On two-dimensional quantum gravity

Luca Victor Iliesiu

A Dissertation
Presented to the Faculty
of Princeton University
in Candidacy for the Degree
of Doctor of Philosophy

Recommended for Acceptance
by the Department of
Physics
Adviser: Silviu Pufu

September 2020

Abstract

This thesis explores topics in two-dimensional quantum gravity, focusing on the specific model of Jackiw-Teitelboim (JT) gravity and its relation to higher-dimensional black holes (BHs). Such a study is motivated by (i) the fact that JT gravity is a full-fledged theory of quantum gravity and (ii) because problematic features in higher-dimensional gravity, such as those related to black holes or wormholes, can be addressed in two-dimensions.

Chapter 2 is based on work with S. Pufu, Y. Wang, and H. Verlinde [1]. We propose an exact quantization of JT gravity by formulating the theory as a gauge theory. We find that this theory's partition function matches that of the Schwarzian theory. Observables are also matched: correlation functions of boundary-anchored Wilson lines in the bulk are given by those of bi-local operators in the Schwarzian.

Chapter 3 is based on work with J. Krutthof, G. Turiaci, and H. Verlinde [2]. We compute the partition function of JT gravity at finite cutoff in two ways: (i) by evaluating the Wheeler-DeWitt wavefunctional and (ii) by performing the path integral exactly. Both results match the partition function in the Schwarzian theory deformed by the analog of the $T\overline{T}$ deformation in $2D$ CFTs, thus, confirming the conjectured holographic interpretation of $T\overline{T}$.

Chapter 4 is based on [3]. We study JT gravity coupled to Yang-Mills theory. When solely focusing on the contribution of disk topologies, we show that the theory is equivalent to the Schwarzian coupled to a particle moving on the gauge group manifold. When considering the contribution from all genera, we show that the theory is described by a novel double-scaled matrix integral.

Chapter 5 is based on work with G. Turiaci [4]. We answer an open question in BH thermodynamics: does the spectrum of BH masses have a "mass gap" between an extremal black hole and the lightest near-extremal state? We compute the partition function of Reissner-Nordström near-extremal BHs at temperature scales comparable to the conjectured gap. We find that the density of states at fixed charge exhibits no gap; instead, we see a continuum of states at the expected gap energy scale.

Acknowledgements

During the past years I had the opportunity to learn from and work with many mentors, collaborators, peers and friends.

I am forever indebted to Silviu Pufu who has advised me since I was an undergrad, carefully guiding me through the realms of quantum field theory and quantum gravity up to today. Silviu, through your effort, patience and dedication, I feel that you have brought out the best version of myself. I could not have asked for a better adviser. I am also indebted to Herman Verlinde who has served as a constant source of inspiration and has permanently encouraged me to explore creative problems in quantum gravity and beyond. I am also especially thankful to David Simmons-Duffin, a mentor ever since he was a postdoc at the IAS, who has helped me explore countless facets of quantum field theory. I am also thankful to all my collaborators from past projects: Simone Giombi (who I would also like to thank for being the second reader of this thesis) and Igor Klebanov who showed great patience in the project that would eventually become my first paper in quantum field theory; Shai Chester, with whom I spent days trying to understand the intricacies of various projects; Ran Yacoby, an endless source of QFT knowledge; Filip Kos, who helped me learn so much about doing proper numerical work; David Poland, who strongly encouraged me to join several bootstrap projects; Mark Mezei, always a joy to talk to about any new physics idea; Murat Kologlu, with whom I spent days exploring the difficulties of conformal field theories; Raghu Mahajan, for countless stimulating discussions; Eric Perlmutter, for introducing me to numerous AdS/CFT mysteries; Yifan Wang, a constant model of mathematical rigor; Rajeev Erramilli, who always knows how to make physics a joyful endeavor; Petr Kravchuk, to whom I am indebted for teaching me so many of the mathematical intricacies of QFT; Jorrit Kruthoff, a constant source of creativity; Zhenbin Yang, who helped me develop better intuition for problems in quantum gravity; Gustavo Joaquin Turiaci, who has taught me so much about vast connections within the world of 2D quantum gravity. My deepest gratitude to the numerous physicists of Jadwin Hall and the IAS who offered up countless hours of their time to discuss countless topics: Juan Maldacena, Clay Cordova, Shu-Heng Shao, Daniel Kapec, Nathan Seiberg, Bernardo Zan, Nathan Benjamin, Vladimir Narovlansky, Xinan Zhou, Matt Heydeman, Damon Binder, Ksenia Bulycheva, Mykola Dedushenko, Yale Fan, Akash Goel, Po-Shen Hsin, Ziming Ji, Himanshu Khanchandani, Ho Tat Lam, Aitor Lewkowycz, Henry Lin, Alexey Milekhin, Fedor Popov, Grigory M. Tarnopolskiy, and Wenli Zhao. Christian Jepsen holds a special role among them for being patient enough to listen to all the bad ideas (physics and otherwise) that I've had in the past five years, and for providing his honest input. Finally, I would also like to thank the people that encouraged me early on to pursue a career in physics for there relentless and unconditional support, Ioana and Valentin Stoica.

I am also thankful to the numerous friends that have kept me joyful and grounded: to the daily lunch crews that I've been part of throughout the years (including the chefs at Thai Village), to the homemade dinner crew, to the movie club, to the pack that make reunions the highlight of the year, to all the amazing roommates that I have had throughout the years and to Andrei, Olek and Alex, my constant gym and tennis partners. I am forever indebted to my girlfriend Nina for being my rock throughout these many years, offering constant and unconditional support, comfort, and joy; not to mention, for helping me correct my grammar in the introduction of this thesis and in numerous past papers. To her family - Nick, Marianna, and Irina - I also extend my sincerest gratitude for their encouragement, kindness, and constant provisions of salmon. I would also be remiss if I didn't thank the Boulet and Bischof families for offering a safehaven (and the freshest seafood) during the coronavirus quarantine and the simultaneous writing of this thesis: Andrea, Beka, Marcus, Winn, Zander, Melanie, Bill & Ruthie, Irwin, Fo, and Mo, I am still stunned by your patience, your bigheartedness, and generosity.

Finally, I would like to thank my family, my sister Ioana, my mother Marilena Iliesiu, and my grandmother Lucia Popa. Without your unconditial love and support it would have been impossible to even start a PhD; not to mention, finish one.

In the memory of Victor Cristian Iliesiu.

Contents

Chapter 1

Introduction

1.1 The path integral in gravity

Reconciling general relativity with quantum mechanics remains one of the foundational open problems in modern physics. To understand the origin of this clash, we first review what goes wrong when trying to naively view the theory of general relativity as a quantum field theory.[1] The Einstein-Hilbert action[2]

$$I_{\text{EH}} = \frac{1}{16\pi G_N} \int d^d x \sqrt{-g}\, R\,, \tag{1.1}$$

governs the dynamics of general relativity through the evolution of the space-time metric, $ds^2 = g_{\mu\nu} dx^\mu dx^\nu$, that describes the geometry of the universe that we inhabit. The problem with the quantization of the action (1.1) stems from dimensional analysis: given that the metric $g_{\mu\nu}$ is dimensionless, the scaling dimension (i.e. how this quantity scales in units of energy) of the Newton constant is given by

$$[G_N] = 2 - d\,. \tag{1.2}$$

While such a coupling does not seem problematic at first sight, the issue appears when expanding the action (1.1) around it's classical saddles, $g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$.[3] Here, $\eta_{\mu\nu}$ is the classical saddle

[1] See [5] for a more detailed perspective on these issues.

[2] Throughout this thesis, we will mostly focus on studying Euclidean gravity. However, since we want to explain why gravity cannot be viewed as a consistent quantum field theory in our own universe, the action (1.1) is expressed in Lorentzian signature.

[3] For concreteness, here we will will consider the expansion around that flat metric solution, $\eta_{\mu\nu} = \text{diag}(-1, 1, 1, 1)$.

while $h_{\mu\nu}$ is meant to capture quantum fluctuations. In such a case, the Einstein-Hilbert action schematically becomes

$$I_{\text{EH, pert.}} = \frac{1}{16\pi G_N} \int d^d x \left[(\partial h)^2 + (\partial h)^2 h + \dots \right] \,, \tag{1.3}$$

where the $\dots$ capture terms with a higher number of derivatives or higher powers of h. Finally, rescaling $\tilde{h}_{\mu\nu} = \sqrt{8\pi G_N} h_{\mu\nu}$ we arrive at

$$I_{\text{EH, pert.}} = \int d^d x \left[\frac{1}{2} (\partial \tilde{h})^2 + \sqrt{2\pi G_N} (\partial \tilde{h})^2 \tilde{h} + \dots \right] \,. \tag{1.4}$$

The first term in the action above is that of a free field theory of spin-2 fields. All higher terms are graviton interactions. However, all such interactions formed from derivatives or higher powers of $\tilde{h}$ are irrelevant for $d > 2$.[4] At a technical level, this implies that the theory is nonrenormalizable: in order for expectation values in the quantum theory to converge, an infinite number of counterterms needs to be introduced in order to cancel all divergences. Consequently, such theories are not "UV complete" as they do not make sense at arbitrary energy thresholds. If taken literally, this means that the path integral $\int Dg_{\mu\nu}\, e^{-I_{\text{EH}}}$ does not make sense for $d > 2$ – rather, the path integrals that are well behaved are those perturbing Gaussian fixed points by relevant operators.

There are two lessons that one can extract from this analysis. The first lesson is that the naive path integral of quantum gravity has the potential to make sense in $d \leq 2$. In this thesis, we will extensively study gravitational path integrals and their application in $d = 2$.[5] The second lesson is that (1.1) is not UV complete but rather is a low-energy effective theory for a complete theory of quantum gravity. The main candidate for such a complete theory of quantum gravity is string theory. There are numerous open problems in this higher dimensional gravitational theory that also exist in $d = 2$. Throughout this thesis, we will make several observations that could provide insights towards resolving these problems through the lens of two-dimensional quantum gravity. We now point out several such open problems in the context of the holographic principle, a fundamental concept in quantum gravity.

[4] There are however gravitational theories in $d = 3$ whose path integral is well defined[6, 7, 8].

[5] Due to diffeomorphism invariance, the case $d = 1$ is, in some sense, too constrained to be worth studying.

1.2 The holographic principle

The holographic principle, or the gauge/gravity, duality is a tenet of quantum gravity which claims that a theory of gravity in some volume of spacetime can be described by a quantum field theory living on the boundary of that volume [9, 10, 11]. As we have explained in the previous section, gravity in d spacetime dimensions cannot be viewed as a consistent renormalizable quantum theory. Instead, at least in the case in which the cosmological constant is negative, it can consistently be viewed as a quantum field theory in one lower dimension. The example that provides the most computational evidence for this conjecture is the AdS/CFT correspondence, claiming that a theory of quantum gravity in $d+1$-dimensional anti-de Sitter space (AdS) is dual to a conformal field theory (CFT) that resides on the d-dimensional boundary. Before describing the details of this duality, it is instructive to review the main ingredients, both in the bulk (in the gravitational theory) and on the boundary (described by a quantum field theory).

Starting on the boundary side, conformal field theories are special types of quantum field theories that exhibit additional spacetime symmetries. Specifically, in addition to invariance under translations and Lorentz transformations, such theories are also invariant under dilatations and special conformal transformations. Put together, such transformations generate the conformal group which, for CFTs in d dimensions (in Lorentzian signature), is isomorphic to $SO(d-1,2)$. As in any quantum field theory, operators transform in various possible representations of the symmetry group. To characterize the representations under which CFT operators transform, one could choose a basis of operators $\{\mathcal{O}\}$ that transform in a finite-dimensional irreducible representations of the Lorentz subgroup, which are also eigenfunctions of the dilation operator (i.e., corresponding to transformations which rescale the coordinates by an arbitrary constant). The eigenvalue under dilatations is denoted by Δ and defines the scaling dimension of each operator $\mathcal{O}$.

On the gravitational side, we are interested in semi-classical solutions in quantum gravity which are described by metrics which, close to the boundary, describe AdS space. AdS_{d+1} is a hyperboloid in $\mathbb{R}^{d,2}$, whose metric can be expressed in Poincaré coordinates as[6]

$$ds^2 = \frac{L^2}{z^2}\left(dz^2 + dx_\mu dx^\mu\right) , \tag{1.5}$$

where the boundary of AdS_{d+1} is located at $z=0$. Since hyperboloids are maximally symmetric spaces, the Ricci scalar R is constant, and fixed to $R = -\frac{(d+1)d}{L^2}$. A key ingredient in the AdS/CFT

[6]In this section, we are referring to AdS_{d+1} as a Loretzian spacetime. In later sections, we will also use AdS_{d+1} or H_{d+1} interchangeably, to define a Euclidean spacetime.

correspondence is the identification between the conformal group of a d-dimensional CFT (described above) and the isometry group of AdS_{d+1}. Explicitly, one could easily observe from the embedding of AdS_{d+1} in $\mathbb{R}^{d,2}$ that the isometry group is $SO(d-1,2)$, since both the ambient metric η_{MN} and the embedding equation $\eta_{MN}X^M X^N = -L^2$ are invariant under $SO(d-1,2)$ transformations. The existence of such an isometry implies that if in the gravitational theory, we introduce a bulk field ϕ, then ϕ should also transform in representations of $SO(d-1,2)$. This fact leads to the identification between fields ϕ in the bulk, and operators $\mathcal{O}_\phi$ in the boundary theory. Fields and operators both transform under the same representation of $SO(d-1,2)$. For instance, by equating the eigenvalue of the quadratic Casimirs on a scalar field ϕ, with mass m in the bulk, to that of a scalar boundary operator $\mathcal{O}_\phi$, with scaling dimension Δ, one finds that

$$\Delta(\Delta - d) = m^2 L^2 \,. \tag{1.6}$$

While at the level of representation theory, the identification of the fields in the bulk with operators on the boundary might appear as a mathematical artifact, the AdS/CFT dictionary starts carrying physical significance once one starts identifying correlation functions in the bulk with those measured on the boundary. To be explicit, we can consider the example of a bulk scalar field in AdS dual to a scalar operator $\mathcal{O}_\phi$ on the boundary side. One can introduce boundary conditions for the bulk scalar field ϕ such that to leading order in the Poincaré coordinates, $\phi \sim j_\phi(x) z^{d-\Delta} + \dots$ as we approach the boundary of AdS_d. On the boundary side, one can source the operator $\mathcal{O}$ by adding $\int d^d x j_\phi(x)\mathcal{O}(x)$ to the action of the CFT. The AdS/CFT dictionary states that the generating functional for connected correlators on the boundary side is identified with the on-shell gravitational action when the field ϕ is sourced on the boundary:

$$W[j_\phi] = -I_{\text{on-shell}}[j_\phi(x)] \qquad \Leftrightarrow \qquad W[j_\phi] = \log\left\langle \exp \int d^d x j_\phi(x)\mathcal{O}(x) \right\rangle \tag{1.7}$$

This statement, in turn, implies that the correlation functions on the operator $\mathcal{O}(x)$ can be matched with correlation functions of the bulk field ϕ, when the field ϕ is placed close to the boundary. To obtain a complete dictionary, one should map correlation functions of *any* field in the bulk to that of *some* operators on the boundary side. Understanding the mapping of all such correlators is the goal of the bulk reconstruction program, whose features and related open problems we describe in the next subsections.

1.2.1 Bulk reconstruction

The bulk reconstruction program aims to find the exact map between the algebra of operators in the bulk and that on the boundary.[7] The field typically studied in the reconstruction program is again a local bulk field $\phi(x, z)$, in an attempt to reconstruct correlation functions of this field anywhere in the bulk from boundary correlators [14]. This match can be done by smearing the boundary operator $\mathcal{O}_\phi$, using an appropriate integration kernel [14]. While this method provides a nice way of reconstructing bulk fields from the boundary, this reconstruction procedure runs into several problems that stem from the fact that field $\phi(x, z)$ is not diffeomorphism invariant (i.e., it depends on the coordinate system that one chooses in the bulk).[8] Instead, the physical operators that one should aim to reconstruct should all be invariant under the diffeomorphism gauge symmetry of the bulk. One can construct a diffeomorphism invariant operator by gravitationally "dressing" the field $\phi(x, z)$ to obtain a non-local operator in the bulk. One problem with the dressing procedure is that it is not unique, and a full classification of all diffeomorphism invariant operators in the bulk is extremely difficult in known, higher-dimensional, holographic examples.

Luckily, as we shall explain shortly in certain models of $2D$ quantum gravity, it is possible to construct a "complete" basis of gauge-invariant operators. We shall compute the possible correlators of such gauge-invariant operators and map them to expectation values of various operators in an equivalent boundary theory.

1.2.2 Black hole microstates and approximate bulk isometries

An important aspect of the gauge/gravity correspondence is the duality between correlators computed in black hole geometries in the bulk and observables computed at finite temperature on the boundary. The most basic example of such equality is between the logarithm of the finite temperature partition function of the CFT and the bulk on-shell action measured in the black hole geometry: $Z_{\text{CFT}}(\beta) = e^{-I_{\text{bulk}}}$. While one can thus view the black hole geometry to correspond to a canonical ensemble in the CFT, a more fine-grained statement in the holographic dictionary is the correspondence between black hole microstates and specific "heavy" (with large scaling dimension) states in the CFT.

One could consequently ask what computation on the boundary side or in the bulk could shed light on the properties of these microstates. For instance, can one compute the spacing between such black hole microstates and their counter-parts on the CFT side, and what is the density of states on

[7] See [12, 13] for recent reviews of the bulk reconstruction program.
[8] In fact, it is straightforward to see that no local fields in quantum gravity are diffeomorphism invariant.

either side? Unfortunately, such computations are difficult both in the bulk and on the boundary. In the bulk, to understand the properties of black hole microstates, one would, in principle, need a better understanding of the UV complete theory in a regime where it is strongly coupled. On the boundary side, it is difficult to compute the spectrum of such heavy states in the absence of an underlying symmetry principle – for instance, if such states are protected by supersymmetry.

As we shall soon explain, one special case where insight can be gained by performing computations in the bulk is that of extremal and near-extremal black holes. Such black holes are special in so much that the geometry in their near-horizon regions is drastically simplified: there is an AdS_2 throat with an internal space that varies slowly as the horizon is approached (see, for example, [15]). Thus, the near-horizon region benefits from an additional AdS_2 $SO(2,1) \sim PSL(2,\mathbb{R})$ isometry. The goal in the later sections of this thesis will be to understand the consequences of this additional isometry for the spectrum of black hole masses and their dual "heavy" CFT states.

1.2.3 Moving the AdS boundary inside the bulk

A question related to that of bulk reconstruction is how to extend the holographic dictionary once the bulk is no longer asymptotically AdS_{d+1}. Rather, one would like to extend the gauge/gravity duality when the bulk ends on a Dirichlet wall yielding a patch of spacetime with finite volume. Equivalently, on the boundary side, one could ponder what the bulk dual is once we move away from the conformal fixed point. While for general spacetime dimensions this is still an intractable problem, the AdS_3/CFT_2 duality benefits from integrability properties that help shed light on this problem. Specifically, there exists a general class of exactly solvable irrelevant deformations of 2D CFTs, the simplest of which is the $T\overline{T}$ deformation (where $T \equiv T_{zz}$ and $\overline{T} \equiv T_{\overline{z}\overline{z}}$ are the left and right moving components of the stress tensor). In the holographic context, turning on this deformation on the boundary side was conjectured to have the following bulk dual [16]:

$$I_{\text{CFT}} + \lambda \int d^2x\, T\overline{T} \qquad \Leftrightarrow \qquad AdS_3 \text{ with a Dirichlet wall at finite cutoff}\,. \tag{1.8}$$

While such a duality has been tested semi-classically, an exact check for a finite value of λ away from the limit in which the bulk path integral is dominated by its saddle has yet to appear. Furthermore, although Zamolodchikov [17] showed that the $T\overline{T}$ operator satisfies some remarkable properties, such as factorization in translation invariant states, the quantum theory does have a peculiar feature. For large enough energies and a fixed deformation parameter, the energy spectrum complexifies. It, therefore, seems that the deformed theory becomes non-unitary. In the bulk, this corresponds to black hole states that fill up more spacetime than available in the finite cutoff geometry. The resolution of this problem has not been fully understood and is further plagued by complications having to do with properly defining the composite $T\overline{T}$ operator in two-dimensional field theories.

Once again, an analysis from the perspective of two-dimensional gravity proves to be fruitful. In such a case, it has been conjectured that placing the two-dimensional bulk in a finite patch of spacetime is equivalent to deforming its one-dimensional dual by an analog of the $T\overline{T}$ deformation, a composite operator formed out of powers of the Hamiltonian. By computing the path integral in two-dimensional dilaton gravity exactly, we are able to provide the first explicit check of this conjecture. Furthermore, through the exact computation of the path integral we provide evidence on the resolution of the complexification of energy levels that plagues the $T\overline{T}$ deformation and its gravity dual.

1.2.4 The problem of Euclidean wormholes

From yet another perspective, we can again discuss the problem of properly identifying the correct boundary dual of certain bulk geometries. As previously emphasized, AdS/CFT is thought to state that the sum over all geometries with fixed boundary conditions is the same as the partition function of a (conformal) field theory living on the boundary. A puzzle arises when considering Euclidean geometries that have n disconnected boundaries (each having the same boundary conditions that people traditionally consider for a single boundary of the bulk) [18]. On the boundary side, according to the holographic dictionary, one should simply consider n decoupled copies of the holographic CFT (which by itself would be dual to a single copy of the bulk). In such a case, one finds multiple solutions for the bulk geometry: the obvious solution is given by a disconnected set of copies of the traditional AdS geometry; the less obvious, more puzzling solutions are the ones which include Euclidean wormholes that connect different (previously disconnected) boundaries. Examples of such Euclidean wormhole solutions were found in [18]. If the contribution of such geometries to the gravitational partition function is not vanishing we arrive at the following puzzle: for a single copy of the CFT an a single bulk copy we have we have that $Z_{\text{bdy.}}(\beta) = \# e^{-I_{\text{one-copy}}}$; for multiple boundaries we should find that $[Z_{\text{bdy.}}(\beta)]^n = \# e^{-n\, I_{\text{one-copy}}} + \# e^{-I_{\text{wormhole}}}$, inconsistent with the result for a single boundary (for which we would get $[Z_{\text{bdy.}}(\beta)]^n = \# e^{-n\, I_{\text{one-copy}}} \neq \# e^{-n\, I_{\text{one-copy}}} + \# e^{-I_{\text{wormhole}}}$) [18]. Fully understanding the resolution to this issue is, as of yet, an open problem.[9]

[9]Of course, one could postulate that geometries that connect different boundaries should not be considered in the gravitational partition function when summing over possible geometries. However, this is unnatural from multiple perspectives. The first is that in string theory (the UV completion of gravity in the bulk) we are told to sum over all possible topologies of the string worldsheet; the second is that there is no local term that one could add the the bulk action that would exclude the contribution of such geometries to the gravitational partition function.

Two-dimensional quantum gravity once again offers a different perspective to the problem [19]. Instead of considering a standard, unitary, quantum mechanical boundary theory, one can consider an ensemble of theories on the boundary. Such an ensemble, can have the property that $\langle Z^n_{\text{bdy.}}\rangle \neq \langle Z_{\text{bdy.}}\rangle^n$ (where $\langle \dots \rangle$ denotes the ensemble average), offering a potential loophole to match the contribution of Euclidean wormholes to the boundary result. As we shall soon review, the ensemble of theories that we have to consider in the simplest example of dilaton gravity is given by a double-scaled matrix model [19].

1.3 The resolutions that two-dimensional gravity provides

While AdS/CFT [9, 10, 11] has provided a broad framework to understand quantum gravity, most discussions are limited to perturbation theory around a fixed gravitational background. The difficulty of going beyond perturbation theory stems from our limited understanding of both sides of the duality: on the boundary side, it is difficult to compute correlators in strongly coupled CFTs, while in the bulk there are no efficient ways of performing computations beyond tree level in perturbation theory. 2D/1D holography provides one of the best frameworks to understand quantum gravity beyond perturbation theory, partly because gravitons or gauge bosons in two dimensions have no dynamical degrees of freedom.[10] Nevertheless, many of the open questions from higher dimensional holography, such as questions related to bulk reconstruction or the physics of black holes and wormholes, persist in 2D/1D holography.

1.3.1 Jackiw-Teitelboim gravity

One of the simplest starting points to discuss 2D/1D holography is the two-dimensional Jackiw-Teitelboim (JT) theory [38, 39], which involves a dilaton field ϕ and the metric tensor $g_{\mu\nu}$. The Euclidean action is given by[11]

$$\begin{aligned} I_{JT}[\phi, g] = & \overbrace{-\frac{1}{16\pi G_N}\int_\Sigma d^2x\sqrt{g}\,\phi_0 R - \frac{1}{8\pi G_N}\int_{\partial\Sigma} du\,\sqrt{\gamma}\phi_0 K}^{\frac{\phi_0\chi(M)}{8G_N}} \\ & -\frac{1}{16\pi G_N}\int_\Sigma d^2x\sqrt{g}\,\phi(R+\Lambda) - \frac{1}{8\pi G_N}\int_{\partial\Sigma} du\,\sqrt{\gamma}\,(\phi|_{\partial\Sigma})\,K\,, \end{aligned} \tag{1.9}$$

[10]See [20, 21, 22, 23, 24, 25, 26, 27, 28, 29, 30, 31, 32, 33, 34, 35, 36, 37] for various discussions about models of 2D/1D holography.

[11]Moving forward we will fix the two-dimensional gravitational constant $G_N = 1/(8\pi)$.

where we have placed the theory on a manifold M with metric $g_{\mu\nu}$ and where the boundary of this manifold, ∂M, is endowed with the induced metric γ and the extrinsic curvature K. The first term is a purely topological term which includes the Euler characteristic $\chi(M)$ of the manifold. For large values of the dilaton displacement ϕ_0 the topological term suppresses the contribution of higher genus manifolds – thus, at first, we will solely focus on manifolds with the topology of a disk and we will ignore the contribution of this term.

The bulk term in (1.9) yields the equations of motion

$$R = -\Lambda\,, \qquad \nabla_\mu \nabla_\nu \phi = \frac{\Lambda}{2} g_{\mu\nu}\phi\,. \tag{1.10}$$

Thus, on-shell, the bulk term in (1.9) vanishes. The remaining degrees of freedom are thus all on the boundary of some connected patch of Euclidean AdS_2 (or, equivalently, of the Poincaré disk). The boundary term in (1.9) is in fact necessary in order to have a well-defined variational principle when studying the theory with Dirichlet boundary conditions: one can fix

$$\phi|_{\partial\Sigma} = \phi_b = \phi_r/\epsilon\,, \qquad\qquad g_{uu} = 1/\epsilon^2\,. \tag{1.11}$$

such that the total proper boundary length is given by $L = \beta/\epsilon$. JT gravity that is typically studied is the asymptotically AdS_2 limit in which $\epsilon \to 0$, therefore making the proper length of the boundary large.

1.3.2 A review: Asymptotic AdS_2 spaces, the Schwarzian theory, and its quantization

We now proceed to study the quantization of JT gravity by better understanding the nature of the boundary degrees of freedom. The path integral of the action (1.9) is given by

$$\begin{aligned} Z_{JT}[\beta,\, \phi_r] &= \int_{\phi=\phi_b + i\mathbb{R}} D\phi D g_{\mu\nu}\, e^{-I_{JT}[\phi, g]} \\ &= \int D g_{\mu\nu} \delta(R+\Lambda) e^{\int du \phi_b K}\,. \end{aligned} \tag{1.12}$$

where in going from the first to the second line we have integrated the dilaton along imaginary values. As can also be seen from the classical equations of motion, the contribution of the bulk action term fully vanishes. We are thus left with a sum over all AdS_2 patches which have a fixed proper perimeter L and fixed metric g_{uu}. This constraint implies that when parametrizing the space

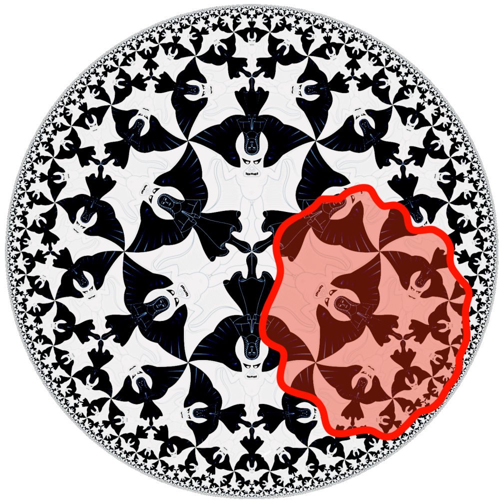


Figure 1.1: Cartoon exemplifying a typical AdS_2 patch with large proper boundary length. In this cartoon of hyperbolic space, sketched by M. C. Escher, each demon/angel has the same proper area.

in terms of Poincaré coordinates and the boundary by some coordinate u such that

$$ds^2 = \frac{d\tau^2 + dx^2}{x^2}\,, \qquad \frac{1}{\varepsilon^2} = \frac{\tau'^2 + x'^2}{x^2}\,, \qquad \tau' = \partial_u \tau\,. \tag{1.13}$$

then one can solve for $x[\tau(u)]$, at least to the first few orders in ε in perturbation theory [29],

$$x[\tau(u)] = \varepsilon\tau' + O(\varepsilon^2)\,. \tag{1.14}$$

Using this result one can thus hope to rewrite the extrinsic curvature in (1.9) in terms of a single field, $\tau(u)$. In order to do that we use the definition of the extrinsic curvature, which simplifies for two-dimensional manifolds to

$$K = -\frac{g(T,\, \nabla_T n)}{g(T,T)} \tag{1.15}$$

where $g(X,Y) = g_{ab}X^aY^b$ with g_{ab} is the metric on M, where n is the normal vector to boundary of M and T is the tangent vector along ∂M. In the coordinate system (1.13), the vectors T^a and n_a are given by

$$T^a = (\tau', x')\,, \qquad n^a = \frac{x}{\sqrt{\tau'^2 + x'^2}}(-x', \tau')\,. \tag{1.16}$$

In such a case, the extrinsic curvarture becomes

$$K = \frac{\tau'(\tau'^2 + x'^2 + x'x'') - xx'\tau''}{(x'^2 + \tau'^2)^{3/2}} = 1 + \varepsilon^2 \mathrm{Sch}(\tau, u) + O(\epsilon^4) \tag{1.17}$$

where to obtain the second equality we have used (1.14). Here, $\mathrm{Sch}(\tau, u)$ denotes the Schwarzian derivative

$$\mathrm{Sch}(\tau, u) = \frac{\tau'''}{\tau'} - \frac{3}{2}\frac{\tau''^2}{\tau'^2} \, . \tag{1.18}$$

The JT gravity action simply reduces to

$$I_{JT} = -\int_0^\beta du \phi_r \left(\mathrm{Sch}(\tau, u) + O(\varepsilon^2)\right) \tag{1.19}$$

Thus, we have reduced the path integral for the two-dimensional gravitational theory (1.9) to that over a theory in one dimension. In that sense, the equivalence between JT gravity and the Schwarzian theory (1.19) can be viewed as a toy example of the AdS/CFT correspondence.

Our next goal is to analyze the classical and quantum behavior of this action. Following from the approximation (1.14), the $SL(2, \mathbb{R})$ isometry of AdS_2 which we have discussed in section 1.2 reduces to a symmetry of the Schwarzian theory

$$\tau \to \frac{a\tau + b}{c\tau + d} \tag{1.20}$$

which acts on the field $\tau(u)$ through a fractional linear transformation. From the perspective of effective field theory, one could, therefore, ponder why we have obtained an action which solely depends on $\mathrm{Sch}(\tau, u)$. It is because it is the action with the lowest derivative order, which is invariant under $SL(2, \mathbb{R})$. As we will explain later in this thesis, higher derivative orders are suppressed in an ε expansion and only become important when studying the theory at finite cutoff.

There are three conserved charges associated to the transformation (1.20) whose Poisson algebra is $\mathfrak{sl}(2, \mathbb{R})$. The Casimir of these charges is also conserved and happens to once again be the Schwarzian derivative $\mathrm{Sch}(\tau, u)$. Therefore, the equation of motions of (1.19) are equivalent to

$$\partial_u \mathrm{Sch}(\tau, u) = 0 \tag{1.21}$$

The solution to this equation which is consistent with the boundary condition $\tau(0) = \tau(\beta)$ is, up to

$SL(2,\mathbb{R})$ transformations, given by[12]

$$\tau(u) = \tan\frac{\pi u}{\beta}\,. \tag{1.22}$$

Using this solution, we can determine the on-shell action to be

$$I_{JT} = -2\pi^2\frac{\phi_b}{\beta}\,. \tag{1.23}$$

Having explained the classical behavior of the theory, we now briefly review its quantization. Specifically, we are interested in performing the path integral

$$Z_{JT}[\beta,\,\phi_r] = \int \frac{d\mu[\tau]}{SL(2,\mathbb{R})} e^{\int du \phi_b \mathrm{Sch}(\tau,u)}\,, \qquad\qquad d\mu[\tau] = \prod_{u\in[0,\beta)} \frac{d\tau}{\tau'}\,. \tag{1.24}$$

The quotient by $SL(2,\mathbb{R})$ is meant to eliminate patches that are identical up to the $SL(2,\mathbb{R})$ transformations in AdS_2. $d\mu[\tau]$ can be straightforwardly obtained by requiring that this measure should be local and invariant under boundary diffeomorphisms. Alternatively, this measure could be obtained by studying the symplectic form of a theory equivalent to JT gravity – an $\mathfrak{sl}(2,\mathbb{R})$ BF theory (we will review this equivalence in the next subsection). The integration space is over all real periodic functions $\tau(u)$, with $\tau(0) = \tau(\beta)$. Performing the rescaling $\tau \to a\tau$ and $\phi_b \to \phi_b/\alpha$ under which the action is invariant, one can conclude that ϕ_b/β serves as the effective coupling in the Schwarzian theory – when ϕ_b/β is large the path integral should be dominated by classical saddle (1.23), while when ϕ_b/β is small quantum fluctuation become relevant.

The result for the path integral (1.24) can be obtained using several methods: from fermionic localization [25], exploiting the fact that $\mathrm{Diff}(S^1)/SL(2,\mathbb{R})$ is a symplectic manifold, by using the equivalence between the Schwarzian and a particle in a magnetic field moving in hyperbolic space [35, 36] (we review this approach in appendix A), by using the equivalence between the Schwarzian and a dimensionally reduced version of Liouville theory or by using an $\mathfrak{sl}(2,\mathbb{R})$ BF theory with a carefully chosen gauge group (we summarize this approach in the next section and present details in

[12]The solution (1.22) is however not unique. The general solution is given by $\tau(u) = \tan\frac{n\pi u}{\beta}$, with $n \in \mathbb{Z}\{0\}$. Such saddles correspond to Euclidean solutions where the boundary is self-intersecting. There are two issues which such solutions. Firstly, one could eliminate such solutions by requiring that manifolds appearing in the path integral of Euclidean gravity have non-intersecting boundaries. Secondly, in [29] such saddles that correspond to higher boundary winding have been proven to be unstable.

section 2). The result for the path integral obtained from any of the approaches mentioned above is

$$Z_{JT}[\beta,\,\phi_r] \sim \int dE\,\sinh(2\pi\sqrt{E})e^{-\frac{\beta}{2\phi_r}E} \sim \left(\frac{\phi_b}{\beta}\right)^{\frac{3}{2}} e^{2\pi^2\frac{\phi_b}{\beta}}\,. \tag{1.25}$$

The first equation emphasizes that the Schwarzian density of states is $\rho(E) = \sinh(2\pi\sqrt{E})$ while the second shows that the path integral reproduces the classical saddle (1.24) to leading order in ϕ_b/β.

1.3.3 A review: Contributions from higher genus topologies

As mentioned in section 1.2.4, we are not only interested in summing over manifolds with a single topology. Instead, in the case of JT gravity, we should sum over manifolds with any topology that could support a hyperbolic geometry and which satisfy the boundary conditions mentioned in the previous subsection. Therefore, we will once again consider the contribution of ϕ_0 in (1.9).

The basic building blocks needed to compute the contribution to the partition function of higher genus manifolds is [37]:

- The path integral over a "trumpet", $\mathcal{M}_T$, which on one side has asymptotically AdS_2 boundary conditions specified by (4.2) and, on the other side, ends on a geodesic of length b.
- The path integral over a bordered Riemann surfaces of constant negative curvature that has n boundaries and genus g. For such surfaces, we fix the lengths of the geodesic boundaries b_1, $\ldots$, b_n, across all n boundaries.
- The correct measure for gluing "trumpets" to the higher genus bordered Riemann surfaces.

By gluing the above geometries along the side where the boundary is a geodesic, we can obtain any orientable geometry with constant negative curvature (with arbitrary genus g, and an arbitrary number of boundaries n), which has asymptotically AdS_2 boundaries.

To start, we compute the "trumpet" partition function which closely follows the computation for the disk partition function. The JT gravity action, again reduces to an integral over the extrinsic curvature K on the boundary where we impose Dirichlet boundary conditions. The boundary that ends on the geodesic of length b requires no associated boundary term and, therefore, does not contribute to the action.[13] Therefore, the path integral over the boundary once again reduces to that of the Schwarzian. What differs from the disk computation is the boundary condition for the

[13] This is because no boundary term is required when fixing K and g_{uu}.

Schwarzian field:

$$Z_{\text{trumpet}} = \int \frac{D\mu[\tau]}{U(1)} e^{\int du \phi_r \text{Sch}(\tau,u)} \tag{1.26}$$

where the boundary condition $\tau(\beta) = \frac{a\tau(0)+b}{c\tau(0)+d}$, can be obtained by identifying two different geodesics on the Poincaré plane. Here, the fractional linear transformation can be related to the length of the geodesic b. The new boundary conditions can be viewed as a fugacity for the $SL(2,\mathbb{R})$ symmetry in the Schwarzian theory. In the presence of such a fugacity, we can no longer quotient the integration space by $SL(2,\mathbb{R})$; rather we should only quotient by the preserved $U(1)$ subgroup. The result for the path integral (1.26) can once again obtained from localization [25] or by a dimensionally reduced version of Liouville theory [40]:

$$Z_{\text{trumpet}} \sim \int \frac{dE}{\sqrt{E}} \cos(b\sqrt{E}) e^{-\frac{\beta E}{2\phi_r}} \sim \frac{\phi_r^{1/2}}{(2\pi)^{1/2}\beta^{1/2}} e^{-\frac{\phi_r}{2}\frac{b^2}{\beta}} \tag{1.27}$$

The next step is to compute the volume of the moduli space of n-bordered Riemann surfaces with constant curvature, denoted by $\text{Vol}_{g,n}(b_1, \ldots, b_n)$. While, we do not describe the exact procedure to obtain these volumes, we will mention that a recursion relation for these volumes was found in [41] (see [42] for a review). It was later showed that this recursion relation can be related to the "topological recursion" seen in the genus expansion of a double-scaled matrix integral [43]. Finally, the integration measure needed in order to glue the "trumpet" to a boundary of a bordered Riemann surfaces one needs to use the Weyl-Peterson measure dbb.[14] Thus, the contribution to the partition function of a higher genus surface is given by:

$$Z_{JT}(\beta, \phi_r) \supset e^{2\pi\phi_0\chi_{g,1}} \int db\, b\, \text{Vol}_{g,1}(b) Z_{\text{trumpet}}(\beta, \phi_r, b)\,. \tag{1.28}$$

Similarly, one can compute the contribution of can compute the contribution of geometries that connect n boundaries:

$$Z_{JT}(\beta_1, \phi_{r,1}, \ldots, \beta_n, \phi_{r,n}) \supset e^{2\pi\phi_0\chi_{g,n}} \underbrace{\int db_1\, b_1 \ldots db_n\, b_n\, \text{Vol}_{g,n}(b_1, \ldots, b_n) \times Z_{\text{trumpet}}(\beta_1, \phi_{r,1}, b_1) \ldots Z_{\text{trumpet}}(\beta_n, \phi_{r,n}, b_n)}_{Z_{g,n}(\beta_j)}\,. \tag{1.29}$$

which for $n \geq 1$ is never vanishing. As emphasized in section 1.2.4, the gravitational path integral

[14]This measure can once again be obtained by considering the symplectic form in an equivalent BF theory.

can receive contributions from such geometries which connect the n disconnected boundaries.

We now describe the equivalent matrix integral that reproduces the results above, following [37]. Consider a Hermitian matrix integral over $N \times N$ Hermitian matrices with some potential $S[H]$:

$$\mathcal{Z} = \int dH e^{-S(H)} , \qquad S[H] \equiv N \left(\frac{1}{2} \text{Tr}_N H^2 + \sum_{j \geq 3} \frac{t_j}{j} \text{Tr}_N H^j \right) , \tag{1.30}$$

where Tr_N is the standard trace over $N \times N$ matrices. An observable that proves important in the genus expansion of the gravitational theory is the correlator of the thermal partition function operator, $Z(\beta) = \text{Tr}_N\, e^{-\beta H}$. Correlators of such operators have an expansion in $1/N$, where each order in N can be computed by looking at orientable double-line graphs of fixed genus [44, 45] (for a review see [46]). Consequently, this is known as the genus expansion of the matrix model (1.30).

For a general set of potentials $S[H]$, each order in the expansion can be determined in terms of a single function $\rho_0(E)$. This function is simply the leading density of eigenvalues in matrices with $N \to \infty$. Consider the double-scaling limit of (1.30), in which the size of the matrix $N \to \infty$ and in which we focus on the edge of the eigenvalue distribution of the matrix H, where the eigenvalue density remains finite and is denoted by e^{S_0}. The expansion of the correlators mentioned above can now be expressed in terms of e^{S_0} instead of the size of the matrix N. In this double-scaled limit the density of eigenvalues $\rho_0(E)$ is not necessarily normalizable and with an appropriate choice of potential $S[H]$, $\rho_0(E)$ can be set to be equal to the energy density in the Schwarzian theory (4.11)

$$\rho_0(E) = \frac{\phi_b}{2\pi^2} \sinh(2\pi \sqrt{2\phi_b E}) . \tag{1.31}$$

As previously emphasized, choosing (1.31) determines all orders (in the double scaled limit) in the e^{-S_0} perturbative expansion for correlators of operators such as $Z(\beta) = \text{Tr}_N\, e^{-\frac{\beta H}{2\phi_r}}$ [47]. The result found by [37], building on the ideas of [43], is that the genus expansion in pure JT gravity agrees with the e^{S_0} genus expansion of the double-scaled matrix integral whose eigenvalue density of states is given by (1.31):

$$Z^n_{\text{JT}}(\beta_1, \dots, \beta_n) = \langle Z(\beta_1) \dots Z(\beta_n) \rangle = \sum_g Z_{g,n}(\beta_j) e^{-S_0 \chi(\mathcal{M}_{g,n})} . \tag{1.32}$$

The density of states (1.31) was shown to arise when considering the matrix integral associated to the $(2, p)$ minimal string. Specifically, this latter theory was shown to be related to a matrix integral whose density of eigenvalues is given by [48, 49, 50, 51, 52]

$$\rho_0(E) \sim \sinh\left(\frac{p}{2}\text{arccosh}\left(1+\frac{E}{\kappa}\right)\right)\,, \tag{1.33}$$

where κ is set by the value of p and by the value of μ from the Liouville theory which is coupled to the $(2,p)$ minimal model [53]. Taking the $p\to\infty$ limit in (1.33) and rescaling E appropriately, one recovers the density of states (1.31). Consequently, one can conclude that the double-scaled matrix integral which gives rise to the genus expansion in pure JT gravity is the same as the matrix integral which corresponds to the $(2,\infty)$ minimal string. Thus, if we view the matrix integral as the equivalent boundary theory we find that by considering ensemble averages, the open problem brought up by Maldacena and Maoz is resolved [18].

1.4 Jackiw-Teitelboim gravity in the first order formalism

So far, we have focused on the partition function in the pure gravitational theory, but we have not yet addressed what happens when we couple the theory to matter fields. To study this problem in the approximation in which such fields are treated as probe particles, it is useful to consider the spectrum of (non-local) operators in the theory.

While our discussion has mostly focused on understanding Jackiw-Teitelboim gravity in terms of the metric and the dilaton, we will show that, if we want to better understand the exact quantization of the theory as well as its operator spectrum, it is convenient to formulate the theory as an equivalent gauge theory. Such a reformulation has the advantage that we can quickly identify the gauge-invariant or, equivalently, diffeomorphism invariant operators in the theory (which, in turn, are equivalent to the aforementioned probe particles). In general relativity, we can always rewrite the dependence of the action on the metric in terms of two additional sets of fields: the frame fields and spin connection. As we will discuss in detail below, when paired together, these fields constitute the necessary component for a $2d$ gauge field appearing in the reformulation.

1.4.1 Classical equivalence

As shown in [54, 55], JT gravity (1.9) can be equivalently written in the first-order formulation, which involves the frame and spin-connection of the manifold, as a 2D BF theory with gauge algebra $\mathfrak{sl}(2,\mathbb{R})$.[15] Let us review this correspondence starting from the BF theory.[16] To realize this equivalence on shell, we only need to rely on the gauge algebra of the BF theory and not on the global structure of the gauge group. Thus, the gauge group could be $PSL(2,\mathbb{R})$ or any of its central extensions. For this reason, we will for now consider the gauge group to be $\mathcal{G}$ and will specify the exact nature of $\mathcal{G}$ in Section 2.3.

To set conventions, let us write the $\mathfrak{sl}(2,\mathbb{R})$ algebra in terms of three generators P_0, P_1, and P_2, obeying the commutation relations

$$[P_0,P_1]=P_2\,,\qquad [P_0,P_2]=-P_1\,,\qquad [P_1,P_2]=-P_0\,. \tag{1.34}$$

For instance, in the two-dimensional representation the generators P_0, P_1, and P_2 can be represented as the real matrices

$$P_0=\frac{i\sigma_2}{2}\,,\qquad P_1=\frac{\sigma_1}{2}\,,\qquad P_2=\frac{\sigma_3}{2}\,. \tag{1.35}$$

An arbitrary $\mathfrak{sl}(2,\mathbb{R})$ algebra element consists of a linear combination of the generators with real coefficients. The field content of the BF theory consists of the gauge field A_μ and a scalar field ϕ, both transforming in the adjoint representation of the gauge algebra. Under infinitesimal gauge transformations with parameter $\epsilon(x)\in\mathfrak{sl}(2,\mathbb{R})$, we have

$$\delta\phi=[\epsilon,\phi]\,,\qquad \delta A_\mu=\partial_\mu\epsilon+[\epsilon,A_\mu]\,. \tag{1.36}$$

Consequently, the covariant derivative is $D_\mu=\partial_\mu-A_\mu$ (because then we have, for instance, $\delta(D_\mu\phi)=[\epsilon,D_\mu\phi]$), and then the gauge field strength is $F_{\mu\nu}\equiv-[D_\mu,D_\nu]=\partial_\mu A_\nu-\partial_\nu A_\mu-[A_\mu,A_\nu]$. In differential form notation, $F=dA-A\wedge A$.

Ignoring any potential boundary terms, the BF theory Euclidean action is

$$S_{\rm BF}=-i\int \mathrm{Tr}\,(\phi F)\,, \tag{1.37}$$

where the trace is taken in the two-dimensional representation (1.35), such that $\mathrm{Tr}\,\phi F=\eta^{ij}\phi_i F_j/2$, where $\eta^{ij}=\mathrm{diag}(-1,1,1)$, with $i,j=1,\,2,\,3$. To show that the action (1.37) in fact describes JT

[15] Similarly, there is an equivalence between a different 2D gravitational model, the CallanGiddingsHarveyStrominger model and a 2D BF-theory with the gauge algebra given by a central extension of $\mathfrak{iso}(1,1)$ [56, 57]. Similar to our work here, it would be interesting to explore exact quantizations of this gauge theory.

[16] Unlike [54, 55], we will work in Euclidean signature.

gravity, let us denote the components of A and ϕ as

$$A(x) = \sqrt{\frac{\Lambda}{2}} e^a(x) P_a + \omega(x) P_0 \,, \qquad \phi(x) = \phi^a(x) P_a + \phi^0(x) P_0 \,, \tag{1.38}$$

where the index $a = 1, 2$ is being summed over, $\Lambda > 0$ is a constant, and e^a and ω are one-forms while ϕ^a and ϕ^0 are scalar functions. An explicit computation using $F = dA - A \wedge A$ and the commutation relations (1.34) gives

$$F = \sqrt{\frac{\Lambda}{2}} \left[de^1 + \omega \wedge e^2 \right] P_1 + \sqrt{\frac{\Lambda}{2}} \left[de^2 - \omega \wedge e^1 \right] P_2 + \left[d\omega + \frac{\Lambda}{2} e^1 \wedge e^2 \right] P_0 \,. \tag{1.39}$$

The action (1.37) becomes

$$S_{\text{BF}} = -\frac{i}{2} \int \sqrt{\frac{\Lambda}{2}} \left[\phi^1 (de^1 + \omega \wedge e^2) + \phi^2 (de^2 - \omega \wedge e^1) \right] - \phi^0 \left(d\omega + \frac{\Lambda}{2} e^1 \wedge e^2 \right) \,. \tag{1.40}$$

The equations of motion obtained from varying ϕ yields $F = 0$. Specifically, the variation of ϕ^1 and ϕ^2 imply $\tau^a = de^a + \omega^a{}_b \wedge e^b = 0$, with $\omega^1{}_2 = -\omega^2{}_1 = \omega$, which are precisely the zero torsion conditions for the frame e^a with spin connection $\omega^a{}_b$. Plugging these equations back into (1.40) and using the fact that for a 2d manifold $d\omega = \frac{R}{2} e^1 \wedge e^2$, with R being the Ricci scalar, we obtain

$$S_{\text{BF}} = \frac{i}{4} \int d^2x \, \sqrt{g} \, \phi^0 \left(R + \Lambda \right) \,, \tag{1.41}$$

which is precisely the bulk part of the JT action with the dilaton ϕ in the second order formalism identified with $-i\phi^0/4$.[17] Here, the 2d metric is $g_{\mu\nu} = e^1_\mu e^1_\nu + e^2_\mu e^2_\nu$, and $d^2x\sqrt{g} = e^1 \wedge e^2$. The equation of motion obtained from varying ϕ_0 implies $R = -\Lambda$, and since $\Lambda > 0$, we find that the curvature is negative. Thus, the on-shell gauge configurations of the BF theory parameterize a patch of hyperbolic space (Euclidean AdS).

Note that the equations of motion obtained from varying the gauge field, namely

$$D_\mu \phi = \partial_\mu \phi - [A_\mu, \phi] = 0 \,, \tag{1.42}$$

[17] One might be puzzled by the fact that when ϕ^0 is real, ϕ is imaginary. However, when viewing ϕ or ϕ_0 as Lagrange multipliers, this is the natural choice for the reality of both fields. However, note that in the second-order formulation of JT-gravity (1.9) one fixes the value of the dilaton (ϕ) along the boundary to be real. As we describe in Section 2.2.1, we do not encounter such an issue in the first-order formulation, since we will not fix the value of ϕ along the boundary.

can be written as

$$
\begin{aligned}
d\phi_0 &= \sqrt{\frac{\Lambda}{2}}\left(-e^1\phi^2 + e^2\phi^1\right) , \\
d\phi^1 &= -\omega\phi^2 + \sqrt{\frac{\Lambda}{2}}e^2\phi_0 , \\
d\phi^2 &= \omega\phi^1 - \sqrt{\frac{\Lambda}{2}}e^1\phi_0 .
\end{aligned}
\tag{1.43}
$$

It is straightforward to check that taking another derivative of the first equation and using the other two gives the equation for ϕ in (1.10).

The spin connection $\omega^a{}_b$ is a connection on the orthonormal frame bundle associated to a principal $SO(2)$ bundle. For a pair of functions ϵ^a transforming as an $SO(2)$ doublet, the covariant differential acts by $D\epsilon^a = d\epsilon^a + \omega^a{}_b\epsilon^b$. With this notation, we see that the infinitesimal gauge transformations (1.36) in the BF theory with gauge parameter $\epsilon = \sqrt{\Lambda/2}\epsilon^a P_a + \epsilon^0 P_0$ take the form

$$
\begin{aligned}
\delta e^1 &= D\epsilon^1 - \epsilon^0 e^2 , \\
\delta e^2 &= D\epsilon^2 + \epsilon^0 e^1 , \\
\delta\omega &= d\epsilon^0 + \frac{\Lambda}{2}(\epsilon^2 e^1 - \epsilon^1 e^2) .
\end{aligned}
\tag{1.44}
$$

The interpretation of these formulas is as follows. The parameters ϵ^i act as local gauge parameters for the $SO(2)$ symmetry. When the gauge connection is flat with $F = 0$, infinitesimal gauge transformation are related to infinitesimal diffeomorphisms generated by a vector fields ξ^μ (via $\delta g_{\mu\nu} = \nabla_\mu\xi_\nu + \nabla_\nu\xi_\mu$)

$$
\epsilon^a = e^a_\mu\xi^\mu , \qquad \epsilon^0(x) = \omega_\mu(x)\xi^\mu(x) . \tag{1.45}
$$

The parameter ϵ^0 generates an infinitesimal frame rotation, and thus it leaves the 2d metric invariant. Note that the gauge transformations in the BF theory preserve the zero-torsion condition and the 2d curvature because these quantities appear in the expression for F in (1.39) and the equation $F = 0$ is gauge-invariant.

1.4.2 Quantum equivalence

So far, we have solely focused on the classical analysis of the equivalent gauge theory – explicitly, we have shown that the on-shell equations of motion in the bulk agree between the gauge theory formulation and the second order gravitational formulation. We have not yet specified the crucial ingredients that are needed to provide an exact description of the quantum theory: specifying the boundary condition along $\partial\Sigma$ in (1.37) or determining the global structure of the gauge group. Thus, in this thesis we will first focus on possible boundary conditions and boundary terms such that the resulting theory has a well defined variational principle, while later, we will discuss the global structure of the gauge group. Putting the two together, we will then study the exact quantization of this theory and study its observables.

We start by reviewing the possible boundary conditions on the gauge theory side. When placing the gauge theory on a disk, the natural Dirichlet boundary conditions are set by fixing the gauge field or, equivalently, the frame e^a and spin connection ω at the boundary of the disk. In such a case, a boundary term like that in (1.9) does not need to be added to the action in order for the theory to have a well-defined variational principle. The resulting system can be shown to be a trivial topological theory which does not capture the boundary dynamics of (1.9). Consequently, we introduce a boundary condition changing defect whose role in the BF-theory is to switch the natural Dirichlet boundary conditions to those needed in order to reproduce the Schwarzian dynamics. With this boundary changing defect the first and second formulations of JT gravity give rise to the same boundary theory:[18]

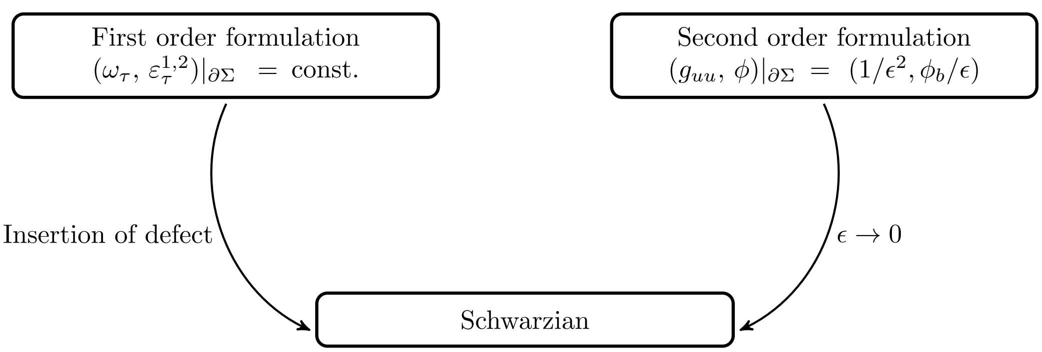


Figure 1.2: Schematic representation showing that the dynamics on the defect in the gauge theory is the same as that in the Schwarzian theory, which in turn describes the boundary degrees of freedom of (1.9).

For the equivalence between the Schwarzian and the gauge theory to continue to hold at the quantum level, we find the gauge group needed to properly capture the global properties of the gravitational theory. As we will show, this is given by an extension of $PSL(2, \mathbb{R})$ by $\mathbb{R}$. This extension is related to the universal cover of the group $PSL(2, \mathbb{R})$, denoted by SL2.[19] With this

[18]Possible boundary conditions for the gauge theory reformulation of JT-gravity were also discussed in [58]. A concrete proposal for the rewriting of the boundary term in (1.9) was also discussed in [59], however the quantization of the theory was not considered.

[19]A similar observation was made in [60]. There it was shown that in order for gravitational diffeomorphisms to be

choice of gauge group, when placing the bulk theory in Euclidean signature on a disk, we find a match between its exact partition function and that computed in the Schwarzian theory [61, 26, 62]. This match is obtained by demanding that the gauge field component along the boundary should vanish.[20]

The first natural observable to consider beyond the partition function is given by introducing probe matter in the gauge theory. On the gauge theory side, introducing probe matter is equivalent to adding a Wilson line anchored at two points on the boundary. In the Schwarzian theory, we expect that this coupling is captured by bilocal operators $\mathcal{O}_\lambda(u_1, u_2)$. We indeed confirm that all the correlation functions of bi-local operators in the Schwarzian theory [40] match the correlation functions of Wilson lines that intersect the defect. More specifically, the time-ordered correlators of bi-local operators in the boundary theory are given by correlators of non-intersecting defect-cutting Wilson lines, while out-of-time-ordered correlators are given by intersecting Wilson line configurations. By computing the expectation value of bulk Wilson lines in the gauge theory, we provide a clear representation of theoretic meaning to their correlators. Furthermore, we provide the combinatorial toolkit needed to compute any such correlator. As we will show, these Wilson lines also have a gravitational interpretation: inserting such Wilson lines in the path integral is equivalent to summing over all possible world-line paths for a particle moving between two fixed points on the boundary of the AdS_2 patch. Furthermore, we discuss the existence of further non-local gauge-invariant operators, which can potentially be used to compute the amplitudes associated with a multitude of scattering problems in the bulk.

1.5 Revisiting the second-order formalism: Going beyond the asymptotic AdS_2 limit

As we have reviewed thus far, traditionally, the JT path integral is computed with Dirichlet boundary conditions, for very large proper boundary lengths and boundary values for the dilaton (see [29]). In that limit, we have shown that the partition function of JT gravity reduces to a simpler path integral over a boundary quantum mechanics theory, the Schwarzian theory. As reviewed, this theory can

mapped to gauge transformations in the BF-theory when placed on a cylinder, one needs to consider a gauge group given by SL2, instead of the typically assumed $PSL(2,\mathbb{R})$.

[20] In a gauge-independent language, here we demand a trivial holonomy around the boundary of the disk. For general boundary holonomy, the dual is given by a non-relativistic particle moving on H_2^+ in a magnetic field, in the presence of an SL2 background gauge field. As we point out in Appendix B.1, this is slightly different than considering the Schwarzian with $SL(2,\mathbb{R})$ twisted boundary conditions, which was considered in [26, 63].

be solved exactly [61, 25, 40].[21] At finite cutoff, the values for the boundary length and dilaton are no longer large, and the computation of the partition function has been an open problem which we resolve in this thesis.

Our computation is naturally motivated by the question of understanding the AdS/CFT correspondence at finite cutoff. As mentioned in section 1.2.3, this question has recently attracted tremendous attention, especially in the context of the AdS/CFT correspondence. In three bulk dimensions, it was conjectured in [16] (see also [66]) that finite patches of asymptotically AdS_3 spacetimes can be obtained by deforming the CFT by the irrelevant $T\overline{T}$ deformation. This duality was only analyzed in the semi-classical limit and, as of yet, its fate in the quantum theory remains unclear. Furthermore, while the $T\overline{T}$ operator satisfies some remarkable properties, its spectrum has numerous unwanted features that we have emphasized in section 1.2.3. To circumvent the complications of 3D/2D holography and yet address some of its unwanted problems, we turn to analyzing the problem in one dimension lower.

We will discuss two possible quantization techniques for the two-dimensional gravitational theory at finite cutoff.

The first is the canonical quantization. In the canonical approach, one foliates the spacetime with a certain, usually time-like, coordinate, and parametrizes the metric in an ADM decomposition [67]. As a result of the diffeomorphism invariance of the action, the action becomes a sum of constraints. These constraints are then uplifted to quantum mechanical constraints, and its solutions are functionals of the data on the chosen foliation. In general, these constraints are difficult to solve unless some approximations, such as the mini-superspace approximation is made. Luckily, for two-dimensional theories of dilaton gravity, the constraints can be reduced to two first-order functional differential equations that can be solved exactly in the quantum theory. This was first demonstrated for JT gravity in the 80s by Henneaux [68] and generalized to general dilaton gravities in [69]. The resulting solutions to the constraints are known as Wheeler-deWitt (WdW) wavefunctionals: they are functionals of diffeomorphism invariant quantities, which in our case are the dilaton profile and boundary length.

The WdW wavefunctionals relevant for our analysis are not the traditional ones, which are constructed for geometries on a constant time slice. Instead, we consider a "radial quantization" in Euclidean signature for which the wavefunctionals are related to the partition function (modulo counterterms) at a finite value for the boundary length and dilaton. This provides a way to compute

[21] For further details about solvability properties of the Schwarzian theory and JT gravity, see [28, 20, 30, 31, 61, 40, 64, 65, 35, 36, 37, 63, 1].

the path integral at finite cutoff.

In the second approach, we compute the Euclidean path integral directly. In general, this is very complicated at finite proper boundary length as some of the gravitational modes are frozen in the large volume limit. After integrating out the dilaton, the JT path integral localizes to one over a boundary action given by the extrinsic curvature K of the boundary. By using constraints from the $SL(2,\mathbb{R})$ isometry of AdS_2, we manage to express K in an expansion solely containing powers of the Schwarzian derivative and its derivatives. This expansion greatly facilitates our computations and allows us to express the partition function as the expectation value of an operator in the Schwarzian theory. Using integrability properties in the Schwarzian theory, we manage to compute the partition function to all orders in a perturbative expansion in the cutoff.

Both the canonical and path integral approaches use widely different techniques to compute the finite cutoff partition function, yet, as expected from the equivalence between the two quantization procedures, the results agree. Luckily we find a perfect agreement between the two approaches with the proposed deformation of the Schwarzian partition function, analogous to the $T\overline{T}$ deformation for $2D$ CFTs [70, 71]. This result thus provides concrete evidence that the duality between the $T\overline{T}$ deformation and the bulk cutoff movement holds not only semi-classically; rather, we show that it is an exact duality.

1.6 Mysteries in black hole thermodynamics

1.6.1 Generic features of extremal and near-extremal black holes

As briefly mentioned in section 1.2.2, extremal, and near-extremal black holes have long offered a simplified set-up to resolve open questions in black hole physics, ranging from analytic studies of mergers to microstate counting. The simplicity of near-extremal black holes comes from the universality of their near-horizon geometry: there is an AdS_2 throat with an internal space that varies slowly as the horizon is approached (see, for example, [15]).

To understand the nature of extremal and near-extremal black holes we start by reviewing the simplest example of Reissner-Nordström, a black hole with electric or magnetic charge. Such black holes are solutions for the Einstein-Hilbert action coupled to electromagnetism[22]

$$I_{EH} = \int d^4x\sqrt{-g}(R - 2\Lambda + F_{\mu\nu}F^{\mu\nu})\,, \tag{1.46}$$

[22] Here, we will neglect the discussion of boundary terms and boundary conditions at the AdS or flat space boundary.

where, in what follows, we will consider black holes in AdS (with $\Lambda < 0$) or in flat space (with $\Lambda = 0$). For a black hole with total electric charge Q and mass M,[23] one finds the solution

$$ds^2_{(4d)} = -f(r)dt^2 + \frac{dr^2}{f(r)} + r^2 d\Omega_2\,, \qquad f(r) = 1 - \frac{2G_N M}{r} + \frac{G_N}{4\pi}\frac{Q^2}{r^2} + \frac{r^2}{L^2}\,. \tag{1.47}$$

Such black holes have two horizons located at r_+ and r_- for which $f(r_\pm) = 0$ (where we will always take $r_+ > r_-$). We can now understand some basic thermal properties of the Reissner-Nordström black hole. Analytically continuing the solution (1.47) to Euclidean signature via a Wick rotation $t \to -i\tau$, we find that the space ends at the location of the exterior horizon r_+. Expanding the metric (1.47) around r_+ we find a patch of flat space; for this patch to be smooth, we need to require that there is no conical singularity which, in turn, implies that the Euclidean time τ needs to be periodically identified, with period $\beta = 4\pi/|f'(r_+)|$. The identification of Euclidean time is equivalent to putting a quantum mechanical system at finite temperature. Thus, β can be identified as the inverse temperature of the black hole, and the radiation emitted at this temperature is called Hawking radiation.

Next, we describe some properties of extremal and near-extremal black holes. Precisely at extremality, the extrior and interior horizons coincide with $r_+ = r_- = r_0$. One can consequently check that for such black holes $|f'(r_+)| = 0$ and consequently such black holes have temperature $T = \beta^{-1} = 0$. Therefore, black holes that are extremal do not emit any Hawking radiation. In the near-extremal limit, black holes have $r_+ - r_- \ll r_h = r_+$; in such a limit, black holes have very small temperatures as compared to their horizon size $\beta \gg r_h$ and therefore radiate slowly. If we consider black holes with fixed charge Q, and temperature T, then one can determine from (1.47) that the mass of the near-extremal black hole can be approximated by

$$M = M_0 + \frac{M_{\text{gap}}}{T^2} \tag{1.48}$$

where one can in principle also capture higher order corrections in T. The meaning of the parameter M_{gap} will be clarified shortly.

As previously mentioned, a crucial property that we will use in the final chapters of this thesis is that the near-horizon geometry simplifies. If one expands $\rho = r - r_h$ for $\delta r \ll r_h$ then one finds

[23] In what follows, we will assume that the black hole solely has an electric charge. However, due to $4d$ electric-magnetic duality, all results are equally applicable to magnetic black holes.

that the metric can be rewritten as

$$ds^2_{(4d)} = \frac{\rho^2 - \delta r_h^2}{L_2^2} d\tau^2 + \frac{L_2^2}{\rho^2 - \delta r_h^2} d\rho^2 + (r_0 + \rho)^2 d\Omega_2 \tag{1.49}$$

where $\delta r_h = r_+ - r_-$. The first two terms capture the geometry of AdS_2, while the second term captures the geometry of an internal space (S^2) whose size is slowly varying as we go away from the horizon. As it turns out, this feature, together with our understanding of JT gravity, will greatly simplify our analysis in what follows.

1.6.2 The problem of the mass gap

While the near-horizon geometry exhibits great simplicity, the thermodynamics of extremal and near-extremal black holes brings up several important open questions. At extremality, black holes have zero temperature, mass M_0, and area A_0. Performing a semiclassical analysis when raising the mass slightly above extremality, one finds that the energy growth of near-extremal black holes scales with temperature as $\delta E = E - M_0 = T^2/M_{\text{gap}}$. Naively, one might conclude that when the temperature, $T < M_{\text{gap}}$, the black hole does not have sufficient mass to radiate even a single Hawking quanta of average energy. Consequently, M_{gap} is considered the energy scale above extremality at which the semiclassical analysis of Hawking must breakdown [72, 73, 74].[24] A possible way to avoid the failings of the semiclassical analysis is to interpret M_{gap} as a literal "mass gap" between the extremal black hole and the lightest near-extremal state in the spectrum of black hole masses. Such a conjecture is, in part, supported by microscopic constructions [75, 76, 77] which suggest that, in the case of black holes with sufficient amounts of supersymmetry, M_{gap} could indeed be literally interpreted as a gap in the spectrum of masses.[25] Nevertheless, it is unclear if such results are an artifact of supersymmetry or whether such a gap truly exists for the most widely-studied non-supersymmetric examples: in Reissner-Nordström (RN) or Kerr-Newman (KN) black holes.

The mass-gap puzzle is related to another critical question of understanding the large zero-temperature entropy of extremal black holes. If a gap exists, and the semiclassical analysis is correct at low temperatures, extremal black holes would exhibit a huge degeneracy proportional to the macroscopic horizon area measured in Planck units. In the absence of supersymmetry, it is unclear

[24] Even at temperatures $T \sim O(M_{\text{gap}})$ there is a breakdown of thermodynamics since a single Hawking quanta with average energy could drastically change the temperature of the black hole.

[25] In [77], it is assumed that the lightest near-extremal state has non-zero spin, in contrast to the extremal Reissner-Nordström. However, in section 5.3, we show that in fact the lightest near-extremal state has zero spin. [75, 76] focus on string constructions for near-extremal black holes in supergravity. Since our analysis depends on the massless matter content in the near-horizon region, we cannot compare our results with the gaped results of [76]. Nevertheless, an analysis of the $2d$ effective theory in the near-horizon region for near-extremal black holes in supergravity is currently underway [78].

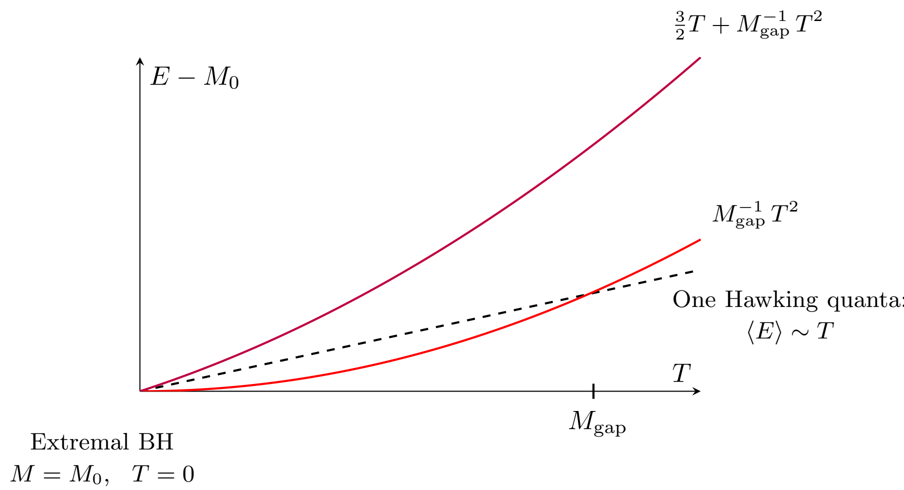


Figure 1.3: Energy above extremality at fixed charge as a function of the temperature when obtained from the semiclassical analysis (in red) and when accounting for the quantum fluctuations in the near-horizon region (in purple). This should be compared to the average energy of one Hawking quanta (dashed line) whose energy is on average $\langle E\rangle \sim T$.

how such a degeneracy could exist without being protected by some other symmetry. Alternatively, if one takes the semiclassical analysis seriously only at temperatures $T \gg M_{\text{gap}}$, then it is possible that the entropy obtained by this analysis, would not count the degeneracy of the ground-state; rather, it could count the total number of states with energy below $E - M_0 \lesssim M_{\text{gap}}$ [74]. We find this solution unsatisfactory; from the Gibbons-Hawking prescription, we should be able to compute the Euclidean path integral at lower temperatures.

In this thesis, we settle the debate about the existence of a mass-gap for $4d$ Reissner-Nordström black holes. We show that such near-extremal black holes do not exhibit a mass gap at the scale M_{gap}.[26] To arrive at this conclusion, we go beyond the semiclassical analysis and account for quantum fluctuations to reliably compute the partition function of such black holes at temperatures $T \sim M_{\text{gap}}$ in the canonical and grand canonical ensembles. By taking the Laplace transform of the partition function, we find the density of states in the spectrum of black holes masses. Due to the presence of $T\log(T/M_{\text{gap}})$ corrections to the free energy, [27] we find that the spectrum looks like a continuum of states and, consequently, exhibits no gap of order $\sim M_{\text{gap}}$. This continuum is observed because our computation is not sensitive enough to distinguish between individual black hole mi-

[26] While in this thesis we will mostly focus on studying $4d$ black holes in an asymptotically flat or AdS_4 space, our analysis could be applied to RN black holes in any number of dimensions.

[27] The $T\log T$ corrections discussed throughout this thesis should not be confused with the logarithmic area corrections to the entropy studied for extremal black holes in [79, 80, 81, 82]. While we did not find any connection between the two corrections (as the logarithmic area correction to the entropy is studied in a specific limit for the mass, charge and temperature; such a limit is not employed in this thesis), it would be interesting to understand whether the results obtained in this thesis can be used to also account for the entropy corrections from [79, 80, 81, 82].

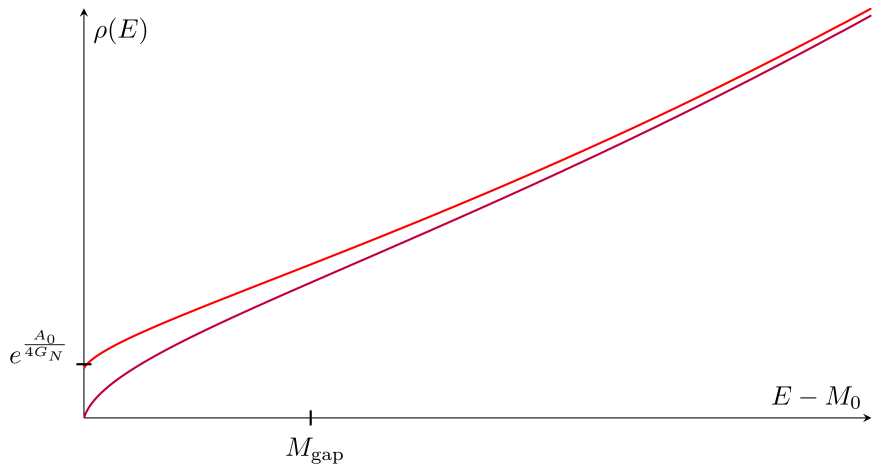


Figure 1.4: Purple: Density of states (at fixed charge) for black holes states as a function of energy above extremality $E - M_0$, including backreaction effects given in (5.72). Red: Plot of the naive density of states $\rho \sim \exp\left(A_{\rm hor}/4G_N\right)$ which starts deviating from the full answer below energies of order $M_{\rm gap}$.

crostates; for that, a UV complete gravitational theory is necessary. Nevertheless, our computation does suggest that, for non-supersymmetric theories, the degeneracy of extremal black holes is much smaller than that obtained from the area-law Bekenstein-Hawking entropy (in figure 1.4 we show the shape of the density of states at fixed charge)[28].

The potential breakdown of Hawking's analysis raised in [72] is also resolved. In figure 1.3, we compare the temperature dependence of the energy above extremality in the classical analysis and when accounting for quantum fluctuations. As opposed to the semiclassical analysis, we find that when only slightly above extremality, $E - M_0 \sim \frac{3}{2} T > T$, therefore resolving the naive failure of thermodynamics at very small temperatures.

A similar analysis was done recently for near-extremal rotating BTZ black holes in AdS_3 [84]. These black holes present a breakdown of their statistical description at low temperatures when restricted to the semiclassical analysis. The breakdown is similarly resolved by including backreaction effects in the Euclidean path integral.

To reliably compute the partition function at such small temperatures, we perform a dimensional reduction to the two dimensional AdS_2 space in the near-horizon region[29]. We find that the only relevant degrees of freedom that affect the density of states are the massless modes coming from the gravitational sector, the electromagnetic gauge field, and the $SO(3)$ gauge fields generated by the

[28] The logic in this thesis is very different from the argument in [83]. The degeneracy of the extremal black hole and the presence of a gap depends on the amount of supersymmetry in the theory (see section 5.5).

[29] The geometry describing the throat is $\mathrm{AdS}_2 \times \mathrm{S}^2$. Even though the size of the transverse sphere r_0 is large, we will consider temperatures well below the KK scale $T \ll M_{\rm KK} \sim 1/r_0$. This is consistent since, in all cases, we study the gap is a parametrically smaller scale $T \sim M_{\rm gap} \ll M_{\rm KK}$.

dimensional reduction. The resulting effective theory turns out to be that of $2d$ Jackiw-Teitelboim (JT) gravity [85, 39] coupled to gauge degrees of freedom. The Euclidean path integral of such an effective theory can be computed exactly by first integrating out the gauge degrees [3, 86] and then by analyzing the boundary modes [87] of the resulting model using the well-studied Schwarzian theory.[30]

The connection between JT gravity and near-extremal black holes has been widely discussed in past literature.[31] In fact, in [88], the scale $M_{\rm gap}$ defined through the thermodynamics was identified as the symmetry breaking scale for the emergent near-horizon AdS_2 isometries, $SL(2,\mathbb{R})$. Moreover, this is also the scale at which the equivalent Schwarzian theory becomes strongly coupled. However, compared to past literature, to compute the partition function at small temperatures, $T \sim M_{\rm gap}$, we had to keep track of all the fields generated through the dimensional reduction and exactly compute the path integral for the remaining massless relevant degrees of freedom. Our qualitative picture is nevertheless similar to that presented in [88] as we show that the semiclassical analysis fails due to the backreaction of the dilaton and gauge fields on the metric.

For the reasons described above, to avoid confusion from now on, we will stop calling the scale in which the semiclassical analysis breaks down $M_{\rm gap}$ since there is no gap at that scale. Instead we will redefine it as $M_{\rm gap} \to \frac{1}{2\pi^2} M_{\rm SL(2)}$.[32] More importantly, we want to stress that the appropriate meaning of this energy is the symmetry breaking scale of the approximate near horizon conformal symmetry.

[30] See [28, 20, 29, 31, 61, 25, 40, 37] for details.
[31] See [88, 89, 90, 91, 92, 93, 94, 95, 96, 97, 98, 99, 100]
[32] The factor of $2\pi^2$ will be useful but is just conventional.

Chapter 2

Dilaton gravity in the first-order formalism

2.1 Outline of results

This section elaborates on the ideas presented in section 1.4 and is organized as follows. In Section 2.2 we show the on-shell equivalence between the equations of motion of the Schwarzian theory and those in the gauge theory description of JT gravity, when boundary conditions are set appropriately. In Section 2.3 we discuss the quantization of the gauge theory. In this process, in order to match results in the Schwarzian theory, or, alternatively in the second order formulation of JT gravity, we determine a consistent global structure for the gauge group and determine potential boundary conditions such that the partition function of the gauge theory agrees with that of the Schwarzian. In Section 2.4, we show the equivalence between Wilson lines in the gauge theory and bi-local operators in the boundary theory. Furthermore, we discuss the role of a new class of gauge invariant non-local operators and compute their expectation value. Finally we discuss future directions of investigation in Section 2.5. In Appendix B.1, we review various properties of the Schwarzian theory and derive at the level of the path integral, its equivalence to a non-relativistic particle moving in hyperbolic space in the presence of a magnetic field. For the readers interested in details, we suggest reading Appendix B.3 and B.4 where we provide a detailed description of harmonic analysis on the SL2 group manifold and derive the fusion coefficients for various representations of SL2. Finally, we revisit the gravitational interpretation of the gauge theory observables in Appendix B.5 and we show that Wilson lines that intersect the defect are equivalent to probe particles in JT-gravity propagating between different points on the boundary.

2.2 Classical analysis of $\mathfrak{sl}(2,\mathbb{R})$ gauge theory

2.2.1 Variational principle, boundary conditions, and string defects

Infinitesimal variations of the action (1.37) yield

$$\delta S_{BF} = \text{(bulk equations of motions)} \;\; - i\int_{\partial\Sigma} \text{Tr}\,\left(\phi\delta A_\tau\right)\,, \tag{2.1}$$

where τ is used to parametrize the boundary $\partial\Sigma$. As is well-known [101] and can be easily seen from the variation (2.1), the BF theory has a well-defined variational principle when fixing the gauge field A_τ along the boundary $\partial\Sigma$. In the first-order formulation of JT gravity, this amounts to fixing the spin connection and the frame and no other boundary term is necessary in order for the variational principle to be well defined.[1] In fact, due to gauge invariance, observables in the theory will depend on A_τ only through the holonomy around the boundary,

$$\tilde{g} = \mathcal{P}\exp\left(\int_{\partial\Sigma} A\right) \in \mathcal{G}\,, \tag{2.2}$$

instead of depending on the local value of A_τ. However, solely fixing the gauge field around the boundary yields a trivial topological theory (see more in Section 2.3). Of course, such a theory cannot be dual to the Schwarzian. In order to effectively modify the dynamics of the theory we consider a defect along a loop I on Σ. A generic way of inserting such a defect is by adding a term S_I, to the BF action,

$$S_E = S_{BF} + S_I\,, \qquad S_I = e\int_0^\beta du\, V(\phi(u))\,. \tag{2.3}$$

where u is the proper length parametrization of the loop I, whose coordinates are given by $x_I(u)$ and whose total length is β measured with the induced background metric from the disk.[2]

Since, the overall action needs to be gauge invariant we should restrict $V(\phi)$ to be of trace-class; as we will prove shortly in order to recover the Schwarzian on-shell we simply set $V(\phi) = -\text{Tr}\,\phi^2/4$, with the trace in the fundamental representation of $\mathfrak{sl}(2,\mathbb{R})$.

[1] This is in contrast with the second-order formulation of JT gravity (1.9), when fixing the metric and the dilaton along the boundary. In such a case the boundary term in (1.9) needs to be added to the action in order to have a well defined variational principle.

[2] Consequently, the defect is not topological.

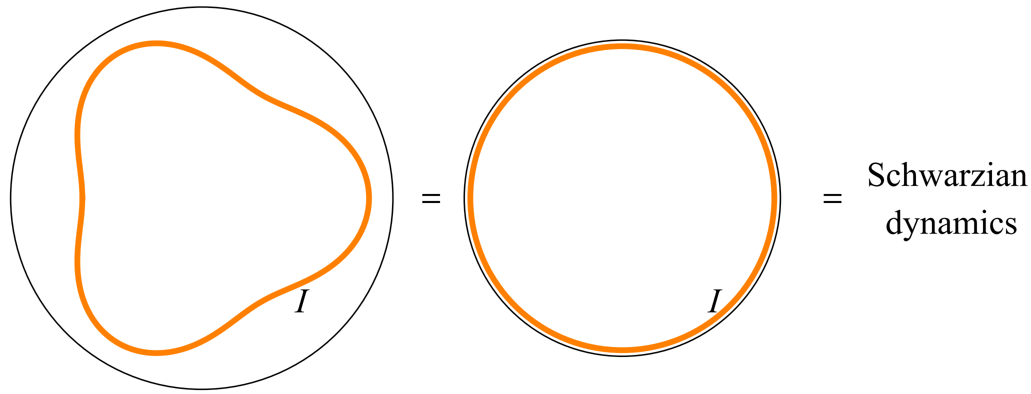


Figure 2.1: Cartoon emphasizing the properties of the string defect. The resulting theory is invariant under perimeter preserving defect diffeomorphisms and thus the defect can be brought arbitrarily close to the boundary of the manifold. Furthermore, the degrees of freedom of the gauge theory defect can be captured by those in the Schwarzian theory.

Note that as a result of the Schwinger-Dyson equation

$$\left\langle d\mathrm{Tr}\,\phi^2(x)\dots\right\rangle_{BF} = -2i\left\langle \mathrm{Tr}\,\left(\phi(x)\frac{\delta}{\delta A(x)}\right)\dots\right\rangle_{BF} = 0 \tag{2.4}$$

$\mathrm{Tr}\,\phi^2$ is a topological operator in the BF theory independent of its location on the spacetime manifold, as long as the other insertions represented by $\dots$ above do not involve A.[3]

As emphasized in Figure 2.1, due to the fact that theory is topological away from I and due to the appearance of the length form in (2.3) the action is invariant under diffeomorphisms that preserve the local length element on I.[4] Thus, one can modify the metric on Σ, away from I, in order to bring it arbitrarily close to the boundary $\partial\Sigma$. This proves convenient for our discussion below since we fix the component A_τ of the gauge field along the boundary and can thus easily use the equations of motion to solve for the value of ϕ along I.

Specifically, we choose

$$A_\tau\Big|_{\text{bdy}} \equiv \omega\,\ell_0 + \sqrt{\frac{\Lambda}{2}}e_+\,\ell_+ + \sqrt{\frac{\Lambda}{2}}e_-\,\ell_-\,, \tag{2.5}$$

[3]In the $\mathfrak{sl}(2,\mathbb{R})$ gravitational theory, $-\mathrm{Tr}\,\phi^2$ is usually interpreted as a black hole mass and its conservation law can be interpreted as an energy conservation law [59].

[4]This is similar to 2d Yang-Mills theory which is invariant under area preserving diffeomorphisms [102, 103, 104].

where

$$\begin{aligned}
\ell_0 &\equiv i P_0 \,, & \ell_+ &\equiv -P_2 - i P_1 \,, & \ell_- &\equiv P_2 - i P_1 \,, \\
\omega &\equiv -i \omega_\tau \Big|_{\text{bdy.}} \,, & e_+ &\equiv \frac{i e^1_\tau - e^2_\tau}{2} \Big|_{\text{bdy.}} \,, & e_- &\equiv \frac{i e^1_\tau + e^2_\tau}{2} \Big|_{\text{bdy.}} \,.
\end{aligned} \tag{2.6}$$

The generators ℓ_0 and $\ell_\pm$ satisfy the commutation relations

$$[\ell_\pm, \ell_0] = \pm \ell_\pm \,, \qquad [\ell_+, \ell_-] = 2\ell_0 \,. \tag{2.7}$$

As previously discussed, all observables can only depend on the value of the holonomy, thus without loss of generality we can set ω and $e_\pm$ to be constants whose value we discuss in the next subsection. Fixing the value of the gauge field, in turn, sets the metric in the JT-gravity interpretation along the boundary to be $g_{\tau\tau} = -4e_+e_-$.

The equation of motion obtained by varying A_τ close to the boundary, $D_\tau \phi = \partial_\tau \phi - [A_\tau|_{\text{bdy}}, \phi] = 0$, can be used to solve for the value of ϕ along I. It is convenient to relate the two parametrizations of the defect I through the function $u(\tau)$, choosing τ in such a way that $e\phi_-(\tau) \equiv \sqrt{\Lambda} e_- / \partial_\tau u(\tau)$, where $\phi = \phi_0 \ell_0 + \phi_+ \ell_+ + \phi_- \ell_-$. Instead of solving the equation of motion for A_τ in terms of $u(\tau)$ it is more convenient to perform a reparametrization and rewrite the equation in terms of $\tau(u)$ using $A_u = A_\tau \tau'(u)$, where $\tau'(u) \equiv \partial_u \tau(u)$. The solution to the equation of motion for the ℓ_- and ℓ_0 components of $D_u \phi = 0$ yields

$$e\phi(u) = \sqrt{2\Lambda} e_- \ell_- \tau' + 2\ell_0 \left(\omega \tau' - \frac{\tau''}{\tau'} \right) + \sqrt{2\Lambda} \ell_+ \left(e_+ \tau' + \frac{\tau'''}{\Lambda e_- (\tau')^2} - \frac{\omega \tau''}{\Lambda e_- \tau'} - \frac{(\tau'')^2}{\Lambda e_- (\tau')^3} \right) , \tag{2.8}$$

where $\tau(u)$ is further constrained from the component of the $D_u \phi = 0$ along ℓ_+,

$$0 = 4 \det A_\tau (\tau')^4 \tau'' + 3(\tau'')^3 - 4\tau' \tau'' \tau''' + (\tau')^2 \tau'''' \,, \tag{2.9}$$

with $\det A_\tau = \left(-\omega^2 + 2\Lambda e_- e_+ \right)/4 = (2\omega_\tau^2 - \Lambda g_{\tau\tau})/8|_{\text{bdy}}$. When considering configurations with $\tau'(u) = 0$ (and $\tau'' \neq 0$ or $\tau''' \neq 0$), $\phi(u)$ becomes divergent and consequently the action also diverges. Thus, we restrict to the space of configurations where $\tau(u)$ is monotonic, and we can set $\tau(\beta) - \tau(0) = L$, where L is an arbitrary length whose meaning we discuss shortly. Using this solution for $\phi(u)$ we can now proceed to show that the dynamics on the defect is described by the Schwarzian.

2.2.2 Recovering the Schwarzian action

We can now proceed to show that the Schwarzian action is a consistent truncation of the theory (2.3). We start by integrating out ϕ inside the defect which sets $F = 0$ and thus the nonvanishing part of the action (2.3) comes purely from the region between (and including) the defect and the boundary. Next we *partially* integrate out A_τ in this region using the equations of motion of $D_u\phi = 0$ along the ℓ_- and ℓ_0 directions, whose solution is given by (2.8). Plugging (2.8) back into the action (2.3), we find that the total action can be rewritten as[5]

$$S_E[\tau] = -\frac{1}{e}\int_0^\beta du\left(\{\tau(u), u\} + 2\tau'(u)^2 \det A_\tau\right) , \qquad \tau(\beta) - \tau(0) = L\,, \tag{2.10}$$

where the determinant is computed in the fundamental representation of $\mathfrak{sl}(2,\mathbb{R})$. The equation of motion obtained by infinitesimal variations $\delta\tau(u)$ in (2.10) yields [29]

$$\partial_u\left[\{\tau(u), u\} + 2\tau'(u)^2 \det A_\tau\right] = 0 \tag{2.11}$$

which is equivalent to (2.9) that was obtained directly from varying all components of A_τ in the original action (2.3). This provides a check that the dynamics on the boundary condition changing defect in the gauge theory is consistent with that of the action (2.10).

Finally, performing a change of variables,

$$F(u) = \tan\left(\sqrt{\det A_\tau}\,\tau(u)\right) , \tag{2.12}$$

we recover the Schwarzian action as written in (1.19),

$$S_E[F] = -\frac{1}{e}\int_0^\beta du\{F(u),\, u\}\,. \tag{2.13}$$

While we have found that the dynamics on the defect precisely matches that of the Schwarzian we have not yet matched the boundary conditions for (2.13) with those typically obtained from the second-order formulation of JT gravity: $\beta = L$ and $F(0) = F(\beta)$.[6] The relation between L and β is obtained by requiring that the field configuration is regular inside of the defect I: this can be

[5] This reproduces the result in [58, 59] where the Schwarzian action was obtained by adding a boundary term similar to that in (2.3), by imposing a relation between the boundary value of the gauge field A_τ and the zero-form field ϕ and by fixing the overall holonomy around the boundary. In our discussion, by using the insertion of the defect, we greatly simplify the quantization of the theory. Our method is similar in spirit to the derivation of the 2D Wess-Zumino-Witten action from 3D Chern-Simons action with the appropriate choice of gauge group [105].

[6] Instead the relation between $F(0)$ and $F(\beta)$ in (2.13), with the boundary conditions set by those in (2.10), is

achieved by requiring that the holonomy around a loop inside of I be trivial. In order to discuss regularity we thus need to address the exact structure of the gauge group instead of only specifying the gauge algebra. To gain intuition about the correct choice of gauge group it will prove useful to first discuss the quantization of the gauge theory and that of the Schwarzian theory.

