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GENERALIZATION OF THE GROSS-PITAEVSKII EQUATION FOR BOSE GAS IN THE PRESENCE OF QUASI-PARTICLES

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The Gross-Pitaevskii equation has been generalized for the case of presence of the Bogolyubov quasi-particles at hydrodynamic stage of their evolution. Hydrodynamic equations for quasi-particle subsystem are constructed by the Chapman-Enskog method. The obtained equations can be considered as a new form of hydrodynamic equations of superfluid Bose gas. In the Landau-Khalatnikov hydrodynamics the total density of the system is used instead of amplitude of the condensate wave function.

1 Introduction

The problem of justification and generalization of the Gross-Pitaevskii equation is widely discussed in the literature (see, for example, [1, 2]). In the present paper it is studied on the basis of the Bogolyubov reduced description method which allows investigating domain of applicability of the theory and building corrections to it. Key issue of our consideration is definition of condensate wave function $\psi(x,t)$ as an average value of the Bose field operator $\psi(x)$ with statistical operator of the system $\rho(t)$

$$\psi(x,t) = \mathrm{Sp}\rho(t)\psi(x) = \eta(x,t)e^{i\varphi(x,t)} \qquad (\eta(x,t) \geq 0, \quad 0 \leq \varphi(x,t) < 2\pi). \tag{1}$$

This definition leaves aside the questions: is the condensate as a subsystem of a Bose gas in a pure state and is this state stable. However, this definition is a fruitful basis for various generalizations of the Gross-Pitaevskii equation. Hamilton operator of the system takes into account short range repulsive interaction between particles $\Phi(r)$. Our consideration is based on set of equations obtained in paper [3] in which nonequilibrium states Bose gas in the presence of condensate are described by amplitude $\eta(x,t)$ of the condensate wave function, local velocity of the condensate $\upsilon_n(x,t) = m^{-1}\partial\varphi(x,t)/\partial x_n$ (m is mass a particle; in this paper $\hbar = 1$), the Wigner distribution function for the Bogolyubov quasi-particles $f_p(x,t)$ in reference system of the condensate rest. Gradients of parameters $\xi_\mu(x,t)$: $f_p(x,t)$, $\eta(x,t)$, $\upsilon_n(x,t)$ are assumed to be small (small parameter g) and therefore states of the system are weak non-uniform. Interaction between particles $\Phi(r)$ is also considered as small one. For summation of contributions of the perturbation theory, which leads to the Bogolyubov quasi-particles spectrum in the leading approximation, estimations $\Phi(r) \sim \lambda^2$, $\eta \sim \lambda^{-1}$ are used (small parameter λ). Equations of the mentioned paper one can consider as a generalization of Gross-Pitaevskii equation for case of presence of quasi-particles at kinetic stage of their evolution. The purpose of the present work is a generalization of the Gross-Pitaevskii equation for case of presence of quasi-particles at hydrodynamic stage of their evolution.

2 Basic equations of the theory

Set of equations obtained in paper [3] has structure

$$\frac{\partial\eta}{\partial t} = -\upsilon_n\frac{\partial\eta}{\partial x_n} - \frac{\eta}{2}\frac{\partial\upsilon_n}{\partial x_n} + L \equiv L_\eta, \qquad \frac{\partial\varphi}{\partial t} = -m\left(\frac{\upsilon^2}{2} + h\right),$$

$$\frac{\partial f_p}{\partial t} = -\frac{\partial f_p}{\partial x_l}\frac{\partial(\varepsilon_p + p_n\upsilon_n)}{\partial p_l} + \frac{\partial f_p}{\partial p_l}\frac{\partial(\varepsilon_p + p_n\upsilon_n)}{\partial x_l} + L_p, \qquad \frac{\partial\upsilon_n}{\partial t} = -\upsilon_l\frac{\partial\upsilon_l}{\partial x_n} - \frac{\partial h}{\partial x_n} \equiv L_{\upsilon_n} \tag{2}$$

where ε_p is spectrum of the Bogolyubov quasi-particles

$$\varepsilon_p = (\alpha_p^2 - \beta_p^2)^{1/2}, \qquad \alpha_p \equiv \varepsilon_p^o + \beta_p, \qquad \varepsilon_p^o \equiv \frac{p^2}{2m}, \qquad \beta_p \equiv \nu_p\eta^2, \qquad \nu_p \equiv \int d^3x\Phi(|x|)e^{ipx}. \tag{3}$$

