x)\gamma^{\mu}_{ab})\theta_b + \sqrt{2}(\bar{\theta}\theta)\psi_a(x) \quad , \tag{A.17}$$

and the observation that the coefficients S and χ_a can be made to vanish by an appropriate gauge transformation.

For completeness, we write out the full change in L due to a gauge transformation U:

$$\Delta L = \frac{i\xi}{12}\left\{ \partial_{\mu} \mathrm{Tr}\, [U^{-1}(\bar{P}_a U)\gamma^{\mu}_{ab} V_b] + \bar{P}_a \gamma^{\mu}_{ab} \mathrm{Tr}\, [U^{-1}(P_b U)V_{\mu} - U^{-1}(\partial_{\mu} U)V_b] \right.$$

$$\left. + \frac{1}{g^2} \mathrm{Tr}\left([U^{-1}(\bar{P}_a U)\gamma^{\mu}_{ab} U^{-1}(P_b U)] U^{-1}\partial_{\mu} U \right) \right\} \quad . \tag{A.18}$$

We do not know at present if the ξ term in L is related to some four-dimensional supersymmetric construct (consult [22] for examples) in any way that might be analogous to the connection (5) between $\mathscr{L}_{\xi}$ and the four-dimensional instanton density.

APPENDIX B: WAVE PROPAGATION AT LARGE TIME IN THE AHARONOV-BOHM SYSTEM

The result whose derivation is sketched here is introduced for application to the question raised in the footnote referred to after equation (18).

The Aharonov-Bohm [16] in-wavefunction of momentum $\vec{k}$ is given almost everywhere at large distance, in Coulomb gauge, by

$$\psi^{in}_{\vec{k}}(\vec{x}) \underset{r\to\infty}{\sim} \left[e^{i\vec{k}\cdot\vec{x}}\right]\left[e^{i\rho\theta_{\vec{k}}(\hat{x})}\right] \quad . \tag{B.1}$$

The extra phase in (B.1) reflects the pure gauge behavior of the vector potential at large r. The parameter ρ is the product of the charge of the scattered particle and the flux of the source, divided by 2π. The symbol $\theta_{\vec{k}}(\hat{x})$ refers to the polar angle of $\vec{x}$, normalized by $\theta_{\vec{k}}(-\hat{k}) \equiv 0$. Equation (B.1) is valid as long as $\vec{x}$ and $\vec{k}$ do not point in the same direction. (The approach to the asymptotic form (B.1) becomes increasingly slow as the angle between $\vec{x}$ and $\vec{k}$ tends to zero.)

We shall argue that for all normalizable test functions f, the state

$$\psi(\vec{x}, t) \equiv \int \psi^{in}_{\vec{k}}(\vec{x}) f(\vec{k}) \left[e^{-ik^2 t/2m}\right] d^2k \quad , \tag{B.2}$$

evolving with time t under the influence of the Aharonov-Bohm vector potential, approaches for large negative time the state

$$\tilde{\psi}(\vec{x}, t) \equiv \int \left[e^{i\vec{k}\cdot\vec{x}}\right] f(\vec{k}) \left[e^{-k^2 t/2m}\right] d^2k \quad , \tag{B.3}$$

propagating according to the laws of free Schrödinger dynamics. "m" is the mass of the scattered particle. (A similar argument can be formulated for out states and large positive times.)

We shall rely on the "scattering into cones" formula [19] which, when applied to the definition (B.3), leads to

$$\tilde{\psi}(\vec{x}, t) \underset{t \to -\infty}{\sim} \left(\frac{\pi}{it}\right) \exp\left(\frac{ir^2 m}{4t}\right) f\left(\frac{\vec{x}}{2t}\right) \quad . \tag{B.4}$$

We shall also have to assume that

$$\int \psi^{in}_{\vec{k}}(\vec{x}) \psi^{in}_{\vec{k}'}(\vec{x})^* d^2x = (2\pi)^2 \delta^2(\vec{k} - \vec{k}') \quad . \tag{B.5}$$

Equation (B.5) seems reasonable, but was not actually proved in [16].

To proceed: Let us assume that the test function $f(\vec{k})$ is nonzero only in some wedge with opening angle less than π, so that we need not be concerned in what follows with the discontinuity of $\theta_{\vec{k}}(\hat{x})$ at $\hat{x} = \hat{k}$. Any general f can always be expressed as a sum of such wedge functions.