2.3 Quantization and choice of gauge group

So far we have focused on the classical equivalence between the $\mathfrak{sl}(2,\mathbb{R})$ gauge theory formulation of JT gravity and the Schwarzian theory. This discussion relied only on the gauge algebra being $\mathfrak{sl}(2,\mathbb{R})$, with the global structure of the gauge group not being important. We will now extend this discussion to the quantum level, where, with a precise choice of gauge group in the 2d gauge theory, we will reproduce exactly the partition function and the expectation values of various operators in the Schwarzian theory.

2.3.1 Quantization with non-compact gauge group $\mathcal{G}$

We would like to consider the theory with action (2.3) and (non-compact) gauge group $\mathcal{G}$ (to be specified below), defined on a disk D with the defect inserted along the loop I of total length β. The quantization of gauge theories with non-compact gauge groups has not been discussed much in the literature,[7] although there is extensive literature on the quantum 2d Yang-Mills theory with compact gauge group [102, 103, 104, 108, 109, 110, 111, 112].[8] Let us start with a brief review of relevant results on the compact gauge group case, and then explain how these results can be extended to the situation of interest to us.

What is commonly studied is the 2d Yang-Mills theory defined on a manifold $\mathcal{M}$ with a compact gauge group G, with Euclidean action

$$S^{\text{2d YM}}[\phi, A] = -i \int_{\mathcal{M}} \text{Tr}\,(\phi F) - g_{\text{YM}}^2 \int_{\mathcal{M}} d^2x\,\sqrt{g}\,V(\phi)\,, \qquad V(\phi) = \frac{1}{4}\text{Tr}\,\phi^2\,. \tag{2.15}$$

After integrating out ϕ, this action reduces to the standard form $-\frac{1}{2g_{\text{YM}}^2}\int_{\mathcal{M}} d^2x\,\sqrt{g}\text{Tr}\,F_{\mu\nu}F^{\mu\nu}$. When quantizing this theory on a spatial circle, it can be argued that due to the Gauss law constraint,

given by,

$$F(\beta) = \frac{\cos(\sqrt{\det A_\tau}L)F(0) + \sin(\sqrt{\det A_\tau}L)}{-\sin(\sqrt{\det A_\tau}L)F(0) + \cos(\sqrt{\det A_\tau}L)}\,. \tag{2.14}$$

[7]See however, [106] and comments about non-unitarity in Yang-Mills with non-compact gauge group in [107].
[8]See also the more recent discussion about the quantization of Yang-Mills theory when coupled to JT gravity [3].

the wave functions are simply functions $\Psi[g]$ of the holonomy $g = P \exp[\oint A^a T_a]$ around the circle that depend only on the conjugacy class of g. Here T^a are anti-Hermitian generators of the group G. The generator T^a are normalized such that $\mathrm{Tr}\,(T^i T^j) = N\eta^{ij}$, where for compact groups we set $\eta^{ij} = \mathrm{diag}(-1, \ldots, -1)$. Thus, the wavefunctions $\Psi[g]$ are class functions on G, and a natural basis for them is the "representation basis" given by the characters $\chi_R(g) = \mathrm{Tr}\,_R g$ of all unitary irreducible representations R of G.

The partition function of the theory (2.15) when placed on a Euclidean manifold $\mathcal{M}$ with a single boundary is given by the path integral,

$$Z_{\mathcal{M}}^{\text{2d YM}}(g, g_{\text{YM}}^2 \mathcal{A}) = \int D\phi\, DA\, e^{-S^{\text{2d YM}}[\phi,\, A]} \tag{2.16}$$

where we impose that overall G holonomy around the boundary of $\mathcal{M}$ be given by g. Note that this partition function depends on the choice of metric for $\mathcal{M}$ only through the total area $\mathcal{A}$ (as the notation in (2.16) indicates, it depends only on the dimensionless combination $g_{\text{YM}}^2 \mathcal{A}$). The partition function can be computed using the cutting and gluing axioms of quantum field theory from two building blocks: the partition function on a small disk and the partition function on a cylinder. For the disk partition function $Z_{\text{disk}}^{\text{2d YM}}(g, g_{\text{YM}}^2 \mathcal{A})$, which in general depends on the boundary holonomy g and $g_{\text{YM}}^2 \mathcal{A}$, the small $\mathcal{A}$ limit is identical to the small g_{YM}^2 limit in which (2.15) becomes topological. In this limit, the integral over ϕ imposes the condition that A is a flat connection, which gives $g = \mathbf{1}$, so [103]

$$\lim_{\mathcal{A}\to 0} Z_{\text{disk}}^{\text{2d YM}}(g, g_{\text{YM}}^2 \mathcal{A}) = \delta(g) = \sum_R \dim R\, \chi_R(g)\,. \tag{2.17}$$

Here, $\delta(g)$ is the delta-function on the group G defined with respect to the Haar measure on G, which enforces that $\int dg\, \delta(g) x(g) = x(\mathbf{1})$.

To determine this partition functions at finite area, note that the action (2.15) implies that the canonical momentum conjugate to the space component of the gauge field $A_1^i(x)$ is $\phi_i(x)$, and thus the Hamiltonian density that follows from (2.15) is just $\frac{g_{\text{YM}}^2}{4} \mathrm{Tr}\,(\phi_i T^i)^2$. In canonical quantization, one find that $\pi_j = -iN\phi_j$ and the Hamiltonian density becomes $H = -\frac{g_{\text{YM}}^2}{4N} \eta^{ij} \pi_i \pi_j$. Using $\pi_j = \frac{\delta}{\delta A_1^j}$, each momentum acts on the wavefunctions $\chi_R(g)$ as $\pi_i \chi_R(g) = \chi_R(T_i g)$. It follows that the Hamiltonian density derived from the action (2.15) acts on each basis element of the Hilbert space $\chi_R(g)$ diagonally with eigenvalue $g_{\text{YM}}^2 C_2(R)/(4N)$ [104], where $C_2(R)$ is the quadratic Casimir, with

$C_2(R) \geq 0$ for compact groups. One then immediately finds

$$Z^{\text{2d YM}}_{\text{disk}}(g, g^2_{\text{YM}}\mathcal{A}) = \sum_R \dim R\, \chi_R(g) e^{-\frac{g^2_{\text{YM}}}{4N}\mathcal{A}C_2(R)}\,. \tag{2.18}$$

From these expressions, sticking with compact gauge groups for now, one can determine the disk partition function of a modified theory

$$S = -i\int_{\mathcal{M}} \text{Tr}\,(\phi F) - e\int_I du\, V(\phi)\,, \qquad V(\phi) = \frac{1}{4}\text{Tr}\,\phi^2\,, \tag{2.19}$$

where I is a loop of length β as in Figure 2.1. Such an action can be obtained by modifying the Hamiltonian of the theory to a time-dependent one and by choosing time-slices to be concentric to the loop I. [9] Applying such a quantization to the theory with a loop defect we obtain

$$Z(g, e\beta) = \sum_R \dim R\, \chi_R(g) e^{-\frac{e\beta\, C_2(R)}{4N}}\,. \tag{2.20}$$

One modification that one can perform in the above discussion is to consider, either in (2.15) or in (2.19) a more general $V(\phi)$ than $\frac{1}{4}\text{Tr}\,\phi^2$. For example, if $V(\phi) = \frac{1}{4}\text{Tr}\,\phi^2 + \frac{1}{4}\alpha(\text{Tr}\,\phi^2)^2$, then one should replace $C_2(R)$ by $C_2(R) + \frac{\alpha}{N}C_2(R)^2$ in all the formulas above.

The discussion above assumed that G is compact, and thus the spectrum of unitary irreps is discrete. The only modification required in the case of a non-compact gauge group $\mathcal{G}$ is that the irreducible irreps are in general part of a continuous spectrum.[10] To generalize the proof above, we have to use the Plancherel formula associated with non-compact groups in (2.17)

$$\delta(g) = \sum_R \dim R\, \chi_R(g) \qquad \to \qquad \delta(g) = \int dR\, \rho(R)\, \chi_R(g)\,, \tag{2.21}$$

where $\rho(R)$ is the Plancherel measure.[11] Then, following the same logic that led to the disk partition function in (2.18), by determining the Hamiltonian density and applying it to the characters in (2.21),

[9] Alternatively, one can consider the gluing of a topological theory with $g^2_{\text{YM}} = 0$ in the regions inside and outside I, and a theory of type (2.15) in a fattened region around I of a small width (so that the region does not intersect with other operator insertions such as Wilson lines).

[10] For the case with non-compact gauge group we will continue to maintain the same sign convention in Euclidean signature as that shown in (2.16).

[11] In the case in which the spectrum of irreps has both continuous and discrete components, $\rho(R)$ will be a distribution with delta-function support on the discrete components.

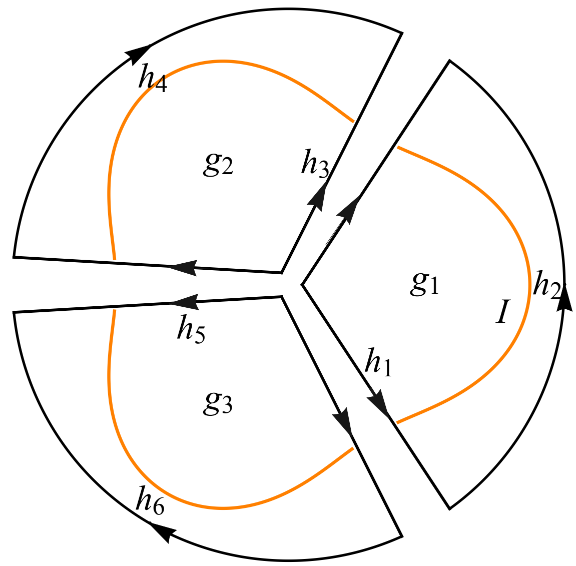


Figure 2.2: Cartoon showing an example of gluing of three disk patches whose overall partition function is given by the gluing rules in (2.23). Each segment has an associated group element h_a and each patch has an associated holonomy g_i. In the case pictured above: $g_1 = h_1 h_2 h_3^{-1}$, $g_2 = h_3 h_4 h_5^{-1}$ and $g_3 = h_5 h_6 h_1^{-1}$. We take all edges to be oriented in the counter-clockwise direction.

we find that the disk partition function of the theory (2.19) reduces to

$$Z(g, e\beta) = \int dR\, \rho(R) \chi_R(g) e^{-\frac{e\beta\, C_2(R)}{4N}} \,. \tag{2.22}$$

where we normalize the generators P^i of the non-compact group by $\mathrm{Tr}\,(P^i P^j) = N\eta^{ij}$, where η^{ij} is diagonal with ± 1 entries. In these conventions we set the Casimir of the group to be given by $\widehat{C}_2 = -\eta^{ij} P_i P_j$. One may worry that if the gauge group is non-compact, then it is possible for the quadratic Casimir $C_2(R)$ to be unbounded from below, and then the integral (2.22) would not converge. If this is the case, we should think of $V(\phi)$ in (2.19) as a limit of a more complicated potential such that the integral (2.22) still converges. For instance, we can add $\frac{1}{4}\alpha(\mathrm{Tr}\,\phi^2)^2$ to (2.19) and consequently $\alpha C_2(R)^2$ to the exponent of (2.22) as described above.

In order to consider more complicated observables, we can glue together different segments of the boundary of the disk. In general, the gluing of n disks, each containing a defect I_j of length β_j, onto a different manifold Σ with a single boundary with holonomy g, will formally be given by integrating over all group elements $h_1, h_2, \ldots, h_m$ associated to the $\mathcal{C}_1, \ldots, \mathcal{C}_m$ segments which need to be glued. Here, $h_i = \mathcal{P}\exp\int_{\mathcal{C}_i} A$. The resulting partition function is given by[12]

$$Z(g,\, e\beta,\, \Sigma) = \frac{1}{\text{Vol}(\mathcal{G})^m} \int \left(\prod_{i=1}^{m} dh_i\right) \left(\prod_{j=1}^{n} Z(g_j(h_a), e\beta_j)\right) \delta\left(g^{-1}\prod_{j=1}^{n} g_j(h_a)\right)\,, \tag{2.23}$$

where the product i runs over all m edges which need to be glued, while the product j runs over the labels of the n disks. Each disk j comes with a total holonomy $g_j(h_a)$ depending on the group elements h_a associated to each segment $\mathcal{C}_a$ along the boundary of disk j. Thus, for instance if the edge of the disk j consists of the segments $\mathcal{C}_1,\, \dots,\, \mathcal{C}_{m_j}$ (in counter-clockwise order), then $g_j(h_a) = h_1 \cdots h_{m_j}$. Furthermore, dh_i denotes the Haar measure on the group $\mathcal{G}$, which is normalized by the group volume. The group δ-function imposes that the total holonomy around the boundary of Σ is fixed to be g. An example of the gluing of three patches is given in Figure 2.2.

While for compact gauge groups (2.23) yields a convergent answer when considering manifolds Σ with higher genus or no boundary, when studying non-compact gauge theories on such manifolds divergences can appear. This is due to the fact that the unitary representations of a non-compact group $\mathcal{G}$ are infinitely dimensional.[13]

2.3.2 The Schwarzian theory and SL2 representations

In order to identify the gauge group $\mathcal{G}$ for which the theory (2.3) becomes equivalent to the Schwarzian theory at the quantum level, let us first understand what group representations are relevant in the quantization of the Schwarzian theory. Specifically, the partition function of the Schwarzian theory at temperature β is given, up to a regularization dependent proportionality constant, by

$$Z_{\text{Schwarzian}}(\beta) \propto \int_0^\infty ds s \sinh(2\pi s) e^{-\frac{\beta}{2C}s^2}\,, \tag{2.24}$$

(computed using fermionic localization in [26]), can be written as an integral of the form

$$Z_{\text{Schwarzian}}(\beta) \propto \int dR\, \rho(R) e^{-\frac{\beta}{2C}\left[C_2(R)-\frac{1}{4}\right]}\,. \tag{2.25}$$

[12] Various formulae useful for gluing in gauge theory are shown in Appendix B.2, where results for compact gauge groups and non-compact gauge groups are compared.

[13] When setting $\mathcal{G}$ to be $PSL(2,\mathbb{R})$ or one of its extensions, these divergences are in tension with the expected answers in the gravitational theory (1.9). This is a reflection of the fact that while the moduli space of Riemann surfaces has finite volume, the moduli space of flat $PSL(2,\mathbb{R})$ (or other group extensions of $PSL(2,\mathbb{R})$) connections does not. Thus, the techniques applied in this chapter are only valid for manifolds with the topology of a disk. See [37, 42, 113, 3] for a detailed discussion about a sum over all topologies.

over certain irreps of the universal cover SL2.[14]

To identify the representations R needed to equate (2.24) to (2.25), let us first review some basic aspects of SL2 representation theory, following [115]. The irreducible representations of $\widetilde{SL}(2,\mathbb{R})$ are labeled by two quantum numbers λ and μ. These can be determined from the eigenvalue $\lambda(1-\lambda)$ of the quadratic Casimir $\widehat{C}_2 = -\eta^{ij}P_iP_j = P_0^2 - P_1^2 - P_2^2 = -\ell_0^2 + (\ell_-\ell_+ + \ell_+\ell_-)/2$, as well as the eigenvalue $e^{2\pi i\mu}$ under the generator $e^{-2\pi i \ell_0}$ of the $\mathbb{Z}$ center of the SL2. Furthermore, states within each irreducible representation are labeled by an additional quantum number m which represents the eigenvalue under ℓ_0. Thus,

$$\begin{aligned} &\widehat{C}_2|\lambda,\mu,m\rangle = \lambda(1-\lambda)|\lambda,\mu,m\rangle\,, \\ &\ell_0|\lambda,\mu,m\rangle = -m|\lambda,\mu,m\rangle \qquad \text{with} \qquad m\in\mu+\mathbb{Z}\,. \end{aligned} \tag{2.26}$$

One can go between states with different values of m using the raising and lowering operators:

$$\begin{aligned} \ell_-|\lambda,\mu,m\rangle &= -\sqrt{(m-\lambda)(m-1+\lambda)}|\lambda,\mu,m-1\rangle\,, \\ \ell_+|\lambda,\mu,m\rangle &= -\sqrt{(m+\lambda)(m+1-\lambda)}|\lambda,\mu,m+1\rangle\,. \end{aligned} \tag{2.27}$$

where the generators satisfy the $\mathfrak{sl}(2,\mathbb{R})$ algebra (2.7). Using these labels and requiring the positivity of the matrix elements of the operators L_+L_- and L_-L_+ one finds that there are four types of irreducible unitary representations:[15]

- *Trivial representation I:* $\mu = 0$ and $m = 0$;
- *Principal unitary series $\mathcal{C}^{\mu}_{\lambda=\frac{1}{2}+is}$:* $\lambda = \frac{1}{2}+is$, $m=\mu+n$, $n\in\mathbb{Z}$, $-1/2\leq\mu\leq 1/2$;
- *Positive/negative discrete series $\mathcal{D}^{\pm}_{\lambda}$:* $\lambda>0$, $\lambda=\pm\mu$, $m=\pm\lambda\pm n$, $n\in\mathbb{Z}^+$, $\mu\in\mathbb{R}$;
- *Complementary series $\mathcal{C}^{\mu}_{\lambda}$:* $|\mu|<\lambda<1/2$, $m=\mu+n$, $n\in\mathbb{Z}$,[16]

Only the principal unitary series and the positive/negative discrete series admit a well defined Hermitian inner-product, so for them one can define a density of states given by the Plancherel measure (up to a proportionality constant given by the regularization of the group's volume).

[14] As already discussed in [40, 35, 114] and as we explain in Appendix B.1, we can interpret $H = \left(\widehat{C}_2 - 1/4\right)/C$ as the Hamiltonian of a quantum system and $\rho(R)$ as the density of states. Such an interpretation can be made precise after noticing that the Schwarzian theory is equivalent to the theory of a non-relativistic particle in 2D hyperbolic space placed in a pure imaginary magnetic field.

[15] The two-dimensional representation (corresponding to $\lambda=-1/2$ and $\mu=\pm1/2$) used in Section 2.2 in order to write down the Lagrangian is not a unitary representation and therefore does not appear in the list below.

[16] Since in the Plancherel inversion formula the complementary series does not appear, we will not include it in any further discussion.

As reviewed in Appendix B.3, the principal unitary series has the Plancherel measure given by

$$\rho(\mu, s)\, d\mu\, ds = \frac{(2\pi)^{-2} s \sinh(2\pi s)}{\cosh(2\pi s) + \cos(2\pi\mu)}\, ds\, d\mu\,, \qquad \text{with } -\frac{1}{2} \leq \mu \leq \frac{1}{2}\,, \tag{2.28}$$

and for the positive and negative discrete series

$$\rho(\lambda) d\lambda = (2\pi)^{-2} \left(\lambda - \frac{1}{2}\right) d\lambda\,, \qquad \text{with } \lambda = \pm\mu, \qquad \lambda \geq \frac{1}{2}\,, \tag{2.29}$$

where $\lambda = \mu$ for the positive discrete series and $\lambda = -\mu$ for the negative discrete series.

Matching (2.25) to (2.24) can be done in two steps:

1. We first restrict the set of R that appear in (2.25) to representations with fixed $e^{2\pi i \mu}$. As mentioned above, this quantity represents the eigenvalue under the generator of the $\mathbb{Z}$ center of SL2. After this step, (2.25) becomes

$$\int_0^\infty ds\, \frac{(2\pi)^{-2} s \sinh(2\pi s)}{\cosh(2\pi s) + \cos(2\pi\mu)} e^{-\frac{\beta}{2C} s^2} + \sum_{n=1}^{n_{\max}} \frac{1}{2\pi^2} \left(\mu + n - \frac{1}{2}\right) e^{-\frac{\beta}{2C}\left[(\mu+n)(1-\mu-n) - \frac{1}{4}\right]}, \tag{2.30}$$

provided that we took $\mu \in [-\frac{1}{2}, \frac{1}{2})$. In writing (2.30) we imposed a cutoff $n_{\max}$ on the discrete series representations. A different regularization could be achieved by adding the square of the quadratic Casimir in the exponent, with a small coefficient. As a function of μ, Eq. (2.30) can be extended to a periodic function of μ with unit period.

2. We analytically continue the answer we obtained in the previous step to $\mu \to i\infty$. When doing so, the sum in (2.30) coming from the discrete series goes as $e^{-\frac{\beta}{C}(\operatorname{Im}\mu)^2}$, and the integral coming from the continuous series goes as $e^{-2\pi|\operatorname{Im}\mu|}$. Thus, when $\operatorname{Im}\mu \to \infty$ the continuous series dominates, and (2.30) becomes proportional to the partition function of the Schwarzian.

As was already discussed in [28, 35, 114, 40] and we review in Appendix B.1, fixing $\mu \to i\infty$ can also be understood in deriving the equivalence between the Schwarzian and a non-relativistic particle in 2D hyperbolic space, as fixing the magnetic field $\tilde{B}$ to be pure imaginary, $\tilde{B} = -\frac{iB}{2\pi} = \mu$, with $B \to \infty$. As we shall see below, on the gauge theory side, fixing the parameter $\mu \to i\infty$ can be done with an appropriate choice of the gauge group $\mathcal{G}$ and boundary conditions.

2.3.3 $PSL(2,\mathbb{R})$ extensions, one-form symmetries, and revisiting the boundary condition

In Section 2.3.2 we have gained some insight about the SL2 representations that are needed in order to write the Schwarzian partition function as in (2.25). We thus seek to choose a gauge group and boundary conditions that automatically isolate precisely the same representations as in Step 1 above. We then choose the defect potential for the 2D gauge theory to achieve the desired analytically continued gauge theory partition function presented in Step 2.

Choice of gauge group

In a pure gauge theory the center of the gauge group gives rise to a one-form symmetry under which Wilson loops are charged [116]. Thus, since an SL2 gauge group gives rise to a $\mathbb{Z}$ one-form symmetry, fixing the charge under the center of the gauge group is equivalent to projecting down to states of a given one-form symmetry charge. A well known way to restrict the one-form symmetry charges in the case of a compact gauge group G is by introducing an extra generator in the gauge algebra and embedding the group G into its central extension [116, 117].

In the case of non-compact groups we proceed in a similar fashion, and consider a new gauge group which is given by the central extension of $PSL(2,\mathbb{R})$ by $\mathbb{R}$,[17]

$$\mathcal{G}_B \equiv \frac{\mathrm{SL2}\times\mathbb{R}}{\mathbb{Z}}\,, \tag{2.31}$$

where the quotient, and, consequently, the definition of the group extension, is given by the identification

$$(\tilde{g},\theta)\sim(h_n\tilde{g},\theta+Bn)\,. \tag{2.32}$$

Above, $\tilde{g}\in\mathrm{SL2}$ and $\theta\in\mathbb{R}$, h_n is the n-th element of $\mathbb{Z}$ and $B\in\mathbb{R}$ is the parameter which defines the extension. The resulting irreducible representations of $\mathcal{G}_B$ can be obtained from irreducible representation of $\mathrm{SL2}\times\mathbb{R}$ which are restricted by the quotient (2.31). The unitary representations of $\mathbb{R}$ are one-dimensional and are labeled by their eigenvalue under the $\mathbb{R}$ generator, $I|k\rangle = k|k\rangle$. In other words, the action of a general $\mathbb{R}$ group element $U^{\mathbb{R}}(\theta)=e^{iI\theta}$ on the state $|k\rangle$ is given by

[17] Such extensions are classified by the Čech cohomology group $\check{\mathrm{H}}^1(SL(2,\mathbb{R}),\mathbb{R})\simeq\mathrm{Hom}(\pi_1(SL(2,\mathbb{R})\to\mathbb{R})\simeq\mathrm{Hom}(\mathbb{Z}\to\mathbb{R})\simeq\mathbb{R}$ where $\mathrm{Hom}(\mathbb{Z}\to\mathbb{R})$ classifies the set of homomorphisms from $\mathbb{Z}$ to $\mathbb{R}$. In other words, all extensions by $\mathbb{R}$ will be given by a push-forward from the elements of $\mathbb{Z}$ center of SL2 to elements of $\mathbb{R}$. A basis of homomorphisms from $\mathbb{Z}$ to $(\mathbb{R},+)$ are given by $f_B(n)=Bn$ for $B\in\mathbb{R}$. Such a homomorphism imposes the identification (2.31) for different elements of the group [118].

multiplication by $U_k^{\mathbb{R}}(\theta) = e^{ik\theta}$.

Considering the representation U_k of $\mathbb{R}$ and a representation $U_{\lambda,\mu}$ of the SL2, evaluated on the group element (h_n, θ) we have $U_{\lambda,\mu,k}^{\mathrm{SL2}\times\mathbb{R}}(h_n,\theta) = U_{\lambda,\mu}(h_n)U_k^{\mathbb{R}}(\theta) = e^{2\pi i\mu n + ik\theta}$. We now impose the quotient identification (2.32) on the representations, $e^{ik\theta} = e^{2\pi i\mu n + ik(\theta+Bn)}$, which implies $k = -2\pi\left(\mu - p\right)/B$, with $p \in \mathbb{Z}$. Thus, $\mathbb{R}$ irreps labeled by k restrict the label μ of representations in (2.26) to be[18]

$$\mu = -\frac{Bk}{2\pi} + p\,, \qquad \text{with } p \in \mathbb{Z}\,. \tag{2.33}$$

Thus, by projecting down to a representation k of $\mathbb{R}$ in the 2D gauge theory partition function, we can restrict to representations with a fixed eigenvalue $e^{2\pi i\mu}$ for the center of the gauge group $\mathbb{Z}$.

In order to understand how to perform the projection to a fixed k (or $e^{2\pi i\mu}$) in the BF theory, it is useful to explicitly write down the $\mathcal{G}_B$ gauge theory action.

To start, we write the gauge algebra $\mathfrak{sl}(2,\mathbb{R}) \oplus \mathbb{R}$,

$$[\tilde{\ell}_\pm, \tilde{\ell}_0] = \pm\tilde{\ell}_\pm\,, \qquad [\tilde{\ell}_+, \tilde{\ell}_-] = 2\tilde{\ell}_0 - \frac{BI}{\pi}\,, \qquad e^{2\pi i\tilde{\ell}_0} = 1\,, \qquad [\tilde{\ell}_{0,\pm}\,, I] = 0\,, \tag{2.34}$$

where the condition $e^{2\pi i\tilde{\ell}_0} = 1$, imposed on the group, enforces the representation restriction (2.33). Of course at the level of the algebra, we can perform the redefinition $\ell_0 = \tilde{\ell}_0 - BI/(2\pi)$ and $\ell_\pm = \tilde{\ell}_\pm$ to still find that $\ell_{0,\pm}$ satisfy an $\mathfrak{sl}(2,\mathbb{R})$ algebra (2.7) from which we can once again define the set of generators P_i using (2.6). Considering a theory with gauge group $\mathcal{G}_B$ in (2.31), we can write the gauge field and zero-form field ϕ as[19]

$$A = e^a P_a + \omega P_0 + \frac{B^2}{\pi^2}A^{\mathbb{R}} I\,, \qquad \phi = \phi^a P_a + \phi^0 P_0 + \phi^{\mathbb{R}} I\,, \tag{2.35}$$

where $a = 1,\,2$ and where α is the $\mathbb{R}$ gauge field. Thus, the gauge invariant action (2.3) can be written as

$$S_E = -i\int_\Sigma \left(\frac{\phi^a F_a + \phi^0 F_0}{2} + \phi^{\mathbb{R}} F^{\mathbb{R}}\right) - e\int_{\partial\Sigma} du\;\; V(\phi^0,\, \phi^\pm)\,. \tag{2.36}$$

Since the $\mathfrak{sl}(2,\mathbb{R})$ generators form a closed algebra, it is clear that under a general gauge transformation the e^a and ω transform under the actions of $\mathfrak{sl}(2,\mathbb{R})$, while α transforms independently under the action of $\mathbb{R}$. Thus one can fix the holonomy of the $\mathfrak{sl}(2,\mathbb{R})$ gauge components independently

[18]For $B = 0$ one simply finds the trivial extension of $PSL(2,\mathbb{R})$ by $\mathbb{R}$ which does not contain SL2 as a subgroup.
[19]Note that the normalization for the $\mathbb{R}$ component of A is such that the BF-action in (2.36) is in a standard form.

from that of $\mathbb{R}$.[20]

Revisiting the boundary condition

Since the two sectors are decoupled, we can independently fix the holonomy $\tilde{g}$ of the $\mathfrak{sl}(2,\mathbb{R})$ components of the gauge field, as specified in Section 2.2, and fix the value of $\phi^{\mathbb{R}} = k_0$ on the boundary. In order to implement such boundary conditions and in order for the overall action to have a well-defined variational principle, one can add a boundary term

$$S_{\text{bdy.}} = i \oint_{\partial\Sigma} \phi^{\mathbb{R}} A^{\mathbb{R}} \,. \tag{2.37}$$

to the action (2.36). The partition function when fixing this boundary condition can be related to that in which the $\mathcal{G}_B$ holonomy $g = (\tilde{g}, \theta)$, is fixed, with $\tilde{g} = \oint_{\partial\Sigma} A^i P_i \in \mathrm{SL2}$ and $\theta = \oint_{\partial\Sigma} A^{\mathbb{R}} \in \mathbb{R}$, as

$$Z_{k_0}(\tilde{g},\, e\beta) = \int d\theta Z((\tilde{g}, \theta),\, e\beta) e^{-ik_0\theta} \,. \tag{2.38}$$

More generally, without relying on (2.38), following the decomposition of the partition function into a sum of irreducible representation of $\mathcal{G}_B$, fixing $\phi^{\mathbb{R}} = k_0$, isolates the contribution of the $\mathbb{R}$ representation labeled by k_0, in the partition function, or equivalently fixes $e^{2\pi i\mu}$ with $\mu = -\frac{Bk_0}{2\pi} + \text{integer}$. This achieves the goal of Step 1 in the previous subsection 2.3.2.

To achieve Step 2, namely sending $\mu \to i\infty$, or equivalently $kB \to i\infty$, we can choose

$$\mathcal{G} \equiv \mathcal{G}_B \qquad \text{with} \qquad B \to \infty \,, \qquad \phi^{\mathbb{R}} = k_0 = -i \,. \tag{2.39}$$

Note that all the groups $\mathcal{G}_B$ with $B \neq 0$ are isomorphic. Therefore, one can make different choices when considering the limits in (2.39) as long as the invariant quantity $kB \to i\infty$.

Alternatively, instead of fixing the value of $\phi^{\mathbb{R}}$ on the boundary, the change in boundary condition (2.38) can be viewed as the introduction of a 1D complexified Chern-Simons term for the $\mathbb{R}$ gauge field component α, $S_{CS} = ik_0 \oint_{\partial\Sigma} A^{\mathbb{R}}$, which is equivalent to the boundary term in (2.37). By adding

[20] We now briefly revisit the equivalence between the gauge theory and JT-gravity, as discussed in Section 1.4.1. One important motivation for this is that Section 1.4.1 solely focused on an $\mathfrak{sl}(2,\mathbb{R})$ gauge algebra while $\mathcal{G}_B$ has an $\mathfrak{sl}(2,\mathbb{R}) \oplus \mathbb{R}$ algebra. The equations of motion for the $\mathfrak{sl}(2,\mathbb{R})$ components are independent from those for the $\mathbb{R}$ components, namely $F^{\mathbb{R}} = 0$ and $\phi^{\mathbb{R}} =$ constant. Thus, the $\mathfrak{sl}(2,\mathbb{R})$ and $\mathbb{R}$ sectors are fully decoupled and, since $F_{\mathbb{R}} = 0$, the $\mathbb{R}$ sector does not contribute to the bulk term in the action. Finally, note that $\mathcal{G}_B$ indeed has a two-dimensional representation with $(\lambda, \mu, k) = (-1/2, \pm 1/2, \mp\pi/B)$, as discussed in Section 2.2.2 when recovering the Schwarzian action. Since we will be considering the limit $B \to \infty$ throughout this chapter, the contribution from the $\mathbb{R}$ component to $\operatorname{Tr}\phi^2$ in this two dimensional representation is suppressed. Thus, the classical analysis in Section 2.2 is unaffected by the extension of the group.

such a term to the action and by integrating over the $\mathbb{R}$ holonomy we once again recover the partition function given by (2.38) .

Thus, the choice of gauge group $\mathcal{G}$ (with $B \to \infty$) together with the boundary condition for the field $\phi^{\mathbb{R}}$ or through the addition of the boundary Chern-Simons discussed above, will isolate the contribution of representations with $k = k_0$ in the partition function.[21] Finally, note that in order to perform the gluing procedure described in Section 2.3.1, one first computes all observables in the presence of an overall $\mathcal{G}$ holonomy. By using (2.38) one can then fix $\phi^{\mathbb{R}} = k_0$ along the boundary and obtain the result with $k_0 = -i$ by analytic continuation.[22]

Higher order corrections to the potential $V(\phi)$

Finally, as shown in Section 2.2 in order to reproduce the Schwarzian on-shell the potential $V(\phi^0, \phi^\pm, \phi^{\mathbb{R}})$ needed to be quadratic to leading order. However, as we shall explain below, one option is to introduce higher order terms, suppressed in $\mathcal{O}(1/B)$, in order to regularize the contribution of discrete series representations whose energies (given by the quadratic Casimir) are arbitrarily negative. Thus, we choose

$$V(\phi^0, \phi^\pm, \phi^{\mathbb{R}}) = \frac{1}{2} + \frac{1}{4}\,\mathrm{Tr}_{\,(\mathbf{2}, -\frac{\pi}{B})}\, \phi^2 \\ + \text{ higher order terms in } \phi \text{ suppressed in } 1/B\,, \qquad (2.40)$$

where $\mathrm{Tr}_{\,(\mathbf{2}, -\frac{\pi}{B})}$ is the trace taken in the two-dimensional representation with $k = -\frac{\pi}{B}$, and the shift in the potential is needed in order to reproduce the shift for the Casimir seen in (2.25). Note that in the limit $B \to \infty$, the trace only involves the $\mathfrak{sl}(2, \mathbb{R})$ components of ϕ. While observables are unaffected by the exact form of these higher order terms, their presence regularizes the contribution of such representations to the partition function.[23]

2.3.4 The partition function in the first-order formulation

Since we have proven that the degrees of freedom in the second-order formulation of JT-gravity can be mapped to those in the first-order gauge theory formulation, we expect that with the appropriate

[21] Note that in such a case the representations of $\mathbb{R}$ with $k \in \mathbb{C} \setminus \mathbb{R}$ are not δ-function normalizable.

[22] This analytic continuation is analogous to the one needed in Chern-Simons gravity when describing Euclidean quantum gravity [119].

[23] An example for such a higher-order term is given by $e^{(2)} \left(\left(\phi^0\right)^2 + 2\phi^+\phi^- + \frac{1}{4} \right) / B$ where $e^{(2)} \sim O(1)$ is a new coupling constant in the potential .

choice of measure and boundary conditions, the two path integrals agree:

$$\int D\phi\, DA\, e^{-S_E[\phi,\, A]} \cong \int Dg_{\mu\nu}\, D\phi\, e^{-S_{JT}[\phi,\, g]}\,. \tag{2.41}$$

Using all the ingredients in Section 2.3.3, we can now show that the partition function of the gravitational theory (2.41) matches that of the Schwarzian. We first compute the partition function in the presence of a fixed $\mathcal{G}$ holonomy is given by

$$Z(g,\, e\beta) \propto \int_{-\infty}^{\infty} dk \int_0^{\infty} ds \frac{s \sinh(2\pi s)}{\cosh(2\pi s) + \cos(Bk)} \chi_{(s,\mu=-\frac{Bk}{2\pi},k)}(g) e^{-\frac{e\beta s^2}{2}}$$
$$+ \text{ discrete series contribution}\,, \tag{2.42}$$

where, we remind the reader that the generators P_i satisfying the $\mathfrak{sl}(2,\mathbb{R})$ algebra are normalized by $\mathrm{Tr}_{\mathbf{2}}(P^i P^j) = -\eta^{ij}/2$ with $\eta^{ij} = \mathrm{diag}(-1,\, 1,\, 1)$. When using the symbol "$\propto$" in the computation of various observables in the gauge theory we mean that the result is given up to a regularization dependent, but β-*independent*, proportionality constant.

Using this result, we can now understand the partition function in the presence of the mixed boundary conditions discussed in the previous subsection. To leading order in B the partition function with a fixed holonomy $\tilde{g}$ and a fixed value of $\phi^{\mathbb{R}} = k_0 = -i$ is dominated by terms coming from the principal series representations,

$$Z_{k_0}(\tilde{g},\, e\beta) \propto e^{-B} \int_0^{\infty} ds\, s\, \sinh(2\pi s) \chi_{s,\mu=-\frac{Bk_0}{2\pi}}(\tilde{g}) e^{-\frac{e\beta s^2}{2}} + O(e^{-2B})\,, \tag{2.43}$$

where $\chi_{s,\mu}(\tilde{g})$ is the character of the SL2 principal series representation labeled by $(\lambda = 1/2 + is, \mu)$ evaluated on the group element $\tilde{g}$, which can be parametrized by exponentiating the generators in (2.7) as $\tilde{g} = e^{\phi P_0} e^{\xi P_1} e^{-\eta P_0}$. For $\phi - \eta \in [2\pi(n-1),\, 2\pi n)$, the character for the continuous series representation s is given by

$$\chi_{s,\mu}(\tilde{g}) = \begin{cases} e^{2\pi i \mu n} \left(\frac{|x|^{1-2\lambda} + |x|^{-1+2\lambda}}{|x - x^{-1}|} \right)\,, & \text{for } \tilde{g} \text{ hyperbolic}, \\ 0\,, & \text{for } \tilde{g} \text{ elliptic}. \end{cases} \tag{2.44}$$

Here, x (and x^{-1}) are the eigenvalues of the group element $\tilde{g}$, when expressed in the two-dimensional representation (see Appendix B.3). Note that for hyperbolic elements, $x \in \mathbb{R}$, with $|x| > 1$, and the character is non-vanishing, while for elliptic elements, we have $|x| = 1$ (with $x \notin \mathbb{R}$) and the character is always vanishing.[24]

Note that since in the partition function only representations with a fixed value of μ contribute, when the holonomy is set to different center elements h_n of $\mathcal{G}$, the partition function will only differ by an overall constant $e^{2\pi i\mu n}$ as obtained from (2.44). For simplicity we will consider $\tilde{g} = \mathbf{1}$. The character in such a case can be found by setting $n = 0$ and taking the limit $x \to 1^+$ from the hyperbolic side in (2.44). In this limit, the character is divergent, yet the divergence is independent of the representation, s. Thus, as suggested in Section 2.3.3, we find that after setting $k_0 = -i$ via analytic continuation in the limit $B \to \infty$,

$$Z_{k_0} \propto \Xi \int_0^\infty ds\, \rho(s) e^{-\frac{e\beta s^2}{2}}\,, \qquad \rho(s) \equiv s\sinh(2\pi s)\,, \tag{2.45}$$

where $\Xi = \lim_{x\to 1^+, n=0} \chi_{s,\mu}(g)$ is the divergent factor mentioned above, which comes from summing over all states in each continuous series irrep $\lambda = 1/2 + is$. Note that we have absorbed the factor of e^{-B} in (2.43) by redefining our regularization scheme, thus changing the partition function by an overall proportionality constant. In the remainder of this chapter we will use this regularization scheme in order to compute all observables.

Performing the integral in (2.45) we find

$$Z_{k_0} = \Xi \left(\frac{2\pi}{e\beta}\right)^{\frac{3}{2}} e^{\frac{2\pi^2}{e\beta}}\,. \tag{2.46}$$

Thus, up to an overall regularization dependent factor, we have constructed a bulk gauge theory whose energies and density of states (2.45) match that of the Schwarzian theory (2.24) for $\frac{1}{C} = e$, reproducing the relationship suggested in the classical analysis.

2.4 Wilson lines, bi-local operators and probe particles

An important class of observables in any gauge theory are Wilson lines and Wilson loops,

$$\widehat{\mathcal{W}}_R(\mathcal{C}) = \chi_R\left(\mathcal{P}\exp\int_{\mathcal{C}} A\right)\,, \tag{2.47}$$

where R is an irreducible representation of the gauge group, $\mathcal{C}$ denotes the underlying path or loop, and $\chi_R(g)$ is the character of $\mathcal{G}$. When placing the theory on a topologically trivial manifold all

[24] In Appendix B.1 we confirm the expectation that (2.43) reproduces the partition function in the Schwarzian theory when twisting the boundary condition for the field $F(u)$ by an SL2 transformation $\tilde{g}$. We expect such configurations with non-trivial holonomy to correspond to singular gravitational configurations.

Wilson loops that do not intersect the defect are contractible and therefore have trivial expectation values. A more interesting class of non-trivial non-local operators in the gauge theory are the Wilson lines that intersect the defect loop and are anchored on the boundary.

To determine the duals of such operators, we start by focusing on Wilson lines in the positive or negative discrete series irreducible representation of $\mathcal{G}$, with $R = (\lambda^{\pm}, \mp\frac{2\pi\lambda}{B})$ where the $\pm$ superscripts distinguish between the positive and negative discrete series. In the $B \to \infty$ limit, this representation becomes $R = (\lambda^{\pm}, 0)$.[25] As we will discuss in detail below, in order to regularize the expectation value of these boundary-anchored Wilson lines, we will replace the character $\chi_R(g)$ in (2.47) by a truncated sum $\overline{\chi}_R(g)$ over the diagonal elements of the matrix associated to the infinite-dimensional representation R.

We propose the duality between such Wilson lines, "renormalized" by an overall constant N_R,

$$\mathcal{W}_\lambda \equiv \widehat{\mathcal{W}}_R(\mathcal{C}_{\tau_1,\tau_2})/N_R = \overline{\chi}_R\left(\mathcal{P}\exp\int_{\mathcal{C}} A\right)/N_R\,, \tag{2.48}$$

and bi-local operators $\mathcal{O}_\lambda(\tau_1, \tau_2)$ in the Schwarzian theory, defined in terms of the field $F(u)$ appearing in (1.19)

$$\mathcal{O}_\lambda(\tau_1, \tau_2) \equiv \left(\frac{\sqrt{F'(\tau_1)F'(\tau_2)}}{F(\tau_1) - F(\tau_2)}\right)^{2\lambda}\,. \tag{2.49}$$

Our goal in this section will thus be to provide evidence that [26]

$$\mathcal{O}_\lambda(\tau_1, \tau_2) \quad \Longleftrightarrow \quad \mathcal{W}_\lambda(\mathcal{C}_{\tau_1,\tau_2})\,, \tag{2.50}$$

for any boundary-anchored path $\mathcal{C}_{\tau_1,\tau_2}$ on the disk D that intersects I at points τ_1 and τ_2 (see the bottom-left diagram in Figure 2.3).[27]

If imposing that gauge transformations are fixed to the identity along the boundary, the group

[25] If choosing the Wilson lines to be in the principal series representations, they would have imaginary correlation functions whose meaning is not clear in the context of a physical theory where we expect the expectation value of observables to be real. From the perspective of a particle moving on a worldline discussed in section 2.4.1 and in appendix B.5, Wilson lines in the principal series representation are equivalent to probe particles whose mass is imaginary.

[26] As we will elaborate on shortly, when using the proper normalization, both Wilson lines in the positive or negative discrete series representation $\mathcal{D}^{\pm}_{\lambda}$ will be dual to insertions of $\mathcal{O}_\lambda(\tau_1, \tau_2)$. For intersecting Wilson-line insertions we will consider the associated representations to be either all positive discrete series or all negative discrete series. Note that the gauge theory has a charge-conjugation symmetry due to the $\mathbb{Z}_2$ outer-automorphism of the $\mathfrak{sl}(2,\mathbb{R})$ algebra that acts as $(P_0, P_1, P_2) \to (-P_0, P_1, -P_2)$. In particular, the principal series representations are self-conjugate, but the positive and negative series representations $\mathcal{D}^{\pm}_{\lambda}$ are exchanged under this $\mathbb{Z}_2$. Since the boundary condition $A_\tau = 0$ preserves the charge-conjugation symmetry, the Wilson lines associated to the representations $\mathcal{D}^{\pm}_{\lambda}$ have equal expectation values.

[27] Similar Wilson lines have been previously considered for compact gauge group [120]. They have also been considered in the context of a dimensional reduction from 3D Chern-Simons gravity [121, 122].

element $g = \mathcal{P}\exp\int_{\mathcal{C}} A$ is itself gauge invariant. While so far it was solely necessary to fix the holonomy around the boundary, to make the boundary-anchored Wilson lines (2.48) well-defined, we have to now specify the value of the gauge field on the boundary.[28] For this reason throughout this section we will set $A_\tau = 0$. With this choice of boundary conditions, we will perform the path integral with various Wilson line insertions and match with the corresponding correlation functions of the bilocal operators computed using the equivalence between the Schwarzian theory and a suitable large c limit of 2D Virasoro CFT [40]. We then generalize our result to any configuration of Wilson lines and reproduce the general diagrammatic 'Feynman rules' conjectured in [40] for correlation functions of bi-local operators in the Schwarzian theory .

2.4.1 Gravitational interpretation of the Wilson line operators

The matching between correlation functions of the bilocal operator and of boundary-anchored Wilson lines should not come as a surprise. On the boundary side, the bilocal operator should be thought of as coupling the Schwarzian theory to matter. After rewriting JT-gravity as the bulk gauge theory, the Wilson lines are described by coupling a point-probe particle to gravity. A similar situation has been studied when describing 3D Einstein gravity in terms of a 3D Chern-Simons theory with non-compact gauge group [8, 123, 124, 125, 126, 127, 128, 129], and the relation is analogous in 2D, in the rewriting presented in Section 1.4.1. Specifically, as we present in detail in Appendix B.5, the following two operator insertions are equivalent in the gauge theory/gravitational theory:[29]

$$\mathcal{W}_\lambda(\mathcal{C}_{\tau_1\tau_2}) \cong \int_{\text{paths}\sim\mathcal{C}_{\tau_1\tau_2}} [dx]\, e^{-m\int_{\mathcal{C}_{\tau_1\tau_2}} ds\,\sqrt{g_{\alpha\beta}\dot{x}^\alpha\dot{x}^\beta}}, \tag{2.51}$$

The right-hand side represents the functional integral over all paths $x(s)$ diffeomorphic to the curve $\mathcal{C}_{\tau_1\tau_2}$ weighted with the standard point particle action (with $\dot{x}^\alpha = \frac{dx^\alpha}{ds}$). In turn, this action is equal to the mass m times the proper length of the path, where the mass m is determined by the representation λ of the Wilson line, $m^2 = -C_2(\lambda) = \lambda(\lambda-1)$. In computing their expectation values, the mapping between the gauge theory and the gravitational theory should schematically yield

$$\int D\phi DA e^{-S_E[A]}\frac{\chi_R(g)}{\mathcal{C}_R} = \int Dg_{\mu\nu}D\phi \int_{\text{paths}\sim\mathcal{C}_{\tau_1\tau_2}}[dx]e^{-S_{JT}[g,\,\phi]-m\int_{\mathcal{C}_{\tau_1\tau_2}} ds\,\sqrt{g_{\alpha\beta}\dot{x}^\alpha\dot{x}^\beta}}\,. \tag{2.52}$$

[28] More precisely we have to specify the holonomy between any two points at which the Wilson lines intersect the boundary.

[29] Note that the discussion in appendix B.5 shows the equivalence of the two insertions beyond the classical level. Typically, in 3D Chern-Simons theory the equivalence has been shown to be on-shell. See for instance [127].

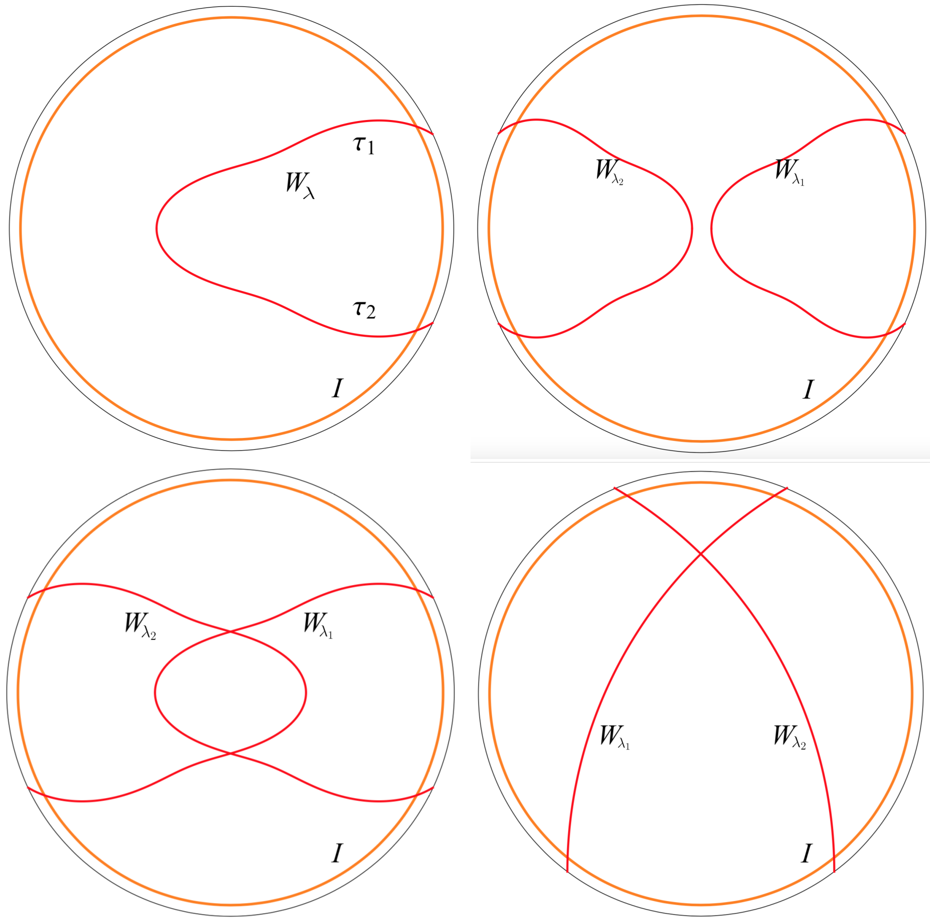


Figure 2.3: Several Euclidean Wilson line configurations, equivalent to different finite temperature correlation functions of the bi-local operator $\mathcal{O}_\lambda(x_1, x_2)$: the top-left figure shows $\langle \mathcal{O}_\lambda(\tau_1, \tau_2)\rangle_\beta = \langle \mathcal{W}_\lambda(\mathcal{C}_{\tau_1,\tau_2})\rangle$, the top-right figure yields the equality of the time-ordered correlators $\langle \mathcal{O}_{\lambda_1}(\tau_1, \tau_2)\mathcal{O}_{\lambda_2}(\tau_3, \tau_4)\rangle_\beta = \langle \mathcal{W}_{\lambda_1}(\mathcal{C}_{\tau_1,\tau_2})\mathcal{W}_{\lambda_2}(\mathcal{C}_{\tau_3,\tau_4})\rangle$, the bottom-left figure shows a pair of intersecting Wilson lines that can be disentangled to the top-right configuration, while the bottom-right figure gives the out-of-time-ordered configurations. Note that the results are independent of the trajectory of the Wilson line inside of the bulk and only depend on the location where the Wilson lines intersect the defect.

Thus, the expectation value of Wilson lines does not only match the expectation value of bi-local operators on the boundary, but it also offers the possibility to compute the exact coupling to probe matter in JT-gravity (see [114] for an alternative perspective).

2.4.2 Two-point function

The correlation function for a single Wilson line that ends on two points on the boundary, in a 2D gauge theory placed on a disk D, is given by the gluing procedure described in Section 2.3.1.

Specifically, for the group $\mathcal{G}$, the un-normalized expectation value is given by

$$\langle \widehat{\mathcal{W}}_{\lambda^\pm,k}(\mathcal{C}_{\tau_1,\tau_2})\rangle(g) = \int dh Z\left(h, e\tau_{21}\right) \chi^{\pm}_{\lambda,k}(h) Z\left(gh^{-1}, e\tau_{12}\right)\,, \tag{2.53}$$

where $\tau_{21} = \tau_2 - \tau_1$ is the length of I enclosed by the boundary-anchored Wilson line $\mathcal{C}_{\tau_1,\tau_2}$ and $\tau_{12} = \beta - \tau_2 + \tau_1$ is the complementary length of I. Here and below, $Z(h, e\tau)$ is the partition function computed in (2.42) on a patch of the disk, in the presence of a defect of length τ inside the patch, when setting the holonomy to be h around the boundary of the patch. The total $\mathcal{G}$ holonomy around the boundary holonomy of the disk is set to g. Since we are interested in the case in which the gauge field along the boundary is trivial, we will want to consider the limit $\tilde{g} \to \mathbf{1}$ at the end of this computation. As was previously mentioned, the Wilson line is in the positive or negative discrete series representation $(\lambda^\pm, k = 0)$ of $\mathcal{G}$, where $k = \mp\frac{2\pi\lambda}{B}$ is the $\mathbb{R}$ representation mentioned in Section 2.3 that becomes 0 due to the $B \to \infty$ limit. Expanding (2.53) in terms of characters by using (2.42), we find

$$\begin{aligned}\langle \widehat{\mathcal{W}}_{\lambda^\pm,k}(\mathcal{C}_{\tau_1,\tau_2})\rangle(g) = &\int dh \int_{-\infty}^{\infty} dk_1 dk_2 \int_0^{\infty} ds_1 ds_2\, \rho\left(\frac{Bk_1}{2\pi},\, s_1\right) \rho\left(\frac{Bk_2}{2\pi},\, s_2\right) \\ &\times \chi_{(s_1,\mu_1=-\frac{Bk_1}{2\pi},k_1)}(h)\, \overline{\chi}^{\pm}_{\lambda,k}(h)\, \chi_{(s_2,\mu_2=-\frac{Bk_2}{2\pi},k_2)}(gh^{-1}) e^{-\frac{e}{2}\left[s_1^2\tau_{21}+s_2^2\tau_{12}\right]} \\ &+ \text{ discrete series contributions }\,.\end{aligned} \tag{2.54}$$

As in the previous sections, we are interested in obtaining observables in the presence of mixed boundary conditions in which we set $\phi^{\mathbb{R}} = k_0 = -i$. This isolates the representations with $k_1 = k_2 = -i$ and, the limit $B \to \infty$ sets the $\mathbb{R}$ representation of the Wilson line $k = \mp 2\pi\lambda/B \to 0$.[30] However, an order of limits issue appears: since the $\mathcal{G}$ representation of the Wilson line is infinite dimensional we have to consider the $B \to \infty$ limit carefully. Thus, instead of inserting the full character in (2.53) we truncate the number of states in the positive or negative discrete series using the cut-off Ξ, with $\Xi \ll B$,

$$\overline{\chi}_{\lambda^\pm,0}(g) = \sum_{k=0}^{\Xi} U^{\pm(\lambda+k)}_{\lambda,\pm(\lambda+k)}(\tilde{g})\,, \tag{2.55}$$

where $g = (\tilde{g}, \theta)$ with $\tilde{g}$ an element of SL2 and θ an element of $\mathbb{R}$, $U^{\pm(\lambda+k)}_{\lambda,\pm(\lambda+k)}(\tilde{g})$ is the SL2 matrix element computed explicitly in Appendix B.3.

[30] In this limit, all contributions appearing as sums over the discrete series representations in (2.54) once again vanish.

Since the values of k_i are fixed and the integral over the $\mathbb{R}$ component of h is trivial, we are thus left with performing the integral over the SL2 components $\tilde{h}$ of h. In order to perform this integral, we use the SL2 fusion coefficients between two continuous series representations and a discrete series representation that we computed in Appendix B.4 in the limit μ_1, $\mu_2 \to i\infty$. When expanding the product of an $\mathcal{C}_{s_1}^{\mu\to i\infty}$ continuous series and a $\mathcal{D}_{\lambda^\pm}$ discrete series character into characters of the continuous series $\mathcal{C}_{s_2}^{\mu\pm\lambda} = \mathcal{C}_{s_2}^{\mu\to i\infty}$, we find the fusion coefficients between the three representations, $N^{\lambda}{}_{s_1,s_2} = N^{s_2}{}_{s_1,\lambda}$. Specifically, as we describe in great detail in Appendix B.4,

$$\int d\tilde{h}\chi_{(s_1,\mu_1\to i\infty)}(\tilde{h})\overline{\chi}_{\lambda^\pm}(\tilde{h})\chi_{(s_2,\mu_2\to i\infty)}(\tilde{g}\tilde{h}^{-1}) = N_{\lambda^\pm}\, N^{s_2}{}_{s_1,\lambda}\,\chi_{(s_2,\mu_2\to i\infty)}(\tilde{g}) \tag{2.56}$$

$$+ \text{ discrete series contributions} \ , \tag{2.57}$$

where $N^{s_2}{}_{s_1,\lambda}$ is given by

$$N^{s_2}{}_{s_1,\lambda} = \frac{|\Gamma(\lambda + is_1 - is_2)\Gamma(\lambda + is_1 + is_2)|^2}{\Gamma(2\lambda)} = \frac{\Gamma(\lambda \pm is_1 \pm is_2)}{\Gamma(2\lambda)} \,, \tag{2.58}$$

where $\Gamma(x \pm y \pm z) \equiv \Gamma(x+y+z)\Gamma(x-y-z)\Gamma(x+y-z)\Gamma(x-y+z)$. The fusion coefficient has an overall normalization coefficient, $N_{\lambda^\pm}$, that appears in (2.56) and is computed in Appendix B.4 and is independent of s_1 and s_2. We can thus properly define the "renormalized" Wilson line, as previously mentioned in (2.48),

$$\mathcal{W}_\lambda(\mathcal{C}_{\tau_1,\tau_2}) \equiv \frac{\widehat{\mathcal{W}}_{\lambda^\pm,k\to 0}(\mathcal{C}_{\tau_1,\tau_2})}{N_{\lambda^\pm}} \,, \tag{2.59}$$

for which the associated fusion coefficient $N^{s_2}{}_{s_1,\lambda}$ is independent of whether the discrete series representation is given $\mathcal{D}_{\lambda^+}$ or $\mathcal{D}_{\lambda^-}$. Furthermore, since all unitary discrete series representations appearing in the partition function are suppressed in the $B \to \infty$ limit, they do not contribute in the thermal correlation function of any number of Wilson lines. Consequently, plugging (2.56) and (2.43) into (2.54) we find

$$\langle \mathcal{W}_\lambda(\mathcal{C}_{\tau_1,\tau_2})\rangle_{k_0}(\tilde{g}) \propto \int ds_1\rho(s_1)ds_2\rho(s_2)N^{s_2}{}_{s_1,\lambda}\,\chi_{s_2}(\tilde{g})e^{-\frac{\epsilon}{2}\left[(\tau_2-\tau_1)s_1^2+(\beta-\tau_2+\tau_1)s_2^2\right]} \,. \tag{2.60}$$

where we have set the value of $\phi^{\mathbb{R}} = -i$ along the boundary. When taking the limit $\tilde{g} \to \mathbf{1}$, one can evaluate the limit of the SL2 characters to find the normalized expectation value

$$\frac{\langle \mathcal{W}_\lambda(\mathcal{C}_{\tau_1,\tau_2})\rangle_{k_0}}{Z_{k_0}} \propto \left(\frac{e\beta}{2\pi}\right)^{3/2} e^{-\frac{2\pi^2}{e\beta}} \int ds_1\rho(s_1)ds_2\rho(s_2)\, N^{s_2}{}_{s_1,\lambda}\, e^{-\frac{e}{2}\left[(\tau_2-\tau_1)s_1^2+(\beta-\tau_2+\tau_1)s_2^2\right]}$$
$$\propto \left(\frac{e\beta}{2\pi}\right)^{3/2} e^{-\frac{2\pi^2}{e\beta}} \int ds_1^2 ds_2^2 \sinh(2\pi s_1)\sinh(2\pi s_2)\frac{\Gamma(\lambda\pm is_1\pm is_2)}{\Gamma(2\lambda)}$$
$$\times\, e^{-\frac{e}{2}\left[(\tau_2-\tau_1)s_1^2+(\beta-\tau_2+\tau_1)s_2^2\right]}\,. \tag{2.61}$$

where $\Gamma(\lambda \pm is_1 \pm is_2)$ was defined after (2.58). Using the correspondence $e = 1/C$, the result agrees precisely with the computation [40] of the expectation value of a single bi-local operator $\langle \mathcal{O}_\lambda(\tau_1,\tau_2)\rangle$ in the Schwarzian theory. The result there was obtained using the equivalence between the Schwarzian theory and a suitable large c limit of 2D Virasoro CFT and had no direct interpretation in terms of SL2 representation theory.[31] Here we can generalize their result and study more complicated Wilson line configurations to reproduce the conjectured Feynman rules [40] in the Schwarzian theory.

2.4.3 Time-ordered correlators

For instance, we can consider n non-intersecting Wilson lines inserted along the contours $\mathcal{C}_{\tau_1,\tau_2}$, ..., $\mathcal{C}_{\tau_{2n-1},\tau_{2n}}$ with $\tau_1 < \tau_2 < \cdots < \tau_{2n}$. As an example, the Wilson line configuration for the time-ordered correlator of two bi-local operators is represented in the top right column of Figure 2.3. The n-point function is given by,

$$\left\langle \prod_{i=1}^n \widehat{\mathcal{W}}_{\lambda_i^\pm,k_i}(\mathcal{C}_{\tau_{2i-1},\tau_{2i}})\right\rangle(g) = \int \left(\prod_{i=1}^n dh_i\right)\left(\prod_{i=1}^n Z\left(h_i,\, e\tau_{2i,\,2i-}\right)\overline{\chi}^{\pm}_{\lambda_i,k}(h_i)\right)$$
$$\times\, Z\left(g(h_1\ldots h_n)^{-1}, e\tau_{1,2n}\right)\,, \tag{2.62}$$

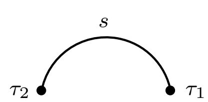
where $\tau_{2i,\,2i-1} = \tau_{2i} - \tau_{2i-1}$ is the length of an individual segment along I enclosed by the contour $\mathcal{C}_{\tau_{2i-1},\tau_{2i}}$, while $\tau_{2n,1} = \beta - \tau_{12} - \ldots - \tau_{2n-1,2n}$ is the length of the segment along I complementary to the union of $\mathcal{C}_{\tau_1,\tau_2}$, $\ldots$, $\mathcal{C}_{\tau_{2n-1},\tau_{2n}}$. Once again, all Wilson lines are in the positive or negative discrete series representation $(\lambda_i^\pm, k_i) = \lim_{B\to\infty}(\lambda_i^\pm, \mp 2\pi\lambda_i/B) = (\lambda_i^\pm, 0)$. Following the procedure presented in the previous subsection, we set the overall holonomy for the $\mathfrak{sl}(2,\mathbb{R})$ components of the

[31] However, the recent paper of [122, 121] offer an interpretation in terms of representations of the semigroup $SL^+(2,\mathbb{R})$.

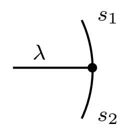
gauge field to $\tilde{g} \to \mathbf{1}$ and isolate the representations with $k_0 = \phi^{\mathbb{R}} = -i$. We find

$$\frac{\langle \prod_{i=1}^n \mathcal{W}_{\lambda_i}(\mathcal{C}_{\tau_{2i-1},\tau_{2i}})\rangle_{k_0}}{Z_{k_0}} = \left(\frac{e\beta}{2\pi}\right)^{3/2} e^{-\frac{2\pi^2}{e\beta}} \int ds_0 \rho(s_0) \left(\prod_{i=1}^n ds_1 \rho(s_1)\right) \left(\prod_{i=1}^n N^{s_0}{}_{s_i,\lambda_i}\right) \\ \times \exp\left\{-\frac{e}{2}\left[\left(\sum_{i=1}^n s_i^2(\tau_{2i}-\tau_{2i-1})\right) + s_0^2\left(\beta - \sum_{i=1}^n(\tau_{2i}-\tau_{2i-1})\right)\right]\right\}. \tag{2.63}$$

This result does not only agree with the time-ordered correlator of two bilocal operators in the Schwarzian theory, but it also reproduces the conjectured Feynman rule for any time-ordered bi-local correlator [40] and gives them an interpretation in terms of SL2 representation theory. Specifically, to each segment between two anchoring points on the boundary we can associate an $\widetilde{SL}(2,\mathbb{R})$ principal series representation labeled by s_i. Furthermore, at each anchoring point of the Wilson line, or at each insertion point of the bi-local operator, we associate the square-root of the fusion coefficient. Diagrammatically [40],



$$= e^{-s^2(\tau_2-\tau_1)},$$



$$= \sqrt{N^{s_1}{}_{s_2,\lambda}}\,. \tag{2.64}$$

Finally, we integrate over all principal series representation labels s_i associated to boundary segments using the Plancherel measure $\rho(s_0)\cdots\rho(s_n)$. Since for time-ordered correlators, both anchoring points of any Wilson line contributes the same fusion coefficient, we square the contribution of the right vertex in (2.64), in agreement with our expression in (2.63).

2.4.4 Out-of-time-ordered correlators and intersecting Wilson lines

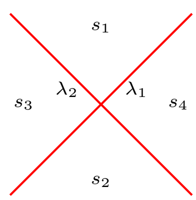
While for time-ordered correlators we have considered disjoint Wilson lines,[32] in order to reproduce correlators of out-of-time-ordered correlators we have to discuss intersecting Wilson line configurations. As an example, we show the Wilson line configuration associated to the correlator of two out-of-time-ordered bi-locals in Figure 2.3 in the bottom-right. The correlator of intersecting Wilson loops in Yang-Mills theory with a compact gauge group has been determined in [103]. Using the gluing procedure, the expectation value of the intersecting Wilson lines in the bottom-right of Figure

[32] We will revisit this assumption shortly.

2.3, when fixing the overall boundary $\mathcal{G}$ holonomy, is given by[33]

$$\begin{aligned}\langle \widehat{\mathcal{W}}_{\lambda_1^\pm,0}(\mathcal{C}_{\tau_1,\tau_2})\widehat{\mathcal{W}}_{\lambda_2^\pm,0}(\mathcal{C}_{\tau_3,\tau_4})\rangle(g) = \int dh_1 dh_2 dh_3 dh_4\, & Z\left(h_1 h_2^{-1},\, e\tau_{31}\right) Z\left(h_2 h_3^{-1},\, e\tau_{32}\right) \times \\ & \times Z\left(h_3 h_4^{-1},\, e\tau_{42}\right) Z\left(g\, h_4 h_1^{-1},\, e\tau_{41}\right) \times \\ & \times \overline{\chi}_{\lambda_1^\pm,0}(h_1 h_3^{-1})\overline{\chi}_{\lambda_2^\pm,0}(h_2 h_4^{-1})\,, \end{aligned} \tag{2.65}$$

where we consider the ordering $0 < \tau_1 < \tau_3 < \tau_2 < \tau_4 < \beta$, with $\tau_{41} = \beta - \tau_4 + \tau_1$, and we are once again interested in the limit $\tilde{g} \to \mathbf{1}$. Using the formula (2.43) for the partition function, one finds that performing the group integrals over $h_1, \ldots, h_4$ gives eight Clebsch-Gordan coefficients associated to the representations of the four areas separated by Wilson lines and to the two representations of the Wilson lines themselves (see Appendix B.4.3 for a detailed account). Collecting the Clebsch-Gordan coefficients associated to the bulk vertex one finds the 6-j symbol of $\widetilde{SL}(2,\mathbb{R})$, which we call $R_{s_a s_b}\left[\begin{smallmatrix} s_2 & \lambda_2 \\ s_1 & \lambda_1 \end{smallmatrix}\right]$, which can schematically be represented as



$$= \quad R_{s_3 s_4}\left[\begin{smallmatrix} s_2 & \lambda_2 \\ s_1 & \lambda_1 \end{smallmatrix}\right]. \tag{2.66}$$

As we discuss in detail in Appendix B.4.3, the 6-j symbol is given by [130, 131]

$$\begin{aligned} R_{s_a s_b}\left[\begin{smallmatrix} s_2 & \lambda_2 \\ s_1 & \lambda_1 \end{smallmatrix}\right] \;&=\; \mathbb{W}(s_a, s_b; \lambda_1 + is_2, \lambda_1 - is_2, \lambda_2 - is_1, \lambda_2 + is_1) \\ &\times\; \sqrt{\Gamma(\lambda_2 \pm is_1 \pm is_a)\Gamma(\lambda_1 \pm is_2 \pm is_a)\Gamma(\lambda_1 \pm is_1 \pm is_b)\Gamma(\lambda_2 \pm is_2 \pm is_b)}\,, \end{aligned} \tag{2.67}$$

where $\mathbb{W}(s_a, s_b; \lambda_1 + is_2, \lambda_1 - is_2, \lambda_2 - is_1, \lambda_2 + is_1)$ denotes the Wilson function which is defined by a linear combination of ${}_4F_3$ functions. Thus, the expectation value of two intersecting Wilson lines when setting the holonomy for the $\mathfrak{sl}(2,\mathbb{R})$ components to $\tilde{g} \to \mathbf{1}$ and setting $\phi^{\mathbb{R}} = -i$ is given by

$$\begin{aligned}\langle \mathcal{W}_{\lambda_1}(\mathcal{C}_{\tau_1,\tau_2})\mathcal{W}_{\lambda_2}(\mathcal{C}_{\tau_3,\tau_4})\rangle_{k_0}(\tilde{g}) \propto \int & R_{s_3\, s_4}\left[\begin{smallmatrix} s_2 & \lambda_2 \\ s_1 & \lambda_1 \end{smallmatrix}\right] \sqrt{N^{s_4}{}_{\lambda_1,s_1} N^{s_3}{}_{\lambda_1,s_2} N^{s_3}{}_{\lambda_2,s_1} N^{s_4}{}_{\lambda_2,s_2}} \\ & \times \chi_{s_b}(\tilde{g})\, e^{-\frac{e}{2}\left[s_1^2(\tau_3-\tau_1)+s_3^2(\tau_3-\tau_2)+s_2^2(\tau_4-\tau_2)+s_4^2(\beta-\tau_4+\tau_1)\right]} \prod_{i=1}^{4} ds_i \rho(s_i) \end{aligned} \tag{2.68}$$

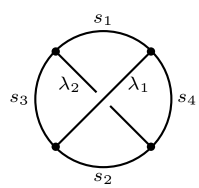
[33] Once again the $\pm$ signs for the two discrete series representation of the two lines are uncorrelated.

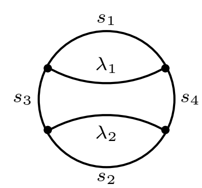
where the exponential factors are those associated to each disk partition function $Z(h,\, e\tau_{ij})$ appearing in (2.65), while the factors ${N^{s_i}}_{\lambda_k, s_k}$ are the remainder from the fusion coefficients after collecting all factors necessary for the 6-j symbol. Evaluating the correlator with a $A_\tau = 0$ on the boundary and dividing by the partition function, we find

$$\frac{\langle \mathcal{W}_{\lambda_1}(\mathcal{S}_{\tau_1,\tau_2}) \mathcal{W}_{\lambda_2}(\mathcal{S}_{\tau_3,\tau_4}) \rangle}{Z_{k_0}} = \left(\frac{e\beta}{2\pi}\right)^{3/2} e^{-\frac{2\pi^2}{e\beta}} \int R_{s_3 s_4}\left[\begin{smallmatrix} s_2 & \lambda_2 \\ s_1 & \lambda_1 \end{smallmatrix}\right] \sqrt{{N^{s_4}}_{\lambda_1,s_1}{N^{s_3}}_{\lambda_1,s_2}{N^{s_3}}_{\lambda_2,s_1}{N^{s_4}}_{\lambda_2,s_2}}$$
$$\times\, e^{-\frac{e}{2}\left[s_1^2(\tau_3-\tau_1)+s_3^2(\tau_3-\tau_2)+s_2^2(\tau_4-\tau_2)+s_4^2(\beta-\tau_4+\tau_1)\right]} \prod_{i=1}^{4} ds_i\, \rho(s_i)\,, \qquad (2.69)$$

which is in agreement with the result for the out-of-time order correlator for two bi-local operators obtained in the Schwarzian theory in [40].

The result (2.69) is easily generalizable to any intersecting Wilson line configuration as one simply needs to associated the symbol $R_{s_3 s_4}\left[\begin{smallmatrix} s_2 & \lambda_2 \\ s_1 & \lambda_1 \end{smallmatrix}\right]$ to any intersection.[34] This reproduces the conjectured Feynman rule for the Schwarzian bi-local operators,

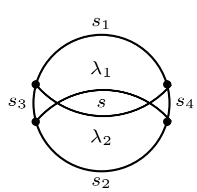


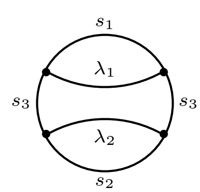
$= R_{s_3 s_4}\left[\begin{smallmatrix} s_2 & \lambda_2 \\ s_1 & \lambda_1 \end{smallmatrix}\right]$


. (2.70)

where one multiplies the diagram on the right by the 6-j symbol before performing the integrals associated to the SL2 representation labels along the edges.[35]

Finally, as a consistency check we verify that correlation functions are insensitive to Wilson lines intersections that can be uncrossed in the bulk, without touching the defect loop I (as that in the bottom-left figure 2.3). Diagrammatically, we want to prove for instance the Feynman rule



$= \frac{\delta(s_3-s_4)}{\rho(s_3,\mu_3)}$


. (2.71)

We will denote the contours of two such Wilson lines as $\tilde{\mathcal{C}}_{\tau_1,\tau_2}$ and $\tilde{\mathcal{C}}_{\tau_3,\tau_4}$, where we assume that

[34] Note that in the compact case discussed in [103] the gauge group 6-j symbol appears squared. This is due to the fact that when considering two Wilson loops which are not boundary-anchored they typically intersect at two points in the bulk.

[35] Note that the right diagram in (2.70) is just a useful mnemonic for performing computations that involve intersecting Wilson lines. It does not correspond to a configuration in the gauge theory since the representations s_3 and s_4 are kept distinct even though they would correspond to the same bulk patch in the gauge theory.

$0 < \tau_1 < \tau_2 < \tau_3 < \tau_4 < \beta$. The expectation value in such a configuration is given by

$$\begin{aligned}\langle \widehat{\mathcal{W}}_{\lambda_1^\pm,0}(\tilde{\mathcal{C}}_{\tau_1,\tau_2})\widehat{\mathcal{W}}_{\lambda_2^\pm,0}(\tilde{\mathcal{C}}_{\tau_3,\tau_4})\rangle(g) = \int dh_1 dh_2 dh_3 dh_4 dh_5 dh_6\, & Z\left(h_1 h_2^{-1},\, e\tau_{41}\right) Z\left(h_5^{-1} h_3^{-1} h_1^{-1},\, e\tau_{12}\right) \\ & \times Z\left(h_6^{-1} h_5,\, e\tau_{23}\right) Z\left(g\, h_2 h_4 h_6,\, e\tau_{43}\right) Z\left(h_3 h_4^{-1}, 0\right) \\ & \times \overline{\chi}_{\lambda_1^\pm,0}(h_1 h_4 h_5)\overline{\chi}_{\lambda_2^\pm,0}(h_2 h_3 h_6)\,. \end{aligned} \tag{2.72}$$

Using (2.43), we will associate the representation labeled by s_4, s_2, s_3, s_1, and s, in this order, to the five disk partition functions in (2.72). Performing all the group integrals we once again obtain a contracted sum of Clebsch-Gordan coefficients each of which is associated to a Wilson line representation and the representations labelling two neighboring regions. Performing the contractions for all of the Clebsch-Gordan coefficients we find two 6-j symbol symbols, $R_{s_3 s}\left[\begin{smallmatrix} s_2 & \lambda_2 \\ s_1 & \lambda_1 \end{smallmatrix}\right]$ and $R_{s_4 s}\left[\begin{smallmatrix} s_2 & \lambda_2 \\ s_1 & \lambda_1 \end{smallmatrix}\right]$, each associated to the 6 representations that go around each of the two vertices. The remaining sums over Clebsch-Gordan coefficients yield the product of four fusion coefficients, $\sqrt{N^{s_4}{}_{\lambda_1,s_2} N^{s_2}{}_{\lambda_1,s_3} N^{s_3}{}_{\lambda_2,s_1} N^{s_1}{}_{\lambda_2,s_4}}$.

Using the orthogonality relation for the 6-j symbol that follows from properties of the Wilson function (see [130, 131])

$$\int ds \rho(s,\mu) R_{s_3 s}\left[\begin{smallmatrix} s_2 & \lambda_2 \\ s_1 & \lambda_1 \end{smallmatrix}\right] R_{s_4 s}\left[\begin{smallmatrix} s_2 & \lambda_2 \\ s_1 & \lambda_1 \end{smallmatrix}\right] + \text{ discrete series contribution} = \frac{\delta(s_3 - s_4)}{\rho(s_3,\mu_3)}\,, \tag{2.73}$$

where $\rho(s,\mu)$ is the Plancherel measure defined in (2.28), we find that if there's a bulk region enclosed by intersecting Wilson that does not overlap with the defect loop, one can always perform the integral over the corresponding representation label s to eliminate this region. The integral over s_3 or s_4 then becomes trivial due to the delta-function in (2.73) and thus the remaining fusion coefficients reproduce those in (2.63) for two non-intersecting Wilson lines.

Thus, putting together (2.71), (2.70), and (2.64), we have re-derived the diagrammatic rules needed to compute the expectation value of any bi-local operator configuration. These rules are simply reproduced combinatorially in the gauge theory starting from the basic axioms presented in Section 2.3.1.

2.4.5 Wilson lines and local observables

While one can recover the correlation functions of some local observables by considering the zero length limit for various loop or line operators, it is informative to also independently compute correlation functions of local operators. In this section, we consider the operator $\operatorname{Tr}\phi^2(x)$ which is topological (see (2.4)). Consequently correlators of $\mathrm{Tr}\,\phi^2(x)$ are independent of the location of insertion. Indeed they can be easily obtained by insertions of the Hamiltonian operator at various points in the path integral, the un-normalized correlation function is given by

$$\begin{aligned}\langle \mathrm{Tr}\,\phi^2(x_1)\dots \mathrm{Tr}\,\phi^2(x_n)\rangle_{k_0} &= (e/4)^{-n}\,\langle H(x_1)\dots H(x_n)\rangle_{k_0}\\ &\propto \Xi \int ds\,\rho(s) s^{2n} e^{-e\beta s^2/2}\,,\end{aligned} \tag{2.74}$$

where we first evaluated the correlator for a generic value of the boundary $\mathcal{G}$ holonomy and then fixed the value of the field $\phi^{\mathbb{R}}$ on the boundary and send $B \to \infty$ as described in Section 2.3. At separated points, the correlator (2.74) agrees with that of n insertions of the Schwarzian operator [25, 40], thus showing that the Schwarzian operator and $\mathrm{Tr}\,\phi^2$ are equivalent, as shown classically in Section 2.2.2.[36] This computation explains why the correlators of the Schwarzian operator at separated points are given by moments of the energy E computed with the probability distribution $\rho(\sqrt{E}/e)$, as first observed in [25].

In the presence of Wilson line insertions, the operator $\mathrm{Tr}\,\phi^2$ remains topological as long as we do not move it across a Wilson line. Consequently the correlation functions of $\mathrm{Tr}\,\phi^2$ depend only on the number of $\mathrm{Tr}\,\phi^2$ insertions within each patch separated by the Wilson lines. For instance, we can consider the insertion of $p = p_0 + p_1 + p_2 + \dots + p_n$ $\mathrm{Tr}\,\phi^2$ operators in the non-intersecting Wilson lines correlator considered in Section 2.4.3, as follows. Let us put p_0 operators in the bulk and outside of the contour of any of the Wilson lines, together with p_1 $\mathrm{Tr}\,\phi^2$, operators enclosed by $\mathcal{C}_{\tau_1,\tau_2}$, p_2 such operators enclosed by $\mathcal{C}_{\tau_3,\tau_4}$, and so on. The separated point correlator is then

$$\begin{aligned}\frac{\langle\left(\prod_{j=1}^p \mathrm{Tr}\,\phi^2(x_j)\right)\left(\prod_{i=1}^n \mathcal{W}_{\lambda_i}(\mathcal{C}_{\tau_{2i-1},\tau_{2i}})\right)\rangle_{k_0}}{Z_{k_0}} &= \left(\frac{e\beta}{2\pi}\right)^{3/2} e^{-\frac{2\pi^2}{e\beta}} \int ds_0\rho(s_0)\left(\prod_{i=1}^n ds_i\rho(s_i)\right)\\ \times\, s_1^{p_1}\dots s_n^{p_n} s_0^{p_{n+1}} \left(\prod_{i=1}^n N^{s_0}{}_{s_i,\lambda_i}\right) & e^{-\frac{e}{2}\left[\left(\sum_{i=1}^n s_i^2(\tau_{2i}-\tau_{2i-1})\right)+s_0^2\left(\beta-\sum_{i=1}^n(\tau_{2i}-\tau_{2i-1})\right)\right]}\,.\end{aligned} \tag{2.75}$$

In the Schwarzian theory, such a correlator is expected to reproduce the expectation value

$$\left\langle \left[\prod_{j=1}^p \{F,u\}|_{u=\tilde{\tau}_j}\right]\left[\prod_{i=1}^n \mathcal{O}_{\lambda_i}(\tau_{2i-1},\tau_{2i})\right]\right\rangle, \tag{2.76}$$

where $\tau_1 < \tilde{\tau}_1 < \dots < \tilde{\tau}_{p_1} < \tau_2 < \dots$. Such a computation can also be performed using the Virasoro CFT following the techniques outlined [40]. Following similar reasoning, one can consider

[36] However, the contact terms associated with these correlators are different. We hope to determine the exact bulk operator dual to the Schwarzian in future work.

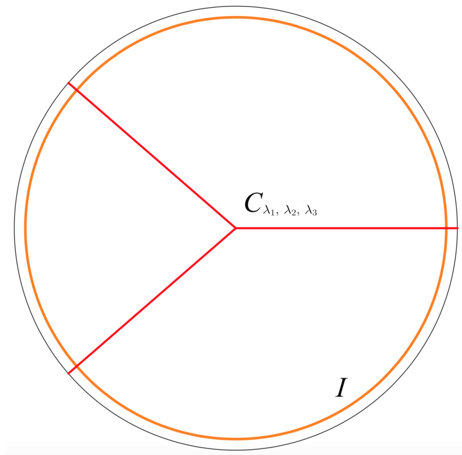


Figure 2.4: An example of a three-particle bulk interaction vertex corresponding to the junction of three Wilson lines defined by a Clebsch-Gordan coefficient at the vertex.

the correlators of the operator $\mathrm{Tr}\,\phi^2$ in the presence of any other Wilson line configurations.

2.4.6 A network of non-local operators

While so far we have focused on Wilson lines that end on the boundary, we now compute the expectation values of more complex non-local operators that are invariant under bulk gauge transformations that approach the identity on the boundary. Such objects, together with the previously discussed Wilson lines, serve as the basic building blocks for constructing "networks" of Wilson lines that capture various scattering problems in the bulk. The simplest such operator that includes a vertex in the bulk is given by the junction of three Wilson lines

$$C_{\lambda_1,\lambda_2,\lambda_3}(g_{\mathcal{C}_{\tau_1,v}}, g_{\mathcal{C}_{\tau_2,v}}, g_{\mathcal{C}_{\tau_3,v}}) = \sum_{\substack{m_1=\lambda_1+\mathbb{Z}^+_{<\Xi},\\ m_2=\lambda_2+\mathbb{Z}^+_{<\Xi}}} \sum_{\substack{n_1=\lambda_1+\mathbb{Z}^+,\\ n_2=\lambda_2+\mathbb{Z}^+}} \frac{C^{\lambda_1^+,\lambda_2^+,\lambda_3^+}_{m_1,m_2,m_1+m_2}(C^{\lambda_1^+,\lambda_2^+,\lambda_3^+}_{n_1,n_2,n_1+n_2})^*}{\mathcal{N}_{\lambda_1^+,\lambda_2^+,\lambda_3^+}}$$
$$\times\, U^{m_1}_{(\lambda_1^+,0),\,n_1}(g_{\mathcal{C}_{\tau_1,v}})U^{m_2}_{(\lambda_2^+,0),\,n_2}(g_{\mathcal{C}_{\tau_2,v}})U^{m_1+m_2}_{(\lambda_3^+,0),\,n_1+n_2}(g_{\mathcal{C}_{\tau_3,v}})\,, \qquad (2.77)$$

with

$$g_{\mathcal{C}_{\tau_i,v}} = \mathcal{P}\exp\left(\int_{\mathcal{C}_{\tau_i,v}} A\right)\,, \qquad (2.78)$$

where $\mathcal{C}_{\tau_i,v}$ is a contour which starts on the boundary, intersects the defect at a point τ_i, and ends at a bulk vertex point v. As indicated in (2.77), the sums over m_1 and m_2 are truncated by the cut-off Ξ. Such a non-local object is schematically represented in Figure 2.4. For simplicity, we assume $0 < \tau_1 < \tau_2 < \tau_3 < \beta$ and we consider λ_1, λ_2, λ_3 labelling the Wilson lines to be positive discrete series representations. Once again, $U^m_{(\lambda^+,0),\,n}(g)$ is the $\mathcal{G}$ matrix element for the discrete representation $(\lambda^+,0)$, $C^{\lambda_1^+,\lambda_2^+,\lambda_3^+}_{m_1,m_2,m_3}$ is the SL2 (or, equivalently, $\mathcal{G}$) Clebsch-Gordan coefficient for the representations λ_1, λ_2, and λ_3, and $\mathcal{N}_{\lambda_1^+,\lambda_2^+,\lambda_3^+}$ is a normalization coefficient for the Clebsch-Gordan coefficients discussed in Appendix B.4. Note that the operator (2.77) is invariant under bulk gauge transformations. This follows from combining the fact that a gauge transformation changes $g_{\mathcal{C}_{\tau_i,v}} \to g_{\mathcal{C}_{\tau_i,v}} h_v$, where h_v is an arbitrary $\mathcal{G}$ element, with the identity

$$\sum_{m_1,m_2} U^{m_1}_{(\lambda_1^+,0),\,n_1}(h_v) U^{m_2}_{(\lambda_2^+,0),\,n_2}(h_v) U^{m_1+m_2}_{(\lambda_3^+,0),\,n_1+n_2}(h_v) C^{\lambda_1^+,\lambda_2^+,\lambda_3^+}_{m_1,m_2,m_1+m_2} = C^{\lambda_1^+,\lambda_2^+,\lambda_3^+}_{n_1,n_2,n_1+n_2}\,. \tag{2.79}$$

Using the gluing rules specified in Section 2.3.1, the expectation value of the operator (2.77) with holonomy g between the defect intersection points 3 and 1, and trivial holonomy between all other intersection points, is given by

$$\begin{aligned}\langle C_{\lambda_1,\lambda_2,\lambda_3}\rangle(g) = \int dh_1 dh_2 dh_3 Z(h_1 h_2^{-1},\, e\tau_{12}) Z(h_2 h_3^{-1},\, e\tau_{12}) Z(g h_3 h_1^{-1},\, e\tau_{12}) \\ \times\, C_{\lambda_1,\lambda_2,\lambda_3}(h_1,h_2,h_3)\,. \end{aligned} \tag{2.80}$$

As before, we are interested in the case where we fix the SL2 component of $\mathcal{G}$ to $\tilde{g} \to \mathbf{1}$. Expanding (2.80) into $\mathcal{G}$ matrix elements we find the product of eight Clebsch-Gordan coefficients. Summing up the Clebsch-Gordan coefficients that have unbounded state indices (those that involve that n_i indices instead of the m_i indices in (2.77)) we obtain the 6-j symbol with all representations associated to the bulk vertex, $R_{\lambda_1 s_1}\left[\begin{smallmatrix}\lambda_2 & \lambda_3\\ s_2 & s_3\end{smallmatrix}\right]$, which is also related to the Wilson function as shown in [131]. Setting the boundary condition $\phi^{\mathbb{R}} = -i$ and take $\tilde{g} \to \mathbf{1}$ we find that the 6-j symbol together with the sum over the remaining four Clebsch-Gordan coefficients yield

$$\begin{aligned}\frac{\langle C_{\lambda_1,\lambda_2,\lambda_3}\rangle}{Z_{k_0}} = \left(\frac{e\beta}{2\pi}\right)^{3/2} e^{-\frac{2\pi^2}{e\beta}} N_{\lambda_1^+,\lambda_2^+,\lambda_3^+} \int ds_1\rho(s_1) ds_2 \rho(s_2) ds_3 \rho(s_3) \sqrt{N^{s_1}{}_{\lambda_1,s_2} N^{s_2}{}_{\lambda_2,s_3} N^{s_3}{}_{\lambda_3,s_1}} \\ \times\, R_{\lambda_1 s_1}\left[\begin{smallmatrix}\lambda_2 & \lambda_3\\ s_2 & s_3\end{smallmatrix}\right] e^{-\frac{e}{2}\left[s_1^2(\tau_2-\tau_1)+s_2^2(\tau_3-\tau_2)+s_3^2(\beta-\tau_3+\tau_1)\right]}\,,\end{aligned} \tag{2.81}$$

where in the limit in which all continuous representations have $\mu_1, \mu_2, \mu_3 \to i\infty$, $N_{\lambda_1^+\lambda_2^+\lambda_3^+}$ is a normalization constant independent of the representations s_1, s_2 or s_3 that can be absorbed in the definition of the operator $\mathcal{C}_{\lambda_1,\lambda_2,\lambda_3}$.

We expect that the same reasoning as that applied for boundary-anchored Wilson lines should show that such a non-local operator corresponds to inserting the world-line action of three particles which intersect at a point in AdS_2 in the gravitational path integral (summing over all possible trajectories diffeomorphic to the initial paths shown in Figure 2.4).[37] Thus, such insertions of non-local operators should capture the amplitude corresponding to a three-particle interaction in the bulk, at tree-level in the coupling constant between the three particles, but exact in the gravitational coupling. Similarly, by inserting a potentially more complex network of non-local gauge invariant operators in the path integral of the BF theory one might hope to capture the amplitude associated to any other type of interaction in the bulk.

2.5 Discussion and future directions

We have thus managed to formulate a comprehensive holographic dictionary between the Schwarzian theory and the $\mathcal{G}$ gauge theory: we have shown that the dynamics of the Schwarzian theory is equivalent to that of a defect loop in the $\mathcal{G}$ gauge theory. Specifically, we have matched the partition function of the two theories, and have shown that bi-local operators in the boundary theory are mapped to boundary-anchored defect-cutting Wilson lines. The gluing methods used to compute the correlators of Wilson lines provide a toolkit to compute the expectation value of any set of bi-local operators and reveal their connections to SL2 representation theory.