Right hand side part of this equations were calculated in [3] with accuracy shown below

$$h = h_0 + h^{(0,2)} + O(g^0\lambda^4, g^1\lambda^2, g^2\lambda^1), \qquad h_0 \equiv -\frac{\Delta\eta}{2m^2\eta} + \frac{1}{m}\int d^3x'\Phi\left(|x'|\right)\eta\left(x+x'\right)^2;$$

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$$L = L^{(0,3)} + L^{(1,1)} + O(g^0\lambda^5, g^1\lambda^2, g^2\lambda^1); \qquad L_p = L_p^{(0,2)} + L_p^{(0,4)} + O(g^0\lambda^5, g^1\lambda^2, g^2\lambda^1), \tag{4}$$

where $L^{(0,3)}, L^{(1,1)}, h^{(0,2)}, L_p^{(0,2)}, L_p^{(0,4)}$ are known functions of parameters $\xi_\mu(x)$, which are given by bulky expressions and omitted here ($A^{(m,n)}$ is contribution to A of the order $g^m\lambda^n$; $L_p^{(0,2)}, L_p^{(0,4)}$ are collision integrals for quasi-particles). The forth equation in (2) is a consequence of the second one. The second equation was given above because it with the first one are equivalent to the Gross-Pitaevskii equation if we restrict ourselves by approximation

$$L = 0, \qquad h = h_0^{(0,0)} + h_0^{(2,0)} \tag{5}$$

The problem of this paper is to construct hydrodynamic equations for quasi-particle subsystem of the Bose gas interacting with the condensate subsystem. The hydrodynamic equations are a consequence of energy and momentum conservation laws for quasi-particles and in the local reference system of the condensate rest according to the third equation in (2) have the form

$$\frac{\partial\varepsilon^o}{\partial t} = -\frac{\partial q_n^o}{\partial x_n} - \frac{\partial\varepsilon^o\upsilon_n}{\partial x_n} - t_{ln}^o\frac{\partial\upsilon_l}{\partial x_n} - a\left(L - \frac{\eta}{2}\frac{\partial\upsilon_n}{\partial x_n}\right) \equiv L_0,$$

$$\frac{\partial\pi_l^o}{\partial t} = -\frac{\partial t_{ln}^o}{\partial x_n} - \frac{\partial\pi_l^o\upsilon_n}{\partial x_n} - \pi_n^o\frac{\partial\upsilon_n}{\partial x_l} + a\frac{\partial\eta}{\partial x_l} \equiv L_l. \tag{6}$$

Here ε^o, π_n^o are densities of energy and momentum, q_n^o, t_{ln}^o are flux densities of energy and momentum in the mentioned reference system, a is thermodynamic force which condensate acts on quasi-particle subsystem with. These values are given by formulas

$$\varepsilon^o = \int d\tau_p\varepsilon_p f_p, \qquad \pi_l^o = \int d\tau_p p_l f_p, \qquad q_n^o = \int d\tau_p\varepsilon_p\frac{\partial\varepsilon_p}{\partial p_n}f_p, \qquad t_{ln}^o = \int d\tau_p p_l\frac{\partial\varepsilon_p}{\partial p_n}f_p,$$

$$a = -\int d\tau_p\frac{\partial\varepsilon_p}{\partial\eta}f_p \qquad (d\tau_p \equiv \frac{d^3p}{(2\pi)^3}). \tag{7}$$

In order to close equations (6) we use the Bogolyubov functional hypothesis on which the Chapman-Enskog method is based [4]. According it distribution function of quasi-particles $f_p(x,t)$ at times of their hydrodynamic evolution ($t \gg \tau_o$) is a functional of parameters $\zeta_\alpha(x,t) : \varepsilon^o(x,t), \pi_n^o(x,t), \eta(x,t), \upsilon_n(x,t)$

$$f_p(x,t)\xrightarrow[t\gg\tau_0]{} f_p(x,\zeta(t)). \tag{8}$$

In hydrodynamics of quasi-particles instead of densities $\varepsilon^o(x,t)$,$\pi_n^o(x,t)$ one can use their local temperature $T(x,t)$ and drift velocity $\omega_n(x,t)$ in the reference system of the condensate rest which are defined by relations

$$\varepsilon^o(x) = \int d\tau_p\varepsilon_p(\eta(x))f_p(x,\zeta) = \int d\tau_p\varepsilon_p(\eta(x))n_p(\chi(x)),$$

$$\pi_n^o(x) = \int d\tau_p p_n f_p(x,\zeta) = \int d\tau_p p_n n_p(\chi(x)). \tag{9}$$