According to (B.1), outside the wedge, at large r (that is, r greater than some large reference radius), (B.2) can be replaced by

$$\begin{aligned} \psi(\vec{x}, t) &\sim \int \left[e^{i\vec{k}\cdot\vec{x}}\right] \left[e^{i\rho\theta_{\vec{k}}(\hat{x})}\right] f(\vec{k}) \left[e^{-ik^2t/2m}\right] d^2k \\ &= \left[e^{i\rho\theta_{\vec{k}_o}(\hat{x})}\right] \int \left[e^{i\vec{k}\cdot\vec{x}}\right]\left[e^{-i\rho\theta_{\vec{k}_o}(\hat{k})}\right] f(\vec{k}) \left[e^{-ik^2t/2m}\right] d^2k \quad , \end{aligned} \tag{B.6}$$

where $\vec{k}_o$ is some reference vector inside the wedge. Immediate application of the scattering-into-cones formula gives

$$\psi(\vec{x}, t) \underset{\substack{r \text{ large} \\ x \text{ outside wedge} \\ t \to -\infty}}{\sim} \left(\frac{\pi}{it}\right) \exp\left(\frac{ir^2m}{4t}\right) f\left(\frac{\vec{x}}{2t}\right) \quad . \tag{B.7}$$

If it were not for the restriction to large r and to $\vec{x}$ outside the wedge, this would be the same as (B.4) and the proof would be accomplished.

But notice two things: First, for large negative time the probability in the right-hand side of (B.7) is automatically concentrated outside the wedge and far from the origin. Thus

$$\int_{\substack{r\ \text{large} \\ \vec{x}\ \text{outside wedge}}} |\psi(x, t)|^2 d^2x \underset{t \to -\infty}{\sim} \int_{\text{all } x} \left(\frac{\pi}{t}\right)^2 \left| f\left(\frac{\vec{x}}{2t}\right)\right|^2 d^2x$$

$$= 4\pi^2 \int |f(\vec{k})|^2 d^2k \quad . \tag{B.8}$$

Second, according to (B.2) and (B.5) we have

$$\int_{\text{all } \vec{x}} |\psi(\vec{x}, t)|^2 d^2x = 4\pi^2 \int |f(\vec{k})|^2 d^2k \tag{B.9}$$

for all time. Putting (B.8) and (B.9) together we learn that for infinitely large negative time, $\psi(\vec{x}, t)$ assigns no probability at all to $\vec{x}$ not obeying the restrictions indicated in (B.7). Thus the restrictions in (B.7) can actually be deleted and the argument is complete.

APPENDIX C: UNITARY REPRESENTATIONS OF THE 2 + 1-DIMENSIONAL POINCARÉ ALGEBRA WITH ANY SPIN

Our representation space will be the set of all square-normalizable complex-valued functions on the positive-energy mass hyperboloid $p_0{}^2 - p_1{}^2 - p_2{}^2 - m^2$, $p_0 > 0$, in three-dimensional momentum space. We seek three differential operators J_μ satisfying the algebra

$$[\, J_\mu , J_\nu] = -i\varepsilon_{\mu\nu\lambda} J^\lambda \quad , \qquad \text{(C.1)}$$

$$[\, J_\mu , P_\nu] = -i\varepsilon_{\mu\nu\lambda} P^\lambda \quad , \qquad \text{(C.2)}$$

and the constraint

$$P_\mu J^\mu = mS \quad , \qquad \text{(C.3)}$$

where the intrinsic spin S is an arbitrary real number. The other generators are related to the J_μ by $M_{\mu\nu} \equiv \varepsilon_{\mu\nu\lambda} J^\lambda$. The momentum operator P_μ is represented by multiplication with the momentum coordinate p_μ.

Equations (C.1-3) for J_μ and P_μ resemble the constraints satisfied by the angular momentum and position operators of a nonrelativistic quantum-mechanical electron in the field of a fixed magnetic monopole in three-dimensional space [20]. Thus, by analogy, we are led to the <u>ansatz</u>

$$J_\mu \equiv i\varepsilon_{\mu\nu\lambda} p^\nu (\partial^\lambda - ia^\lambda) + \left(\frac{s}{m}\right) p_\mu / \sqrt{p^2} \quad , \qquad \text{(C.4)}$$

where the a_μ, functions of momentum, satisfy

$$\partial^{\mu} a^{\nu} - \partial^{\nu} a^{\mu} = \left(\frac{s}{2} \right) \varepsilon^{\mu\nu\sigma} p_{\sigma} / (p^2)^{3/2} \quad . \tag{C.5}$$

We can solve equation (C.5) by adapting one of the published solutions for the vector potential of the monopole [21]. Our result is

$$a^{\mu} = \left(\frac{s}{2m\sqrt{p^2}} \right) \frac{\varepsilon^{\mu\rho\lambda} n_{\rho} p_{\lambda}}{p \cdot n + \sqrt{p^2}} \quad , \tag{C.6}$$

where n is an arbitrary but fixed three-vector with unit Lorentz norm.