There are numerous directions that we wish to pursue in the future. As emphasized in Section 2.2, while the choice of gauge algebra was sufficient to understand the on-shell equivalence between the gauge theory and JT-gravity, a careful analysis about the global structure of the gauge group was necessary in order to formulate the exact duality between the bulk and the boundary theories. While we have resorted to the gauge group $\mathcal{G}$ with a simple boundary potential for the scalar field ϕ, it is possible that there are other gauge group choices which reproduce observables in the Schwarzian theory or in related theories. For instance, it would be instructive to further study the reason for the apparent equivalence between representations of the group $\mathcal{G}$ in the $B \to \infty$ limit and representations of the non-compact subsemigroup $SL^+(2, \mathbb{R})$ which was discussed in [121, 122, 132]. Both gauge theory choices seemingly reproduce correlation functions in the Schwarzian theory. However, in the latter case the exact formulation of a two-dimensional action seems, as of yet, unclear. Another interesting direction is to study the role of q-deformations for the 2d gauge theory associated to a non-compact group, which have played an important role in the case of compact groups [133]. Such a deformation is also relevant from the boundary perspective, where [134] have shown that correlation functions in the large-N double-scaled limit of the SYK model can be described in terms

[37] It would be interesting to understand if this can be proven rigorously following an analogous approach to that presented in Appendix B.5.

of representations of q-deformed $SU(1,1)$.

It is likely that one can generalize the 2D gauge theory/1D quantum mechanics duality for different choice of gauge groups and scalar potentials [59]. A semi-classical example was given in [22], where various 1D topological theories were shown to be semi-classically equivalent to 2D Yang-Mills theories with more complicated potentials for the field strength. It would be interesting to further understand the exact duality between such systems [135].

Finally, one would hope to generalize our analysis to the two other cases where the BF-theory with an $\mathfrak{sl}(2,\mathbb{R})$ gauge algebra is relevant: in understanding the quantization of JT-gravity in Lorentzian AdS_2 and in dS_2.[38] By making appropriate choices of gauge groups and boundary conditions in the two cases, one could once again hope to exactly compute observables in the gravitational theory by first understanding their descriptions and properties in the corresponding gauge theory. We hope to address some of these above problems in the near future.

[38] See [34, 136] for a recent analysis of the quantization of the two gravitational systems. Furthermore, recently a set of gauge invariant operators was identified in the Schwarzian theory whose role is to move the bulk matter in the two-sided wormhole geometry relative to the dynamical boundaries [137]. It would be interesting to identify the existence of such operators in the gauge theory context.

Chapter 3

Dilaton gravity in the second-order formalism

3.1 Outline of results

This chapter expands on the ideas presented in section 1.5. Before outlining the main results of this chapter, in order to further motivate our computation, it is useful to review some details about the $1d$ $T\overline{T}$ deformation.

3.1.1 Review $1d$ $T\overline{T}$

In the past work of [70], a particular deformation of the Schwarzian quantum mechanics was shown to be classically equivalent to JT gravity with Dirichlet boundary conditions for the metric and dilaton. The deformation on the Schwarzian theory follows from a dimensional reduction of the $T\overline{T}$ deformation in $2D$ CFTs [1]. Explicitly the deformation involves a flow of the action S of the quantum mechanical theory,

$$\partial_\lambda S = \int_0^1 d\theta \frac{T^2}{1/2 - 2\lambda T} \tag{3.1}$$

where T is the trace of the stress-'scalar' of the quantum mechanical theory and λ is the deformation parameter. By going from the Lagrangian to the Hamiltonian formulation, we can write an equivalent

[1] This reduction is valid in the classical limit and should be seen as a motivation for the proposed deformation. It would be interesting to extend it to a precise statement using the methods of [84].

flow for the Hamiltonian instead of S and find the flow of the energy eigenvalues,[2]

$$\partial_\lambda H = \frac{H^2}{1/2 - 2\lambda H} \quad \Rightarrow \quad \mathcal{E}_\pm(\lambda) = \frac{1}{4\lambda}\left(1 \mp \sqrt{1-8\lambda E}\right). \tag{3.2}$$

Here E are the energy levels of the undeformed theory and matching onto the original spectrum as $\lambda \to 0$ results in picking the minus sign for the branch of the root in (3.2). In section 3.4 we will see that the other branch of the root will also make its appearance. In the case of the Schwarzian theory, which has a partition function that can be exactly computed [25],[3]

$$Z(\beta) = \int_0^\infty dE\, \frac{\sinh(2\pi\sqrt{2CE})}{\sqrt{2C\pi^3}} e^{-\beta E} = \frac{e^{2C\pi^2/\beta}}{\beta^{3/2}}, \tag{3.3}$$

of the deformed partition function is,

$$Z_\lambda(\beta) = \int_0^\infty dE\, \frac{\sinh(2\pi\sqrt{2CE})}{\sqrt{2C\pi^3}} e^{-\beta\mathcal{E}_+(\lambda)}. \tag{3.4}$$

Let us make two observations. First, the integral over E runs over the full positive real axis and therefore will also include complex energies $\mathcal{E}_+(\lambda)$ when $\lambda > 0$, i.e. for $E > 1/8\lambda$ the deformed spectrum complexifies. This violates unitarity and needs to be dealt with. We will come back to this issue in section 3.4. Second, given that there is a closed from expression of the original Schwarzian partition function, one can wonder whether this is also the case for the deformed partition function. This turns out to be the case. For the moment let us assume $\lambda < 0$ so that there are no complex energies, then it was shown in [70] that the deformed partition function is given by an integral transform of the original one, analogous to the result of [138] in 2d. The integral transform reads,

$$Z_\lambda(\beta) = \frac{\beta}{\sqrt{-8\pi\lambda}} \int_0^\infty \frac{d\beta'}{\beta'^{3/2}} e^{\frac{(\beta-\beta')^2}{8\lambda\beta'}} Z(\beta'), \tag{3.5}$$

Plugging (3.3) into this expression and performing the integral over β' yields,

$$Z_\lambda(\beta) = \frac{\beta e^{-\frac{\beta}{4\lambda}}}{\sqrt{-2\pi\lambda(\beta^2 + 16C\pi^2\lambda)}} K_2\left(-\frac{1}{4\lambda}\sqrt{\beta^2 + 16C\pi^2\lambda}\right). \tag{3.6}$$

[2] Since the deformation is a function of the Hamiltonian, the eigenfunctions do not change under the flow.

[3] In the gravitational theory C is equal to ϕ_r, the renormalised boundary value of the dilaton. Furthermore, here we picked a convenient normalisation of the partition function.

with the associated density of states given by

$$\rho_\lambda(E) = \frac{1-4\lambda E}{\sqrt{2\pi^3 C}} \sinh\left(2\pi\sqrt{2CE(1-2\lambda E)}\right) \tag{3.7}$$

Although we have derived this formula assuming that $\lambda < 0$, we will simply analytically continue to $\lambda > 0$ to obtain the partition function of the deformed Schwarzian theory that describes JT gravity at finite cutoff. One might be worried that this would not yield the same as (3.4) and indeed there are a few subtleties involved in doing that analytic continuation as discussed in the end of section 3.3 and in section 3.4.

3.1.2 Summary of results

The purpose of this chapter is to give two independent bulk computation that reproduce the partition function (3.4). In section 3.2 we present a derivation of the partition function of JT gravity (with negative cosmological constant) at finite cutoff by computing the radial Wheeler-de Witt (WdW) wavefunctional. Due to Henneaux it is known since the 80's that the contraints of 2d dilaton gravity can be solved exactly in the full quantum theory [68]. We will review this computation and fix the solution by imposing Hartle-Hawking boundary conditions. In particular we find that

$$\Psi_{\text{HH}}[\phi_b(u), L] = \int_0^\infty dM \ \sinh(2\pi\sqrt{M}) \ e^{\int_0^L du \left[\sqrt{\phi_b^2 - M - (\partial_u\phi_b)^2} - \partial_u\phi \tan^{-1}\left(\sqrt{\frac{\phi_b^2-M}{(\partial_u\phi_b)^2}-1}\right)\right]}. \tag{3.8}$$

This wavefunction is computed in a basis of fixed dilaton $\phi_b(u)$, where u corresponds to the proper length along the boundary, and L the total proper length of the boundary. The above results obtained through the WdW constraint are non-perturbative in both L and $\phi_b(u)$.

When considering a constant dilaton profile $\phi_b(u) = \phi_b$, the wavefunction (3.8) reproduces the $T\overline{T}$ partition function in (3.4), with the identification

$$M \to 2CE, \quad \phi_b^2 \to \frac{C}{4\lambda}, \quad L \to \frac{\beta}{\sqrt{4C\lambda}}, \tag{3.9}$$

In terms of these variables, (3.8) matches with $T\overline{T}$ up to a shift in the ground state energy, which can be accounted for by a boundary counterterm $e^{-I_{\text{ct}}} = e^{-\phi_b L}$ added to the gravitational theory. An important aspect that this analysis emphasizes if the fact that, for JT gravity, studying boundary conditions with a constant dilaton is enough. As we explain in section 3.2.2, if the wavefunction for a constant dilaton is known, the general answer (3.8) is fixed by the constrains and does not constain any further dynamical information [139].

The partition function (3.4) is also directly computed from the path integral in JT gravity at finite cutoff in section 3.3. We will impose dilaton and metric Dirichlet boundary conditions, in terms of ϕ_b and the total proper length L. For the reasons explain in the previous paragraph, it is enough to focus on the case of a constant dilaton. It is convenient to parametrize these quantities in the following way

$$\phi_b = \frac{\phi_r}{\varepsilon}, \qquad L = \frac{\beta}{\varepsilon}, \tag{3.10}$$

in terms of a renormalized length β and dilaton ϕ_r. We will refer to ε as the cutoff parameter [4]. When comparing with the $T\overline{T}$ approach this parameter is $\varepsilon = \sqrt{2\lambda}$ (in units for which we set $\phi_r \to 1/2$). In order to compare to the asymptotically AdS_2 case previously studied in the literature [29, 37], we need to take $\phi_b, L \to \infty$ with a fixed renormalized length L/ϕ_b. In terms of the cutoff parameter, this limit corresponds to $\varepsilon \to 0$, keeping ϕ_r and β fixed.

We will solve this path integral perturbatively in the cutoff ε, to all orders. We integrate out the dilaton and reduce the path integral to a boundary action comprised of the extrinsic curvature K and possible counter-terms. We find an explicit form of the extrinsic curvature valid to all orders in perturbation theory in ε. A key observation in obtaining this result is the realisation of a (local) $SL(2,\mathbb{R})$ invariance of K in terms of lightcone coordinates $z = \tau - ix$, $\overline{z} = \tau + ix$:

$$K[z,\overline{z}] = K\left[\frac{az+b}{cz+d}, \frac{a\overline{z}+b}{c\overline{z}+d}\right]. \tag{3.11}$$

Solving the Dirichlet boundary condition for the metric allows us to write K as a functional of the Schwarzian derivative of the coordinate z.[5] As we will explain in detail, the remaining path integral can be computed exactly using integrability properties in the Schwarzian theory to all orders in ε^2.

Thus, by the end of section 3.3, we find agreement between the WdW wavefunctional, the Euclidean partition function and the $T\overline{T}$ partition function from (3.4):

$$e^{-I_{\rm ct}}\Psi_{\rm HH}[\phi_b,\, L] \overset{\rm non-pert.}{=} Z_\lambda(\beta) \overset{\rm pert.}{=} Z_{\rm JT}[\phi_b,\, L]\,. \tag{3.12}$$

Here we emphasize again that we show that the first equality is true non-perturbatively in ε (respectively in λ), whereas we prove the second equality to all orders in perturbation theory.

[4] In Poincaré coordinates ε corresponds semiclassically to the bulk coordinate of the cutoff surface.

[5] This generalizes the computation of [29] which found the relation between the extrinsic curvature and the Schwarzian derivative in the infinite cutoff limit.

In section 3.4 we discuss various extensions of the deformed partition function including further corrections. In particular we discuss two types of corrections in the path integral and in the integral over energies in (3.4): first, we analyze non-perturbative terms in ε coming from contributions that cannot be written as a path integral on the disk (the contracting branch of the wavefunction) and second, we speculate about non-perturbative corrections coming from the genus expansion. Related to the first kind of ambiguity, given the exact results we obtained for the wavefunctional and partition function, we explore how the complexification of the energy levels (that we mentioned above) can be cured. In particular, we propose that it requires the inclusion of the other branch of the root in (3.2), but still results in a negative density of states. The structure of the negative density of states suggests that the (unitary) partition function is not an ordinary one, but one with a chemical potential turned on. Related to the second type, we compute the partition function of the finite cutoff "trumpet" which is a necessary ingredient when constructing higher genus hyperbolic surfaces. Finally, we speculate about the range of the remaining Weil-Petersson integral which is needed in order to compute the finite cutoff partition function when including the contribution of surfaces with arbitrary topology.

Section 3.5 applies the computation from section 3.2 to the case of JT gravity with a positive cosmological constant and finds the wavefunctional on a de Sitter time-slice at finite time. This wavefunctional has some interesting behaviour, similar to the Hagedorn divergence present in (3.6). We finish with a discussion of our results and future directions in section 3.6.

3.2 Wheeler-deWitt wavefunction

In this section, we will start by reviewing the canonical quantization of $2D$ dilaton-gravity following the approach of [68, 69]. In these references, the authors find the space of exact solutions for both the momentum and Wheeler-deWitt constrains. Later, in subsections 3.2.3 and 3.2.4, we will focus on JT gravity, and we will explain how to impose the Hartle-Hawking condition appropriately to pick a solution corresponding to finite cutoff AdS_2.

Let us consider the more general two dimensional dilaton gravity in Lorentzian signature,

$$I = \frac{1}{2}\int_M d^2x\sqrt{g}[\phi R - U(\phi)] + \int_{\partial M} du\sqrt{\gamma_{uu}}\,\phi K, \tag{3.13}$$

with an arbitrary potential $U(\phi)$. g is the two-dimensional space-time metric on M and γ the induced metric on its boundary ∂M. The boundary term in (3.13) is necessary in order for the variational principle to be satisfied when imposing Dirichlet boundary conditions for the metric and dilaton. In (3.13) we could also add the topological term $\frac{1}{2}\int_M d^2x\sqrt{g}\,\phi_0 R + \int_{\partial M} du\sqrt{\gamma_{uu}}\;\phi_0 K = 2\pi\phi_0$ which will be relevant in section 3.4.3.

It will be useful to define also the prepotential $W(\phi)$ by the relation $\partial_\phi W(\phi) = U(\phi)$. In the case of JT gravity with negative (or positive) cosmological constant we will pick $U(\phi) = -2\phi$ (or $U(\phi) = 2\phi$) and $W(\phi) = -\phi^2$ ($W(\phi) = \phi^2$), which has as a metric solution AdS_2 (dS_2) space with unit radius.

We will assume the topology of space to be a closed circle, and will use the following ADM decomposition of the metric

$$ds^2 = -N^2 dt^2 + h(dx + N_\perp dt)^2, \qquad h = e^{2\sigma} \tag{3.14}$$

where N is the lapse, $N_\perp$ the shift, h the boundary metric (which in this simple case is an arbitrary function of x) and we identify $x \sim x+1$. After integrating by parts and using the boundary terms, the action can then be written as

$$\begin{aligned} I \;\; = \;\; & \int d^2x\; e^\sigma \Big[\frac{\dot\phi}{N}\,(N_\perp \partial_x\sigma + \partial_x N_\perp - \dot\sigma) \\ & + \frac{\partial_x\phi}{N}\left(\frac{N\partial_x N}{e^{2\sigma}} - N_\perp\partial_x N_\perp + N_\perp\dot\sigma - N_\perp^2\partial_x\sigma\right) - \frac{1}{2}NU(\phi)\Big] \end{aligned} \tag{3.15}$$

where the dots correspond to derivatives with respect to t. As usual the action does not involve time derivatives of fields N and $N_\perp$ and therefore

$$\Pi_N = \Pi_{N_\perp} = 0, \tag{3.16}$$

which act as primary constrains. The momenta conjugate to the dilaton and scale factor are

$$\Pi_\phi = \frac{e^\sigma}{N}(N_\perp\partial_x\sigma + \partial_x N_\perp - \dot\sigma), \qquad \Pi_\sigma = \frac{e^\sigma}{N}(N_\perp\partial_x\phi - \dot\phi). \tag{3.17}$$

With these equations we can identify the momentum conjugate to the dilaton with the extrinsic curvature $\Pi_\phi \sim K$, and the momentum of σ with the normal derivative of the dilaton $\Pi_\sigma \sim \partial_n\phi$. The classical Hamiltonian then becomes

$$H = \int dx\,\left[N_\perp \mathcal{P} + e^{-\sigma} N \mathcal{H}_{\rm WdW}\right] \tag{3.18}$$

where

$$\begin{aligned} \mathcal{P} &\equiv \Pi_\sigma \partial_x \sigma + \Pi_\phi \partial_x \phi - \partial_x \Pi_\sigma, && (3.19)\\ \mathcal{H}_{\text{WdW}} &\equiv -\Pi_\phi \Pi_\sigma + \frac{1}{2} e^{2\sigma} U(\phi) + \partial_x^2 \phi - \partial_x \phi \partial_x \sigma, && (3.20) \end{aligned}$$

and classically the momentum and Wheeler-deWitt constrains are respectively $\mathcal{P} = 0$ and $\mathcal{H}_{\text{WdW}} = 0$.

So far the discussion has been classical. Now we turn to quantum mechanics by promoting field to operators. We will be interested in wavefunctions obtained from path integrals over the metric and dilaton, and we will write them in configuration space. The state will be described by a wave functional $\Psi[\phi, \sigma]$ and the momentum operators are replaced by

$$\widehat{\Pi}_\sigma = -i \frac{\delta}{\delta \sigma(x)}, \quad \widehat{\Pi}_\phi = -i \frac{\delta}{\delta \phi(x)}, \tag{3.21}$$

The physical wavefunctions will only depend on the boundary dilaton profile and metric.

Usually, when quantizing a theory, one needs to be careful with the measure and whether it can contribute Liouville terms to the action. Such terms only appear when in conformal gauge, which is not what we are working in presently. Actually, the ADM decomposition (3.14) captures a general metric and is merely a parametrization of all 2d metrics and so we have not fixed any gauge. The quantum theory is thus defined through the quantum mechanical version of the classical constraints (3.19) and (3.20) [6]. As a result, we do not need to include any Liouville term in our action in the case of pure gravity. If matter would have been present, there could be Liouville terms coming from integrating out the matter, but that is beyond the scope of this thesis.

3.2.1 Solution

In references [68, 69], the physical wavefunctions that solve the dilaton gravity constrains are constructed as follows. The key step is to notice that the constraints $\mathcal{P}$ and $\mathcal{H}_{\text{WdW}}$ are simple enough that we can solve for Π_σ and Π_ϕ separately. For instance, by combing $\Pi_\sigma \mathcal{P}$ with the WdW constraint, we get

$$\partial_x (e^{-2\sigma} \Pi_\sigma^2) = \partial_x (e^{-2\sigma} (\partial_x \phi)^2 + W(\phi)) \;\Rightarrow\; \Pi_\sigma = \pm \sqrt{(\partial_x \phi)^2 + e^{2\sigma} [M + W(\phi)]}, \tag{3.22}$$

[6]From the path integral perspective, we are assuming an infinite range of integration over the lapse. Different choices for the contour of integration can drastically modify the constrains after quantization. We thank S. Giddings for discussions on this point.

with M an integration constant that is proportional to the ADM mass of the system as we will see momentarily. It is then straightforward to plug this into the WdW constraint to find an expression for Π_ϕ. Quantum mechanically, we want the physical wavefunction to satisfy,

$$\widehat{\Pi}_\sigma \Psi_{\text{phys}} = \pm Q[M;\phi,\sigma]\Psi_{\text{phys}}, \qquad \widehat{\Pi}_\phi \Psi_{\text{phys}} = \pm \frac{g[\phi,\sigma]}{Q[M;\phi,\sigma]}\Psi_{\text{phys}}, \tag{3.23}$$

where we defined the functions

$$Q[E;\phi,\sigma] \equiv \sqrt{(\partial_x\phi)^2 + e^{2\sigma}[M+W(\phi)]}, \qquad g[\phi,\sigma] \equiv \frac{1}{2}e^{2\sigma}U(\phi) + \partial_x^2\phi - \partial_x\phi\partial_x\sigma. \tag{3.24}$$

Wavefunctions that solve these constrains also solve the momentum and Wheeler de Witt constrains as explained in [68, 69]. In particular they solve the following WdW equation with factor ordering,[7]

$$\left(g - \widehat{Q}\widehat{\Pi}_\phi \widehat{Q}^{-1}\widehat{\Pi}_\sigma\right)\Psi_{\text{phys}} = 0 \tag{3.25}$$

The most general solution can be written as

$$\Psi = \Psi_+ + \Psi_-, \quad \Psi_\pm = \int dM \rho_\pm(M)\Psi_\pm(M), \tag{3.26}$$

where we will distinguish the two contributions

$$\Psi_\pm(M) = \exp\left[\pm i\int dx \left(Q[M;\phi,\sigma] - \partial_x\phi \tanh^{-1}\left(\frac{Q[M;\phi,\sigma]}{2\partial_x\phi}\right)\right)\right], \tag{3.27}$$

with the function Q defined in (3.24) which depends on the particular dilaton potential. We will refer in general to Ψ_+ (Ψ_-) as the expanding (contracting) branch.

This makes explicit the fact that solutions to the physical constrains reduce the naive Hilbert space from infinite dimensional to two dimensional with coordinate M (and its conjugate). The most general solution of the Wheeler-deWitt equation can then be expanded in the base $\Psi_\pm(M)$ with coefficients $\rho_\pm(M)$. The new ingredient in this thesis will be to specify appropriate boundary conditions to pick $\rho_\pm(M)$ and extract the full Hartle-Hawking wavefunction. We will see this is only possible for JT gravity for reasons that should will be clear in the next section.

It will be useful to write the physical wavefunction in terms of diffeomorphism invariant quantities. This is possible thanks to the fact that we are satisfying the momentum constrains. In order

[7]Here we think of $\widehat{Q}$ as well as $\widehat{M}$ as operators. The physical wavefunctions can be written as linear combinations of eigenfunctions of the operator $\widehat{M}$ with eigenvalue M.

to do this we will define the proper length u of the spacelike circle as

$$du = e^{\sigma} dx, \qquad L \equiv \int_0^1 e^{\sigma} dx, \tag{3.28}$$

where L denotes the total length. The only gauge invariant data that the wavefunction can depend on is then L and $\phi(u)$, a dilaton profile specified as a function of proper length along the boundary. The wavefunction (3.27) can be rewritten as

$$\Psi_{\pm}(M) = e^{\pm i \int_0^L du \left[\sqrt{W(\phi)+M+(\partial_u\phi)^2} - \partial_u \phi \, \tanh^{-1}\left(\sqrt{1+\frac{W(\phi)+M}{(\partial_u\phi)^2}} \right) \right]}, \tag{3.29}$$

which is then manifestly diffeomorphism invariant.

The results of this section indicate the space of physical states that solve the gravitational constrains is one dimensional, labeled by M. In the context of radial quantization of AdS_2 that we will analyze in the next section, this parameter corresponds to the ADM mass of the state, while in the case of dS_2, it corresponds to the generator of rotations in the spatial circle. Phase space is even-dimensional, and the conjugate variable to E is given by

$$\Pi_M = -\int dx \frac{e^{2\sigma}\Pi_\rho}{\Pi_\rho^2 - 2(\partial_x \phi)^2} \tag{3.30}$$

such that $[M, \Pi_M] = i$.[8]

3.2.2 Phase space reduction

Having the full solution to the WdW equation, we now study the minisuperspace limit. In this limit, the dilaton ϕ and boundary metric $e^{2\sigma}$ are taken to be constants. In a general theory of gravity, minisuperspace is an approximation. In JT gravity, as we saw above, the physical phase space is finite-dimensional (two dimensional to be precise). Therefore giving the wavefunction in the minisuperspace regime encodes all the dynamical information of the theory, while the generalization to varying dilaton is fixed purely by the constrains. In this section, we will directly extract the equation satisfied by the wavefunction as a function of constant dilaton and metric, from the more general case considered in the previous section.

If we start with the WdW equation and fix the dilaton and metric to be constant, the functional

[8] The simplicity of the phase space of dilaton gravity theories was also noted in [140].

derivatives then become ordinary derivatives and the equation reduces to

$$\left(\frac{1}{2}e^{2\sigma}U(\phi) - \widehat{Q}\partial_\phi \widehat{Q}^{-1}\partial_\sigma\right)\Psi(\phi,\sigma) = 0. \tag{3.31}$$

with $\widehat{Q} = (\widehat{M} + W(\phi))^{1/2}$. Due to the factor ordering, this differential equation still depends on the operator $\widehat{M}$, which is a bit unsatisfactory. Fortunately, we know that a σ derivative acting on Ψ is the same as acting with $Q^2/g\partial_\phi$. In the minisuperspace limit, we can therefore write (3.31) as

$$(LU(\phi) - 2L\partial_L(L^{-1}\partial_\phi))\Psi(\phi, L) = 0, \tag{3.32}$$

where L is the total boundary length. This equation is the exact constrain that wavefunctions with a constant dilaton should satisfy even though it was derived in a limit. We can explicitly check this by using (3.27) and noticing that any physical wavefunction, evaluated in the minisuperspace limit, will satisfy precisely this equation.

This equation differs from the one obtained in [139] by $\Psi_{\text{here}} = L\Psi_{\text{there}}$ and, therefore, changes the asymptotics of the wavefunctions, something we will analyze more closely in the next subsection.

3.2.3 Wheeler-deWitt in JT gravity: radial quantization

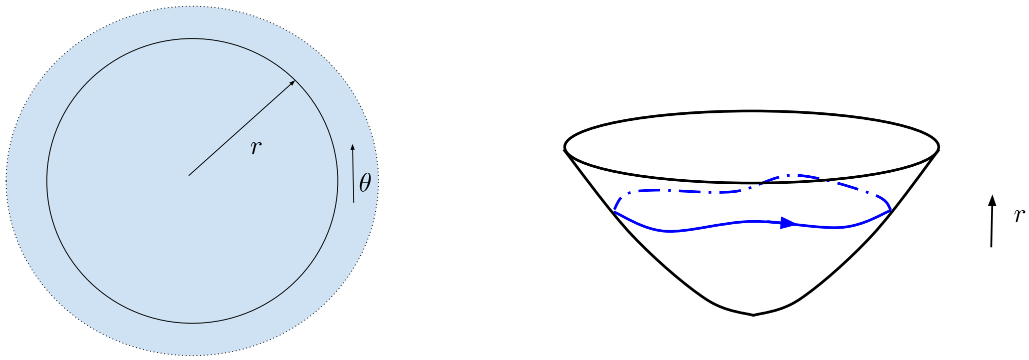
In this section, we will specialize the previous discussion to JT gravity with a negative cosmological constant. We fix units such that $U(\phi) = -2\phi$. We will analytically continue the results of the previous section to Euclidean space and interpret them in the context of radial quantization, such that the wavefunction is identified with the path integral in a finite cutoff surface. Then, we will explain how to implement Hartle-Hawking boundary conditions, obtaining a proposal for the exact finite cutoff JT gravity path integral that can be compared with results for the analog of the $T\overline{T}$ deformation in $1d$ [70, 71].

Lets begin by recalling some small changes that appear when going from Lorenzian to Euclidean radial quantization. The action we will work with is

$$I_{\text{JT}} = -\frac{1}{2}\int_M \sqrt{g}\phi(R+2) - \int_{\partial M}\sqrt{\gamma}\phi K, \tag{3.33}$$

and the ADM decomposition of the metric we will use is

$$ds^2 = N^2 dr^2 + h(d\theta + N_\perp dr)^2, \quad h = e^{2\sigma}\,, \tag{3.34}$$



(a) (b)

Figure 3.1: (a) We show the slicing we use for Euclidean JT gravity in asymptotically AdS_2, which has disk topology (but not necessarily rigid hyperbolic metric). (b) Frame where the geometry is rigid $EAdS_2$ with r increasing upwards and a wiggly boundary denoted by the blue curve.

where r is the radial direction while $\theta \sim \theta + 1$ corresponds to the angular direction that we will interpret as Euclidean time. We show these coordinates in figure 3.1. In terms of holography we will eventually interpret θ as related to the Euclidean time of a boundary quantum mechanical theory.

As shown in figure 3.1, and as we will explicitly show in section 3.3, the radial quantization wavefunction is identified with the gravitational path integral at a finite cutoff (inside the black circle) with Dirichlet boundary conditions

$$\Psi[\phi_b(u), \sigma(u)] = \int \mathcal{D}g\mathcal{D}\phi \; e^{-I_{\rm JT}[\phi,g]}, \qquad \text{with} \qquad \phi|_\partial = \phi_b(u), \qquad g|_\partial = \gamma_{uu} = e^{2\sigma(u)}. \tag{3.35}$$

The geometry inside the disk in figure 3.1 is asymptotically $EAdS_2$. From this path integral we can derive the WdW and momentum constrains and therefore solving the latter with the appropriate choice of state should be equivalent to doing the path integral directly.

The result of previous section implies that this path integral is given by a linear combination of

$$\text{Expanding branch:} \quad \Psi_+(M) = e^{\int_0^L du \left[\sqrt{\phi_b^2 - M - (\partial_u\phi_b)^2} - \partial_u\phi_b \tan^{-1}\left(\sqrt{\frac{\phi_b^2 - M}{(\partial_u\phi_b)^2} - 1}\right)\right]}, \tag{3.36}$$

$$\text{Contracting branch:} \quad \Psi_-(M) = e^{-\int_0^L du \left[\sqrt{\phi_b^2 - M - (\partial_u\phi_b)^2} - \partial_u\phi_b \tan^{-1}\left(\sqrt{\frac{\phi_b^2 - M}{(\partial_u\phi_b)^2} - 1}\right)\right]}. \tag{3.37}$$

We will focus on the purely expanding branch of the solution (3.36), as proposed in [141] and [139] to correspond to the path integral in the disk and therefore set $\rho_-(M) = 0$. We will go back to possible effects coming from turning on this term later. Thus, we will study the solutions

$$\Psi_{\rm disk}[\phi_b(u),\sigma(u)] = \int dM\rho(M)\ e^{\int_0^L du\left[\sqrt{\phi_b^2-M-(\partial_u\phi_b)^2}-\partial_u\phi_b\tan^{-1}\left(\sqrt{\frac{\phi_b^2-M}{(\partial_u\phi_b)^2}-1}\right)\right]}. \tag{3.38}$$

To make a choice of boundary conditions that fix the boundary curve very close to the boundary of the disk we will eventually take the limit of large L and ϕ_b.

3.2.4 Hartle-Hawking boundary conditions and the JT wavefunctional

To determine the unknown function $\rho(M)$, we will need to impose a condition that picks the Hartle-Hawking state. For this, one usually analyses the limit $L \to 0$ [142]. Such a regime is useful semiclassically but not in general. From the no-boundary condition, $L \to 0$ should reproduce the path integral over JT gravity inside tiny patches deep inside the hyperbolic disk; performing such a calculation is difficult. Instead, it will be simpler to impose the Hartle-Hawking condition at large $L \to \infty$. In this case, we know how to do the path integral directly using the Schwarzian theory. The derivation of the Schwarzian action from [29] explicitly uses the no-boundary condition, so we will take this limit instead, which will be enough to identify a preferred solution of the WdW equation.

To match the wavefunction with the partition function of the Schwarzian theory, it is enough to consider the case of constant dilaton and metric. Then, the wavefunction simplifies to

$$\Psi[\phi_b,\,\sigma] = \int dM\rho(M)\ e^{\int_0^1 d\theta e^{\sigma}\sqrt{\phi_b^2-M}} = \int dM\rho(M)\ e^{\int_0^L du\sqrt{\phi_b^2-M}} \tag{3.39}$$

with ϕ_b and σ constants. Expanding the root at large ϕ_b and large $L = e^{\sigma}$ gives,

$$\Psi[\phi_b,\sigma] = e^{L\phi_b}\int dM\rho(M)\ e^{-L\frac{M}{2\phi_b}+\dots} \tag{3.40}$$

We find the usual divergence for large L and ϕ, which can be removed by adding to (3.33) the counter term, $I_{\rm ct} = \int_0^L du\,\phi_b$. In fact, we will identify the JT path integral with this counter term as computing the thermal partition function at a temperature specified by the boundary conditions. At large L and ϕ_b we know that the gravity partition function is given by the Schwarzian theory:

$$\int \mathcal{D}g\mathcal{D}\phi\ e^{-I_{\rm JT}[\phi,g]} \to e^{L\phi_b}\int \frac{\mathcal{D}f}{SL(2,\mathbb{R})}\ e^{\phi_b\int_0^L du\,{\rm Sch}(\tan\frac{\pi}{L}f,u)}, \tag{3.41}$$

where ${\rm Sch}(F(u),u) \equiv \frac{F'''}{F'} - \frac{3}{2}\left(\frac{F''}{F'}\right)^2$. By rescaling time we can see the path integral only depends on L/ϕ_b which we will sometimes refer to as renormalized length. This result can be derived by first integrating over the dilaton over an imaginary contour, localizing the geometry to rigid AdS_2. Then the remaining degree of freedom is the shape of the boundary curve, from which the Schwarzian theory arises.

The Schwarzian partition function can be computed exactly and gives

$$Z_{\rm Sch}(\ell) \equiv \int \frac{\mathcal{D}f}{SL(2,\mathbb{R})} e^{\int_0^\ell du\, {\rm Sch}(\tan\frac{\pi}{\ell}f,u)} = \left(\frac{\pi}{\ell}\right)^{3/2} e^{\frac{2\pi^2}{\ell}} = \int dk^2 \sinh(2\pi k) e^{-\ell k^2/2} \tag{3.42}$$

Applying this result to the JT gravity path integral with the replacement $\ell \to L/\phi_b$ gives the partition function directly in the form of equation (3.40) where we can straightforward identify the Schwarzian density of states with the function of M as

$$\rho_{\rm HH}(M) = \sinh(2\pi\sqrt{M}), \tag{3.43}$$

where the subscript indicates that we picked the Hartle-Hawking state. It is important that we are able to compute the path integral of JT gravity for ϕ_b, $L \to \infty$ but fixed L/ϕ_b. This involves an exact treatment of the Schwarzian mode since otherwise we would only obtain $\rho_{\rm HH}(M)$ in some limits. This ingredient was missing in [68, 69] making them unable to identify the HH state from the full space of physical states.

To summarize, the solution of the gravitational constrains gives the finite cutoff JT gravity path integral as

$$\Psi_{\rm HH}[\phi_b(u), L] = \int_0^\infty dM \sinh(2\pi\sqrt{M})\; e^{\int_0^L du\left[\sqrt{\phi_b^2 - M - (\partial_u\phi_b)^2} - \partial_u\phi \tan^{-1}\left(\sqrt{\frac{\phi^2 - M}{(\partial_u\phi)^2}-1}\right)\right]}. \tag{3.44}$$

By construction, this matches the Schwarzian limit when ϕ_b and σ are constant.

When the dilaton is constant but $\sigma(u)$ is not, it is clear that we can simply go to coordinates $d\tilde{\theta} = e^\sigma d\theta$ in both the bulk path integral and the WdW wavefunction and see that they give the same result. Since we can always choose time-slices with a constant value for the dilaton, this situation will suffice for comparing our result to the analog of the $T\overline{T}$ deformation in the next subsection.

The more non-trivial case is for non-constant dilaton profiles. We provide a further check of our result in appendix A.1.1, where we compare the wavefunctional (3.44) to the partition function of JT gravity with a non-constant dilaton profile when the cutoff is taken to infinity.

3.2.5 Comparison to $T\overline{T}$

Let us now compare the wavefunctional (3.44) to the partition function obtained from the $1D$ analog of the $T\overline{T}$ deformation (3.4). First of all, let us consider configurations of constant ϕ_b, so $\partial_u\phi_b = 0$. This will simplify $\Psi_{\rm HH}$ to

$$\Psi_{\rm HH}[\phi_b, L] = \int_0^\infty dM\ \sinh(2\pi\sqrt{M})e^{\phi_b L\sqrt{1-M/\phi_b^2}}. \tag{3.45}$$

The partition function is then obtained by multiplying this wavefunction by $e^{-I_{\rm ct}} = e^{-L\phi_b}$. The resulting partition function agrees with (3.4) with identifications:

$$M \to 2CE, \quad \phi_b^2 \to \frac{C}{4\lambda}, \quad L \to \frac{\beta}{\sqrt{4C\lambda}}, \tag{3.46}$$

up to an unimportant normalization. In fact, we can say a little more than just mapping solution onto each other. In section 3.2.2 we showed that in the minisuperspace approximation the wave*functions* satisfy (3.32). With the identifications made above and the inclusion of the counter term, the partition function $Z_\lambda(\beta)$ satisfies

$$\left[4\lambda\partial_\lambda\partial_\beta + 2\beta\partial_\beta^2 - \left(\frac{4\lambda}{\beta} - 1\right)\partial_\lambda\right] Z_\lambda(\beta) = 0. \tag{3.47}$$

This is now purely written in terms of field theory variables and is precisely the flow equation as expected from (3.1), i.e. solutions to this differential equation have the deformed spectrum (3.2). This is also the flow of the partition function found in two dimensions in [143], specialised to purely imaginary modular parameter of the torus. We will analyze the associated non-perturbative ambiguities associated to this flow in section 3.4.

Let us summarise. We have seen that the partition function of the deformed Schwarzian theory is mapped to the exact dilaton gravity wavefunctions for constant ϕ_b and γ_{uu}. In fact, *any* quantum mechanics theory that is deformed according to (3.1) will obey the quantum WdW equation (for constant ϕ_b and σ). This principle can be thought of as the two-dimensional version of [141]. It is only the boundary condition at $\lambda \to 0$ (or large $\phi_b L$), where we know the bulk JT path integral gives the Schwarzian theory, that tells us that the density of states is $\sinh(2\pi\sqrt{M})$. Next, we will show that the wavefunction for constant ϕ_b and γ_{uu} can be reproduced by explicitly computing the Euclidean path integral in the bulk, at finite cutoff.

3.3 The Euclidean path integral

We will once again consider the JT gravity action, (3.33), and impose Dirichlet boundary conditions for the dilaton field $\phi|_{\partial M_2} \equiv \phi_b \equiv \phi_r/\varepsilon$, boundary metric γ_{uu}, and proper length $L \equiv \beta/\varepsilon$ and with the addition the counter-term,

$$I_{\text{ct}} = \int du \sqrt{\gamma}\phi \,, \tag{3.48}$$

whose addition leads to an easy comparison between our results and the infinite cutoff results in JT gravity. As in the previous section we will once again focus on disk topologies.

As discussed in section 3.2.4, the path integral over the dilaton ϕ yields a constrain on the curvature of the space, with $R = -2$. Therefore, in the path integral we are simply summing over different patches of AdS_2, which we parametrize in Euclidean signature using Poincaré coordinates as $ds^2 = (d\tau^2 + dx^2)/x^2$. To describe the properties which we require of the boundary of this patch we choose a proper boundary time u, with a fixed boundary metric $\gamma_{uu} = 1/\varepsilon^2$ (related to the fix proper length $L = \int_0^\beta du \sqrt{\gamma_{uu}}$). Fixing the intrinsic boundary metric to a constant, requires:

$$\frac{\tau'^2 + x'^2}{x^2} = \frac{1}{\varepsilon^2}\,, \qquad \frac{-t'^2 + x'^2}{x^2} = \frac{1}{\varepsilon^2}\,, \qquad \tau = -it\,. \tag{3.49}$$

If choosing some constant $\varepsilon \in \mathbb{R}$ then we require that the boundary has the following properties:

- If working in Euclidean signature, the boundary should never self-intersect. Consequently if working on manifolds with the topology of a disk this implies that the Euler number $\chi(M_2) = 1$.
- If working in Lorentzian signature, the boundary should always remain time-like since (3.49) implies that $-(t')^2 + (x')^2 = (x' - t')(t' + x') > 0$.[9] From now on we will assume without loss of generality that $t' > 0$.

Both conditions are important constraints which we should impose at the level of the path integral. Such conditions are not typical if considering the boundary of the gravitational theory as the worldline of a particle moving on H^2 or AdS_2: in Euclidean signature, the worldline could self-intersect, while in Lorentzian signature the worldline could still self-intersect but could also become space-like. These are the two deficiencies that [35, 36] encountered in their analysis, when viewing the path integral of JT gravity as that of a particle moving in an imaginary magnetic field on H^2.

[9]While fixing the metric γ_{uu} to be a constant is not diffeomorphism invariant, the notion of the boundary being time-like ($\text{sgn}\,\gamma_{uu}$) is in fact diffeomorphism invariant.

For the purposes of this chapter it will also prove convenient to introduce the light-cone coordinates (with $z = -ix + \tau\,, \overline{z} = ix + \tau$), for which fixing the intrinsic boundary metric implies:

$$-\frac{4z'\overline{z}'}{(z-\overline{z})^2} = \frac{1}{\varepsilon^2}\,. \tag{3.50}$$

In Euclidean signature $z = \overline{z}^*$, while in Lorentzian signature $z,\, \overline{z} \in i\mathbb{R}$. The constraint that the boundary is time-like implies that $iz' > 0$ and $i\overline{z}' < 0$ (alternatively, if assuming $t' < 0$, $iz' < 0$ and $i\overline{z}' > 0$). In order to solve the path integral for the remaining boundary fluctuations in the 1D system it will prove convenient to use light-cone coordinates and require that the path integral obeys the two properties described above.

3.3.1 Light-cone coordinates and $SL(2,\mathbb{R})$ isometries in AdS_2

As is well known, AdS_2, even at finite cutoff, exhibits an $SL(2,\mathbb{R})$ isometry. This isometry becomes manifest when considering the coordinate transformations:

$$\begin{aligned}
&\text{E \& L}: \qquad z \to \frac{az+b}{cz+d}\,, \qquad\qquad \overline{z} \to \frac{a\overline{z}+b}{c\overline{z}+d}\,,\\
&E: \qquad x+i\tau \to \frac{a(x+i\tau)+b}{c(x+i\tau)+d}\,, \qquad L: t+x \to \frac{a(t+x)+b}{c(t+x)+d}\,,
\end{aligned} \tag{3.51}$$

It is straightforward to check that under such transformations the boundary metrics, (3.49) and (3.50), both remain invariant. The same is true of the extrinsic curvature, which is the light-cone parametrization of the boundary degrees of freedom can be expressed as

$$K[z(u),\overline{z}(u)] = \frac{2z'^2\overline{z}' + (\overline{z}-z)\overline{z}'z'' + z'(2\overline{z}'^2 + (z-\overline{z})\overline{z}'')}{4(z'\overline{z}')^{3/2}}\,. \tag{3.52}$$

Consequently, invariance under $SL(2,\mathbb{R})$ transformations gives:

$$K[z,\overline{z}] = K\left[\frac{az+b}{cz+d}, \frac{a\overline{z}+b}{c\overline{z}+d}\right]\,, \tag{3.53}$$

Therefore, if solving for $\overline{z}[z(u)]$ (as a functional of $z(u)$) we will find that

$$\overline{z}[z(u)] \qquad \Rightarrow \qquad K[z] = K\left[\frac{az+b}{cz+d}\right] \tag{3.54}$$

As we will see in the next subsection, such a simple invariance under $SL(2,\mathbb{R})$ transformations will be crucial to being able to relate the path integral of the boundary fluctuations to that of some deformation of the Schwarzian theory. An important related point is that when solving for $\tau[x(u)]$ as a functional of $x(u)$, the resulting extrinsic curvature is not invariant under the $SL(2,\mathbb{R})$ transformations, $\tau \to \frac{a\tau+b}{c\tau+d}$. Rather this is only a valid symmetry in the $\varepsilon \to 0$ limit, for which $x \to 0$, while τ is kept finite. It is only in the asymptotically AdS_2 limit that the transformation in the second line of (3.51) can be identified with $\tau \to \frac{a\tau+b}{c\tau+d}$. If keeping track of higher orders in ε, the transformation on τ would involve a growing number of derivatives on the τ field which should be proportional to the order of the ε-expansion.

3.3.2 Restricting the extrinsic curvature

Next, we discuss the expansion of the extrinsic curvature $K[z]$ to all orders in perturbation in ε:

$$K[z] = \sum_{n=0}^{\infty} \varepsilon^n K_n[z]\,, \qquad K_n[z] = K_n\left[\frac{az+b}{cz+d}\right]\,, \tag{3.55}$$

We could in principle explicitly solve for $\overline{z}[z(u)]$ to first few orders in perturbation theory in ε and then plug the result into (3.63). The first few orders in the expansion can be solved explicitly and yield:

$$\begin{aligned} &K_0[z] = 1, \qquad K_1[z] = 0, \qquad K_2[z] = \mathrm{Sch}(z,u), \\ &K_3[z] = -i\,\partial_u \mathrm{Sch}(z,u)\,, \qquad K_4[z] = -\frac{1}{2}\mathrm{Sch}(z,u)^2 + \partial_u^2\,\mathrm{Sch}(z,u)\,. \end{aligned} \tag{3.56}$$

The fact that all orders in $K_n[z(u)]$ solely depend on the Schwarzian and its derivatives is not a coincidence. In fact, one generally finds that:

$$K_n[z] = \mathcal{K}_n[\mathrm{Sch}(z,u),\, \partial_u]\,. \tag{3.57}$$

The reason for this is as follows. $K_n[z]$ is a local function of $z(u)$ since solving for $\overline{z}[z(u)]$ involves only derivatives of $z(u)$. The Schwarzian can be written as the Casimir of the $\mathfrak{sl}(2,\mathbb{R})$ transformation, $z \to \frac{a\,z+b}{c\,z+d}$ [29]. Because the rank of the $\mathfrak{sl}(2,\mathbb{R})$ algebra is 1, higher-order Casimirs of $\mathfrak{sl}(2,\mathbb{R})$ can all be expressed as a polynomial (or derivatives of powers) of the quadratic Casimir. Since local functions in u that are $SL(2,\mathbb{R})$ invariant, can also only be written in terms of the Casimirs of $\mathfrak{sl}(2,\mathbb{R})$ this implies that they should also be linear combinations of powers (or derivatives of powers) of the quadratic Casimir, which is itself the Schwarzian.

Alternatively, we can prove that $K_n[z(u)]$ is a functional of the Schwarzian by once again noting that $K_n[z(u)]$ only contains derivatives of $z(u)$ up to some finite order. Then we can check explicitly how each infintesimal $SL(2,\mathbb{R})$ transformation constrains $K_n[z(u)]$. For instance, translation transformations $z \to z+b$ imply that K_n solely depends on derivatives of $z(u)$. The transformation $z(u) \to az(u)$ implies that $K_n[z(u)]$ depends solely on ratios of derivatives with a matching order in z between the numerator and denominator of each ratio, of the type $(\prod_k z^{(k_i)})/(\prod_k z^{(\tilde{k}_i)})$. Finally considering all possible linear combinations between ratios of derivatives of the type $(\prod_k z^{(k_i)})/(\prod_k z^{(\tilde{k}_i)})$ and requiring invariance under the transformation $z(u) \to 1/z(u)$, fixes the coefficients of the linear combination to those encountered in arbitrary products of Schwarzians and of its derivatives.

Once again, we emphasize that this does not happen when using the standard Poincaré parametrization (3.49) in τ and x. When solving for $\tau[x]$ and plugging into $K[\tau(u)]$, since we have that $K[\tau(u)] \neq K[a\tau(u)+b/(c\tau(u)+d)]$ and consequently $K[\tau(u)]$ is not a functional of the Schwarzian; it is only a functional of the Schwarzian at second-order in ε. This can be observed by going to fourth order in the ε-expansion, where

$$K_4[\tau(u)] = \frac{\tau^{(3)}(u)^2}{\tau'(u)^2} + \frac{27\tau''(u)^4}{8\tau'(u)^4} + \frac{\tau^{(4)}(u)\tau''(u)}{\tau'(u)^2} - \frac{11\tau^{(3)}(u)\tau''(u)^2}{2\tau'(u)^3}\,, \tag{3.58}$$

which cannot be written in terms of $\mathrm{Sch}(\tau(u),u)$ and of its derivatives.

3.3.3 Finding the extrinsic curvature

Perturbative terms in $K[z(u)]$

The previous subsection identified the abstract dependence of the extrinsic curvature as a function of the Schwarzian. To quantize the theory, we need to find the explicit dependence of K_n on the Schwarzian. To do this, we employ the following trick. Consider the specific configuration for $z(u)$:[10]

$$z(u) = \exp(au)\,, \qquad \mathrm{Sch}(z,u) = -\frac{a^2}{2}\,. \tag{3.59}$$

Since $K[z(u)]$ is a functional of the $\mathrm{Sch}(z,u)$ and of its derivatives to all orders in perturbation theory in ε, then $K_n[z(u) = \exp(au)] = \mathcal{K}_n[\mathrm{Sch}(z,u),\,\partial_u] = \mathcal{K}_n[a]$. On the other hand, when using a specific configuration for $z(u)$ we can go back to the boundary metric constraint (3.50) and explicitly solve for $\bar{z}(u)$. Plugging-in this solution together with (3.59) into the formula for the extrinsic curvature $K[z(u),\bar{z}(u)]$ (3.52), we can find $\mathcal{K}_n[a]$ and, consequently, find the powers of the

[10] While (3.59) is, in fact, a solution to the equation of motion for the Schwarzian theory it is not necessarily a solution to the equation of motion in the theory with finite cutoff.

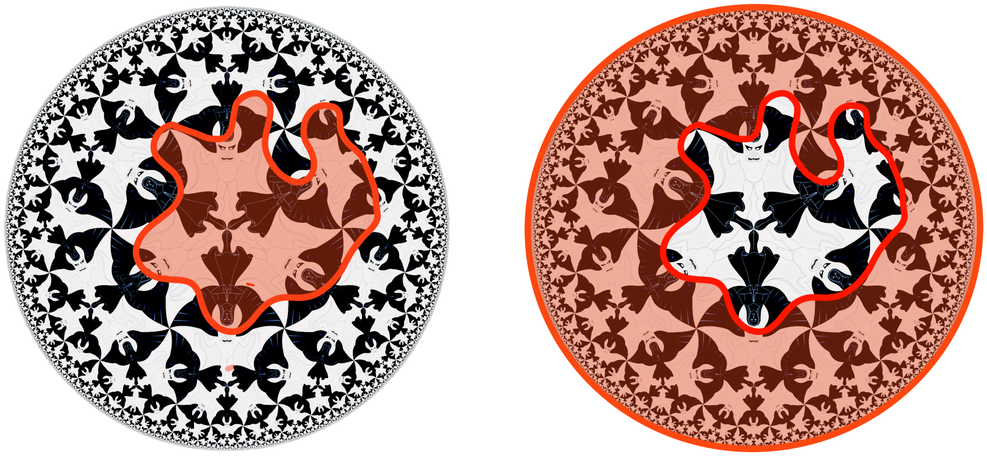


Figure 3.2: Cartoon exemplifying typical AdS_2 patches with a finite proper boundary length. The surface on the left represents the $K > 0$ solution and the surface on the right corresponds to the $K < 0$ solution.

Schwarzian in $\mathcal{K}_n[\text{Sch}(z, u),\, \partial_u]$.

The metric constraint involves solving the first order differential equation

$$-\frac{4\, a\, e^{au}\overline{z}'}{(e^{au} - \overline{z})^2} = \frac{1}{\varepsilon^2}\,, \tag{3.60}$$

whose solution, to all orders in perturbation theory in ε, is given by

$$\overline{z}(u) = e^{au}\left(1 - 2a^2\varepsilon^2 - 2a\varepsilon\sqrt{-1 + a^2\varepsilon^2}\right)\,. \tag{3.61}$$

We can plug this solution for $\overline{z}(u)$ together with the configuration $z(u) = \exp(au)$ to find that

$$K\left[z(u) = \exp(au)\right] = \sqrt{1 - \varepsilon^2 a^2}\,. \tag{3.62}$$

Depending on the choice of branch one can reverse the sign of (3.62) to find that $K\left[z(u) = \exp(au)\right] = -\sqrt{1 - \varepsilon^2 a^2}$ which corresponds to the considering the exterior of an AdS_2 patch as our surface (instead of a regular AdS_2 patch). This is analogous to the contracting branch in of the WdW functional in (3.37).

Consequently, it follows that in a perturbative series in ε we find:[11]

$$K_{\pm}[z(u)] = \pm\left(\sqrt{1 + 2\varepsilon^2\, \text{Sch}(z, u)} \;+\; \text{derivatives of Sch.}\right)\,, \tag{3.63}$$

[11]The terms containing derivatives of the Schwarzian are not necessarily total derivatives and thus we need to explain why they do not contribute to the path integral.

where we find that the quadratic term in ε for the + branch of (3.63) agrees with the expansion of K in terms of ε in JT gravity in asymptotic AdS_2 [29] (which found that $K[z(u)] = 1+\varepsilon^2\text{Sch}(z,u)+\dots$). The + branch in (3.63) corresponds to compact patches of AdS_2 for which the normal vector points outwards; the $-$ branch corresponds to non-compact surfaces (the complement of the aforementioned AdS_2 patches) for which the normal vector is pointing inwards. While the + branch has a convergent path integral for real values of ϕ_r, for a normal choice of countour for $z(u)$, the path integral of the $-$ branch will be divergent. Even for a potential contour choice for which the path integral were convergent, the $-$ branch is non-perturbatively suppressed by $O(e^{-\int_0^\beta du\,\phi_b/\varepsilon}) = O(e^{-1/\varepsilon^2})$. Therefore, for now, we will ignore the effect of this different branch $(-)$ and set $K[z(u)] \equiv K_+[z(u)]$; we will revisit this problem in section 3.4 when studying non-perturbative corrections in ε.

In principle, one can also solve for the derivative of the Schwarzian in (3.63) following a similar strategy to that outlined above. Namely, it is straightforward to find that when $\text{Sch}(z,u) = au^n$, for some $n \in \mathbb{Z}$, then $z(u)$ is related to a Bessel function. Following the steps above, and using the fact that $\partial^{n+1}\text{Sch}(z,u) = 0$ for such configurations, one can then determine all possible terms appearing in the extrinsic curvature. However, since we are interested in quantizing the theory in a constant dilaton configuration, we will shortly see that we can avoid this more laborious process.

Therefore, the JT action that we are interested in quantizing is given by:

$$I_{JT} = -\int_0^\beta \frac{du}{\varepsilon^2}\phi_r\left(\sqrt{1+2\varepsilon^2\,\text{Sch}(z,u)} - 1 + \text{derivatives of Sch.}\right), \tag{3.64}$$

where we have added the correct counter-term needed in order to cancel the $1/\varepsilon^2$ divergence in the $\varepsilon \to 0$ limit.

While we have found $K[z(u)]$ and I_{JT} to all orders in perturbation theory in ε, we have not yet studied other non-perturbative pieces in ε (that do not come from the $-$ branch in (3.63)). Such corrections could contain non-local terms in u since all terms containing a finite number of derivatives in u are captured by the ε-perturbative expansion. The full solution of (3.60) provides clues that such non-perturbative corrections could exist and are, indeed, non-local (as they will not be a functional of the Schwarzian). The full solution to (3.60) is

$$\overline{z}(u) = e^{au}\left(1 - 2a^2\varepsilon^2 + 2a\varepsilon\left(\sqrt{-1+a^2\varepsilon^2} - \frac{2\varepsilon}{\frac{\varepsilon}{\sqrt{-1+a^2\varepsilon^2}} + \mathcal{C}_1 e^{\frac{u}{\varepsilon}\sqrt{-1+a^2\varepsilon^2}}}\right)\right), \tag{3.65}$$

for some integration constant $\mathcal{C}_1$. When $\mathcal{C}_1 \neq 0$, note that the correction to $\overline{z}(u)$ in (3.65) are exponentially suppressed in $1/\varepsilon$ and do not contribute to the series expansion $\mathcal{K}_n$. However, when taking $\mathcal{C}_1 \neq 0$, (3.65) there is no way of making $\overline{z}(u)$ periodic (while it is possible to make $z(u)$ periodic). While we cannot make sure that every solution has the feature that non-perturbative corrections are inconsistent with the thermal boundary conditions, for the remainder of this section we will only focus on the perturbative expansion of $K[z(u)]$ with the branch choice for the square root given by (3.63). We will make further comments about the nature of non-perturbative corrections in section 3.4.

3.3.4 Path integral measure

Before we proceed by solving the path integral of (3.64), it is important to discuss the integration measure and integration contour for $z(u)$. Initially, before imposing the constraint (3.50) on the boundary metric, we can integrate over both $z(u)$ and $\overline{z}(u)$, with the two variables being complex conjugates in Euclidean signature. However, once we integrate out $\overline{z}(u)$ we are free to choose an integration contour consistent with the constraint (3.50) and with the topological requirements discussed at the beginning of this section. Thus, for instance if we choose $z(u) \in \mathbb{R}$ then the constraint (3.50) would imply that $z'(u) > 0$ (or $z'(u) < 0$); this, in turn, implies that we solely need to integrate over strictly monotonic functions $z(u)$. The boundary conditions for $z(u)$ should nevertheless be independent of the choice of contour; therefore we will impose that $z(u)$ is periodic, $z(0) = z(\beta)$. Of course, this implies that $z(u)$ has a divergence. In order to impose that the boundary is never self-intersecting we will impose that this divergence occurs solely once.[12] Such a choice of contour therefore satisfies the following two criteria:

- That the boundary is not self-intersecting.
- The boundary is time-like when going to Lorentzian signature. This is because redefining $z(u) \to z^{\text{Lor.}}(u) = -iz(u) \in \mathbb{R}$ leaves the action invariant and describes the boundary of a Lorentzian manifold. Since $i(z^{\text{Lor.}})' > 0$, it then follows that the boundary would be time-like.

Furthermore, while we have chosen a specific diffeomorphism gauge which fixes $\gamma_{uu} = 1/\varepsilon^2$, the path integral measure (as opposed to the action) should be unaffected by this choice of gauge and should rather be diffeomorphism invariant. The only possible local diffeomorphism invariant path integral measure is that encountered in the Schwarzian theory [144, 61, 25] and, in JT gravity at

[12]All this is also the case in the Schwarzian theory whose classical solution is $\tau(u) = \tan(\pi u/\beta)$. [29] has found that if considering solutions where $\tau(u)$ diverges multiple times ($\tau(u) = \tan(n\pi u/\beta)$ with $n \in \mathbb{Z}$) then the fluctuations around such solutions are unbounded, and the path integral is divergent (one can still make sense of this theory though, as explained in [63]).

infinite cutoff [37]:

$$D\mu[z] = \prod_{z\in[0,\beta)} \frac{dz(u)}{z'(u)}\,. \tag{3.66}$$

In principle, one should also be able to derive (3.66) by considering the symplectic form for JT gravity obtained from an equivalent $\mathfrak{sl}(2,\mathbb{R})$ BF-theory. In [37] this symplectic form (which in turn yields the path integral measure (3.66)) was derived in the limit $\varepsilon \to 0$. It would however be interesting to rederive the result of [37] at finite ε in order to find a more concrete derivation of (3.66).

To summarize, we have therefore argued that both the path integration measure, as well as the integration contour, in the finite-ε theory, can be taken to be the same as those in the pure Schwarzian theory.

3.3.5 Finite cutoff partition function as a correlator in the Schwarzian theory

The path integral which we have to compute is given by

$$\begin{aligned} Z_{JT}[\phi_b, L] = \int_{z'(u)>0} D\mu[z] \exp\Bigg[\int_0^\beta \frac{du}{\varepsilon^2}\phi_r\Bigg(&\sqrt{1+2\varepsilon^2 \text{Sch}(z,u)} - 1 + \\ &+ \text{ derivatives of Sch.}\Bigg)\Bigg]\,, \end{aligned} \tag{3.67}$$

Of course, due to the agreement of integration contour and measure, we can view (3.67) as the expectation value of the operator in the pure Schwarzian theory with coupling ϕ_r:

$$\begin{aligned} &Z_{JT}[\phi_b, L] = \langle \mathcal{O}_{\text{deformation}}\rangle \equiv \\ &\equiv \left\langle \exp\left[\int_0^\beta \frac{du}{\varepsilon^2}\phi_r\left(\sqrt{1+2\varepsilon^2\text{Sch}(z,u)} - 1 - \varepsilon^2\,\text{Sch}(z,u) + \text{derivatives of Sch.}\right)\right]\right\rangle. \end{aligned} \tag{3.68}$$

A naive analysis (whose downsides will be mention shortly) would conclude that, since in the pure Schwarzian theory, the Schwarzian can be identified with the Hamiltonian of the theory ($-\frac{H}{2\phi_r^2} = \text{Sch}(z,u)$), then computing (3.68) amounts to computing the expectation value for some function of the Hamiltonian and of its derivatives. In the naive analysis, one can use that the Hamiltonian is conserved and therefore all derivatives of the Schwarzian in (3.68) can be neglected. The conservation of the Hamiltonian would also imply that the remaining terms in the integral in the exponent (3.68) are constant. Therefore, the partition function simplifies to

$$Z_{JT}[\phi_b, L] =_{\text{naive}} \left\langle \exp\left[\frac{\beta\phi_r}{\varepsilon^2}\left(\sqrt{1-\frac{\varepsilon^2}{\phi_r^2}H}-1+\frac{\varepsilon^2}{2\phi_r^2}H\right)\right]\right\rangle. \tag{3.69}$$

which can be conveniently rewritten in terms of the actual boundary value of the dilaton $\phi_b = \phi_r/\varepsilon$ and the proper length $L = \beta/\varepsilon$ as

$$Z_{JT}[\phi_b, L] =_{\text{naive}} \left\langle \exp\left[L\phi_b\left(\sqrt{1-\frac{H}{\phi_b^2}}-1+\frac{H}{\phi_b}\right]\right\rangle. \tag{3.70}$$

The result for this expectation value in the Schwarzian path integral is given by

$$Z_{JT}[\phi_b, L] =_{\text{naive}} \int ds\, s\sinh(2\pi s)e^{L\phi_b\left(\sqrt{1-\frac{s^2}{\phi_b^2}}-1\right)} \tag{3.71}$$

where we have identified the energy of the Schwarzian theory in terms of the $\mathfrak{sl}(2,\mathbb{R})$ Casimir for which (for the principal series) $E = C_2(\lambda = is+\frac{1}{2})+\frac{1}{4} = s^2$ (see [115, 35, 36, 1]). The result (3.71) agrees with both the result for the WdW wavefunctional presented in section 3.2 (up to an overall counter-term) and with the results of [70, 71] (reviewed in the introduction), obtained by studying an analogue of the $T\overline{T}$ deformation in $1d$.[13]

As previously hinted, the argument presented above is incomplete. Namely, the problem appears because correlation functions of the $\text{Sch}(z, u)$ are not precisely the same as those of a quantum mechanical Hamiltonian. While at separated points correlation functions of the Schwarzian are constant (just like those of $1d$ Hamiltonians), the problem appears at identical points where contact-terms are present. Therefore, the rest of this section will be focused on a technical analysis of the contribution of these contact-terms, and we will show that the final result (3.71) is indeed correct even when including such terms.

The generating functional

To organize the calculation we will first present a generating functional for the Schwarzian operator in the undeformed theory. This generating functional is defined by

$$Z_{\text{Sch}}[j(u)] \equiv \int \frac{D\mu[z]}{SL(2,\mathbb{R})}\, e^{\int_0^\beta du j(u)\text{Sch}(z(u),u)}, \tag{3.72}$$

[13] We identify the deformation parameter $\lambda = \frac{\varepsilon^2}{4\phi_r}$ in [70, 71].

for an arbitrary function $j(u)$ which acts as a source for Schwarzian insertions. This path integral can be computed repeating the procedure in [25], which we also review in appendix A.1.1. The final answer is given by

$$Z_{\rm Sch}[j(u)] \sim e^{\int_0^\beta du \frac{j'(u)^2}{2j(u)}} \int ds\, s \sinh(2\pi s) e^{-\frac{s^2}{2}\int_0^\beta \frac{du}{j(u)}} \,. \tag{3.73}$$

We will use (3.73) to evaluate the integrated correlator (3.68), by rewriting it as

$$\langle \mathcal{O}_{\rm deformation}\rangle = \left[\exp\left(\int_0^\beta \frac{du}{\varepsilon^2}\phi_r : \left(\sqrt{1+2\varepsilon^2\frac{\delta}{\delta j(u)}} - 1 + \mathcal{K}\left[\partial_u \frac{\delta}{\delta j(u)}\right]\right) : \right) \times Z_{\rm Sch}[j(u)]\right]\Bigg|_{j(u)=0} , \tag{3.74}$$

where $\mathcal{K}\left[\partial_u \frac{\delta}{\delta j(u)}\right]$ is a placeholder for terms containing derivative terms of the Schwarzian and, equivalently, for terms of the from $\ldots \partial_u \frac{\delta}{\delta j(u)} \ldots$. Finally, $: \mathcal{O} :$ is a point-splitting operation whose role we will clarify shortly.

Computing the full path integral

To understand the point splitting procedure necessary in (3.77), we start by analyzing the structure of correlators when taking functional derivatives of $Z_{JT}[j(u)]$. Schematically, we have that

$$\left(\frac{\delta}{\delta j(u_1)} \cdots \frac{\delta}{\delta j(u_n)} Z_{\rm Sch}[j(u)]\right)\Bigg|_{j(u)=\phi_r} = a_1 + a_2[\delta(u_{ij})] + a_3[\partial_u \delta(u_{ij})] + \ldots \,, \tag{3.75}$$

where a_1 is a constant determined by the value of the coupling constant ϕ_r and $a_2[\delta(u_{ij})]]$ captures terms which have δ-functions in the distances $u_{ij} = u_i - u_j$, while $a_3[\partial_u \delta(u_{ij})]$ contains terms with at least one derivative of the same δ-functions for each term.[14] The $\ldots$ in (3.75) capture potential higher-derivative contact-terms.

If in the expansion of the square root in the exponent of (3.74) one takes the functional derivative $\delta/\delta j(u)$ at identical points then the contact terms in (3.75) become divergent (containing $\delta(0)$, $\delta'(0)$, $\ldots$). An explicit example about such divergences is given in appendix A.3 when evaluating the contribution of $K_4[z]$ in the perturbative series. In order to eliminate such divergences we define the point-splitting procedure

$$: \frac{\delta^n}{\delta j(u)^n} : \equiv \lim_{(u_1,\,\ldots,\,u_n)\to u} \frac{\delta}{\delta j(u_1)} \cdots \frac{\delta}{\delta j(u_n)} \,. \tag{3.76}$$

[14]For example, when $n = 2$ the exact structure of (3.75) is computed in [25] and is reviewed in appendix A.3.

Such a procedure eliminates the terms containing $\delta(0)$ or its derivatives since we first evaluate the functional derivatives in the expansion of (3.77) at separated points.

The structure of the generating functional also suggests that when integrating the correlator (3.75) the contribution of the derivatives of $\delta(u_{ij})$ vanish after integration by parts since we will be evaluating (3.77) for constant dilaton values. As we explain in more detail in appendix A.3, the origin of the derivatives of $\delta(u_{ij})$ is two-fold: they either come by taking functional derivatives $\delta/\delta j(u)$ of the term $\exp\left(\int_0^\beta du \frac{j'(u)^2}{2j(u)}\right)$ in $Z_{Sch}[j(u)]$, or they come from the contribution of the derivative terms $\mathcal{K}\left[\partial_u \frac{\delta}{\delta j(u)}\right]$. In either case, both sources only contribute terms containing derivatives of δ-functions (no constant terms or regular δ-functions). Thus, since such terms vanish after integration by parts, neither $\mathcal{K}\left[\partial_u \frac{\delta}{\delta j(u)}\right]$ nor $\exp\left(\int_0^\beta du \frac{j'(u)^2}{2j(u)}\right)$ contribute to the partition function. Consequently, we have to evaluate

$$\langle \mathcal{O}_{\text{deformation}} \rangle$$
$$= \left(\int ds\, s \sinh(2\pi s) \, \exp\left[\int_0^\beta \frac{du}{\varepsilon^2} \phi_r \left(: \sqrt{1 + 2\varepsilon^2 \frac{\delta}{\delta j(u)}} : -1 \right) \right] e^{-\frac{s^2}{2} \int_0^\beta du \frac{1}{j(u)}} \right)\Bigg|_{j(u)=0} . \quad (3.77)$$

To avoid having to deal with the divergences eliminated by the point-splitting discussed in the continuum limit, we proceed by discretizing the thermal circle into β/δ units of length δ (and will ultimately consider the limit $\delta \to 0$).[15] Divergent terms containing δ in the final result correspond to terms that contain $\delta(0)$ in the continuum limit and thus should be eliminated by through the point-splitting procedure (3.76). Therefore, once we obtain the final form of (3.77), we will select the universal diffeomorphism invariant δ-independent term.

To start, we can use that

$$e^{-\frac{s^2 \delta}{2j(u)}} = \frac{1}{2\pi i} \int_{-c-i\infty}^{-c+i\infty} d\alpha_u \left[-\frac{\pi Y_1(2\sqrt{\alpha_u})}{\sqrt{\alpha_u}} \right] e^{-\frac{2\alpha_u j_u}{s^2 \delta}} \quad (3.78)$$

where we have introduced a Lagrange multiplier α_u for each segment in the thermal circle. The integration contours for all α_u are chosen along the imaginary axis for some real constant c. The

[15] Sums and products of the type $\sum_{u\in[0,\beta)}$ and $\prod_{u\in[0,\beta)}$ will iterate over all β/δ intervals.

next step is to apply the differential operator in the exponent in (3.77) to (3.78),

$$\left(e^{\int_0^\beta du \frac{\phi_r}{\varepsilon^2}\left(:\sqrt{1+2\varepsilon^2\frac{\delta}{\delta j(u)}}:-1\right)}\right)\prod_{u\in[0,\beta)} e^{-\frac{2\alpha_u j_u}{s^2\delta}}\Bigg|_{j_u=0} =$$
$$= \left(e^{\int_0^\beta du \frac{\phi_r}{\varepsilon^2}\left(:\sqrt{1+2\varepsilon^2\frac{\delta}{\delta j(u)}}:-1\right)}\right) e^{-\int_0^\beta du\, \frac{2\alpha_u j_u}{s^2\delta^2}}\Bigg|_{j_u=0}$$
$$=: \exp\left[\sum_{u\in[0,\beta)} \frac{\delta\phi_r}{\varepsilon^2}\left(\sqrt{1-\frac{4\alpha_u\varepsilon^2}{s^2\delta^2}}-1\right)\right]:, \tag{3.79}$$

where $:\cdots:$ indicates that we will be extracting the part independent of the UV cutoff, δ, when taking the limit $\delta\to 0$. Thus, we now need to compute

$$Z_{JT}[\phi_b,\, L] = \frac{1}{2\pi i} : \int_0^\infty ds\, s\sinh(2\pi s)\int_{-c-i\infty}^{-c+i\infty}\left(\prod d\alpha_u\right)\left[-\frac{\pi Y_1(2\sqrt{\alpha_u})}{\sqrt{\alpha_u}}\right]$$
$$\times\, e^{\sum_{u\in[0,\beta)}\frac{\delta\phi_r}{\varepsilon^2}\left(\sqrt{1-\frac{4\alpha_u\varepsilon^2}{s^2\delta^2}}-1\right)} :. \tag{3.80}$$

In order to do these integrals we introduce an additional field σ_u, such that

$$e^{\frac{\delta\phi_r}{\varepsilon^2}\left(\sqrt{1-\frac{4\alpha_u\varepsilon^2}{s^2\delta^2}}-1\right)} = \int_0^\infty \frac{d\sigma_u}{\sigma_u^{3/2}}\sqrt{-\frac{\delta\phi_r}{2\pi\varepsilon^2}}\, e^{-\frac{2\sigma_u\alpha_u\phi_r}{s^2\delta}+\frac{\delta\phi_r}{2\sigma_u\varepsilon^2}(1-\sigma_u)^2}, \tag{3.81}$$

where in order for the integral (3.81) to be convergent, we can analytically continue ϕ_r to complex values. We can now perform the integral over α_u using (3.78), since α_u now appears once again in the numerator of the exponent:

$$Z_{JT}[\phi_b,\, L] = : \int_0^\infty ds\, s\sinh(2\pi s)$$
$$\times \int_0^\infty\left(\prod_{u\in[0,\beta)}\frac{d\sigma_u}{\sigma_u^{3/2}}\sqrt{-\frac{\delta\phi_r}{2\pi\varepsilon^2}}\right) e^{\sum_{u\in[0,\beta)}\left[-\frac{s^2\delta}{2\sigma_u\phi_r}+\frac{\delta\phi_r}{2\sigma_u\varepsilon^2}(1-\sigma_u)^2\right]} : . \tag{3.82}$$

We now change variable in the equation above from $\sigma_u\to 1/\tilde{\sigma}_u$ and perform the Laplace transform, once again using (3.81). We finally find that (when keeping the finite terms in δ) the partition function is given by:[16]

$$Z_{JT}[\phi_b,\, L] \sim \int_0^\infty ds\, s\sinh(2\pi s)\, e^{\frac{\beta\phi_r}{\varepsilon^2}\left(\sqrt{1-\frac{s^2\varepsilon^2}{\phi_r^2}}-1\right)}$$
$$\sim \int_0^\infty ds\, s\sinh(2\pi s)\, e^{\frac{\beta}{4\lambda}\left(\sqrt{1-4\lambda s^2/\phi_r}-1\right)}, \tag{3.83}$$

[16] Once again to integrate over $\tilde{\sigma}_u$ we have to analytically continue ϕ_r to complex values. Finally, to perform the integral over s in (3.83) we analytically continue back to real values of ϕ_r and, equivalently, ϕ_b.

where we defined $\lambda = \varepsilon^2/(4\phi_r)$. This partition function agrees with the naive result (3.71) obtained by replacing the Schwarzian with the Hamiltonian of the pure theory. Consequently, we arrive to the previously mentioned matching between the Euclidean partition function, the WdW wavefunctional and the partition function of the $T\overline{T}$ deformed Schwarzian theory,

$$e^{-I_{\rm ct}}\Psi_{HH}[\phi_b,\, L] = Z_{\lambda=\varepsilon^2/(4\phi_r)}(\beta) = Z_{JT}[\phi_b,\, L]\,. \tag{3.84}$$

As a final comment, the Euclidean path integral approach hides two ambiguities. First, as we briefly commented in section 3.3.3, the finite cutoff expansion of the extrinsic curvature might involve terms that are non-perturbatively suppressed in ε. As we have mentioned before, such terms can either come from considering non-local terms in the extrinsic curvature $K[z(u)]$ or by considering the contribution of the negative branch in (3.63). Second, even if these terms would vanish, the perturbative series is only asymptotic. Performing the integral (3.83) over energies explicitly gives a finite cutoff partition function

$$Z_{JT}[\phi_b, L] = \frac{L\phi_b^2 e^{-L\phi_b}}{L^2+4\pi^2} K_2\Big(-\sqrt{\phi_b^2(L^2+4\pi^2)}\Big). \tag{3.85}$$

This formal result is not well defined since the Bessel function is evaluated at a branch cut [17]. The ambiguity related to the presence of this branch cut can be regulated by analytic continuation; for example, in $L \to Le^{i\epsilon}$, and the $\epsilon \to 0$ limit we find different answers depending on the sign of ϵ. The ambiguity given by the choice of analytic continuation can be quantified by the discontinuity of the partition function Disc Z for real ϕ and L.

A similar effect is reproduced by the contracting branch of the wavefunction from the canonical approach, there are two orthogonal solutions to the gravitational constraint $\Psi_\pm$, defined by their small cutoff behavior $\Psi_\pm \sim e^{\pm\phi L} Z_\pm$, where $Z_\pm$ is finite. In the language of the Euclidean path integral, the different choice of wavefunctionals correspond to different choices for the square root in the extrinsic curvature (3.63). Imposing Hartle-Hawking boundary conditions fixes Ψ_+, which matches the perturbative expansion of the Euclidean path integral. The corrections to the partition function from the other branch are exponentially suppressed $\Psi_-/\Psi_+ \sim e^{-\frac{1}{\varepsilon^2}}$.

As previously hinted, contributions from turning on Ψ_- are not only related to the choice of branch for $K[z(u)]$, but is the same as the branch-cut ambiguity mentioned above for (3.85). To

[17] This can be tracked to the fact that we are sitting at a Stokes line. It is curious that this explicit answer gives a complex function even though the perturbative terms we found from the path integral are all real (this phenomenon also happens in more familiar setups like WKB).

see this, we can notice that Disc Z is a difference of two functions that separately satisfy the WdW equation and goes to zero at small cutoff. Therefore it has to be of the same form as the Ψ_- branch given in (3.37).

3.4 The contracting branch and other topologies

In this section, we will analyze two different kinds of non-perturbative corrections to the partition function. First we will study corrections that are non-perturbative in the cutoff parameter ε in sections 3.4.1 and 3.4.2, which come from turning on the contracting branch of the wavefunction. Then, we will comment on non-perturbative corrections coming from non-trivial topologies in section 3.4.3.

3.4.1 Unitarity at finite cutoff

Given the exact form of the wavefunction for general cutoff surfaces, we can study some of the more detailed questions about $T\overline{T}$ in AdS_2. One such question is whether the theory can be corrected to become unitarity. As can be seen from the expression for the dressed energy levels (3.2), the energies go complex whenever $\lambda > 1/(8E)$. This is unsatisfactory if we want to interpret the finite cutoff JT gravity partition function as being described by a $0+1$ dimensional theory, just like the Schwarzian theory describes the full AdS_2 bulk of JT gravity. There are a few ways in which one can go around this complexification.

Firstly, we can truncate the spectrum of the initial theory so that E is smaller than some $E_{\max}$. This is totally acceptable, but if we want to have an initial theory that describes the full AdS_2 geometry, we cannot do that without making the flow irreversible. In other words, the truncated Schwarzian partition function is not enough to describe the entire JT bulk. The second option is to accept there are complex energies along the flow but truncate the spectrum to real energies after one has flowed in the bulk. In $1D$ this was emphasized in [70] (and in [16, 145] for $2D$ CFTs). The projection operator that achieves such a truncation will then depend on λ and, in general, will not solve the flow equation (3.47) of the partition function. A third option is that we use the other branch of the deformed energy levels $\mathcal{E}_-$ (see (3.2)) to make the partition function real. In doing so, we will be guaranteed a solution to the Wheeler-de-Witt equation. Let us pursue option three in more detail and show that we can write down a real partition function $Z_\lambda(\beta)$ with the correct (Schwarzian) boundary condition at $\lambda \to 0$.

The solution to the $T\overline{T}$ flow equation (3.47) that takes the form of a partition function is,

$$Z_\lambda^{\text{non-pert.}}(\beta) = \int_0^\infty dE \rho_+(E) e^{-\beta \mathcal{E}_+(E,\lambda)} + \int_{-\infty}^\infty dE \rho_-(E) e^{-\beta \mathcal{E}_-(E,\lambda)}. \tag{3.86}$$

Here, we took the ranges of E to be such that $\mathcal{E}_\pm$ are bounded from below. As $\lambda \to 0$, we see that the first term goes to some constant (as we already saw previously), but the second term goes to zero non-perturbatively in λ as $e^{-\beta/(2\lambda)}$. From the boundary condition $\lambda \to 0$ we can therefore not fix the general solution, but only $\rho_+(E) = \sinh(2\pi\sqrt{2CE})$. If we demand the partition function to be real, then both integrals over E in (3.86) should be cutoff at $E = 1/(8\lambda)$ and it will therefore not be a solution to (3.47) anymore, because the derivatives with respect to λ can then act on the integration limit. However, by picking

$$\rho_- = \begin{cases} -\sinh(2\pi\sqrt{2CE}) & 0 < E < \frac{1}{8\lambda} \\ \widehat{\rho}(E) & E < 0 \end{cases}, \tag{3.87}$$

with $\widehat{\rho}(E)$ an arbitrary function of E, the boundary terms cancel and we obtain a valid solution to (3.47) and the associated wavefunction $\Psi = e^{L\phi_b} Z$ will solve the WdW equation (3.32). The final partition function is then given by (see appendix A.2 for details),

$$Z_\lambda^{\text{non-pert.}}(\beta) = \frac{\pi \beta e^{-\frac{\beta}{4\lambda}}}{\sqrt{2\lambda}(\beta^2 + 16C\pi^2\lambda)} I_2\left(\frac{1}{4\lambda}\sqrt{\beta^2 + 16C\pi^2\lambda}\right) + \int_{-\infty}^0 dE \widehat{\rho}(E) e^{-\beta\mathcal{E}_-(E,\lambda)}. \tag{3.88}$$

Notice that when we redefine E such that we have the canonical Boltzman weight in the second term of (3.88), the support of $\widehat{\rho}$ is for $E > \frac{1}{2\lambda}$, because for this redefined energy $E = 0$ maps to $\frac{1}{2\lambda}$. Let us comment on this partition function. First, because of the sign in (3.87), the first part of (3.88) has a negative density of states and turns out the be equal to (3.7) with support between $0 \le E \le \frac{1}{2\lambda}$, see Fig. 3.3. Second, there is a whole function worth of non-perturbative ambiguities coming from the second term in (3.88) that cannot be fixed by the Schwarzian boundary condition. From the Euclidean path integral approach, assuming that the extrinsic curvature does not receive non-perturbative corrections, we could fix $\widehat{\rho}(E) = 0$ by choosing an appropriate analytic continuation on L when defining the partition function.

3.4.2 Relation to 3d gravity

The analysis in the previous section can be repeated in the context of 3D gravity and $T\overline{T}$ deformations of 2D CFTs on a torus of parameters τ and $\overline{\tau}$. The deformed partition function satisfies an equation

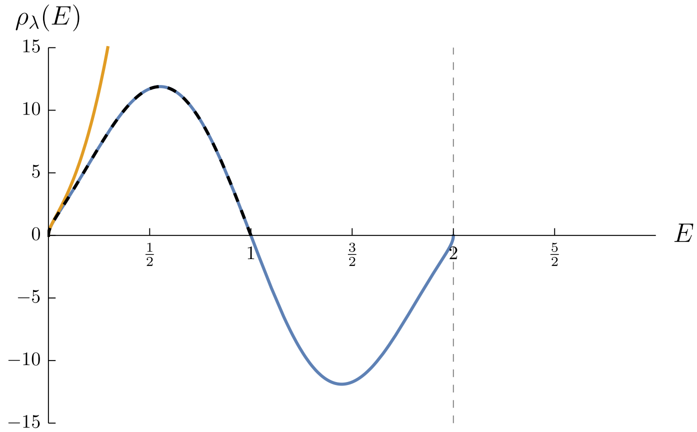


Figure 3.3: In orange, we show the undeformed density of states $\sinh(2\pi\sqrt{E})$ of JT gravity at infinite cutoff. In dashed black, we show the density of states of the theory with just the branch of the root, $\mathcal{E}_-$, that connects to the undeformed energies, until the energy complexifies. In blue, we show the density of states $\rho_\lambda(E)$ of the deformed partition function (3.88) which includes non-perturbative corrections in λ. Above we have set $\widehat{\rho}(E) = 0$ and $\lambda = 1/4$ and $C = 1/2$, the black line therefore ends at $E = \frac{1}{4\lambda} = 1$. The vertical dashed line indicates the energy beyond which $\widehat{\rho}$ has support.

similar to (3.47) derived in [143]. This is given by

$$-\partial_\lambda Z_\lambda = \left[8\tau_2\partial_\tau\partial_{\overline{\tau}} + 4\left(i(\partial_\tau - \partial_{\overline{\tau}}) - \frac{1}{\tau_2}\right)\lambda\partial_\lambda\right] Z_\lambda \tag{3.89}$$

The solutions of this equation, written in a form of a deformed partition function, can be written as

$$Z(\tau, \overline{\tau}, \lambda) = \sum_{\pm,\, k} \int_{E_0}^{\infty} dE \rho_\pm(E) e^{-\tau_2 \mathcal{E}_\pm(E,k) + 2\pi i k \tau_1} \tag{3.90}$$

where $\tau = \tau_1 + i\tau_2$ and $\overline{\tau} = \tau_1 - i\tau_2$. Here we have set the radius to one and

$$\mathcal{E}_\pm(E, k) = \frac{1}{4\lambda}\left(1 \mp \sqrt{1 - 8\lambda E + 64\pi^2 k^2 \lambda^2}\right). \tag{3.91}$$

As usual we pick the minus sign of the root as that connects to the undeformed energy levels at $\lambda = 0$. The energy levels of the deformed partition function complexify when $E_c = \frac{1}{8\lambda} + 8k^2\pi^2\lambda^2$. So we would like to cutoff the integral there. Similarly, a hard cutoff in the energy will not solve the above differential flow equation anymore. We can resolve this by subtracting the same partition function but with the other sign of the root in (3.91). This is again a solution, but (again) with negative density of states.

3.4.3 Comments about other topologies

Finally, we discuss the contribution to the path integral of manifolds with different topologies. The contribution of such surfaces is non-perturbatively suppressed by $e^{-\phi_0\chi(M)}$, where $\chi(M)$ is the Euler characteristic of the manifold.

We start with surfaces with two boundaries of zero genus, where one boundary has the Dirichlet boundary conditions (3.10) and the other ends on a closed geodesic with proper length b. The contribution of such surfaces to the partition function, referred to as "trumpets", has been computed in the infinite cutoff limit in [37]. We can repeat the method of section 3.2.4 to a spacetime with the geodesic hole of length b by applying the WdW constraints to the boundary on which we have imposed the Dirichlet boundary conditions. This constraint gives the trumpet finite cutoff partition function

$$Z_{\text{trumpet}}[\phi_b, L, b] = \frac{\phi L}{\sqrt{L^2-b^2}} K_1\Big(-\sqrt{\phi_b^2(L^2-b^2)}\Big)\,. \tag{3.92}$$

The partition function diverges as $L \to b$, indicating the fact that the boundary with Dirichlet boundary conditions overlaps with the geodesic boundary.

In order to construct higher genus surfaces or surfaces with more Dirichlet boundaries one can naviely glue the trumpet to either a higher genus Riemann bordered surface or to another trumpet. In order to recover the contribution to the partition function of such configurations we have to integrate over the closed geodesic length b using the Weil-Petersson measure, $d\mu[b] = db\, b$. However, if integrating over b in the range from 0 to ∞ for a fixed value of L we encounter the divergence at $L = b$.

One way to resolve the appearance of this divergence is to once again consider the non-perturbative corrections in ε discussed in section 3.4.1 for the trumpet partition function (3.92). We can repeat the same procedure as in 3.4.1 by accounting for the other WdW branch thus making the density of states of the "trumpet" real. Accounting for the other branch we find that

$$Z^{\text{non-pert.}}_{\text{trumpet}}[\phi_b, L, b] = \frac{2\pi\phi L}{\sqrt{L^2-b^2}} I_1\Big(\sqrt{\phi_b^2(L^2-b^2)}\Big)\,, \tag{3.93}$$

where we set the density of states for negative energies for the contracting branch to 0. Interestingly, the partition function (3.93) no longer has a divergence at $L = b$ which was present in (3.92) and precluded us previously from performing the integral over b. We could now integrate [18]

$$Z_{\text{cyl.}}^{\text{non-pert.}}[\phi_{b_1}, L_1, \phi_{b_2}, L_2] =_{\text{naive}} \int_0^\infty db\, b\, Z_{\text{trumpet}}^{\text{non-pert.}}[\phi_{b_1}, L_1, b] Z_{\text{trumpet}}^{\text{non-pert.}}[\phi_{b_2}, L_2, b]\,, \tag{3.94}$$

to obtain a potential partition function for the cylinder.[19]

Besides the ambiguity related to the non-perturbative corrections, there is another issue with the formula for the cylinder partition function (3.94). Specifically, for any value of the proper length L_1 and L_2 and for a closed geodesic length b (with $b < L_1$ and $b < L_2$) there exist cylinders for which the Dirichlet boundaries intersect with the closed geodesic of length b. Such surfaces cannot be obtained by gluing two trumpets along a closed geodesic as (3.94) suggests when using the result (3.92). Given that the partition function (3.93) does not have a clear geometric interpretation when including the contributions from the contracting branch, it is unclear if (3.94) accounts for such geometries. Given these difficulties, we hope to revisit the problem of summing over arbitrary topologies in the near future.

As another example of non-trivial topology, one can study the finite cutoff path integral in a disk with a conical defect in the center. Such defects were previously studied at infinite cutoff in [63]. The answer from the canonical approach is given by

$$Z_{\text{defect}}[\phi_b, L] = \frac{\phi L}{\sqrt{L^2 + 4\pi^2\alpha^2}} K_1\Big(-\sqrt{\phi_b^2(L^2 + 4\pi^2\alpha^2)}\Big), \tag{3.95}$$

where α is the opening angle, and $\alpha = 1$ gives back the smooth disk wavefunction. This function is finite for all L.

3.5 de Sitter: Hartle-Hawking wavefunction

As a final application of the results in this chapter, we will study JT gravity with positive cosmological constant, in two-dimensional nearly dS spaces. We will focus on the computation of the Hartle-Hawking wavefunction, see [139], and [147]. The results in these references focus on wavefunctions at late times, with an accurate Schwarzian description. Using the methods in this chapter, we will be able to compute the exact wavefunction at arbitrary times.

[18] Alternatively, one might hope to directly use WdW together with the results of [37] for arbitrary genus to directly compute the partition function at finite cutoff. However, as pointed out in [146], the WdW framework is insufficient for such a computation; instead, computing the full partition function requires a third-quantized framework which greatly complicates the computation.

[19] While unfortunately we cannot compute the integral over b exactly it would be interesting to check whether the partition function for the cylinder can be reproduced by a matrix integral whose leading density of states is given by the one found from the disk contribution.

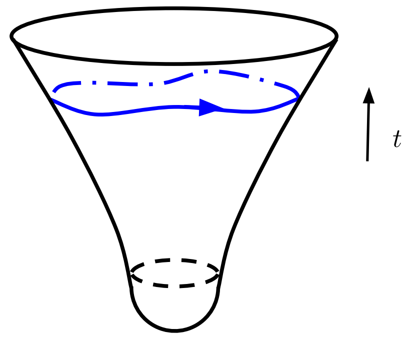


Figure 3.4: Frame in which geometry is rigid dS_2. Time runs upwards. We show the wiggly curve where we compute the wavefunction in blue (defined by its length and dilaton profile).