Here

$$n_p(\chi) = [e^{\frac{\varepsilon_p(\eta) - p_n\omega_n}{T}} - 1]^{-1} \tag{10}$$

is the Planck distribution (in this paper $k_B = 1$); $\omega_n(x,t) \equiv u_n(x,t) - \upsilon_n(x,t)$; $u_n(x,t)$ is local velocity of quasi-particle subsystem in the lab reference system; $\chi_\alpha(x)$ denotes parameters: $T(x), \omega_n(x), \eta(x), \upsilon_n(x)$. According to (2), (8) functional $f_p(x,\zeta)$ satisfies equation

$$\sum_\alpha\int d^3x'\frac{\delta f_p}{\delta\zeta_\alpha(x')}L_\alpha(x') = -\frac{\partial f_p}{\partial x_l}\frac{\partial(\varepsilon_p + p_n\upsilon_n)}{\partial p_l} + \frac{\partial f_p}{\partial p_l}\frac{\partial(\varepsilon_p + p_n\upsilon_n)}{\partial x_l} + L_p \tag{11}$$

($L_\alpha : L_0, L_n, L_\eta, L_{\upsilon_n}$) with additional conditions (9).

3 Hydrodynamic equations for quasi-particles

Distribution function of quasi-particles $f_p(x,\zeta)$ at the reduced description is calculated from equation (11) in perturbation theory in g and λ taking into account relations (9). As a result it has been obtained in the form

$$f_p = n_p + f_p^{(1,-2)} + O(g^0\lambda^4, g^1\lambda^{-1}),$$

$$f_p^{(1,-2)} = n_p(1 + n_p)\{A_n(p)\frac{\partial T}{\partial x_n} + B_{nl}(p)\frac{\partial u_n}{\partial x_l} + C(p)\frac{\partial\omega_n\eta^2}{\partial x_n}\} \tag{12}$$

where functions $A_n(p), B_{nl}(p), C(p)$ are solution of integral equations

$$a_n(p) = \int d^3p' K(p,p') A_n(p'), \qquad b_{nl}(p) = \int d^3p' K(p,p') B_{nl}(p'), \qquad c(p) = \int d^3p' K(p,p') C(p') \tag{13}$$

($a_n(p), b_{nl}(p), c(p)$ are known functions) with additional conditions

$$\overline{A_n(p)p_l} = 0, \qquad \overline{B_{nl}(p)\varepsilon_p} = 0, \qquad \overline{C(p)\varepsilon_p} = 0 \tag{14}$$

where for arbitrary function $F(p)$ notation

$$\overline{F(p)} \equiv \int d\tau_p n_p (1+n_p) F(p)$$

is introduced. These conditions are a consequence of definition of the temperature T and drift velocity ω_n (9). Kernel $K(p,p')$ of integral equations (13) is defined by linearized collision integral

$$M(p,p') \equiv \left.\frac{\delta L_p^{(0,2)}}{\delta \mathrm{f}_{p'}}\right|_{f=n}, \qquad M(p,p') n_{p'}(1+n_{p'}) = -n_p(1+n_p) K(p,p'). \tag{15}$$

Solution of equations (13) will be discussed in another paper.