In the case of the monopole, the singularity corresponding to $\sqrt{p^2} = -p \cdot n$ is a problem and leads directly to the quantization of electric charge. Here we can choose the singular line to miss the positive-energy mass hyperboloid altogether (for example, $n_{\mu} \equiv (1,0,0)$), so that nothing forces quantization of spin.

By contrast, one should still expect quantization of the spins of <u>fields</u> with finitely many components. Fields are defined throughout spacetime so that a singularity like that in (C.6) cannot be made inaccessible.

FOOTNOTES

F1 In addition, there is a direct connection between three-dimensional Euclidean quantum field theory and the high-temperature limit of four-dimensional real-time quantum field theory. See [3].

F2 Couplings involving the completely antisymmetric symbols in spacetimes of other dimensionalities are known to occur naturally in the context of dual and supergravity models. See [6]. We note that with the definition of the trace extended from matrix operators on finite-dimensional vectors to more general linear operators on functions of spacetime coordinates and internal indices, a purely formal manipulation gives

$$\int d^3x\, \mathscr{L}_\xi = -(\xi/3g^2)\mathrm{Tr}\ \epsilon_{\mu\nu\lambda}(D^\mu D^\nu D^\lambda) \qquad ,$$

where D_μ is $\partial_\mu - igA_\mu$, the gauge-covariant derivative (R. Pearson, private communication). Since D_μ becomes $U^{-1}(x)D_\mu U(x)$ under gauge transformation, the expression on the right-hand side is manifestly gauge invariant, while the left-hand side is definitely sensitive to topological characteristics of $U(x)$. Thus, this formal equality has its limitations. Perhaps, because of manifest gauge-invariance the right-hand side represents a better definition of the ξ-interaction. This would permit application of the usual functional methods [8] to prove gauge invariance of nonabelian ξ-theories, which we are unable to show otherwise. Nevertheless, we are hesitant to use this definition as a basis for our analysis because of mathematical complications overlooked in the naive introduction of the functional trace, and because it is not clear that such a structure can be quantized canonically.

F3 Actually, this assumption is naive. It breaks down because of Feynman diagrams in which a self-energy subgraph is attached to one end of an open charged line (Fig. 1). The contribution of such a diagram to the current corresponding to this line cannot be defined when the external particles are on shell because then the propagator associated with the internal segment that connects the self-energy insertion to the rest of the diagram is singular. This problem was recognized and resolved for QED in 3 + 1 dimensions by Bialynicki-Birula [10]. Superficially, it would seem that Bialynicki-Birula's arguments cannot be adapted to ξ QED because they assume that the difference between $D^C_{\mu\nu}$ and $D^L_{\mu\nu}$ is of the form $p_\mu g_\nu(p) + g_\mu(p)p_\nu$ (once the instantaneous Coulomb interaction has been cancelled), while the antisymmetric terms in (9) and (10) appear not to conform to this pattern. However, because of (11) and (12), even with the ξ QED antisymmetric terms we still have $J_e{}^\mu(D^C_{\mu\nu} - D^L_{\mu\nu})J_a{}^\mu = 0$ for all J_e and J_a orthogonal to p (after cancellation of the instantaneous term), and this implies that the difference between $D^C_{\mu\nu}$ and $D^L_{\mu\nu}$ in ξ QCD is of the form $p_\mu h_\nu(p) + k_\mu(p)p_\nu$. So actually ξ QED differs from the situation discussed in [10] only in that h_μ and k_ν are not equal (in fact they are complex conjugate). It seems to us that the methods in [10] are flexible enough to accommodate this slightly more general situation.

F4 Here is how to interpret this singularity, at least when one of the charges is infinitely heavy and the other is nonrelativistic: If the light charge approaches the heavy one with momentum-space wavefunction $\psi_{in}(\vec{p})$ in the distant past, then its state in the far future has wavefunction

$$\psi_{out}(p, \phi) = \psi_{in}(p, \phi) - 2\pi i m \lim_{\delta \to 0^+}\left[\int_{\phi-\pi}^{\phi-\delta} + \int_{\phi+\delta}^{\phi+\pi}\right] d\phi_1 t(p, \phi - \phi_1)\psi_{in}(p, \phi_1) \quad ,$$