The Lorenzian action for positive cosmological constant JT gravity is given by

$$I_{\rm JT} = \frac{1}{2}\int_M \sqrt{g}\phi(R-2) - \int_{\partial M}\sqrt{\gamma}\phi K. \tag{3.96}$$

Following section 3.2.3, we use the ADM decomposition of the metric

$$ds^2 = -N^2 dt^2 + h(d\theta + N_\perp dt)^2, \qquad h = e^{2\sigma} \tag{3.97}$$

where now t is Lorenzian time and θ the spatial direction. We will compute the wavefunction of the universe $\Psi[L, \phi_b(u)]$ as a function of the total proper length of the universe L and the dilaton profile $\phi_b(u)$ along a spatial slice. The proper spatial length along the boundary is defined by $du = e^\sigma d\theta$. The solution satisfying the gravitational constrains is given by

$$\Psi_+[\phi_b(u), L] = \int dM\rho(M) e^{-i\int_0^L du\left[\sqrt{\phi_b^2 - M + (\partial_u\phi_b)^2} - \partial_u\phi_b \tanh^{-1}\left(\sqrt{1+\frac{\phi_b^2-M}{(\partial_u\phi_b)^2}}\right)\right]}, \tag{3.98}$$

where the index $+$ indicates we will focus on the expanding branch of the wavefunction. This is defined by its behavior $\Psi_+ \sim e^{-i\int_0^L du\ \phi_b(u)}$ in the limit of large universe (large L).

To get the wavefunction of the universe, we need to impose the Hartle-Hawking boundary condition. We will look again to the limit of large L, and for simplicity, we can evaluate it for a constant dilaton setting $\partial_u\phi_b = 0$ (this is enough to fix the expanding branch of the wavefunction completely).

As explained in [139], one can independently compute the path integral with Hartle-Hawking boundary conditions in this limit by integrating out the dilaton first. This fixes the geometry to be rigid dS_2, up to the choice of embedding of the boundary curve inside rigid dS_2, see figure 3.4. Then the result reduces to a Schwarzian path integral parametrizing boundary curves, just like in AdS_2.

The final result for a constant dilaton and total length L is given by

$$\Psi_+[\phi_b, L] \sim e^{-i\phi_b L} \int dM \sinh(2\pi\sqrt{M}) e^{iL\frac{M}{2\phi_b}}, \qquad L\phi_b \to \infty, \quad \phi_b/L \text{ fixed.} \tag{3.99}$$

This boundary condition fixes the function $\rho(M)$ in (3.98), analogously to the procedure in section 3.2.4.

Then the final answer for the expanding branch of the Hartle-Hawking wavefunction of JT gravity is

$$\Psi_+[\phi_b, L] = \int_0^\infty dM \sinh(2\pi\sqrt{M}) e^{-i\int_0^L du \left[\sqrt{\phi_b^2 - M + (\partial_u\phi_b)^2} - \partial_u\phi_b \tanh^{-1}\left(\sqrt{1 + \frac{\phi_b^2 - M}{(\partial_u\phi_b)^2}}\right)\right]}. \tag{3.100}$$

The same result can be reproduced for constant values of $\phi_b(u) = \phi_b$ by following the procedure in section 3.3, writing the extrinsic curvature along the spatial slice as a functional of the Schwarzian derivative. Following the same steps as in section 3.3.5, one could then recover the wavefunction (3.100) by computing the Lorentzian path integral exactly, to all orders in cutoff parameter ε.

The procedure outlined so far parallels the original method of Hartle and Hawking [142]. First, we solve the WdW equation, which for this simple theory can be done exactly. Then, we impose the constrains from the no-boundary condition. The only subtlety is that, while Hartle and Hawking impose their boundary conditions in the past, we are forced to impose the boundary condition at late times. This is a technical issue since the limit $L \to 0$ is strongly coupled. Nevertheless, we could, in principle, do it at early times if we would know the correct boundary condition in that regime.

A different procedure was proposed by Maldacena [148]. The idea is to compute the no-boundary wavefunction by analytic continuation, where one fills the geometry with '$-AdS$' instead of dS.[20] We can check now in this simple model that both prescriptions give the same result. For simplicity, after fixing the dilaton profile to be constant, one can easily check that the result (3.100) found following Hartle and Hawking matches with the analytic continuation of the finite cutoff Euclidean path integral in AdS computed in section 3.3.

For a constant dilaton profile, we can perform the integral to compute the wavefunction

$$\Psi_+[\phi_b, L] = \frac{L\phi_b^2}{L^2 - 4\pi^2 - i\epsilon} K_2\left(i\sqrt{\phi_b^2(L^2 - 4\pi^2 - i\epsilon)}\right), \tag{3.101}$$

where the $i\epsilon$ prescription is needed to make the final answer well defined (see also section 3.4). This

[20]For a review in the context of JT gravity see section 2.3 of [139].

wavefunction satisfies the reduced WdW equation [21]

$$(L\phi - L\partial_L(L^{-1}\partial_\phi))\Psi[L,\phi] = 0. \tag{3.102}$$

One interesting feature of this formula is the fact that it also satisfies the naive no-boundary condition since $\Psi_+[L \to 0, \phi_b] \to 0$. Nevertheless, even though it behaves as expected for small lengths, it has a divergence at $L_{\text{div}} = 2\pi$ (the Bessel function blows up near the origin). Semiclassically, the geometry that dominates the path integral when $L = 2\pi$ is the lower hemisphere of the Euclidean S^2 (dashed line in figure 3.4). This is reasonable from the perspective of the JT gravity path integral since this boundary is also a geodesic, but it would be nice to understand whether this divergence is unique to JT gravity, or would it also be present in theories of gravity in higher dimensions.

We can also comment on the $T\overline{T}$ interpretation of dS gravity. For large L it was argued in [139] that a possible observable in a dual QM theory computing the wavefunction can be $\Psi_+[L] \sim \text{Tr}[e^{iLH}]$, with an example provided after summing over non-trivial topologies by a matrix integral (giving a dS version of the AdS story in [37]). We can extend this (before summing over topologies) to a calculation of the wavefunction at finite L by $T\overline{T}$ deforming the same QM system. This is basically an analytic continuation of the discussion for AdS_2 given in previous sections.

So far we focused on the expanding branch of the wavefunction following [139]. We can also find a real wavefunction analogous to the one originally computed by Hartle and Hawking [142], which we will call $\Psi_{\text{HH,real}}$. This is easy to do in the context of JT gravity and the answer is

$$\Psi_{\text{HH,real}}[L,\phi_b] = \frac{\pi L \phi_b^2}{L^2 - 4\pi^2} I_2\Big(i\sqrt{\phi_b^2(L^2-4\pi^2)}\Big) \tag{3.103}$$

This wavefunction is real, smooth at $L = 2\pi$ and also satisfies $\Psi_{\text{HH,real}}[L \to 0, \phi_b] \to 0$. We plotted the wavefunction in Fig. 3.5. For large universes this state has an expanding and contracting branch with equal weight.

Finally, the results of this section can be extended to pure 3D gravity with positive cosmological constant $\Lambda = 2/\ell^2$. Using Freidel reconstruction kernel, the wavefunction $\Psi[e^{\pm}]$ satisfying WdW, as a function of the boundary frame fields $e^{\pm}$, is given by

$$\Psi_+[e^{\pm}] = e^{i\frac{\ell}{16\pi G_N}\int e} \int \mathcal{D}E \; e^{-i\frac{\ell}{8\pi G_N}\int E^+\wedge E^-} Z(E+e). \tag{3.104}$$

[21] This differs from the wavefunction written in [139] since we found a modification in the WdW equation. The solutions are related by $\Psi_{\text{here}} = L\Psi_{\text{there}}$. The Klein-Gordon inner product defined in [139] should also be modified accordingly.

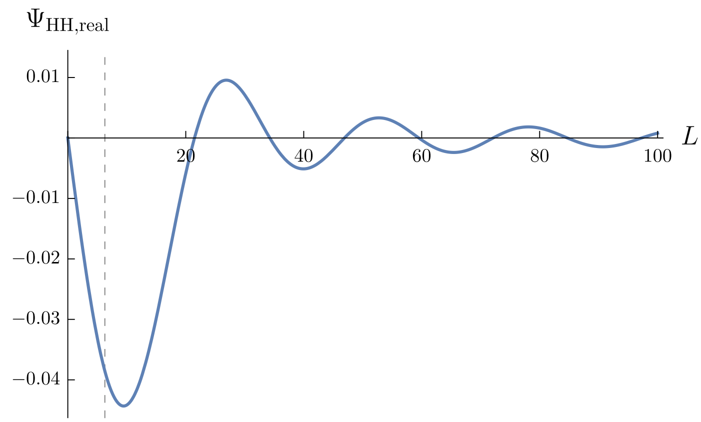


Figure 3.5: Plot of the wavefuntion $\Psi_{\text{HH,real}}$ for $\phi_b = 1/4$. The vertical dashed line indicates the location, $L = 2\pi$, where the expanding branch Ψ_+ of the wavefunction (3.101) diverges, but $\Psi_{\text{HH,real}}$ remains finite.

This is the most general, purely expanding, solution of WdW up to an arbitrary function of the boundary metric $Z(E)$. We can fix Ψ_+ uniquely by looking at the late time limit, or more accurately, boundary metrics with large volume. In this limit Freidel formula gives $\Psi_+[Te^{\pm}] \sim e^{iS_{\text{c.t.}}(T,e)} Z(e)$ for large T. The first term is rapidly oscillating with the volume T at late times and we see the finite piece is precisely the boundary condition we need $Z(e)$. The path integral calculation of the finite piece $Z(e)$ was done in [147] for the case of a boundary torus (see their equation 4.121 and also [149]) and gives a sum over $SL(2,\mathbf{Z})$ images of a Virasoro vacuum character. We leave the study of the properties of this wavefunction for future work.

3.6 Outlook

JT gravity serves as an essential toolbox to probe some universal features of quantum gravity. In the context of this chapter, we have shown that the WdW wavefunctional at finite cutoff and dilaton value in AdS_2 agrees with an explicit computation of the Euclidean path integral; this, in turn, matches the partition function of the Schwarzian theory deformed by a $1D$ analog of the $T\overline{T}$ deformation. Consequently, our computation serves as a check for the conjectured holographic duality between a theory deformed by $T\overline{T}$ and gravity, in AdS, at a finite radial distance.

Finite cutoff unitarity

Beyond providing a check, our computations indicate paths to resolve several open problems related to this conjectured duality. One such issue is that of complex energies that were present when deforming by $T\overline{T}$ (both in 1 and $2D$), and were also present in the WdW wavefunctional when solely accounting for the expanding branch. However, from the WdW perspective, one could also consider the contribution of the contracting branch, and, equivalently, in the Euclidean path integral, one could also account for the contribution of non-compact geometries. In both cases, such corrections are non-perturbative in the cutoff parameter ε or, in the context of $T\overline{T}$, in the coupling of the deformation λ. Nevertheless, we have shown that there exists a linear combination between the two wavefunctional branches that leads to a density of states which is real for all energies. Thus, this suggests that a natural resolution to the problem of complex energy levels is the addition of the other branch, instead of the proposed artificial cutoff for the spectrum once the energies complexify [16]. While the problem of complex energy levels is resolved with the addition of the contracting branch, a new issue appears: the partition function now has a negative density of states. This new density of states implies that, even with such a resolution, the partition function is not that of a single unitary quantum system. In three bulk dimensions, one has a similar state of affairs. The energy levels again complexify, and the other branch of the solution space can cure this, with the caveat that the density of states will become negative. A possible resolution consistent with unitarity would be that the finite cutoff path integral is not computing a boundary partition function but something like an index, where certain states are weighted with a negative sign.

A related issue that leads to the ambiguity in the choice of branches is that the non-perturbative piece of the partition function that cannot be fixed by the $\lambda \to 0$ boundary condition. This ambiguity can be cured by putting additional conditions on the partition function. Fixing the λ-derivative of $Z_\lambda^{\text{non-pert.}}(\beta)$ does not work, but for instance $Z_\lambda^{\text{non-pert.}}(\beta) \to 0$ as $\beta \to 0$ would be enough to fix the partition function completely. One other possibility, motivated by the bulk, is to fix the extrinsic curvature K at $\varepsilon \to 0$. This will eliminate one of the two branches and, therefore, also $\widehat{\rho}$ in (3.88).[22] One can also try to foliate the spacetime with different slices, for instance, by taking constant extrinsic curvature slices.[23] In $3D$, this was done explicitly in [151] for a toroidal boundary and in [152] for more general Rieman surfaces. In particular, for the toroidal boundary, it was found that the wavefunction in the mini-superspace approximation inherits a particular modular invariance, and it would be interesting to compare that analysis to the one done in [143].

In the AdS_3/CFT_2 context, it would also be interesting to understand the non-perturbative corrections to the partition function purely from the field theory. As the $T\overline{T}$ deformation is a particular irrelevant coupling, it is not unreasonable to suspect that such corrections are due to

[22]However, such a resolution appears to bring back the complex energies.

[23]Appendix A.1.2, in fact, provides a non-trivial check of the form of the extrinsic curvature $K[z(u)]$ by considering boundary conditions with fixed extrinsic curvature slices. We will provide further comments about such boundary conditions in [150].

instanton effects contributing at $O(e^{-1/\lambda})$. The fate of such instantons can be studied using, for example, the kernel methods [138, 153] or the various string interpretation of $T\overline{T}$ [154, 155]; through such an analysis, one could hope to shed some light on the complexification of the energy levels.

Application: Wavefunction of the universe

The techniques presented in this chapter also apply to geometries with constant positive curvature. We do this calculation in two ways. On one hand we solve the WdW constraint that this wavefunction satisfies, imposing the Hartle-Hawking boundary condition. On the other hand, we compute the wavefunction as an analytic continuation from the Euclidean path integral on '-AdS'. As expected, we find that both results match. We also analyze two possible choices to define the wavefunction. The first solely includes the contribution of the expanding branch and has a pole when the size of the universe coincides with the dS radius. The second is a real wavefunctional, which includes the non-perturbative contribution of the contracting branch and is now smooth at the gluing location. It would be interesting to identify whether this divergence is present in higher dimensions or if it is special to JT gravity. We also leave for future work a better understanding of the appropriate definition of an inner product between these states.[24] Finally, we outlined how a similar analysis can be used to find the no-boundary wavefunction for pure 3D gravity with a positive cosmological constant, the simplest example corresponding to a toroidal universe.

Sum over topologies

An important open question that remains unanswered is the computation of the JT gravity partition function when including the contribution of manifolds with arbitrary topology. While we have determined the partition function of finite cutoff trumpets using the WdW constraint, this type of surface is insufficient for performing the gluing necessary to obtain any higher genus manifold with a fixed proper boundary length. It would be interesting to understand whether the contribution of such manifolds to the path integral can be accounted for by using an alternative gluing procedure that would work for any higher genus manifold.

For the cylinder, we can actually avoid the gluing. From a third quantisation point of view, one way to think about the cylinder partition function, or double trumpet, is as the propagator associated to the WdW equation in mini-superspace,

$$\left[-L\phi + L\partial_L(L^{-1}\partial_\phi)\right]\Psi_{\text{cylinder}}(\phi,\phi',L,L') = \delta(L-L')\delta(\phi-\phi'). \tag{3.105}$$

[24] In the limit of large universes, some progress in this direction was made in [139].

This avoids the integral over b and since the WdW equation (3.105) is just the propagator of a massive particle in a constant electric field[25], we can solve it with standard methods. The resulting propagator is proportional to a Hankel function of the geodesic distance on mini-superspace, but does not have the same form as the double trumpet computed in [37] once L, L', ϕ and ϕ' are taken large. In fact, it vanishes in that limit. Furthermore, there is a logarithmic divergence when the geodesic distance in mini-superspace vanishes, i.e. when $L = L'$ and/or $\phi = \phi'$. There are several reasons for this discrepancy. The obvious one would be that the cylinder is not the propagator in third quantisation language, but this then raises the question, what is this propagator? Does it have a geometric interpretation? It would be interesting to understand this discrepancy better and what the role of the third quantised picture is.

Coupling to matter & generalizations

Finally, it would be interesting to understand the coupling of the bulk theory to matter. When adding gauge degrees of freedom to a $3D$ bulk and imposing mixed boundary conditions between the graviton and the gauge field, the theory is dual to a $2D$ CFT deformed by the $J\overline{T}$ deformation [156].[26] In $2D$, the partition function of the theory coupled to gauge degrees of freedom can be computed exactly even at finite cutoff; this can be done by combining the techniques presented in this chapter with those in [3] [27] . It would be interesting to explore the possibility of a $1D$ deformation, analogous to the $J\overline{T}$ deformation in $2D$, which would lead to the correct boundary dual for the gravitational gauge theory. Since gauge fields do not have any propagating degrees of freedom in $2D$, it would also be interesting to explore the coupling of JT gravity to other forms of matter.[28] In the usual finite cutoff $AdS_3/T\overline{T}$ deformed CFT correspondence, adding matter results in the dual gravitational theory having mixed boundary conditions for the non-dynamical graviton [158]. Only when matter fields are turned off are these mixed boundary conditions equivalent to the typical finite radius Dirichlet boundary conditions. In $2D$ this was done for the matterless case in [71] and it would be interesting to generalise this to include matter.

---

[25] In the coordinates $u = \phi^2$ and $v = L^2$, (3.32) reduces to $\left(\partial_u \partial_v - \frac{1}{4} - \frac{1}{2v}\partial_u\right)\Psi = 0$. This is the KG equation for $m^2 = 1/2$ and external gauge field $A = \frac{i}{2v}dv$. Notice that the mini-superspace is Lorentzian, whereas the geometries Ψ describes are Euclidean.

[26] Here, $J\overline{T}$ is a composite operator containing J, a chiral $U(1)$ current, and $\overline{T}$, a component of the stress tensor.

[27] Another possible direction could be to understand the result for 2D gravity as a limit of 3D (either for near extremal states [84] or in relation to SYK-like models [157]).

[28] One intriguing possibility is to couple JT gravity to a $2D$ CFT. The effect of the CFT on the partition function has been studied in [36, 4] through the contribution of the Weyl anomaly in the infinite cutoff limit. It would be interesting to see whether the effect of the Weyl anomaly can be determined at finite cutoff solely in terms of the light-cone coordinate $z(u)$.

Chapter 4

Coupling to gauge fields

4.1 Outline of results

As emphasized in the introduction, the geometry of the near-horizon region in near-extremal black holes is universal: as we approach the horizon there is an AdS_2 throat with a slowly varying internal space. The low-energy behavior of such black holes is expected to arise from the near-horizon region which, in turn, can be captured by a two-dimensional effective gravitational action coupled to Yang-Mills theory[1]

$$\begin{aligned} S^E_{JTYM} =& -\frac{1}{2}\phi_0 \int_{\mathcal{M}} d^2x\sqrt{g}\,\mathcal{R} - \frac{1}{2}\int_{\mathcal{M}} d^2x\sqrt{g}\,\phi(R+2) \\ & - \int_{\mathcal{M}} d^2x\sqrt{g}\,g^{\mu\eta}g^{\nu\rho}\left(\frac{1}{4e^2} + \frac{\phi_0+\phi}{4e_\phi^2\phi_0}\right)\mathrm{Tr}\,F_{\mu\nu}F_{\eta\rho} + S_{\mathrm{boundary}}(g,\phi,A)\,. \end{aligned} \tag{4.1}$$

As we will rigorously show in the next chapter, the action (4.1) captures all the massless degrees of freedom that can generically arise in such an effective description.[2] The first line in (4.1) describe the bulk terms in pure Jackiw-Teitleboim (JT) gravity [38, 39], with a cosmological constant normalized to $\Lambda = -2$. The dilaton $\phi_0 + \phi$ parametrizes the size of the internal space and is split into two-

[1] It is instructive to consider how the action (4.1) arises from the dimensional reduction to AdS_2 in a specific example of near-extremal black holes. The dimensional reduction of the near-horizon region in Reissner-Nordström black holes in flat space is discussed extensively in the review [90]. The inclusion (in asymptotically flat or AdS_4 space) of the Maxwell field under which the black hole is charged is discussed in [88, 92, 97], while the addition of the massless gauge degrees of freedom appearing due to the isometry of the S^2 internal space is discussed in [89, 92].

[2] Through-out this chapter we solely work with the action (4.1) written in Euclidean signature. Above, $g_{\mu\nu}$ is the metric, $\mathcal{R}$ is the scalar curvature (here we use the notation $\mathcal{R}$ for the scalar curvature which should not be confused with the notation R for unitary irreducible representations of G which will be used shortly) and $F_{\mu\nu}$ is the field strength associated to the gauge field A_μ. Further details about the conventions in (4.1) will be discussed in the beginning of section 4.2. Details about the integration contours for the fields are also discussed in that section and the meaning of the ϕ integral contour in the context of the low energy effective action of near-extremal black holes is discussed in footnote [10].

parts: ϕ_0 parametrizes the size of the internal space at extremality, while ϕ gives the deviation from this values. While generically, the dimensional reduction on the internal space gives rise to a more complicated dependence in the action of the dilaton field $\phi_0 + \phi$, because we are solely interested in describing the near-horizon region close to extremality, we may assume that $\phi \ll \phi_0$. Consequently, we can linearize the potential for the dilaton field to obtain the effective gravitational action (4.1) which is linear in the deviation ϕ.

The gauge fields that appear in (4.1) through the field strength $F = dA - A \wedge A$ have two possible origins: (i) they are present in the higher dimensional gravitational theory, and the near-extremal black hole could, for instance, be charged under them; for example, for Reissner-Nordström black holes in AdS_4 or in flat space, the U(1) Maxwell field under which the black hole is charged is also present in the dimensionally reduced theory; (ii) the fields can arise from the dimensional reduction on the internal space, in which case, the gauge group is given by the isometry of this space; including such degrees of freedom in the effective action describes the behavior of the black hole beyond the S-wave sector [92].[3]

As mentioned in the introduction, beyond appearing in the effective action that describes the dimensional reduction of the near-horizon region of such black holes, pure JT gravity serves as a testbed for ideas in 2D/1D holography and quantum gravity. For instance, when solely isolating contributions from surfaces with disk topology, the quantization of pure JT gravity can be shown to be equivalent to that of the Schwarzian theory [29, 37]; in turn, this 1d model arises as the low-energy limit of the SYK model [159, 160, 28, 20]. When considering the quantization of the gravitational theory on surfaces with arbitrary topology, the partition function of the theory can be shown to agree with the genus expansion of a certain double-scaled matrix integral [37, 113]. The solubility of pure JT gravity is due, in part, to the fact that the bulk action can be re-expressed as a topological field theory [54, 55, 161, 37, 1]. Consequently, all bulk observables in the purely gravitational theory are invariant under diffeomorphisms and can oftentimes be shown to be equivalent to boundary observables directly at the level of the path integral. The addition of the Yang-Mills term in (4.1) provides an additional layer of complexity for a theory of 2d quantum gravity since the bulk action is no longer topological. Consequently, there is a richer set of diffeomorphism invariant observables that could be explored in the bulk.

In this chapter, we present an exact quantization of the gravitational theory (4.1), for an arbitrary choice of gauge group G and gauge couplings, e, and e_ϕ.[4] By combining techniques used to quantize

[3]Depending on the origin of the gauge fields, the couplings e and e_ϕ can be related to the value of the dilaton at extremality ϕ_0.

[4]The bulk term in (4.1) is equivalent in the first order formalism to a Poisson-sigma model. The quantization

pure JT gravity and the Schwarzian theory [38, 39, 37, 1], together with known results from the quantization of 2D Yang-Mills [108, 109, 102, 103, 110, 111, 104, 112], we derive the partition function of the new gravitational gauge theory (4.1) for surfaces with arbitrary genus. While in this chapter we mainly focus on performing the gravitational path integrals over orientable manifolds, our derivation can be easily generalized to the unorientable cases discussed in [113], and we outline the ingredients necessary for this generalization.

The derivation of the partition function depends on the choice of boundary conditions for the metric, dilaton and gauge field. In turn, this choice fixes the boundary term $S_{\text{boundary}}(g, \phi, A)$ needed in order for (4.1) to have a well-defined variational principle. For the metric and dilaton field, we solely set asymptotically AdS_2 Dirichlet boundary conditions [25],

$$g_{uu}|_{\text{bdy.}} = \frac{1}{\epsilon^2}, \qquad \phi|_{\text{bdy.}} = \frac{\phi_b}{\varepsilon}, \tag{4.2}$$

where $u \in [0, \beta]$ is a variable that parametrizes the boundary, whose total proper length is fixed, $\int_0^\beta du\sqrt{g_{uu}} = \beta/\varepsilon$. In this chapter, we analyze the limit $\varepsilon \to 0$ which implies that we are indeed considering surfaces which are asymptotically AdS_2.[5] However, for the gauge field, we study a variety of boundary conditions for which the gravitational gauge theory (4.1) will prove to be dual to different soluble 1d systems.

Specifically, when solely focusing on the contribution to the path integral of surfaces with disk topology, we find that with the appropriate choice of boundary conditions for the gauge field, the theory (4.1) is equivalent to the Schwarzian theory coupled to a particle moving on the gauge group manifold. Based on symmetry principles, one expects such a theory to arise in the low energy limit of SYK or tensor models with global symmetries [166, 167, 168, 169, 170, 171, 172, 173, 174]. For instance, the low-energy limit of the complex SYK model with a U(1) global symmetry can be described by the Schwarzian coupled to a U(1) phase-mode [166, 167, 97]; on the gravitational side, such a theory arises from (4.1) when fixing the gauge group to be U(1) [89, 92, 97].

When considering the path integral over surfaces with arbitrary genus, we find that the partition function of the gravitational gauge theory can equivalently be described in terms of a collection of double-scaled matrix integrals. Each matrix is associated with a unitary irreducible representation of the gauge group, and the size of that matrix is related to the dimension of its associated representation. Yet another equivalent description of this matrix integral, and consequently of the

of such theories was studied in the first order formalism in [162, 163] and in [164] for manifolds with boundary. Nevertheless, the quantization of the Euclidean gravitational theory (in which a sum over all genera is required) or its relation to matrix integrals has not been previously discussed in the literature.

[5] An analysis for any value of ϵ is forthcoming [165].

gravitational theory, can be obtained by considering Hermitian matrices whose elements are not regular complex numbers,[6] but instead are functions which map group elements of G to complex numbers. Such matrix elements are given by the complex group algebra $\mathbb{C}[G]$.[7] This construction can easily be extended to include the contribution of unorientable manifolds by studying the same matrix integral, this time considering symmetric matrices whose elements are functions mapping group elements of G to real numbers (i.e., the real group algebra $\mathbb{R}[G]$).

Beyond, our computation of partition functions, we construct several diffeomorphism invariant bulk observables, compute their expectation value in the weakly coupled limit and discuss their boundary dual. One such observable is obtained by coupling the gauge field to the world-line action of a charged particle (for instance, a quark) moving on the surface $\mathcal{M}$ in (4.1). The resulting operator is a generalization of the Wilson lines from pure Yang-Mills theory to a non-local diffeomorphism invariant operator in the gravitational gauge theory (4.1). Studying such observables is crucial for understanding the coupling of (4.1) to charged matter. From the perspective of the effective theory describing the aforementioned black holes, such charged matter fields can arise from the Kaluza-Klein reduction on the internal space and can play an essential role in the low-energy behavior of near-extremal black holes [92].

The remainder of this chapter is organized as follows. In section 4.2 we discuss the preliminaries needed for the quantization of the theory with action (4.1). As a warm-up problem which emphasizes the role of boundary conditions in the gauge theory, we start by discussing the simple case in which the gauge theory is weakly coupled. In section 4.3 we move on to discuss the case of general coupling, compute the partition function of the gravitational gauge theory on surfaces with disk topology, and describe the dual boundary theory. In section 4.4, we compute the partition function of the gravitational theory on surfaces with arbitrary genus, g, and an arbitrary number of boundaries, n. Next, we show how this result can be obtained from the genus expansion of the previously introduced matrix integrals. We discuss the construction of several diffeomorphism invariant observables in section 4.5 and compute their expectation values in a variety of scenarios. Finally, in section 4.6 we summarize our results and discuss future research directions.

[6] In the case in which the path integral is solely over orientable manifolds.

[7] We thank H. Verlinde for providing the unpublished pre-thesis work of A. Solovyov [175] and for suggesting the useful mathematical references [176, 177, 178]. While these works focus on an analysis of matrix integrals in the case of discrete groups, they proved to be a valuable source of inspiration for our analysis of gauge theories whose gauge groups are compact Lie groups.

4.2 Preliminaries and a first example

Before, proceeding with the quantization of theory (4.1), we first recast the bulk action into a more convenient form, by introducing the field ϕ as a G-adjoint valued zero-form [103]. The path integral associated to the action (4.1) can be rewritten as:

$$
\begin{aligned}
Z_{JTYM} &= \int Dg_{\mu\nu} D\phi DA\, e^{-S_E[\phi, g^{\mu\nu}, A]} \\
&= \int Dg_{\mu\nu} D\phi D\phi DA \;\; \exp\left[\frac{1}{2}\phi_0 \int_{\mathcal{M}} d^2x\sqrt{g}\, R + \frac{1}{2}\int_{\mathcal{M}} d^2x \sqrt{g}\,\phi(\mathcal{R}+2)\right. \\
&\quad \left. + \int_{\mathcal{M}} i\mathrm{Tr}\,\phi F + \frac{1}{2}\int_{\mathcal{M}} d^2x\sqrt{g}\left(\frac{e^2 e_\phi^2}{e^2(1+\frac{\phi}{\phi_0}) + e_\phi^2}\right)\mathrm{Tr}\,\phi^2 + S_{\mathrm{boundary}}(g,\phi,A)\right]. \qquad (4.3)
\end{aligned}
$$

Throughout the chapter, we use $\mathrm{Tr}\,(\dots)$ to denote the trace in the fundamental representation of the group G. The trace in the fundamental representation can be explicitly expressed in terms of the G generators T^i, normalized such that $\mathrm{Tr}\,(T^i T^j) = \mathcal{N}\eta^{ij}$, where $\mathcal{N}$ is the Dynkin index and η^{ij} is chosen such that $\eta^{ij} = \mathrm{diag}(-1, \dots, -1)$. The trace in all representations R of the gauge group G is denoted by $\chi_R(\dots)$.

After (once again) considering the limit in which $\phi \ll \phi_0$, the action appearing in (4.3) can be rewritten as,

$$
\begin{aligned}
S^E_{JTYM} &= -\frac{1}{2}\phi_0 \int_{\mathcal{M}} \mathcal{R} - \frac{1}{2}\int_{\mathcal{M}} \phi(\mathcal{R}+2) - \int_{\mathcal{M}} i\mathrm{Tr}\,\phi F - \frac{1}{2}\int_{\mathcal{M}} d^2x\sqrt{g}\,(\tilde{e} - \tilde{e}_\phi \phi)\,\mathrm{Tr}\,\phi^2 \\
&\quad + S_{\mathrm{boundary}}(g,\phi,A)\,, \qquad (4.4)
\end{aligned}
$$

where,

$$
\tilde{e} \equiv \frac{e^2 e_\phi^2}{e^2 + e_\phi^2}\,, \qquad \tilde{e}_\phi \equiv \frac{e_\phi^2 e^4}{\phi_0 (e^2 + e_\phi^2)^2}\,. \qquad (4.5)
$$

In the remainder of this chapter we solely use $\tilde{e}$ and $\tilde{e}_\phi$ and we will quantize the theory (4.4) without making any assumptions about these two gauge couplings.

As previously mentioned, in order to compute the partition function (4.3) we need to specify the boundary term $S_{\mathrm{boundary}}(g,\phi,A)$ which is needed in order for the theory to have a well-defined variational principle. When considering the boundary condition (4.2) for the metric and the dilaton field, one needs to include a Gibbons-Hawking term in $S_{\mathrm{boundary}}(g,\phi,A) \supseteq -\left[\phi_0 \int_{\partial\mathcal{M}} du\sqrt{g_{uu}}\,\mathcal{K} + \int_{\partial\mathcal{M}} du\sqrt{g_{uu}}\,\phi(\mathcal{K}-1)\right]$. Here, $\mathcal{K}$ is the boundary extrinsic curvature.

For the gauge field, we can, for instance, consider Dirichlet boundary conditions, in which we fix the value of the gauge field along the boundary, $\delta A_u = 0$. Equivalently, due to the invariance of the partition function under large gauge transformations, instead of fixing $A_u|_{\partial\mathcal{M}} = \mathcal{A}_u(u)$ all along the boundary,[8] we solely need to fix the holonomy around the boundary[9]

$$h \equiv \mathcal{P}\exp\left(\oint_{\partial\mathcal{M}} \mathcal{A}^a\, T_a\right) \qquad\qquad (\text{Dirichlet})\,. \tag{4.6}$$

As we will explain shortly, the states obtained by performing the path integral on surfaces with disk topology and fixed boundary holonomy h, span the entire Hilbert space associated to Yang-Mills theory; as we exemplify shortly, we can always compute correlators in the presence of a different set of boundary conditions for the gauge field, by inserting a boundary condition changing defect [1] in the theory with Dirichlet boundary.

With Dirichlet boundary conditions for the gauge field and the boundary conditions (4.2) for the metric and dilaton, no other boundary term besides the Gibbons-Hawking term is needed in order for the theory to have a well-defined variational principle. Thus, the action (4.4) can finally be recasted as,

$$\begin{aligned} S^E_{\substack{\text{JTYM}\\ \text{Dirichlet}}} =& -2\pi\,\phi_0\chi(\mathcal{M}) - \left[\frac{1}{2}\int_{\mathcal{M}} d^2x\sqrt{g}\,\phi(\mathcal{R}+2) + \int_{\partial\mathcal{M}} du\sqrt{g_{uu}}\phi(\mathcal{K}-1)\right] \\ & - \left[\int_{\mathcal{M}} i\text{Tr}\,\phi F + \frac{1}{2}\int_{\mathcal{M}} d^2x\sqrt{g}\,(\tilde{e}-\tilde{e}_\phi\phi)\,\text{Tr}\,\phi^2\right]\,, \end{aligned} \tag{4.7}$$

where $\chi(\mathcal{M})$ is the Euler characteristic of the manifold $\mathcal{M}$, which appears due to the Gauss-Bonnet relation $\frac{1}{2}\int_{\mathcal{M}}\sqrt{g}\mathcal{R} + \int_{\partial\mathcal{M}}\mathcal{K} = 2\pi\chi(\mathcal{M})$. From here on, we denote $S_0 = 2\pi\phi_0$ and e^{S_0} serves as the genus expansion parameter when discussing path integral over surfaces with arbitrary genus.

Our goal is thus to quantize the theory with action (4.7) and theories related to (4.7) by a change of boundary conditions for the gauge field. Towards that scope, it is first useful to discuss the symmetries of the problem in the weak gauge coupling limit $\tilde{e}$ and $\tilde{e}_\phi \to 0$. In this case the theory becomes topological: the third-term in the action (4.4) describes a BF topological theory and in fact, as previously mentioned, the bulk JT gravity action itself, can also be recast as a BF theory whose gauge algebra is $\mathfrak{sl}(2,\mathbb{R})$ [54, 55, 161, 37, 1]. This limit proves useful for understanding the boundary dual of the gravitational theory in a simpler setting and for the computation of various diffeomorphism invariant observables in section 4.5. Therefore, as a warm-up, we discuss it first in the next subsection.

[8] Here, we take $\mathcal{A}_u(u)$ to be an arbitrary periodic function on the thermal circle.

[9] We, however, need to fix gauge transformations on the boundary in section 4.5.3, when discussing correlators of boundary anchored Wilson lines.

4.2.1 A warm up: the weakly coupled limit on the disk topology

Because in the weakly coupled limit, the gauge theory is topological, we can proceed by separately computing the path integral for the pure JT sector and the gauge theory sector. Thus, we first review the computation of the path integral in JT gravity following [29, 37]. By integrating out the dilaton field ϕ along the contour $\phi = \phi_b/\varepsilon + i\mathbb{R}$,[10] we find that the curvature of the surfaces considered in the path integral is constrained:

$$Z_{JT} = \int Dg_{\mu\nu} e^{\int_{\partial\mathcal{M}} du\sqrt{g_{uu}}\frac{\phi_b}{\epsilon}\mathcal{K}[g_{\mu\nu}]}\,\delta(\mathcal{R}+2)\,. \tag{4.8}$$

The remaining path integral is thus solely over the boundary degrees of freedom of AdS_2 patches. In order to simplify the path integral over the boundary degrees of freedom, we consider parametrizing the AdS_2 patches by using Poincaré coordinates, under which the boundary condition for the metric becomes

$$ds^2 = \frac{dF^2 + dz^2}{z^2}\,, \qquad g_{uu}|_{\text{bdy.}} = \frac{(F')^2 + (z')^2}{z^2} = \frac{1}{\varepsilon^2}\,, \tag{4.9}$$

where the boundary is parametrized using the variable u, with $F' = \partial F/\partial u$. Solving the latter equation to first order in ϵ, we find $z = \varepsilon F' + O(\epsilon^2)$. Since $z(u)$ is small in the $\varepsilon \to 0$ limit, the path integral is thus indeed dominated by asymptotically AdS_2 patches. In this set of coordinates, the extrinsic curvature can be expressed as

$$\mathcal{K}[F(u), z(u)] = \frac{F'(F'^2 + z'^2 + zz'') - zz'F''}{(F'^2 + z'^2)^{3/2}} = 1 + \varepsilon^2\,\mathrm{Sch}(F, u) + O(\epsilon^3)\,. \tag{4.10}$$

Thus, (4.8) can be rewritten as a path integral over the boundary coordinate $F(u)$

$$Z_{\text{JT}}^{\text{disk}}(\phi_b, \beta) = Z_{\text{Schw.}}(\phi_b, \beta) = e^{S_0}\int DF\, e^{\phi_b\int_0^\beta \{F(u),u\}}\,, \qquad DF = \prod_{u\in\partial\mathcal{M}} \frac{dF(u)}{F'(u)}\,. \tag{4.11}$$

[10] To understand the meaning of this contour in the context of the near-extremal black hole effective action it is useful to review how the integral over ϕ behaves in Lorentzian signature. In that case, the contour for ϕ is restricted from $-\phi_0$ to ∞, due to the fact that the internal space should have a positive volume ($\phi + \phi_0 > 0$). In the limit considered in this chapter, $\phi_0 \to \infty$, the integral over ϕ indeed converges to $\delta(\mathcal{R}+2)$ in a distributional sense. To make this statement precise we could keep track of the higher powers of the dilaton in the action, whose coefficients are suppressed in ϕ_0, and vanish in the limit $\phi \to \pm\infty$. Then, the path integral over ϕ would be peaked around the configurations where $\mathcal{R} = -2 + O(1/\phi_0)$. When in Euclidean signature, we have to analytically continue ϕ along the complex axis in order to get a convergent answer, still peaked around $\mathcal{R} = -2 + O(1/\phi_0)$. While such a contour for ϕ does not have a nice geometric meaning when relating $\phi + \phi_0$ to the volume of the internal space, it isolates the same type of constant curvature configurations in Euclidean signature as those that dominate in the Lorentzian path integral. We thank R. Mahajan and D. Kapec for useful discussions about this point.

where the measure DF is obtained by using the symplectic form over flat gauge connections in the $\mathfrak{sl}(2,\mathbb{R})$ BF theory rewriting of JT gravity [37]. The path integral (4.11) can be computed by using localization and has been found to be one-loop exact [25]. The solution obtained from localization is given by

$$Z_{\text{JT}}^{\text{disk}}(\phi_b,\beta) = Z_{\text{Schw.}}(\phi_b,\beta) = e^{S_0}\int ds\,\frac{s}{2\pi^2}\sinh(2\pi s)e^{-\frac{\beta s^2}{2\phi_b}} = e^{S_0}\frac{\phi_b^{3/2}e^{\frac{2\pi^2\phi_b}{\beta}}}{(2\pi)^{1/2}\beta^{3/2}}\,, \tag{4.12}$$

where one can consequently read-off the density of states for the Schwarzian theory:

$$\rho_0(E) = \frac{\phi_b}{2\pi^2}\sinh(2\pi\sqrt{2\phi_b E})\,. \tag{4.13}$$

We now move on to describing the gauge theory side. With Dirichlet boundary conditions, the disk partition function is trivial, $Z_{BF}(h) = \delta(h)$ and, consequently, $Z_{JTBF}(h) = Z_{\text{Schw.}}\delta(h)$. In order to obtain a non-trivial result, the boundary conditions imposed on the gauge field need to explicitly break invariance under arbitrary diffeomorphisms in the topological theory. One such boundary condition is obtained by relating the value of the gauge field on the boundary to the zero-form field ϕ

$$A_u|_{\partial\mathcal{M}} - \sqrt{g_{uu}}\,i\varepsilon\tilde{e}_b\phi|_{\partial\mathcal{M}} = \mathcal{A}_u \qquad\qquad (\text{mixed})\,, \tag{4.14}$$

for some constant $\mathcal{A}_u$. We label this class of boundary conditions as "mixed".

In order for the action to have a well-defined variational principle, one needs to add

$$S_{\text{boundary}}^{\text{gauge}}[\phi, A] = \frac{i}{2}\int_{\partial\mathcal{M}} du\,\text{Tr}\,\phi A_u\,, \tag{4.15}$$

to the aforementioned Hawking-Gibbons term specified in (4.7). As in pure JT gravity, we can reduce the BF path integral to an integral over boundary degrees of freedom, whose action is given by (4.15). The integral over the zero-form field ϕ in the bulk, restricts the path integral to flat gauge connections, with $A = q^{-1}dq$, where q is a function mapping $\mathcal{M}$ to group elements of G. Plugging in this solution for A into the boundary term (4.15) and using the boundary condition (4.14), we find that

$$Z^{\text{disk}}_{\substack{\text{BF}\\ \text{mixed}}}(\beta, h) = Z_G(\beta, h) = \int Dq\, e^{\frac{1}{2\varepsilon\tilde{e}_b}\int_0^\beta du \sqrt{g_{uu}} g^{uu} \text{Tr}\left[(q^{-1}\partial_u q)^2 + \mathcal{A}_u (q^{-1}\partial_u q)\right]},$$
$$Z^{\text{disk}}_{\substack{\text{JTBF}\\ \text{mixed}}}(\phi_b, \beta, h) = Z_{\text{Schw.}}(\phi_b, \beta) Z_G(\beta, h)\,. \tag{4.16}$$

Just like in the case of the pure JT gravity path integral, the measure for the boundary degree of freedom Dh is obtained from the symplectic form in the BF theory with gauge group G.

The path integral in (4.16) describes a particle moving on the G group manifold, whose partition function we denote as $Z_G(\beta, \mathcal{A}_u)$; as we will explain shortly, $\mathcal{A}_u$ serves as a background gauge field for one of the G symmetries present in this theory.

4.2.2 Reviewing the quantization of a particle moving on a group manifold

To proceed, we briefly review the quantization of a particle moving on a group manifold G [179, 180, 181], in the presence of an arbitrary 1d background metric and of a G background gauge field. In order to do so it is again useful to introduce a Lagrange multiplier $\boldsymbol{\alpha}$, valued in the adjoint representation of G. The path integral (4.15) can be rewritten as

$$Z_G(\beta, h) = \int Dq D\boldsymbol{\alpha}\, e^{\int_0^\beta du \left(i \text{Tr}\,(\boldsymbol{\alpha}\, q^{-1} D_{\mathcal{A}} q) + \sqrt{g_{uu}} \frac{\varepsilon\, \tilde{e}_b}{2} \text{Tr}\, \boldsymbol{\alpha}^2\right)}, \qquad D_{\mathcal{A}} q = \partial_u q + q\mathcal{A}_u\,. \tag{4.17}$$

At this point it proves useful to turn-off the background $\mathcal{A}_u$ and analyze the symmetries of the action appearing in (4.17). Firstly, we note that (4.17) is invariant under reparametrizations, $u \to F(u)$ and thus, instead of using the variable u we can also use the AdS_2 boundary coordinate $F(u)$ to describe the action in (4.17).[11] Furthermore, for an arbitrary choice of parametrization of the boundary, such that $g_{uu}(u)$ is an arbitrary function of u, we can always perform a diffeomorphism and assume a constant boundary metric g_{uu}, as in the boundary condition (4.2). Invariance under such diffeomorphisms also implies that the temperature dependence of the partition function appears as $Z_G(\tilde{e}_b,\, \beta,\, \mathcal{A}_u) = Z_G(\tilde{e}_b\beta,\, \mathcal{A}_u)$.

Expanding $q(u)$ around a base-point, with $q(u) = e^{x^a(u)T_a} q(u_0)$ we find that the canonical

[11] This is oftentimes done when discussing the low energy behavior of SYK models with global symmetries. For instance, this appears when coupling the Schwarzian to a phase mode [89, 171, 92, 97, 174].

momenta associated to $x^a(u)$ in the action in (4.17) are given by

$$\pi_{x^i} = \mathrm{Tr}\,(T_i q \boldsymbol{\alpha} q^{-1})\,, \tag{4.18}$$

which are in fact the generators of the G symmetry which acts by left multiplication on q, as $q \to Uq$ and $\boldsymbol{\alpha} \to \boldsymbol{\alpha}$. Similarly, one finds that the generators of the G symmetry that acts by right multiplication on q, as $q \to qU$ and $\boldsymbol{\alpha} \to U^{-1}\boldsymbol{\alpha}U$ are simply given by $\boldsymbol{\alpha}_i$. The background $\mathcal{A}_u$, which appeared in the choice of mixed boundary conditions (4.14), gauges the right acting copy of the symmetry group G (alternatively, we could choose to background gauge the left acting copy).

The Hamiltonian is time dependent and is given by $H(u) = \varepsilon \tilde{e}_b \sqrt{g_{uu}}\mathrm{Tr}\,\boldsymbol{\alpha}^2/4$. In turn, this is proportional to the quadratic Casimir associated to G, given by

$$\frac{\widehat{C}_2}{\mathcal{N}} = -\frac{\eta^{ij}\pi_{x^i}\pi_{x^j}}{\mathcal{N}} = \mathrm{Tr}\,(\boldsymbol{\alpha}^2) = \frac{4\,H(u)}{\varepsilon\tilde{e}_b\sqrt{g_{uu}}}\,. \tag{4.19}$$

The Hilbert space of the theory, $\mathcal{H}^G$, is given by normalizable functions on the group manifold that are spanned by the matrix element of all unitary irreducible representations R, $U^n_{R,m}(h)$. By definition, such states of course transform correctly under the action of the left- and right- acting G symmetry groups. Namely, we take the generators of the G symmetry that acts by left multiplication to act on the left index, n, and those of the right-acting symmetry to act on m. Such states are also eigenstates of the Hamiltonian with $\widehat{C}_2\, U^n_{R,m}(h) = C_2(R) U^n_{R,m}(h)$. Thus, the thermal partition function at inverse-temperature β associated to the action (4.17) is given by,[12]

$$Z_G(\beta) = \mathrm{Tr}_{\,\mathcal{H}^G} e^{-\int_0^\beta H(u)du} = \sum_R (\dim R)^2 e^{-\frac{\varepsilon\tilde{e}_b C_2(R)}{4\mathcal{N}}\int_0^\beta du\sqrt{g_{uu}}} = \sum_R (\dim R)^2 e^{-\frac{\beta\tilde{e}_b C_2(R)}{4\mathcal{N}}}\,. \tag{4.20}$$

Here, the sum is over all unitary irreducible representations R of the gauge group G. Because we will encounter this situation when discussing the boundary dual of gravitational Yang-Mills theory, we note that if we replace $\mathrm{Tr}\,\boldsymbol{\alpha}^2$ by a general function $\widehat{V}(\boldsymbol{\alpha})$ (that preserves the G symmetries by being a trace-class function) in the action in (4.17), the resulting theory has a Hamiltonian that can always be expressed in terms of the Casimirs of the group G. Thus, in the partition function, the eigenvalue $C_2(R)$ of the quadratic Casimir is replaced by a function $V(R)$ that can be easily be related to $\widehat{V}(\boldsymbol{\alpha})$.[13]

[12]Note that the path ordering which is needed in (4.20) does not affect the exponentiated integral since the Hamiltonian is always proportional to the Casimir of G and, therefore, commutes with itself at any time.

[13]For instance, when $G = \mathrm{SU}(2)$ or $\mathrm{SO}(3)$, all higher-order Casimirs can be expressed in terms of powers of the quadratic Casimir and, consequently, the potential can always be expressed as $\widehat{V}(\boldsymbol{\alpha}) \equiv \tilde{V}(\mathrm{Tr}\,\boldsymbol{\alpha}^2)$. In this case $V(R) = \tilde{V}(C_2(R))$.

We now re-introduce the background gauge field $\mathcal{A}$ which appeared through the boundary condition (4.14), to obtain the partition function of (4.17) in the more general case. Just like in the case of Yang-Mills theory with Dirichlet boundary conditions, the action in (4.17) is invariant under background gauge transformations and, consequently, the partition function depends solely on the holonomy of the background $\mathcal{A}$, $h = \mathcal{P}\exp(\oint \mathcal{A})$ through trace-class functions. The insertion of such a background is equivalent to adding a chemical potential for the left-acting G-symmetry, that exponentiates the associated charges of the left G-symmetry to a G group element in the same conjugacy class as h. Thus, the partition function (4.17) becomes

$$Z_G(\beta,\, h) = \mathrm{Tr}_{\mathcal{H}}\left(h\, e^{-\int_0^\beta H(u)du}\right) = \sum_R (\dim R)\, \chi_R(h) e^{-\frac{\tilde{e}_b \beta C_2(R)}{4N}}\,, \tag{4.21}$$

where $\chi_R(h)$ are the characters of the group element h associated to the representation R. Similarly, in the theory whose potential is $\widehat{V}(\boldsymbol{\alpha})$, the partition function is given by

$$Z_G^{\widehat{V}}(\beta,\, h) = \sum_R (\dim R)\, \chi_R(h) e^{-\tilde{e}_b V(R) \int_0^\beta du \sqrt{g_{uu}}} = \sum_R (\dim R)\, \chi_R(h) e^{-\tilde{e}_b \beta V(R)}\,. \tag{4.22}$$

Thus, to summarize, in the weak gauge coupling limit, we have found that the gravitational gauge theory (4.1) is equivalent to the Schwarzian theory decoupled from a particle moving on the gauge group manifold. Its partition function, with boundary conditions (4.2) for the metric and dilaton and (4.14) for the gauge field, is given by

$$Z^{\text{disk}}_{\substack{\text{JTBF}\\ \text{mixed}}}(\phi_b, \beta, h) = e^{S_0}\left(\int ds\, \frac{s}{2\pi^2}\sinh(2\pi s) e^{-\frac{\beta s^2}{2\phi_b}}\right)\left[\sum_R \dim R\, \chi_R\left(\mathcal{P}e^{\int \mathcal{A}_u}\right) e^{-\frac{\tilde{e}_b \beta C_2(R)}{4N}}\right]. \tag{4.23}$$

4.2.3 Reviewing the quantization of 2d Yang-Mills

While in the weakly coupled limit we were able to directly reduce the bulk path integral to a boundary path integral, since the theory is not topological at non-zero gauge coupling, this cannot be easily done more generally. Thus, it proves instructive to reproduce the partition function (4.23) by performing the path integral directly in the bulk.

Before performing the bulk path integral, it is useful to review the well known quantization of the gauge theory [108, 109, 102, 103, 110, 111, 104, 112], when fixing the metric $g_{\mu\nu}$ and the dilaton as backgrounds. Thus, we seek to quantize Yang-Mills theory, $S^E_{YM} = -\int_{\mathcal{M}} i\mathrm{Tr}\,\phi F - \frac{1}{2}\int_{\mathcal{M}} d^2x\sqrt{g}j(x)\mathrm{Tr}\,\phi^2$, where $j(x) \equiv \tilde{e} - \tilde{e}_\phi \phi(x)$ is an arbitrary source for the operators $\mathrm{Tr}\,\phi^2$.[14]

[14] In this chapter, we omit the possibility of adding a θ-angle for the gauge field. This will be discussed in the study of the weak gauge coupling limit [182].

The source $j(x)$ can be absorbed by changing the surface form $d^2x\sqrt{g}$. Due to the fact that the theory is invariant under local area preserving diffeomorphisms, the partition function can thus solely depend on the dimensionless quantity $a = \int_{\mathcal{M}} d^2x\sqrt{g}\, j(x)$. It is therefore sufficient to review the quantization of the theory on a flat manifold with area $\tilde{a}$ and coupling $e_{\rm YM}^2$, such that $a = e_{\rm YM}^2\tilde{a}$.

The quantization of this theory is similar to that of the particle moving on the gauge group manifold discussed in the previous subsection and, for pedagogical purposes, it is useful to emphasize these similarities. When using the Dirichlet boundary conditions (4.6) the partition function of the gauge theory is a trace-class function of h and thus it is spanned by characters of the group $\chi_R(h)$. Consequently, the characters $\chi_R(h)$ can be viewed as a set of wavefunctions which span the Hilbert space $\mathcal{H}^{\rm YM}$ of the gauge theory.

The partition function on a manifold with arbitrary genus g and an arbitrary number of boundaries n can be computed using the cutting and gluing axioms of quantum field theory and by solely using the partition function of the gauge theory on the disk with the Dirichlet boundary condition (4.6). As previously mentioned, in the limit $a \to 0$ the gauge theory becomes topological. In this limit, the integral over ϕ imposes the condition that A is a flat connection, which yields $h = e$ (where e is the identity element of G), so [103]

$$\lim_{a\to 0} Z_{\rm YM}^{\rm disk}(a, h) = \delta(h) = \sum_R \dim R\, \chi_R(h)\,, \tag{4.24}$$

where $\delta(h)$ is the delta-function on the group G defined with respect to the Haar measure on G, which enforces that $\int dh\, \delta(h)x(h) = x(e)$. This is the same as the partition function of the particle moving on the G group manifold (4.20) in the limit $\tilde{e}_b \to 0$.

For non-zero a, note that the canonical momentum conjugate to the space component of the gauge field $A_1^i(x)$ is $\phi_i(x)$, and thus the Hamiltonian density is just $H = \frac{e_{\rm YM}^2}{4}\mathrm{Tr}\,(\phi_i T^i)^2$. It then follows, from $\pi_i = -i\mathcal{N}\phi_i$, that $H = -\frac{e_{\rm YM}^2}{4\mathcal{N}}\eta^{ij}\pi_i\pi_j$. Using $\pi_j = \frac{\delta}{\delta A_1^j}$, each momentum acts on the wavefunctions $\chi_R(g)$ as $\pi_i\chi_R(h) = \chi_R(T_i h)$. It follows that the Hamiltonian density acts on each basis element of the Hilbert space $\chi_R(g)$ diagonally with eigenvalue $e_{\rm YM}^2 C_2(R)/(4\mathcal{N})$ [104], where $C_2(R)$ is the quadratic Casimir, with $C_2(R) \geq 0$ for compact groups. Note that the Hamiltonian of the gauge theory is therefore closely related to that of the particle moving a group manifold (4.19). One then immediately finds

$$Z_{\rm YM}^{\rm disk}(a, h) = \sum_R \dim R\, \chi_R(h) e^{-\frac{e_{\rm YM}^2\tilde{a}C_2(R)}{4\mathcal{N}}} = \sum_R \dim R\, \chi_R(h) e^{-\frac{C_2(R)}{4\mathcal{N}}\int d^2x\sqrt{g}j(x)}\,. \tag{4.25}$$

Following from the relation between the Hamiltonian of the gauge theory and that of a particle moving on the G group manifold, we of course find that (4.25) agrees with (4.21) for the appropriate choice of $\tilde{e}_b$ or $j(x)$.

The partition function of Yang-Mills theory on an orientable manifold $\mathcal{M}_{g,n}$ of genus g, with n boundaries, can be obtained by gluing different segments on the boundary of the disk [102, 103, 110, 111, 104]. This is given by

$$Z_{\rm YM}^{(g,n)}(a, h_1, \ldots, h_n) = \sum_R (\dim R)^{\chi(\mathcal{M}_{g,n})} \chi_R(h_1)\chi_R(g_2)\ldots\chi_R(h_n) e^{-\frac{C_2(R)}{4N}\int d^2x\sqrt{g}j(x)}\,. \qquad (4.26)$$

With these results in mind, we can therefore proceed with the analysis of the simplified case of obtaining the contribution to the path integral of the disk topology in the weakly coupled limit by directly performing the path integral in the bulk.

4.2.4 Quantization with a boundary condition changing defect

To determine the partition function with the boundary condition (4.14) we consider a boundary changing defect

$$S^E_{\rm Defect}[g,\phi] = -\frac{\varepsilon\tilde{e}_b}{2}\int_I du\sqrt{g_{uu}}{\rm Tr}\,\phi^2\,, \qquad (4.27)$$

which we can insert along a contour I which is arbitrarily close to the boundary $\partial\mathcal{M}$. We now show that the boundary condition changing defect indeed implements the change of boundary conditions from Dirichlet to those listed in (4.14). By integrating the equation of motion obtained from the variation of ϕ at the location of defect on an infinitesimal interval in the direction perpendicular to the defect we find,

$$A_u|_{\partial\mathcal{M}} - A_u|_I = -i\sqrt{g_{uu}}\varepsilon\tilde{e}_b\phi|_I\,, \qquad (4.28)$$

where $A_u|_{\partial\mathcal{M}}$ is the gauge field on the boundary on $\mathcal{M}$ that is fixed when using Dirichlet boundary conditions for the action, $A_u|_I$ is the gauge field in the immediate neighborhood inside of the defect and $\phi|_I$ is the value of the zero-form field on the defect. Moving $A_u|_I$ to the RHS and setting $A_u|_{\partial\mathcal{M}} = \mathcal{A}_u$, we reproduce the boundary condition (4.14). Thus, the theory with the defect and Dirichlet boundary conditions should reproduce the results in the theory without the defect and with the boundary condition (4.14) for the gauge field.

As we further exemplify in section 4.5, the advantage of using the description of the BF theory in the presence of the defect (4.27) is that the expectation value of any observable can easily be computed by using standard techniques in 2d Yang-Mills theory. For example, when computing the partition function of the theory with the defect (4.27) on a disk, we can use (4.25) setting $j(x) \sim \delta(x - x_I)$ and $h = \mathcal{P}\exp(\int_{\partial\mathcal{M}} \mathcal{A})$, to find that

$$Z^{\text{disk}}_{\substack{\text{BF}\\ \text{mixed}}}(\beta, h) = \sum_R \dim(R)\chi_R(h)e^{-\frac{\tilde{e}_b \beta C_2(R)}{4\mathcal{N}}}\,. \tag{4.29}$$

Using this result together with the reduction of the JT gravity path integral on a disk to that of the Schwarzian, we find the result (4.23). Moving forward, we fix the normalization of the Casimir by fixing the Dynkin index, $\mathcal{N} \equiv 1/2$.

More generally, we can consider adding a defect which depends on a general gauge invariant potential $\widehat{V}(\phi)$, $S_{\text{Defect}}[g,\phi] = -\int_I du\sqrt{g_{uu}}\,\varepsilon\widehat{V}(\phi)$. In this case, the boundary condition which the gauge field needs to satisfy is again given by the ϕ equation of motion, which implies that $(A_u - i\varepsilon\partial\widehat{V}(\phi)/\partial\phi)|_{\partial\mathcal{M}} = \mathcal{A}_u$. The quantization of Yang-Mills theory with such a general potential was discussed in [111, 112] and closely follows the quantization of a particle moving on a group manifold with the general potential $\widehat{V}(\boldsymbol{\alpha})$ discussed in the previous subsection. In fact the result for the bulk partition function

$$Z^{\text{disk}}_{\substack{\text{BF}\\ \text{mixed}\,\widehat{V}(\phi)}}(\beta, h) = \sum_R \dim(R)\chi_R(h)e^{-\tilde{e}_b\beta V(R)} \tag{4.30}$$

agrees with the partition function (4.22) obtained by considering a particle moving on the G group manifold with a potential $\widehat{V}(\boldsymbol{\alpha})$ and in the presence of the background gauge field $\mathcal{A}_u$. Therefore, we obtain the first general equivalence which we schematically present in figure 4.1.

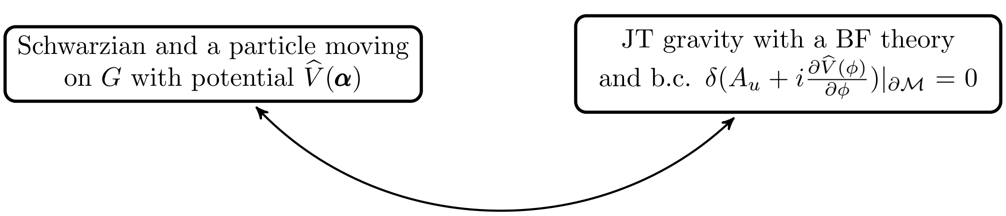


Figure 4.1: Schematic representation of the equivalence between the gravitational gauge theory at weak gauge coupling and the Schwarzian decoupled from a particle moving on the group manifold G.

4.3 Disk partition function

4.3.1 2D Yang-Mills theory with Dirichlet boundary conditions

We finally arrive at the quantization of the theory (4.1) for arbitrary gauge group and gauge couplings, when fixing the boundary conditions to (4.2) for the metric and dilaton and when using Dirichlet boundary conditions for the gauge field $A_u|_{\text{bdy.}} = \mathcal{A}_u$. Using (4.26) for $\chi(\mathcal{M}) = 1$, $j(x) = \tilde{e} - \tilde{e}_\phi \phi(x)$ and setting $h = \mathcal{P}\exp(\int_{\partial\mathcal{M}} \mathcal{A})$, we find that after integrating out the gauge field A_μ and the zero-form field ϕ, the partition function is given by[15]

$$\begin{aligned} Z_{\substack{\text{disk}\\ \text{JTYM}\\ \text{Dirichlet}}}(\phi_b, \beta, h) &= \int Dg_{\mu\nu} D\phi e^{-S_{JT}[g_{\mu\nu},\phi]} \left(\sum_R \dim(R) \chi_R(h) e^{-\frac{C_2(R)\int_{\mathcal{M}} d^2x\,\sqrt{g}\left[\tilde{e}-\tilde{e}_\phi \phi\right]}{2}} \right) \\ &= e^{S_0} \sum_R \dim(R)\chi_R(h) \int Dg_{\mu\nu} D\phi\, e^{\frac{1}{2}\int_{\mathcal{M}} d^2x \sqrt{g}\,\phi(\mathcal{R}+2+\tilde{e}_\phi C_2(R))} \\ &\qquad \times e^{-\frac{\tilde{e}C_2(R)}{2}\int_{\mathcal{M}} d^2x\,\sqrt{g} + \int_{\partial\mathcal{M}} du \sqrt{g_{uu}}\phi(\mathcal{K}-1)}, \end{aligned} \tag{4.31}$$

where the couplings $\tilde{e}$ and $\tilde{e}_\phi$ are related to the initial couplings by (4.5). We can now view the terms in the exponent in (4.31) as coming from an effective action for each representation R of the gauge group.

Integrating out the dilaton field ϕ, we once again find that the path integral localizes to AdS_2 patches, whose cosmological constant is now given by $\tilde{\Lambda} = -2 - \tilde{e}_\phi C_2(R)$ and whose boundary degrees of freedom is the sole remaining dynamical degrees of freedom in the path integral. Thus, we are summing over AdS_2 patches whose curvatures depend on the representation sector from the sum in (4.31).

After integrating out the dilaton field ϕ one can rewrite the remaining area term $\tilde{e}\int_{\mathcal{M}} d^2x\sqrt{g}$ using the Gauss-Bonnet theorem

$$\tilde{e}\int_{\mathcal{M}} d^2x \sqrt{g} = -\frac{\tilde{e}}{2+\tilde{e}_\phi C_2(R)} \int d^2x \sqrt{g}\,\mathcal{R} = \frac{\tilde{e}}{1+\frac{\tilde{e}_\phi C_2(R)}{2}} \left[\int_{\partial\mathcal{M}} \sqrt{h}\,\mathcal{K} - 2\pi\chi(\mathcal{M}) \right], \tag{4.32}$$

where for the disk, the Euler characteristic is $\chi(\mathcal{M}) = 1$. Thus, the path integral becomes,

$$\begin{aligned} Z_{\substack{\text{disk}\\ \text{JTYM}\\ \text{Dirichlet}}}(\phi_b, \beta, h) = e^{S_0} \sum_R \dim(R)\chi_R(h) \int D\mu[F] \exp\Bigg[\frac{2\pi\tilde{e}\,C_2(R)}{2+\tilde{e}_\phi C_2(R)} \\ + \left(\frac{\phi_b}{\epsilon} - \frac{\tilde{e}\,C_2(R)}{2+\tilde{e}_\phi C_2(R)} \right) \int_{\partial\mathcal{M}} du \sqrt{g_{uu}} \mathcal{K}[F(u)] - \frac{\phi_b}{\epsilon} \int_{\partial\mathcal{M}} du \sqrt{g_{uu}} \Bigg]. \end{aligned} \tag{4.33}$$

[15] Here we assume the path integral over the gauge degrees of freedom can always be made convergent with the proper choice of integration contour for the field ϕ.

where we have used the fact that the path integral over the gauge degrees of freedom does not affect the measure for the Schwarzian field, $D\mu[F]$, and we have added a counter-term $-\frac{\phi_b}{\epsilon}\int_{\partial\mathcal{M}} du\sqrt{g_{uu}}$ to cancel the leading divergence appearing in the exponent. It is convenient to define a "renormalized" Casimir

$$\tilde{C}_2(R) \equiv \frac{C_2(R)}{2\left(1+\frac{\tilde{e}_\phi C_2(R)}{2}\right)}\,, \tag{4.34}$$

to capture the dependence on the G-group second-order Casimir appearing in (4.33). The origin of this modified Casimir comes from the R dependence of the cosmological constant that can be seen through (4.32). Note that for compact Lie groups, when choosing the coupling e and e_ϕ to be real, $\tilde{C}_2(R)$ is a real positive function of R, which for representations with growing dimensions, asymptotes to a constant value.

The path integral can then be rewritten using the relation (4.10) between the extrinsic curvature and the Schwarzian derivative

$$Z^{\text{disk}}_{\substack{\text{JTYM}\\ \text{Dirichlet}}}(\phi_b,\beta,h) = \sum_R \dim(R)\chi_R(h)\int D\mu[F] e^{\left[2\pi\tilde{e}\tilde{C}_2(R)-\frac{\phi_b\beta}{\epsilon^2}+\left(\phi_b-\epsilon\tilde{e}\tilde{C}_2(R)\right)\int_0^\beta du\left(\frac{1}{\epsilon^2}+\text{Sch}(F,u)\right)\right]}\,. \tag{4.35}$$

For now, let's ignore the fact that the coupling in front of the Schwarzian might be negative for sufficiently large ϵ and assume that $\phi_b > \varepsilon\tilde{e}\tilde{C}_2(R)$. Once again using the computation for the Schwarzian path integral, which is one-loop exact, we find

$$\begin{aligned} Z^{\text{disk}}_{\substack{\text{JTYM}\\ \text{Dirichlet}}}(\phi_b,\beta,h) &= \sum_R \dim(R)\chi_R(g)\int ds\frac{s}{2\pi^2}\sinh(2\pi s)e^{-\frac{\beta}{(\phi_b-\epsilon\,\tilde{e}\tilde{C}_2(R))}s^2+\tilde{e}\tilde{C}_2(R)\left(2\pi-\frac{\beta}{\epsilon}\right)} \\ &= \sum_R \dim(R)\chi_R(h)\frac{1}{(2\pi)^{1/2}}\left(\frac{\tilde{\phi}_b(R)}{\beta}\right)^{3/2} e^{\frac{\pi^2\tilde{\phi}_b(R)}{\beta}+\tilde{e}\tilde{C}_2(R)\left(2\pi-\frac{\beta}{\epsilon}\right)}\,, \end{aligned} \tag{4.36}$$

where we have defined

$$\tilde{\phi}_b(R) \equiv \phi_b - \epsilon\,\tilde{e}\,\tilde{C}_2(R)\,, \tag{4.37}$$

which can be seen as the "renormalization" of the boundary value of the dilaton ϕ_b. Thus, the addition of the Yang-Mills term to the JT gravity action has the effect of "re-normalizing" all the dimensionful quantities appearing in JT gravity by a representation dependent factor.

As previously mentioned, our result is reliable only in the regime in which $\phi_b > \epsilon\tilde{e}\tilde{C}_2(R)$ for which the coupling in the Schwarzian action in (4.35) is positive. If this was not the case than the path integral over the field $F(u)$ would no longer be convergent, at least when considering a contour along which $F(u)$ is real. From the perspective of near-extremal black holes, this inequality is indeed obeyed: namely, for representations with very large dimensions one expects $C_2(R) \to \infty$ and thus $\tilde{C}_2(R) \to 2/\tilde{e}_\phi$. Since $\tilde{e}_\phi > 0$ when the couplings e and e_ϕ are real in (4.1) , $\tilde{C}_2(R)$ asymptotes to a negative constant and therefore satisfies $\phi_b > \epsilon\tilde{e}\tilde{C}_2(R)$ for sufficiently small ε.

In the $(\epsilon/\tilde{e} \to 0,\, \tilde{e}_\phi \to 0)$ limit the singlet representation dominates in the sum in (4.36). This $1/\epsilon$ divergence in the exponent appears due to a divergence in the area of the nearly AdS_2 patches that dominate in the gravitational gauge theory path integral. In the upcoming subsection, we show that such a divergence can be eliminated using a change in boundary conditions for the gauge field, which amounts to adding the appropriate boundary counter-term that cancels the divergence in the action. In the limit $(\epsilon \to 0,\, \tilde{e}_\phi \to 0)$, with $\epsilon/\tilde{e}$ kept finite, the partition function of the theory matches the one we have found in section 4.2 when coupling JT gravity to a BF theory.

Going away from the strict $\epsilon \to 0$ limit and instead viewing (4.36) in an ε expansion we note that if we keep the next order terms in ϵ in the extrinsic curvature in (4.10) they would only contribute $O(\varepsilon^2)$ in the exponent.[16] Thus, the Casimir dependent terms shown in (4.36), which are $O(1/\varepsilon)$ to $O(\epsilon)$, are the most important contributions in the ϵ expansion of the partition function of the gravitational gauge theory (4.1).

4.3.2 Counter-terms from a change in boundary conditions

As is typical when analyzing theories in AdS in the holographic context, the action of the theory under consideration is generically not finite on-shell and needs to be supplemented by boundary terms, a procedure referred to as holographic renormalization. Given the appropriate boundary terms, one could then use the variational principle to check what boundary conditions can be consistently imposed in order for the variational problem to be well defined and in order for the overall on-shell action to be finite. Although various boundary terms supplementing the Maxwell or Yang-Mills actions have been considered in the past in the context of 2d/1d holography (for example, see [183, 184, 185, 186, 30, 22]), here we take a different approach and show that, in order to cancel the

[16] This can be easily seen by computing the next order in the ε expansion in the solution of (4.9), $\tau = \epsilon F' + \epsilon^3 \frac{(F'')^2}{2F'} + O(\epsilon^5)$. Plugging this result in the extrinsic curvature formula (4.10), we find that

$$\mathcal{K}[F(u)] = 1 + \varepsilon^2 \mathrm{Sch}(F,u) + \varepsilon^4 \left(\frac{27}{8}\frac{(F'')^4}{(F')^4} + \frac{(F^{(3)})^2}{(F')^2} + \frac{F^{(4)}F''}{(F')^2} - \frac{11(F'')^2F^{(3)}}{2(F')^3} \right) + O(\varepsilon^6) \tag{4.38}$$

Consequently the first correction on the gravitational side coming from $\phi_b\mathcal{K}[F(u)]/\varepsilon^2$ is $O(\varepsilon^2)$. Work on computing the partition function in pure JT gravity to all perturbative orders in ε is currently underway [165]. A similar perspective can be gained by studying an analog of the $T\overline{T}$ deformation in 1d [70].

divergence in the exponent in (4.36), it is sufficient to add a boundary condition changing defect similar to the one considered in section 4.2.4. After stating the proper form of the boundary condition changing defect, we can immediately derive the necessary boundary conditions that the gauge theory needs to satisfy.

Namely, we consider adding

$$S_{\text{defect}} = \frac{1}{2}\int_I du\sqrt{g_{uu}}\left[\frac{\tilde{e}\,\mathrm{Tr}\phi^2}{1+\frac{\tilde{e}_\phi\,\mathrm{Tr}\phi^2}{2}} - \varepsilon\,\tilde{e}_b\mathrm{Tr}\phi^2\right], \tag{4.39}$$

to the action (4.7) where, once again, I is a contour which is arbitrarily close to the boundary $\partial\mathcal{M}$ and $\tilde{e}_b$ is an arbitrary constant. Similar to our analysis in subsection 4.2.4, multiplying $\tilde{e}_b$ by $\mathrm{Tr}\,\phi^2$ instead of a more general trace-class function $V(\phi)$ is an arbitrary choice that is only meant to regularize the sum over all irreducible representations appearing in the partition function. Integrating the equation of motion on the defect yields

$$A_u|_{\partial\mathcal{M}} - A_u|_I = -i\sqrt{g_{uu}}\left[\frac{\tilde{e}\phi}{1+\frac{\tilde{e}_\phi}{2}\mathrm{Tr}\phi^2} - \frac{\tilde{e}\,\tilde{e}_\phi\,\phi\,\mathrm{Tr}\phi^2}{2\left(1+\frac{\tilde{e}_\phi}{2}\mathrm{Tr}\phi^2\right)^2} - \varepsilon\tilde{e}_b\phi\right]\Bigg|_I. \tag{4.40}$$

Once again moving $A_u|_I$ to the right hand side and denoting $A_u|_{\partial\mathcal{M}} = \mathcal{A}_u$, we find that by inserting the defect the new "mixed" boundary condition in the resulting theory is given by

$$\delta\left(A_u - i\sqrt{g_{uu}}\left[\frac{\tilde{e}\phi}{1+\frac{\tilde{e}_\phi}{2}\mathrm{Tr}\phi^2} - \frac{\tilde{e}\,\tilde{e}_\phi\,\phi\,\mathrm{Tr}\phi^2}{2\left(1+\frac{\tilde{e}_\phi}{2}\mathrm{Tr}\phi^2\right)^2} - \varepsilon\tilde{e}_b\phi\right]\right)\Bigg|_I = 0\,. \tag{4.41}$$

Adding this defect modifies the path integral computation at the step (4.35). Following the procedure presented in subsection 4.2.4, we find that after integrating out the gauge field degrees of freedom we get

$$\begin{aligned} Z^{\text{disk}}_{\substack{\text{JTYM,}\\ \text{mixed}}}(\phi_b,\beta,h) = &\sum_R \dim(R)\chi_R(g) \\ &\times\int DF e^{\left[\tilde{e}\tilde{C}_2(R)\left(2\pi+\frac{\beta}{\varepsilon}\right)-\beta\tilde{e}_bC_2(R)-\frac{\phi_b\beta}{\epsilon^2}+\left(\phi_b-\epsilon\tilde{C}_2(R)\right)\int_0^\beta du\left(\frac{1}{\epsilon^2}+\{F,u\}\right)\right]}\,. \end{aligned} \tag{4.42}$$

After performing the integral over $F(u)$ by following the steps in (4.36), we find

$$Z^{\text{disk}}_{\substack{\text{JTYM,}\\ \text{mixed}}}(\phi_b,\beta,h) = \sum_R \dim(R)\chi_R(g)\frac{1}{(2\pi)^{1/2}}\left(\frac{\tilde{\phi}_b(R)}{\beta}\right)^{3/2} e^{\frac{\pi^2\tilde{\phi}_b(R)}{\beta}+2\pi\tilde{e}\tilde{C}_2(R)-\tilde{e}_b\beta C_2(R)}\,. \tag{4.43}$$

Note that, the $1/\varepsilon$ divergence present in the exponent in (4.36) has vanished, the singlet representation is no longer the dominating representation and the sum over all irreducible representations R is generically convergent for $\tilde{e}_b \geq 0$. With these results in mind, we now discuss the boundary dual of the 2d gravitational Yang-Mills theory (4.1), both with Dirichlet boundary conditions and the mixed conditions discussed in this subsection.

4.3.3 Equivalent boundary theory

As extensively discussed in subsections 4.2.1–4.2.4, when adding a BF theory to the JT gravity action, and using mixed boundary conditions between the gauge field and the zero-form scalar ϕ, the gravitational theory can be equivalently expressed as the Schwarzian theory decoupled from a particle moving on the group manifold G. Here, we show how, by going to finite gauge coupling, the two boundary theories become coupled.

To find the dual of JT gravity coupled to Yang-Mills theory it is useful to interpret the partition functions (4.36) (Dirichlet) or (4.43) (mixed) in terms of the path integral of a particle moving on a group manifold with a time dependent metric g_{uu}. Towards that aim, we use this particle's path integral to reproduce the intermediate steps (4.35) and (4.42) in which we have integrated out the gauge degrees of freedom, but have not yet integrate out the Schwarzian field $F(u)$. To do this we set $\sqrt{g_{uu}(u)} \equiv j(u)$ for the particle moving on the group manifold G:[17]

$$\begin{cases} j_{\text{Dirichlet}}(u) = \frac{1}{\varepsilon} - \frac{2\pi}{\beta} + \varepsilon\,\text{Sch}(F,u)\,, & \text{for dual of Dirichlet b.c. from (4.36)}\,, \\ j_{\text{mixed}}(u) = -\frac{2\pi}{\beta} + \varepsilon\,\text{Sch}(F,u) & \text{for dual of mixed b.c. from (4.43)}\,. \end{cases} \tag{4.44}$$

Fixing the action of the particle moving on a group manifold coupled to the Schwarzian theory to be given by

$$\begin{cases} S_{\substack{\text{Schw}\rtimes G \\ \text{Dirichlet}}} \equiv \int_0^\beta du \left[\left(\frac{\phi_b}{2} - \frac{\varepsilon\,\tilde{e}\,\text{Tr}\,\boldsymbol{\alpha}^2}{2(1+\tilde{e}_\phi \text{Tr}\,\boldsymbol{\alpha}^2)} \right) \text{Sch}(F,u) - i\text{Tr}\left(\boldsymbol{\alpha}\, h^{-1} D_A h\right) + \frac{\tilde{e}\left(\frac{1}{\varepsilon} - \frac{2\pi}{\beta}\right)\text{Tr}\,\boldsymbol{\alpha}^2}{2(1+\tilde{e}_\phi \text{Tr}\,\boldsymbol{\alpha}^2)} \right]\,, \\ S_{\substack{\text{Schw}\rtimes G \\ \text{mixed}}} \equiv \int_0^\beta du \left[\left(\frac{\phi_b}{2} - \frac{\varepsilon\,\tilde{e}\,\text{Tr}\,\boldsymbol{\alpha}^2}{2(1+\tilde{e}_\phi \text{Tr}\,\boldsymbol{\alpha}^2)} \right) \text{Sch}(F,u) - i\text{Tr}\left(\boldsymbol{\alpha}\, h^{-1} D_A h\right) + \frac{2\pi\tilde{e}\,\text{Tr}\,\boldsymbol{\alpha}^2}{\beta(1+\tilde{e}_\phi \text{Tr}\,\boldsymbol{\alpha}^2)} - \frac{\tilde{e}_b}{2}\text{Tr}\,\boldsymbol{\alpha}^2 \right]\,. \end{cases} \tag{4.45}$$

[17] One should not be concerned about the invertibility of the 1d metric in (4.44). Rather one can view this metric as an arbitrary source for the potential $\widehat{V}(\boldsymbol{\alpha})$ in the path integral of the particle moving on the G group manifold.

After integrating out h and $\boldsymbol{\alpha}$ that the partition function of this theory is given by,

$$Z_{\substack{\mathrm{Schw}\rtimes G\\ j(u)}}(\beta,h)=\sum_R(\dim R)\chi_R(h)\int D\mu[F]\,e^{-\frac{\beta\tilde{e}_b C_2(R)}{2}-\left(\tilde{e}\tilde{C}_2(R)\int_0^\beta du\, j(u)\right)+\left(\phi_b\int_0^\beta du\,\mathrm{Sch}(F,u)\right)}\,. \tag{4.46}$$

where $j(u)$ is the source in (4.44). Comparing this partition function to (4.35) for Dirichlet boundary conditions in the bulk or with (4.42) for mixed boundary conditions, we conclude that the partition function of the particle moving on the group manifold coupled to the Schwarzian theory matches the partition function of gravitational Yang-Mills theory, for an arbitrary G holonomy h: $Z^{\mathrm{disk}}_{\substack{\mathrm{JTYM},\\ \mathrm{Dirichlet}}}(h)=Z_{\substack{\mathrm{Schw}\rtimes G,\\ \mathrm{Dirichlet}}}(h)$ and $Z^{\mathrm{disk}}_{\substack{\mathrm{JTYM},\\ \mathrm{mixed}}}(h)=Z_{\substack{\mathrm{Schw}\rtimes G,\\ \mathrm{mixed}}}(h)$. Based on this result we conjecture the result presented in figure 4.2.

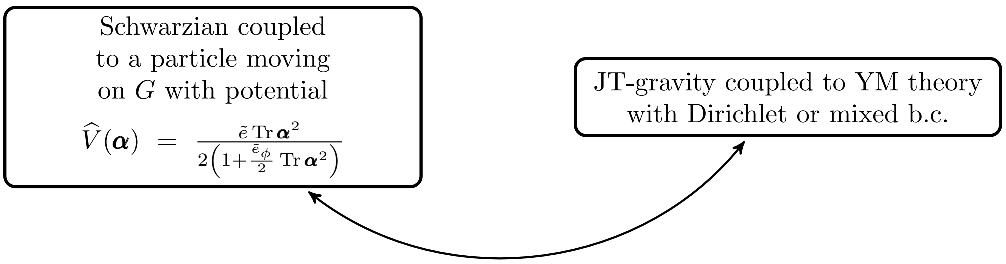


Figure 4.2: Schematic representation of the equivalence between the gravitational gauge theory and the Schwarzian coupled to a particle moving on the group manifold G.

More generally, one can replace $\tilde{e}\mathrm{Tr}\,\phi^2$ and $\tilde{e}_\phi\mathrm{Tr}\,\phi^2$ in the action (4.7) by generic gauge-invariant functions of ϕ.[18] In such a case we expect that the dual quantum mechanical theory be given by

$$S_{\substack{\mathrm{Schw}\rtimes G\\ \mathrm{General}}}=\int_0^\beta du\left[-i\mathrm{Tr}\,(\boldsymbol{\alpha}\,h^{-1}D_A h)-\widehat{\mathcal{W}}(\boldsymbol{\alpha})+\widehat{\mathcal{V}}(\boldsymbol{\alpha})\,\mathrm{Sch}(F(u),u)\right]\,. \tag{4.47}$$

The functions $\widehat{\mathcal{V}}(\boldsymbol{\alpha})$ and $\widehat{\mathcal{W}}(\boldsymbol{\alpha})$ are invariant under adjoint transformations of $\boldsymbol{\alpha}$ and can be straightforwardly related to the functions of ϕ that appear in the generalization of the action (4.7).[19]

The action (4.47) is a generic effective action with a $G\times SL(2,\mathbb{R})$ symmetry.[20] Based on symmetry principles, we expect that such an effective action, preserving $G\times SL(2,\mathbb{R})$, appears in the low energy

[18] Such functions could appear when keeping tracks of higher field-strength powers in the effective action for higher-dimensional near-extremal black holes.

[19] Explicitly if considering replacing the terms in the action of the gravitational gauge theory (4.4)

$$S_{\mathrm{JTYM}}\supseteq\frac{1}{2}\int_{\mathcal{M}}d^2x\sqrt{g}\,(\tilde{e}-\tilde{e}_\phi\phi)\,\mathrm{Tr}\,\phi^2\qquad\to\qquad\int_{\mathcal{M}}d^2x\sqrt{g}\,\left(\mathcal{V}_1(\phi)-\phi\mathcal{V}_2(\phi)\right) \tag{4.48}$$

and considering the boundary condition $\delta(A_u+i\sqrt{g_{uu}}\,\widehat{\mathcal{V}}_b(\phi))=0$, we find that the the functions $\widehat{\mathcal{V}}(\boldsymbol{\alpha})$ and $\widehat{\mathcal{W}}(\boldsymbol{\alpha})$ in (4.47) are given by

$$\widehat{\mathcal{V}}(\boldsymbol{\alpha})=\phi_b-\frac{\varepsilon\widehat{\mathcal{V}}_1(\boldsymbol{\alpha})}{1+2\widehat{\mathcal{V}}_2(\boldsymbol{\alpha})}\,,\qquad\widehat{\mathcal{W}}(\boldsymbol{\alpha})=\left(\frac{1}{\varepsilon}-\frac{1}{\beta}\right)\frac{\widehat{\mathcal{V}}_1(\boldsymbol{\alpha})}{1+2\widehat{\mathcal{V}}_2(\boldsymbol{\alpha})}-\frac{\widehat{\mathcal{V}}_b(\boldsymbol{\alpha})}{\varepsilon}\,. \tag{4.49}$$

[20] In fact, the global symmetry group in this action is enhanced to $G\times G\times SL(2,\mathbb{R})$.

limit of a modification of SYK models which have a global symmetry G [166, 167, 168, 169, 170, 171, 172, 173, 174]. For instance, when $G = \mathrm{U}(1)$, (4.47) should appear in the low-energy limit of the complex SYK model studied in [166, 167]; it would be interesting to derive the functions $\mathcal{V}(\boldsymbol{\alpha})$ and $\mathcal{W}(\boldsymbol{\alpha})$ directly in this model.

4.4 Higher genus partition function

Following the same strategy of firstly integrating out the gauge field degrees of freedom and rewriting the resulting area dependence from the Yang-Mills path integral in terms of the extrinsic curvature, we determine the partition function of the gravitational gauge theory for surfaces of arbitrary genus.

4.4.1 The building blocks

In computing the contribution of the gravitational degrees of freedom to the higher genus partition function, we follow the strategy presented in [37]. The basic building blocks needed in order to obtain the genus expansion of the gravitational gauge theory are given by [37]:

- The disk partition functions computed in sections 4.2 or 4.3.
- The path integral over a "trumpet", $\mathcal{M}_T$, which on one side has asymptotically AdS_2 boundary conditions specified by (4.2) and, on the other side, ends on a geodesic of length b. For the gauge field, we first consider Dirichlet boundary conditions by fixing the holonomy on both sides of "trumpet": we denote h_{nAdS_2} to be the holonomy of the side with asymptotically AdS_2 boundary conditions and h_b to be the holonomy on the other side. Following our analysis in section 4.3.2 we then consider mixed boundary conditions on the asymptotically AdS_2 boundary.
- The path integral over a bordered Riemann surfaces of constant negative curvature that has n boundaries and genus g. For such surfaces, we fix the holonomies h_1, h_2, $\dots$, h_n and the lengths of the geodesic boundaries b_1, $\dots$, b_n, across all n boundaries.

By gluing the above geometries along the side where the boundary is a geodesic, we are able to obtain any constant negative curvature geometry that is orientable (with arbitrary genus g and an arbitrary number of boundaries n) and has asymptotically AdS_2 boundaries.

We start by computing the path integral over the trumpet geometry, by integrating out the gauge field. Using (4.26) we find

$$Z^{\text{trumpet}}_{\substack{\text{JTYM}\\ \text{Dirichlet}}} = \int Dg_{\mu\nu} D\phi e^{-S_{JT}[g_{\mu\nu},\,\phi]} \left(\sum_R \chi_R(g_{nAdS_2}) \chi_R(g_b) e^{-\frac{C_2(R)\int_{\mathcal{M}_T} d^2x\sqrt{g}\left[\tilde{e}-\tilde{e}_\phi \phi(x)\right]}{2}} \right) \tag{4.50}$$

where the area term depends on the bulk metric configuration. Integrating out the dilaton field ϕ in each representation sector R, we localize over trumpets with constant negative curvature (once again, with $\tilde{\Lambda} = -2 - \tilde{e}_\phi C_2(R)$), whose boundary degrees of freedom are given by Schwarzian field describing the wiggles on the nearly-AdS_2 boundary. The trumpet area term is given by Gauss-Bonnet:

$$\int_{\mathcal{M}_T} d^2x \tilde{e}\sqrt{g} = -\frac{\tilde{e}}{2+\tilde{e}_\phi C_2(R)} \int_{\mathcal{M}_T} d^2x \sqrt{g}\mathcal{R} = \frac{\tilde{e}}{1+\frac{\tilde{e}_\phi C_2(R)}{2}} \int_{\partial\mathcal{M}_T} du \sqrt{g_{uu}}\, \mathcal{K}\,, \tag{4.51}$$

where, for the trumpet, we have used the Euler characteristic $\chi(\mathcal{M}_T) = 0$ and the fact that the extrinsic curvature has $\mathcal{K} = 0$ along the geodesic boundary. Above we have denoted $\partial\mathcal{M}_T$ to be the boundary of the trumpet with asymptotically AdS_2 boundary conditions. Thus, the path integral becomes

$$Z^{\text{trumpet}}_{\substack{\text{JTYM}\\ \text{Dirichlet}}} = \sum_R \chi_R(h_{nAdS_2}) \chi_R(h_b) \int \frac{d\mu(\tau)}{\text{U}(1)} e^{\left(\frac{\phi_b}{\epsilon} - \tilde{e}\tilde{C}_2(R)\right)\int_{\partial\mathcal{M}_T} du\, \sqrt{g_{uu}}\, \mathcal{K} - \frac{\phi_b}{\epsilon}\int_{\partial\mathcal{M}_T} du\, \sqrt{g_{uu}}}\,, \tag{4.52}$$

The metric can be parametrized as $ds^2 = d\sigma^2 + \cosh^2(\sigma) d\tau^2$, with the periodic identification $\tau(u) \sim \tau(u) + b$. Writing the extrinsic curvature (4.10) in these coordinates, the path integral becomes [37]

$$Z^{\text{trumpet}}_{\substack{\text{JTYM}\\ \text{Dirichlet}}} = \sum_R \chi_R(h_{nAdS_2}) \chi_R(h_b) \int \frac{d\mu(\tau)}{\text{U}(1)} e^{-\frac{\phi_b \beta}{\epsilon^2} + \left(\phi_b - \epsilon\, \tilde{e}\, \tilde{C}_2(R)\right)\int_0^\beta du \left(\frac{1}{\epsilon^2} + \{\exp[-\tau(u)], u\}\right)}\,, \tag{4.53}$$

where we note that the periodic identification of τ breaks the $SL(2,\mathbb{R})$ isometry of the disk down to U(1) translations of τ. Once again performing the one-loop exact path integral over the Schwarzian field $\tau(u)$ [25, 37], we find

$$\begin{aligned} Z^{\text{trumpet}}_{\substack{\text{JTYM}\\ \text{Dirichlet}}} &= \pi \sum_R \chi_R(h_{nAdS_2})\chi_R(h_b) e^{-\frac{\tilde{C}_2(R)\beta}{\epsilon}} \int \frac{ds}{\pi^{1/2}} \cos(bs) e^{-\frac{\beta}{2(\phi_b - \epsilon \tilde{C}_2(R))} s^2} \\ &= \sum_R \chi_R(h_{nAdS_2})\chi_R(h_b) \left(\frac{\phi_b - \epsilon\tilde{C}_2(R)}{2\pi\beta}\right)^{1/2} e^{-\frac{\phi_b b^2}{2\beta} - \tilde{C}_2(R)\left(\frac{\beta}{\epsilon} - \frac{\epsilon b^2}{2\beta}\right)} \\ &= \sum_R \chi_R(h_{nAdS_2})\chi_R(h_b) \left(\frac{\tilde{\phi}_b(R)}{2\pi\beta}\right)^{1/2} e^{-\frac{\tilde{\phi}_b(R) b^2}{2\beta} - \frac{\beta\,\tilde{e}\,\tilde{C}_2(R)}{\epsilon}}\,, \end{aligned} \tag{4.54}$$

where $\tilde{C}_2(R)$ is given by (4.34) and $\tilde{\phi}_b(R)$ is given by (4.37). We again encounter a $1/\varepsilon$ divergence appearing in the exponent in (4.54) which is due to the divergence of the area of the trumpet at finite values of b.