Main contribution in gradients to fluxes can be calculated using approach of our paper [5]

$$q_n^{o[0,0]} = \int d\tau_p \varepsilon_p \frac{\partial \varepsilon_p}{\partial p_n} n_p = (\varepsilon^o - \omega)\omega_n, \qquad t_{ln}^{o[0,0]} = \int d\tau_p p_l \frac{\partial \varepsilon_p}{\partial p_n} n_p = -\omega \delta_{ln} + \omega_n \pi_l^o \tag{16}$$

where ω is thermodynamic potential of quasi-particles defined by the formula

$$\omega = T \int d\tau_p \ln(1 - e^{\frac{p_s\omega_s - \varepsilon_p}{T}}). \tag{17}$$

Entropy density of their subsystem s is given by definition

$$s = s(n), \qquad s(f) \equiv \int d\tau_p \{(1+f_p)\ln(1+f_p) - f_p \ln f_p\} \tag{18}$$

which leads to thermodynamic relations

$$sT = \varepsilon^o - \omega - \pi_n^o \omega_n, \qquad d\omega = -s dT - \pi_n^o d\omega_n - a_0 d\eta. \tag{19}$$

Distribution function (12) gives equations

$$\frac{\partial \eta}{\partial t} = -u_n \frac{\partial \eta}{\partial x_n} - \frac{\eta}{2}\frac{\partial u_n}{\partial x_n} + \frac{1}{2\eta}\frac{\partial \omega_n \eta^2}{\partial x_n} + L^{[1,1]} + O(g^1\lambda^2, g^2\lambda^{-1}),$$

$$\frac{\partial \varphi}{\partial t} = -\frac{mv^2}{2} - m(h_0 + h^{[0,2]} + h^{[1,0]}) + O(g^0\lambda^4, g^1\lambda^1), \tag{20}$$

which generalize the Gross-Pitaevskii equation, and connected with them hydrodynamic equations for quasi-particles

$$\frac{\partial \varepsilon^o}{\partial t} = -\frac{\partial \varepsilon^o u_n}{\partial x_n} + (\omega + \frac{1}{2}a_0\eta)\frac{\partial u_n}{\partial x_n} - s\omega_n \frac{\partial T}{\partial x_n} - \frac{a_0}{2\eta}\frac{\partial \omega_n \eta^2}{\partial x_n} - \frac{\partial q_n^{o[1,-2]}}{\partial x_n} - t_{ln}^{o[1,-2]}\frac{\partial \upsilon_l}{\partial x_n} + a^{[1,-1]}\frac{\eta}{2}\frac{\partial \upsilon_n}{\partial x_n} + O(g^1\lambda^1, g^2\lambda^{-1}),$$

$$\frac{\partial \pi_l^o}{\partial t} = -\frac{\partial \pi_l^o u_n}{\partial x_n} - s\frac{\partial T}{\partial x_l} - \pi_n^o \frac{\partial u_n}{\partial x_l} - \frac{\partial t_{ln}^{o[1,-2]}}{\partial x_n} + a^{[1,-1]}\frac{\partial \eta}{\partial x_l} + O(g^1\lambda^0, g^2\lambda^{-1}),$$

$$\frac{\partial \upsilon_n}{\partial t} = -\upsilon_l \frac{\partial \upsilon_l}{\partial x_n} - \frac{\partial}{\partial x_n}(h_0 + h^{[0,2]} + h^{[1,0]}) + O(g^1\lambda^4, g^2\lambda^1). \tag{21}$$

In contrast to standard Chapman-Enskog method we took into account approximate nature of the equation set (2) ($A^{[m,n]}$ is contribution to A of the order $g^m\lambda^n$ connected with expansion of distribution function $f_p(x,\zeta)$ in series of the perturbation theory; $a^{[0.0]} \equiv a_0$). Equations (21) contain also dissipative values which have the structure

$$q_n^{o[1,-2]} = -\kappa_{nl}\frac{\partial T}{\partial x_l} - \alpha_{n,lm}\frac{\partial u_l}{\partial x_m} - \lambda_n \frac{\partial \omega_l \eta^2}{\partial x_l}, \quad t_{nl}^{o[1,-2]} = -\beta_{nl,m}\frac{\partial T}{\partial x_m} - \eta_{nl,ms}\frac{\partial u_m}{\partial x_s} - \mu_{nl}\frac{\partial \omega_s \eta^2}{\partial x_s},$$

$$h^{[1,0]} = -\chi_n \frac{\partial T}{\partial x_n} - \varphi_{nl} \frac{\partial u_n}{\partial x_l} - \psi \frac{\partial \omega_n \eta^2}{\partial x_n}, \quad a^{[1,-1]} = -\sigma_n \frac{\partial T}{\partial x_n} - \nu_{nl} \frac{\partial u_n}{\partial x_l} - \gamma \frac{\partial \omega_n \eta^2}{\partial x_n}. \tag{22}$$