where p and ϕ are the polar coordinates of the two-dimensional momentum vector $\vec{p}$, m is the mass of the light charge, and $t(p,\phi)$ is the nonrelativistic scattering amplitude, behaving like $1/\phi$ for small ϕ. Thus, when applied to wavepackets, the singularity in t does not lead to singular asymptotic time evolution. The principal value rule above follows from the general formula,

$$<g|S|f> = <g|f> - \frac{i}{4\pi^2} \lim_{\varepsilon\to 0^+} \int d^2p\, d^2q \; \frac{2\varepsilon}{\varepsilon^2 + (p^2 - q^2)^2/4m^2} \; g^*(\vec{p}) <\vec{p}|V|\vec{q}> f(\vec{q}) \; ,$$

adapted from chapter 8 of ref. [12], for the Born approximation to the amplitude for a particle of mass m to scatter from state $|f>$ to state $|g>$ under the influence of the perturbation V. In the case at hand, V corresponds to $eA_0 - \frac{ie}{2m}[(\partial_j A_j) + 2(A_j \partial_j)]$ with $A_\mu(\vec{x})$ given by (15). ("e" here means the charge of the light particle, "e" in (15) is the charge of the heavy one.) When the Fourier transform $<\vec{p}|V|\vec{q}>$ is nonsingular, one can take ε to zero before doing the integral by replacing the function in square brackets by $2\pi\delta(\frac{p^2}{2m} - \frac{q^2}{2m})$. In the present case the matrix element $<\vec{p}|V|\vec{q}>$ is singular at $\vec{p} = \vec{q}$ and one must proceed more cautiously. Can the idea of the Mandelstam representation be generalized so that, in models such as ξ QED, it can accommodate both this kind of singularity and the absence of massless particles?

F5 The presence of a pure gauge at infinity raises a question of principle: At large times before and after the interaction of two charges, do their wavefunctions evolve freely, or must some kind of compensating gauge transformation be applied before propagation can be regarded as free? There is a mathematical literature on nonrelativistic Schrödinger scattering past a fixed and localized source of magnetic field (see [17] and references therein); but as far as we know the only vector potentials for which this question has been analyzed rigorously fall off too rapidly at large distances to support (according to Stokes' theorem) nonzero total flux in two spatial dimensions. Aharonov and Bohm [16] produced exact expressions for the stationary scattering states corresponding to an impenetrable zero-width source of nonzero flux; in appendix B we show (up to a technical assumption) how their expressions imply that propagation at large times is free in their system. This is our plausibility argument for the parallel statement in ξ QED. If this fails, our results on scattering amplitudes would be called into question because they were obtained without taking this complication into account. One might wonder whether a non-null Aharonov-Bohm effect could spell some kind of inconsistency for ξ QED. Following (18), this would lead to the quantization rule $(e_1 e_2/\xi) = 2\pi n_{12}$ (plus possible quantum corrections) for all charges e_1 and e_2, where the n_{12} are integers. We have so far encountered no evidence for such a phenomenon.

F6 The Noether energy-momentum current is given in terms of $T^{\mu\nu}$ and $T^{\sigma\mu\nu}$ by $\mathscr{T}^{\mu\nu} = T^{\mu\nu} + \partial_\sigma T^{\sigma\mu\nu}$.

F7 This possibility is being considered by E. Witten (private communication).

F8 These matrices satisfy tr $\gamma_\mu \gamma_\nu \gamma_\lambda = 2i\varepsilon_{\mu\nu\lambda}$ (in four dimensions, zero would be the only possibility). Because of this, when there are charged two-component spinor fields, the lowest-order vacuum polarization can have an antisymmetric piece even for $\xi = 0$.

F9 This was suggested to us by E. Witten.

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FIGURE CAPTION

Fig. 1: Example of a Feynman diagram in ξ QED with self-energy insertion at an end of an open charged-particle line.

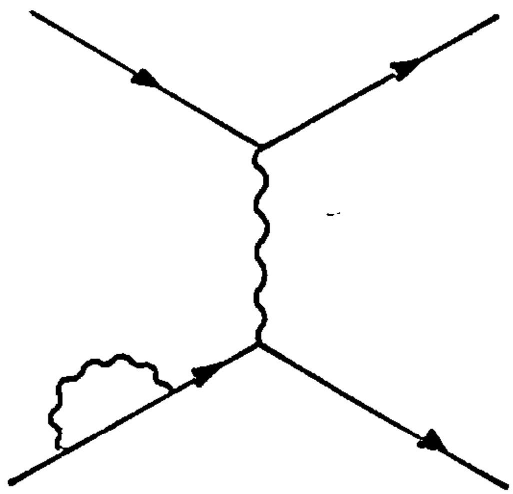

Fig. 1