In order to eliminate such a divergence we consider the change of boundary conditions for the gauge field given by (4.40) at the nearly-AdS_2 boundary. As explained in section 4.3.2 this change can be implemented by inserting the boundary condition changing defect. The insertion of such a defect indeed leads to a convergent term in the exponent in (4.54), as can be seen from the resulting partition function

$$Z^{\text{trumpet}}_{\substack{\text{JTYM}\\ \text{mixed}}}(\phi_b, \beta, b, h_{nAdS_2}, h_b) = \sum_R \chi_R(h_{nAdS_2})\chi_R(h_b) \left(\frac{\tilde{\phi}_b(R)}{2\pi\beta}\right)^{1/2} e^{-\frac{\tilde{\phi}_b(R) b^2}{2\beta}} e^{-\tilde{e}_b \beta C_2(R)}\,. \tag{4.55}$$

We now compute the partition function associated to the n-bordered Riemann surface of genus g, which we denote by $Z^{(g,n)}_{\substack{\text{JTYM}\\ \text{Dirichlet}}}(b_j,\, h_j)$. Integrating out the gauge field by using (4.26) and then integrating out the dilaton, we find

$$\begin{aligned} Z^{(g,n)}_{\substack{\text{JTYM}\\ \text{Dirichlet}}}(b_j,\, h_j) &= \sum_R (\dim R)^{2-n-2g} \chi_R(h_1) \dots \chi_R(h_n)\, e^{\chi(\mathcal{M}_{g,n}) S_0} \\ &\quad \times \int Dg^{\mu\nu} \delta\left(R + 2 + \tilde{e}_\phi C_2(R)\right) e^{-\frac{\tilde{e}\, C_2(R) \int_{\mathcal{M}_{g,\,n}} d^2x\sqrt{g}}{2}}\,, \end{aligned} \tag{4.56}$$

where $\int_{\mathcal{M}_{g,\,n}} d^2x\sqrt{g}$ is the area of the constant curvature manifold. From Gauss-Bonnet, we find

$$\int_{\mathcal{M}_{g,\,n}} d^2x\sqrt{g} = -\frac{1}{2 + \tilde{e}\, C_2(R)} \int_{\mathcal{M}_{g,\,n}} d^2x\sqrt{g}\mathcal{R} = \frac{2\pi(2g+n-2)}{1 + \frac{\tilde{e} C_2(R)}{2}}\,, \tag{4.57}$$

where we have used $\chi(\mathcal{M}_{g,n}) = 2 - 2g - n$ and have used the fact that the extrinsic curvature vanishes on the geodesic borders of this Riemann surface. Thus, the partition function of the n-

bordered Riemann surface is given by

$$Z^{(g,n)}_{\substack{\text{JTYM}\\ \text{Dirichlet}}}(b_j,\, h_j) = \sum_R \chi_R(h_1)\dots\chi_R(h_n)\text{Vol}_{g,n}(b_1,\,\dots,\,b_n)\left(\dim R\, e^{S_0}\, e^{2\pi\tilde{e}\tilde{C}_2(R)}\right)^{\chi(\mathcal{M}_{g,n})}\,, \quad (4.58)$$

where $\text{Vol}_{g,n}(b_1,\,\dots,\,b_n)$ is the volume of the moduli space of n-bordered Riemann surfaces with constant curvature. A recursion relation for these volumes was found in [41] (see [42] for a review). It was later showed that this recursion relation can be related to the "topological recursion" seen in the genus expansion of a double-scaled matrix integral [43]. As we discuss later, this relation proves important when discussing the matrix integral interpretation of the genus expansion in pure and gauged JT gravity.

Using (4.54) or (4.55), together with (4.58) we now determine the partition function on surfaces with arbitrary genus.

4.4.2 The genus expansion

Using the gluing rules outlined above, the partition function when summing over all orientable manifold is given by the genus expansion,

$$Z^{n=1}_{\substack{\text{JTBF}\\ \text{mixed}}}(\phi_b,\beta,h) = Z^{\text{disk}}_{\substack{\text{JTBF}\\ \text{mixed}}}(\phi_b,\beta,h) + \sum_{g=1}^{\infty}\int d\tilde{h}\int db\, b\, Z^{\text{trumpet}}_{\substack{\text{JTBF}\\ \text{mixed}}}(\phi_b,\beta,b,h,\tilde{h}) Z^{(\text{g},\,1)}_{\substack{\text{JTBF}\\ \text{Dirichlet}}}(b,\tilde{h})\,. \quad (4.59)$$

Putting (4.36), (4.54) and (4.58) together, we find the genus expansion for the gravitational partition function for surfaces with a single boundary on which we fix Dirichlet boundary conditions for the gauge field:

$$Z^{n=1}_{\substack{\text{JTYM}\\ \text{Dirichlet}}}(\phi_b\,,\beta,\,h) = \sum_R \chi_R(h) e^{-\frac{\tilde{C}_2(R)\beta}{\varepsilon}}\Bigg[\left(\dim(R)e^{2\pi\tilde{e}\tilde{C}_2(R)}e^{S_0}\right)\frac{1}{(2\pi)^{1/2}}\left(\frac{\tilde{\phi}_b(R)}{\beta}\right)^{3/2} e^{\frac{2\pi^2\tilde{\phi}_b(R)}{\beta}} \quad (4.60)$$
$$+\sum_{g=1}^{\infty}\left(\dim(R)e^{2\pi\tilde{e}\tilde{C}_2(R)}e^{S_0}\right)^{\chi(\mathcal{M}_{g,\,1})}\left(\frac{\tilde{\phi}_b(R)}{2\pi\beta}\right)^{\frac{1}{2}}\int_0^{\infty} db\, b\, e^{-\frac{\tilde{\phi}_b(R)b^2}{2\beta}}\text{Vol}^{\alpha}_{g,1}(b)\Bigg].$$

It is instructive to express this result in terms of $Z_{g,1}(\phi_{b_1},\dots,\phi_{b_n},\beta_1,\dots,\beta_n)$, the contribution of surfaces of genus g with n asymptotically AdS_2 boundaries to the pure JT gravity partition function.

Thus (4.60) can be compared to the result in pure JT gravity:

$$
\begin{aligned}
&Z_{\mathrm{JT}}^{n=1}(\phi_b,\beta)=\sum_{g=0}^{\infty}e^{S_0\chi(\mathcal{M}_{g,1})}Z_{g,1}(\beta/\phi_b)\\
&\xRightarrow[\text{term}]{\text{adding Yang-Mills}}\quad Z^{n=1}_{\substack{\mathrm{JTYM}\\ \text{Dirichlet}}}(\phi_b\,,\beta,\,h)=\sum_R\chi_R(h)e^{-\frac{\tilde{C}_2(R)\beta}{\varepsilon}}\\
&\qquad\qquad\times\left[\sum_{g=0}^{\infty}\left(\dim(R)e^{2\pi\tilde{e}\tilde{C}_2(R)}e^{S_0}\right)^{\chi(\mathcal{M}_{g,1})}Z_{g,1}\left(\beta/\tilde{\phi}_b(R)\right)\right],
\end{aligned}
\tag{4.61}
$$

where we have absorbed the entropy dependence $e^{\chi(\mathcal{M}_{g,n})S_0}$, in $Z_{g,n}(\phi_{b_1},\dots,\phi_{b_n},\beta_1,\dots,\beta_n)$: $Z_{\mathrm{JT}}^{(g,n)}(\phi_{b_1},\dots,\phi_{b_n},\beta_1,\dots$ $e^{\chi(\mathcal{M}_{g,n})S_0}Z_{g,n}(\beta_1/\phi_{b_1},\dots,\beta_n/\phi_{b_n})$ (from the partition function on trumpet geometries, one immediately deduces that $Z_{g,n}$ solely depends on the ratios β_j/ϕ_{b_j}). The coefficients $Z_{g,n}(\beta_j/\phi_{b_j})\equiv Z_{g,n}(\beta_1/\phi_{b_1},\dots,\beta_n/\phi_{b_n})$ are in fact those encountered in the genus expansion of correlators of the partition function operator in the double-scaling of the certain matrix integral that we have previously mentioned.

We can also determine the partition function of the space which has n boundaries,

$$
\begin{aligned}
Z^{n}_{\substack{\mathrm{JTYM}\\ \text{Dirichlet}}}(\phi_{b,j},\beta_j,h_j)&=\sum_R\chi_R(h_1)\dots\chi_R(h_n)e^{-\frac{\tilde{e}\tilde{C}_2(R)\sum_{j=1}^n\beta_j}{\varepsilon}}\left[\sum_{g=0}^{\infty}(\dim R e^{2\pi\tilde{e}\tilde{C}_2(R)}e^{S_0})^{\chi(\mathcal{M}_{g,\,n})}\right.\\
&\times\left.\left(\frac{\tilde{\phi}_{b,1}(R)\dots\tilde{\phi}_{b,1}(R)}{\pi^n\beta_1\dots\beta_n}\right)^{\frac{1}{2}}\int_0^{\infty}db_1b_1\dots\int_0^{\infty}db_nb_n\,\mathrm{Vol}^{\alpha}_{g,n}(b_{1,\dots,n})e^{-\sum_{i=1}^n\frac{\tilde{\phi}_{b,i}(R)b_i^2}{\beta_i}}\right].
\end{aligned}
\tag{4.62}
$$

In terms of the coefficients $Z_{g,n}(\beta_j/\phi_{b_j})$, this becomes

$$
\begin{aligned}
Z^{n}_{\substack{\mathrm{JTYM}\\ \text{Dirichlet}}}(\phi_{b_j}\,,\beta_j,\,h_j)&=\sum_R\chi_R(h_1)\dots\chi_R(h_n)e^{-\frac{\tilde{e}\tilde{C}_2(R)\sum_{j=1}^n\beta_j}{\varepsilon}}\\
&\times\left[\sum_{g=0}^{\infty}\left(\dim(R)e^{2\pi\tilde{e}\tilde{C}_2(R)}e^{S_0}\right)^{\chi(\mathcal{M}_{g,n})}Z_{g,n}\left(\beta_j/\tilde{\phi}_{b_j}(R)\right)\right].
\end{aligned}
\tag{4.63}
$$

In the $\varepsilon\to 0$ limit, $\tilde{\phi}_{b_j}(R)=\phi_{b_j}$ for all j and, in the square parenthesis in (4.61) and (4.63), the dependence on the irreducible representation R can be absorbed in the overall entropy on the disk $S_0\to S_0-\tilde{e}\tilde{C}_2(R)-\log\dim R$; thus, the density of states associated to each representation sector is the same as in pure JT gravity. As we explain shortly, this serves as a useful guide in determining the matrix integral derivation of (4.60).

With Dirichlet boundary conditions and in the limit $\varepsilon\to 0$, the singlet representation dominates in the sum over representations due to the $1/\varepsilon$ divergence in the first exponent of (4.60) or (4.62). This behavior can be altered by the change of boundary conditions (4.41) presented in section 4.3.2

or, equivalently, by the addition of a defect close to each one of the n boundaries of the manifold. When using the boundary condition changing defect, the result in each representation sector gets regularized such that

$$Z^n_{\substack{\text{JTYM}\\\text{mixed}}}(\phi_{b_j},\beta_j,h_j) = \sum_R Z^n_{\substack{\text{JTYM}\\\text{Dirichlet}}}(\phi_{b_j},\beta_j,h_j)_R \; e^{\left(\frac{\tilde{e}\tilde{C}_2(R)}{\varepsilon}\right)\left(\sum_{j=1}^n \beta_j\right)-\frac{1}{2}C_2(R)\left(\sum_{j=1}^n \tilde{e}_{b_j}\beta_j\right)}\,, \tag{4.64}$$

where $Z^n_{\substack{\text{JTYM}\\\text{Dirichlet}}}(\phi_{b_j},\beta_j,h_j)_R$ is the contribution of the representation R to the sum in (4.63). Above, the mixed boundary condition obtained from (4.41) with a coupling $\tilde{e}_{b_j}$ is considered for each of the n boundaries.

The result (4.64) simplifies further in the (topological) weak gauge coupling limit

$$\begin{aligned} Z^n_{\substack{\text{JTBF}\\\text{mixed}}}(\phi_{b_j},\beta_j,h_j) = \sum_R \chi_R(h_1)\dots\chi_R(h_n) e^{-\frac{C_2(R)\sum_{i=j}^n \tilde{e}_{b_j}\beta_j}{2}} \\ \times\left[\sum_{g=0}^{\infty}\left(\dim(R)e^{S_0}\right)^{\chi(\mathcal{M}_{g,n})} Z_{g,n}(\beta_j/\phi_{b_j})\right], \end{aligned} \tag{4.65}$$

where we have used the boundary condition (4.28)

$$\delta(A_u + i\sqrt{g_{uu}}\, e_{b_j}\,\phi)|_{(\partial\mathcal{M})_j} = 0\,, \tag{4.66}$$

for each of the n-boundaries.

It is worth pondering the interpretation of (4.65). While for the disk contribution to the partition function (4.29), the gravitational and topological theories were fully decoupled, the topological theory of course couples to JT gravity through the genus expansion.

One case in which the sum over R can be explicitly computed is when $\tilde{e}_b = 0$, for which the sum over irreducible representations evaluates to the volume of flat G connection on each surface of genus g. For instance, in the case when $G = \text{SU}(2)$ all such volumes have been computed explicitly in [103]. More generally for any G, when focusing on surfaces with a single boundary ($n = 1$) and setting $h \neq e$, the contribution from surfaces with disk topology to (4.65) vanishes, and the leading contribution is given by surfaces with the topology of a punctured torus. In this limit, the contribution of non-trivial topology is, in fact, visible even at large values of e^{S_0}. In the limit in which $h \to e$, the contribution from surfaces with the topology of a disk or a punctured torus are divergent; in the case when $G = \text{SU}(2)$ such divergences behave as $O(1/\tilde{e}_b^{3/2})$ and $O(1/\tilde{e}_b^{1/2})$ respectively. The leading contribution for all other surfaces behaves as $O(1)$. In other words, this limit further isolates the contribution of surfaces with disk and punctured torus topology in the partition function.

4.4.3 Matrix integral description

Reviewing the correspondence between pure JT gravity and matrix integrals

In order to understand how to construct the matrix integral that reproduces the genus expansion in the gravitational gauge theory (4.1) we first briefly review this correspondence in the case of pure JT gravity, following [37]. Consider a Hermitian matrix integral over $N \times N$ Hermitian matrices with some potential $S[H]$:

$$\mathcal{Z} = \int dH e^{-S(H)}\,, \qquad S[H] \equiv N\left(\frac{1}{2}\text{Tr}_N H^2 + \sum_{j\geq 3}\frac{t_j}{j}\text{Tr}_N H^j\right)\,, \tag{4.67}$$

where Tr_N is the standard trace over $N \times N$ matrices. An observable that proves important in the genus expansion of the gravitational theory is the correlator of the thermal partition function operator, $Z(\beta) = \text{Tr}_N\, e^{-\beta H}$. Correlators of such operators have an expansion in $1/N$, where each order in N can be computed by looking at orientable double-line graphs of fixed genus [44, 45] (for a review see [46]). Consequently, this is known as the genus expansion of the matrix model (1.30).

For a general set of potentials $S[H]$, each order in the expansion can be determined in terms of a single function $\rho_0(E)$. This function is simply the leading density of eigenvalues in matrices with $N \to \infty$. Consider the double-scaling limit of (1.30), in which the size of the matrix $N \to \infty$ and in which we focus on the edge of the eigenvalue distribution of the matrix H, where the eigenvalue density remains finite and is denoted by e^{S_0}. The expansion of the correlators mentioned above can now be expressed in terms of e^{S_0} instead of the size of the matrix N. In this double-scaled limit the density of eigenvalues $\rho_0(E)$ is not necessarily normalizable and with an appropriate choice of potential $S[H]$, $\rho_0(E)$ can be set to be equal to the energy density in the Schwarzian theory (4.11)

$$\rho_0(E) = \frac{\phi_b}{2\pi^2}\sinh(2\pi\sqrt{2\phi_b E})\,. \tag{4.68}$$

In the remainder of this subsection, we follow [37] and normalize

$$\phi_{b_j} \equiv 1/2\,, \qquad Z_{g,n}(\beta_j) \equiv Z_{g,n}(\beta_j/\phi_{b_j})\,, \tag{4.69}$$

for all the n boundaries of the theory, and use the short-hand notation in (4.69). As previously emphasized, choosing (4.68) determines all orders (in the double scaled limit) in the e^{-S_0} perturbative expansion for correlators of operators such as $Z(\beta) = \mathrm{Tr}_N\, e^{-\beta H}$ [47]. The result found by [37], building on the ideas of [43], is that the genus expansion in pure JT gravity agrees with the e^{S_0} genus expansion of the double-scaled matrix integral whose eigenvalue density of states is given by (4.68):

$$Z^n_{\mathrm{JT}}(\beta_1, \dots, \beta_n) = \langle Z(\beta_1)\dots Z(\beta_n)\rangle = \sum_g Z_{g,n}(\beta_j) e^{-S_0\chi(\mathcal{M}_{g,n})}\,. \tag{4.70}$$

The density of states (4.68) was shown to arise when considering the matrix integral associated to the $(2,p)$ minimal string. Specifically, this latter theory was shown to be related to a matrix integral whose density of eigenvalues is given by [48, 49, 50, 51, 52]

$$\rho_0(E) \sim \sinh\left(\frac{p}{2}\mathrm{arccosh}\left(1+\frac{E}{\kappa}\right)\right)\,, \tag{4.71}$$

where κ is set by the value of p and by the value of μ from the Liouville theory which is coupled to the $(2,p)$ minimal model [53]. Taking the $p \to \infty$ limit in (4.71) and rescaling E appropriately, one recovers the density of states (4.68). Consequently, one can conclude that the double-scaled matrix integral which gives rise to the genus expansion in pure JT gravity is the same as the matrix integral which corresponds to the $(2,\infty)$ minimal string.

Our goal is to extend this analysis and find a modification of the matrix integral presented in (1.30) such that the partition function includes the contributions from the gauge field that appeared in the genus expansion of JT gravity coupled to Yang-Mills theory. As we will show below, there are two possible equivalent modifications of the matrix integral (1.30):

- As shown in subsection 4.4.2, in the $\varepsilon \to 0$ limit, the contribution of the gauge degrees of freedom to the partition function can be absorbed in each representation sector R by an R-dependent shift of the entropy S_0. This indicates that instead of obtaining the gravitational gauge theory partition function from a single double-scaled matrix integral, one can obtain the contribution of the gauge degrees of freedom from a collection of double-scaled matrix integrals, where each matrix H^R is associated to a different irreducible representation R of G. The size of H^R is proportional to the dimension of the representation R.

- In order to obtain such a collection of random matrix ensembles in a natural way, we consider a different modification of the matrix integral (1.30). Specifically, instead of considering a Hermitian matrix whose elements are complex, we rather consider matrices whose elements are complex functions on the group G (equivalently, they are elements of the group algebra $\mathbb{C}[G]$).

Equivalently, as we will discuss shortly, one can consider matrices that in addition to the two discrete labels characterizing the elements, have two additional labels in the group G and are invariant under G transformations. By defining the appropriate traces over such matrices, we show that such matrix integrals are equivalent to the previously mentioned collection of matrix integrals, which in turn reproduce the genus expansion in the gravitational gauge theory. This latter model serves as our starting point.

In our analysis, we first consider the necessary modifications of the matrix integral (1.30) which reproduce the results from the weak gauge coupling limit and, afterward, we discuss the case of general coupling.

Modifying the matrix integral: the weakly coupled limit

We start by modifying the structure of the Hermitian matrix H, by supplementing the discrete indices $i, j \in 1, \ldots, N$ that label the elements H_{ij}, by two additional elements $g, h \in G$.[21] Thus, elements of the matrix are given by $H_{(i,g),(j,h)}$. For such matrices, their multiplication is defined by

$$(HM)_{(i,g),(j,\tilde{g})} = \sum_{k=1}^{N} \int dh H_{(i,g),(k,h)} M_{(k,h),(j,\tilde{g})} \tag{4.72}$$

where dh is the Haar measure defined on the group, normalized by the volume of group such that $\int dh = 1$.

The (left) action of the group element $f \in G$ on the matrix $H_{(i,g),(j,h)}$ is defined as $H_{(i,g),(j,h)} \to H_{(i,fg),(j,fh)}$, where we emphasize that the integer indices remain unaltered. In order to reproduce the collection of matrix integrals that we have previously mentioned, in this work we are interested in G-invariant matrices [175], defined by the property

$$H_{(i,g),\,(j,h)} = H_{(i,fg),\,(j,fh)}\,, \tag{4.73}$$

for any $f \in G$. For such matrices one can therefore, define $H_{i,j}(g)$ by using [175]

$$H_{(i,g),\,(j,h)} = H_{(i,e),\,(j,g^{-1}h)} \equiv H_{i,j}(g^{-1}h) \in \mathbb{C}[G] \tag{4.74}$$

where $\mathbb{C}[G]$ is the complex group algebra associated to G. In other words, each element $H_{i,j}$, instead of being viewed as a complex element, can be viewed as a function on the group $G \to \mathbb{C}$.

[21] Here we consider the case when G is a compact Lie group, while the past discussion of matrix integrals of this type focused solely on the case when G is a finite group [175, 176, 177, 178].

For G-invariant matrices, the product (4.72) simplifies to

$$(HM)_{ij}(g) = \sum_{k=1}^{N} \int dh H_{ik}(h) M_{kj}(h^{-1}g)\,, \tag{4.75}$$

where the integral over h simply gives the convolution of functions defined on the group G.

We wish to understand the free energy of a matrix model whose action is given by [175]

$$S[H] = N\left[\frac{1}{2}\,\chi_{\rm el}(H^2) + \sum_{j\geq 3}\frac{t_j}{j}\chi_{\rm el}(H^j)\right]\,, \tag{4.76}$$

where H is a G-invariant matrix defined through (4.74) and $\chi_{\rm el}$ is the trace which, at first, we take to be in the elementary representation of the group G. The trace in the (reducible) elementary representation of the group is given by evaluating the H in (4.74) on the identity element e of the group G,[22]

$$\begin{aligned}\chi_{\rm el}(H) \equiv \sum_{i=1}^{N} H_{i,i}(e) &= \int d\tilde{h}\,\delta(\tilde{h}) \sum_{i=1}^{N} H_{i,i}(\tilde{h}) = \sum_{i=1}^{N}\sum_{R}\int d\tilde{h}\,(\dim R)\chi_R(\tilde{h}^{-1})H_{i,i}(\tilde{h}) \\ &= \sum_R (\dim R) \sum_{i=1}^{N}\sum_{j=1}^{\dim R} (H_{i,i})^j_{R,j} = \sum_R (\dim R) {\rm Tr}_{(\dim R)\,N}(H_R)\,,\end{aligned} \tag{4.78}$$

Here, we have used the decomposition $H_{i,j}(g) = \sum_R \sum_{k,l=1}^{\dim R} (\dim R) U^k_{R,l}(g)(H_{i,j})^k_{R,l}$ where $U^k_{R,l}(g)$ are the matrix elements of G.[23] Thus, we can view H_R as an $(\dim R\, N)\times(\dim R\, N)$ matrix and, above, ${\rm Tr}_{\dim R\, N}(\dots)$ is the standard trace over such matrices. Furthermore, to evaluate the trace in the elementary representation for products of such matrices we can use

$$(H^k)_{i_1,i_{k+1}}(h) = \sum_R \sum_{\substack{j_1,\dots,\\ j_{k+1}=1}}^{\dim R} (\dim R) \sum_{\substack{i_2,\dots,\\ i_k=1}}^{N} (H_{i_1,i_2})^{j_1}_{R,j_2}\dots(H_{i_k,i_{k+1}})^{j_k}_{R,j_{k+1}} U^{j_1}_{R,j_{k+1}}(h)\,, \tag{4.79}$$

[22] One might contemplate whether (4.78) is indeed a well-defined trace. We, in fact, show that the trace is still valid when replacing $\delta(\tilde{h})$ in (4.83) by an arbitrary trace-class function, $\sigma(\tilde{h}^{-1})$. This can, of course, be viewed as a trace in an arbitrary (most often) reducible representation of G. To show this, we have

$$\begin{aligned}\chi_f(HM) = \int d\tilde{h}\,dh\,\sigma(\tilde{h}^{-1}) \sum_{i,k=1}^{n} H_{ik}(h)M_{ki}(h^{-1}\tilde{h}) &= \sum_R \sigma_R \int dh \sum_{i,k=1}^{n} H_{ik}(h)U_R(h^{-1})(M_{ki})_R \\ = \sum_R \sigma_R \sum_{i,k=1}^{n}\sum_{m,p=1}^{\dim R} (H_{ik})^p_{R,m}(M_{ki})^m_{R,p} &= \chi_f(MH) \implies \chi_f([H,M]) = 0\,,\end{aligned} \tag{4.77}$$

which indeed implies that $\chi_{\rm el}(\dots)$ is a well-defined trace. Above, we have used the fact that for trace-class function $\sigma(\tilde{h}^{-1})$, there is a decomposition $\sigma(\tilde{h}^{-1}) = \sum_R \sigma_R\,\chi_R(\tilde{h}^{-1})$.

[23] Note that $H_{i,j}(g)$ is generically not trace class since $H_{i,j}(h^{-1}gh) \neq H_{i,j}(g)$, for generic group elements g and h. Thus, $H_{i,j}(g)$ should be decomposed in the matrix elements of G, $U^k_{R,l}(g)$, instead of its characters $\chi_R(g)$.

which yields

$$\chi_{\rm el}(H^k) = \sum_R (\dim R)(H_{i_1,i_2})^{j_2}_{R,j_1}\dots(H_{i_k,i_1})^{j_1}_{R,j_k} = \sum_R (\dim R)\mathrm{Tr}_{(\dim R)N}(H_R^k)\,. \tag{4.80}$$

Thus, the action (4.76) becomes [175]

$$S[H] = \sum_R N(\dim R)\left[\frac{1}{2}\,\mathrm{Tr}_{(\dim R)N}(H_R^2) + \sum_{j\geq 3}\frac{t_j}{j}\,\mathrm{Tr}_{(\dim R)N}(H_R^j)\right]\,, \tag{4.81}$$

which is the same as a collection of decoupled GUE-like matrix integrals, where each matrix H_R is Hermitian, is associated to the representation R, and has dimension $(\dim R\, N)\times(\dim R\, N)$. Such matrix integrals are truly decoupled if the measure for the path integral in (4.76) associated to $H(g)$ is chosen such that it reduces to the standard measure for GUE-like matrix integrals associated to dH_R. To summarize, this result simply comes from the harmonic decomposition onto different representation sectors of our initial Hermitian matrices whose elements were in $\mathbb{C}[G]$.

We now compare correlation functions in the standard Hermitian matrix model with $N\times N$ matrices, to those in the model whose matrix elements are part of the group algebra $\mathbb{C}[G]$, when having the same couplings in both models. Equivalently, we can compare such correlators to those in the collection of matrix models in (4.81). In order to do this we compare correlation functions of the trace of $e^{-\beta H}$ to the gravitational answer. When H is an $N\times N$ Hermitian matrix the trace is the standard $\mathrm{Tr}_N e^{-\beta H}$. However, when H has elements in $\mathbb{C}[G]$ the trace needs to be modified :

$$Z(\beta) = \mathrm{Tr}_N\left(e^{-\beta H}\right) \qquad\Rightarrow\qquad Z_{\rm cyl.}(h,E) = \chi_{{\rm cyl.},\,h}(e^{-\beta H})\,, \tag{4.82}$$

where,

$$\begin{aligned}\chi_{{\rm cyl.},\,h}(H) &= \int d\tilde h\, Z^{(0,\,2)}_{\substack{\rm BF\\ \rm mixed}}(\tilde h^{-1},h)\sum_{i=1}^N H_{i,i}(\tilde h) = \int d\tilde h \sum_R \chi_R(\tilde h^{-1})\chi_R(h)e^{-\frac{\tilde e_b\beta C_2(R)}{2}}\sum_{i=1}^N H_{i,i}(\tilde h)\\ &= \sum_R \chi_R(h)e^{-\frac{\tilde e_b\beta C_2(R)}{2}}\mathrm{Tr}_{(\dim R)N}(H_R)\,,\end{aligned} \tag{4.83}$$

where $Z^{(0,\,2)}_{\substack{\rm BF\\ \rm mixed}}(g^{-1},h)$ is the partition function of BF theory on the cylinder given by (4.26), where on one of the edges we use Dirichlet boundary conditions and on the other we impose the mixed boundary conditions discussed for BF theory.

Consequently, using the multiplication properties for the $\mathbb{C}[G]$ matrices (4.80), we find

$$\chi_{\text{cyl.},\,h}(e^{-\beta H}) = \sum_R \chi_R(h) e^{-\frac{\tilde{e}_b \beta C_2(R)}{2}} \text{Tr}_{(\dim R)N}(e^{-\beta H_R})\,. \tag{4.84}$$

When $h = e$ and $\tilde{e}_b = 0$, one finds that $\chi_{\text{cyl.},\,h}(H) = \chi_{\text{el}}(H)$ and this will correspond to imposing Dirichlet boundary conditions on the boundary on the gravitational gauge theory. The role of the trace (4.82) is to reproduce results when setting mixed boundary conditions for each boundary of $\mathcal{M}_{g,n}$ in the genus expansion of the partition function in the gravitational gauge theory.

We start by checking that by using the matrix ensemble given by (4.76), or equivalently (4.81), together with the new definition of the trace we are able to reproduce this expansion for surfaces with a single boundary ($n = 1$). Using (4.81), we find that in comparison to the initial regular matrix integral the one-point function of $Z_{\text{cyl.}}(h, \beta)$ becomes

$$\begin{aligned}
\langle Z(\beta)\rangle_{\text{conn.}} &\simeq \sum_{g=0}^{\infty} \frac{\tilde{Z}_{g,1}(\beta)}{N^{\chi(\mathcal{M}_{g,n})}} \\
&\xRightarrow[\substack{H_{ij}\to H_{(i,g),(j,h)} \\ \text{Tr}(\dots)\to\chi_{\text{el}}(\dots)}]{} \langle Z_{\text{cyl.}}(h,\beta)\rangle_{\text{conn.}} = \sum_R \chi_R(h) e^{-\frac{\tilde{e}_b\beta C_2(R)}{2}} \langle \text{Tr}_{(\dim R)N} e^{-\beta H_R}\rangle \\
&\simeq \sum_{g=0}^{\infty}\sum_R (\dim R\, N)^{\chi(\mathcal{M}_{g,1})} \chi_R(h) e^{-\frac{\tilde{e}_b\beta C_2(R)}{2}} \tilde{Z}_{g,1}(\beta)\,,
\end{aligned} \tag{4.85}$$

where $\tilde{Z}_{g,n}(\beta_j)$ are the factors appearing in the genus expansion of the regular matrix integral (1.30). Replacing $N \to e^{S_0}$ as the expansion parameter in the double-scaling limit, and using the matrix integral discussed in [37], the coefficients $\tilde{Z}_{g,1}(\beta_j)$ in (4.85) become $Z_{g,1}(\beta_j)$ which gives the contribution of surfaces of genus g with n-boundaries to the JT gravity path integral. Thus, we find that in the double-scaling limit the perturbative expansion (4.85) matches the genus expansion in the weakly coupled gravitational gauge theory (4.65) when $n = 1$.

Next, we check that the genus expansion of the gravitational gauge theory and the matrix integral matches for surfaces with an arbitrary number of boundaries. In order to obtain a match, we need to specify what to do with the holonomies appearing in the traces (4.82). The procedure is to associate each holonomy to the boundary of a separate disk; in order to obtain a single surface with n-boundaries it is necessary to glue the boundaries of the n-disks, such that the holonomy of the resulting n-boundaries are $h_1, \dots, h_n$. This is precisely the same procedure used to glue n disks into an n-holed sphere in Yang-Mills or BF-theory. Such a gluing implies that instead of having a separate sum over irreducible representations for each insertion of $Z_{\text{cyl.}}(h_j, \beta_j)$, we obtain a unique sum over R. We denote correlation functions after performing such a gluing as $\langle \dots \rangle^{\text{glued}}(h_1, \dots, h_n)$.

Thus, we find that the matrix integral results from pure JT gravity are modified such that[24]

$$
\begin{aligned}
&\xRightarrow[\substack{H_{ij}\to H_{(i,g),(j,h)} \\ \mathrm{Tr}(\dots)\to\chi_{\mathrm{el}}(\dots)}]{} \quad \langle Z_{\text{cyl.}}(\beta_1)\dots Z_{\text{cyl.}}(\beta_n)\rangle^{\text{glued}}_{\text{conn.}}(h_1,\,\dots,\,h_n) \simeq \\
&\qquad \simeq \sum_R \chi_R(h_1)\dots\chi_R(h_n)\, e^{-\frac{C_2(R)\sum_{i=j}^n \tilde{e}_{b_j}\beta_j}{2}} \langle \mathrm{Tr}_{(\dim R)N} e^{-\beta_1 H_R}\dots \mathrm{Tr}_{(\dim R)N} e^{-\beta_n H_R}\rangle \\
&\qquad = \sum_{g=0}^{\infty}\sum_R (\dim R\, e^{S_0})^{\chi(\mathcal{M}_{g,n})}\chi_R(h_1)\dots\chi_R(h_n)\, e^{-\frac{C_2(R)\sum_{i=j}^n \tilde{e}_{b_j}\beta_j}{2}} Z_{g,n}(\beta_1,\dots,\beta_n)\,, \quad (4.86)
\end{aligned}
$$

where the dependence on ϕ_{b_j} is realized through the overall re-scaling of the proper length β_j associated to each boundary. Of course one can use the second line in (4.86) as the definition of the observable in the collection of matrix integrals (4.81).

Thus, if we consider the matrix integral associated to the $(2,p)$ minimal string [48] in the $p\to\infty$ limit [37] and if we promote the matrix H to be of the form (4.73), we find we can reproduce the genus expansion in the gravitational gauge theory with the mixed boundary conditions (4.66) for the gauge field (or with Dirichlet boundary conditions when $\tilde{e}_{b_j}=0$ for all j).

Modifying the matrix integral: arbitrary gauge couplings

Similarly, we can reproduce the genus expansion with arbitrary gauge couplings $\tilde{e}$ and $\tilde{e}_\phi$ for asymptotically AdS_2 ($\varepsilon\to 0$) boundaries by modifying the matrix integral (4.76). We start by considering mixed boundary conditions for the gauge field. Instead of taking the trace in the elementary representation we can consider the more general trace for the matrix H:

$$
\begin{aligned}
\chi_{\text{YM}}(H) &\equiv \int dg\, Z^{\text{disk}}_{\text{YM}}(g^{-1})\sum_{i=1}^N (H)_{i,i}(g) = \sum_{i=1}^N\sum_R \int dg\,(\dim R)\chi_R(g^{-1})(H)_{i,i}(g)\, e^{2\pi\tilde{e}\tilde{C}_2(R)} \\
&= \sum_R (\dim R)\, e^{2\pi\tilde{e}\tilde{C}_2(R)}\, \mathrm{Tr}_{(\dim R)N}(H_R)\,, \qquad (4.87)
\end{aligned}
$$

[24] In (4.86) when referring to the correlator $\langle Z_{\text{cyl.}}(\beta_1)\dots Z_{\text{cyl.}}(\beta_n)\rangle^{\text{glued}}_{\text{conn.}}$ we have omitted to specify the holonomies associated to the traces $\chi_{\text{cyl.}}(\dots)$ appearing in $Z_{\text{cyl.}}$. That is because there are multiple gluing procedures that can be chosen to obtain a surface with the topology of the n-holed sphere starting from n-disks. We thus only specify the final holonomies $h_1,\,\dots,\,h_n$ along the n-boundaries of $\mathcal{M}_{g,n}$.

where $\tilde{C}_2(R)$ is given by (4.34). In such a case the action of the associated matrix model can be rewritten as,

$$\begin{aligned} S[H] &= N\left[\frac{1}{2}\,\chi_{\rm YM}(H^2)+\sum_{j\geq 3}\frac{t_j}{j}\chi_{\rm YM}(H^j)\right] \\ &= \sum_R N(\dim R)e^{2\pi\tilde{e}\tilde{C}_2(R)}\left[\frac{1}{2}\,{\rm Tr}_{(\dim R)N}(H_R^2)+\sum_{j\geq 3}\frac{t_j}{j}\,{\rm Tr}_{(\dim R)N}(H_R^j)\right]\,, \end{aligned} \tag{4.88}$$

Once again, this is a collection of decoupled matrix models, whose expansion parameter is given by $N(\dim R)e^{\tilde{e}\tilde{C}_2(R)}$. In order to produce correlators with mixed boundary conditions, we again use the operator insertion $\chi_{{\rm cyl},\,h}(e^{-\beta_j H})$. Thus, compared to the standard $(2,p)$ double-scaled matrix integral in the $p\to\infty$ limit, correlation functions of $Z_{\rm cyl.}(\beta_j)$ become

$$\begin{aligned} &\xRightarrow[\substack{H_{ij}\to H_{(i,g),(j,h)}\\ {\rm Tr}(\dots)\to\chi_{\rm YM}(\dots)}]{}\quad \langle Z_{\rm cyl.}(\beta_1)\dots Z_{\rm cyl.}(\beta_n)\rangle^{\rm glued}_{\rm conn.}(h_1,\,\dots,\,h_n)\simeq \\ &\simeq\sum_{g=0}^{\infty}\sum_R(\dim R\,e^{2\pi\tilde{e}\tilde{C}_2(R)}e^{S_0})^{\chi(\mathcal{M}_{g,n})}\chi_R(h_1)\dots\chi_R(h_n)\,e^{-\frac{C_2(R)\sum_{i=j}^n\tilde{e}_{b_j}\beta_j}{2}}Z_{g,n}(\beta_1,\,\dots,\,\beta_n)\,. \end{aligned} \tag{4.89}$$

Thus, the matrix integral (4.96) together with the cylindrical trace (4.83), describe the partition function of JT gravity coupled to Yang-Mills on surfaces whose boundaries are asymptotically AdS_2 ($\varepsilon\to 0$). However, in section 4.4.2 we have computed the first order correction in ε which has led to the renormalization of the dilaton boundary value (4.37), $\phi_b\Rightarrow\tilde{\phi}_b(R)=\phi_b-\varepsilon\,\tilde{e}\,\tilde{C}_2(R)$. This renormalization changes the density of states that appears in the contribution of disk topologies in each representation sector R, $\rho_0(E)=\frac{\phi_b}{2\pi^2}\sinh(2\pi\sqrt{2\phi_b E})\Rightarrow\rho_0^R(E)=\frac{\tilde{\phi}_b(R)}{2\pi^2}\sinh\left(2\pi\sqrt{2\tilde{\phi}_b(R)E}\right)$. This implies that when setting $\phi_b\equiv 1/2$, if rescale the temperature β_j in each representation sector, such that in the cylindrical trace (4.84) we replace ${\rm Tr}_{R(\dim N)}e^{-\beta H_R}\Rightarrow{\rm Tr}_{R(\dim N)}e^{-\beta\frac{H_R}{1-\varepsilon\tilde{e}\tilde{C}_2(R)}}$, we can reproduce the genus expansion of the partition functions (4.60) and (4.62); as previously mentioned, this accounts for the first order correction in ε to correlators of $Z_{{\rm cyl.},\,h}(\beta)$. Therefore, including this correction in ε simply amounts to correcting the trace (4.83) for the matrix integral operator insertion.

Thus, the equivalence between the genus expansion of correlators in the gravitational gauge theory and the genus expansion of the matrix integral is schematically summarized in figure 4.3.

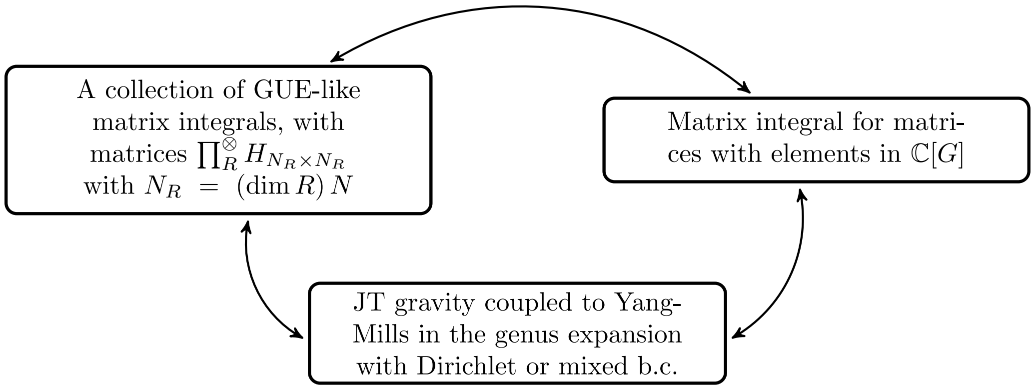


Figure 4.3: Schematic representation of the equivalence between the gravitational gauge theory in the genus expansion, a collection of Hermitian random matrix ensembles $\prod_R^{\otimes} H_{N_R\times N_R}$ and a single Hermitian random matrix ensemble with elements in $\mathbb{C}[G]$.

4.4.4 An interlude: the theory on orientable and unorientable manifolds

In subsection 4.4.3 we have reviewed the relation between the gravitational genus expansion on orientable manifolds and matrix integrals over complex Hermitian matrices [37], for which the symmetry group that acts on the ensemble of such matrices is $\mathrm{U}(N)$ (this is known as the $\beta = 2$ Dyson-ensemble [187], also referred to as GUE). Furthermore, we have shown how these matrix integrals account for the gauge degrees of freedom when considering Hermitian matrices with elements in $\mathbb{C}[G]$ (i.e., G-invariant matrices (4.73) whose complex elements are labeled by two discrete labels and two group elements).

To conclude our discussion about the equivalence between the genus expansions in the gravitational gauge theories and the random matrix ensemble, it is worth schematically mentioning how the results in the previous sections can be modified when also summing over unorientable manifolds. Considering such manifolds in the path integral is relevant whenever the boundary theory has time-reversal symmetry, T [113]. Thus, for pure JT gravity, the matrix integral which reproduces the correct genus expansion should be over matrices in which time-reversal is assumed. The contribution of such surfaces to the partition function and the relation to matrix integrals with time-reversal was studied in [113]. Depending on the way in which one accounts for cross-cap geometries, one obtains two different bulk theories (whose partition function differs by a factor of $(-1)^c$ factor for the contribution of surfaces that include c cross-caps)[25] which are related to two different random

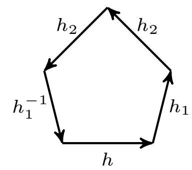
[25] As mentioned in [113], the gravitational computation in fact involves the factor $(-1)^{\chi(\mathcal{M})}$, however, it is convenient to replace the factor $(-1)^{\chi(\mathcal{M})}$ by $(-1)^c$. As noted in [113], the factors $(-1)^{\chi(\mathcal{M})}$ by $(-1)^c$ differs by a minus sign for each boundary component, since $2-2g$ is always an even number. This replacement serves to make a more clear map between JT gravity and random matrix resolvents.

matrix ensembles [187]: (i) if $\mathsf{T}^2 = 1$ then the integral was shown to be over real symmetric matrices ($H_{ij} = H_{ji}$) for which the associated group is $\mathrm{O}(N)$ (labeled as the $\beta = 1$ Dyson-ensemble or as GOE-like); (ii) if $\mathsf{T}^2 = -1$ then the associated group is $\mathrm{Sp}(N)$ (labeled as the $\beta = 4$ Dyson-ensemble or as GSE-like).

As was shown in [188, 189, 113], the volume of the moduli space of unorientable manifolds has a divergence appearing from the contribution of geometries that include small cross-caps. A similar divergence is found in the relevant double-scaled matrix integral, predicting the correct measure for the cross-caps, but impeding the study of arbitrary genus correlators [113]. Nevertheless, when coupling the gravitational theory to Yang-Mills theory, we can still determine the contribution of the gauge degrees of freedom in the genus expansion of partition function even if the volume of the moduli space is divergent. On the matrix integral side, we can also understand how to modify the random matrix ensembles (i) or (ii) to account for this contribution (however, for matrix integrals we will focus on (i)).

We start by analyzing the path integral in the gravitational gauge theory over both orientable and unorientable surfaces. As before, the contribution of the gauge degrees of freedom to the partition function of the gravitational gauge theory is simply given by dressing the gravitational contribution $Z_{\mathcal{M}}^{(\beta=1,4)}$ by the appropriate representation dependent factors. Here, $Z_{\mathcal{M}}^{(\beta=1,4)}$ is the contribution of manifolds with the topology of $\mathcal{M}$ to the pure JT gravity path integral. Since we are also summing over orientable manifolds, the partition function already includes all the terms in (4.62), but also includes the contributions from unorientable manifolds which can always be obtained by gluing together surfaces with the topology of trumpets, three-holed spheres, punctured Klein bottles and cross-cap geometries (punctured $\mathbb{RP}^2$) [103]. Thus, we label such surfaces by $\mathcal{M}_{g,n,s,c}$, where s is the number of Klein bottles and c is the number of cross-caps.

When gluing together only trumpets, three-holed spheres, and Klein bottles, the contribution of the gauge fields exactly follows from (4.26) [103], accounting for the contribution of the Klein bottles to the Euler characteristic and only including the sum over representations that are isomorphic to their complex conjugates, $R = \overline{R}$ (real or quaternionic). The non-trivial contribution comes from the gluing of cross-cap geometries. Therefore, we first consider the example of a trumpet geometry, glued to a cross-cap and will then generalize our derivation to surfaces with arbitrary topology. To understand the contribution to the path integral in pure Yang-Mills theory of a surface with the topology of a cross-cap, it is useful to understand how to construct such a surface by gluing a 5-edged polygon [103]. Specifically, introducing the holonomies h_1 and h_2, the cross-cap can be constructed by gluing the edges of the polygon [103]:



(4.90)

Above, h is the holonomy on the resulting boundary of the cross-cap. Thus, the contribution of a single cross-cap glued to a trumpet whose boundary is asymptotically AdS_2 is schematically given by

$$Z^{(0,1,0,1)}_{\substack{\text{JTYM}\\ \text{mixed}}}(\phi_b,\beta,h) = e^{S_0\chi(\mathcal{M}_{0,1,0,1})}\int Dg^{\mu\nu}\delta\left(\mathcal{R}+2+\tilde{e}_\phi C_2(R)\right)e^{\int du\sqrt{g_{uu}}\phi\mathcal{K}}$$
$$\times\left(\sum_R(\dim R)\,e^{-\tilde{e}_b\beta C_2(R)-\frac{\tilde{e}\,C_2(R)\int_{\mathcal{M}_{0,1,0,1}}d^2x\sqrt{g}}{2}}\int dh_1dh_2\chi_R(hh_1h_2^2h_1^{-1})\right), \tag{4.91}$$

where $\mathcal{M}_{0,1,0,1}$ are surfaces with cross-cap topology (equivalent to $\mathbb{RP}^2$ with a puncture) that has genus 0, 1 boundary, 0 Klein bottles and, of course, 1 cross-cap component. Consequently, $\chi(\mathcal{M}_{0,1,0,1})=0$. Above, the measure over the gravitational degrees of freedom of course depends on whether the bulk theory is defined to weight cross-cap geometries by a factor of $(-1)^c$.

After integrating out h_1 we are left with the group integral $\int dh_2\chi_R(h_2^2)$. Thus, in order to compute (4.91) we need to identify the Frobenius-Schur indicator for the representations R of the compact Lie group G:

$$f_R=\int dh\,\chi_R(h^2)\,,\qquad f_R=\begin{cases}1 & \exists\text{ symm. invar. bilinear form } R\otimes R\to\mathbb{C}\,,\\ -1 & \exists\text{ anti-symm. invar. bilinear form } R\otimes R\to\mathbb{C}\,,\\ 0 & \nexists\text{ invar. bilinear form } R\otimes R\to\mathbb{C}\,.\end{cases} \tag{4.92}$$

Such an invariant bilinear form exists if and only if $R=\overline{R}$. The representation is real, $R\in\widehat{G}_1$, if $f_R=1$ and quaternionic (equivalent, to a pseudo-real irreducible representation), $R\in\widehat{G}_4$, if $f_R=-1$. When the representation R is complex, $R\in\widehat{G}_2$ and $f_R=0$.

Integrating out the the gauge field we thus find that the contribution of a single cross-cap-trumpet, with holonomy h, is given by

$$Z^{(0,1,0,1)}_{\substack{\text{JTYM}\\ \text{Dirichlet}}}(\phi_b,\beta,h)=\sum_R f_R\,\chi_R(h)\left(\dim R e^{S_0}e^{2\pi\tilde{e}\tilde{C}_2(R)}\right)^{\chi(\mathcal{M}_{0,1,0,1})}e^{-\tilde{e}_b\beta C_2(R)}Z_{0,1,0,1}(\beta/\phi_b)\,, \quad (4.93)$$

where $Z_{0,1,0,1}(\beta/\phi_b)$ is the (divergent) contribution of the cross-cap topologies to the partition function [113]. As previously mentioned, depending on the definition of the bulk theory $Z_{0,1,0,1}(\beta/\phi_b)$ could differ by an overall sign for this cross-cap geometry.

Thus, when gluing this cross-cap geometry to other surfaces, we dress the gravitational results by the factors appearing in (4.93). Thus, the result in the gravitational gauge theory can be obtained from the result in pure JT gravity, by introducing a sum over representations, dressing the entropy factor $e^{S_0}\to\dim R e^{S_0}e^{\tilde{e}C_2(R)}$, introducing a factor $(f_R)^c$ for geometries with c cross-caps, replacing the boundary value of the dilaton $\phi_b\to\tilde{\phi}_b(R)$ and adding the terms corresponding to the introduction of the boundary condition changing defect (or to the use of mixed bounday conditions) introduced in section 4.3. Thus, the result from pure JT gravity over orientable and unorientable manifolds becomes

$$\begin{aligned}
&Z^{n,\,(\beta=1,4)}_{\text{JT}}(\phi_{b_j},\beta_j)=\sum_{\substack{\mathcal{M}_{g,n,s,c}\\ n\text{ fixed}}}e^{S_0\chi(\mathcal{M}_{g,n,s,c})}Z^{(\beta=1,4)}_{g,n,s,c}(\beta_j/\phi_{b_j})\\
&\xRightarrow[\text{term}]{\text{adding Yang-Mills}}\quad Z^{n,\,(\beta=1,4)}_{\text{JTYM}}(\phi_{b_j},\beta_j)=\sum_R\Bigg[\Bigg\{\sum_{\substack{\mathcal{M}_{g,n}\\ n\text{ fixed}}}(\dim R e^{2\pi\tilde{e}\tilde{C}_2(R)}e^{S_0})^{\chi(\mathcal{M}_{g,n})}e^{-\frac{C_2(R)}{2}\left(\sum_{j=1}^n\tilde{e}_{b_j}\beta_j\right)}\\
&\qquad\times Z^{(\beta=1,4)}_{g,n,s,c}(\beta_j/\tilde{\phi}_{b_j}(R))\Bigg\}+\Bigg\{\sum_{\substack{\mathcal{M}_{g,n,s,c}\\ \text{unorientable}\\ n\text{ fixed}}}(f_R)^c(\dim R e^{2\pi\tilde{e}\tilde{C}_2(R)}e^{S_0})^{\chi(\mathcal{M}_{g,n,s,c})}\\
&\qquad\times e^{-\frac{C_2(R)}{2}\left(\sum_{j=1}^n\tilde{e}_{b_j}\beta_j\right)}Z^{(\beta=1,4)}_{g,n,s,c}(\beta_j/\tilde{\phi}_{b_j}(R))\Bigg\}\Bigg]\,,
\end{aligned}\quad (4.94)$$

where the first sum in the first parenthesis is over all orientable manifold $\mathcal{M}_{g,n}$ and the sum in the second parenthesis is over all distinct topologies among the manifolds $\mathcal{M}_{g,n,s,c}$ which are unorientable. Above, the number of boundaries n is kept fixed.

Only real and quaternionic representations appear in the contribution of unorientable manifolds to the path integral since $f_R=0$ for complex representations. In fact, due to the factor $(f_R)^c$, switching between the $\beta=1$ and $\beta=4$ bulk definitions is equivalent to switching the role of real and quaternionic representations.

As mentioned previously, the contributions from all geometries which contain a cross-cap have a divergence appearing from small cross-caps, and thus, in practice, the contribution of higher genus or demigenus unorientable surfaces is impossible to compute. Nevertheless, we can still formally reproduce the genus expansion over orientable and unorientable surfaces from matrix integrals. For simplicity, we only discuss the limit $\varepsilon \to 0$, in which we consider $\tilde{\phi}_b(R) = \phi_b \equiv 1/2$. Once again, for this normalization, we use the shorthand notation $Z^{(\beta=1,4)}_{g,n,s,c}(\beta_j) \equiv Z^{(\beta=1,4)}_{g,n,s,c}(\beta_j/\phi_{b_j})$. We also focus on the case in which we start from a GOE-like matrix integral ($\beta = 1$), for which matrices are real and symmetric.

Our starting point is once again the same general matrix potential from subsection 4.4.3, however, we now consider matrices whose elements are real functions on the group manifold G (describing the real group algebra, $\mathbb{R}[G]$), instead of complex functions; i.e. they are G-invariant matrices (4.73) that have real elements which are labeled by two discrete labels and two group elements. Similar to our derivation for $\mathbb{C}[G]$, we wish to decompose $\mathbb{R}[G]$, accounting for the contribution of each representation R. Using the trace (4.87), we conclude that the decomposition is given by[26]

$$\chi_{\mathrm{YM}}(H) \equiv \int dh\, Z^{\mathrm{disk}}_{\mathrm{YM}}(h^{-1}) \sum_{i=1}^{N} (H)_{i,i}(h) = \sum_{\substack{R_i \in \widehat{G}_1 \\ i=1,2,4}} (\dim R_i)\, e^{2\pi \tilde{e} \tilde{C}_2(R_i)}\, \mathrm{Tr}_{(\dim R_i)N}(H_{R_i})\,, \qquad (4.95)$$

where $\widehat{G}_1$ are all the real unitary irreducible representations of G, $\widehat{G}_2$ are all the complex ones and $\widehat{G}_4$ are all the quaternionic (pseudo-real) representations of G. Consequently, the symmetry groups associated to the matrices H_{R_i} follow from the properties of $U^{k_i}_{R_i,l_i}(h)$: H_{R_1} is GOE-like, H_{R_2} is GUE-like and H_{R_4} is GSE-like (also known as a quaternionic matrix) [176, 177, 178, 190]. Similarly, the same decomposition follows for any power of H, following the convolution properties (4.80). The matrix model (4.96) thus becomes

$$\begin{aligned} S[H] &= N \left[\frac{1}{2}\, \chi_{\mathrm{YM}}(H^2) + \sum_{j \geq 3} \frac{t_j}{j} \chi_{\mathrm{YM}}(H^j) \right] \\ &= \sum_{\substack{R_i \in \widehat{G}_i \\ i=1,2,4}} N (\dim R_i) e^{2\pi \tilde{e} \tilde{C}_2(R_i)} \left[\frac{1}{2}\, \mathrm{Tr}_{(\dim R_i)N}(H^2_{R_i}) + \sum_{j \geq 3} \frac{t_j}{j}\, \mathrm{Tr}_{(\dim R_i)N}(H^j_{R_i}) \right]. \end{aligned} \qquad (4.96)$$

The appropriate choice of measure for the initial path integral $dH(g)$ decomposes to give the standard GOE-like matrix integral measure for H_{R_1}, the GUE-like measure for H_{R_2} and the GSE-like measure for H_{R_4}. Once again we find that the matrix integral over $H_{ij}(g)$ is equivalent to a collection of matrix integrals, where each integral is associated to a unitary irreducible representation R and the associated symmetry group to each matrix is set by the reality of this representation. As was the case

[26] Once again, [176, 177, 178] list a similar decomposition to (4.95) for finite groups.

for $\mathbb{C}[G]$, all the results presented so far in this subsection are due to the harmonic decomposition of our matrices whose elements in $\mathbb{R}[G]$.

Compared to the (formal) topological expansion of correlators of $Z(\beta_j)$ in the matrix integral associated to pure JT gravity, the expansion of correlators of the thermal partition sum $Z_{\text{cyl.}}(\beta_j) = \chi_{\text{cyl.}}(e^{-\beta_j H})$ becomes,

$$\begin{aligned}
&\langle Z(\beta_1)\dots Z(\beta_n)\rangle^{(\beta=1)} = \sum_{\substack{\mathcal{M}_{g,n,s,c}\\ n \text{ fixed}}} e^{S_0 \chi(\mathcal{M}_{g,n,s,c})} Z^{(\beta=1)}_{g,n,s,c}(\beta_j/\phi_{b_j}) \\
&\xrightarrow[\substack{H_{ij}\to H_{(i,g),(j,h)}\\ \text{Tr}(\dots)\to\chi_{\text{YM}}(\dots)}]{} \quad \langle Z_{\text{cyl.}}(\beta_1)\dots Z_{\text{cyl.}}(\beta_n)\rangle^{\text{glued},\,(\beta=1)}_{\text{conn.}}(h_1,\,\dots,\,h_n) \simeq \\
&\simeq \Bigg[\sum_{\substack{\mathcal{M}_{g,n,s,c}\\ \text{orientable \&}\\ \text{unorientable}}} \sum_{\substack{R_i\in\widehat{G}_i\\ i=1,4}} (f_{R_i})^c (\dim R_i\, e^{2\pi\tilde{e}\tilde{C}_2(R_i)} e^{S_0})^{\chi(\mathcal{M}_{g,n,s,c})} \chi_{R_i}(h_1)\dots\chi_{R_i}(h_n) \\
&\times e^{-\frac{C_2(R_i)\sum_{i=j}^n \tilde{e}_{b_j}\beta_j}{2}} Z^{\beta=1}_{g,n,s,c}(\beta_1,\,\dots,\,\beta_n)\Bigg] + \Bigg[\sum_{\mathcal{M}_{g,n}} \sum_{R_2\in\widehat{G}_2} (\dim R_i\, e^{2\pi\tilde{e}\tilde{C}_2(R_i)} e^{S_0})^{\chi(\mathcal{M}_{g,n})} \\
&\times \chi_{R_2}(h_1)\dots\chi_{R_2}(h_n) e^{-\frac{C_2(R_2)\sum_{i=j}^n \tilde{e}_{b_j}\beta_j}{2}} Z^{\beta=1}_{g,n,s,c}(\beta_1,\,\dots,\,\beta_n)\Bigg]\,. \qquad (4.97)
\end{aligned}$$

Since the matrix integrals over H_{R_1} and H_{R_4} are GOE-like and GSE-like respectively, the sum in the first parenthesis is over all distinct topologies among both the orientable and unorientable manifolds $\mathcal{M}_{g,n,s,c}$. The factor of $(f_{R_i})^c$ precisely accounts for the $(-1)^c$ factor for the GOE and GSE ensembles associated to the integrals over H_{R_1} and, respectively, H_{R_4}. Because H_{R_2} is hermitian, the sum in the second square parenthesis is solely over orientable manifolds. Noting that $f_{R_2} = 0$, for complex representation R_2 it is straightforward to realize that the sums in (4.97) reduce to those in (4.94), in the limit in which $\tilde{\phi}_b(R) = \phi_b$. Thus, we indeed find a (formal) agreement between the matrix integral and the gravitational gauge theory genus expansion. A similar proof is straightforward to derive when starting with a GSE-like matrix integral (and, consequently, using the other definition for the bulk theory).

Thus, we suggest the equivalence between the Euler characteristic expansion of correlators in the gravitational gauge theory, on both orientable and unorientable surfaces, and the expansion in the matrix integral discussed above. This relation is summarized through diagram 4.4. With this generalization in mind, we now return to the usual situation in which we sum solely over orientable manifolds, with the goal to analyze the diffeomorphism and gauge-invariant operators of the theory.

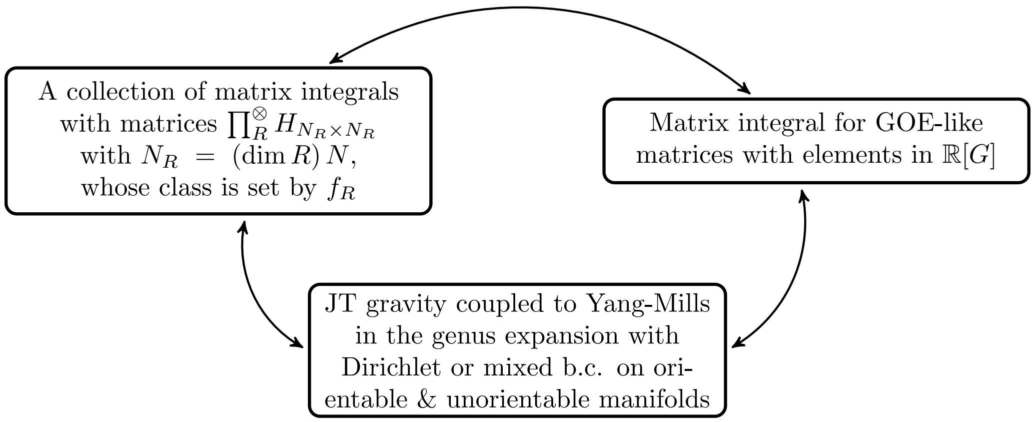


Figure 4.4: Schematic representation of the equivalence between the gravitational gauge theory in the genus expansion on orientable and unorientable surfaces, a collection of random matrix ensembles $\prod_R^{\otimes} H_{N_R \times N_R}$ whose class is specified by f_R and a single GOE-like random matrix ensemble with elements in $\mathbb{R}[G]$.

4.5 Observables

4.5.1 Diffeomorphism and gauge invariance

The goal in this section is to define a set of diffeomorphism and gauge invariant observables in the gravitational BF or Yang-Mills theories. In order to do this it useful to first review how diffeomorphisms act on the zero-form and one-form fields in the theory. Under a diffeomorphism defined by an infinitesimal vector field ξ, the zero form field and the one form field transform as,

$$\begin{aligned} &\phi \to \phi + i_\xi d\phi\,, \\ &A \to A + i_\xi dA + d(i_\xi A) = A + i_\xi F + D_A(i_\xi A)\,, \end{aligned} \tag{4.98}$$

where i_ξ represents the standard map from a p-form to a $(p-1)$-form. Since we are fixing the metric along the boundary, we fix diffeomorphisms on $\partial\mathcal{M}$ to vanish, $\xi|_{\partial\mathcal{M}} = 0$.

To start, we first analyze the possible set of local operators. In Yang-Mills theory, the local operator $\operatorname{Tr}\phi^2(x)$ (which is also proportional to the quadratic Casimir of the gauge group G) is indeed a good diffeomorphism invariant operator since $d\operatorname{Tr}\phi^2(x) = 0$ (also valid as an operator equation). Similarly, all other local gauge-invariant operators are given by combinations of Casimirs of the group G. Since all other Casimirs are constructed by considering the trace of various powers of ϕ, they are also conserved on the entire manifold. Consequently, they also serve as proper diffeomorphism and gauge-invariant observables in the gravitationally coupled Yang-Mills theories.

We also analyze the insertion of non-local operators of co-dimension 1: i.e. Wilson lines and loops,

$$\mathcal{W}_R(\mathcal{C}) = \chi_R\left(\mathcal{P}e^{\int_{\mathcal{C}} A}\right) \tag{4.99}$$

where the meaning of the contour $\mathcal{C}$ will be specified shortly.

Before moving forward with the analysis of correlators for (4.99), we have to require that non-local observables are also diffeomorphism invariant. In the weak gauge coupling limit (BF theory) the path integral localizes to the space of flat connections, and thus the infinitesimal diffeomorphism (4.98) is, in fact, equivalent to an infinitesimal gauge transformation with the gauge transformation parameter given by $\Lambda = i_\xi A$. Since Wilson loops or lines are invariant under bulk gauge transformations, in BF theory they are also invariant under diffeomorphisms (which, of course, also follows from the fact that in BF theory the expectation value of Wilson loops or lines only depends on their topological properties rather than on the exact choice of contour). When computing such correlators in the genus expansion, one has to also specify the homotopy class of the Wilson line or loop. Since the manifolds that we are summing over in the genus expansion, have different fundamental groups and, therefore, different homotopy classes for the Wilson loops(or lines), there is no way to specify the fact that the contour of the loop or line belongs to a particular class within the genus expansion. Of course, the exceptions are the trivial classes in which the contour can always be smoothly contracted to a segment of the boundary (for boundary anchored lines) or to a single point (for closed loops).

An even more pronounced problem appears in Yang-Mills theory where the observable (4.99) is not diffeomorphism invariant, even when placing the theory on a disk; because the path integral no longer localizes to the space of flat connections, the infinitesimal diffeomorphism in (4.98) is no longer equivalent to a gauge transformation. Rather, the expectation value of a Wilson line or loop is affected by performing the infinitesimal diffeomorphism (4.98). Therefore, we are forced to consider generalizations of (4.99) which should be diffeomorphism invariant. Thus, we define the generalized Wilson loops, by summing over all contours (either closed or anchored at two boundary points) on the manifolds $\mathcal{M}_{g,n}$, included in the genus expansion in (4.60) or (4.62):

$$\mathcal{W}_R \equiv \int [d\mathcal{C}]\,\chi_R\left(\mathcal{P}e^{\int_{\mathcal{C}} A}\right)\,, \qquad \mathcal{W}_{\lambda,R} \equiv \int [d\mathcal{C}]\, e^{im\int_{\mathcal{C}} ds\sqrt{g_{\mu\nu}\dot{x}^\mu\dot{x}^\nu}}\chi_R\left(\mathcal{P}e^{\int_{\mathcal{C}} ds\dot{x}^\mu A_\mu}\right)\,. \tag{4.100}$$

where $m^2 = \lambda(1-\lambda)$ the measure $[d\mathcal{C}]$ is chosen such that (4.100) is diffeomorphism invariant.[27]

[27] Instead of expressing our results in terms of the mass m of the particle, it proves convenient to use the SL2 representation λ [115, 165], which is the charge of the particle under AdS_2 isometries.

When considering lines that are anchored, we can fix gauge transformations on the boundary in order for (4.100) to be gauge invariant. When fixing gauge transformations on the boundary, we can consider the more general diffeomorphism and gauge invariant operators[28]

$$\mathcal{U}^{m_2}_{R,m_1} \equiv \int [dC]\, U^{m_2}_{R,m_1}\left(\mathcal{P}e^{\int_C A}\right)\,, \quad \mathcal{U}^{m_2}_{(\lambda,\,R),m_1} \equiv \int [dC]\, e^{im\int_C ds\sqrt{g_{\mu\nu}\dot{x}^\mu\dot{x}^\nu}}\, U^{m_2}_{R,m_1}\left(\mathcal{P}e^{\int_C A}\right)\,, \quad (4.101)$$

where $U^{m_2}_{R,m_1}(h)$ is the a matrix element of the R representation.

The first operators (4.100) and (4.101) can be associated to the worldline path integrals of massless particle charged in the R representation, while the second corresponds to the worldline of a massive particle. Because of this connection, we refer to these operators as "quark worldline operators". In (4.101), we not only specify the representation R but we also specify the states m and n within the representation R in which the quark should be at the two end-points on the boundary; (4.100) is insensitive to the states of the particle at the end-points as long as the two are the same. When the worldlines are boundary anchored and the end-points of the contours C are both kept fix to u_1 and u_2, we denote such operators by $\mathcal{W}_{\lambda,R}(u_1,u_2)$ or by $\mathcal{U}^{m_2}_{R,m_1}(u_1,u_2)$.

For simplicity, in this chapter, we solely focus on the expectation values of the quark worldline operators when the theory is in the weak gauge coupling limit. Moreover, we take the contours associated to the worldlines to be anchored at two fixed points on the boundary and to be smoothly contractable onto the boundary segment in between the two anchoring points.

4.5.2 Local operators

To start, we consider correlation functions of local operators first on surfaces with disk topology, then in the genus expansion, and, in both cases, we determine the equivalent observables on the boundary side.

In section 4.3.3, we have proven that $Z^{\text{disk}}_{\text{JTYM}}(\phi_b,\beta,h) = Z_{\text{Schw}\rtimes G}(\beta,h)$ for both Dirichlet and mixed boundary conditions, for any choice of holonomy of the gauge field $\mathcal{A}_u$. Given this equality, it is straightforward to determine how to reproduce boundary correlators of G-symmetry charges from the bulk perspective. By using functional derivatives with respect to the background gauge field on the boundary side and derivatives with respect to the gauge field $\mathcal{A}_u$ appearing in the boundary

[28] In fact, one only needs to fix gauge transformations at the anchoring points in order for (4.100) and (4.101) to be gauge invariant. The expectation value of such operators in depends on the group elements $h_{j,j+1} = \mathcal{P}e^{\int_{u_j}^{u_{j+1}} A}$, where u_j and u_{j+1} are all the pairs of neighboring anchoring points.

condition for the bulk gauge field, we find the following match:

$$\frac{\delta^k Z^{\text{disk}}_{\text{JTYM}}(\phi_b,\beta,\mathcal{P}e^{\int_{\partial\mathcal{M}}\mathcal{A}})}{\delta\mathcal{A}_u^{a_1}(u_1)\dots\delta\mathcal{A}_u^{a_k}(u_k)} \longleftrightarrow \frac{\delta^k Z_{\text{Schw}\rtimes G}(\beta,\mathcal{P}e^{\int_{\partial\mathcal{M}}\mathcal{A}})}{\delta\mathcal{A}_u^{a_1}(u_1)\dots\delta\mathcal{A}_u^{a_k}(u_k)} = i^k\langle\boldsymbol{\alpha}_{a_1}(u_1)\dots\boldsymbol{\alpha}_{a_k}(u_k)\rangle\,. \tag{4.102}$$

The equivalence above holds when choosing both Dirichlet or mixed boundary conditions for the bulk gauge field and, as presented in subsection 4.3.3, when choosing the appropriate boundary theory. Note that since $\boldsymbol{\alpha}(u)$ is not invariant under background gauge transformations, in (4.102) we should fix $\mathcal{A}_u(u)$ at every point and not only its overall holonomy for any choice of gauge field boundary conditions.

Similarly, we find a match between the conserved G quadratic Casimir in Yang-Mills theory and the conserved G quadratic Casimir on the boundary side:

$$\text{Tr}\phi^2 \longleftrightarrow \text{Tr}\,\boldsymbol{\alpha}^2\,. \tag{4.103}$$

The correlators or such operators are obtained by inserting the G quadratic Casimir in the path integral, to find that[29]

$$\begin{aligned}\langle\text{Tr}\phi^2(x_1)\dots\text{Tr}\phi^2(x_k)\rangle(h) &\propto \sum_R \dim(R)\chi_R(h)(2\,C_2(R))^n\left(\frac{\tilde{\phi}_b(R)}{\beta}\right)^{3/2}\\ &\quad\times e^{\frac{\pi^2\tilde{\phi}_b(R)}{\beta}+2\pi\tilde{e}\tilde{C}_2(R)-\tilde{e}_b\beta C_2(R)} =\\ &= \langle\text{Tr}\,\boldsymbol{\alpha}^2(u_1)\dots\text{Tr}\,\boldsymbol{\alpha}^2(u_n)\rangle\,,\end{aligned} \tag{4.104}$$

where we note that the correlator is independent of the bulk insertion points $x_1, \dots, x_n$ and of the boundary insertion points $u_1, \dots, u_n$.[30] Following the same reasoning, the correlation functions of any gauge invariant operators match:

$$\widehat{V}(\phi) \longleftrightarrow \widehat{V}(\boldsymbol{\alpha})\,. \tag{4.105}$$

Correlation functions such as $\langle\widehat{V}_1(\phi(u_1))\dots\widehat{V}_n(\phi(u_n))\rangle$ can be matched by replacing the factor of the Casimir $(C_2(R))^n$ in (4.104) by $V_1(R)\dots V_n(R)$. Since all diffeomorphism and gauge invariant operators are of the form (4.105) we conclude that the correlation functions of local operators on surfaces with disk topology match those in the boundary theory (4.45).

[29]For brevity, we use $\propto$ to denote the solution to correlators, un-normalized by the partition function in the associated theories.

[30]The factor of 2 in front of the Casimir comes from the normalization $\mathcal{N} \equiv 1/2$.

We now consider such correlators in the genus expansion of orientable surfaces. With mixed boundary conditions for the gauge field in the gravitational gauge theory, such correlators are given by

$$\langle \mathrm{Tr}\phi^2(x_1)\dots \mathrm{Tr}\phi^2(x_k)\rangle(\phi_{b_j},\, \beta_j,\, h_j) \propto \sum_R \chi_R(h_1)\dots\chi_R(h_n) e^{-\frac{\tilde{e}_b C_2(R)\sum_{j=1}^n \beta_j}{2}}$$
$$\times \left[\sum_{g=0}^{\infty}\left(\dim(R) e^{2\pi\tilde{e}\tilde{C}_2(R)} e^{S_0}\right)^{\chi(\mathcal{M}_{g,n})} (2C_2(R))^k\, Z_{g,n}^{(\phi_{b_j}(R))}(\beta_j)\right], \qquad (4.106)$$

when considering surfaces with n-boundaries. For simplicity we assume $\varepsilon \to 0$ such that we take $\tilde{\phi}_{b_j}(R) = \phi_{b_j}$. This result can be reproduced from the random matrix ensemble (4.96) by considering correlators of the operator

$$\begin{aligned}\chi_{\mathrm{Tr}\phi^2,h}(e^{-\beta_j H}) &\equiv \int d\tilde{h}\, \langle \mathrm{Tr}\phi^2\rangle^{(0,\,2)}_{\substack{\mathrm{BF}\\ \mathrm{mixed}}}(\tilde{h}^{-1}, h) \sum_{i=1}^N \left(e^{-\beta_j H}\right)_{i,i}(\tilde{h}) \\ &= \int d\tilde{h} \sum_R \chi_R(\tilde{h}^{-1})\chi_R(h)(2\,C_2(R)) e^{-\frac{\tilde{e}_b\beta C_2(R)}{2}} \sum_{i=1}^N \left(e^{-\beta_j H}\right)_{i,i}(\tilde{h}) \\ &= \sum_R \chi_R(h)(-C_2(R)) e^{-\frac{\tilde{e}_b\beta C_2(R)}{2}} \mathrm{Tr}_{(\dim R)N}(e^{-\beta_j H_R})\,, \end{aligned} \qquad (4.107)$$

where $\langle \mathrm{Tr}\phi^2\rangle^{(0,\,2)}_{\substack{\mathrm{BF}\\ \mathrm{mixed}}}$ is the expectation value of the operator $\mathrm{Tr}\phi^2$ on the cylinder ($\mathcal{M}_{(0,2)}$) in the BF-theory with the mixed boundary condition (4.14) on one of the sides of the cylinder and with Dirichlet boundary conditions on the other. Plugging the above into the "glued" matrix integral correlator, we indeed find that[31]

$$\langle \mathrm{Tr}\phi^2(x_1)\,\dots\,\mathrm{Tr}\phi^2(x_k)\rangle(\phi_{b_j},\,\beta_j,\,h_j) = \langle \chi_{\mathrm{Tr}\phi^2}\,(e^{-\beta_1 H})\,\dots\,\chi_{\mathrm{Tr}\phi^2}\,(e^{-\beta_k H})\rangle^{\mathrm{glued}}_{\mathrm{conn.}}(\phi_{b_j},\,\beta_j,\,h_j) \qquad (4.108)$$

Similarly, by modifying the trace function in (4.107) by replacing $\mathrm{Tr}\phi^2$ by the arbitrary function $V(\phi)$, we can prove that for all gauge and diffeomorphism invariant observables on the boundary side one can construct the equivalent set of operators on the matrix integral side.

4.5.3 Quark worldline operators in the weakly coupled limit

Since we have discussed the correlators of all gauge-invariant local operators, we can now move-on to computing the expectation value of the aforementioned quark worldline operators (4.100) and (4.101). As previously stated, in this subsection we solely consider boundary anchored quark

[31] Once again we omit to specify the holonomies associated to the traces $\chi_{\mathrm{Tr}\phi^2}(\dots)$. See footnote[24].

worldlines in the weakly coupled topological limit, with the mixed boundary conditions studied in section 4.2. We again start by studying surfaces with disk topology and then discuss correlators of such operators in the genus expansion. For higher genus manifolds, we only consider massless quark worldline operators whose contours have both endpoints on the same boundary. Moreover, we solely consider contours that can be smoothly contractible to a segment on the boundary when keeping these boundary endpoints fixed.

Considering the weak gauge coupling limit offers two advantages.

The first is that the expectation value of operators with self-intersecting contours C is the same as the expectation value of operators with contours $\tilde{C}$ that have the same endpoints and are not self-intersecting; i.e., there is a smooth transformation taking C and $\tilde{C}$ which vanishes at the endpoints.[32] Therefore, in the weak gauge coupling limit, we only have to consider the expectation value of lines that are not self-intersecting.

The second advantage of the weak gauge coupling limit is that on surfaces with disk topology, the contribution of the gauge field in the worldline operators (4.100) and (4.101) can be factorized:

$$\mathcal{U}^{m_2}_{(\lambda,\,R),m_1} = \int [dC]\, e^{im\int_C ds\sqrt{g_{\mu\nu}\dot{x}^\mu\dot{x}^\nu}}\, U^{m_2}_{R,m_1}\left(\mathcal{P}e^{\int_C A}\right) = \left(\int [dC]\, e^{im\int_C ds\sqrt{\dot{x}_\mu\dot{x}^\mu}}\right) U^{m_2}_{R,m_1}\left(\mathcal{P}e^{\int_{\tilde{C}} A}\right). \tag{4.109}$$

The above equation holds for any contour choice $\tilde{C}$ which has the same end-points as the contours C. Correlators of $\mathcal{O}_\lambda(C) \equiv \left(\int [dC]\, e^{im\int_C ds\sqrt{g_{\mu\nu}\dot{x}^\mu\dot{x}^\nu}}\right)$ have been studied in pure JT gravity on disk topologies in [121, 1]. Such operators were shown to be equivalent to Wilson lines in a BF theory with $\mathfrak{sl}(2,\mathbb{R})$ gauge algebra. In turn, the expectation value of such lines were shown to match correlation functions of bi-local operators in the Schwarzian theory [40, 64, 191, 1],

$$\mathcal{O}_\lambda(\mathcal{C}) \equiv \left(\int [dC]\, e^{im\int_C ds\sqrt{g_{\mu\nu}\dot{x}^\mu\dot{x}^\nu}}\right) \quad \leftrightarrow \quad \mathcal{O}_\lambda(u_1,u_2) \equiv \left(\frac{F'(u_1)F'(u_2)}{|F(u_1)-F(u_2)|^2}\right)^\lambda, \tag{4.110}$$

where $F(u)$ is the Schwarzian field and u_1 and u_2 are the locations of the end-points for the countours C.

Thus, by using the correlator functions of Wilson lines in $\mathfrak{sl}(2,\mathbb{R})$ BF theory,[33] together with

[32] As previously discussed, when quantizing BF-theory each patch has an associated irreducible representation R. As we will summarize shortly, for each Wilson line intersection, one associates a $6j$-symbol of the group G which includes the four representation associated to the patches surrounding the intersection and the two representations associated to the two lines. When the line is self-intersecting, one instead uses two copies of the representation associated with that line. The fact that Wilson lines with the contour $\mathcal{C}$ and $\tilde{\mathcal{C}}$ have the same expectation value follows from orthogonality properties of the $6j$-symbol.

[33] Or, equivalently, the expectation value of bi-local operators in the Schwarzian theory [40].

the expectation value of boundary anchored non-intersecting Wilson lines in G-BF theory,[34] we determine arbitrary correlators of quark worldlines in the weak gauge coupling limit on surfaces with disk topology. Using closely related techniques, we then move-on to the genus expansion when setting the mass of the quark to $m = 0$.

A single line on the disk

When fixing the boundary conditions for the gauge field to be given by (4.14), the expectation value of a boundary anchored quark worldline operator can be computed in two different ways.

The first follows the reasoning presented in subsection 4.2.1: we reduce the bulk path integral in the presence of a quark worldline operator to a boundary path integral. Such a reduction was studied in the case of pure BF theory in [120, 192, 121]. As mentioned in subsection 4.2.1 the path integral over the zero-form field ϕ imposes a restriction to the space of flat connections, $A = q^{-1}dq$. For such configuration the path-ordered integral that appears in the Wilson-line becomes $\mathcal{P}e^{\int_C A} = q^{-1}(u_2)q(u_1)$, for any contour C whose end-points are u_1 and u_2. Similarly, one can show that the the Wilson line in the $\mathfrak{sl}(2,\mathbb{R})$ BF-theory reduces to the bi-local operator (4.110). Thus in the boundary path-integral (4.17), we need to insert the operator $U^{m_2}_{R,m_1}(q^{-1}(u_2)q(u_1))$:[35]

$$\langle \mathcal{U}^{m_2}_{(\lambda,R),m_1}(u_1,u_2)\rangle \propto \left[\int DqD\boldsymbol{\alpha}\, U^{m_2}_{R,m_1}(q^{-1}(u_2)q(u_1))\; e^{\int du\left(i\mathrm{Tr}\,(\boldsymbol{\alpha}\, q^{-1}D_{\mathcal{A}}q)+\sqrt{g_{uu}}\frac{\varepsilon\tilde{e}_b}{2}\mathrm{Tr}\;\boldsymbol{\alpha}^2\right)}\right] \\ \times\left[\int DF\,\mathcal{O}_\lambda(u_1,u_2)e^{\int_0^\beta du\mathrm{Sch}(F,u)}\right]. \tag{4.111}$$

The path integral in the first parenthesis was computed in [121, 120] when the background gauge field $\mathcal{A}_u = 0$. Nevertheless, we follow the same reasoning as in [121, 120] to solve the path integral for an arbitrary background. By using the quantization procedure from subsection 4.2.2 and using $U^{m_2}_{R,m_1}(q^{-1}(u_2)q(u_1)) = U^{m_2}_{R,p}(q^{-1}(u_1))U^p_{R,m_1}(q(u_2))$, we find that the first square parenthesis can be rewritten as [121, 120]

$$\langle U^{m_2}_{R,m_1}\rangle_G \equiv \mathrm{Tr}_{\,\mathcal{H}^G} U^{m_2}_{R,p}(q^{-1}(u_1))h_{12}e^{-u_{12}H}U^p_{R,m_1}(q(u_2))h_{21}e^{-u_{21}H}\,, \tag{4.112}$$

where $h_{12} = \mathcal{P}e^{\int_{u_1}^{u_2}\mathcal{A}}$ and h_{21} is given by the integral along the complementary segment. Furthermore, we have simplified notation by denoting $u_{ij} = |u_i - u_j|$ for $i > j$ and $u_{ji} = |\beta - u_j + u_i|$. By

[34] The expectation value of boundary anchored Wilson lines in the more general Yang-Mills theory with gauge group G were studied in [120, 192, 121].

[35] Note that because the action in the first path integral in (4.111) is invariant under 1d diffeomorphisms, one can equivalently use the AdS_2 coordinate given by the Schwarzian field $F(u)$ to parametrize the boundary and the anchoring points.

inserting the complete basis of eigenstates of the Hamiltonian H at various locations in (4.112) one can easily compute the expression above [120, 121].

Before, discussing the final result (4.116) of the path integral in (4.111), we briefly summarize how one can compute the expectation value of $\mathcal{U}^{m_2}_{(\lambda,R),m_1}(u_1,u_2)$ by directly performing the bulk path integral. By using the fact that the mixed boundary conditions are equivalent to the insertion of the boundary condition changing defect (4.27), we find that the contribution of the gauge field is given by the gluing formula

$$\langle U^{m_2}_{R,m_1}\rangle_G = \int dh Z^{(0,1)}_{\substack{\text{BF}\\ \text{mixed}}}(u_{12},h_{12}h) U^{m_2}_{R,m_1}(h) Z^{(0,1)}_{\substack{\text{BF}\\ \text{mixed}}}(u_{21},h_{21}h^{-1})\,. \tag{4.113}$$

This, or equivalently (4.112), yields[36]

$$\begin{aligned}\langle U^{m_2}_{R,m_1}\rangle_{\substack{\text{BF}\\ \text{mixed}}} = \langle U^{m_2}_{R,m_1}\rangle_G = \sum_{R_1,\,R_2} (\dim R_1)(\dim R_2)\, e^{-\frac{1}{2}\tilde{e}_b(u_{12}\,C_2(R_1)+u_{21}\,C_2(R_2))} \\ \times \sum_{\substack{p_j,\,q_j=1\\ j=1,2}}^{\dim R_j} \begin{pmatrix} R_1 & R & R_2 \\ p_1 & m_1 & -p_2\end{pmatrix}\begin{pmatrix} R_1 & R & R_2 \\ q_1 & m_2 & -q_2\end{pmatrix} U^{q_1}_{R_1,p_1}(h_{12}) U^{q_2}_{R_2,p_2}(h_{21})\,, \end{aligned}\tag{4.115}$$

where $\begin{pmatrix} R_1 & \tilde{R} & R_2 \\ p_1 & m_1 & -p_2\end{pmatrix}$ is the 3j-symbol for the representations R_1, R and R_2 of the group G.

Putting this together with the result for the expectation value of the bi-local operator in the Schwarzian theory [40] or, equivalently, for the expectation value of a Wilson line in an $\mathfrak{sl}(2,\mathbb{R})$ BF-theory [121, 1], we find that

$$\begin{aligned}\langle \mathcal{U}^n_{(\lambda,R),m}(u_1,u_2)\rangle \propto \int ds_1\rho_0(s_1)\, ds_2\rho_0(s_2)\tilde{N}^{s_2}{}_{s_1,\lambda} \sum_{R_1,\,R_2} (\dim R_1)(\dim R_2)\, e^{-u_{12}\left(\frac{s_1^2}{2\phi_b}+\frac{\tilde{e}_b C_2(R_1)}{2}\right)} \\ \times\, e^{-u_{21}\left(\frac{s_2^2}{2\phi_b}+\frac{\tilde{e}_b C_2(R_2)}{2}\right)} \sum_{\substack{p_j,\,q_j=1\\ j=1,2}}^{\dim R_j} \begin{pmatrix} R_1 & R & R_2 \\ p_1 & m_1 & -p_2\end{pmatrix}\begin{pmatrix} R_1 & R & R_2 \\ q_1 & m_2 & -q_2\end{pmatrix} U^{q_1}_{R_1,p_1}(h_{12}) U^{q_2}_{R_2,p_2}(h_{21}), \end{aligned}\tag{4.116}$$

where $\tilde{N}^{s_2}{}_{s_1,\lambda}$ can be viewed as the fusion coefficient for the principal series repesentations $\lambda_1 =$

[36] Here we have normalized the $3-j$ symbol following [121], such that

$$\int dh U^{m_1}_{R_1,n_1}(h) U^{m_2}_{R_2,n_2}(h) U^{m_3}_{R_3,n_3}(h^{-1}) = \begin{pmatrix} R_1 & R_2 & R_3 \\ m_1 & m_2 & -m_3\end{pmatrix}\begin{pmatrix} R_1 & R_2 & R_3 \\ n_1 & n_2 & -n_3\end{pmatrix}. \tag{4.114}$$

$1/2 + is_1$ and $\lambda_2 = 1/2 + is_2$ and the discrete series representation λ in SL2, given by[37]

$$\tilde{N}^{s_2}{}_{s_1,\lambda} = \frac{|\Gamma(\lambda + is_1 - is_2)\Gamma(\lambda + is_1 + is_2)|^2}{\Gamma(2\lambda)} = \frac{\Gamma(\lambda \pm is_1 \pm is_2)}{\Gamma(2\lambda)} \,. \tag{4.117}$$

A simplifying limit for (4.116) appears when considering the operator $\mathcal{W}_{\lambda,\,\tilde{R}}(u_1, u_2)$, with $\mathcal{A}_u = 0$ all along the boundary ($h_{12} = h_{21} = e$):

$$\begin{aligned}\langle\, \mathcal{W}_{\lambda,\,R}(u_1, u_2)\,\rangle \propto \sum_{R_1,\,R_2} (\dim R_1)(\dim R_2) \int ds_1 \rho_0(s_1)\, ds_2 \rho_0(s_2) \\ \times N^{R_2}{}_{R_1,R}\, \tilde{N}^{s_2}{}_{s_1,\lambda} e^{-\frac{1}{2\phi_b}\left[(u_2-u_1)\left(\frac{s_1^2}{2\phi_b}+\frac{\tilde{e}_b C_2(R_1)}{2}\right)+(\beta-u_2+u_1)\left(\frac{s_2^2}{2\phi_b}+\frac{\tilde{e}_b C_2(R_2)}{2}\right)\right]}\,,\end{aligned} \tag{4.118}$$

where $N^{R_2}{}_{R_1,\tilde{R}}$ is the fusion coefficient for the tensor product of representations, $R_1 \otimes \tilde{R} \to N^{R_2}{}_{R_1,\tilde{R}}\, R_2$.

Following the same techniques presented so far we can compute correlation functions of an arbitrary number of quark worldline operators, $\mathcal{U}^n_{(\lambda,R),m}(u_j, u_{j+1})$ or $\mathcal{W}_{\lambda,\,\tilde{R}}(u_j, u_{j+1})$, by performing the bulk path integral directly; alternatively, we can compute the expectation value of operators such as $U^n_{R,m}(q^{-1}(u_j)q(u_{j+1}))\mathcal{O}_\lambda(u_j, u_{j+1})$ on the boundary side. To better exemplify the power of these techniques we give results for two other examples of quark worldline correlators on surfaces with disk topology.