Here κ_{nl} is heat conductivity and $\eta_{nl,ms}$ is viscosity of quasi-particle subsystem. For calculation of these kinetic coefficients we obtained the following expressions

$$\kappa_{nl} = -\overline{\varepsilon_p \frac{\partial \varepsilon_p}{\partial p_n} A_l(p)}, \qquad \alpha_{n,lm} = -\overline{\varepsilon_p \frac{\partial \varepsilon_p}{\partial p_n} B_{lm}(p)}, \qquad \lambda_n = -\overline{\varepsilon_p \frac{\partial \varepsilon_p}{\partial p_n} C(p)};$$

$$\beta_{ln,m} = -\overline{p_l \frac{\partial \varepsilon_p}{\partial p_n} A_m(p)}, \qquad \eta_{ln,ms} = -\overline{p_l \frac{\partial \varepsilon_p}{\partial p_n} B_{ms}(p)}, \qquad \mu_{ln} = -\overline{p_l \frac{\partial \varepsilon_p}{\partial p_n} C(p)};$$

$$\chi_n = -\frac{1}{m} \overline{(\frac{1}{2\eta} \frac{\partial \varepsilon_p}{\partial \eta} + \nu_0 \frac{\alpha_p}{\varepsilon_p}) A_n(p)}, \qquad \varphi_{nl} = -\frac{1}{m} \overline{(\frac{1}{2\eta} \frac{\partial \varepsilon_p}{\partial \eta} + \nu_0 \frac{\alpha_p}{\varepsilon_p}) B_{nl}(p)}, \qquad \psi = -\frac{1}{m} \overline{(\frac{1}{2\eta} \frac{\partial \varepsilon_p}{\partial \eta} + \nu_0 \frac{\alpha_p}{\varepsilon_p}) C(p)};$$

$$\sigma_n = \overline{\frac{\partial \varepsilon_p}{\partial \eta} A_n(p)}, \qquad \nu_{nl} = \overline{\frac{\partial \varepsilon_p}{\partial \eta} B_{nl}(p)}, \qquad \gamma = \overline{\frac{\partial \varepsilon_p}{\partial \eta} C(p)}. \tag{23}$$

These relations allow to investigate dependence of kinetic coefficients of quasi-particle subsystem on temperature. The result will be presented in another paper. This investigation can be simplified in the small drift velocity approximation. In this limit non-zero kinetic coefficients are scalar functions.

Additional information about properties of dissipative fluxes and kinetic coefficients can be obtained from evolution equation for entropy. This equation follows from relations (19) and has the form

$$\frac{\partial s}{\partial t} = -\frac{\partial s_n}{\partial x_n} + R \tag{24}$$

where s_n is flux and R is production of the entropy. In the considered approximation these values can be written as

$$s_n = s u_n + \frac{\varphi_n^{[1,-2]}}{T} + O(g^0 \lambda^2, g^1 \lambda^1),$$

$$R = -\varphi_n^{[1,-2]} \frac{1}{T^2} \frac{\partial T}{\partial x_n} - \tau_{ln}^{[1,-2]} \frac{1}{T} \frac{\partial u_l}{\partial x_n} - a^{[1,-1]} \frac{1}{2\eta T} \frac{\partial \eta^2 \omega_n}{\partial x_n} + O(g^2 \lambda^1) \tag{25}$$

where

$$\varphi_n^{[1,-2]} \equiv q_n^{o[1,-2]} - \omega_l t_{ln}^{o[1,-2]}, \qquad \tau_{ln}^{[1,-2]} \equiv t_{ln}^{o[1,-2]} - \frac{\eta}{2} a^{[1,-1]} \delta_{ln}. \tag{26}$$

According to the Onsager principle this result allows to establish symmetry of kinetic coefficients (23) and to prove inequality $R \geq 0$.

4 Conclusions

The reduced description method allows to justify domain of applicability of the Gross-Pitaevskii equation and to find various corrections to it. The obtained in the paper equations can be applied to investigation of possibility of quasi-particles creation at evolution of the condensate and to problem of stability of equilibrium quasi-particle subsystem.

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