Time-ordered correlators

First we consider the case of multiple boundary anchored lines whose end-points are u_j and u_{j+1} (with $j = 1, 3, \ldots 2n-1$) and the points are ordered as $u_1 \le u_2 \le \cdots \le u_{2n}$. In such a configuration, we find that when setting $\mathcal{A}_u = 0$, the correlation function of $\mathcal{W}_{m_i,\tilde{R}_i}$ is given by

$$\begin{aligned}\langle \prod_{i=1}^n \mathcal{W}_{m_i,\tilde{R}_i}(u_{2i-1}, u_{2i})\,\rangle \propto \sum_{\substack{R_1,\ldots,R_n,\\ R_0}} \int ds_0 \rho(s_0)(\dim R_0) \left(\prod_{i=1}^n ds_i \rho(s_i)(\dim R_i)\right) \\ \times \left(\prod_{i=1}^n N^{R_0}_{R_i,\tilde{R}_i} \tilde{N}^{s_0}{}_{s_i,\lambda_i}\right) e^{-\left(\sum_{i=1}^n u_{2i,2i-1}\left(\frac{s_i^2}{2\phi_b}+\frac{\tilde{e}_b C_2(R_i)}{2}\right)\right)-\left(\beta-\sum_{i=1}^n u_{2i,2i-1}\right)\left(\frac{s_0^2}{2\phi_b}+\frac{\tilde{e}_b C_2(R_0)}{2}\right)}\,.\end{aligned} \tag{4.119}$$

This case corresponds to studying time-ordered correlators of the equivalent boundary operators, $\chi_R(h^{-1}(u_j)h(u_{j+1}))\mathcal{O}_\lambda(u_j, u_{j+1})$.

[37]For details about the computation of (4.117), see [1].

Multiple intersecting lines and out of time-ordered correlators

As our second example we consider the case of two set of boundary anchored worldlines whose end-points are u_1, u_2 and u_3, u_4 and the points are ordered as $u_1 \leq u_3 \leq u_2 \leq u_4$. The Wilson lines associated to the two quark worldlines operators are in a configuration that is homotopically equivalent (when fixing the endpoints) to the case in which the contours of the two lines intersect solely once. Therefore, we solely consider this latter configuration to compute the contribution of the gauge degrees of freedom to the correlator. Once again, we find that when setting $\mathcal{A}_u = 0$ the result simplifies. In particular, the correlation function is given by:

$$\begin{aligned}\langle \mathcal{W}_{m_1,\tilde{R}_1}(u_1,u_2)\mathcal{W}_{m_2,\tilde{R}_2}(u_3,u_4)\rangle \propto & \sum_{R_1,\dots,R_4}\int\left(\prod_{i=1}^{4} ds_i\rho_0(s_i)\dim R_i\right) \\ & \times\sqrt{\tilde{N}^{s_4}{}_{\lambda_1,s_1}\tilde{N}^{s_3}{}_{\lambda_1,s_2}\tilde{N}^{s_3}{}_{\lambda_2,s_1}\tilde{N}^{s_4}{}_{\lambda_2,s_2}}R_{s_3\,s_4}\begin{bmatrix} s_2 & \lambda_2 \\ s_1 & \lambda_1\end{bmatrix}\begin{Bmatrix} R_3 & R_2 & \tilde{R}_2 \\ R_4 & R_1 & \tilde{R}_1\end{Bmatrix}^2 \\ & \times e^{-\left[\left(\frac{s_1^2}{2\phi_b}+\frac{\tilde{e}_bC_2(R_1)}{2}\right)u_{13}+\left(\frac{s_3^2}{2\phi_b}+\frac{\tilde{e}_bC_2(R_3)}{2}\right)u_{32}+\left(\frac{s_2^2}{2\phi_b}+\frac{\tilde{e}_bC_2(R_2)}{2}\right)u_{24}+\left(\frac{s_4^2}{2\phi_b}+\frac{\tilde{e}_bC_2(R_4)}{2}\right)u_{41}\right]},\end{aligned} \tag{4.120}$$

where $\begin{Bmatrix} R_3 & R_2 & \tilde{R}_2 \\ R_4 & R_1 & \tilde{R}_1\end{Bmatrix}$ is the $6-j$ symbol for the representations of the group G and $R_{s_3\,s_4}\begin{bmatrix} s_2 & \lambda_2 \\ s_1 & \lambda_1\end{bmatrix}$ is the $6-j$ symbol for 4 principal and two discrete series representation of SL2.[38]

Moving to higher genus: massless quark worldlines in the genus expansion

To conclude our discussion about non-local operators in the gravitational gauge theory, we move away from the disk topology and compute an example of a quark worldline correlator on the bulk-side. Finally, we again show how this correlator can be reproduced through a matrix integral. Specifically, we consider a boundary anchored quark massless ($m = 0$ and, consequently $\lambda = 0$ or 1) worldline operators with homotopically trivial contours in the weak coupling.[39] By using the gluing procedure described above we find that the correlator for a single quark worldline on a surface with

[38] Once again, for details about the appearance of the SL2 $6j$-symbol, see [1, 130, 131].

[39] The reason we are solely considering correlation functions of massless field is due to the divergence observed in [37] when considering correlation functions of matter fields on higher genus surfaces for which the length of the closed geodesic along which the trumpet is glued has $b \to 0$.

n-boundaries is given by,

$$\langle \mathcal{U}^n_{(0,R),m}(u_1,u_2)\rangle(h_{12},h_{21},h_2,\dots,h_n) \propto \sum_{g=0}^{\infty} Z_{g,n} e^{S_0\chi(\mathcal{M}_{g,n})}\int dh Z^{(0,1)}_{\substack{\text{BF}\\ \text{mixed}}}(h_{12}h) Z^{(g,n)}_{\substack{\text{BF}\\ \text{mixed}}}(h^{-1}h_{21})$$
$$\times U^n_{R,m}(h) = \sum_{g=0}^{\infty} Z_{g,n}\sum_{R_1,R_2}(\dim R_1)\left(\dim R_2\, e^{S_0}\right)^{\chi(\mathcal{M}_{g,n})}\chi_R(h_2)\dots\chi_R(h_n) e^{-\frac{C_2(R)\sum_{j=2}^n e_{b_j}\beta_j}{2}}$$
$$\times e^{-\frac{\tilde{e}_b u_{12}C_2(R_1)}{2}-\frac{\tilde{e}_b u_{21}C_2(R_2)}{2}}\sum_{\substack{p_j,\,q_j=1\\ j=1,2}}^{\dim R_j}\begin{pmatrix} R_1 & R & R_2\\ p_1 & m_1 & -p_2\end{pmatrix}\begin{pmatrix} R_1 & R & R_2\\ q_1 & m_2 & -q_2\end{pmatrix} U^{q_1}_{R_1,p_1}(h_{12})U^{q_2}_{R_2,p_2}(h_{21})\,. \tag{4.121}$$

Here, when $g > 0$, the contours are contractible to the segment of the boundary whose length is u_{12}, with $u_{12}+u_{21}=\beta$. Once again, while on the disk the the contribution of the gauge and gravitational degrees of freedom are factorized, the two theories which are topological in the bulk are once again coupled through the genus expansion. The gluing procedure in (4.121) is easily generalized for any number of quark worldlines whose contours are each contractible to a boundary segment. Specifically, results for time-ordered and out-of-time order correlators easily follow from (4.119) and (4.120), respectively.

It is instructive to understand how such correlators can be reproduced from matrix integrals. For simplicity, we focus on reproducing (4.121) for a single boundary ($n=1$). Once again, we rely on modifying the trace of of operator $e^{-\beta H}$ that we have previously used in the correlator of matrix integrals. Therefore we define

$$\chi_{U^{m_2}_{R,m_1},\,h_{12},\,h_{21}}(e^{-\beta H}) \equiv \int d\tilde{h}\,\langle \mathcal{U}^{m_2}_{R,m_1}\rangle_{\substack{\text{BF}\\ \text{mixed}}}(h_{12},h_{21}\tilde{h}^{-1})\sum_{i=1}^{N}\left(e^{-\beta H}\right)_{i,i}(\tilde{h})$$
$$= \sum_{R_1,R_2} e^{-\frac{\tilde{e}_b}{2}(u_{12}C_2(R_1)+u_{21}C_2(R_2))}\mathrm{Tr}_{(\dim R_2)N}\left(e^{-\beta H_{R_2}}\right) \tag{4.122}$$
$$\times \sum_{\substack{p_j,\,q_j=1\\ j=1,2}}^{\dim R_j}\begin{pmatrix} R_1 & R & R_2\\ p_1 & m_1 & -p_2\end{pmatrix}\begin{pmatrix} R_1 & R & R_2\\ q_1 & m_2 & -q_2\end{pmatrix} U^{q_1}_{R_1,p_1}(h_{12})U^{q_2}_{R_2,p_2}(h_{21})\,,$$

where $\langle \mathcal{U}^n_{R,m}\rangle_{\substack{\text{BF}\\ \text{mixed}}}(h_{12},h_{21}\tilde{h}^{-1})$ is the expectation value of the boundary anchored Wilson line $\mathcal{U}^n_{R,m}(h)$ inserted in a G-BF theory with the mixed boundary conditions (4.14). Using the matrix integral whose action is given by (4.96), it quickly follows that

$$\langle \mathcal{U}^{m_2}_{(0,R),m_1}(u_1,u_2)\rangle^{n=1}_{\substack{\text{JTBF}\\ \text{mixed}}}(h_{12},h_{21}) = \langle \chi_{U^{m_2}_{R,m_1},\,h_{12},\,h_{21}}(e^{-\beta H})\rangle\,. \tag{4.123}$$

The construction of the traces in (4.107), corresponding to the insertion of the local operator $\mathrm{Tr}\phi^2$, and the trace (4.122), corresponding to the insertion of the massless quark worldline operator suggest the general prescription needed in order to reproduce any gauge theory observable in the weak gauge coupling limit. For an operator $\mathcal{O}$, that can be entirely contracted to the boundary of the gauge theory, one can schematically construct the operator

$$\chi_{\mathcal{O}}(e^{-\beta H}) = \int d\tilde{h} \langle \mathcal{O} \rangle_{\substack{\mathrm{BF} \\ \mathrm{mixed}}} (\tilde{h}^{-1}) \sum_{i=1}^{N} \left(e^{-\beta H}\right)_{i,i} (\tilde{h}) \,, \qquad \langle \mathcal{O} \rangle_{\substack{\mathrm{JTBF} \\ \mathrm{mixed}}} = \langle \chi_{\mathcal{O}}(e^{-\beta H}) \rangle \,. \tag{4.124}$$

Of course, it would be interesting to extend this construction and the analysis performed in this subsection to worldline operators which cannot necessarily be contracted to the boundary and when the gauge theory is not necessarily weakly coupled. We hope to report in the future on progress in this direction.

4.6 Outlook

We have managed to quantize JT gravity coupled to Yang-Mills theory, both through the metric and through the dilaton field, when the theory has an arbitrary gauge group G and arbitrary gauge couplings.

When solely looking at surfaces with disk topology, we have found that the theory is equivalent to the Schwarzian coupled to a particle moving on the gauge group manifold. Explicitly, we have computed a great variety of observables in the gravitational gauge theory, ranging from the partition functions presented in section 4.3, to correlators of quark worldline operators discussed in section 4.5. We matched each of them with the proper boundary observable. This boundary theory (the Schwarzian coupled to a particle moving on a group manifold) is expected to arise in the low-energy limit of several disordered theories and tensor models that have a global symmetry G; the argument primarily relies on the fact that the resulting effective theory needs to have an $SL(2,\mathbb{R})\times G$ symmetry.[40] Nevertheless, it would be interesting to understand whether one can derive the potential and coupling to the Schwarzian theory that we have encountered for the particle moving on the group manifold G directly from a specific disordered theory or a particular tensor model.[41]

In parallel to our analysis of surfaces with disk topology, we also computed the same correlators in the genus expansion, when considering orientable surfaces with an arbitrary number of boundaries. For all such correlators, we have found two equivalent matrix integral descriptions. In both, the

[40] See [166, 167, 168, 169, 170, 171, 172, 173, 174] for details

[41] We thank G. Tarnopolskiy for useful discussions about this direction.

starting point was to consider the matrix integral description of the $(2, p)$ minimal string, in the $p \to \infty$ limit. In the first matrix integral description, one promotes the matrix elements $H_{i,j}$ from complex numbers to complex group algebra elements in $\mathbb{C}[G]$. Keeping the couplings in the associated matrix model to be the same, but redefining the traces appearing in the model, after some algebraic manipulation, we obtain the second equivalent matrix integral description.

This description is given by a collection of random matrix ensembles, where each matrix is Hermitian, is associated to a unitary irreducible representation R of G, and has its size is simply proportional to the dimension of the irreducible representation R. Using this latter matrix description, we have found that the genus expansion of correlators in the gravitational gauge theory on surfaces with n boundaries matches the expectation value of n operator insertions $e^{-\beta H}$ in the matrix integral ensemble. Depending on which operators we include in the correlator on the gravitational side, we have shown that one can construct the appropriate trace for the operator $e^{-\beta H}$ on the matrix integral side.

Besides considering correlators in the gravitational gauge theory defined on orientable surfaces, we have also briefly discussed the computation of the partition function of the theory on both orientable and unorientable surfaces. In this case, we have recovered the partition function from a GOE-like matrix integral with matrix elements in $\mathbb{R}[G]$. It would, of course, be interesting to analyze the same more general correlators as those studied in this chapter, both in this gravitational gauge theory and in its associated random matrix ensemble. However, as mentioned in [113], when studying unorientable surfaces, all computations are limited by the logarithmic divergence encountered due to small cross-cap geometries.

Relation to SYK-like models

As discussed in [37], the random matrix statistics encountered when studying pure JT gravity only qualitatively describe some aspects of the SYK model. Similarly, the random matrix ensembles that we have encountered when analyzing the gravitational gauge theory reproduce the same features of SYK models with global symmetries but do not adequately describe the disordered theory. One example in which the matrix integral provides a qualitative description is for the ramp saddle point encountered in SYK [19] which was found to be analogous to the double trumpet configuration from pure JT gravity. When studying an SYK model with global symmetry, one expects similar ramp saddle points in each representation sector; as can be inferred from our results, the contribution of each representation sector to the double trumpet configuration in the gravitational gauge theory indeed reproduces the linearly growing “ramp” contribution to the spectral form factor.

Rewriting $2D$ Yang-Mills theory as a string theory

One significant development in the study of 2d Yang-Mills has been its reformulation as a theory of strings [193, 194]. Furthermore, as presented in [19] and [113], and as reviewed in this chapter, the genus expansion of pure JT gravity is related to the matrix integral obtained from the $(2, p)$ minimal string, in the $p \to \infty$ limit. Consequently, it is natural to ask whether, when coupling 2d Yang-Mills to JT gravity, it is possible to rewrite the partition function or the diffeomorphism and gauge-invariant correlators in this theory as a sum over the branched covers considered in [193, 194].[42]

A further study of correlators

Regarding the classification of all diffeomorphism and gauge-invariant operators in the gravitational gauge theory and the computation of their associated correlators, we have managed to understand all local observables coming from pure Yang-Mills theory and have computed their expectation values. For non-local operators we have defined a set of quark worldline operators which generalize the Wilson lines from pure Yang-Mills theory. The purpose of this generalization was to obtain observables which are diffeomorphism invariant. We have, however, only studied these operators when considering worldlines that are boundary anchored and are smoothly contractible to a segment on the boundary. It would, of course, be interesting to understand how to perform computations for more general topological configurations. This brings up two problems. The first is to determine a way to assign weights in the path integral to the different homotopy classes in which the contours of the boundary anchored worldlines can belong. Such an assignment is well known for worldline path integrals in quantum mechanics [195], however, considering worldlines in the genus expansion in $2d$ quantum gravity adds a layer of complexity. This is because the first homotopy group for surfaces with different genera is, of course, different. The second problem with studying worldline path integrals with topologically non-trivial contours is that for certain homotopy classes such contours are necessarily self-intersecting.[43] Consequently, one needs to develop a bookkeeping device for tracking the $6j$-symbols associated with each intersection that would necessarily appear in the genus expansion.

A further research direction that would lead to a better understanding of quark worldline opera-

[42] In fact, investigating the behavior of 2d Yang-Mills coupled to 2d quantum gravity is an open research direction suggested in the review [104].

[43] For instance, consider a closed curve on the torus $\mathcal{M}_{1,0}$, for which $\pi_1(\mathcal{M}_{1,0}) = \mathbb{Z} \times \mathbb{Z}$. Consider a curve that winds p times around one cycle and q times around the other with $(p, q) \in \pi_1(\mathcal{M}_{1,0})$. Then the minimum number of self-intersections for such a curve is $\gcd(p,\, q) - 1$ [196] for $p, q > 0$.

tors would be to compute their associated correlators beyond the weak gauge coupling limit. Perhaps one can use diffeomorphism invariance to simplify this computation. For instance, by working in a diffeomorphism gauge where the metric determinant $\sqrt{g}$ is concentrated around the boundary and is almost vanishing in the bulk, it might be possible to reduce the computation at arbitrary gauge coupling to the computation at weak gauge coupling.

Chapter 5

Relation to near-extremal black holes

5.1 Outline of results

This chapter expands on the ideas presented in section 1.6 and is organized as follows. In section 5.2, we describe the set-up for Reissner-Nordström black holes, discuss details about the dimensional reduction, dynamics and boundary conditions for massless fields in the near-horizon region. In section 5.3, we reduce the dynamics in the near-horizon region to that of a $1d$ system, the Schwarzian theory coupled to a particle moving on a $U(1) \times SO(3)$ group manifold. We compute the partition function and density of states in such a system in the canonical and grand canonical ensembles, thus obtaining the main result of this chapter in section 5.3.2 and 5.3.3. In section 5.3.4, we also account for deviations from the spinless Reissner-Nordström solution to Kerr-Newman solutions with small spin, in a grand canonical ensemble that includes a chemical potential for the angular momentum (or equivalently, fixing the boundary metric). More details about the connection between the $SO(3)$ gauge field appearing from the dimensional reduction and the angular momentum of the black hole are discussed in appendix C. In section 5.4, we revisit the contribution of massive Kaluza-Klein modes to the partition function. We show their effect is minimal and does not modify the shape of the density of states. Finally, in section 5.5 we summarize our results and discuss future research directions, focusing on possible non-perturbative corrections to the partition function and speculating about the role that geometries with higher topology have in the near-horizon region.

5.2 Near-extremal black hole and JT gravity

In this chapter, we will focus on several kinds of $4d$ black hole solutions. Specifically, in this section, we will consider the Reissner-Nordström black holes solutions and Kerr-Newman solutions of low spin, in both asymptotically AdS_4 spaces and flat spaces. While here we focus on black holes in $D = 4$, the techniques used here apply to a broader set of near-extremal black holes in any number of dimensions.

5.2.1 Setup

In this section we will study Einstein gravity in asymptotically AdS$_4$ coupled to a $U(1)$ Maxwell field. The Euclidean action is given by

$$\begin{aligned} I_{EM} = &-\frac{1}{16\pi G_N}\left[\int_{M_4} d^4x\sqrt{g_{(M_4)}}\,(R+2\Lambda) - 2\int_{\partial M_4}\sqrt{h_{\partial M_4}}K\right] \\ &- \frac{1}{4e^2}\int_{M_4} d^4x\sqrt{g_{(M_4)}}\,F_{\mu\nu}F^{\mu\nu}\,, \end{aligned} \tag{5.1}$$

where $F = dA$ and where we take A to be purely imaginary. The coupling constant of the gauge field is given by e, and $\Lambda = 3/L^2$ denotes the cosmological constant with corresponding AdS radius L. It will be more intuitive to sometimes keep track of G_N by using the Planck length instead, $G_N = \ell_{\text{Pl}}^2$.

The focus of this chapter will be to compute the Euclidean path integral (fixing boundary conditions in the boundary of flat space or AdS_4) around certain background geometries. Throughout this chapter, we fix the boundary metric h_{ij} of the manifold M_4, which requires the addition of the Gibbons-Hawking-York term in (5.1). For the gauge field, we will pick boundary conditions dominated by solutions with a large charge at low temperatures, in the regime where the black hole will be close to extremality. Specifically, the two boundary conditions that we will study will be:

- Fixing the components of A_i along the boundary ∂M_4. With such boundary conditions, (5.1) is a well defined variational problem. As we will see shortly, dimensionally reducing the action (5.1) to $2d$, amounts to fixing the holonomy around the black hole's thermal circle; in turn, this amounts to studying the system in the charge grand canonical ensemble with the holonomy identified as a chemical potential for the black hole's charge.

- We will also be interested in fixing the charge of the black hole, which corresponds to studying the charge microcanonical ensemble. Fixing the charge amounts to fixing the field strength

F_{ij} on the boundary. In this case, we need to add an extra boundary term for (5.1) to have a well defined variational principle [197, 198]

$$\tilde{I}_{EM} = I_{EM} - \frac{1}{e^2} \int_{\partial M_4} \sqrt{h} F^{ij} \, \widehat{n}_i A_j \,, \tag{5.2}$$

where $\widehat{n}$ is outwards unit vector normal to the boundary. To compute the free energy in the case of black holes in AdS_4, we could alternatively add the usual holographic counterterms in the AdS_4 boundary [199, 200]. A detailed analysis of all possible saddles was done in [201]. For our purposes, we will focus on the charged black hole contribution.

To start, we review the classical Reissner-Nordström solution of (5.1), obtained when fixing the field strength on the boundary and consequently the overall charge of the system. The metric is given by

$$ds^2_{(4d)} = f(r)d\tau^2 + \frac{dr^2}{f(r)} + r^2 d\Omega_2^2 \,, \qquad f(r) = 1 - \frac{2G_N M}{r} + \frac{G_N}{4\pi}\frac{Q^2}{r^2} + \frac{r^2}{L^2} \,, \tag{5.3}$$

For concreteness we will pick the pure electric solution with $F = \frac{eQ}{4\pi} * \epsilon_2$, with ϵ_2 the volume form on S^2, while the magnetic solution has $F = \frac{eQ}{4\pi}\epsilon_2$.[1] Such black holes have two horizons r_+ and r_- located at the zeroes of $f(r_\pm) = 0$. We will refer to the larger solution as the actual horizon radius $r_h = r_+$. As a function of the charge, the temperature and chemical potential are given by

$$\beta = \frac{4\pi}{|f'(r_h)|}, \qquad \mu = \frac{e}{4\pi}\frac{Q}{r_h} \,. \tag{5.4}$$

In terms of the chemical potential the vector potential can be written as $A = i\mu \left(1 - \frac{r_h}{r}\right) d\tau$ such that its holonomy is $e^{\mu\beta}$ along the boundary thermal circle. The Bekenstein-Hawking entropy for these black holes is given by

$$S = \frac{A}{4G_N} = \frac{\pi r_h^2}{G_N} \,. \tag{5.5}$$

However, as we will see below, if the entropy is defined through the Gibbons-Hawking procedure instead, the result can be very different due to large fluctuations in the metric. To enhance this effects we will consider the regime of low temperatures and large charge next.

Near-extremal Limits

[1] As we will show shortly, in these units the charge is quantized as $Q \in e \cdot \mathbb{Z}$.

In the extremal limit, both radii become degenerate and $f(r)$ develops a double zero at r_0 (which can be written in terms of for example the charge). In this case, the extremal mass, charge and Bekenstein-Hawking entropy are given by

$$Q^2 = \frac{4\pi}{G_N}\left(r_0^2 + \frac{3r_0^4}{L^2}\right), \quad M_0 = \frac{r_0}{G_N}\left(1 + \frac{2r_0^2}{L^2}\right), \quad S_0 = \frac{\pi r_0^2}{G_N}. \tag{5.6}$$

This is the naive zero temperature extremal black hole. As we will see below, the small temperature limit of the entropy will not be given by the extremal area S_0 but it will still be a useful parameter to keep track of.

Since the semiclassical description breaks down at sufficiently small temperatures, we will study near-extremal large black holes with very large $\beta = T^{-1}$. We will first review its semiclassical thermodynamics in this limit. To be concrete, we will do it here by fixing the charge and the temperature. We will write the horizon radius as $r_h = r_0 + \delta r_h$ where r_0 is the extremal size for the given charge. Then the temperature is related to δr_h as

$$r_h = r_0 + \delta r_h, \quad \delta r_h = \frac{2\pi}{\beta} L_2^2 + \ldots, \quad L_2 \equiv \frac{L r_0}{\sqrt{L^2 + 6r_0^2}}, \tag{5.7}$$

where the dots denote sub-leading terms in the large β limit and the physical interpretation of the quantity $L_2(r_0)$ will become clear later. The energy and Bekenstein-Hawking entropy if we fix the charge behave as

$$E(\beta, Q) = M_0 + \frac{2\pi^2}{M_{\mathrm{SL}(2)}} T^2 + \ldots, \quad S(\beta, Q) = S_0 + \frac{4\pi^2}{M_{\mathrm{SL}(2)}} T + \ldots, \tag{5.8}$$

where the dots denote terms suppressed at low temperatures, and where we define the gap scale

$$M_{\mathrm{SL}(2)}^{-1} \equiv \frac{r_0 L_2^2}{G_N}, \tag{5.9}$$

where r_0 is a function of the charge given by (5.11). Due to this scaling with temperature, as reviewed in the introduction, the statistical description breaks down at low temperatures $\beta \gtrsim M_{\mathrm{SL}(2)}^{-1}$ so we identify this parameter with the proposed gap scale of [72] (as anticipated in the introduction, we will see in the next section that this intuition is wrong). A similar analysis to the one above can be done for fixed chemical potential.

Two limits of this near-extremal black hole will be particularly useful. The first is the limit $L \to \infty$ where we recover a near-extremal black hole in flat space, and large Q. In this case the mass and entropy scale with the charge as

$$r_0 \sim \ell_{\rm PL} Q, \quad M_0 \sim \frac{Q}{\ell_{\rm PL}}, \quad S_0 \sim Q^2. \tag{5.10}$$

We will take the limit also of large charge Q for two reasons. First, we want the black hole to be macroscopic with a large size compared with Planck's length. Second, we want $S_0 \gg 1$. As we will see below, this will suppress topology changing processes near the horizon [37]. In this limit $M_{\rm SL(2)} \sim G_N/Q^3$.

The second limit we will consider is a large black hole in AdS, keeping L fixed. Following [91] we will take large charges such that $r_0 \gg L$. We achieve this by choosing boundary conditions such that $Q \gg L/\ell_{Pl}$ (or $\mu \gg e/\ell_{Pl}$). In this regime the charge and mass are approximately

$$Q^2 = \frac{4\pi}{G_N} \frac{3r_0^4}{L^2}, \qquad M_0 = \frac{2r_0^3}{G_N L^2} \sim Q^{3/2}, \qquad S_0 = \frac{\pi r_0^2}{G_N} \sim Q\,. \tag{5.11}$$

For a bulk of dimension $D = d+1$, the mass of the extremal state scales as $M_0 \sim Q^{\frac{d}{d-1}}$ for large charge. This scaling is dual to the thermodynamic limit of the boundary CFT_d in a state with finite energy and charge density, see for example [202]. Since $L \gg \ell_{Pl}$, then $r_0 \gg L$ implies $r_0 \gg \ell_{Pl}$ and therefore $S_0 \gg 1$, suppressing topology changing processes near the horizon. In this limit $M_{\rm SL(2)} \sim G_N^{3/4}/Q^{1/2}$.

Near-extremal Geometry

Finally, in the near-extremal limit we will divide the bulk geometry in a physically sensible way that will be very useful below [91]. We will separately analyze the near-horizon region and the far region, as depicted in figure 5.1. They are described as:

Near-horizon region (NHR): This is located at radial distances $r - r_0 \ll r_0$ and is approximately $\text{AdS}_2 \times S^2$ with an AdS_2 and S^2 radius given by

$$L_2 = \frac{L r_0}{\sqrt{L^2 + 6r_0^2}}, \quad R_{S^2} = r_0. \tag{5.12}$$

Indeed from the metric (5.3) we can approximate, defining $\rho = r - r_0$, in the near-horizon region

$$ds^2_{(4d)} = \frac{\rho^2 - \delta r_h^2}{L_2^2} d\tau^2 + \frac{L_2^2}{\rho^2 - \delta r_h^2} d\rho^2 + (r_0 + \rho)^2 d\Omega_2 \tag{5.13}$$

where the first two terms correspond to the thermal AdS_2 factor with AdS radius L_2 and the second

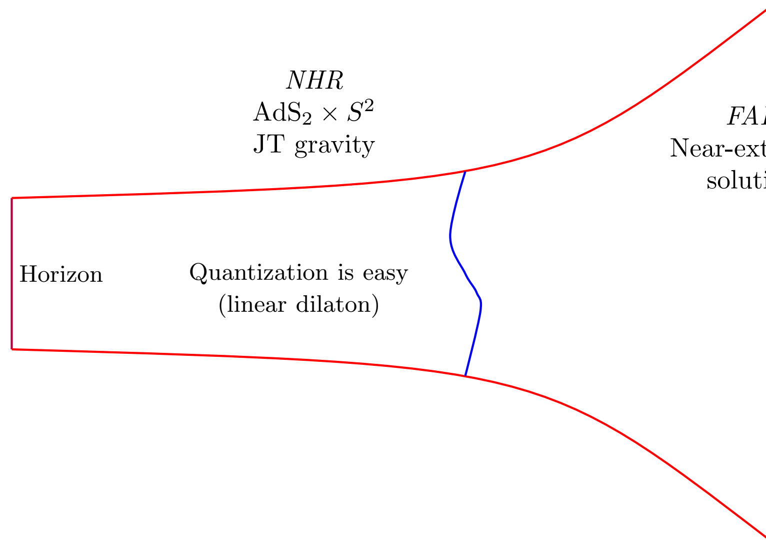


Figure 5.1: A cartoon of the near-horizon region (NHR) and the far-away region (FAR) separated by a boundary at which the boundary term of JT gravity will need to be evaluated. In the throat quantization is easy and necessary to account for at low temperatures. In the FAR quantization is hard but quantum corrections are suppressed.

factor is a sphere with an approximately constant radius r_0. For a black hole in flat space limit the radius of AdS_2 is $L_2 \approx r_0$ while for a large black hole in AdS it is given by $L_2 \approx L/\sqrt{6}$.

We kept the slowly varying term in the size of the transverse S^2 since this small correction breaks the AdS_2 symmetries and dominates the low-temperature dynamics [87, 29]. As indicated in figure 5.1, we will review how the four-dimensional theory reduces to JT gravity in this region. At positions $\rho \gg \delta r_h$, the finite temperature effects can be neglected, and the geometry becomes vacuum AdS_2. Since we will take very low temperatures $\delta r_h \ll L_2$ and therefore the geometry becomes approximately vacuum AdS_2 before we reach the asymptotic AdS_2 regime $\rho \gg L_2$.

We also look at the behavior of the $U(1)$ field strength in this region $F_{\tau\rho} \sim Q/(4\pi r_0^2)$. Therefore the throat is supported by a constant electric field.

Far-away region (FAR): This is located instead at large r, where the metric can be approximated by the extremal AdS_4 metric

$$ds^2_{(4d)} = f_0(r)d\tau^2 + \frac{dr^2}{f_0(r)} + r^2 d\Omega_2, \quad f_0(r) = \frac{(r-r_0)^2}{r^2 L^2}(L^2 + 3r_0^2 + 2rr_0 + r^2) \tag{5.14}$$

with the identification $\tau \sim \tau + \beta$. As the temperature is taken to zero this region keeps being well approximated by the semiclassical geometry. This is appropriate for the case of large black hole limit in AdS_4. For the case of black holes in the flat space limit, we take $L \to \infty$ of the metric above, finding the extremal geometry in asymptotically flat space.

Both the NHR and the FAR region overlap inside the bulk. We will match the calculations in each region at a surface included in the overlap, denoted by the blue line in figure 5.1. This happens at radial distances such that $L_2 \ll r - r_0 \ll r_0$. We will denote the gluing radius by $r_{\partial M_{\rm NHR}} = r_0 + \delta r_{\rm bdy}$, but as we will see below, the leading low-temperature effects are independent of the particular choice of $r_{\partial M_{\rm NHR}}$ as long as its part of the overlapping region.

5.2.2 Dimensional reduction

So far, we analyzed the semiclassical limit of large near-extremal black holes. We explained how the full four-dimensional geometry decomposes in two regions near the horizon throat (NHR) and far from the horizon (FAR). The parameter controlling quantum effects in the FAR region is G_N which we always keep small, while in the throat the parameter becomes the inverse temperature $\beta M_{\rm SL(2)}$ (due to the pattern of symmetry breaking). Since the geometry in the throat is nearly $\text{AdS}_2 \times S^2$ we can do a KK reduction on the transverse sphere, and the dominant effects become effectively two dimensional.

In this section, we will work out the dimensional reduction from four dimensions to two dimensions. With respect to [91], our new ingredients will be to point out that the reduction works for low temperatures $\beta M_{\rm SL(2)} \gtrsim 1$ where the semiclassical approximation breaks down, and to include the $SO(3)$ gauge mode associated to diffeomorphisms of the transverse sphere. We will begin by analyzing the reduction of the metric and will include the gauge fields afterwards. The ansatz for the four dimensional metric that we will use, following [203], is

$$ds^2_{(4d)} = \frac{r_0}{\chi^{1/2}} g_{\mu\nu} dx^\mu dx^\nu + \chi \, h_{mn}(dy^m + \mathbf{B}^a \xi_a^m)(dy^n + \mathbf{B}^b \xi_b^n)\,, \tag{5.15}$$

where $x^\mu = (\tau, \rho)$ label coordinates on AdS_2 and $y^m = (\theta, \phi)$ coordinates on S^2 with metric $h_{mn} = \text{diag}(1, \sin^2\theta)$. At this point r_0 is a constant parameter which will later be chosen to coincide with the extremal radius introduced above, when we look at solutions. The size of the transverse sphere is parametrized by the dilaton $\chi(x)$ while we also include the remaining massless mode from sphere fluctuations $\mathbf{B}$. We can use diffeomorphisms to make the gauge field independent of the coordinates on S^2, so $\mathbf{B}^a = B^a_\mu(x)dx^\mu$. Here $\xi_a = \xi_a^n \partial_n$ are the (three) Killing vectors on S^2 given by

$$
\begin{aligned}
\xi_1 &= \cos\varphi\partial_\theta - \cot\theta\sin\varphi\partial_\varphi, \\
\xi_2 &= -\sin\varphi\partial_\theta - \cot\theta\cos\varphi\partial_\varphi, \\
\xi_3 &= \partial_\varphi,
\end{aligned}
\tag{5.16}
$$

and via the Lie bracket $[\xi_a, \xi_b] = \varepsilon_{abc}\xi_c$ they generate the Lie algebra of the $SU(2)$ isometry group. The consistency of this reduction was analyzed perturbatively in [204]. Some useful technical results involving this ansatz were derived in [205]. The Einstein action after the reduction, keeping only massless fields, is

$$
\begin{aligned}
I_{EH}^{(2d)} =& -\frac{1}{4G_N}\left[\int_{M_2} d^2x\sqrt{g}[\chi R - 2U(\chi)] + 2\int_{\partial M_2} du\sqrt{h}\chi K\right] \\
& - \frac{1}{12G_N r_0}\int_{M_2} d^2x\sqrt{g}\chi^{5/2}\ \mathrm{Tr}(H_{\mu\nu}H^{\mu\nu}),
\end{aligned}
\tag{5.17}
$$

which has the form of a two dimensional dilaton-gravity theory coupled to $SO(3)$ Yang-Mills field with dilaton potential and field strength

$$
U(\chi) = -r_0\left(\frac{3\chi^{1/2}}{L^2} + \frac{1}{\chi^{1/2}}\right),
\tag{5.18}
$$

We also defined a $SO(3)$ valued field $B = B^a_\mu T^a dx^\mu$, with T^a antihermitian generators in the adjoint representation normalized such that $[T^a, T^b] = \varepsilon_{abc}T^c$ and $\mathrm{Tr}(T^aT^b) = -\frac{1}{2}\delta^{ab}$, and field strength $H = dB - B\wedge B$. We will see below how in the state corresponding to a large near-extremal black hole this reduces to Jackiw-Teitelboim gravity [39, 85].

Finally, we can reduce the Maxwell term to the massless s-wave sector. In order to do this, we decompose the gauge field as [204][2]

$$
A_\mu(x,y) = a_\mu(x)\frac{1}{\sqrt{4\pi}} + \sum_{\ell\geq 1,m} a_\mu^{(\ell,m)}(x)Y_\ell^m(y),
\tag{5.19}
$$

$$
A_n(x,y) = \sum_{\ell\geq 1,m} a^{(\ell,m)}(x)\epsilon_{np}\nabla^p Y_\ell^m(y) + \sum_{\ell\geq 1,m} \tilde{a}^{(\ell,m)}(x)\nabla_n Y_\ell^m(y),
\tag{5.20}
$$

where in the first line $Y_\ell^m(y)$ are the scalar spherical harmonics in S^2, and in the second line we wrote the vector spherical harmonics in terms of the scalar ones. This decomposition shows that the only massless field after reduction is the two dimensional s-wave gauge field $a_\mu(x)$. In the second

[2] The expansion in (5.19) assumes that no overall magnetic flux is thread through S^2.

line we see there is no component for A_n that is constant on S^2 (since such configurations would yield a singular contribution to the action from the poles of S^2) and therefore no other massless field is generated. Therefore the s-wave massless sector of the Maxwell action becomes

$$I_M^{(2d)} = -\frac{1}{4e^2 r_0}\int_{M_2} d^2x\sqrt{g}\chi^{3/2} f_{\mu\nu}f^{\mu\nu}, \qquad f = da \tag{5.21}$$

Putting everything together, the massless sector of the dimensionally reduced Einstein-Maxwell action (5.1) is given by

$$\begin{aligned} I_{EM}^{(2d)} = &-\frac{1}{4G_N}\left[\int_{M_2} d^2x\sqrt{g}[\chi R - 2U(\chi)] + 2\int_{\partial M_2} du\sqrt{h}\chi K\right] \\ &- \frac{1}{12G_N r_0}\int_{M_2} d^2x\sqrt{g}\chi^{5/2}\ \mathrm{Tr}(H_{\mu\nu}H^{\mu\nu}) - \frac{1}{4e^2 r_0}\int_{M_2} d^2x\sqrt{g}\chi^{3/2} f_{\mu\nu}f^{\mu\nu}\,, \end{aligned} \tag{5.22}$$

where the first terms corresponds to two dimensional gravity, the second to the $SO(3)$ gauge theory generated from the KK reduction and the third to the reduction of the four dimensional $U(1)$ gauge field. The contribution of the remaining massive fields coming from the $U(1)$ gauge field, metric or other potential matter couplings is summarized in section 5.4 and their contribution to the partition function is discussed in section 5.4.3. As explained in the introduction, such modes are shown to have a suppressed contribution at low temperatures and, therefore, in order to answer whether or not there is an energy gap for near-extremal black holes it is sufficient to study the contribution of the massless fields from (5.22). Consequently, we proceed by studying the quantization of the $2d$ gauge field in (5.22), neglecting the coupling of the $SO(3)$ gauge field to the massive Kaluza-Klein modes and coupling of the $U(1)$ gauge field to other potential matter fields that can be present in (5.1).

5.2.3 Two dimensional gauge fields

In order to proceed with the quantization of the gauge field in (5.22) it is necessary to introduce two Lagrange multipliers zero-form fields, $\phi^{U(1)}$ and $\phi^{SO(3)}$, with the latter valued in the adjoint representation of $SO(3)$. The path integral over the gauge fields with action (5.22) can be related to the path integral over A, B and $\phi^{U(1),\,SO(3)}$ for the action

$$\begin{aligned} \tilde{I}_{EM} = &-\frac{1}{4G_N}\left[\int_{M_2} d^2x\sqrt{g}[\chi R - 2U(\chi)] + 2\int_{\partial M_2} du\sqrt{h}\chi K\right] \\ &- i\int_{M_2}\left(\phi^{U(1)} f + \mathrm{Tr}\,\phi^{SO(3)} H\right) - \int_{M_2} d^2x\sqrt{g}\left[\frac{3G_N r_0}{2\chi^{5/2}}\ \mathrm{Tr}(\phi^{SO(3)})^2 + \frac{e^2 r_0}{2\chi^{3/2}}(\phi^{U(1)})^2\right], \end{aligned} \tag{5.23}$$

by integrating out the Lagrange multipliers $\phi^{U(1),\, SO(3)}$. One subtlety arises in going between (5.23) and (5.22). When integrating-out $\phi^{U(1),\, SO(3)}$ there is a one-loop determinant which depends on the dilaton field χ which yields a divergent contribution to the measure (behaving as $\exp 4\delta(0)\int_{M_2} du \log \chi(u))$ for the remaining dilaton path integral. There are two possible resolutions to this problem. The first is to define the measure for the dilaton path integral for the action (5.22) in such a way that it cancels the contribution of the one-loop determinant coming from (5.23). The second resolution is to rely on the fact that logarithmic corrections to the free energy (that are of interest in this chapter) solely come from integrating out fields in the near-horizon region. However, as we will see shortly, in the near-horizon region, the dilaton field χ is dominated by its value at the horizon and consequently the one-loop determinant is simply a divergent constant which can be removed by the addition of counterterms to the initial action (5.22). Regardless, of which resolution we implement, the gauge degrees of freedom in two dimensional Yang-Mills theory coupled to dilaton gravity as in (5.23) can be easily integrated-out [3].

To begin, we fix the gauge field along the three-dimensional boundary which implies that we are also fixing the holonomy at the boundary ∂M_2, $e^{\mu} = \exp \oint a$ and take $e^{i\beta\mu_{\mathrm{SO}(3)}\sigma_3} \sim [\mathcal{P}\exp(\oint B)]$.[3] In such a case we find that by integrating out the gauge degrees of freedom yields an effective theory of dilaton gravity for each $U(1)$ charge Q and each $SO(3)$ representation j:

$$Z_{\mathrm{RN}}[\mu,\beta] = \sum_{Q\in e\cdot\mathbb{Z},\ j\in\mathbb{Z}} (2j+1)\chi_j(\mu_{\mathrm{SO}(3)}) e^{\beta\mu\frac{Q}{e}} \int Dg_{\mu\nu} D\chi e^{-I_{Q,j}[g_{\mu\nu},\chi]}, \tag{5.24}$$

where $\chi_j(\theta) = \frac{\sin(2j+1)\theta}{\sin\theta}$ is the $SO(3)$ character, and the gravitational action includes extra terms in the dilaton potential from the integrated out gauge fields

$$\begin{aligned} I_{Q,j}[g,\chi] &= -\frac{1}{4G_N}\int_{M_2} d^2x\sqrt{g}\,[\chi R - 2U_{Q,j}(\chi)] - \frac{1}{2G_N}\int_{\partial M_2} du\sqrt{h}\chi K, && (5.25)\\ U_{Q,j}(\chi) &= r_0\left[\frac{G_N}{4\pi\chi^{3/2}}Q^2 + \frac{3G_N^2}{\chi^{5/2}}j(j+1) - \frac{3\chi^{1/2}}{L^2} - \frac{1}{\chi^{1/2}}\right]. && (5.26)\end{aligned}$$

Fixing the field strength (which corresponds to studying the system in the canonical ensemble) instead of the gauge field holonomy (the grand canonical ensemble) simply isolates individual terms in the sum over Q and j which corresponds to fixing the black hole charge and, as we will show shortly, to its angular momentum.

[3]Here, and throughout the rest of this chapter, $\sim$ specifies equality of conjugacy classes. The meaning of the holonomy for the $SO(3)$ gauge field arising from the dimensional reduction is that as one observer travels along ∂M_2 the internal space S^2 is rotated by an angle $\mu_{\mathrm{SO}(3)}$ around a given axis.

The equations of motion corresponding to this theory are given by [206]

$$\nabla_\mu \nabla_\nu \chi - g_{\mu\nu}\nabla^2\chi - g_{\mu\nu}U_{Q,j}(\chi) = 0 \tag{5.27}$$

$$R - 2\partial_\chi U_{Q,j}(\chi) = 0. \tag{5.28}$$

By fixing part of the gauge freedom, the most general static solution can be put into the following form

$$\chi = \chi(r), \qquad ds^2 = \frac{\chi^{1/2}}{r_0}\left[f(r)d\tau^2 + \frac{dr^2}{f(r)}\right]. \tag{5.29}$$

The equation for the dilaton gives $\partial_{r^2}\chi = $ constant, and using remaining gauge freedom the solution can be put in the form $\chi(r) = r^2$. For this choice the metric equation becomes

$$f(\chi) = \frac{1}{\chi^{1/2}}\left[C - \frac{1}{2r_0}\int^{\chi} d\chi U_{Q,j}(\chi)\right], \tag{5.30}$$

where C is an integration constant that can be fixed by the boundary conditions. This gives the complete solution of the dilaton gravity equations. After analyzing some particular cases, we will see why the specific ansatz (5.29) that we chose is convenient.

First, the simplest case is to study states with $j = 0$. Then the equation of motion for the metric and dilaton for each effective action (5.25) yields

$$f(\chi) = \frac{1}{\chi^{1/2}}\left[C - \frac{1}{2r_0}\int^{\chi} d\chi U_{Q,0}(\chi)\right] = 1 + \frac{\chi}{L^2} + \frac{G_N}{4\pi}\frac{Q^2}{\chi} + \frac{C}{\chi^{1/2}}. \tag{5.31}$$

Using $\chi = r^2$ and the boundary conditions at large r we can fix the integration constant $C = -2G_N M$. Replacing this in the equation above, and replacing the two dimensional metric (5.29) into the four dimensional (5.15), we see that this precisely agrees with the Reinsner-Nordström solution (5.3) described in section 5.2.1 for fixed charge Q.

We can now discuss the case of arbitrary small j. Up to subtleties about the backrection of the $SO(3)$ gauge field on the g_{rr} and $g_{\tau\tau}$ metric components, the states with fixed j can be identified as the KN solutions reviewed in appendix C. Specifically, as we show in appendix C, the deformation from Reissner-Nordström (5.3) is given by $SO(3)$ gauge field solutions, plugged into the metric ansatz (5.15):

$$g_{\mu\nu} = g^{\text{RN}}_{\mu\nu} + \delta g_{\mu\nu}\,, \qquad \delta g_{\mu\nu}dx^\mu dx^\nu = 4ir^2\sin^2\theta\Big(\alpha_1 + \frac{\alpha_2}{r^3}\Big)d\phi d\tau\,. \tag{5.32}$$

α_1 and α_2 are two constants which are determining by the boundary conditions on the $SO(3)$ gauge field and by requiring that the gauge field be smooth at the black hole horizon. Turning on a non-trivial profile for the $SO(3)$ gauge field as in (5.32) breaks the $SO(3)$ rotational isometry down to $U(1)$. This is the same as in the well-known KN solution reviewed in appendix C. Solving the equations of motion in the semiclassical limit when fixing the field strength on the boundary to $H^3_{r\tau}|_{\partial M_2} = i\frac{6G_N j^2}{\sqrt{2}r^4}|_{\partial M_2}$, corresponds to fixing j in the sum in (5.24), and using that $\alpha_2 = \frac{1}{\sqrt{2}}G_N j^2$ yields a solution with a fixed 4d total angular momentum $J = j/\sqrt{2}$.[4] Since the KN solution is the unique solution with a $U(1)$ rotation isometry and with fixed angular moment and charge, this makes the metric ansatz that includes the deformation (5.32) agree (for sufficiently small j) with the KN solution up to diffeomorphisms.

We can now address the subtlety about the $SO(3)$ gauge field backreacting on the g_{rr} and $g_{\tau\tau}$ components of the metric. The reason why we need to account for such backreaction is that it can source other massive Kaluza-Klein modes of the metric, which are not accounted for in the action (5.25). In order to understand the $SO(3)$ gauge field backreaction, we can repeat the analysis above in which we studied the backreaction of the $U(1)$ gauge field on $f(r)$. For $j \neq 0$ we get a correction to the metric $\delta_j f \sim \frac{G_N^2 j(j+1)}{r^4}$. Since we do not want to source further backreaction on the massive Kaluza-Klein modes, we will require that this correction is small everywhere far from the horizon and require that the spin of the black hole satisfy $j(j+1) \ll (r_h/\ell_{Pl})^4$.

5.2.4 New boundary conditions in the throat

While quantizing the action (5.25) directly is out of reach, we can do better by separating the integral in the action in the NHR and FAR. To conveniently manipulate the action into a form where quantization can be addressed, we follow the strategy of [91]. Namely we choose the NHR and FAR to be separated by an arbitrary curve with a fixed dilaton value $\chi|_{\partial M_{\text{NHR}}} = \chi_b$ and fixed intrinsic boundary metric $h_{uu} = 1/\varepsilon^2$ and proper length $\ell = \int du\sqrt{h}$.

In the NHR, the equations of motion fixes the value of the dilaton at the horizon to be

$$\phi_0 \equiv \frac{\chi(r_h)}{G_N} = \frac{r_0^2(Q)}{G_N}\,, \tag{5.33}$$

which acts as a very large constant background. The function $r_0(Q)$ obtained from dilaton-gravity is equivalent to solving (5.6). In the NHR where $r - r_h \ll r_h$ we can study small fluctuations around

[4] Where J is normalized as in the KN solution (C.1).

this value $\chi(r) = [\phi_0 + \phi(r)]/G_N$. Expanding the action to first order we find that

$$I^{Q,j}_{\text{NHR}}[g_{\mu\nu}, \chi] = \frac{1}{4}\int_{M_{\text{NHR}}} d^2x\sqrt{g}\left[-\phi_0 R - \phi\Big(R + \frac{2}{L_2^2}\Big) + O\left(\frac{\phi^2}{\phi_0}\right)\right] , \tag{5.34}$$

where the two dimensional AdS radius is $L_2 = \frac{Lr_0}{\sqrt{L^2+6r_0^2}}$, which in general (except for the case of large black holes in AdS_4) also depends on the charge of the black hole through $r_0(Q)$. From now, L_2 and r_0 should be understood as functions of the charge. The last term captures a quadratic correction in the dilaton variation. The quantization of the above action has been widely discussed in the presence of an appropriate boundary term.

We will see next how this boundary term arises from including fluctuations in the FAR region. We proceed by expanding the near-extremal metric and dilaton in the FAR region into their contribution from the extremal metric and their fluctuation:

$$g_{\mu\nu} = g^{\text{ext}}_{\mu\nu} + \delta g^{\text{near-ext}}_{\mu\nu} , \qquad \chi = \chi^{\text{ext}} + \delta\chi^{\text{near-ext}} . \tag{5.35}$$

Both the extremal and near-extremal $4d$ metrics are solutions to the equations of motion at fixed β, i.e. with periodic Euclidean time $\tau \sim \tau + \beta$. The extremal solution however contains a singularity at the horizon if imposing any periodicity for the Euclidean time. Nevertheless, if separating the space into the NHR and the FAR, the singularity would not be present in the latter region and we can safely expand the action around the extremal solution. If expanding around the the extremal metic, following from the variational principle the first order term in the expansion is solely a total derivative term which when integrated by parts results in a total boundary term. Explicitly, the action is given by

$$\begin{aligned} I^{Q,j}_{\text{FAR}}[g_{\mu\nu}, \chi] &= I^{Q,j}_{\text{FAR}}[g^{\text{ext}}_{\mu\nu}, \chi^{\text{ext}}] - \frac{1}{2G_N}\int_{\partial M_{\text{NHR}}} du\sqrt{h}\left[\chi\delta K - (\partial_n\chi - \chi K)\delta\sqrt{h_{uu}}\right] , \\ \delta K &\equiv K_{\text{NHR}} - K_{\text{ext}} , \qquad \delta\sqrt{h_{uu}} = 0 . \end{aligned} \tag{5.36}$$

The last equality follows from the fact that we have imposed Dirichlet boundary conditions for the intrinsic boundary metric. Consequently, as sketched in figure 5.2, we obtained a surface which has a small discontinuity precisely on the curve that separates the NHR from the FAR. Above, K_{NHR} is the extrinsic curvature evaluated on the boundary of the NHR (defined with respect to the direction of the normal vector $\widehat{n}_{\text{NHR}}$) and K_{ext} is the extrinsic curvature evaluated on the boundary of the FAR with the extremal metric on it (wrt the normal vector $\widehat{n}_{\text{FAR}}$).

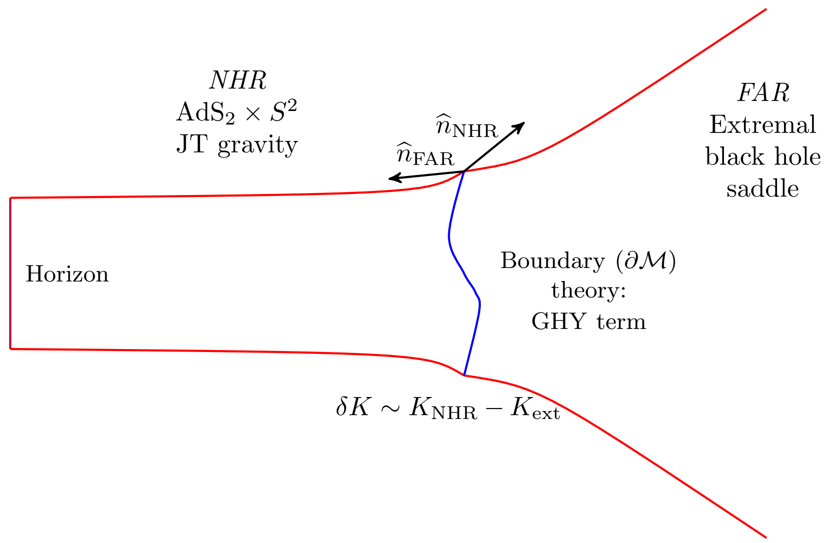


Figure 5.2: A cartoon of the near-horizon region (NHR) and the far-away region (FAR) separated by a curve along which the boundary term of JT gravity will need to be evaluated.

We can now understand the effect of the Dirichlet boundary conditions for the dilaton $\chi_b = G_N(\phi_0 + \phi_b/(2\varepsilon))$ and proper boundary length $\ell = \int du\sqrt{h} = \beta L_2/\varepsilon$. Here, ε is some parameter fixed by the value of ℓ and β whose role we will understand shortly. Curves of constant dilaton in the extremal solution are fixed to have a constant value of $r_{\partial M_{\rm NHR}} \equiv r_0 + \delta r_{\rm bdy}$ and are parametrized by τ when using the coordinate system in (5.14). In the extremal solution, the dilaton value, proper length and extrinsic curvature $K_{\rm ext}$ on the extremal side are all fixed by the value of $\delta r_{\rm bdy}$:

$$
\begin{aligned}
&\chi_b = G_N\left(\phi_0 + \frac{\phi_{b,Q}}{2\varepsilon}\right), \qquad \text{with} \qquad \frac{\phi_{b,Q}}{2\varepsilon} = \frac{r_0\delta r_{\rm bdy}}{G_N}\,, \\
&\ell = \int du\sqrt{h} = \frac{\beta L_2}{\varepsilon}\,, \qquad \text{with} \qquad \varepsilon = \frac{L_2^2}{\delta r_{\rm bdy}}\,, \qquad \phi_{b,Q} = M_{\rm SL(2)}^{-1} = \frac{r_0 L_2^2}{G_N}\,, \\
&K_{\rm ext} = \frac{1}{L_2}\left(1 - \frac{4}{3}\frac{\delta r_{\rm bdy}}{r_0} + \frac{(L^2 + 25\delta r_{\rm bdy}^2)}{(12 r_0^2)} + O\left(\frac{\delta r_{\rm bdy}^3}{r_0^3}\right)\right),
\end{aligned} \tag{5.37}
$$

where we computed the extremal extrinsic curvature using the metric (5.14). In the near-extremal limit we have that $\beta \gg \varepsilon$ and $\phi_b \gg \varepsilon$. These inequalities will prove important in relating (5.37) to a boundary Schwarzian theory.

We see here explicitly that the renormalized value of the dilaton is precisely given by the inverse mass gap scale in the way defined previously by thermodynamic arguments. Consequently, the overall action is given by

$$
\begin{aligned}
I^{Q,j}_{EM} = &-\frac{1}{4}\int_{M_{\rm NHR}} d^2x\sqrt{g}\left[\phi_0 R+\phi\Big(R+\frac{2}{L_2^2}\Big)\right] \\
&-\frac{1}{2}\int_{\partial M_{\rm NHR}} du\sqrt{h}(\phi_0+\phi)\Big[K_{\rm NHR}-\frac{1}{L_2}\Big(1+\frac{4}{3}\frac{\delta r_{\rm bdy}}{r_0}\Big)\Big]+I^{Q,j}_{\rm FAR}[g^{\rm ext}_{\mu\nu},\chi^{\rm ext}]\,. \qquad (5.38)
\end{aligned}
$$

The quadratic fluctuations in the FAR region are suppressed compared to the contribution of the first two NHR terms in (5.38).[5] Therefore, we will neglect the possible quadratic (or higher order) fluctuations around the extremal metric in the FAR region and proceed by evaluating the contribution of FAR action on-shell. To simplify the computation, we will, for now, focus on the $j=0$ sector where there is no backreation from the $SO(3)$ gauge field on the other components of the metric. On-shell, the bulk term in the FAR action evaluates to

$$
\begin{aligned}
I^{Q,j=0}_{\text{FAR, bulk}}[g^{\rm ext}_{\mu\nu},\chi^{\rm ext}] &= -\frac{1}{4G_N}\int d^2x\sqrt{g_{\rm ext}}\left[\chi R-2U_{Q,0}[g^{\mu\nu}_{\rm ext},\chi^{ext}]\right] \\
&= -\frac{3r_{\partial M_2}\beta}{4G_N}\Big(1+\frac{r^2_{\partial M_2}}{12L^2}\Big)+\frac{2r_0\beta}{G_N}\Big(1+\frac{2r_0^2}{L^2}\Big)-\frac{\beta\delta r_{\rm bdy}}{2G_N}\Big(1+\frac{6r_0^2}{L^2}\Big)\,. \qquad (5.39)
\end{aligned}
$$

where, as we will see shortly, the divergent terms can be canceled by adding counter-terms to the boundary term in the action (5.1) (which we have so far neglected). We now include this boundary term from (5.1) (associated to the Dirichlet boundary conditions on ∂M_2) together with possible counter-terms. This evaluates to:

$$
\begin{aligned}
I^{Q,j=0}_{\text{FAR, bdy.}}[g^{\rm ext}_{\mu\nu},\chi^{\rm ext}] &= \frac{1}{2G_N}\int_{\partial M_2} du\sqrt{h}\Big(\chi K+\mathcal{C}_1\frac{\chi^{3/4}}{r_0^{3/2}}+\mathcal{C}_2\frac{r_0^{1/2}}{\chi^{1/4}}\Big) \\
&= \frac{\beta r^3_{\partial M_2}\left(2\mathcal{C}_1 L-3r_0^2\right)}{4G_N L^2 r_0^2}+\frac{\beta r_{\partial M_2}\left(\mathcal{C}_1L^2+2\mathcal{C}_2 r_0^2-Lr_0^2\right)}{4G_N L r_0^2}-\frac{\beta\mathcal{C}_1\left(L^2+2r_0^2\right)}{2G_N L r_0}\,, \qquad (5.40)
\end{aligned}
$$

where the terms including $\mathcal{C}_1$ and $\mathcal{C}_2$ are the counterterms necessary to cancel the divergence in (5.39). In order to cancel the divergence in (5.39) we set,

$$
\mathcal{C}_1=\frac{2r_0^2}{L}\,,\qquad \mathcal{C}_2=L\,. \qquad (5.41)
$$

We can also find precisely the same terms with the right prefactors by dimensionally reducing the holographic counterterm of [207], reproducing the same overall on-shell action. In total we thus find

[5] Even when we will integrate over order one fluctuations of the Schwarzian mode in the next section, the fluctuations in the metric near the boundary of AdS_2 is suppressed by the cut-off. For example $\delta g_{\tau\tau}\sim\epsilon^2\text{Sch}(\tau,u)$. Therefore fluctuations in the FAR region are always small, and become large only very close to the horizon far inside the throat.

that

$$
\begin{aligned}
I^{Q,j=0}_{\text{FAR}} &= I^{Q,j=0}_{\text{FAR, bulk}}[g^{\text{ext}}_{\mu\nu},\chi^{\text{ext}}] + I^{Q,j=0}_{\text{FAR, bdy.}}[g^{\text{ext}}_{\mu\nu},\chi^{\text{ext}}] = \frac{r_0\beta}{G_N}\Big(1+\frac{2r_0^2}{L^2}\Big) - \frac{\beta\delta r_{\text{bdy}}}{2G_N}\Big(1+\frac{6r_0^2}{L^2}\Big) \\
&= \beta M_0(Q) - \frac{\beta\delta r_{\text{bdy}}}{2G_N}\Big(1+\frac{6r_0^2(Q)}{L^2}\Big)\,, \qquad (5.42)
\end{aligned}
$$

where in the last line we emphasize the charge dependence of the extremal mass and horizon radius, given by (5.11). The δr_{bdy} dependent term in the action (5.42), $\frac{2\sqrt{6}}{3G_N}\int_{\partial M_{\text{NHR}}} du\sqrt{h}\frac{\chi\delta r_{\text{bdy}}}{r_0}$, also precisely cancels the δr_{bdy} term in (5.39). This is simply a consequence of the fact that the parameter δr_{bdy} is chosen arbitrarily to separate M_2 into the NHR and the FAR and, consequently, the fact that all our results are independent of δr_{bdy} can be seen as a consistency check.

Next, we can consider the contribution to the action of the $SO(3)$ gauge fields and of the backreaction of the field on other components of the metric. Corrections could appear in the contribution to the partition function in the extremal area term or in the extremal energy. The former is of order $\delta\phi_0 \sim \frac{G_N L^2}{r_0^4}j(j+1)$ (for a large black hole in AdS) or $\delta\phi_0 \sim \frac{G_N}{r_0^2}j(j+1)$ (for a black hole in flat space) and therefore is very small and can be neglected in either case. The term coming from the correction to the extremal mass, originating from the backreaction on the metric and by the $SO(3)$ Yang-Mills term in the action, is multiplied by a large factor of β and gives the leading correction

$$
M_0(Q,j) = M_0(Q,j=0) + \frac{G_N}{2r_0^3}j(j+1) + O(j^4), \qquad (5.43)
$$

where $r_0(Q)$ is the extremal horizon size for the RN black hole given by (5.6). In principle, the backreaction of the $SO(3)$ gauge field also affects the boundary value of the dilaton $\phi_b/(2\epsilon)$. However, such a contribution appears at the same order as other $O(1/\phi_0)$ corrections, which we have ignored in the NHR. Therefore, we will solely track the Q-dependence of $\phi_b(Q,j)\to\phi_{b,Q}$.

We find this result reliable for the case of large black holes in AdS with $r_0 \gg L$. For temperatures of order the gap, the correction to the partition function is $\delta\log Z \sim \beta\delta M \sim \delta M/M_{\text{SL}(2)} = \frac{L^2}{r_0^2}j(j+1)$. This way we can take large order one values of j while still not affecting the answer considerably. We can check this by comparing with the result from the KN black hole since we know that the $SO(3)$ gauge field sources angular momentum. We get

$$
\delta M_0^{\text{KN}}(Q,j) = G_N J^2 \frac{(L^4+5L^2r_0^2+8r_0^4)}{2r_0^3(L^2+2r_0^2)^2} \sim \frac{G_N J^2}{r_0^3}\,. \qquad (5.44)
$$

This matches in the limit $r_0 \gg L$ with the result we found from the dimensional reduction when

$J \gg 1$. For black holes in flat space or for smaller black holes in AdS$_4$ one has to in principle account for the backreaction of the $SO(3)$ gauge field on other Kaluza-Klein modes in (5.43), to recover the exact correction (5.44).

Thus, in total we find that the dynamics of the near-extremal black hole is described by

$$I_{EM}^{Q,j}[g_{\mu\nu}^{\text{ext}}, \chi^{\text{ext}}] = \beta M_0(Q,j) - \frac{1}{4}\int_{M_{\text{NHR}}} d^2x\sqrt{g}\left[\phi_0(Q,j)R + \phi\Big(R + \frac{2}{L_2^2}\Big) + O\left(\frac{\phi^2}{\phi_0^2}\right)\right]$$
$$- \frac{1}{2}\int_{\partial M_{\text{NHR}}} du\sqrt{h}\left[\phi_0(Q,j)\, K_{\text{NHR}} + \frac{\phi_{b,Q}}{\varepsilon}\Big(K_{\text{NHR}} - \frac{1}{L_2}\Big)\right], \tag{5.45}$$

where the on-shell contribution of the FAR action can be seen as an overall shift of the ground state energy of the system. We can now proceed by using (5.39) to determine the exact ground state energy of the system, and then by quantizing the remaining degrees of freedom in (5.38).

Before moving on, we can briefly comment on corrections coming from non-linearities in the dilaton potential present in the first line of (5.45). To leading order, we get the JT gravity action written above. The next correction behaves like $\delta U \sim \phi^2/\phi_0$. The contribution to the partition function from such a term was computed in [208] and scales as $\delta \log Z \sim \phi_b^2/(\beta^2\phi_0)$. Such a contribution is suppressed by the large extremal area $\phi_0 \gg 1$. Higher-order corrections to the dilaton potential are further suppressed by higher powers of ϕ_0 and, more importantly, decay faster at low temperatures. Therefore, they can all be neglected.

5.3 The partition function for near-extremal black holes

5.3.1 An equivalent 1D boundary theory

We will now evaluate the contribution to the partition function of the quantum fluctuations from the remaining graviton and dilaton fields present in the effective action of the NHR (5.45). We briefly review this procedure by first reducing the path integral of (5.45) to that of a boundary Schwarzian theory.

Integrating out the dilaton enforces that the curvature is fixed to $R = -2/L_2^2$.[6] Thus, each near-horizon region configuration that contributes to the path integral is a patch of AdS_2 cut along a curve

[6] In order to enforce such a condition, the contour for dilaton fluctuation $\phi(x)$ needs to go along the imaginary axis such that

$$\int Dg_{\mu\nu}\int_{\phi_b - i\infty}^{\phi_b+i\infty} D\phi\; e^{\int_{M_{NHR}} d^2x\sqrt{g}\phi\left(R+\frac{2}{L_2^2}\right)} = \int Dg_{\mu\nu}\delta\Big(R - \frac{2}{L_2^2}\Big). \tag{5.46}$$

This choice of contour for ϕ isolates the same type of constant curvature configurations in Euclidean signature as those that dominate in the Lorentzian path integral. More details about this choice of countour in the context of near-extremal black holes are discussed in footnote 9 of [3].

with a fixed proper length ℓ. Following [29], we can write the AdS_2 metric by $ds^2_{AdS_2} = L_2^2 \frac{dF^2+dz^2}{z^2}$ and parametrize the boundary with a proper time u, with $u \in [0, \beta)$ and $h_{uu} = 1/\varepsilon^2$. In this case, one can solve for the value of $z(u)$ in terms of $F(u)$ on the boundary, in the limit in which $\beta \gg \varepsilon$ to find that $z(u) = \varepsilon F'(u)$. The extrinsic curvature can then be written in terms of the Schwarzian derivative [29]:

$$K_{\rm NHR} = \frac{1}{L_2}\left[1 + \varepsilon^2\,{\rm Sch}(F,u) + O(\varepsilon^4)\right]\,, \qquad {\rm Sch}(F,u) = \frac{F'''}{F'} - \frac{3}{2}\left(\frac{F''}{F'}\right)^2\,. \tag{5.47}$$

The geometry we are working with in the NHR after reducing on S^2 is actually the hyperbolic disk. We can easily go from the Poincare coordinates to the disk by replacing

$$F(u) = \tan\frac{\pi\tau(u)}{\beta}, \qquad \tau(u+\beta) = \tau(u) + \beta \tag{5.48}$$

in the Schwarzian action. Here τ parametrizes the Euclidean circle at the boundary of the NHR which we glue to the FAR region. For simplicity we will mostly write the Schwarzian action in terms of $F(u)$ instead.

The path integral over the the metric reduces to an integral over the field $F(u)$ and the partition function becomes:[7]

$$Z_{\rm RN}[\beta,\mu,\mu_{\rm SO(3)}] = \sum_{Q\in e\cdot\mathbb{Z},\, j\in\mathbb{Z}} (2j+1)\chi_j(\mu_{\rm SO(3)}) e^{-\frac{Q}{e}\beta\mu} e^{\pi\phi_0(Q,j)} e^{-\beta M_0(Q,j)} \times \int \frac{\mathcal{D}\mu[F]}{{\rm SL}(2,\mathbb{R})} e^{\phi_{b,Q}\int_0^\beta du\ {\rm Sch}(F,u)}\,. \tag{5.49}$$

This relation shows that we can identify the term giving the extremal area $S_0 = \pi\phi_0$ coming from the topological part of the dilaton gravity NHR action. The extremal mass term comes from the action in the FAR region. The path integral over the Schwarzian theory includes finite temperature near-extremal effects. The effective coupling of this mode depends on the charge and spin of each black hole in the ensemble.

Before reviewing the quantization of (5.49), it is also interesting to study the possibility that the sum over all the possible representations is reproduced by a single 1d theory. Reproducing the sum over charges can be done by coupling the Schwarzian theory to a theory having a $U(1)\times SO(3)$ symmetry. As explained in [89, 3], the theory that exhibits this symmetry and correctly captures the sum over charges is that of a particle moving on a $U(1) \times SO(3)$ group manifold. To obtain

[7] Above, the path integral measure $\mathcal{D}\mu[F]$ over the field $F(u)$ can be determined from the symplectic form associated to an $SL(2,\mathbb{R})$ BF-theory which is equivalent on-shell to JT gravity.

this model, we introduce four additional fields: a compact scalar $\theta(u) \sim \theta(u) + 2\pi$ together with a Lagrange muliplier $\alpha(u)$ and a field $h(u) \in SO(3)$ together with another Lagrange multiplier $\boldsymbol{\alpha}(u) \in SO(3)$. The general coupling between the particle moving on a group manifold and the Schwarzian theory is given by:

$$I_{\text{Sch}\times U(1)\times SO(3)} = -\int_0^\beta du\Big[i\alpha\theta' + i\text{Tr}\,\left(\boldsymbol{\alpha}h^{-1}h'\right) + \mathcal{V}(\alpha, \text{Tr}\,\boldsymbol{\alpha}^2) - \mathcal{W}(\alpha)\,\text{Sch}\,(F,u)\Big]\,, \tag{5.50}$$

where the potential $\mathcal{W}(\alpha)$ is independent of the the $SO(3)$ degrees of freedom since we are neglecting the effect of angular momentum of the boundary value of the dilaton $\phi_{b,Q}$.

When the generic potential $\mathcal{V}(\alpha, \text{Tr}\,\boldsymbol{\alpha}^2)$ is of trace-class, the theory has a $U(1)$ symmetry $\theta \to \theta + a$ and two $SO(3)$-symmetries generated by the transformations $h \to g_L h g_R$ and $\boldsymbol{\alpha} \to g_R^{-1}\boldsymbol{\alpha} g_R$, with g_L, $g_R \in SO(3)$. Consequently, the Hilbert space arranges itself in representations of $U(1) \times SO(3) \times SO(3)$. However, the quadratic Casimir of both $SO(3)$-symmetries is in fact the same. Therefore, the Hilbert space arranges itself in representations of $U(1)$ and two copies of the same $SO(3)$-representation. If we are interested in reproducing the near-extremal black hole partition function with Dirichlet boundary conditions for the $U(1)$ and $SO(3)$ gauge fields, then we need to introduce a chemical potential for the $U(1)$ symmetry of (5.50) and for one of its $SO(3)$ symmetries. This can be done by introducing a $U(1)$ background gauge field, $\mathcal{A}$ with $\exp(\oint \mathcal{A}) = e^{\beta\mu}$, and an $SO(3)$ background gauge field, $\mathcal{B}$ with $\mathcal{P}\exp(\oint \mathcal{B}) \sim e^{i\beta\mu_{\text{SO}(3)}\sigma_3}$, coupling the first background to the $U(1)$ charge through $-i\int_0^\beta du\alpha\mathcal{A}_u$ and the second background to the $SO(3)$ charges through $-i\int_0^\beta du\,\text{Tr}(\boldsymbol{\alpha}\mathcal{B}_u)$. In such a case, the partition function of the general theory (5.50) can be shown to be [3]:[8]

$$Z_{Sch\times U(1)\times SO(3)} = \sum_{Q\in e\cdot\mathbb{Z}, j\in\mathbb{Z}} e^{\beta\mu\frac{Q}{e}}(2j+1)\chi_j(\mu_{\text{SO}(3)})e^{-\beta\mathcal{V}(\frac{Q}{e},j(j+1))}e^{\mathcal{W}(\frac{Q}{e},j(j+1))\int_0^\beta du\ \text{Sch}(F,u)}\,, \tag{5.51}$$

which up to an overall proportionality constant corresponding to the extremal black hole entropy agrees with the form of (5.49). Therefore, the potentials $\mathcal{V}(\alpha, \boldsymbol{\alpha})$ and $\mathcal{W}(\alpha, \boldsymbol{\alpha})$ need to be tuned in order for the partition function of the theory (5.50) to reproduce the charge dependence in the sum in (5.49). For example, for large black holes in AdS_4 we find that:

$$\mathcal{V}(\alpha, \boldsymbol{\alpha}) = \frac{|\alpha|^{3/2}}{(3\pi)^{3/4}(2L)^{1/2}G_N^{1/4}} + \frac{\sqrt{2}G_N^{1/4}(3\pi)^{3/4}}{L^{3/2}|\alpha|^{3/2}}\text{Tr}\,\boldsymbol{\alpha}^2\,, \qquad \mathcal{W}(\alpha) = \frac{|\alpha|^{1/2}L^{5/2}}{6\sqrt{2}(3\pi G_N^3)^{1/4}}\,. \tag{5.52}$$

[8]When taking the trace over the Hilbert space of the theory (5.50) and summing over states within the two copies of some $SO(3)$ representation j then the sum over the gauged copy yields $\chi_R(\theta)$ while the sum over the other copy yields the degeneracy $\dim R$ in (5.51).

For black holes in flat space we find:

$$\mathcal{V}(\alpha,\boldsymbol{\alpha}) = \frac{|\alpha|}{2(\pi G_N)^{1/2}} + \frac{4\pi^{3/2}}{G_N^{1/2}|\alpha|^3}\mathrm{Tr}\,\boldsymbol{\alpha}^2\,, \qquad \mathcal{W}(\alpha) = \frac{|\alpha|^3 G_N^{1/2}}{8\pi^{3/2}}\,. \tag{5.53}$$

We will see in the next section that for fluctuations around extremality the action for the U(1) and SO(3) mode further simplifies.

5.3.2 The partition function at $j = 0$

We have identified the effects that dominate the temperature dependence in the near-extremal limit. In this section, we will put everything together to find a final answer for the partition function. To at first simplify the discussion, we will pick boundary conditions in the four-dimensional theory that fix the angular momentum j to zero. In the dimensional reduced theory this is equivalent to picking only the $j = 0$ sector of expression (5.49). We will analyze fixed $U(1)$ charge and chemical potential separately.

Fixed Charge

This is the simplest case to consider where we fix the temperature, $U(1)$ charge Q and angular momentum to zero. From a Laplace transform of equation (5.49) the partition function is given by

$$Z_{\mathrm{RN}}[\beta, Q] = e^{\pi\phi_0(Q)} e^{-\beta M_0(Q)} \int \frac{\mathcal{D}\mu[F]}{\mathrm{SL}(2,\mathbb{R})} e^{\phi_{b,Q}\int_0^\beta du\ \mathrm{Sch}(F,u)}. \tag{5.54}$$

This means that for boundary conditions of fixed charge, the $U(1)$ mode is effectively frozen and does not contribute to the partition function, leaving only the Schwarzian mode. The path integral of the Schwarzian theory can be computed exactly and gives

$$Z_{\mathrm{Sch}}(\phi_{b,Q}, \beta) \equiv \int \frac{\mathcal{D}\mu[F]}{\mathrm{SL}(2,\mathbb{R})} e^{\phi_{b,Q}\int_0^\beta du\ \mathrm{Sch}(F,u)} = \left(\frac{\phi_{b,Q}}{\beta}\right)^{3/2} e^{\frac{2\pi^2}{\beta}\phi_{b,Q_0}}. \tag{5.55}$$

Then the final expression for the canonical partition function is

$$Z_{\mathrm{RN}}[\beta, Q] = \left(\frac{\phi_{b,Q}}{\beta}\right)^{3/2} e^{\pi\phi_0(Q) - \beta M_0(Q) + \frac{2\pi^2}{\beta}\phi_{b,Q_0}}. \tag{5.56}$$

Here the first term comes from the gravitational one-loop correction from the JT mode which dominates at low temperatures. This gives a correction $-\frac{3}{2}T\,\log T$ to the free energy (equivalently a $\frac{3}{2}\log T$ correction to $\log Z$). The terms in the exponential are first the extremal entropy through

$S_0 = \pi\phi_0$, the extremal mass term $-\beta M_0(Q)$ and the third gives the leading semiclassical correction near extremality. The temperature dependence of this expression is exact even for $\phi_{b,Q}/\beta$ finite. The result is valid as long as, stringy effects are not important, $r_0 \gg \ell_{\rm Pl}$ (equivalently, $Q \gg 1$) and when the black hole is near-extremal, $\beta \gg r_0$ $\left(\text{equivalently, } \beta^2 \gg \frac{L^2}{6}\left[\sqrt{1+\frac{3G_N Q^2}{\pi L^2}}-1\right]\right)$.

With this expression we can analyze the thermodynamics of the system. The entropy is given by

$$S(\beta,Q) = (1-\beta\partial_\beta)\log Z = S_0 + \frac{4\pi^2\phi_{b,Q}}{\beta} - \frac{3}{2}\log\frac{\beta}{e\phi_{b,Q}}, \tag{5.57}$$

$$E(\beta,Q) = M_0 + \frac{2\pi^2\phi_{b,Q}}{\beta^2} + \frac{3}{2\beta} \tag{5.58}$$

This gives a resolution of the "thermodynamic gap scale" puzzle. At very low temperatures the energy goes as $E - M_0 \sim \frac{3}{2}T$ (as opposed to $\sim T^2$). Therefore the energy is always bigger than the temperature and the argument of [72] does not apply. We will see this again in the next section when we work directly with the density of states, showing explicitly that there is no gap in the spectrum.

Finally, there are well-known corrections to the partition function of an extremal black hole computed by Sen [81] coming from integrating out matter fields. Those effects can correct the extremal entropy S_0 at subleading orders. These corrections are significant compared to the ones coming from the Schwarzian mode but are temperature-independent in the limit we are taking (see also the results of [100]) and can be absorbed by a shift of S_0. As previously stated, the goal of this chapter is to study the leading temperature-dependent contributions to the free energy. Therefore, we can neglect these possible shifts of S_0.

Fixed Chemical Potential

The partition function with fixed $U(1)$ chemical potential μ and zero angular momentum is given by

$$Z_{\rm RN}[\beta,\mu] = \sum_{Q\in e\cdot\mathbb{Z}} e^{\beta\mu\frac{Q}{e}} e^{\pi\phi_0(Q)} e^{-\beta M_0(Q)} Z_{\rm Sch}(\phi_{b,Q},\beta) \tag{5.59}$$

As previously mentioned the terms in the sum for which the near-extremal black hole approximations made above are those with $Q \gg 1$ and with $\frac{4\pi}{G_N}\left(\beta^2 + \frac{3\beta^4}{L^2}\right) \gg Q^2$ (this is equivalent to $\beta \gg r_0(Q)$). Consequently, in order for the sum (5.59) to be valid we need it to be dominated by charges within this (very large) range. This problem is only well-defined when the sum converges, which only happens at finite L (in flat space the integrand grows too fast with the charge). Therefore, when fixing the chemical potential we will only consider finite L.

In order to make contact with previous work in the literature and simplify the equivalent boundary theory, it is interesting to study the dominating charge within this sum and the charge fluctuations around it.

In the large charge limit the Schwarzian contribution is order one and balancing only the chemical potential and mass term gives

$$\partial_Q\Big(\mu\frac{Q}{e}-M_0\Big)\Big|_{Q_0}=0 \quad \Rightarrow \quad Q_0^2=\frac{(4\pi)^2L^2\mu^2}{3e^4}(4\pi G_N\mu^2-e^2). \tag{5.60}$$

The near-extremal approximation is valid as long as $\mu \ll \frac{e}{2L}\sqrt{\frac{L^2+3\beta^2}{G_N}}$. This formula is consistent with (5.4) but now the extremal charge Q_0 should be thought of as a function of μ. This extremal value of the charge is not the true saddle point of the full partition function in (5.59). It is useful anyways to expand around it $Q=Q_0+eq$, such that $q\in\mathbb{Z}$. Then keeping terms up to quadratic order in q we obtain

$$Z_{\rm RN}[\beta,\mu]=e^{\beta\mu\frac{Q_0}{e}}e^{\pi\phi_0(Q_0)}e^{-\beta M_0(Q_0)}\sum_{q\in\mathbb{Z}}e^{2\pi\mathcal{E}q-\beta\frac{q^2}{2K}}Z_{\rm Sch}(\phi_{b,Q_0+eq},\beta), \tag{5.61}$$

where following [97], we defined the coefficients

$$K\equiv\frac{4\pi(L^2+6r_0^2)}{3e^2r_0}=\frac{4\pi L^2r_0}{3e^2L_2^2},\quad \mathcal{E}\equiv\frac{eLr_0\sqrt{L^2+3r_0^2}}{\sqrt{4\pi G_N}(L^2+6r_0^2)}=\frac{L_2^2}{4\pi}\frac{Q_0}{r_0^2}\,. \tag{5.62}$$

It is easy to understand in general the origin of these terms. The chemical potential and mass terms do not produce linear pieces since Q_0 is chosen for them to cancel. Then the linear piece in Q comes purely from expanding $S_0=\pi\phi_0(Q_0+eq)$ to linear order. This gives

$$2\pi\mathcal{E}=e\Big(\frac{\partial S_0}{\partial Q}\Big)_{T=0}, \tag{5.63}$$

which we can verify also directly from (5.62) and matches with Sen's relation between the charge dependence of the extremal entropy and the electric field near the horizon [209]. A similar argument gives the prefactor of the quadratic piece (coming to leading order from the $\beta\mu\frac{Q}{e}-\beta M$ term) as

$$K=\frac{1}{e}\Big(\frac{\partial Q}{\partial\mu}\Big)_{T=0}, \tag{5.64}$$

which also is consistent with (5.62) and with the results of [97].

The first three terms of (5.61) give the extremal contribution to the partition function while the sum includes energy fluctuations (through the Schwarzian) and charge fluctuations. These are not decoupled since the Schwarzian coupling depends on the charge. Nevertheless it is easy to see that corrections from the charge dependence of the dilaton are suppressed in the large Q_0 limit and can be neglected (this can be checked directly from (5.55)). Then we have

$$Z_{\rm RN}[\beta,\mu] = e^{\beta\mu\frac{Q_0}{e}} e^{\pi\phi_0(Q_0)} e^{-\beta M_0(Q_0)} Z_{\rm Sch}(\phi_{b,Q_0},\beta) \sum_{q\in\mathbb{Z}} e^{2\pi\mathcal{E}q-\beta\frac{q^2}{2K}} \tag{5.65}$$

The partition function in this limit can be reproduced by a one dimensional theory that is a simplified approximation of the one presented in the previous section for small charge fluctuations around the extremal value

$$I_{\rm Sch\times U(1)} = \phi_{b,Q_0}\int_0^\beta du\ {\rm Sch}\Big(\tan\frac{\pi\tau}{\beta},u\Big) + \frac{K}{2}\int_0^\beta du\Big(\theta'(u)+i\frac{2\pi\mathcal{E}}{\beta}\tau'(u)\Big)^2, \tag{5.66}$$

written in terms of the field $\tau(u)$. This matches the result of [97] obtained from a different perspective. As explained in the introduction the main point of this chapter is to present a derivation that clarifies the fact that this analysis is true at energies lower than the gap scale. Therefore we conclude that besides matching the semiclassical thermodynamics, the quantum corrections of this theory are also reliable. The exact partition function of the Schwarzian mode was given in (5.55) and besides the semiclassical term it only contributes an extra one-loop exact $\frac{3}{2}\log T$ to the partition function. On the other hand the contribution from the $U(1)$ mode is

$$Z_{\rm U(1)}(K,\mathcal{E},\beta) = \sum_{q\in\mathbb{Z}} e^{2\pi\mathcal{E}q-\beta\frac{q^2}{2K}} = \theta_3\Big(i\frac{\beta}{2\pi K}, i\mathcal{E}\Big) \tag{5.67}$$

so the total partition function is given by

$$Z_{\rm RN}[\beta,\mu] = e^{\beta\mu\frac{Q_0}{e}+S_0(Q_0)-\beta M_0(Q_0)}\Big(\frac{\phi_{b,Q_0}}{\beta}\Big)^{3/2} e^{\frac{2\pi^2}{\beta}\phi_{b,Q_0}}\ \theta_3\Big(i\frac{\beta}{2\pi K}, i\mathcal{E}\Big), \tag{5.68}$$

where θ_3 is the Jacobi theta function. In this formula Q_0 is seen as a function of the chemical potential.

In general we do not need the full result for the U(1) mode. The partition function is dominated by a charge $q = 2\pi K\mathcal{E}/\beta$ giving a saddle point contribution $\log Z^{\rm s.p.}_{U(1)} = 2\pi^2\mathcal{E}^2K/\beta$. We can define a $U(1)$ scale by[9]

$$M_{U(1)} \equiv 2K^{-1} = M_{\mathrm{SL}(2)} \frac{3}{2\pi} \frac{e^2 L_2^4}{L^2 G_N}. \tag{5.69}$$

For $T \ll M_{U(1)}$ charge fluctuations are frozen since their spectrum does have a gap of order $M_{U(1)}$ and thermal fluctuations are not enough to overcome it. For $T \gg M_{U(1)}$ the $U(1)$ mode becomes semiclassical and its one-loop correction can contribute an extra factor of $\frac{1}{2}\log T$ to the partition function (see [63] for more details of these limits) from its approximate continuous spectrum.

For large black holes in AdS, $M_{U(1)} \sim M_{\mathrm{SL}(2)} \frac{e^2 L^2}{G_N}$ and therefore is a tunable parameter depending on e. If e is small but order one, then $M_{U(1)} \gg M_{\mathrm{SL}(2)}$ and for $T \sim M_{\mathrm{SL}(2)}$ there is no $\frac{1}{2}\log T$ contribution and charge fluctuations are frozen. If the theory is supersymmetric then $e^2 \sim G_N$ and $M_{U(1)} \sim M_{\mathrm{SL}(2)}$.

5.3.3 Density of states at $j = 0$

In the previous section we computed the partition function and free energy of the black hole. We can also look at the density of states directly as a function of energy and charge, for states of vanishing angular momentum. For this we can start from (5.59) and solve the Schwarzian theory first. This gives

$$Z_{\mathrm{RN}}[\beta, \mu] = \sum_{Q \in e\cdot\mathbb{Z}} e^{\beta\mu\frac{Q}{e}} e^{\pi\phi_0(Q)} e^{-\beta M_0(Q)} \int_0^\infty ds^2 \ \sinh(2\pi s) e^{-\beta\frac{s^2}{2\phi_{b,Q}}} \tag{5.70}$$

This can be used to automatically produced the Legendre transform of the partition function giving the density of states. Now we can define the energy as $E = M_0(Q) + \frac{s^2}{2\phi_{b,Q}}$ to rewrite this expression in a more suggestive way as

$$Z_{\mathrm{RN}}[\beta, \mu] = \sum_{Q \in e\cdot\mathbb{Z}} \int_{M_0(Q)}^\infty dE \ e^{S_0(Q)} \sinh\left[2\pi\sqrt{2\phi_{b,Q}(E - M_0(Q))}\right] \ e^{\beta\mu\frac{Q}{e} - \beta E}. \tag{5.71}$$

From this expression we can read off the density of states for each fixed charge Q sector as

$$\rho(E, Q) = e^{S_0(Q)} \sinh\left[2\pi\sqrt{2\phi_{b,Q}(E - M_0(Q))}\right] \Theta(E - M_0(Q)), \tag{5.72}$$

where $M_0(Q)$ and $S_0(Q)$ are the mass and entropy associated to an extremal black hole of charge Q while $\phi_{b,Q} = M_{\mathrm{SL}(2)}^{-1}$. At large energies we can match with semiclassical Bekenstein-Hawking

[9] If we consider a black hole in flat space $L \to \infty$ and $M_{U(1)} \to 0$, leading to large charge fluctuations. This is related to the fact that the sum over charges is divergent in flat space.

expanded around extremality, while for $E \lesssim M_{\text{SL}(2)}$ the density of states goes smoothly to zero as $E - M_0(Q) \to 0$. Therefore there is no gap of order $M_{\text{SL}(2)}$ in the spectrum. Finally, as we commented above, the path integral over the matter fields can only produce temperature-independent shifts of S_0 and M_0 in the partition function. This means that the energy dependence of the expression (5.72) is reliable in this limit.

This result is not inconsistent with the analysis of Maldacena and Strominger [77]. In that paper, the authors claim the first excited black hole state corresponds to a state with $j = 1/2$, with an energy above extremality that coincides with the gap scale. Here, we have shown that a more careful analysis of the Euclidean path integral shows the presence of excited black holes states of energy smaller than $M_{\text{SL}(2)}$, and they are all within the $j = 0$ sector.

5.3.4 The grand canonical ensemble with fixed boundary metric

Finally we will comment on the situation when we fix the metric in the boundary of AdS_4. For simplicity we will consider the case of a large black hole in AdS_4 with $r_0 \gg L$. In this case, the dimensional reduction produces a partition function given by (5.49) setting the SO(3) chemical potential to zero $\mu_{\text{SO}(3)} \to 0$[10]. This gives

$$Z_{\text{RN}}[\beta,\mu] = \sum_{Q\in e\cdot\mathbb{Z}, j\in\mathbb{Z}} (2j+1)^2 e^{-\beta\mu\frac{Q}{e}} e^{\pi\phi_0(Q,j)} e^{-\beta M_0(Q,j)} Z_{\text{Sch}}(\phi_{b,Q},\beta)\,. \tag{5.73}$$

After repeating the analysis of section 5.3.2 we can obtain the following expression

$$Z_{\text{RN}}[\beta,\mu] = e^{\beta\mu\frac{Q_0}{e}+\pi\phi_0(Q_0)-\beta M_0(Q_0)} Z_{\text{Sch}}(\phi_{b,Q},\beta) Z_{\text{U}(1)}(K,\mathcal{E},\beta) \sum_{j\in\mathbb{Z}} (2j+1)^2 e^{-\beta\frac{G_N j(j+1)}{2r_0^3}}\,. \tag{5.74}$$

Since the correction in the energy from spin $\delta M = G_N j(j+1)/2r_0^3$ is very small for large macroscopic black holes (being suppressed by G_N and also by r_0) we can approximate the contribution of the SO(3) gauge field by

$$Z_{\text{RN}}[\beta,\mu] = e^{\beta\mu\frac{Q_0}{e}+\pi\phi_0(Q_0)-\beta M_0(Q_0)} Z_{\text{Sch}}(\phi_{b,Q},\beta) Z_{\text{U}(1)}(K,\mathcal{E},\beta) \Big(\frac{G_N}{2r_0^3\beta}\Big)^{3/2}\,. \tag{5.75}$$

[10]the result for the general case can be found in appendix C.2.

Therefore at low temperatures, $T \ll T_{\mathrm{U}(1)}$, the non trivial temperature dependence of the partition functions is given by

$$Z_{\mathrm{RN}}[\beta,\mu] \sim e^{\beta\mu\frac{Q_0}{e}+S_0-\beta M_0}\Big(\frac{\phi_{b,Q_0}}{\beta}\frac{G_N}{2r_0^3\beta}\Big)^{3/2} e^{\frac{2\pi^2}{\beta}\phi_{b,Q_0}}\,. \tag{5.76}$$

As a final comment, in a similar manner to the previous section, we can write a simplified, approximate, one dimensional theory capturing the physics of these states. We need to add an extra term

$$\begin{aligned} I_{\mathrm{Sch}\times\mathrm{U}(1)\times\mathrm{SO}(3)} &= \phi_{b,Q_0}\int_0^\beta du\ \mathrm{Sch}\Big(\tan\frac{\pi\tau}{\beta},u\Big)+\frac{K}{2}\int_0^\beta du\Big(\theta'+i\frac{2\pi\mathcal{E}}{\beta}\tau'\Big)^2 \\ &\quad +\frac{K_{\mathrm{SO}(3)}}{2}\int \mathrm{Tr}\Big[h^{-1}h'+i\frac{\mu_{\mathrm{SO}(3)}}{\beta}\tau'\Big]^2, \end{aligned} \tag{5.77}$$

where $K_{\mathrm{SO}(3)} = r_0^3/G_N$. This is a simplification of the more general action written down previously in equation (5.50) since it only captures fluctuations around the angular momentum saddle-point in the sum (5.74). From the discussion here its clear that the prefactor of the $SO(3)$ action is given by

$$K_{\mathrm{SO}(3)} = \frac{1}{2}\Big(\frac{\partial J^2}{\partial E}\Big)_{T=0}. \tag{5.78}$$

Finally the gap scale for the $SO(3)$ mode is given by

$$M_{\mathrm{SO}(3)} = 2\frac{G_N}{r_0^3} = M_{\mathrm{SL}(2)}\frac{L^2}{r_0^2} \ll M_{\mathrm{SL}(2)}\,, \tag{5.79}$$

for large black holes in AdS_4. Therefore when we fix the boundary metric the sphere modes produce an extra factor of $\frac{3}{2}\log T$ as long as $T \gg M_{\mathrm{SO}(3)}$. For $T \ll M_{\mathrm{SO}(3)}$ the thermal energy is not large enough to overcome the gap of this sector, the angular momentum is frozen, and it does not contribute to $\log T$ factors. If we are interested in scales of order, $M_{\mathrm{SL}(2)}$ then we are always above the gap for the $SO(3)$ mode.

5.4 Contributions from massive Kaluza-Klein modes

In the previous section we neglected the contribution from massive Kaluza-Klein modes to the the partition function at low temperatures $T \sim M_{\mathrm{SL}(2)}$. We will argue that this is correct in this section. First, we will summarize the spectrum of masses for the remaining Kaluza-Klein modes in the Reissner-Nordström solution, following the analysis of [204]. As an example, we perform the dimensional reduction of the $4d$ scalar field in the theory, to obtain the contribution of the Kaluza-Klein modes to the action of the $2d$ theory. Then, we will argue that the partition function of massive fields does not contribute to the leading temperature dependence close to extremality.

5.4.1 A summary of the Kaluza-Klein spectrum of masses

The full analysis involving the metric KK modes and the gauge field KK modes is very complicated. Instead, since we will be most interested in the spectrum of masses, a linearized analysis is enough. This was done in detail by Michelson and Spradlin [204] (see also [210]). As we will explicitly show for the case of a $4d$ scalar field, the dimensional reduction can be performed by decomposing the fields into scalar or vector spherical harmonics (labeled by the spin ℓ) on the internal S^2 space.

At the $\ell = 0$ level [204] found two relevant modes. One is the dilaton and two-dimensional metric, which combine into JT gravity and also the s-wave of the gauge field, which gives a massless $2d$ U(1) field, as pointed out in section 5.2.2. At $\ell = 1$ level, we have a massive $2d$ scalar and vector coming from the gauge field and a massless field from the metric which coincides with the $2d$ gauge field B related to the $SO(3)$ symmetry of S^2 (which we also already identified in 5.2.2). Finally, for $\ell \geq 2$, [204] found massive graviton KK modes (although they point out they are not independent degrees of freedom on-shell) and massive vector degrees of freedom from KK modes of the dilaton and $U(1)$ gauge field. Therefore, besides the massless modes that we have already considered in section 5.2 and 5.3, we solely have massive fields whose minimum mass is given by $m^2 = 1/\chi^2$.

5.4.2 An example: the dimensional reduction of a $4d$ scalar

To clarify the summary, we will give the simplest example of a massive mode appearing in the KK reduction of a scalar field in four dimensions. The action for a scalar field X of mass m is

$$I_X = \int d^4x \sqrt{g_4}(g_4^{AB}\partial_A X \partial_B X + m^2 X^2). \tag{5.80}$$

In order to carry out the KK reduction we wrote an ansatz for the metric (5.15). To compute the action of the KK modes it is useful to write explicitly the inverse metric in this notation, which is given by

$$g_4^{\mu\nu} = \chi^{1/2} g_2^{\mu\nu}, \quad g_4^{m\mu} = -\chi^{1/2} B^{a\mu}\xi_a^m, \quad g_4^{mn} = \frac{1}{\chi}h^{mn} + \chi^{1/2} B_\mu^a B^{b\mu}\xi_a^m \xi_b^n, \tag{5.81}$$

where the μ index of B is raised with the $2d$ metric. Also, the determinant of the metric is $g_4 = \chi g_2 h$. We will expand the scalar field into spherical harmonics as

$$X(x,y) = \sum_{\ell} \mathbf{X}_\ell(x) \cdot \mathbf{Y}^\ell(y) \tag{5.82}$$

where we use the (uncommon) notation of denoting the scalar spherical harmonics of spin ℓ as a vector $\mathbf{Y}^\ell(y) = [Y^\ell_{-\ell}(y), Y^\ell_{-\ell+1}(y), \ldots, Y^\ell_\ell(y)]$. Correspondingly, we denoted the modes of the scalar field also a vector in a similar way $\mathbf{X}_\ell(x) = [X^{-\ell}_\ell, X^{-\ell+1}_\ell, \ldots, X^\ell_\ell]$. Then the inner product above denotes $\mathbf{X}_\ell(x) \cdot \mathbf{Y}^\ell(y) \equiv \sum_m X^m_\ell Y^m_\ell$.

We will begin by reducing the kinetic term. For this we need the inverse metric and its clear it will produce terms linear and quadratic in the gauge field B. The following formulas for integrating spherical harmonics will be useful

$$\int_{S^2} dy \sqrt{h} \mathbf{Y}^\dagger_\ell (\xi_a \cdot \partial) \mathbf{Y}_{\ell'} = i T^a \delta_{\ell'\ell}, \quad \int_{S^2} dy \sqrt{h} (\xi_a \cdot \partial) \mathbf{Y}^\dagger_\ell (\xi_b \cdot \partial) \mathbf{Y}_{\ell'} = -T^a T^b \delta_{\ell'\ell}, \tag{5.83}$$

where ξ_a denote the Killing vectors of the sphere and since this is a matrix equation the T^a are matrices giving the spin ℓ representation of the rotation group. Then we can obtain the reduction of the kinetic term as

$$\int d^4x \sqrt{g_4} (\partial X)^2 = \sum_\ell \int d^2x \sqrt{g} \chi^{1/2} \left[(D_\mu \mathbf{X}_\ell)^\dagger (D^\mu \mathbf{X}_\ell) + \frac{\ell(\ell+1)}{\chi} \mathbf{X}^\dagger_\ell \mathbf{X}_\ell \right], \tag{5.84}$$

where we also used the fact that $\Box_{S^2} \mathbf{Y} = -\ell(\ell+1)\mathbf{Y}$, where $\Box_{S^2}$ is the laplacian on the two-sphere. We also defined the covariant derivative

$$D_\mu \mathbf{X} = \partial_\mu \mathbf{X} - i B^a_\mu (T^a)_\ell \mathbf{X}, \tag{5.85}$$

where $(T^a)_\ell$ are the spin ℓ representation matrices acting on the vector $\mathbf{X}$. Adding the mass term, we can obtain the full $2d$ action for the KK reduction of the scalar field as

$$I_X = \sum_\ell \int d^2x \sqrt{g} \chi^{1/2} (|D\mathbf{X}_\ell|^2 + m^2_\ell |\mathbf{X}_\ell|^2), \quad m^2_\ell = m^2 + \frac{\ell(\ell+1)}{\chi}. \tag{5.86}$$

To summarize, a single scalar field KK reduces to a tower of massive fields $\mathbf{X}_\ell$ of dimension $(2\ell+1)$ with $\ell = 0, 1, 2, \ldots$ with increasing mass.

This is a complicated action: besides being coupled to the two-dimensional metric, it is also coupled to the dilaton. The dilaton coupling is not particularly useful in the FAR region since the dilaton varies with the radius. Of course, in this region, the picture of a single scalar in the $4d$ black hole background is more appropriate. In the NHR this becomes very useful since $\chi \approx \chi_0 = r_0^2$. Then we end up, after rescaling $\mathbf{X}_\ell \to r_0^{-1/2}\mathbf{X}_\ell$ in the NHR with a tower of KK modes with action

$$I_X = \sum_\ell \int d^2x\sqrt{g}(|D\mathbf{X}_\ell|^2 + m_\ell^2|\mathbf{X}_\ell|^2), \quad m_\ell^2 = m^2 + \frac{\ell(\ell+1)}{r_0^2}, \tag{5.87}$$

fixing the KK mode scale $\Lambda_{\text{KK}} \sim 1/r_0$. Naively it seems the correction to the mass is small, but we will take such low temperatures that $\beta\Lambda_{\text{KK}} \gg 1$. Then we end up with a tower of canonically normalized free fields.

We can see what happens when turning on scalar field interactions in the initial $4d$ theory. To simplify lets consider self interactions of the scalar field $I_n = \lambda_n \int d^4x\sqrt{g_4}X^n$. After KK reducing, this produces a term of order $\lambda_n r_0$. After rescaling the scalar field by $r_0^{-1/2}$ to make the $2d$ action canonically normalized the effective two dimensional coupling becomes $\lambda_n^{\text{2d}} = \lambda_n r_0^{1-n/2}$. Therefore even if selfinteractions are large in four dimensions, the reduction to two dimensions gives $\lambda_n^{\text{2d}} \to 0$ (for large r_0) and therefore, in the NHR, its enough to consider free fields. Moreover, since we will only consider states for which fluctuations in the gauge field are small $B \sim j/r_0^3$ we will also neglect its coupling to $2d$ matter.

5.4.3 The massive Kaluza-Klein modes in the partition function

As we have summarized in the previous subsection, besides the $2d$ massless gravitational and gauge degrees of freedom, all other modes generated by the dimensional reduction have masses given by the value of the dilaton field at the horizon $1/\chi^2$. Furthermore, as we observed in section 5.2, the dominating background for the $SO(3)$ gauge fields is that in which they are turned off, $B^a = 0$. Therefore, we will assume that the massive modes are decoupled from the $SO(3)$ gauge field. With this set-up in mind, we can now proceed to compute the contribution to the partition function of the massive KK modes. To show that such fields do not yield any correction to the $\log(T)$ term, we will solely focus on scalar fields and compute their contribution in the NHR. As discussed in preceding subsections, in such a region, their mass is constant and given by $m^2 = 1/r_0^2$. We will also ignore the fluctuations of the Schwarzian boundary mode because the contribution of these fluctuations to the massive modes is suppressed by the scale ϵ/r_0 from (5.37).

Therefore, we will compute the contribution of the massive modes in a circular patch of the Poincaré disk, where the proper length of the boundary is $\ell = \beta L_2/\varepsilon$ and its extrinsic curvature is constant. We will choose Dirichlet boundary conditions for the scalar field $\mathbf{X}|_{\partial M_{\rm NHR}} = 0$ at the boundary $\partial M_{\rm NHR}$; this is consistent with the classical solution $\mathbf{X}$ for the field in the FAR when fixing $\mathbf{X}|_{M_2} = 0$. The contribution of a KK mode in the NHR is then abstractly given by $Z_{KK} = \det(g^{\rm NHR}_{\mu\nu}\partial^\mu\partial^\nu + r_0^{-2})^{-1/2}$.

To compute the β-dependence of this determinant we will us the Gelfand-Yaglom method [211], studying the assymptotics of solutions to the Klein-Gordon equation $(\Box_{\rm NHR}+m^2)\psi = 0$.[11] Parametrizing the AdS_2 coordinates by $ds^2_{\rm NHR} = L_2^2\left(dr^2 + \sinh^2(r)d\phi^2\right)$, we find that the boundary is located at $r_{\partial \rm NHR} = \log\left(\frac{\beta}{\pi\varepsilon}\right) + O(\beta^2/\varepsilon^2) \to \infty$. Expanding $\psi(r,\phi) = \psi_k(r)e^{ik\phi}$ with $k \in \mathbb{Z}$, the Klein-Gordon equation becomes

$$\frac{1}{\sinh r}\partial_r(\sinh r\partial_r\psi_k) - \frac{k^2}{\sinh^2 r}\psi_k + (mL_2)^2\psi_k = 0\,, \tag{5.88}$$

whose regular solution at the horizon ($r = 0$) is given by [12]

$$\psi_k(r) = \frac{(\tanh r)^{|k|}}{(\cosh r)^{\Delta_+}}\,{}_2F_1\left(\frac{1}{4} + \frac{|k|}{2} + \frac{\nu}{2}, \frac{3}{4} + \frac{|k|}{2} + \frac{\nu}{2}, 1 + |k|, \tanh^2 r\right)\,, \tag{5.89}$$

where we define

$$\Delta_\pm \equiv \frac{1}{2} \pm \sqrt{\frac{1}{4} + (mL_2)^2}\,, \qquad \nu = \sqrt{\frac{1}{4} + (mL_2)^2}\,. \tag{5.90}$$

The Gelfand-Yaglom method requires that we normalize ψ_k such that its derivative at $r = 0$ is independent of m; this is indeed the case, when expanding (5.89) to first order in r around the horizon. Asymptotically, for $r = r_{\partial \rm NHR} \to \infty$, the solution is given by

$$\psi_k = \frac{\Gamma(1+|k|)2^{|k|}}{\sqrt{\pi}}\left[\frac{1}{(2\cosh r_{\partial \rm NHR})^{\Delta_-}}\frac{\Gamma(\Delta_+ - 1/2)}{\Gamma(\Delta_+ + |k|)} + \frac{1}{(2\cosh r_{\partial \rm NHR})^{\Delta_+}}\frac{\Gamma(\Delta_- - 1/2)}{\Gamma(\Delta_- + |k|)}\right]\,. \tag{5.91}$$

The Gelfand-Yaglom theorem states that the determinant with Dirichlet boundary conditions for the scalar field is given by $\det(\Box_{\rm NHR} + m^2) = \mathcal{N}(\beta,\varepsilon)\prod_k \psi_k(r_{\partial \rm NHR})$, where $\mathcal{N}(\beta,\varepsilon)$ is a mass-independent proportionality constant.

The contribution to the free energy coming from the determinant is then given by,

$$\log Z_{KK} = -\frac{1}{2}\log\mathcal{N}(\beta,\varepsilon) - \frac{1}{2}\sum_{k\in\mathbb{Z}}\log\left[\frac{\Gamma(1+|k|)2^{|k|}}{\sqrt{\pi}(2\cosh r_{\partial \rm NHR})^{\Delta_-}}\frac{\Gamma(\Delta_+ - 1/2)}{\Gamma(\Delta_+ + |k|)}\right]\,. \tag{5.92}$$

[11] This strategy was previously used to study the mass-dependence of the determinant [136].
[12] The other solution diverges at the horizon.

To determine $\mathcal{N}(\beta, \varepsilon)$ we use the result for the partition function of a massless scalar on a circular patch of the Poincaré disk [36]. Since the massless scalar can be treated as a $2d$ CFT, the result can be determined by computing the Weyl anomaly when mapping a unit-disk in flat-space to the circular AdS_2 patch of interest. The first few orders in the large β expansion of the free energy obtained from the Weyl-anomaly are given by,

$$\log Z_{m^2=0} = \frac{c}{24}\frac{\beta}{\pi\varepsilon} + \frac{c}{6}\Big[\log(2L_2) - \frac{1}{2}\Big] + O\Big(\frac{\varepsilon}{\beta}\Big), \tag{5.93}$$

where $c = 1$ is the central charge of one free boson [13]. The term at order $O(\beta/\varepsilon)$ can in principle be canceled by adding a cosmological constant counter-term to the boundary of the NHR, $I_{\text{counter-term},\,CFT} = \int_0^\beta du\ c\sqrt{h_{uu}}/(24\pi)$. However, since we are solely interested in reproducing the $\log\beta$ dependence of the free energy we will not delve into how this term is reproduced by studying the coupling of these scalars to the FAR.

At such low temperatures, the Schwarzian mode is strongly coupled, so we might be worried that it can affect the answer. In [36] it was observed that the boundary Schwarzian fluctuations lead to correction of $O(\varepsilon)$ to the partition function (5.93). Since we expect the same to be true when turning on a mass, the contribution of the Schwarzian fluctuations to the partition function of the Kaluza-Klein fields can be safely ignored.

Therefore, up to terms proportional to β/ε obtained from the counter-term, this fixes

$$\log Z_{KK} = \frac{1}{6}\Big[\log(2L_2) - \frac{1}{2}\Big] - \frac{1}{2}\sum_{k\in\mathbb{Z}}\log\left[\frac{1}{(2\cosh r_{\partial\text{NHR}})^{\Delta_-}}\frac{\Gamma(1+|k|)}{\Gamma(\Delta_+ + |k|)}\right]. \tag{5.94}$$

The sum in (5.92) needs to be regularized in order for it to converge; in principle, this can be done by accounting for the divergent non-universal terms in the massless partition function (5.93). The β-dependent factor in the sum appears through the relation $r_{\partial\text{NHR}} = \log\left(\frac{\beta}{\pi\varepsilon}\right)$; consequently, the sum is given by $-\sum_{k\in\mathbb{Z}}\Delta_-\log(\cosh r_{\partial\text{NHR}})$ which vanishes in ζ-function regularization. Therefore, the contribution of the KK-modes to the partition function is given by

$$\log Z_{KK} = \frac{1}{6}\Big[\log(2L_2) - \frac{1}{2}\Big] - \frac{1}{2}\sum_{k\in\mathbb{Z}}\log\frac{\Gamma(1+|k|)}{\Gamma(\Delta_+ + |k|)}, \tag{5.95}$$

which, to leading order, is β-independent. In conclusion, to leading order in $O(1/\phi_0)$, the KK modes only affect the entropy of the black hole and not the shape of the density of states. Consequently,

[13] To get this result, we write the metric of the hyperbolic disk at finite cut-off g as $g = e^{2\rho}\widehat{g}$ where $\widehat{g}$ is the flat unit disk metric. Then we evaluate the Liouville action for the particular choice of ρ associated to the hyperbolic disk and expand for small ε.

they also to do not change our prior conclusion about the absence of near-extremal black hole gap.

Finally, we will quickly go over a more direct (yet less rigurous) method to compute the functional determinant following [79] [14]. The starting point is again $ds^2_{\rm NHR} = L_2^2\left(dr^2 + \sinh^2(r)d\phi^2\right)$ with a cutoff at $r_{\partial\rm NHR}$ (for simplicity we turn off the Schwarzian mode). We will first take the large cut-off limit for the matter fields and impose $\psi \sim (\cosh r_{\partial\rm NHR})^{-\Delta_+}$ giving eigenvalues that depend only on L_2. Then the contribution from the matter field to the partition function is [79]

$$\log Z_{\rm matter} = (\cosh r_{\partial\rm NHR} - 1)\int_{\epsilon_{\rm UV}}^{\infty} ds \frac{1}{s}\int_0^{\infty} d\lambda(\lambda\tanh\pi\lambda)e^{-s\left[\frac{\lambda^2+\frac{1}{4}}{L_2^2}+m^2\right]}. \tag{5.96}$$

The whole temperature dependence comes then from the prefactor through $\sinh(r_{\partial\rm NHR}) = \frac{\beta}{2\pi\epsilon}$ and this is true regardless of the mass. Expanding at large $r_{\partial\rm NHR}$ gives

$$\cosh(r_{\partial\rm NHR}) - 1 = \frac{\beta}{2\pi\epsilon} - 1 + \mathcal{O}(\epsilon). \tag{5.97}$$

From this expression we can easily see the matter contribution is only a shift of the extremal mass, or a temperature independent (L_2 dependent) finite correction to the partition function which potentially can only correct S_0. Following [79] one could even resum the whole tower of KK modes and reach the same conclusion. One might wonder whether imposing boundary conditions for ψ at a finite cut-off might affect the temperature dependence. However, we have already checked through the Gelfand-Yaglom theorem that this does not happen.

5.5 Outlook

In this chapter, we have computed the partition function of $4d$ near-extremal charged and of slowly-spinning black holes, in the canonical and grand canonical ensembles. By showing that we can reliably neglect all massive Kaluza-Klein modes and by solving the path integral for the remaining massless mode in the near-horizon region, we have shown that our result can be trusted down to low-temperatures, smaller than the scale $\sim M_{\rm SL(2)}$. At this energy scale, we find a continuum of states, disproving the conjecture that near-extremal black holes exhibit a mass gap of order $M_{\rm SL(2)}$ above the extremal state. The existence of a continuum of states suggests that the degeneracy of the extremal state is not given by the naive extremal entropy, fixed by the horizon area. Instead, the horizon area fixes the scaling of the density of states and the level spacing of the states. However,

[14] We would like to thank A. Castro for discussions about the relation between the calculation in this chapter and the previously studied $\log A$ terms [79, 80, 81, 82].

as we will discuss in the following subsection, to make a quantitative statement about the scale of this extremal degeneracy, we need to discuss possible non-perturbative contributions to the $2d$ path integral.

The process of solving the path integral for the massless modes in the $2d$ dimensionally reduced theory, involved obtaining an equivalent $1d$ theory which can be thought to live on a curve at the boundary of the throat, between the near-horizon region and the far-away region. This equivalent $1d$ theory is given by the Schwarzian coupled to a particle moving on a $U(1) \times SO(3)$ group manifold. Generally, the potential of the particle moving on the $U(1) \times SO(3)$ is quite complicated. However, when looking at the theory that approximates the charge and angular momentum fluctuations in the grand canonical ensemble for black holes in AdS_4, the theory is simply given by:

$$I_{\text{Sch}\times\text{U}(1)\times\text{SO}(3)} = I_{\text{Sch}}[\tau] + I_{\text{U}(1)}[\theta,\tau] + I_{\text{SO}(3)}[h,\tau] \tag{5.98}$$

where we defined the Schwarzian, $U(1)$ and $SO(3)$ contributions of the action as

$$\begin{aligned} I_{\text{Sch}}[\tau] &= \frac{1}{M_{\text{SL}(2)}}\int_0^\beta du\ \text{Sch}\Big(\tan\frac{\pi\tau}{\beta},u\Big), & (5.99)\\ I_{\text{U}(1)}[\theta,\tau] &= \frac{1}{M_{\text{U}(1)}}\int_0^\beta du\Big(\theta' + i\frac{2\pi\mathcal{E}}{\beta}\tau'\Big)^2, & (5.100)\\ I_{\text{SO}(3)}[h,\tau] &= \frac{1}{M_{\text{SO}(3)}}\int_0^\beta du\ \text{Tr}\Big(h^{-1}h' + i\frac{\mu_{\text{SO}(3)}}{\beta}\tau'\Big)^2, & (5.101)\end{aligned}$$

where $\theta(u)$ is a compact scalar and $h(u)$ is an element of $SO(3)$ and the mass scales $M_{\text{SL}(2)}$, $M_{\text{U}(1)}$ and $M_{\text{SO}(3)}$ are fixed by thermodynamic relations. Additionally, $M_{\text{SL}(2)}$, $M_{\text{U}(1)}$ and $M_{\text{SO}(3)}$ can be viewed as the breaking scales for each of their associated symmetries ($SL(2,\mathbb{R})$, $U(1)$ and, respectively, $SO(3)$) for the near-horizon region of an ensemble of near-extremal black holes.

Beyond the goal of resolving the mass-gap puzzle for near-extremal Reissner-Nordström black holes, the effective $2d$ dimensionally reduced theory of dilaton gravity (and its equivalent boundary theory) provides a proper framework to resolve several future questions, some of which we discuss below.

Other black holes and different matter contents

While we have successfully analyzed the case of Kerr-Newman black holes with small spin, for which we could neglect the sourcing of massive Kaluza-Klein modes for some of the metric components, it would be instructive to compute the partition function of Kerr-Newman black holes for arbitrary spin. An effective $1d$ boundary theory capturing the dynamics of such black holes was recently described in [89, 96, 99]; however, the quantum fluctuations relevant for understanding the mass-gap puzzle were not analyzed. In the framework described above, resolving such a puzzle for Kerr-Newman black holes amounts to studying how the massive Kaluza-Klein modes are sourced and whether their fluctuations could significantly affect the partition function. If the analysis in section 5.4 follows even in when such fields have a non-trivial classical saddle-point, then it is likely that near-extremal Kerr-Newman black holes do not exhibit a gap for arbitrary angular momenta.

Perhaps an even more intriguing case is that of near-extremal (and, at the same time, near-BPS) black holes in $4d$ $\mathcal{N}=2$ supergravity. As mentioned in the introduction, in such cases, microscopic string theory constructions [75, 76] suggest that the scale $M_{\mathrm{SL}(2)}$ should genuinely be identified as the gap scale in the spectrum of near-extremal black holes masses. While an analysis of the proper effective theory describing such black holes is underway [78], perhaps some intuition can be gained by looking at a related theory that has less supersymmetry: the $\mathcal{N}=2$ super-Schwarzian. In such a theory, the partition function was computed [25, 40] and its resulting spectrum indeed exhibits a gap whose scale is fixed by the inverse of the super-Schwarzian coupling. Since the inverse of the super-Schwarzian coupling coincides with the conjectured gap [72, 74, 88], it is tantalizing to believe that the thermodynamic mass-gap observed in [75, 76] is indeed an artifact of supersymmetry [15].

It would also be interesting to study the contribution of charged scalar or fermionic fields to the partition function of the near-extremal Reissner-Nordström black holes. In AdS, the presence of such fields has been widely used to study the holographic dual for several phases of matter [212, 213, 214, 215]. For black holes in flat space, it would be nice to compute the contribution from charged matter with $q/m > 1$ and see its effect at the level of the microstates.

Finally, it would be interesting to consider black holes in AdS_D, which have known CFT duals. The result of this chapter can be interpreted as a universality of their spectrum when looking at large charges and low temperatures. Those degrees of freedom should be properly described by the effective theory found in this chapter. One approach to this problem can be to apply the conformal bootstrap at large charge for higher dimensional CFT (this was done for the case of rotating BTZ in [84]). Another, perhaps more ambitious, approach is to start directly with the boundary theory and try to derive an equivalent quantum mechanical system in the extremal limit. Such a theory would be similar to SYK (would reduce to the Schwarzian and be maximally chaotic) but would be dual to a local bulk (as opposed to also other higher dimensional versions of SYK [216, 157, 217]).

[15] The exact density of states of the $\mathcal{N}=2$ Schwarzian presents a delta function at extremality with weight e^{S_0} which would be consistent with a highly degenerate extremal black hole. This degeneracy is also consistent with previous microscopic counting and shows that it also relies on supersymmetry to work.

Non perturbative effects

It was recently made precise how including non-trivial topologies in the Euclidean path integral of $2d$ dilaton gravity can fix certain problems with unitarity [37] (the price to pay when accounting for such non-trivial topologies is to allow for disorder in the boundary theory). In the case of JT gravity the non-perturbative completion is given by a random matrix and one has to sum over all two-dimensional topologies consistent with the boundary conditions. It would be tempting to trust these corrections in the context of a near-extremal black hole. Then the spectrum would be random, with an averaged level spacing of order e^{-S_0} and a non-degenerate ground state (moreover there is an exponentially suppressed probability of lying below the extremality bound, but this can be avoided by considering supersymmetry).

Of course, this is too optimistic in the case of $4d$ near-extremal black holes. Other non-perturbative effects can appear from the $4d$ perspective, which are not captured by JT gravity. For example, one can consider multi-black hole solutions [73] or topology changes that involve the whole $4D$ space.

Even within JT gravity, there can be configurations with conical defects in two dimensions, which are smooth when uplifted to the higher dimensional metric. These can be important and hint into solving problems with pure $3d$ gravity [218]. For near-extremal black holes in higher dimensions, one would need to include similar geometries.

The replica ensemble and the Page curve

A procedure was recently found to reproduce the Page curve from the gravitational Euclidean path integral in JT gravity. In order to reproduce the Page curve [219, 220] computed the radiation Renyi entropy, including replica wormholes. In those calculations, one couples JT gravity in AdS_2 with a bath in flat space, making the evaporation of the black hole possible. This setup can be directly understood as an approximate description of an evaporating near-extremal black hole in four dimensions (we can consider this at temperatures $T \gg M_{SL(2)}$ to simplify the problem so that backreaction around each semiclassical saddle is suppressed).

To turn the recent calculations into a justified approximation, we have to make the following changes. First, the gravitational part of the theory should be JT gravity coupled to the appropriate gauge fields (both KK and the ones sourcing extremality) and coupled to a matter CFT. This theory should then be glued to the $2d$ s-wave reduction of the four-dimensional extremal black hole metric in asymptotically flat space (we assume in this region gravity is weak). This is justified as long as the dominant evaporation channel happens through s-waves (if higher angular momenta are exponentially suppressed). Since this is usually the case, the calculation of [219] can be repeated in the context of $4d$ near-extremal black holes. The main complication is to account for the contribution from all the matter fields in this new geometry, and we hope to address this in more detail in future work.

Appendix A

The analysis of JT gravity at finite cut-off

A.1 Additional checks

A.1.1 WdW with varying dilaton

In this section we will check our formula (3.44) in the case of a varying dilaton with an arbitrary profile $\phi_b(u)$. We will still work in the limit of large L and ϕ_b such that we are working near the boundary of AdS_2. Expanding the solution of the WdW equation gives

$$\Psi_{\rm HH}[\phi_b(u), L] = \int dM \rho_{\rm HH}(M)\ \exp\left[\int_0^L du \left(\phi_b - \frac{M}{2\phi_b} + \frac{(\partial_u \phi_b)^2}{2\phi_b} + \dots\right)\right] \tag{A.1}$$

where the dots denote terms that are subleading in this limit. The first term produces the usual divergence piece $\int_0^L du\, \phi_b(u)$. The second term after integrating over M would produce the Schwarzian partition function with an effective length given by $\ell = \int_0^L \frac{du}{\phi_b(u)}$, which can be interpreted as a renormalized length. The final answer is then

$$\Psi_{\rm HH}[L, \phi] = e^{\int_0^L du\, \phi_b(u)} Z_{\rm Sch}\left(\int_0^L \frac{du}{\phi_b(u)}\right)\ e^{\frac{1}{2}\int_0^L du \frac{(\partial_u \phi_b)^2}{\phi_b}}. \tag{A.2}$$

Now we will show the full answer, including the last term in (A.2), $\frac{1}{2}\int_0^L du \frac{(\partial_u \phi)^2}{\phi}$, can be reproduced by the Euclidean path integral through the Schwarzian action.

For a varying dilaton the bulk path integral of JT gravity can be reduced to

$$\int \mathcal{D}g\mathcal{D}\phi\ e^{-I_{\rm JT}[\phi,g]} \to e^{\int_0^L du\phi_b(u)} \int \frac{\mathcal{D}f}{SL(2,\mathbb{R})}\ e^{\int_0^L du\phi_b(u)\,{\rm Sch}(F(u),u)}, \quad F = \tan\pi f \tag{A.3}$$

For simplicity we will assume that $\phi_b(u) > 0$. Following [25] we can compute this path integral using the composition rule of the Schwarzian derivative

$$\mathrm{Sch}(F(\tilde{u}(u)), u) = \mathrm{Sch}(F, \tilde{u})(\partial_u \tilde{u})^2 + \mathrm{Sch}(\tilde{u}, u). \tag{A.4}$$

We can pick the reparametrization to be $\partial_u \tilde{u} = 1/\phi_b(u)$. This implies in terms of the coordinate $\tilde{u}$ the total proper length is given by $\tilde{L} = \int_0^L du/\phi_b(u)$. This simplifies the Schwarzian term and we can write the second term as

$$\int_0^L du\ \phi_b(u)\,\mathrm{Sch}(\tilde{u}, u) = \frac{1}{2}\int_0^L du \frac{(\partial_u \phi_b)^2}{\phi_b} \tag{A.5}$$

up to total derivative terms that cancel thanks to the periodicity condition of the dilaton. Then we can rewrite the path integral as

$$\int \mathcal{D}g\mathcal{D}\phi\ e^{-I_{\rm JT}[\phi,g]} \quad \to \quad e^{\int_0^L du\,\phi_b(u) + \frac{1}{2}\int_0^L du \frac{(\partial_u\phi_b)^2}{\phi_b}} \int \frac{\mathcal{D}f}{SL(2,\mathbb{R})}\ e^{\int_0^{\tilde{L}} d\tilde{u}\,\mathrm{Sch}(F,\tilde{u})}, \tag{A.6}$$

$$= e^{\int_0^L du\,\phi_b(u) + \frac{1}{2}\int_0^L du \frac{(\partial_u\phi_b)^2}{\phi_b}} Z_{\rm Sch}\left(\tilde{L} = \int_0^L \frac{du}{\phi_b}\right) \tag{A.7}$$

which matches with the result coming from the WdW wavefunction (A.2). This is a nontrivial check of our proposal that $\Psi_{\rm HH}$ in (3.44) computes the JT gravity path integral at finite cutoff.

A.1.2 JT gravity with Neumann boundary conditions

To provide a further check of the form of the extrinsic curvature K at finite cutoff (3.64), we can study the theory with Neumann boundary conditions, when fixing the extrinsic curvature $K[z(u)] = K_{\rm b}$ instead of the boundary dilaton value ϕ_r and when fixing the proper length L to be finite in both cases.[1] We will work in Poincaré coordinates (3.49). Since $K_b > 0$ it means (in our conventions) that we are considering a vector encircling a surface with genus 0 (normal vector pointing outwards). On the Poincaré plane, curves of constant K_b are circles, semi-circles (that intersect the H_2 boundary)

[1] A more detailed analysis of the theory with such boundary conditions will be presented in [150].

or lines. All of them can be parametrized in the Poincaré boundary coordinates $\tau(u)$ and $x(u)$ as:

$$\tau(u) = a + b\cos(u)\,, \qquad x(u) = d + b\sin(u)\,, \qquad K_b = \frac{d}{b}\,, \qquad \sqrt{\gamma_{uu}} = \frac{b}{d + b\sin u}\,, \tag{A.8}$$

with b, $d \in \mathbb{R}$. Note that if we want the circle above to be fully contained within the Poincaré half-plane (with $x > 0$) we need to require that $d > 0$ and $d \geq b$ which implies $K_b \geq 1$. Thus, for contractible boundaries which contain the surface inside of them we must have $K_b \geq 1$.

For this value of K_b, the boundary proper length is restricted to be

$$\frac{\beta}{\varepsilon} = \int du \sqrt{\gamma_{uu}} = \frac{2\pi}{\sqrt{(K_b+1)(K_b-1)}}\,. \tag{A.9}$$

Therefore, the partition function with Neuman boundary conditions should solely isolate configurations which obey (A.9). A non-trivial check will be to recover this geometric constraint by going from the partition function with Dirichlet boundary conditions (for which we obtained the action (3.64)) and the partition function with Neumann boundary conditions.

In the phase space of JT gravity $K[z(u)]$ and $\phi(u)$ are canonical conjugate variables on the boundary. Therefore, in order to switch between the two boundary conditions at the level of the path integral, we should be able to integrate out $\phi_r(u)$ to obtain the partition function with Neumann boundary conditions. Explicitly we have that,[2]

$$\begin{aligned} Z_{\rm N}[K_b(u),\, L] &= \int_{\tilde{\phi}_b - i\infty}^{\tilde{\phi}_b + i\infty} D\phi_b(u) Z_{JT}[\phi_b(u),\, L]\, e^{\frac{1}{\varepsilon}\int_0^\beta du\, \phi_b(u)(1-K_b(u))} \\ &= \int_{\tilde{\phi}_b - i\infty}^{\tilde{\phi}_b + i\infty} D\phi_b(u) \int D\phi D g_{\mu\nu}\, e^{\phi_0 \chi(\mathcal{M}) - S_{\rm bulk}[\phi, g_{\mu\nu}] + \frac{1}{\varepsilon}\int du\, \phi_b(u)(K - K_b(u))} \\ &\sim \int D\phi D g_{\mu\nu}\, e^{\phi_0 \chi(\mathcal{M}) - S_{\rm bulk}[\phi, g_{\mu\nu}]} \prod_{u \in \partial\mathcal{M}} \delta(K(u) - K_b(u))\,. \end{aligned} \tag{A.10}$$

which of course fixes the extrinsic curvature on the boundary. To simplify our computation, we will work with the "renormalized" extrinsic curvature $K_{b,r}$, defined as $K_b \equiv 1 + \varepsilon^2 K_{b,r}$ and choose a constant value for $K_{b,r}$.

Using the formula (3.64) for $K[z(u)]$ in (A.10) we can rewrite the second line in terms of a path

[2]Where $\tilde{\phi}_b$ is some arbitrary constant which is used to shift the contour along the real axis.

integral for the Schwarzian mode $z(u)$:

$$Z_N\left[K_b = 1+\varepsilon^2 K_{b,r},\, L=\beta/\varepsilon\right] = \int \frac{d\mu[z(u)]}{SL(2,\mathbb{R})} \prod_{u\in\partial\mathcal{M}} \delta\Bigg(\sqrt{1+2\varepsilon^2 \text{Sch}(z(u),u)} - 1 - \varepsilon^2 K_{b,r} + \text{ derivatives of Sch.}\Bigg). \tag{A.11}$$

One set of solutions for which the δ-function in (A.11) are the configurations for which the Schwarzian is a constant (related to $K_{b,r}$) for which all the derivatives of the Schwarzian vanish.[3] Specifically, for such configurations which obey $z(0) = z(\beta)$, we have that $z(u) = \tan(\pi u/\beta)$, which yields:

$$\sqrt{1+4\varepsilon^2\frac{\pi^2}{\beta^2}} - 1 = \varepsilon^2 K_{b,r} \qquad \Rightarrow \qquad \frac{\beta}{\varepsilon} = \frac{2\pi}{\sqrt{\varepsilon^2 K_{b,r}(2+\varepsilon^2 K_{b,r})}} = \frac{2\pi}{\sqrt{(K_b+1)(K_b-1)}} \tag{A.12}$$

which exactly matches the constraint (A.9). This is a strong consistency check that the relation between the deformed Schwarzian action (3.64) and the extrinsic curvature when moving to finite cutoff.

A.2 General solution to (3.47)

In this appendix we present a more general analysis of the differential equation (3.47), which we reproduce here for convenience,

$$\left[4\lambda\partial_\lambda\partial_\beta + 2\beta\partial_\beta^2 - \left(\frac{4\lambda}{\beta} - 1\right)\partial_\lambda\right] Z_\lambda(\beta) = 0. \tag{3.47}$$

In particular, since (3.4) appears (at least naively) to not converge and the integral transform (3.5) is not well-defined for the sign of λ, i.e. $\lambda > 0$, which is appropriate for JT gravity at finite cutoff, the solution to the differential equation provides a solution for the partition function for that sign.

To solve the differential equation (3.47) it is useful to decouple λ and β. This can be done by defining $R = \beta/(8\lambda)$ and $e^\sigma = \beta/(2C)$ and writing the problem in terms of R and σ. The differential equation becomes,

$$-R^2(\partial_R^2 + 4\partial_R)Z + (\partial_\sigma^2 - \partial_\sigma)Z = 0 \tag{A.13}$$

[3] It is possible that there are other solutions which we do not account for in (A.11) that do not have $\text{Sch}(z(u),u)$ constant but have the sum between the non-derivative terms and derivative terms in (A.11) still yield the overall constant $1+\varepsilon^2 K_{b,r}$. While we do not analyze the possible existence of these configuration, it is intriguing that they do not affect the result of (A.12). We will once again ignore non-perturbative corrections in ε.

By using seperation of variables we find that the general solution is,

$$Z(R,\sigma) = \int_{-\infty}^{\infty} d\nu e^{-\nu\sigma} \sqrt{R} e^{-2R} \left(a_\nu K_{1/2+\nu}(-2R) + b_\nu K_{1/2+\nu}(2R) \right) \tag{A.14}$$

where ν is the related to the seperating contant. We are interested in find the solution with the Schwarzian boundary condition at $R \to \infty$. Expanding the above general solution for $R \to \infty$ we find

$$Z_0 = \lim_{R\to\infty} Z(R,\sigma) = -i\frac{\sqrt{\pi}}{2} \int_{-\infty}^{\infty} d\nu e^{-\nu\sigma} a_\nu. \tag{A.15}$$

Notice that the b_ν coefficients do not play any role, since the Bessel function with positive argument goes as e^{-4R}. The function Z_0 is given by Schwarzian partition function,

$$Z_0 = \left(\frac{1}{2Ce^\sigma} \right)^{3/2} e^{\pi^2 e^{-\sigma}}, \tag{A.16}$$

Expanding this in e^σ fixes the coefficients a_ν and after resumming using the multiplicative theorem for the Bessel $K_s(z)$ functions,

$$\alpha^{-s} K_s(\alpha z) = \sum_{n=0}^{\infty} \frac{(-1)^n}{2^n n!} (\alpha^2 - 1)^n z^n K_{s+n}(z), \tag{A.17}$$

we find the solution with the boundary condition (A.15) to be,

$$\begin{aligned} Z(R,\sigma) = i\frac{1}{\sqrt{2\pi C^3}} \frac{R^{3/2} e^{-2R-\sigma/2}}{Re^\sigma + \pi^2} K_2 \left(-2\sqrt{R^2 + \pi^2 Re^{-\sigma}} \right) \\ + \int_0^\infty d\nu e^{-\nu\sigma} b_\nu \sqrt{R} e^{-2R} K_{\nu+1/2}(2R). \end{aligned} \tag{A.18}$$

The first term is precisely the deformed Schwarzian partition function found in [70]. The second term is there because the boundary condition at $R \to \infty$ is not enough to fully fix the solution. They are non-perturbative corrections to the partition function, discussed in 3.4. In that same section a proposal is presented how to fix, or at least partially, the b_ν. In particular, by requiring $Z(R,\sigma)$ to be real. We know that $K_s(z)$ is real for $z > 0$ and since $R > 0$, we need b_ν to be complex in general. The Bessel $K_s(z)$ functions have a branch cut at the negative real axis and furthermore for integer s we have,

$$K_s(-z) = (-1)^s K_s(z) + (\log(z) - \log(-z)) I_s(z) \Rightarrow K_2(-z) = K_2(z) - i\pi I_2(z), \tag{A.19}$$

where we used $z > 0$ and real after the implication arrow. Notice that here we also picked a particular branch of the logarithm so that $\log(-z) = \log(z) + i\pi$. This choice is motivated by the fact that as $R \to \infty$ the density of states of the corresponding partition function is positive. Consequently, to make $Z(R, \sigma)$ real we need the imaginary part of b_ν, b_ν^{Im}, to satisfy.

$$\frac{1}{\sqrt{2\pi C^3}} \frac{R^{3/2} e^{-2R-\sigma/2}}{Re^\sigma + \pi^2} K_2\left(2\sqrt{R^2 + \pi^2 Re^{-\sigma}}\right) + \int_0^\infty d\nu e^{-\nu\sigma} b_\nu^{\mathrm{Im}} \sqrt{R} e^{-2R} K_{\nu+1/2}(2R) = 0. \quad \text{(A.20)}$$

But this is the same matching as we did to implement the boundary condition (A.15), up to some signs. In fact, picking $b_\nu^{\mathrm{Im}} = -(-1)^\nu a_\nu$ does the job and we get

$$Z(R, \sigma) = \sqrt{\frac{\pi}{2C^3}} \frac{R^{3/2} e^{-2R-\sigma/2}}{Re^\sigma + \pi^2} I_2\left(2\sqrt{R^2 + \pi^2 Re^{-\sigma}}\right) + \tilde{Z}(R, \sigma), \quad \text{(A.21)}$$

where

$$\tilde{Z}(R, \sigma) = \int_{-\infty}^{\infty} d\nu e^{-\nu\sigma} \sqrt{R} e^{-2R} c_\nu K_{1/2+\nu}(2R) \quad \text{(A.22)}$$

with c_ν real. Going back to the λ and β variables, we find

$$Z_\lambda(\beta) = \sqrt{\frac{\pi}{2\lambda}} \frac{\beta e^{-\frac{\beta}{4\lambda}}}{\beta^2 + 16C\pi^2\lambda} I_2\left(\frac{1}{4\lambda}\sqrt{\beta^2 + 16C\pi^2\lambda}\right) + \tilde{Z}(\beta, \lambda). \quad \text{(A.23)}$$

If one insists on getting a partiton function as a solution, i.e a solution that can be written as a sum over energies weighted by some Boltzman factor, we can find solution in a simpler way. The ansatz is then

$$Z_\lambda(\beta) = \sum_E g(\lambda) e^{-\beta\mathcal{E}_\lambda(E)}. \quad \text{(A.24)}$$

Plugging this in the differential equation (3.47) we precisely find the energy levels in (3.2) and $g(\lambda) = 1$, i.e. the density of states is not changed under the flow. If we consider a continuous spectrum we thus find (3.86).

A.3 Details about regularization

Some explicit perturbative calculations for $K[z(u)]$

Since the discussion is section 3.3.5 is mostly formal, in this appendix we will compute the finite cutoff partition function to leading order in the cutoff ε. The unrenormalized quantities are $L = \beta/\varepsilon$ and $\phi_b = \phi_r/\varepsilon$. We want to reproduce the answer from WdW or $T\overline{T}$ which is given in (3.85). Expanding at small ε gives

$$\log Z_{T\overline{T}} = \frac{2\pi^2\phi_r}{\beta} + \frac{3}{2}\log\Big(\frac{\phi_r}{\beta}\Big) - \varepsilon^2\Big(\frac{2\phi_r\pi^4}{\beta^3} + \frac{5\pi^2}{\beta^2} + \frac{15}{8\phi_r\beta}\Big) + \mathcal{O}(\varepsilon^4) \tag{A.25}$$

We want to reproduce the ε^2 term evaluating directly the path integral over the mode $z(u)$. Removing the leading $1/\varepsilon^2$ divergence we need to compute

$$Z_{\rm JT}[\varepsilon] = \int \frac{\mathcal{D}z}{SL(2,\mathbb{R})} e^{\int_0^\beta du\phi_r K_2} e^{\varepsilon\int_0^\beta du\phi_r K_3 + \varepsilon^2\int_0^\beta du\phi_r K_4 + \dots}, \tag{A.26}$$

where $K_2[z(u) = \text{Sch}(z,u)$ gives the leading answer and $K_3[z(u)]$ and $K_4[z(u)]$ are both given in (3.56) and contribute to subleading order. This integral is easy to do perturbatively. First we know that the expectation value of an exponential operator is equal to the generating function of connected correlators. Then any expectation value over the Schwarzian theory gives

$$\log\left\langle e^{\varepsilon\mathcal{O}[z]}\right\rangle_{\rm Sch} = \log Z_0 + \varepsilon\langle\mathcal{O}[z]\rangle + \frac{\varepsilon^2}{2}\langle\mathcal{O}[z]\mathcal{O}[z]\rangle_{\rm conn} + \dots. \tag{A.27}$$

Using this formula we can evaluate the logarithm of the partition function to order ε^2 in terms of K_3 and K_4 as

$$\log Z_{\rm JT} = \log Z_{\rm Sch} + \varepsilon\int_0^\beta du\,\phi_r\langle K_3\rangle + \frac{\varepsilon^2}{2}\int_0^\beta dudu'\,\langle K_3K_3'\rangle + \varepsilon^2\int_0^\beta du\,\phi_r\langle K_4\rangle + \mathcal{O}(\varepsilon^3). \tag{A.28}$$

The first correction is $K_3 = -i\partial_u\text{Sch}(z,u)$, which is a total derivative. This guarantees that, for a constant dilaton profile, the first two terms vanish since $\int du\langle K_3\rangle = 0$ and $\int\int dudu'\,\langle K_3K_3'\rangle = 0$. The second correction is

$$K_4 = -\frac{1}{2}\text{Sch}(z,u)^2 + \partial_u^2\text{Sch}(z,u) \tag{A.29}$$

Then, since the second term in K_4 is a total derivative it can be neglected, giving

$$\log Z_{\rm JT} = \log Z_{\rm Sch} - \frac{\varepsilon^2}{2}\phi_r \int \langle {\rm Sch}(z,u)^2\rangle + \mathcal{O}(\varepsilon^3). \tag{A.30}$$

Using point-splitting we can regulate the Schwarzian square. Schwarzian correlators can be obtained using the generating function. The one-point function is

$$\langle {\rm Sch}(z,u)\rangle = \frac{1}{\beta}\partial_{\phi_r}\log Z = \frac{2\pi^2}{\beta^2} + \frac{3}{2\phi_r\beta} \tag{A.31}$$

The two point function is given by

$$\langle {\rm Sch}(z,u){\rm Sch}(z,0)\rangle = -\frac{2}{\phi_r}\langle {\rm Sch}(z,0)\rangle\delta(u) - \frac{1}{\phi_r}\delta''(u) + \langle : {\rm Sch}(z,u)^2 :\rangle \tag{A.32}$$

where we define the renormalized square schwarzian expectation value as

$$\langle : {\rm Sch}(z,u)^2 :\rangle = \frac{4\pi^4}{\beta^4} + \frac{10\pi^2}{\beta^3\phi_r} + \frac{15}{4\beta^2\phi_r^2}. \tag{A.33}$$

This term only gives the right contribution matching the term in the $T\overline{T}$ partition function

$$\frac{\varepsilon^2}{2}\phi_r \int \langle : {\rm Sch}(z,u)^2 :\rangle = \varepsilon^2\Big(\frac{2\phi_r\pi^4}{\beta^3} + \frac{5\pi^2}{\beta^2} + \frac{15}{8\phi_r\beta}\Big). \tag{A.34}$$

If evaluating $K_4[z(u)]$ without using the point-splitting procedure prescribed in section 3.3.5 then one naviely evaluates (A.33) at identical points. The divergent contributions can precisely be eliminated with the point-splitting prescription (3.76).

Why derivatives of the Schwarzian don't contribute to the partition function

Here we discuss in more detail why terms in $K[z(u)]$ containing derivatives of the Schwarzian do not contribute to the partition function (with constant dilaton value ϕ_r) after following the point-splitting procedure (3.76). As mentioned in section 3.3.5 the schematic form of Schwarzian correlators is given by

$$\left(\frac{\delta}{\delta j(u_1)}\cdots\frac{\delta}{\delta j(u_n)} Z_{\rm Sch}[j(u)]\right)\Bigg|_{j(u)=\phi_r} = a_1 + a_2[\delta(u_{ij})] + a_3[\partial_u\delta(u_{ij})] + \dots , \tag{A.35}$$

where the derivatives in the δ-function terms above come by taking functional derivatives of the term $\exp\left(\int_0^\beta du \frac{j'(u)^2}{2j(u)}\right)$ in $Z_{\mathrm{Sch}}[j(u)]$. After following the point-splitting prescription (3.76) none of the functional derivatives of the form (A.35) that we will have to consider in the expansion of the exponential will be evaluated at identical points and therefore (A.35) will not contain terms containing $\delta(0)$ or its derivatives.

Consequently, note that when series-expanding the exponential functional derivative in (3.74), terms that contain derivatives in $\mathcal{K}\left[\partial_u \frac{\delta}{\delta j(u)}\right]$ would give terms with contributions of the form

$$\begin{aligned}
&\int_0^\beta du_1 \ldots \int_0^\beta du_a \ldots \int_0^\beta du_N \left(\ldots \partial_{u_a} \frac{\delta}{\delta j(u_a)} \ldots Z_{JT}[j(u)] \right) \Big|_{j(u)=\phi_r} = \\
&\quad = \int_0^\beta du_1 \ldots \int_0^\beta du_a \ldots \int_0^\beta du_N \left[a_2[\partial_u \delta(u_{ai})] + a_3[\partial_u^2 \delta(u_{ai}), \partial_u \delta(u_{ai}) \partial_u \delta(u_{ak})] + \ldots \right] \\
&\quad = 0 \,, \qquad \text{(A.36)}
\end{aligned}$$

where we note that a_1 vanishes after taking the derivative ∂_{u_a}.

In the second to last line we have that $a_2[\partial_u \delta(u_{ai})]$ contains first order derivatives in $\delta(u_{ai})$ and $a_3[\partial_u^2 \delta(u_{ai}), \partial_u \delta(u_{ai}) \partial_u \delta(u_{ak})]$ contains second-order derivatives acting on δ-functions involving u_a. Since the functions above only contain δ-functions involving other coordinates than u_a, all terms in the integral over u_a vanish after integration by parts; consequently, the last line of (A.36) follows. Note that if we consider dilaton profiles that are varying $\phi_r(u)$ such derivative of δ-function in fact would contribute after integration by parts. Consequently, it is only in the case of constant dilaton where such derivative terms do not give any contribution.

A very similar argument leads us to conclude that all other terms containing derivatives of δ-functions in (A.35), vanish in the expansion of the exponential functional derivative from (3.74) when the δ-function is evaluated at non-coincident points. Therefore, since the term $\exp\left(\int_0^\beta du \frac{j'(u)^2}{2j(u)}\right)$ only gives rise to terms containing derivatives of $\delta(u)$, this term also does not contribute when evaluating (3.74).

Appendix B

The analysis of JT gravity in the second-order formalism

B.1 A review of the Schwarzian theory

In this section, we review the Schwarzian theory, its equivalence to the particle on the hyperbolic plane H_2^+ placed in a magnetic field and the computation of observables in both theories. The partition function for the Schwarzian theory on a Euclidean time circle of circumference β is given by

$$Z_{\text{Schw.}}(\beta) = \int_{f \in \frac{\text{Diff}(S^1)}{SL(2,\mathbb{R})}} \frac{D\mu[f]}{SL(2,\mathbb{R})} \exp\left[C \int_0^\beta du \left(\{f,u\} + \frac{2\pi^2}{\beta^2}(f')^2\right)\right], \tag{B.1}$$

where C is a coupling constant with units of length, $\{f,u\}$ denotes the Schwarzian derivative, $f' = \partial_u f(u)$ and the path integral measure $D\mu[f]$ will be defined shortly. The field $f(u)$ obeying $f(u+\beta) = f(u)+\beta$ parameterizes the space $\text{Diff}(S^1)$ of diffeomorphisms of the circle. By performing the field redefinition $F(u) = \tan(\pi f(u)/\beta)$ with the consequent boundary condition $F(0) = F(\beta)$, as suggested in (1.19), one can rewrite (B.1) as

$$S[F] = -C \int_0^\beta du\{F,u\}\,. \tag{B.2}$$

Classically, the action in (B.1) can be seen to be invariant under $SL(2,\mathbb{R})$ transformations[1]

$$F \to \frac{aF+b}{cF+d}\,. \tag{B.3}$$

In the path integral (1.19) one simply mods out by $SL(2,\mathbb{R})$ transformations (1.19) which are constant in time (the $SL(2,\mathbb{R})$ zero-mode). As we will further discuss in Section B.1, such a quotient in the path integral is different from dynamically gauging the $SL(2,\mathbb{R})$ symmetry. An appropriate choice for the measure on $\text{diff}(S^1)/SL(2,\mathbb{R})$ which can be derived from the symplectic form of the Schwarzian theory is given by,

$$D\mu[f] = \prod_u \frac{df(u)}{f'(u)} = \prod_u \frac{dF(u)}{F'(u)}\,. \tag{B.4}$$

where the product is taken over a lattice that discretizes the Euclidean time circle.

Finally, the Hamiltonian associated to the action (B.2) is equal to the $\mathfrak{sl}(2,\mathbb{R})$ quadratic Casimir, $H = 1/C\left[-\ell_0^2 + (\ell_-\ell_+ + \ell_+\ell_-)/2\right]$, where ℓ_0 and $\ell_\pm$ are the $\mathfrak{sl}(2,\mathbb{R})$ charges associated to the transformation (B.3), which can be written in terms of $F(u)$ as

$$\begin{aligned}
\ell_0 &= \frac{iC}{\sqrt{2}}\left[\frac{F'''F}{F'^2} - \frac{FF''^2}{F'^2} - \frac{F''}{F'}\right],\\
\ell_+ &= \frac{iC}{\sqrt{2}}\left[\frac{F'''F^2}{F'^2} - \frac{F''^2F^2}{F'^3} - \frac{2FF''}{F'} + 2F'\right],\\
\ell_- &= \frac{iC}{\sqrt{2}}\left[\frac{F'''}{(F')^2} - \frac{(F'')^2}{(F')^3}\right],
\end{aligned} \tag{B.5}$$

The equality between the Hamiltonian and the Casimir suggests a useful connection between the Schwarzian theory and a non-relativistic particle on the hyperbolic upper-half plane, H_2^+, placed in a constant magnetic field B. In the latter the system, the Hamiltonian is also given by an $\mathfrak{sl}(2,\mathbb{R})$ quadratic Casimir. Below we discuss the equivalence of the two models at the path integral level.

An equivalent description

The quantization of the non-relativistic particle on the hyperbolic plane, H_2^+, placed in a constant magnetic field $\tilde{B}$ was performed in [221, 222]. Writing the H_2^+ metric as $ds^2 = d\phi^2 + e^{-2\phi}dF^2$ where

[1] $SL(2,\mathbb{R})$ is the naive symmetry when performing the transformation (B.3) at the level of the action. We will discuss the exact symmetry at the level of the Hilbert space shortly.

both ϕ and F take values in $\mathbb{R}$, the non-relativistic action in Lorentzian time[2]

$$S_{\tilde{B}} = \int dt \left(\frac{1}{4}(\dot{\phi})^2 + \frac{1}{4}e^{-2\phi}(\dot{F})^2 + \tilde{B}\dot{F}e^{-\phi} + \tilde{B}^2 + \frac{1}{4} \right) . \tag{B.6}$$

The Hamiltonian written in terms of the canonical variables (ϕ, π_ϕ) and (F, π_F), is given by[3]

$$H_{\tilde{B}} = \pi_\phi^2 + \pi_F^2 e^{2\phi} - 2\tilde{B}\pi_F e^{\phi} - \frac{1}{4} . \tag{B.7}$$

The thermal partition function at temperature $T = 1/\beta$ can be computed by analytically continuing (B.6) to Euclidean signature by sending $t \to -iu$ and computing the path integral on a circle of circumference β with periodic boundary conditions $\phi(0) = \phi(\beta)$ and $F(0) = F(\beta)$. At the level of the path integral, the partition function with such boundary conditions is given by

$$Z_{\tilde{B}}(\beta) = \int_{\phi(0)=\phi(\beta), F(0)=F(\beta)} D\phi DF \, e^{-\int_0^\beta du \left(\frac{1}{4}\phi'^2 + \frac{1}{4}(e^{-\phi}F' - 2i\tilde{B})^2 \right)} . \tag{B.8}$$

with the $\mathfrak{sl}(2,\mathbb{R})$ invariant measure,

$$D\phi Df \equiv \prod_{u \in [0,\beta]} d\phi(u) dF(u) e^{-\phi(u)} \tag{B.9}$$

For the purpose of understanding the equivalence between this system and the Schwarzian we will be interested in the analytic continuation to an imaginary background magnetic field $\tilde{B} = -\frac{iB}{2\pi}$ with $B \in \mathbb{R}$,

$$\begin{aligned} Z_B(\beta) &= \int_{\phi(0)=\phi(\beta), F(0)=F(\beta)} D\phi DF \, e^{-\int_0^\beta du \left(\frac{1}{4}\phi'^2 + \frac{1}{4}(e^{-\phi}F' - B/\pi)^2 \right)} . \\ &\sim \int_{\phi(0)=\phi(\beta),\, F(0)=F(\beta)} D\phi DF \, e^{-\int_0^\beta du \left(\frac{1}{4}\phi'^2 + \frac{B^2}{4\pi^2} e^{-2\phi}(F' - e^{\phi})^2 \right)} , \end{aligned} \tag{B.10}$$

where we have shifted $\phi \to \phi - \log \frac{B}{\pi}$ in the second line above and dropped an overall factor that only depends on B.

The Schwarzian theory emerges as an effective description of this quantum mechanical system in the limit $B \to \infty$. Indeed, we can apply a saddle point approximation in this limit to integrate out ϕ. This sets $F' = e^{\phi}$ and gives, after taking into account the one-loop determinant for ϕ around

[2] For convenience, we distinguish Lorentzian time derivative $\dot{f}$ from Euclidean time derivatives f'.

[3] We have shifted both the Lagrangian and the Hamiltonian by a factor of $\pm B^2$ in order to set the zero level for the energies of the particle on H_2^+ to be at the bottom of the continuum.

the saddle,

$$Z_B(\beta) \sim \int_{F(0)=F(\beta)} \prod_u \frac{dF(u)}{F'(u)}\, e^{-\int_0^\beta du\left(\frac{1}{4}\left(\frac{F''}{F'}\right)^2\right)} = \int_{F(0)=F(\beta)} D\mu[F]\, e^{\frac{1}{2}\int_0^\beta du\{F,u\}}\,, \tag{B.11}$$

where to obtain the second equality we have shifted the action by a total derivative.

Thus, as promised, we recover the Schwarzian partition function with the same measure for the field $F(u)$ in the $B \to \infty$ limit (and $\tilde{B} \to i\infty$), when setting the coupling $C = \frac{1}{2}$.[4] However, the space of integration for $F(u)$ in (B.11) is different from that in the Schwarzian path integral (1.19). This is most obvious after we transform to the other field variable $f(u) = \frac{\beta}{\pi}\tan^{-1} F(u)$ and

$$Z_B(\beta) \sim \sum_{n\in\mathbb{Z}} \int_{f(0)=f(\beta)+n\beta} D\mu[f]\, e^{\frac{1}{2}\int_0^\beta du\left(\{f,u\}+\frac{2\pi^2}{\beta^2}(f')^2\right)}\,. \tag{B.12}$$

While for the Schwarzian action $f(u) \in \mathrm{Diff}(S^1)$, obeying the boundary condition $f(u+\beta) = f(u) + \beta$, the path integral (B.12) consists of multiple topological sectors labeled by a winding number $n \in \mathbb{Z}$ such that $f(u+\beta) = f(u) + \beta\, n$. In other words, the (Euclidean) Schwarzian theory is an effective description of the quantum mechanical particle in the $n = 1$ sector.

Reproducing the partition function of the Schwarzian theory from the particle of magnetic field thus depends on the choice of integration cycle for $F(u)$ (or $f(u)$). As we explain below, the integration cycle needed in order for the partition function of the particle of magnetic field to be convergent is given by $\tilde{B} = iB \to i\infty$. In order to do this it is useful to consider how the wavefunctions in this theory transform as representations SL2.

When quantizing the particle on H_2^+ in the absence of a magnetic field, the eigenstates of the Hamiltonian transform as irreducible representations of $PSL(2,\mathbb{R})$ [222]. When turning on a magnetic field, the Hamiltonian eigenstates transform as projective representations of $PSL(2,\mathbb{R})$, which are the proper representations of $\widetilde{SL}(2,\mathbb{R})$ mentioned in Section 2.3.3 [222].[5] Specifically, the wavefunctions for the particle in magnetic field $\tilde{B} \in \mathbb{R}$ transform in a subset of irreducible representations of $\widetilde{SL}(2,\mathbb{R})$ with fixed eigenvalues under the center of the group $e^{2\pi i\mu} = e^{2\pi i\tilde{B}}$.[6] Such unitary representations admit a well-defined associated Hermitian inner-product and the Hamiltonian is a

[4]Note that the meaningful dimensionless parameter $\frac{\beta}{C}$ is unconstrained.

[5]Note that not all unitary irreducible representations of SL2 need to appear in the decomposition of the Hilbert space under SL2. While there exist states transforming in any continuous series representation of SL2, there are also states transforming in the discrete series representations as long as $\lambda = -\tilde{B} + n$ with, $n \in \mathbb{Z}$ and $0 \leq n \leq |\tilde{B}| - 1$.

[6]The fact that states transform in projective representations of the classical global symmetry can be understood as an anomaly of the global symmetry. An straightforward example of this phenomenon happens when studying a charged particle on a circle with a θ-angle with $\theta = \pi$ [117]. Note that when $B = \frac{p}{q} \in \mathbb{Q}$ states transform in absolute irreps of the q-cover of $PSL(2,\mathbb{R})$, which are also abolute irreps of $\widetilde{SL}(2,\mathbb{R})$. It is only when $B \in \mathbb{R}\setminus\mathbb{Q}$ that these irreps are absolute for the univesal cover $\widetilde{SL}(2,\mathbb{R})$.

Hermitian operator. Up to a constant shift, their energies are real and are given by the SL2 Casimir in (2.26), $E_\lambda = -(\lambda - 1/2)^2$.

When making $\tilde{B} \in \mathbb{C} \setminus \mathbb{R}$ the Hamiltonian is no longer Hermitian and the representations of SL2 do not admit a well defined Hermitian inner-product. However, the partition function defined by the path-integral (B.6) is convergent. As we explain in Section 2.3.3, if we analytically continue the Plancherel measure and Casimir to imaginary $\tilde{B} \to i\infty$, the thermal partition function in this limit reproduces that of the Schwarzian theory (2.24). Thus, the theory makes sense in Euclidean signature where the correlation function of different observables is convergent, but a more careful treatment is needed in Lorentzian signature.

An SL2 chemical potential

While the classical computation performed in Section 2.2.2 suggests the equivalence between imposing a non-trivial $PSL(2,\mathbb{R})$ twist for the Schwarzian field and the gauge theory (2.3) with a non-trivial holonomy around its boundary this equivalence does not persist quantum mechanically. Instead, in the presence of a non-trivial holonomy, the gauge theory is equivalent to the non-relativistic particle in the magnetic field (B.6) with $\tilde{B} \to i\infty$ and in the presence of an SL2 chemical potential. Note that in the derivation performed above, in order to prove the equivalence between the Schwarzian and the action (B.6) with $\tilde{B} \to i\infty$, we have assumed that the field $F(u)$ is periodic: specifically, if one assumes a $PSL(2,\mathbb{R})$ twist around the thermal circle for the field $F(u)$, one can no longer use the equality in (B.11). Specifically, (B.11) assumes that when adding a total derivative to the action, the integral of that derivative around the thermal circle vanishes – this is no longer true in the presence of a non-trivial twist for the Schwarzian field.

In order to study (B.6) with $\tilde{B} \to i\infty$ in the presence of an SL2 chemical potential, we start by considering the case of $\tilde{B} \in \mathbb{R}$ and then we analytically continue to an imaginary magnetic field $\tilde{B} \in i\mathbb{R}$. The partition function is given by

$$\begin{aligned} Z_{iB}(\tilde{g}, \beta) &\sim \int ds \rho_B(s) e^{-\frac{\beta}{2C}s^2} \sum_{m=-\infty}^{\infty} \langle \frac{1}{2} + is,\, m|\tilde{g}|\frac{1}{2} + is,\, m\rangle + \text{discrete series contributions} \\ &= \int_0^\infty ds \rho_B(s) \chi_s(\tilde{g}) e^{-\frac{\beta}{2C}s^2} + \text{disrete series contributions}\,, \end{aligned} \tag{B.13}$$

where $\chi_s(\tilde{g}) = \text{Tr}_s(\tilde{g})$ is the SL2 character of the principal series representation labelled by $\lambda = 1/2 + is$ (see Appendix B.3 for the explicit character $\chi_s(\tilde{g})$). To recover the partition function when $\tilde{B} = -\frac{iB}{2\pi} \to i\infty$ we again perform the analytic continuation used to obtain (2.24). Once again the discrete series states have a contribution of $\mathcal{O}(Be^{-\beta B^2/C})$ and can be neglected. Thus, up to a proportionality factor

$$Z_{iB}(\tilde{g},\beta) \propto \int_0^\infty ds\rho(s)\chi_s(\tilde{g})e^{-\frac{\beta}{2C}s^2}\,. \tag{B.14}$$

This formula generalizes (2.24) for any $\tilde{g}$ and matches up to an overall proportionality factor, with the result obtained in the gauge theory in Section 2.3.3 (see (2.43)).

B.2 Comparison between compact and non-compact groups

For convenience, we review the schematic comparison between various formulae commonly used for compact gauge groups (which we will denote by G) with finite dimensional unitary irreducible representations and the analogous formulae that need to be used in the non-compact case (which we denote by $\mathcal{G}$) with infinite-dimensional unitary irreducible representations:

$\delta(g)=\sum_R \dim R\,\chi_R(g)$	$\delta(g)=\int dR\rho(R)\,\chi_R(g)$
$\int \frac{dg}{\mathrm{vol}G}U^n_{R,m}(g)U^{m'}_{R',n'}(g^{-1})=\frac{\delta_{RR'}\delta_{mm'}\delta_{nn'}}{\dim R}$	$\int \frac{dg}{\mathrm{vol}\mathcal{G}}U^n_{R,m}(g)U^{m'}_{R',n'}(g^{-1})=\frac{\delta(R,R')\delta_{mm'}\delta_{nn'}}{\rho(R)}$
$\int \frac{dg}{\mathrm{vol}G}\chi_R(g)\chi_{R'}(g^{-1})=\delta_{RR'}$	$\int \frac{dg}{\mathrm{vol}\mathcal{G}}\chi_R(g)\chi_{R'}(g^{-1})=\frac{\Xi\,\delta(R,R')}{\rho(R)}$
$\int \frac{dg}{\mathrm{vol}G}\chi_R(gh_1g^{-1}h_2)=\frac{\chi_R(h_1)\chi_R(h_2)}{\dim R}$	$\int \frac{dg}{\mathrm{vol}\mathcal{G}}\chi_R(gh_1g^{-1}h_2)=\frac{\chi_R(h_1)\chi_R(h_2)}{\Xi}$
$\int \frac{dg}{\mathrm{vol}G}\chi_{R_1}(gh_1)\chi_{R_2}(g^{-1}h_1)=\frac{\delta_{R_1,R_2}\chi_{R_1}(h_1h_2)}{\dim R_1}$	$\int \frac{dg}{\mathrm{vol}\mathcal{G}}\chi_{R_1}(gh_1)\chi_{R_2}(g^{-1}h_2)=\frac{\delta(R_1,R_2)\chi_{R_1}(h_1h_2)}{\rho(R)}$

where $\chi_R(g)$ are the characters of the group G or $\mathcal{G}$, $U^n_{R,m}(g)$ are the associated matrix elements and Ξ is a divergent factor, which can be evaluated by considering the limit $\lim_{g\to\mathbf{1}}\chi_R(g)=\Xi$. In the case of SL2 and $\mathcal{G}$ the limit needs to be taken from the direction of hyperbolic elements and for the group $\mathcal{G}_B$ we have shown that Ξ is independent of the representation R. We consider an in-depth discussion of the above formulae and their consequences in 2D gauge theories with the non-compact gauge group $\mathcal{G}_B$ below.

B.3 Harmonic analysis on SL2 and $\mathcal{G}_B$

We next describe how to work with the characters of SL2 and its $\mathbb{R}$ extension, $\mathcal{G}_B$ (and consequently the group $\mathcal{G}\equiv\mathcal{G}_B$ when taking the limit $B\to\infty$). In order to get there we first need to discuss the meaning of the Fourier transform on the group manifold of SL2 or $\mathcal{G}_B$. Given a finite function

$x(\tilde{g})$ with $\tilde{g} \in$ SL2,[7] for every unitary representation U_R of the continuous and discrete series we can associate an operator

$$U_R(x) = \int x(\tilde{g}) U_R(\tilde{g}) d\tilde{g} \,. \tag{B.15}$$

The operator $U_R(x)$ is called the Fourier transform of $x(\tilde{g})$. Just like in Fourier analysis on $\mathbb{R}$, our goal will be to find the inversion formula for (B.15) and express $x(\tilde{g})$ in terms of its Fourier transform. To start, we can express the Delta-function $\delta(\tilde{g})$ on the group manifold, in terms of its Fourier components

$$\delta(\tilde{g}) = \int \rho(R) \mathrm{tr}(U_R(\tilde{g})) dR \,, \tag{B.16}$$

where as we will see later in the subsection that $\rho(R)$ is the Plancherel measure on the group and $\chi_R(\tilde{g}) \equiv \mathrm{Tr}\,(U_R(\tilde{g}))$ will define the character of the representation R. The integral over R is schematic here (see later section for explicit definitions) and represents the integral over the principal and discrete series of the group. The Delta-function is defined such that,

$$\int x(\tilde{g}\tilde{g}_0) \delta(\tilde{g}) d\tilde{g} = x(\tilde{g}_0) \,. \tag{B.17}$$

Multiplying (B.16) by $x(\tilde{g}\tilde{g}_0)$ and integrating over the group manifold we find that

$$x(\tilde{g}_0) = \int \rho(R) \mathrm{tr}(U_R(x) U_R(\tilde{g}_0^{-1})) dR \,. \tag{B.18}$$

We will review the calculation of the matrix elements $U^m_{R,\,n}(\tilde{g})$, characters $\chi_R(\tilde{g})$ and of the Plancherel measure $\rho(R)$ in the next subsections.

B.3.1 Evaluation of the matrix elements and characters

As explained in [115], one can parameterize $\widetilde{SL}(2,\mathbb{R})$ using the coordinates (ξ, ϕ, η), where we can restrict $\phi + \eta \in [0, 4\pi)$. The $\widetilde{SL}(2,\mathbb{R})$ element $\tilde{g}$ takes the form $\tilde{g} = e^{\phi P_0} e^{\xi P_1} e^{-\eta P_0}$, where the generators P_i are given by (2.7). In this parameterization, the metric is

$$ds^2 = d\xi^2 - d\phi^2 - d\eta^2 + 2\cosh\xi \, d\phi d\eta \tag{B.19}$$

[7] Here finite means that it is infinitely differentiable if the group manifold is connected and is constant in a sufficiently small domain if the group manifold is disconnected.

and the Haar measure is

$$d\mu = \sinh\xi \, d\xi \, d\phi \, d\eta \, . \tag{B.20}$$

For the full group $\mathcal{G}_B$, we normalize the measure by,

$$d\tilde{g} \equiv d\mu d\theta \tag{B.21}$$

As shown in [115], the matrix elements in the representation with quantum numbers λ and μ are given by

$$U^m_{\lambda,n}(\tilde{g}) = e^{i(n\phi - m\eta)}(1-u)^\lambda u^{\frac{n-m}{2}} \sqrt{\frac{\Gamma(n-\lambda+1)\Gamma(n+\lambda)}{\Gamma(m-\lambda+1)\Gamma(m+\lambda)}} \, \mathbf{F}(\lambda - m, n+\lambda, -m+n+1; u) \, , \tag{B.22}$$

where, $\mathbf{F}(a,b,c,z) = \Gamma(c)^{-1} {}_2F_1(a,b,c;z)$, $u = \tanh^2(\xi/2)$ and $m, n \in \mu + \mathbb{Z}$. We can similarly parametrize elements $\mathcal{G}_B$ by $g = (\theta, \tilde{g})$ where x is an element of $\mathbb{R}$. The matrix element for the representation $(\lambda,\, \mu = -\frac{Bk}{2\pi} + q,\, k)$ in $\mathcal{G}_B$ is thus given by,

$$U^m_{(\lambda,\, \mu = -\frac{Bk}{2\pi} + q,\, k),\, n}(g) = e^{ik\theta} U^m_{\lambda,n}(\tilde{g}) \, . \tag{B.23}$$

Once again, this expression depends on μ only in that $m,\, n,\, k \in \mu + \mathbb{Z}$. The diagonal elements are thus given by

$$U^m_{(\lambda,\mu,k),m}(g) = (1-u)^\lambda e^{im(\phi-\eta)} e^{ik\theta} \, {}_2F_1(\lambda - m, \lambda + m; 1; u) \, . \tag{B.24}$$

The characters of the various representations are obtained by summing (B.24) over m. Because the characters are class functions, they must be functions of the eigenvalues $x,\, x^{-1}$ of the SL2 matrix $\tilde{g}$, when $\tilde{g}$ is expressed in the two-dimensional representation. x can be obtained from the angles ϕ,

η and ξ for any representation to be[8]

$$x = \begin{cases} \dfrac{\cos\frac{\phi-\eta}{2} \pm \sqrt{u - \sin^2\frac{\phi-\eta}{2}}}{\sqrt{1-u}}\,, & \text{if } u \geq \sin^2\frac{\phi-\eta}{2}\,, \\ \dfrac{\cos\frac{\phi-\eta}{2} \pm i\sin\frac{\phi-\eta}{2}\sqrt{1-\frac{u}{\sin^2\frac{\phi-\eta}{2}}}}{\sqrt{1-u}}\,, & \text{if } u < \sin^2\frac{\phi-\eta}{2}\,, \end{cases} \tag{B.25}$$

where one of the solutions represents x and the other x^{-1}. Note that for hyperbolic elements, $x \in \mathbb{R}$, which happens whenever $u > \sin^2\frac{\phi-\eta}{2}$. Simple examples of hyperbolic elements have $\phi = \eta = 0$, and in this case $x = e^{\pm\xi/2}$. For elliptic elements, we have $|x| = 1$ (with $x \notin \mathbb{R}$), which happens whenever $u < \sin^2\frac{\phi-\eta}{2}$. Simple examples of elliptic elements have $u = \eta = 0$, and in this case $x = e^{\pm i\phi/2}$. Lastly, for parabolic elements, we have $x = \pm 1$, and in this case $u = \sin^2\frac{\phi-\eta}{2}$. For convenience, from now on we choose x such that $|x| > 1$ and $|x^{-1}| < 1$ for hyperbolic elements. For elliptic elements, we choose x to be associated with the negative sign in the 2nd equation of (B.25).

Continuous series

To obtain the characters for the continuous series, we should set $\lambda = \frac{1}{2} + is$ and sum over all values of $m = \mu + p$ with $p \in \mathbb{Z}$. The sum is given by

$$\chi_{s,\mu,k}(g) = (1-u)^{\frac{1}{2}+is} e^{ik\theta} \sum_{p\in\mathbb{Z}} e^{i(\mu+p)(\phi-\eta)}{}_2F_1(\frac{1}{2} + is - \mu - p, \frac{1}{2} + is + \mu + p; 1; u)\,, \tag{B.26}$$

where we consider $\phi - \eta \in [2\pi(n-1), 2\pi n)$, with $n \in \mathbb{Z}$. This sum can be evaluated using the generating formula for the ${}_2F_1$ hypergeometric function. Evaluating the sum defined in (B.25) yields, in terms of the eigenvalue x associated to $\tilde{g}$ group element, the $\mathbb{R}$ element θ and the branch number, n, for the angle $\phi - \eta$,

$$\chi_{s,\mu,k}(g) = \begin{cases} e^{ik\theta} e^{2\pi i\mu n}\left(\frac{|x|^{1-2\lambda} + |x|^{-1+2\lambda}}{|x - x^{-1}|}\right)\,, & \text{for } \tilde{g} \text{ hyperbolic}, \\ 0\,, & \text{for } \tilde{g} \text{ elliptic}, \end{cases} \tag{B.27}$$

where $\lambda = \frac{1}{2} + is$ and, we remind the reader about the restriction that $\mu = \frac{-Bk}{2\pi} + \mathbb{Z}$.

[8](We wrote two distinct formulas depending on whether u is greater or smaller than $\sin^2\frac{\phi-\eta}{2}$ in order to make explicit the choice of branch cut we use for the square root.)

Discrete series

For the positive discrete series, we have $\mu = \lambda$ and the sum over m goes over values equal to $\lambda + p$ with $p \in \mathbb{Z}^+$:

$$\chi^+_{\lambda,k}(g) = e^{ik\theta} \sum_{p=0}^{\infty} U^{\lambda+p}_{\lambda,\lambda+p}(g) = (1-u)^{\lambda} e^{ik\theta} e^{i\lambda(\phi-\eta)} \sum_{p=0}^{\infty} e^{ip(\phi-\eta)} P_p^{(0,2\lambda-1)}(1-2u)\,, \tag{B.28}$$

where $P_n^{(\alpha,\beta)}(x)$ are the Jacobi polynomials. We can once again evaluate the sum using the generating formula for the Jacobi polynomial to find that in terms of the eigenvalue x, the character is given by

$$\chi^+_{k,\lambda}(g) = \frac{e^{ik\theta} x^{1-2\lambda}}{x - x^{-1}} \tag{B.29}$$

for both hyperbolic and elliptic elements. This expression is identical to the first term in (B.27). For the negative discrete series, we have $\mu = -\lambda$ and so we should take $m = -\lambda - p$, with $p \in \mathbb{Z}^+$, and sum over p:

$$\chi^-_{k,\lambda}(g) = (1-u)^{\lambda} e^{ik\theta} e^{-i\lambda(\phi-\eta)} \sum_{p=0}^{\infty} e^{-ip(\phi-\eta)} P_p^{(0,2\lambda-1)}(1-2u)\,. \tag{B.30}$$

Comparing (B.30) with (B.28), we conclude that

$$\chi^-_{\lambda,k}(g) = e^{ik\theta} \left(\chi^+_{\lambda}(\tilde{g})\right)^* = e^{ik\theta} \left(\frac{x^{1-2\lambda}}{x - x^{-1}}\right)^* . \tag{B.31}$$

This expression is identical to the second term in (B.27).

Before we end this subsection, we summarize a few identities satisfied by the characters above. We have

$$\overline{\chi_R(g)} = \chi_R(g^{-1}) \tag{B.32}$$

which follows from the unitarity of the representations. We also have

$$\chi_{s,\mu,k}(g^{-1}) = \chi_{s,-\mu,-k}(g), \quad \chi^+_{k,\lambda}(g^{-1}) = \chi^-_{-k,\lambda}(g) \quad . \tag{B.33}$$

B.3.2 The Plancherel inversion formula

The normalization of the matrix elements U_R given by (B.22) - (B.24) can be computed following [115]. For the continuous series one finds that,

$$\begin{aligned}\langle U^m_{(\frac{1}{2}+is,\mu,k),n}|U^{m'}_{(\frac{1}{2}+is',\mu',k'),n'}\rangle &= \int dg\, U^m_{(\frac{1}{2}+is,\mu,k),n}(g)U^{n'}_{(\frac{1}{2}+is',\mu',k'),m'}(g^{-1})\\ &= 4\pi^2 B\frac{\cosh(2\pi s)+\cos(Bk)}{s\sinh(2\pi s)}\delta(s-s')\delta(\mu-\mu')\delta_{kk'}\delta_{nn'}\delta_{mm'}\,,\\ &\text{with } s,\, s'>0, \qquad \frac{-1}{2}\le\mu\le\frac{1}{2}\,,\\ &\quad k,\, k'\in -\frac{2\pi(\mu+\mathbb{Z})}{B}\,, \qquad m,\, n,\, m',\, n'\in\mu'+\mathbb{Z}\,.\end{aligned} \tag{B.34}$$

Similarly, for the positive/negative discrete series one finds that,

$$\begin{aligned}&\langle U^m_{(\lambda,k),n)}|U^{m'}_{(\lambda',k'),n'}\rangle = \frac{8\pi^2 B}{2\lambda-1}\delta(\lambda-\lambda')\delta_{kk'}\delta_{mm'}\delta_{nn'}\\ &\quad\text{with } \lambda,\, \lambda'>\frac{1}{2}, \qquad k,\, k'\in -\frac{2\pi(\pm\lambda+\mathbb{Z})}{B}\,, \qquad m,\, n,\, m',\, n'\in\pm(\lambda+\mathbb{Z}^+)\,.\end{aligned} \tag{B.35}$$

Given the orthogonality of the matrix elements one can then write the δ-function in (B.16) as,

$$\begin{aligned}\delta(g) &= \frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{dk\,ds}{(2\pi)^2}\frac{s\sinh(2\pi s)}{\cosh(2\pi s)+\cos(Bk)}\chi_{(s,\mu=-\frac{Bk}{2\pi},k)}(g)+\\ &+\int_{\frac{1}{2}}^{\infty}\frac{d\lambda}{(2\pi)^2 B}\left(\lambda-\frac{1}{2}\right)\sum_{q=-\infty}^{\infty}\left(\chi^+_{(\lambda,k=-\frac{2\pi(\lambda+q)}{B}))}(g)+\chi^-_{(\lambda,k=-\frac{2\pi(-\lambda+q)}{B})}(g)\right)\,,\end{aligned} \tag{B.36}$$

For the purpose of evaluating the partition function of the gauge theory in Section 2.2 it is more convenient to write all the terms in (B.36) under a single k-integral. To do this one can perform a contour deformation [223] to find that $\delta(g)$ can also be expressed as

$$\delta(g) = -i\sum_{p\in\mathbb{Z}}\int_{-\infty}^{\infty}dk\int_{-\infty}^{\infty}ds\left(\frac{Bk}{2\pi}+p+is\right)\tanh(\pi s)\,U^{\frac{Bk}{2\pi}+p}_{(\frac{Bk}{2\pi}+p+is+\frac{1}{2},\frac{Bk}{2\pi}+q,k)\frac{Bk}{2\pi}+p}(\tilde{g})\,, \tag{B.37}$$

with $q\in\mathbb{Z}$. Using $\delta(g)$ from (B.36), the Plancherel inversion formula for SL2 can be generalized to functions acting on the group $\mathcal{G}_B$,

$$\begin{aligned}x(\mathbf{1}) &= \frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{dk\,ds}{(2\pi)^2}\frac{s\sinh(2\pi s)}{\cosh(2\pi s)+\cos(2\pi k)}\chi_{(s,\mu=-\frac{Bk}{2\pi},k)}(x)+\\ &+\int_{\frac{1}{2}}^{\infty}\frac{d\lambda}{(2\pi)^2 B}\left(\lambda-\frac{1}{2}\right)\sum_{q=-\infty}^{\infty}\left(\chi^+_{(\lambda,k=-\frac{2\pi(\lambda+q)}{B}))}(x)+\chi^-_{(\lambda,k=-\frac{2\pi(-\lambda+q)}{B})}(x)\right)\,,\end{aligned} \tag{B.38}$$

with

$$\chi_R(x) \equiv \int d\tilde{g} \int_0^B d\theta x(g) \chi_R(g^{-1})\,. \tag{B.39}$$

In practice, in order to keep track of divergences evaluating the characters on a trivial we introduce the divergent factor Ξ, for which $\chi_{(s,\mu=-\frac{Bk}{2\pi},k)}(x) = \Xi$. One can check this s-independent divergence by taking the limit

$$\lim_{\tilde{g}\to e} \chi_{(s,\mu)}(\tilde{g}) = \lim_{x\to 1,\,\theta\to 0} e^{ik\theta} \frac{x^{2is} + x^{-2is}}{x - x^{-1}} = \lim_{x\to 1} \frac{1}{|x - x^{-1}|} = \Xi\,. \tag{B.40}$$

Similarly, for $n \in \mathbb{Z}$,

$$\lim_{\tilde{g}\to e^{2\pi i n \ell_0}} \chi_{(s,\mu)}(\tilde{g}) = e^{2\pi i \mu n} \lim_{x\to \pm 1} \frac{1}{|x - x^{-1}|} = e^{2\pi i \mu n}\Xi\,. \tag{B.41}$$

Another operation that proves necessary for the computations performed in Section 2.2 is performing the group integral

$$\frac{1}{\mathrm{vol}\mathcal{G}_B} \int dg \chi_{s,k=i}(g h_1 g^{-1} h_2) = \frac{1}{\Xi} \chi_{s,k=i}(h_1) \chi_{s,k=i}(h_2)\,, \tag{B.42}$$

for principal series representation s and for group elements h_1 and h_2. The normalization of this formula is set by taking the limit $h_1 \to e$ and $h_2 \to e$ and using the normalization for the matrix elements U_R, (B.34) and (B.35).

B.3.3 An example: Isolating the principal series representation

The goal of this appendix is to use the techniques presented in the previous subsections to show that we can isolate the contribution of principal series representations in the partition function. Specifically, we want to show that the regularization procedure suggested in Section 2.3.3 by adding higher powers of the quadratic Casimir leads to suppression of the discrete series. Using the rewriting of $\delta(g)$ as in (B.37) we find that the partition function with an overall $\mathcal{G}_B$ holonomy g is given by,

$$\begin{aligned} Z(g, e\beta) \sim -i \sum_{p\in\mathbb{Z}} \int_{-\infty}^{\infty} dk \int_{-\infty}^{\infty} ds \left(-\frac{Bk}{2\pi} + p + is\right) \tanh(\pi s)\, U^{-\frac{Bk}{2\pi}+p}_{-\frac{Bk}{2\pi}+p+is+\frac{1}{2},-\frac{Bk}{2\pi}+p}(\tilde{g}) \\ \times e^{ik\theta} e^{\frac{e\beta}{2}\left[(p+is)^2 - \cdots\right]}\,, \end{aligned} \tag{B.43}$$

where $g = (\tilde{g}, \theta)$ and $\cdots$ captures the contribution of higher powers of the quadratic Casimir. Setting the boundary condition $\phi^{\mathbb{R}} = k_0 = -i$, we find that the partition function becomes

$$Z_{k_0}(\tilde{g}, e\beta) \sim -i \sum_{p\in\mathbb{Z}} \int_{-\infty}^{\infty} ds\ (p+is) \tanh\left(\pi s - \frac{B}{2}\right) U^{\frac{Bi}{2\pi}+p}_{p+is+\frac{1}{2},\frac{Bi}{2\pi}+p}(\tilde{g}) \\ \times e^{\frac{e\beta}{2}\left[(p+is)^2 - \cdots\right]}, \tag{B.44}$$

where, in order to obtain (B.44), we have also performed the contour re-parametrization $s \to s - \frac{B}{2\pi}$. The form of higher order terms captured by $\cdots$ is given by higher powers of the quadratic Casimir: thus, for instance the first correction given by the square of the quadratic Casimir is given by $\sim (p+is)^4/B$. For each term in the sum, we can now deform the contour as $s \to s - ip$. Such a deformation only picks up poles located at $s_* = \frac{1}{2\pi}B - \frac{(2n+1)i}{2}$ with $n \in \mathbb{Z}$ and $2n+1 < p$.[9] The residue of each such pole gives rise to the contribution of the discrete series representations to the partition function. However, by choosing the negative sign for the fourth order and higher order terms in the potential the resulting contribution is suppressed as $\mathcal{O}(Be^{-\frac{e\beta B^2}{2}})$. This is the reason why the partition function is finite and is solely given by the contribution of principal unitary series representations.

$$Z_{k_0}(\tilde{g}, e\beta) \sim \sum_{p\in\mathbb{Z}} \int_{-\infty}^{\infty} ds\, s \tanh\left(\pi s - \frac{B}{2}\right) U^{\frac{Bi}{2\pi}+p}_{is+\frac{1}{2},\frac{Bi}{2\pi}+p}(\tilde{g}) e^{-\frac{e\beta s^2}{2}} + \\ + \mathcal{O}(Be^{-\frac{e\beta B^2}{2}}). \tag{B.45}$$

Note that the integral is even in s and that $\tanh\left(\pi s - \frac{B}{2}\right) = (\sinh(2\pi s) - \sinh(B))/(\cosh(2\pi s) + \cosh(B))$. Thus, when considering the $B \to \infty$ limit the Plancherel measure becomes $dss \sinh(2\pi s)/e^{-B}$. Thus, summing up all matrix coefficients in (B.45) we recover the fact that the partition function only depends on characters, and we recover the result in Section 2.3.4.

B.4 Clebsch-Gordan coefficients, fusion coefficients and 6-j symbols

The purpose of this section is to derive the fusion coefficients and the 6-j symbols needed in the main text. To do so, we find it convenient to represent the states in the unitary representation (μ, λ) of

[9] The only poles in (B.44) come from the measure factor $\tanh(\pi s - B/2)$.

SL2 as functions $f(\phi)$ on the unit circle obeying the twisted periodicity condition

$$f(\phi + 2\pi) = e^{2\pi i\mu} f(\phi)\,, \tag{B.46}$$

with the rule that under a diffeomorphisms $V \in \widetilde{\text{Diff}}_+(S^1)$ of the unit circle, these functions transform as

$$(Vf)(\phi) = \left(\partial_\phi V^{-1}(\phi)\right)^\lambda f(V^{-1}(\phi))\,. \tag{B.47}$$

Such a transformation property can be thought of arising from a "μ-twisted λ-form," namely an object formally written as $f(\phi)(d\phi)^\lambda$. We denote the space of such forms as $\mathcal{F}^\mu_\lambda$. In infinitesimal form, a diffeomorphism is described by a vector field $v(\phi) = v^\phi(\phi)\partial_\phi$, which acts on f via the infinitesimal from of (B.47):

$$vf = -v^\phi \partial_\phi f - \lambda(\partial_\phi v^\phi) f\,. \tag{B.48}$$

To see why the space $\mathcal{F}^\mu_\lambda$ is isomorphic with the representation (μ, λ) of SL2, note that (B.48) implies that the vector fields $L^\phi_n = -ie^{in\phi}$ with $n = -1, 0, 1$ obey the commutation relations

$$[L_{\pm1}, L_0] = \pm L_{\pm1}\,, \qquad [L_1, L_{-1}] = 2L_0 \tag{B.49}$$

so the transformations (B.48) corresponding to them generate an SL2 subalgebra of $\widetilde{\text{Diff}}_+(S^1)$. By comparison with (2.7), we can identify $\ell_0 = L_0$, $\ell_+ = L_1$, $\ell_- = L_{-1}$ when acting on $\mathcal{F}^\mu_\lambda$. From (B.48), we can also determine the action of the quadratic Casimir

$$\widehat{C}_2 f = \left(-L_0^2 + \frac{L_1 L_{-1} + L_{-1} L_1}{2}\right) f = \lambda(1-\lambda) f\,. \tag{B.50}$$

This fact, together with $e^{-2\pi i L_0} f(\phi) = e^{2\pi\partial_\phi} f(\phi) = f(\phi + 2\pi) = e^{2\pi i\mu} f(\phi)$ implies that $\mathcal{F}^\mu_\lambda$ should be identified with the representation (λ, μ) (or with the isomorphic representation $(1-\lambda, \mu)$) of SL2.

Let us now identify the function corresponding to the basis element $|m\rangle$ in the (μ, λ) representation. This basis element has the property that $L_0|m\rangle = -m|m\rangle$, which becomes $i\partial_\phi f = -mf$, so it should be proportional to $f_{\lambda,m} = e^{im\phi}$. (Recall that $m \in \mu + \mathbb{Z}$ for the irrep (μ, λ), so $f_{\lambda,m}$ obeys the twisted periodicity (B.46).) In other words

$$|m\rangle \qquad \text{corresponds to} \qquad c_{\lambda,m} f_{\lambda,m}(\phi) \equiv \langle \phi | m \rangle \tag{B.51}$$

for some constant $c_{\lambda,m}$. To determine $c_{\lambda,m}$, note that from (B.48), we obtain

$$L_n f_{\lambda,m} = -(m + n\lambda) f_{\lambda,m+n}. \tag{B.52}$$

By comparison with the action (2.27) of the raising and lowering operators on the states $|m\rangle$, we conclude that $c_{m,\lambda}$ obeys the recursion relation

$$c_{\lambda,m+1} = c_{\lambda,m} \frac{(\lambda + m)}{\sqrt{(\lambda + m)(1 - \lambda + m)}} \tag{B.53}$$

with the solution[10]

$$c_{\lambda,m} = \frac{\Gamma(\lambda + m)}{\sqrt{\Gamma(\lambda + m)\Gamma(1 - \lambda + m)}} . \tag{B.54}$$

Note that this expression holds both for the continuous series which we will denote as $c_{\lambda,m}$ and for the positive discrete series $c^+_{\lambda,m}$. For negative discrete series we have instead

$$c^-_{\lambda,m-1} = c^-_{\lambda,m} \frac{(m - \lambda)}{\sqrt{(m - \lambda)(m - 1 + \lambda)}} , \tag{B.55}$$

which leads to

$$c^-_{\lambda,m} = (-1)^{m-\mu} \frac{\sqrt{\Gamma(1 - m - \lambda)\Gamma(\lambda - m)}}{\Gamma(1 - \lambda - m)} \tag{B.56}$$

for $m = -\lambda, -\lambda - 1, -\lambda - 2, \ldots$.

From these expressions and $\langle m|n\rangle = \delta_{mn}$, we can infer the inner product on the space $\mathcal{F}^\mu_\lambda$. Indeed, any two functions f and g obeying (B.46) can be expanded in Fourier series as

$$\begin{aligned} f(\phi) &= \sum_m a_m e^{im\phi} \qquad &\Longleftrightarrow \qquad a_m &= \frac{1}{2\pi} \int d\phi \, e^{-im\phi} f(\phi) , \\ g(\phi) &= \sum_m b_m e^{im\phi} \qquad &\Longleftrightarrow \qquad b_m &= \frac{1}{2\pi} \int d\phi \, e^{-im\phi} g(\phi) . \end{aligned} \tag{B.57}$$

[10]The recursion formula only fixes $c_{\lambda,m}$ (similarly for $c^-_{\lambda,m}$ in (B.56)) up to an m independent constant that could depend on λ. Here we have chosen a particular normalization for convenience. The physical observables we compute are however independent of such normalizations.

Then we can write

$$\langle f|g\rangle = \sum_{m,n} \frac{a_m^* b_n}{c_{\lambda,m}^* c_{\lambda,n}} \langle m|n\rangle = \sum_m \frac{a_m^* b_m}{|c_{\lambda,m}|^2} \,. \tag{B.58}$$

Writing a_m and b_m in terms of f_1 and f_2 using the Fourier series inversion formula, we obtain

$$\langle f|g\rangle = \int d\phi_1\, d\phi_2\, f(\phi_1)^* g(\phi_2) G(\phi_1 - \phi_2) \tag{B.59}$$

where $G(\phi)$ given by

$$G(\phi) = \frac{1}{4\pi^2} \sum_m \frac{e^{im\phi}}{|c_{\lambda,m}|^2} \,. \tag{B.60}$$

For the continuous series, $|c_{\lambda,m}|^2 = 1$, and the sum is over $m \in \mu + \mathbb{Z}$. We obtain

$$\text{continuous series:} \qquad G(\phi) = \frac{1}{4\pi^2} e^{i\mu(\phi_1-\phi_2)} D\left(\frac{\phi_1 - \phi_2}{2\pi}\right) \,, \tag{B.61}$$

where $D(x) = \sum_{k\in\mathbb{Z}} \delta(x-k)$ is a Dirac comb with unit period. For the positive discrete series, $m \in \lambda + \mathbb{Z}_+$ and $\mu = \lambda > 0$. We find that (B.60) evaluates to

$$\text{positive discrete series:} \qquad G(\phi) = \frac{e^{i\lambda\phi}{}_2F_1(1,1,2\lambda,e^{i\phi})}{4\pi^2\Gamma(2\lambda)} \,. \tag{B.62}$$

To obtain the fusion coefficients, we need to consider tensor products of representations. As a warm-up, let us consider the tensor product

$$\mathcal{C}_{\frac{1}{2}+is,\mu} \otimes \mathcal{C}_{\frac{1}{2}+is,-\mu} \tag{B.63}$$

and identify the state corresponding to the identity representation. This state is

$$\sum_{m\in\mu+\mathbb{Z}} (-1)^m |m\rangle|-m\rangle \,, \tag{B.64}$$

and it can be obtained as the unique state invariant under $L_n^{(1)} + L_n^{(2)}$, where the $L_n^{(i)}$ (with $n = -1, 0, 1$ and $i = 1, 2$) are the SL2 generator acting on the ith factor of the tensor product.

The state (B.64) can also be found in a more indirect way by first constructing the two-variable function $Y(e^{i\phi_1}, e^{i\phi_2})$ representing it. This function obeys the conditions

$$\sum_{i=1}^{2} \partial_{\phi_i} Y(e^{i\phi_1}, e^{i\phi_2}) = 0\,, \qquad \sum_{i=1}^{2} \left(ie^{\pm i\phi_i}\partial_{\phi_i} \mp \lambda e^{\pm i\phi_i}\right) Y(e^{i\phi_1}, e^{i\phi_2}) = 0\,, \tag{B.65}$$

(with $\lambda = \frac{1}{2} + is$) representing the invariance under the SL2 generators, as well as the periodicity conditions (B.46) in ϕ_1 and ϕ_2 individually. When $0 < \phi_1 - \phi_2 < 2\pi$, the solution of the equations (B.65) is

$$Y(e^{i\phi_1}, e^{i\phi_2}) = C \sin\left(\frac{\phi_1 - \phi_2}{2}\right)^{-2\lambda} \tag{B.66}$$

for some constant C. Away from this interval, the expression (B.66) should be extended using the periodicity condition (B.46). The state corresponding to this function is generally of the form $\sum_{m_1 \in \mu+\mathbb{Z}} \sum_{m_2 \in -\mu+\mathbb{Z}} C_{m_1,m_2} |m_1\rangle |m_2\rangle$, with coefficients C_{m_1,m_2} obtained by taking the inner product with the basis elements:

$$C_{m_1,m_2} = \frac{1}{4\pi^2} \int d\phi_1 \int d\phi_2\, c^*_{\lambda,m_1} c^*_{\lambda,m_2} e^{-im_1\phi_1} e^{-im_2\phi_2} Y(e^{i\phi_1}, e^{i\phi_2}) \tag{B.67}$$

Because Y depends only on $\phi_1 - \phi_2$, the only non-zero C_{m_1,m_2} are those with $m_1 = -m_2$. Using

$$\int_0^{2\pi} d\phi\, e^{-im\phi} \left(\sin\frac{\phi}{2}\right)^{-2\lambda} = \frac{-2e^{-im\pi}\sin(m\pi)\Gamma(1-2\lambda)\Gamma(\lambda-m)}{\Gamma(1-\lambda-m)}\,, \tag{B.68}$$

and $\lambda = \frac{1}{2} + is$, the expression (B.67) with $m_1 = -m_2 = m$ evaluates to

$$C_{m,-m} = e^{-i\pi m} C \frac{\sin(\pi\mu)}{2s\sin(\pi(\mu-\lambda))\sinh(2\pi s)\Gamma(2is)} \sqrt{\frac{\cos(2\pi\mu) + \cosh(2\pi s)}{2}}\,. \tag{B.69}$$

We see that up to an m-independent constant, $C_{m,-m} \propto (-1)^m$, so (B.69) agrees with (B.64).

B.4.1 Clebsch-Gordan coefficients: $\mathcal{C}^{\mu_1}_{\lambda_1=\frac{1}{2}+is_1} \otimes \mathcal{D}^{\pm}_{\lambda_2} \to \mathcal{C}^{\mu}_{\lambda=\frac{1}{2}+is}$

In [115] a general recipe was outlined for obtaining the "Clebsch-Gordan" coefficients for SL2.[11] and, in particular, Ref. [115] constructed the decomposition of the tensor products $\mathcal{D}^+_{\lambda_1} \otimes \mathcal{D}^+_{\lambda_2}$ and $\mathcal{D}^+_{\lambda_1} \otimes \mathcal{D}^-_{\lambda_2}$. Here we follow the same recipe to determine the Clebsch-Gordan coefficients and fusion coefficients between two continuous series representations and a positive/negative discrete series

[11] Alternatively, see [224] and [225] for a more mathematical approach.

representation:

$$\mathcal{C}^{\mu_1}_{\lambda_1=\frac{1}{2}+is_1} \otimes \mathcal{D}^{\pm}_{\lambda_2} \to \mathcal{C}^{\mu}_{\lambda=\frac{1}{2}+is}\,, \tag{B.70}$$

with $\mu = \mu_1 \pm \lambda$. The state $|s,m\rangle$ that is part of $\mathcal{C}^{\mu}_{\lambda=\frac{1}{2}+is}$ in the tensor product (B.70) must take the form

$$|s,m\rangle = \sum_{m_2=\pm(\lambda+\mathbb{Z}^+)} C^{s_1,\,\lambda_2^{\pm},\,s}_{m-m_2,m_2,m}|m-m_2\rangle|m_2\rangle \tag{B.71}$$

where $C^{s_1,\,\lambda_2^{\pm},\,s}_{m-m_2,m_2,m}$ is the Clebsch-Gordan coefficient and the range of m_2 depends on whether it comes from the positive or negative discrete series.

As in the previous section, we determine $C^{s_1,\,\lambda_2^{\pm},\,s}_{m-m_2,m_2,m}$ in a rather indirect way by first constructing the functions $Y_{s,m}(e^{i\phi_1},e^{i\phi_2})$ that represent the state (B.71). This function can be found using the conditions that

$$\begin{aligned} L_0 Y_{s,m} &= -mY_{s,m}\,,\\ \left(-L_0^2 + \frac{L_1L_{-1}+L_{-1}L_1}{2}\right)Y_{s,m} &= \lambda(1-\lambda)Y_{s,m}\,, \end{aligned} \tag{B.72}$$

where $L_n = L_n^{(1)} + L_n^{(2)}$ and $\lambda = \frac{1}{2}+is$. Let us first solve these equations for $0<\phi_1<2\pi$ and $0<\phi_1-\phi_2<2\pi$. (The expression for Y can then be continued away from this range using the appropriate periodicity condition (B.46) in both ϕ_1 and ϕ_2.)

The first equation in (B.72) implies that $Y_{s,m}$ equals $e^{im\phi_1}$ times a function of $\phi_1-\phi_2$. The second condition gives a second order differential equation for this function of $\phi_1-\phi_2$ with two linearly independent solutions

$$\begin{aligned} Y^-_{s,m}(e^{i\phi_1},e^{i\phi_2}) = B^-_{s,m}&e^{im\phi_1}e^{i\lambda_2(\phi_1-\phi_2)}\left(1-e^{i(\phi_1-\phi_2)}\right)^{\lambda-\lambda_1-\lambda_2}\\ &\times {}_2F_1(\lambda-\lambda_1+\lambda_2,\lambda+m,1+m-\lambda_1+\lambda_2,e^{i(\phi_1-\phi_2)})\,. \end{aligned} \tag{B.73}$$

and

$$\begin{aligned} Y^+_{s,m}(e^{i\phi_1},e^{i\phi_2}) = B^+_{s,m}&e^{im\phi_2}e^{i(\lambda_2-m)(\phi_2-\phi_1)}\left(1-e^{i(\phi_2-\phi_1)}\right)^{\lambda-\lambda_1-\lambda_2}\\ &\times {}_2F_1(\lambda-\lambda_1+\lambda_2,\lambda-m,1-m-\lambda_1+\lambda_2,e^{i(\phi_2-\phi_1)})\,. \end{aligned} \tag{B.74}$$

for some constant $B^{\pm}_{s,m}$. Both of the solutions are linearly dependent under $s\to -s$, thus from now on, we will restrict to $s > 0$. As suggested by the notation, this specific basis of solutions correspond precisely to the generating functions for Clebsch-Gordon coefficients for the tensor product $\mathcal{C}^{\mu_1}_{\lambda_1=\frac{1}{2}+is_1} \otimes \mathcal{D}^{\pm}_{\lambda_2}$. This is fixed by requiring that $Y^{+}_{s,m}(z,w)w^{-\lambda_2}$ and $Y^{-}_{s,m}(z,1/w)w^{-\lambda_2}$ to be holomorphic inside the unit disk $|w| < 1$, as suggested by the one-side bounded sum in m_2 in (B.71), with $m_2 = \pm(\lambda_2 + \mathbb{Z}^+)$ [115].

The dependence of $B^{-}_{s,m}$ on m is fixed by requiring $Y^{-}_{s,m}$ to transform appropriately under the action of the raising and lowering operators. Explicit computation shows that

$$L_1 Y^{-}_{s,m} = -\frac{(\lambda+m)(1-\lambda+m)}{1+m-\lambda_1+\lambda_2}\frac{B^{-}_{s,m}}{B^{-}_{s,m+1}}Y^{-}_{s,m+1}\,. \tag{B.75}$$

Comparing with the desired relation $L_1 Y^{-}_{s,m} = -\sqrt{(\lambda+m)(1-\lambda+m)}Y^{-}_{s,m+1}$, we obtain the recursion formula

$$B^{-}_{s,m+1} = \frac{\sqrt{(\lambda+m)(1-\lambda+m)}}{1+m-\lambda_1+\lambda_2}B^{-}_{s,m}\,. \tag{B.76}$$

Up to an overall constant which we denote by B^{-}_{s}, this recursion is solved by

$$B^{-}_{s,m} = \frac{\sqrt{\Gamma(\lambda+m)\Gamma(1-\lambda+m)}}{\Gamma(1-\lambda_1+\lambda_2+m)}B^{-}_{s}\,. \tag{B.77}$$

Similarly we can determine $B^{+}_{s,m}$ by recursion relations

$$L_1 Y^{+}_{s,m} = -(\lambda_1-\lambda_2+m)\frac{B^{+}_{s,m}}{B^{+}_{s,m+1}}Y^{+}_{s,m+1}\,. \tag{B.78}$$

to be

$$B^{+}_{s,m} = \frac{\Gamma(\lambda_1-\lambda_2+m)}{\sqrt{\Gamma(\lambda+m)\Gamma(1-\lambda+m)}}B^{+}_{s}\,. \tag{B.79}$$

Normalization

We would like to compute the normalization constant $\mathcal{N}(s)$ for the inner product of states (B.71),

$$\langle s,m|s',m'\rangle = \mathcal{N}(s)\delta(s-s')\delta_{mm'} \tag{B.80}$$

For this purpose, it is sufficient to consider $m = m'$, and take the inner product of the functions representing the LHS of (B.80). Using (B.71), we can write this inner product as

$$\langle s, m|s', m\rangle = \sum_{m_2} (C^{s_1,\lambda_2^\pm,s}_{m-m_2,m_2,m})^* C^{s_1,\lambda_2^\pm,s'}_{m-m_2,m_2,m} \tag{B.81}$$

(The answer should be independent of m.) The expected delta functions in (B.80) arise from the large m_2 terms in the sum. Thus, let us compute

$$C^{s_1,\lambda_2^\pm,s}_{m-m_2,m_2,m} = \langle m-m_2|\langle m_2,^\pm|s,m\rangle = \frac{1}{c^\pm_{\lambda_2,m_2} e^{im_2\phi_2}} \frac{1}{2\pi}\int d\phi_1\, c^*_{\lambda_1,m-m_2} e^{-i(m-m_2)\phi_1} Y^\pm_{s,m}(e^{i\phi_1}, e^{i\phi_2}) \tag{B.82}$$

at large m_2.

We first start by considering λ_2 in the negative discrete series. After plugging in the expression for Y and writing $\phi_1 = \phi_2 + \phi$, we obtain

$$\begin{aligned} C^{s_1,\lambda_2^-,s}_{m-m_2,m_2,m} = B_s \frac{c^*_{\lambda_1,m-m_2}}{2\pi c^-_{\lambda_2,m_2}} \frac{\sqrt{\Gamma(\lambda+m)\Gamma(1-\lambda+m)}}{\Gamma(1-\lambda_1+\lambda_2+m)} \int_0^{2\pi} d\phi\, e^{im_2\phi} e^{i\lambda_2\phi} \left(1-e^{i\phi}\right)^{\lambda-\lambda_1-\lambda_2} \\ \times {}_2F_1(\lambda-\lambda_1+\lambda_2, \lambda+m, 1+m-\lambda_1+\lambda_2, e^{i\phi})\,. \end{aligned} \tag{B.83}$$

The large m_2 behavior of the ϕ integral comes from the regions where the integrand is singular or non-analytic (because the ϕ integral extracts a Fourier coefficient, and in general, Fourier coefficients with large momenta come from singularities in position space). In this case, the singularities of the integrand are at $e^{i\phi} = 1$, where the integrand is approximately

$$e^{i(m_2+\lambda_2)\phi}\left[(1-e^{i\phi})^{\lambda-\lambda_1-\lambda_2} \frac{\Gamma(1-2\lambda)\Gamma(1+m-\lambda_1+\lambda_2)}{\Gamma(1+m-\lambda)\Gamma(1-\lambda-\lambda_1+\lambda_2)} + \ (\lambda \leftrightarrow 1-\lambda)\right]. \tag{B.84}$$

The integral $\int_0^{2\pi} d\phi\, e^{-ik\phi}(1-e^{i\phi})^\alpha$ has the same large k asymptotics as the integral

$$\int_0^\infty d\phi\, e^{-ik\phi}(-i)^\alpha\, |\phi|^\alpha + \int_{-\infty}^0 d\phi\, e^{-ik\phi} i^\alpha\, |\phi|^\alpha\ . \tag{B.85}$$

Using the formula $\int_0^\infty d\phi\, \phi^\alpha e^{-ik\phi-\epsilon\phi} = \frac{\Gamma(1+\alpha)}{(\epsilon+ik)^{1+\alpha}}$, the integral in (B.83) gives, approximately at large m_2,

$$\begin{aligned} &-2|m_2|^{\lambda_1+\lambda_2-\lambda-1}\sin\pi(\lambda-\lambda_1-\lambda_2)\Gamma(\lambda-\lambda_1-\lambda_2+1) \\ &\times \frac{\Gamma(1-2\lambda)\Gamma(1+m-\lambda_1+\lambda_2)}{\Gamma(1+m-\lambda)\Gamma(1-\lambda-\lambda_1+\lambda_2)} + \ (\lambda \leftrightarrow 1-\lambda)\,. \end{aligned} \tag{B.86}$$

The prefactor in (B.83) gives

$$\frac{B_s^- c^*_{\lambda_1,m-m_2} e^{-i\pi(m_2+\lambda_2)}}{2\pi} \frac{\sqrt{\Gamma(\lambda+m)\Gamma(1-\lambda+m)}}{\Gamma(1-\lambda_1+\lambda_2+m)} |m_2|^{-\lambda_2+\frac{1}{2}} \tag{B.87}$$

In total, we have

$$\begin{aligned}\lim_{m_2\to-\infty} C^{s_1,\lambda_2^-,s}_{m-m_2,m_2,m} = -\frac{c^*_{\lambda_1,m-m_2}}{\pi} B_s^- e^{-i\pi(m_2+\lambda_2)} \sin\pi(\lambda-\lambda_1-\lambda_2)\Bigg(\frac{\Gamma(\lambda-\lambda_1-\lambda_2+1)}{\Gamma(1-\lambda-\lambda_1+\lambda_2)} \\ \times\Gamma(1-2\lambda)\frac{\sqrt{\Gamma(\lambda+m)}}{\sqrt{\Gamma(1-\lambda+m)}}|m_2|^{\lambda_1-\lambda-\frac{1}{2}} + \ (\lambda\leftrightarrow 1-\lambda)\Bigg)\end{aligned} \tag{B.88}$$

Thus, the large m_2 asymptotics of the product $(C^{s_1,\lambda_2^-,s}_{m-m_2,m_2,m})^* C^{s_1,\lambda_2^-,s'}_{m-m_2,m_2,m}$ are,

$$|B_s|^2\left[|m_2|^{i(s-s')-1}\left|\frac{\Gamma(-2is)}{\Gamma(is_1-is+\lambda_2)\Gamma(-is-is_1+\lambda_2)}\right|^2 + \begin{pmatrix} s\to -s \\ s'\to -s'\end{pmatrix}\right], \tag{B.89}$$

where we kept $s\neq s'$ only in the power of m_2, anticipating that the sum over m_2 gives a term proportional to $\delta(s-s')$. To see why the sum $\sum_{m_2}(m_2)^{-1+i\alpha}$ gives a delta function, note that we can regularize the sum by taking $\epsilon>0$, thus writing $\sum m_2^{-1+i\alpha-\epsilon} = \zeta(1-i\alpha-\epsilon)$. Close to $\alpha=0$, this becomes $\frac{i}{\alpha+i\epsilon}\to P\frac{i}{\alpha}+\pi\delta(\alpha)$ as $\epsilon\to 0$. The $P\frac{i}{\alpha}$ cancels from the final answer. We finally find

$$\mathcal{N}_{s_1,\lambda_2^-,s} = 2\left|B_s^-\right|^2\left|\frac{\Gamma(-2is)}{\Gamma(-is\pm is_1+\lambda_2)}\right|^2 . \tag{B.90}$$

Similarly, we compute $\mathcal{N}^+$ by focusing on the large m_2 limit of $(C^{s_1,\lambda_2^+,s}_{m-m_2,m_2,m})^* C^{s_1,\lambda_2^+,s}_{m-m_2,m_2,m}$ with

$$\begin{aligned}C^{s_1,\lambda_2^+,s}_{m-m_2,m_2,m} = B_s^+ \frac{c^*_{\lambda_1,m-m_2}}{2\pi c^+_{\lambda_2,m_2}} \frac{\Gamma(\lambda_1-\lambda_2+m)}{\sqrt{\Gamma(\lambda+m)\Gamma(1-\lambda+m)}}\int_0^{2\pi} d\phi\, e^{i(\lambda_2-m_2)\phi}\left(1-e^{i\phi}\right)^{\lambda-\lambda_1-\lambda_2} \\ \times {}_2F_1(\lambda-\lambda_1+\lambda_2,\lambda-m,1-m-\lambda_1+\lambda_2,e^{i\phi})\,.\end{aligned} \tag{B.91}$$

We find after similar manipulations that when fixing λ_2 to be in the positive discrete series,

$$\mathcal{N}_{s_1,\lambda_2^+,s} = 2\left|B_s^+\right|^2\left|\frac{\Gamma(-2is)}{\Gamma(-is\pm is_1+\lambda_2)}\frac{\sin(\pi(\mu_1+\lambda_2+\lambda))}{\sin(\pi(\mu_1+\lambda_1))}\right|^2 . \tag{B.92}$$

Clebsch-Gordan coefficients in the $\mu_1\to i\infty$ limit

In order to compute the expectation of Wilson lines value once fixing the the value of $\phi^{\mathbb{R}}=-i$ along the boundary we are interested in analytically continuing the product of Clebsch-Gordan coefficients for imaginary values of μ_1. Specifically, we would like to compute

$$I^{s_1,\,\lambda_2^{\pm},\,s}_{m-m_2,m_2,m} \equiv (\mathcal{N}_{s_1,\,\lambda_2^{\pm},\,s})^{-1} C^{s_1,\,\lambda_2^{\pm},\,s}_{m-m_2,m_2,m} (C^{s_1,\,\lambda_2^{\pm},\,s}_{m-m_2,m_2,m})^* \tag{B.93}$$

in the limit $\mu_1 \to i\infty$, with $m - m_2 = \mu_1 + \mathbb{Z}$. Note that in the above expression we will first take conjugate, and then take the limit $\mu_1 \to i\infty$.

We start with (B.83) and (B.91) and use

$$\lim_{x\to\infty} {}_2F_1(a, b + x, c + x, z) = (1 - z)^{-a}\,, \tag{B.94}$$

which holds away from $z = 1$. In this limit the Clebsch-Gordan coefficients become

$$C^{s_1,\,\lambda_2^-,\,s}_{m-m_2,m_2,m} \sim B_s^- \frac{c^*_{\lambda_1,m-m_2}}{2\pi c^-_{\lambda_2,m_2}} \frac{\sqrt{\Gamma(\lambda+m)\Gamma(1-\lambda+m)}}{\Gamma(1-\lambda_1+\lambda_2+m)} \int_0^{2\pi} d\phi\, e^{im_2\phi} e^{i\lambda_2\phi} \left(1 - e^{i\phi}\right)^{-2\lambda_2}\,,$$

$$(C^{s_1,\,\lambda_2^-,\,s}_{m-m_2,m_2,m})^* \sim (B_s^-)^* \frac{c_{\lambda_1,m-m_2}}{2\pi (c^-_{\lambda_2,m_2})^*} \frac{\sqrt{\Gamma(\lambda+m)\Gamma(1-\lambda+m)}}{\Gamma(\lambda_1+\lambda_2+m)} \int_0^{2\pi} d\phi\, e^{-im_2\phi} e^{-i\lambda_2\phi} \left(1 - e^{-i\phi}\right)^{-2\lambda_2}\,, \tag{B.95}$$

Now using

$$\int_0^{2\pi} d\phi\, e^{ia\phi} \left(1 - e^{i\phi}\right)^b = \frac{i(1 - e^{2\pi i a})\Gamma(a)\Gamma(b+1)}{\Gamma(1+a+b)} = \frac{2\pi e^{\pi i a}\Gamma(b+1)}{\Gamma(1-a)\Gamma(1+a+b)}\,, \tag{B.96}$$

valid by analytic continuation in b, and

$$\lim_{z\to\infty, z\notin\mathbb{R}_-} \Gamma(z) \sim e^{-z} z^z \sqrt{\frac{2\pi}{z}} (1 + \mathcal{O}(1/z))\,, \tag{B.97}$$

we have that in the limit $\mu_1 \to i\infty$, and consequently in the limit $m \to i\infty$,

$$\begin{aligned} C^{s_1,\,\lambda_2^-,\,s}_{m-m_2,m_2,m} &\sim \frac{B_s^-}{c^-_{\lambda_2,m_2}} m^{-\lambda_2} e^{\pi i(m_2+\lambda_2)} \frac{\Gamma(1-2\lambda_2)}{\Gamma(1-\lambda_2 \pm m_2)}\,, \\ (C^{s_1,\,\lambda_2^-,\,s}_{m-m_2,m_2,m})^* &\sim \frac{(B_s^-)^*}{(c^-_{\lambda_2,m_2})^*} m^{-\lambda_2} e^{\pi i(m_2+\lambda_2)} \frac{\Gamma(1-2\lambda_2)}{\Gamma(1-\lambda_2 \pm m_2)}\,. \end{aligned} \tag{B.98}$$

Putting this together, we obtain

$$\begin{aligned} I^{s_1,\,\lambda_2^-,\,s}_{m-m_2,m_2,m} &\sim= \mu_1^{-2\lambda_2} \frac{\Gamma(1-m_2-\lambda_2)}{\Gamma(\lambda_2-m_2)} \frac{\Gamma(1-2\lambda_2)^2}{\Gamma(1-\lambda_2 \pm m_2)^2} \mathcal{I} \\ &= \mu_1^{-2\lambda_2} (-1)^{m_2+\lambda_2} \frac{\Gamma(1-2\lambda_2)}{\Gamma(2\lambda_2)\Gamma(1-\lambda_2 \pm m_2)} \mathcal{I}\,, \end{aligned} \tag{B.99}$$

with

$$\mathcal{I} = \frac{1}{2}\left|\frac{\Gamma(-is \pm is_1 + \lambda_2)}{\Gamma(-2is)}\right|^2 = \frac{s\sinh(2\pi s)}{\pi}\Gamma(\pm is \pm is_1 + \lambda_2)\,. \tag{B.100}$$

Similarly we have in this limit

$$C^{s_1,\,\lambda_2^+,\,s}_{m-m_2,m_2,m} \sim B_s^+ \frac{c^*_{\lambda_1,m-m_2}}{2\pi c_{\lambda_2,m_2}} \frac{\Gamma(\lambda_1-\lambda_2+m)}{\sqrt{\Gamma(\lambda+m)\Gamma(1-\lambda+m)}} \int_0^{2\pi} d\phi\, e^{i(\lambda_2-m_2)\phi}\left(1-e^{i\phi}\right)^{-2\lambda_2}\,, \tag{B.101}$$

which, together with the conjugate relation, yields in the limit $\mu_1 \to i\infty$ and $m - m_2 \to i\infty$,

$$\begin{aligned} C^{s_1,\,\lambda_2^+,\,s}_{m-m_2,m_2,m} &\sim \frac{B_s^+}{c_{\lambda_2,m_2}} m^{-\lambda_2} e^{\pi i(m_2-\lambda_2)} \frac{\Gamma(1-2\lambda_2)}{\Gamma(1-\lambda_2\pm m_2)} \\ (C^{s_1,\,\lambda_2^+,\,s}_{m-m_2,m_2,m})^* &\sim \frac{(B_s^+)^*}{c^*_{\lambda_2,m_2}} m^{-\lambda_2} e^{\pi i(m_2-\lambda_2)} \frac{\Gamma(1-2\lambda_2)}{\Gamma(1-\lambda_2\pm m_2)}\,, \end{aligned} \tag{B.102}$$

and

$$\begin{aligned} I^{s_1,\,\lambda_2^+,\,s}_{m-m_2,m_2,m} \sim& \mu_1^{-2\lambda_2} \frac{\Gamma(m_2+1-\lambda_2)}{\Gamma(m_2+\lambda_2)} \frac{\Gamma(1-2\lambda_2)^2}{\Gamma(1-\lambda_2\pm m_2)^2}\mathcal{I} \\ =& \mu_1^{-2\lambda_2}(-1)^{m_2-\lambda_2} \frac{\Gamma(1-2\lambda_2)}{\Gamma(2\lambda_2)\Gamma(1-\lambda_2\pm m_2)}\mathcal{I}\,, \end{aligned} \tag{B.103}$$

which is identical to $I^{s_1,\,\lambda_2^-,\,s}_{m+m_2,-m_2,m}$.

B.4.2 Fusion coefficient as $\mu \to i\infty$

We are interested in generalizing the simple Clebsch-Gordan decomposition of the product of matrix element for some group element g (given by $U^m_{R_1,\,n}(g)U^{m'}_{R_1,\,n'}(g)$) for compact groups, to the case of SL2. To do this we start by inserting two complete set of states to re-express the product of two SL2 matrix elements

$$\begin{aligned} U^{m_1}_{\left(\lambda_1=\frac{1}{2}+is_1,\,\mu_1\right),\,n_1}(g)U^{m_2}_{\lambda_2^\pm,\,n_2}(g) &= \langle(\lambda_1,\mu_1),m_1;\lambda_2^\pm,m_2|g|(\lambda_1,\mu_1),n_1;\lambda_2^\pm,n_2\rangle = \\ &= \int \frac{ds}{\mathcal{N}_{s_1,\,\lambda_2^\pm,\,s}}\frac{ds'}{\mathcal{N}_{s_1,\,\lambda_2^\pm,\,s'}}\langle(\lambda_1,\mu_1),m_1;\lambda_2^\pm,m_2|(\lambda,\mu_1\pm\lambda_2),m_1+m_2\rangle \\ &\times\langle(\lambda_1,\mu_1),n_1;\lambda_2^\pm,n_2|(\lambda',\mu_1\pm\lambda_2),n_1+n_2\rangle^*\langle\lambda,m_1+m_2|g|\lambda',n_1+n_2\rangle \\ &+\ \text{discrete series contributions}\,. \end{aligned} \tag{B.104}$$

Thus, the product of two matrix elements is given by

$$U^{m_1}_{(\lambda_1=\frac{1}{2}+is_1,\,\mu_1),\,n_1}(g)U^{m_2}_{\lambda_2,\,n_2}(g) = \int \frac{ds}{\mathcal{N}_{s_1,\,\lambda_2^+,\,s}} C^{s_1,\lambda_2^\pm,s}_{m_1,m_2,m_1+m_2}(C^{s_1,\lambda_2^\pm,s}_{n_1,n_2,n_1+n_2})^* \, U^{m_1+m_2}_{(\lambda=\frac{1}{2}+is,\mu+\lambda_2),\,n_1+n_2}(g)$$
$$+ \text{ discrete series contributions} \,. \tag{B.105}$$

In the limit $\mu_1 \to i\infty$ we are interested in computing the fusion between the regular character $\chi_{(s_1,\mu_1)}(g)$ and the truncated character $\overline{\chi}_{\lambda_2^\pm}(g)$ defined in (2.55). Thus, the product of characters is given by

$$\chi_{(s_1,\mu_1)}(g)\overline{\chi}_{\lambda_2^\pm}(g) = \int ds \left(\sum_{k=0}^{\Xi} I^{s_1,\,\lambda_2,\,s}_{\mu_1+\tilde{k},\,\pm(\lambda_2+k),\,\mu_1+\tilde{k}\pm(\lambda_2+k)} \right) \chi_{(s,\mu_1+\lambda_2)}(g)$$
$$+ \text{ discrete series contributions} \,, \tag{B.106}$$

where we identify $m_1 = \mu_1 + \tilde{k}$ and $m_2 = \pm(\lambda_2 + k)$ with $\tilde{k} \in \mathbb{Z}$ and $k \in \mathbb{Z}^+$. We note that the sum over k yields a result that is independent of $\tilde{k}$, therefore leading to the separation of the sums in (B.105). Alternatively, the results above can be recasted as the group integral of three matrix elements given by

$$\int dg \, U^{m_1}_{(is_1,\,\mu_1),\,n_1}(g)U^{m_2}_{\lambda_2,\,n_2}(g)U^{n_1+n_2}_{(s,\mu_1+\lambda_2),\,m_1+m_2}(hg^{-1})$$
$$= \frac{C^{s_1,\lambda_2^\pm,s}_{m_1,m_2,m_1+m_2}(C^{s_1,\lambda_2^\pm,s}_{n_1,n_2,n_1+n_2})^* U^{n+n'}_{(\lambda=\frac{1}{2}+is,\mu+\lambda_2),\,m+m'}(h)}{\rho(s,\mu+\lambda_2)\mathcal{N}_{s_1,\,\lambda_2^+,\,s}} \,, \tag{B.107}$$

where $\rho(s,\mu+l_2)$ is the SL2 Plancherel measure in (B.38), and where we note that the product $\rho(s,\mu+\lambda_2)\mathcal{N}_{s_1,\,\lambda_2^+,\,s}$ is symmetric under the exchange of s_1 and s. Consequently, the product of two regular continuous series characters and a regularized discrete series character is given by

$$\int dg \, \chi_{(s_1,\,\mu_1)}(g)\overline{\chi}_{\lambda_2^\pm}(g)\chi_{(s,\mu_1+\lambda_2)}(hg^{-1}) = \frac{\chi_{(s,\mu_1+\lambda_2)}(h)}{\rho(s,\mu_1+\lambda_2)} \sum_{m_1-m_2} I^{s_1,\lambda_2^\pm,s}_{m_1,m_2,m_1+m_2} \,. \tag{B.108}$$

Using Eq. (B.99) and (B.103) we thus find that by taking the $\mu_1 \to i\infty$ limit and truncating the sum over $m_1 - m_2$,

$$\lim_{\mu_1\to i\infty} \int dg \, \chi_{(s_1,\,\mu_1)}(g)\bar{\chi}_{\lambda_2^\pm}(g)\chi_{(\lambda=\frac{1}{2}+is,\mu_1+\lambda_2)}(hg^{-1}) = \frac{N_{\lambda_2^\pm}N^s{}_{s_1,\lambda_2^\pm}}{\rho(s,\mu_1+\lambda_2)}\chi_{(s,\mu_1+\lambda_2)}(h) \,, \tag{B.109}$$

where we define the fusion coefficient $N^{s_1,\lambda_1}{}_s$ in the $\mu_1 \to i\infty$ limit,

$$N^s{}_{s_1,\lambda_2^\pm} \equiv \frac{|\Gamma(\lambda_2 + is_1 - is)\Gamma(\lambda_2 + is_1 + is)|^2}{\Gamma(2\lambda_2)}\,, \tag{B.110}$$

up to a $\lambda_2^\pm$ dependent normalization constant,

$$N_{\lambda_2^\pm} = \sum_{k=0}^{\Xi} \mu_1^{-2\lambda_2}(-1)^k \frac{\Gamma(1-2\lambda_2)}{\Gamma(1+k)\Gamma(1-k-2\lambda_2)} = \frac{(-1)^\Xi \mu_1^{-2\lambda_2}\Gamma(-2\lambda_2)}{\Xi!\Gamma(-\Xi-2\lambda_2)}\,, \tag{B.111}$$

As we take the cut-off, $\Xi \to \infty$, the normalization constant becomes

$$N_{\lambda_2^\pm} = \frac{\mu_1^{-2\lambda_2}\Xi^{2\lambda_2}}{\Gamma(1+2\lambda_2)} \tag{B.112}$$

Using the fusion coefficient, together with the normalization factor, we compute the expectation value of the Wilson lines in Section 2.4.

B.4.3 6-j symbols

To obtain the OTO-correlator in Section 2.4.4 we need to consider the integral of six characters in (2.65),

$$\begin{aligned}
&\int dh_1 dh_2 dh_3 dh_4\, \chi_{s_1}(h_1 h_2^{-1})\chi_{s_2}(h_2 h_3^{-1} s)\chi_{s_3}(h_3 h_4^{-1})\chi_{s_4}(g h_4 h_1^{-1})\overline{\chi}_{\lambda_1^\pm}(h_1 h_3^{-1})\overline{\chi}_{\lambda_2^\pm}(h_2 h_4^{-1}) = \\
&= \int dh_1 dh_2 dh_3 dh_4 \sum_{m_i,n_i,q_i,\tilde{m}_i} U^{m_1}_{s_1,n_1}(h_1) U^{n_1}_{s_1,m_1}(h_2^{-1}) U^{m_2}_{s_2,n_2}(h_2) U^{n_2}_{s_2,m_2}(h_3^{-1}) U^{m_3}_{s_3,n_3}(h_3) \\
&\times U^{n_3}_{s_3,m_3}(h_4^{-1}) U^{m_4}_{s_4,n_4}(g) U^{n_4}_{s_4,q_4}(h_4) U^{q_4}_{s_4,m_4}(h_1^{-1}) U^{\tilde{m}_1}_{\lambda_1^\pm,\tilde{n}_1}(h_1) U^{\tilde{n}_1}_{\lambda_1^\pm,\tilde{m}_1}(h_3^{-1}) U^{\tilde{m}_2}_{\lambda_2^\pm,\tilde{n}_2}(h_2) U^{\tilde{n}_2}_{\lambda_2^\pm,\tilde{m}_2}(h_4)\,,
\end{aligned} \tag{B.113}$$

where, for the case of interest in Section 2.4.4, s_1, s_2, s_3, and s_4 label continuous series representations, and $\lambda_1^\pm$ and $\lambda_2^\pm$ label representations in the positive/negative discrete series. As in the case of computing the time-ordered correlators of the Wilson lines we first consider the result when $\mu_1 \in \mathbb{R}$ and only afterwards analytically continue the final result to $\mu_1 \to i\infty$.

The sums over $\tilde{m}_1$ and $\tilde{n}_1$, as well as that over $\tilde{m}_2$ and $\tilde{n}_2$ are truncated according to the regularization prescription for the characters associated to the Wilson lines. Evaluating the integrals we find

$$\sum_{m_i,n_i,\tilde{m}_i,\tilde{n}_i,q_4} U^{m_4}_{s_4,q_4}(g)\frac{C^{s_1,\lambda_1^\pm,s_4}_{m_1,\,\tilde{m}_1,\,m_4}(C^{s_1,\lambda_1^\pm,s_4}_{n_1,\,\tilde{n}_1,\,q_4})^*}{\rho(s_4,\,\mu_4)\mathcal{N}_{s_1,\,\lambda_1^\pm,\,s_4}}\frac{C^{s_2,\lambda_2^\pm,s_1}_{m_1,\,\tilde{m}_2,\,m_1}(C^{s_2,\lambda_2^\pm,s_1}_{n_2,\,\tilde{n}_1,\,n_1})^*}{\rho(s_1,\,\mu_1)\mathcal{N}_{s_2,\,\lambda_2^\pm,\,s_1}}$$

$$\times\frac{C^{s_3,\lambda_1^\pm,s_2}_{m_3,\,\tilde{m}_1,\,m_2}(C^{s_3,\lambda_1^\pm,s_2}_{n_3,\,\tilde{n}_1,\,n_2})^*}{\rho(s_2,\,\mu_2)\mathcal{N}_{s_3,\,\lambda_1^\pm,\,s_2}}\frac{C^{s_4,\lambda_2^\pm,s_3}_{n_4,\,\tilde{n}_2,\,m_3}(C^{s_4,\lambda_2^\pm,s_3}_{q_4,\,\tilde{m}_2,\,n_3})^*}{\rho(s_3,\,\mu_3)\mathcal{N}_{s_4,\,\lambda_2^\pm,\,s_3}}\,, \tag{B.114}$$

Performing the sums over the n_1, n_2, n_3, $\tilde{n}_1$ and $\tilde{n}_2$ states we obtain the 6-j symbol associated to the six representations s_1, s_2, s_3, s_4, $\lambda_1^\pm$, and $\lambda_2^\pm$. Furthermore, the sum also imposes the constraint $m_4 = q_4$. The remaining sum over four Clebsch-Gordan coefficient yields the square root for the factor present in (B.109). Specifically, we obtain that (B.114) equals

$$N_{\lambda_1^\pm}N_{\lambda_2^\pm}\chi_{s_4}(g)\sqrt{N^{s_4}{}_{\lambda_1^\pm,s_1}N^{s_3}{}_{\lambda_1^\pm,s_2}N^{s_3}{}_{\lambda_2^\pm,s_1}N^{s_4}{}_{\lambda_2^\pm,s_2}}R_{s_3s_4}\begin{bmatrix} s_2 & \lambda_2 \\ s_1 & \lambda_1 \end{bmatrix}\,. \tag{B.115}$$

The 6-j symbol for SL2 is given by [131]

$$R_{s_3s_4}\begin{bmatrix} s_2 & \lambda_2 \\ s_1 & \lambda_1 \end{bmatrix} = \mathbb{W}(s_3,s_4;\lambda_1+is_2,\lambda_1-is_2,\lambda_2-is_1,\lambda_2+is_1) \tag{B.116}$$

$$\times\sqrt{\Gamma(\lambda_2\pm is_1\pm is_3)\Gamma(\lambda_1\pm is_2\pm is_3)\Gamma(\lambda_1\pm is_1\pm is_4)\Gamma(\lambda_2\pm is_2\pm is_4)}\,,$$

where the Wilson function $\mathbb{W}(s_a,s_b;\lambda_1+is_2,\lambda_1-is_2,\lambda_2-is_1,\lambda_2+is_1)$ is given by [130]

$$\mathbb{W}(\alpha,\beta,a,b,c,d)\equiv\frac{\Gamma(d-a)_4F_3\left[\begin{smallmatrix} a+i\beta & a-i\beta & a-i\beta & \tilde{a}+i\alpha & \tilde{a}-i\alpha \\ a+b & a+c & a+c & 1+a-d & \end{smallmatrix};\,1\right]}{\Gamma(a+b)\Gamma(a+c)\Gamma(d\pm i\beta)\Gamma(\tilde{d}\pm i\alpha)}+(a\leftrightarrow d)\,, \tag{B.117}$$

with $\tilde{a}=(a+b+c-d)/2$ and $\tilde{d}=(b+c+d-a)/2$. The normalization for the 6-j symbol in (B.116) is obtained by imposing the orthogonality relation (2.73) using the orthogonality properties of the Wilson function [130, 131]. Such an orthogonality condition on the 6-j symbol follows from its definition in terms of a sum of Clebsch-Gordan coefficients as that shown in (B.114).

Firstly we note that the result is the same when considering λ_1 or λ_2 in the positive or negative discrete series. Furthermore, since the result is explicitly independent of μ_1, μ_2, μ_3 and μ_4 one can easily perform the analytic continuation to $\mu_1,\,\mu_2,\,\mu_3,\mu_4\to i\infty$ as required by our boundary conditions on the field $\phi^{\mathbb{R}}$. Putting this together with the analytic continuation of the fusion coefficients presented in the previous sub-section we find the final results from Section 2.4.

B.5 Wilson lines as probe particles in JT gravity

As mentioned in Section 2.4, the insertion of a Wilson loop in 3D Chern-Simons theory with gauge algebra $\mathfrak{so}(2,2)$ (or an isomorphic algebra) can be interpreted as the effective action of a massive probe in AdS$_3$ (or other spaces with an isomorphic symmetry algebra) [8, 123, 124, 125, 126, 127]. In this Appendix we extend this interpretation to 2D. Specifically, we outline the proof of the equivalence, as stated in section 2.4.1, between the boundary-anchored Wilson line observables $\mathcal{W}_\lambda(\mathcal{C}_{\tau_1\tau_2})$ in the $\mathcal{G} = \mathcal{G}_B$ BF theory formulation, and the boundary-to-boundary propagator of a massive particle in the metric formulation of JT gravity. The latter is given by the functional integral over all paths $x(s)$ diffeomorphic to the curve $\mathcal{C}_{\tau_1\tau_2}$ weighted with the standard point particle action (here $\dot{x}^\mu = \frac{dx^\mu}{ds}$)

$$S[x, g_{\mu\nu}] = m \int_{\mathcal{C}_{\tau_1\tau_2}} ds\, \sqrt{g_{\mu\nu}\dot{x}^\mu \dot{x}^\nu}\,. \tag{B.118}$$

Concretely, we would like to demonstrate that

$$\widehat{\mathcal{W}}_{\lambda,k=0}(\mathcal{C}_{\tau_1\tau_2}) = \mathrm{Tr}_{\,\lambda,k=0}\Big(\mathcal{P}\exp\int_{\mathcal{C}_{\tau_1\tau_2}} A\Big) \cong \int_{\text{paths}\sim\mathcal{C}_{\tau_1\tau_2}} [dx]\, e^{-S[x,g_{\mu\nu}]}, \tag{B.119}$$

where the mass of the particle is determined by the $\mathcal{G}_B$ representation $(\lambda, k = -2\pi\lambda/B)$ as $m^2 = \lambda(\lambda-1) = -C_2(\lambda)$.[12] From now on we assume $|\lambda| > 1$ in order for $m^2 > 0$. In the equation above, we have taken the limit $B \to \infty$ thus set $k = 0$. Consequently the Wilson line $\widehat{\mathcal{W}}_{\lambda,k=0}(\mathcal{C}_{\tau_1\tau_2})$ only couples to the $\mathfrak{sl}(2,\mathbb{R})$-components of the $\mathcal{G}_B$ gauge field. In the rest of this Appendix, we will implicitly assume that A take values in $\mathfrak{sl}(2,\mathbb{R})$. For notation convenience, we will refer to these Wilson lines as $\widehat{\mathcal{W}}_\lambda(\mathcal{C}_{\tau_1\tau_2})$ from now on.[13]

The congruence symbol $\cong$ in (B.119) indicates that we want to prove an operator equivalence inside the functional integral of JT gravity. Indeed, the right-hand side of (B.119) depends only on the diffeomorphism class of the path $\mathcal{C}_{\tau_1\tau_2}$, whereas the Wilson line operator $\widehat{\mathcal{W}}_\lambda(\mathcal{C}_{\tau_1\tau_2})$ on the left-hand side follows some given path. So in writing (B.119), we implicitly assume that $\widehat{\mathcal{W}}_\lambda(\mathcal{C}_{\tau_1\tau_2})$ is evaluated inside the functional integral of a diffeomorphism invariant BF gauge theory.

To start proving (B.119), following [226, 227], we rewrite the Wilson line $\widehat{\mathcal{W}}_\lambda(\mathcal{C}_{\tau_1\tau_2})$ around a given space-time contour $\mathcal{C}_{\tau_1\tau_2}$, parametrized by an auxiliary variable s, as a functional integral over

[12] For notational simplicity, we take all Wilson lines to be in the positive discrete series representations in this section. We also emphasize that the Wilson line in the representation (λ, k) is a defect operator (external probe), thus k is not constrained to be k_0.

[13] Equivalently, one can think of the boundary-anchored Wilson lines $\widehat{\mathcal{W}}_\lambda(\mathcal{C}_{\tau_1\tau_2})$ as $PSL(2,\mathbb{R})$ Wilson lines in the discrete series representation λ (projective for $\lambda \notin \mathbb{Z}$) of $PSL(2,\mathbb{R})$.

paths $g(s) \in PSL(2,\mathbb{R})$ via [14]

$$\mathrm{Tr}_{\lambda}\left(\mathcal{P}\exp\oint_{\mathcal{C}_{\tau_1\tau_2}} A\right) = \int_{\mathcal{C}_{\tau_1\tau_2}} [dg]_{\boldsymbol{\alpha}}\, e^{-S_{\boldsymbol{\alpha}}[g,A]} \tag{B.120}$$

where $S_{\boldsymbol{\alpha}}[g,A]$ denotes the (first order) coadjoint orbit action of the representation λ, coupled to a background $\mathfrak{sl}(2,\mathbb{R})$ gauge field $A_s(s) \equiv A_\mu(x(s))\dot{x}^\mu(s)$

$$S_{\boldsymbol{\alpha}}[g,A] \;=\; \int_{\mathcal{C}_{\tau_1\tau_2}} ds\,\mathrm{Tr}\,\left(\boldsymbol{\alpha}\, g^{-1}D_A g\right) \;=\; \int_{\mathcal{C}_{\tau_1\tau_2}} ds\,\left(\mathrm{Tr}\,(\boldsymbol{\alpha}\, g^{-1}\partial_s g) - \mathrm{Tr}\,(A_s g\boldsymbol{\alpha} g^{-1})\right). \tag{B.121}$$

Here $\boldsymbol{\alpha} = \alpha_i P^i \in \mathfrak{sl}(2,\mathbb{R})$ denotes some *fixed* Lie algebra element with specified length squared equal to the second Casimir

$$\mathrm{Tr}\,(\boldsymbol{\alpha}^2) = -C_2(\lambda) = -\lambda(\lambda-1) \tag{B.122}$$

The classical phase space in (B.121) is over the (co)adjoint orbit of the Lie algebra element $\boldsymbol{\alpha}$

$$\mathcal{O}_{\boldsymbol{\alpha}} \equiv \{g\boldsymbol{\alpha} g^{-1} | g \in PSL(2,\mathbb{R})\} \tag{B.123}$$

Consequently the path integral is over maps from $\mathcal{C}_{\tau_1\tau_2} \to \mathcal{O}_{\boldsymbol{\alpha}}$ which can be equivalently described by their lift $g : \mathcal{C}_{\tau_1\tau_2} \to PSL(2,\mathbb{R})$ up to an identification due to local right group action by the stabilizer of $\boldsymbol{\alpha}$ on g. This is the meaning of path integral measure $[dg]_{\boldsymbol{\alpha}}$ in (B.120).

Let us briefly recall why equation (B.120) holds. Expanding g around a base-point, with $g = e^{x^a(s)P_a} g(s_0)$, we find from (B.121) that the canonical momenta associated to $x^a(s)$ are give by

$$\pi_{x^i} = \mathrm{Tr}\,(P^i g\boldsymbol{\alpha} g^{-1}), \tag{B.124}$$

which are in fact the generators of the $PSL(2,\mathbb{R})$ symmetry which acts by left multiplication on g, as $g \to Ug$. The Casimir associated to $\mathfrak{sl}(2,\mathbb{R})$ component of $\mathcal{G}_B$ is given by $\widehat{C}_2^{\mathfrak{sl}(2,\mathbb{R})} = -\eta^{ij}\pi_{x^i}\pi_{x^j} = -\mathrm{Tr}\,(\boldsymbol{\alpha}^2)$. The Hilbert space of the theory is spanned by functions on the group $\mathcal{G}_B$ which are invariant under right group actions that stabilize $\boldsymbol{\alpha}$. The Hilbert space of the quantum mechanics model on $\mathcal{O}_{\boldsymbol{\alpha}}$ thus forms an irreducible (projective) $PSL(2,\mathbb{R})$ representation λ. Since the functional integral around a closed path $g(s) \in PSL(2,\mathbb{R})$ amounts to taking the trace over the Hilbert space,

[14]Note that coadjoint orbits of a connected semisimple Lie group are identical with those of the universal cover groups, as evident from the definition (B.123) for the $PSL(2,\mathbb{R})$ case and its coverings.

we arrive at the identity (B.120).[15]

Since the identity (B.120) holds for any choice of Lie algebra element $\boldsymbol{\alpha}$ with length squared given by (B.122), we are free to include in the definition of $\mathcal{W}_\lambda(\mathcal{C}_{\tau_1\tau_2})$ a functional integral over all Lie algebra elements of the form

$$\boldsymbol{\alpha}(s) = \alpha_a(s)P^a = \alpha_1(s)P^1 + \alpha_2(s)P^2 \tag{B.125}$$

subject to the constraint (B.122). This leads to the identity (up to an overall factor that does not depend on A)

$$\widehat{\mathcal{W}}_\lambda(\mathcal{C}_{\tau_1\tau_2}) \sim \int [d\alpha_{1,2} dg d\Theta] \, e^{-S_{\boldsymbol{\alpha}}[g,\Theta,A]} \tag{B.126}$$

with

$$S_{\boldsymbol{\alpha}}[g,\Theta,A] = \oint_{\mathcal{C}_{\tau_1\tau_2}} ds \, \left(\text{Tr}\, \left(\boldsymbol{\alpha}\, g^{-1} D_A g\right) + i\Theta(\eta^{ab}\alpha_a\alpha_b - m^2)\right) \, . \tag{B.127}$$

Here $m^2 = \lambda(\lambda-1)$ and Θ denotes a Lagrange multiplier that enforces the constraint (B.122). This already looks closely analogous to the world line action of a point particle of mass m.

So far we have considered a general background gauge field A in the bulk. In the context in which we make A dynamical and perform the path integral in the BF-theory in the presence of a defect (2.3), the path integral (after integrating out the adjoint scalar ϕ) localizes to configurations of flat A, away from the defect. Similarly, on the JT gravity side (in the metric formulation), integrating out the dilaton ϕ forces the ambient metric on the disk to be that of AdS_2. Thus for the purpose of proving (B.119), we can take A to be flat on the BF theory side, and the metric to be AdS_2 on the JT gravity side.

The action (B.127) is invariant under gauge transformations $U(s)$ for which $g \to U(s)g$, together with the corresponding gauge transformation of A which leaves the connection flat. Note however the gauge transformation mixes the components of A associated to the frames and spin connection. We can always (partially) gauge fix by setting $g = \mathbf{1}$ by choosing $U(s) = g^{-1}(s)$ along the curve $\mathcal{C}_{\tau_1\tau_2}$ and smoothly extending this gauge transformation onto the entire disk.[16] After such a gauge

[15] This is because we are considering a boundary condition with $A_\tau = 0$. Consequently, the boundary-anchored Wilson line has the same expectation value as a Wilson loop that touches the boundary.

[16] There's no obstruction for such extensions since $\mathcal{G}_B$ is simply connected.

fixing, the action (B.127) simply becomes,

$$S_1[x,k,\lambda,g^{\mu\nu}] \equiv \int_{\mathcal{C}_{\tau_1\tau_2}} ds(k_\mu \dot{x}^\mu + i\Theta(g^{\mu\nu}k_\mu k_\nu - m^2)) \\ = \int_{\mathcal{C}_{\tau_1\tau_2}} ds\ \left(\eta_{ab}\alpha^a \tilde{e}^b_\mu\, \dot{x}^\mu + i\Theta(\eta^{ab}\alpha_a\alpha_b - m^2)\right)\ , \tag{B.128}$$

where $g_{\mu\nu} = \eta_{ab}e^a_\mu e^b_\nu$ is the AdS_2 metric associated to the background flat connection A and $k_\mu \equiv \alpha_a e^a_\mu$. The action (B.128) agrees with the first order action for a particle moving on the world-line $\mathcal{C}_{\tau_1\tau_2}$. To finish the proof, we need to show that the path integral over flat A in the BF theory reproduces the integral over paths diffeomorphic to $\mathcal{C}_{\tau_1\tau_2}$ for the particle in the JT gravity.

As mentioned in Section 1.4.1, space-time diffeomorphisms can be identified with field dependent gauge transformations in the BF theory when the gauge field is flat

$$\delta^{\text{diff}}_\xi = \delta^{\text{gauge}}_\epsilon\,, \tag{B.129}$$

where the the vector field $\xi^\mu(x)$ generating the diffeomorphism transformation and the infinitesimal gauge transformation parameter $\epsilon^a(x)$ (vanish on the boundary) are related by

$$\epsilon^a(x) = e^a_\mu(x)\xi^\mu(x)\,, \qquad \epsilon^0(x) = \omega_\mu(x)\xi^\mu(x)\,. \tag{B.130}$$

Since flat connections A are generated by gauge transformations, the equivalence (B.129) acting on $\tilde{e}^b_\mu$ implies that,

$$\int_{\mathcal{C}_{\tau_1\tau_2}} ds\ \left(\eta_{ab}\alpha^a(\tilde{e}^b_\mu)_\epsilon\, \dot{x}^\mu + i\Theta(\eta^{ab}\alpha_a\alpha_b - m^2)\right) = \int_{\mathcal{C}^\xi_{\tau_1\tau_2}} ds\ \left(\eta_{ab}\alpha^a \tilde{e}^b_\mu\, \dot{x}^\mu + i\Theta(\eta^{ab}\alpha_a\alpha_b - m^2)\right)$$

where $(\tilde{e}^b_\mu)_\epsilon$ denotes the finite gauge transformation of $\tilde{e}^b_\mu$ generated by ϵ, and $\mathcal{C}^\xi_{\tau_1\tau_2}$ denotes a path diffeomorphic to $\mathcal{C}_{\tau_1\tau_2}$ generated by displacement vector field ξ. Consequently integrating over flat connections A of the BF theory in the presence of the Wilson line insertion is equivalent to integrating over all paths diffeomorphic to the curve $\mathcal{C}_{\tau_1\tau_2}$, which precisely gives the first order form of (B.118) that describes a particle propagating between boundary points in AdS_2.[17]

Alternatively, to get the second order formulation for the world-line action we can directly perform the Gaussian integration over α_a in (B.127) and then integrate out the Lagrange multipler Θ. The

[17]Note that in the world-line action (B.128), the fields $(x^\nu, k_\mu(x))$ take values in the co-tangent bundle $T^*\Sigma$. The path integration measure is the natural one induced by the symplectic structure of $T^*\Sigma$.

world-line path integral (B.126) becomes (up to an A independent factor),

$$\widehat{\mathcal{W}}_\lambda(\mathcal{C}_{\tau_1\tau_2}) \sim \int [dg]\, e^{-S_2[g,A]}\,, \tag{B.131}$$

where the action $S_2[g,A]$ is specified by

$$S_2[g,A] = m \int_{\mathcal{C}_{\tau_1\tau_2}} ds\, \sqrt{\eta_{ab}(g^{-1}D_A g)^a (g^{-1}D_A g)^b}\,. \tag{B.132}$$

Due to the integration over $g(s)$, this is a gauge invariant observable as expected. Note that while (B.132) is exact on-shell in order for the path-integral (B.131) to agree with (B.120) one has to appropriately modify the measure $[dg]$ in (B.131).

Once again performing the gauge transformation with $U(s) = g^{-1}(s)$ along the curve $\mathcal{C}_{\tau_1\tau_2}$ to gauge fix $g(s) = 1$ and smoothly extending the gauge transformation onto the entire disk, the action (B.132) simply becomes,

$$S_2[g,A] = m \int_{\mathcal{C}_{\tau_1\tau_2}} ds\, \sqrt{\eta_{ab} e^a_\alpha e^b_\beta \dot{x}^\alpha \dot{x}^\beta} = m \int_{\mathcal{C}_{\tau_1\tau_2}} ds\, \sqrt{g_{\alpha\beta}\dot{x}^\alpha \dot{x}^\beta}\,, \tag{B.133}$$

which agrees with the 2nd order action (B.118) for a particle moving on the world-line $\mathcal{C}_{\tau_1\tau_2}$. Following the same reasoning as before, the gauge transformation can be mapped to a diffeomorphism, and integrating over flat connections in the BF theory path integral with the Wilson line insertion, is once again equivalent to integrating over all paths diffeomorphic to the curve $\mathcal{C}_{\tau_1\tau_2}$. Using this, we finally arrive at the desired equality between the Wilson line observable and the worldline representation of the boundary-to-boundary propagator given by (B.119).[18]

[18] As usual in AdS/CFT, the worldline observable (boundary-to-boundary propagator) requires appropriate regularization and renormalization due to the infinite proper length near the boundary of AdS_2. Here in the gauge theory description, we also require a proper renormalization of the boundary-anchored Wilson line to remove the divergence due to the infinite dimensional representation carried by the Wilson line (see (2.55)). It would be interesting to understand the precise relation between the two renormalization schemes.

Appendix C

Gravitational interpretation of the $SO(3)$ gauge fields for near-extremal black holes

C.1 The Kerr-Newman solution

When reducing the Einstein action in four dimensions to two dimensions a $SO(3)$ gauge field emerges from the symmetries of the transverse sphere S^2. We denoted the charges associated to this field by J. In this appendix we will explicitly check that two dimensional solutions with charge J can be uplifted to KN solution in four dimensions. In the approximation where all $SO(3)$ charged fields can be neglected, the angular momentum J on the black hole is directly related to the value of the $SO(3)$ field strength given by the $SO(3)$ Casimir [89].

The KN solution in AdS_4 with radius L is given by

$$(ds^{KN})^2 = \frac{\rho^2 \Delta_{\tilde{r}} \Delta_{\theta}}{\Sigma} d\tilde{\tau}^2 + \frac{\rho^2}{\Delta_{\tilde{r}}} d\tilde{r}^2 + \frac{\rho^2}{\Delta_{\tilde{\theta}}} d\tilde{\theta}^2 + \sin^2\tilde{\theta} \frac{\Sigma}{\rho^2 \Xi^2} (d\tilde{\phi} + \mathcal{B}_{\tilde{\tau}}\, d\tilde{t})^2, \tag{C.1}$$

where the mass, angular momentum and charge are parametrized as

$$M = \frac{m}{G_N \Xi^2}\,, \quad J = \frac{ma}{G_N \Xi^2}\,, \quad Q = \frac{q}{\Xi}\,, \quad \Xi \equiv 1 - \frac{a^2}{L^2}\,, \tag{C.2}$$

and the functions appearing in the metric are

$$\rho^2 = \tilde{r}^2 + a^2\cos^2\tilde{\theta}\,, \qquad \Delta_{\tilde{r}} = (\tilde{r}^2 + a^2)\left(1 + \frac{\tilde{r}^2}{L^2}\right) - 2m\tilde{r} + q^2\,, \qquad \Delta_{\tilde{\theta}} = 1 - \frac{a^2}{L^2}\cos^2\tilde{\theta}\,,$$
$$\Sigma = (\tilde{r}^2 + a^2)^2\Delta_{\tilde{\theta}} - a^2\Delta_{\tilde{r}}\sin^2\tilde{\theta}\,, \qquad \mathcal{B}_{\tilde{\tau}} = i\frac{a\Xi[(a^2+\tilde{r}^2)\Delta_{\tilde{\theta}} - \Delta_{\tilde{r}}]}{\Sigma}\,. \tag{C.3}$$

For small a the relation between the angular momentum is given (to first order) by $J = Ma$. At small a the metric (C.1) can be seen as a deformation of the RN solution from (5.3) in which one turns on a non-trivial profile for the $SO(3)$ gauge field with

$$\delta g_{\mu\nu}dx^\mu dx^\nu = \frac{2\mathcal{B}_{\tilde{\tau}}\Sigma\sin^2\tilde{\theta}}{\rho^2\Xi^2}d\tilde{\phi}d\tilde{\tau} = 2ia\sin^2\tilde{\theta}(1 - f(\tilde{r}))d\phi d\tilde{\tau}, \tag{C.4}$$

where $f(\tilde{r})$ is the function appearing in equation (5.3).

As we will show the deformation in (C.4) does not precisely match with the solution for the $SO(3)$ gauge fields inserted into the dimensional reduction ansatz (5.15). Nevertheless, as we will explain in the next subsection, the perturbed solution for the KN metric $g_{\mu\nu}^{KN} = g_{\mu\nu}^{RN} + \delta g_{\mu\nu}$ will turn to be equivalent, up to diffeomorphisms, with the solution for the $SO(3)$ gauge fields inserted into the dimensional reduction ansatz.

Thus, to first order at small J (or equivalently in small a), the partition function is well approximated by considering the quantization of the $SO(3)$ gauge field coupled to the standard RN metric given in each sector with fixed Q. In the next subsections we further show that this approximation is valid by studying the solutions to the equations of motion for the $SO(3)$ gauge field. Furthermore, we show that the average value of angular momentum contributing to the grand canonical partition function does not strongly backreact on the metric (i.e. its contribution is much smaller than that of the $U(1)$ charge).

C.2 Classical $SO(3)$ gauge field configurations

In order to compare the perturbed RN solution to the ansatz for the dimensional reduction (5.15) we need to solve the equations of motion for the $2d$ $SO(3)$ gauge fields whose contribution to the action is given by (5.22),

$$I_{EM}^{SO(3)} = -\frac{1}{12G_N r_0}\int_{M_4}\sqrt{g}\chi^{5/2}\mathrm{Tr}(H_{\mu\nu}H^{\mu\nu})\,. \tag{C.5}$$

We first start with the case in which we fix the boundary holonomy of the $SO(3)$ gauge fields (which corresponds to fixing the boundary metric on ∂M_4) rather than the overall charge of the system.[1] For practical purposes, it proves convenient to choose the boundary component of the gauge field to be constant with $B|_{\partial\mathcal{M}_2} = i\frac{\mu_{\text{SO(3)}}}{\beta}T^3 d\tau$ such that the holonomy is given by $\exp(\oint_{\partial\mathcal{M}_2} B) = \exp(i\mu_{\text{SO(3)}}\sigma^3)$ with $\mu_{\text{SO(3)}} \in [0, 2\pi)$ (according to our conventions $T^a = \frac{1}{2}\sigma^a$ with σ the Pauli matrices).

We can find the solution in the gauge in which $B_r = 0$ and make the ansatz that $B = i\frac{\mu_{\text{SO(3)}}T^3}{\beta}\xi(r)d\tau$ for some function $\xi(r)$ satisfying $\xi(r_{\partial M_2}) = 1$. Then, the field strength is $H = i\frac{\mu_{\text{SO(3)}}T^3}{\beta}\partial_r\xi(r)dr\wedge d\tau$ and the equation of motion $d^*H = 0$ implies that $\left[\xi'(r)/\sqrt{g}\chi^{-5/2}\right]' = 0$. Using the solution $\chi(r) = r^2$, this implies that

$$\xi(r) = \alpha_1 + \frac{\alpha_2}{r^3}\,, \qquad H_{r\tau} = -i\frac{\mu_{\text{SO(3)}}}{\beta}\frac{3\alpha_2}{r^4}T^3$$
$$\delta g^{SO(3)}_{\mu\nu}dx^\mu dx^\nu = 2r^2 i\sin^2(\theta)\frac{\mu_{\text{SO(3)}}}{\beta}\left(\alpha_1 + \frac{\alpha_2}{r^3}\right)d\tau d\phi\,. \tag{C.6}$$

Demanding that the gauge field has unit holonomy around the point with $r = r_0$ imposes that $\mu_{\text{SO(3)}}\left(\alpha_1 + \alpha_2/r_0^3\right) = 2\pi n$ with $n \in \mathbb{Z}$. Furthermore imposing that $\xi(r|_{\partial M_2}) = 1$ implies that $\alpha_1 = 1$, and, consequently, $\alpha_2 = r_0^3\left(\frac{2\pi n}{\mu_{\text{SO(3)}}} - 1\right)$. Consequently, we have that

$$\begin{aligned} B_\tau &= \frac{iT^3}{\beta}\left[\mu_{\text{SO(3)}} + \frac{r_0^3}{r^3}\left(2\pi n - \mu_{\text{SO(3)}}\right)\right] \\ \delta g^{SO(3)}_{\mu\nu}dx^\mu dx^\nu &= 2ir^2\frac{\sin^2(\theta)}{\beta}\left[\mu_{\text{SO(3)}} + \frac{r_0^3}{r^3}\left(2\pi n - \mu_{\text{SO(3)}}\right)\right]d\tau d\phi\,. \end{aligned} \tag{C.7}$$

Gauge field configurations with different n correspond to different instanton configurations for the $SO(3)$ gauge field and different metric solutions, all obeying the same boundary condition on ∂M_2.

As a consistency check, when adding the metric defomation in (C.6) to the RN metric as in the ansatz (5.15) Einstein's field equations are still satisfied to first order in an expansion in $1/\beta$, meaning that the action (5.22) resulting from the dimensional reduction is correct. The total action (C.6) evaluates to

$$\begin{aligned} H^3_{r\tau} &= -i\frac{1}{\beta}\frac{3r_0^3}{r^4}\left(2\pi n - \mu_{\text{SO(3)}}\right)\,, \\ I^{SO(3)}_{EM} &= \frac{1}{6G_N}\int_0^\beta d\tau\int_{r_0}^\infty dr\,\frac{9r_0^6}{\beta^2 r^4}\left(2\pi n - \mu_{\text{SO(3)}}\right)^2 = \frac{2r_0^3\left(2\pi n - \mu_{\text{SO(3)}}\right)^2}{G_N\beta} \end{aligned} \tag{C.8}$$

[1] We thank Silviu Pufu and Yifan Wang for sharing notes during a past project about instanton solutions in $2d$ $SO(3)$ Yang-Mills theory.

in each instanton sector. To make contact with the effective action in each j sector in the sum over $SO(3)$ representations we can evaluate the sum over j for the contribution of each representation to the partition function (5.25) for the on-shell solution $\chi(r) = r^2$:

$$
\begin{aligned}
Z_{RN}^{SO(3)} &= \sum_{j\geq 0}(2j+1)\chi_j(\mu_{\mathrm{SO(3)}})e^{-\frac{G_N\beta}{2r_0^3}j(j+1)} = -\frac{e^{\frac{G_N\beta}{8r_0^3}}}{4\sin\mu_{\mathrm{SO(3)}}}\vartheta_3'\Big(\zeta/2, e^{-\frac{G_N\beta}{8r_0^3}}\Big) \\
&= \sum_{n\in\mathbb{Z}} \frac{2\sqrt{\pi}(\mu_{\mathrm{SO(3)}}-2\pi n)}{\left(\frac{G_N\beta}{2r_0^3}\right)^{3/2}\sin(\mu_{\mathrm{SO(3)}}-2\pi n)} e^{-\frac{2r_0^3}{G_N\beta}(\mu_{\mathrm{SO(3)}}-2\pi n)^2}\,, \qquad \text{(C.9)}
\end{aligned}
$$

where $\vartheta_3'(u,q)$ is the derivative with respect to u of $\vartheta_3(u,q)$ and where to obtain the final equation we have used the expansion in terms $\frac{G_N\beta}{r_0^3}$. Consequently, we find that the sum over instanton saddle in the partition function (C.8) precisely agrees with the sum over $SO(3)$ representations appearing in the partition function associated to the action (5.25).[2]

To find the relation between the $SO(3)$ representation j and the angular momentum it proves convenient to also analyze the classical solutions in the case in which we fix the field strength at the boundary (or equivalently the Lagrange multiplier zero-form $\phi^{SO(3)}$). In this case, we will fix gauge such that $H_{r\tau}dr\wedge d\tau|_{\partial M_2} = i\sqrt{g}\sigma_3\frac{3G_N r_0}{\sqrt{2}\chi^{5/2}}j|_{\partial M_2} = i\frac{3G_N\sigma_3 j}{\sqrt{2}r^4}|_{\partial M_2}$, for some constant j. The resulting gauge field, field strength and $4d$ metric perturbation is given by

$$
\begin{aligned}
&B_\tau = iT^3\left(\frac{\sqrt{2}G_N j}{r_h^3} + \frac{2\pi n}{\beta} - \frac{\sqrt{2}G_N j}{r^3}\right), \qquad H_{r\tau} = i\frac{3G_N\sigma_3 j}{\sqrt{2}r^4}\,, \\
&\delta g_{\mu\nu}^{SO(3)}dx^\mu dx^\nu = 4ir^2\sin^2(\theta)\left[\frac{G_N j}{\sqrt{2}r_0^3} + \frac{2\pi n}{\beta} - \frac{G_N j}{\sqrt{2}r^3}\right]d\tau d\phi\,, \qquad \text{(C.10)}
\end{aligned}
$$

where we have fixed gauge such that $B_r = 0$ and have once again obtained the first r-independent term in B_τ by requiring unit holonomy around the point with $r = r_h$ (i.e. nowhere is H singular).

Next we determine the contribution of the $SO(3)$ gauge field to the action. As for the $U(1)$ gauge field (5.2), in order for to have a well defined variational principle we need to add a boundary term to the action: $I_{EM}^{SO(3),N} = I_{EM}^{SO(3)} + \frac{1}{12G_N r_0}\int du\sqrt{g}\,n_\mu \mathrm{Tr}\,H^{\mu\nu}B_\nu$. Accounting for this boundary term we find that the

$$
I_{EM}^{SO(3),N} = \frac{1}{6}\int_0^\beta d\tau\int_{r_0}^\infty dr\frac{G_N j^2}{r^4} - \int_0^\beta d\tau\left(\frac{G_N j}{3r_0^3} + \frac{2\pi n}{\beta}\right)j \sim \frac{G_N\beta j^2}{r_0^3} + 2\pi n j\,. \qquad \text{(C.11)}
$$

We need to be careful about the n-dependent term appearing in the final result in (C.11). If the

[2]The prefator in front of the exponent in (C.9) can in fact be obtained by computing the one-loop correction to each instanton saddle. The fact that the one loop expansion recovers the complete result is related to the fact that the path integral in $2d$ Yang-Mills theory can be obtained using localization techniques.

solutions (C.11) are gauge inequivalent then, in order to obtain the partition function, we truly have to sum over all different instanton solutions; since the sum over n is unbounded, the partition function would be ill defined. Consequently, the only possibility is that the gauge field solutions in (C.10) are in fact all gauge equivalent. This can only happen if the holonomies around any closed curve on M_2 are the same for all solutions. This, in turn, implies that $j \in \mathbb{Z}$ and we can fix gauge transformations on the boundary in such a way that we only get contributions from the solution with $n = 0$.

Consequently, there is a unique $SO(3)$ gauge field solution for which the action is given by $I_{EM}^{SO(3),N} = \frac{G_N \beta j^2}{r_0^3}$. For sufficiently large $j \gg 1$, this agrees with terms in the exponent in the sum over j (C.9). Since the r-dependence of the gauge field in (C.10) is the same as that in (C.7) we can once again check that when j is sufficiently small that it does not backreact on $f(r)$,[3] then Einstein's equations are indeed satisfied for the $4d$ metric ansatz when using the solution (C.10).

C.3 Uplift of the $SO(3)$ solution

In this section we will take the solution for the $SO(3)$ gauge field and show that it can be understood as a solution of the higher dimensional metric for small angular momentum. The KN solution is the unique solution with fixed $U(1)$ charge and angular momentum that also has a $U(1)$ spatial isometry [228, 229]. Therefore, by finding the angular momentum for the solutions analyzed in C.3 in which we either fix the $SO(3)$ holonomy or the $SO(3)$ field strength we will determine the diffeomorphic equivalent KN solution.

We saw above a solution for the gauge fields appears in the metric as

$$\delta g_{\mu\nu}^{SO(3)} dx^\mu dx^\nu = 2ir^2 \sin^2(\theta) \left(\alpha_1 + \frac{\alpha_2}{r^3}\right) d\tau d\phi \tag{C.12}$$

to linear order in the angular momentum (i.e. no backreaction to $f(r)$), with respect to the charged black hole solution. The equation of motion for the four dimensional Einstein Maxwell theory is $G_{AB} \equiv R_{AB} - \frac{1}{2} g_{AB} R - \frac{3}{L^2} g_{AB} = 8\pi G_N T_{AB}$ where $T_{AB} = \frac{1}{4e^2}(F_{AC}F_B^C - g_{AB}F^2)$ is the stress tensor of the $U(1)$ gauge field. Expanding this to linear order in α_1 and α_2 we can check this corrections satisfies the equation of motion to linear order

$$\frac{1}{8\pi G}\delta G_{\tau\phi} = \delta T_{\tau\phi} = \frac{Q^2}{32\pi^2 r^2}\left(\alpha_1 + \frac{\alpha_2}{r^3}\right)\sin^2\theta, \tag{C.13}$$

[3] $j(j+1) \ll (r_h/\ell_{Pl})^4$ as discussed in section 5.2.3.

and all other components for both δG and δT vanish. The uniqueness of KN solution suggests (C.1) is the correct non-linear completion of this correction, written in a different gauge.

Bibliography

[1] L. V. Iliesiu, S. S. Pufu, H. Verlinde, and Y. Wang, "An exact quantization of Jackiw-Teitelboim gravity," arXiv:1905.02726 [hep-th].

[2] L. V. Iliesiu, J. Kruthoff, G. J. Turiaci, and H. Verlinde, "JT gravity at finite cutoff," arXiv:2004.07242 [hep-th].

[3] L. V. Iliesiu, "On 2D gauge theories in Jackiw-Teitelboim gravity," arXiv:1909.05253 [hep-th].

[4] L. V. Iliesiu and G. J. Turiaci, "The statistical mechanics of near-extremal black holes," arXiv:2003.02860 [hep-th].

[5] A. Shomer, "A pedagogical explanation for the non-renormalizability of gravity," *arXiv preprint arXiv:0709.3555* (2007) .

[6] A. Achucarro and P. K. Townsend, "A chern–simons action for three-dimensional anti-de sitter supergravity theories," in *Supergravities in Diverse Dimensions: Commentary and Reprints (In 2 Volumes)*, pp. 732–736. World Scientific, 1989.

[7] E. Witten, "2+ 1 dimensional gravity as an exactly soluble system," *Nuclear Physics B* **311** no. 1, (1988) 46–78.

[8] E. Witten, "Topology-changing amplitudes in 2+ 1 dimensional gravity," *Nuclear Physics B* **323** no. 1, (1989) 113–140.

[9] J. M. Maldacena, "The Large N limit of superconformal field theories and supergravity," *Int. J. Theor. Phys.* **38** (1999) 1113–1133, arXiv:hep-th/9711200 [hep-th]. [Adv. Theor. Math. Phys.2,231(1998)].

[10] E. Witten, "Anti-de Sitter space and holography," *Adv. Theor. Math. Phys.* **2** (1998) 253–291, arXiv:hep-th/9802150 [hep-th].

[11] S. S. Gubser, I. R. Klebanov, and A. M. Polyakov, "Gauge theory correlators from noncritical string theory," *Phys. Lett.* **B428** (1998) 105–114, arXiv:hep-th/9802109 [hep-th].

[12] T. De Jonckheere, "Modave lectures on bulk reconstruction in ads/cft," *arXiv preprint arXiv:1711.07787* (2017) .

[13] D. Harlow, "Tasi lectures on the emergence of the bulk in ads/cft," *arXiv preprint arXiv:1802.01040* (2018) .

[14] A. Hamilton, D. Kabat, G. Lifschytz, and D. A. Lowe, "Local bulk operators in ads/cft correspondence: A boundary view of horizons and locality," *Physical Review D* **73** no. 8, (2006) 086003.

[15] H. K. Kunduri, J. Lucietti, and H. S. Reall, "Near-horizon symmetries of extremal black holes," *Class. Quant. Grav.* **24** (2007) 4169–4190, arXiv:0705.4214 [hep-th].

[16] L. McGough, M. Mezei, and H. Verlinde, "Moving the CFT into the bulk with $T\overline{T}$," *JHEP* **04** (2018) 010, arXiv:1611.03470 [hep-th].

[17] A. B. Zamolodchikov, "Expectation value of composite field T anti-T in two-dimensional quantum field theory," arXiv:hep-th/0401146 [hep-th].

[18] J. Maldacena and L. Maoz, "Wormholes in ads," *Journal of High Energy Physics* **2004** no. 02, (2004) 053.

[19] P. Saad, S. H. Shenker, and D. Stanford, "A semiclassical ramp in SYK and in gravity," arXiv:1806.06840 [hep-th].

[20] J. Maldacena and D. Stanford, "Remarks on the Sachdev-Ye-Kitaev model," *Phys. Rev.* **D94** no. 10, (2016) 106002, arXiv:1604.07818 [hep-th].

[21] J. Polchinski and V. Rosenhaus, "The Spectrum in the Sachdev-Ye-Kitaev Model," *JHEP* **04** (2016) 001, arXiv:1601.06768 [hep-th].

[22] M. Mezei, S. S. Pufu, and Y. Wang, "A 2d/1d Holographic Duality," arXiv:1703.08749 [hep-th].

[23] D. J. Gross and V. Rosenhaus, "A line of CFTs: from generalized free fields to SYK," *JHEP* **07** (2017) 086, arXiv:1706.07015 [hep-th].

[24] D. J. Gross and V. Rosenhaus, "All point correlation functions in SYK," *JHEP* **12** (2017) 148, arXiv:1710.08113 [hep-th].

[25] D. Stanford and E. Witten, "Fermionic Localization of the Schwarzian Theory," *JHEP* **10** (2017) 008, arXiv:1703.04612 [hep-th].

[26] E. Witten, "An SYK-Like Model Without Disorder," arXiv:1610.09758 [hep-th].

[27] I. R. Klebanov and G. Tarnopolsky, "Uncolored random tensors, melon diagrams, and the Sachdev-Ye-Kitaev models," *Phys. Rev.* **D95** no. 4, (2017) 046004, arXiv:1611.08915 [hep-th].

[28] A. Kitaev, "Talks given at the Fundamental Physics Prize Symposium and KITP seminars,".

[29] J. Maldacena, D. Stanford, and Z. Yang, "Conformal symmetry and its breaking in two dimensional Nearly Anti-de-Sitter space," *PTEP* **2016** no. 12, (2016) 12C104, arXiv:1606.01857 [hep-th].

[30] K. Jensen, "Chaos in AdS_2 Holography," *Phys. Rev. Lett.* **117** no. 11, (2016) 111601, arXiv:1605.06098 [hep-th].

[31] J. Engelsy, T. G. Mertens, and H. Verlinde, "An investigation of AdS_2 backreaction and holography," *JHEP* **07** (2016) 139, arXiv:1606.03438 [hep-th].

[32] J. Maldacena, D. Stanford, and Z. Yang, "Diving into traversable wormholes," *Fortsch. Phys.* **65** no. 5, (2017) 1700034, arXiv:1704.05333 [hep-th].

[33] J. Maldacena and X.-L. Qi, "Eternal traversable wormhole," arXiv:1804.00491 [hep-th].

[34] D. Harlow and D. Jafferis, "The Factorization Problem in Jackiw-Teitelboim Gravity," arXiv:1804.01081 [hep-th].

[35] A. Kitaev and S. J. Suh, "Statistical mechanics of a two-dimensional black hole," arXiv:1808.07032 [hep-th].

[36] Z. Yang, "The Quantum Gravity Dynamics of Near Extremal Black Holes," *JHEP* **05** (2019) 205, arXiv:1809.08647 [hep-th].

[37] P. Saad, S. H. Shenker, and D. Stanford, "JT gravity as a matrix integral," arXiv:1903.11115 [hep-th].

[38] C. Teitelboim, "Gravitation and hamiltonian structure in two spacetime dimensions," *Physics Letters B* **126** no. 1-2, (1983) 41–45.

[39] R. Jackiw, "Lower Dimensional Gravity," *Nucl. Phys.* **B252** (1985) 343–356.

[40] T. G. Mertens, G. J. Turiaci, and H. L. Verlinde, "Solving the Schwarzian via the Conformal Bootstrap," *JHEP* **08** (2017) 136, arXiv:1705.08408 [hep-th].

[41] M. Mirzakhani, "Simple geodesics and weil-petersson volumes of moduli spaces of bordered riemann surfaces," *Inventiones mathematicae* **167** no. 1, (2007) 179–222.

[42] R. Dijkgraaf and E. Witten, "Developments in Topological Gravity," *Int. J. Mod. Phys.* **A33** no. 30, (2018) 1830029, arXiv:1804.03275 [hep-th].

[43] B. Eynard and N. Orantin, "Weil-Petersson volume of moduli spaces, Mirzakhani's recursion and matrix models," arXiv:0705.3600 [math-ph].

[44] E. Brezin, C. Itzykson, G. Parisi, and J. B. Zuber, "Planar Diagrams," *Commun. Math. Phys.* **59** (1978) 35.

[45] G. 't Hooft, "A Planar Diagram Theory for Strong Interactions," *Nucl. Phys.* **B72** (1974) 461. [,337(1973)].

[46] É. Brézin, V. Kazakov, D. Serban, P. Wiegmann, and A. Zabrodin, *Applications of random matrices in physics*, vol. 221. Springer Science & Business Media, 2006.

[47] B. Eynard, "Topological expansion for the 1-Hermitian matrix model correlation functions," *JHEP* **11** (2004) 031, arXiv:hep-th/0407261 [hep-th].

[48] G. W. Moore, N. Seiberg, and M. Staudacher, "From loops to states in 2-D quantum gravity," *Nucl. Phys.* **B362** (1991) 665–709.

[49] J. D. Edwards and I. R. Klebanov, "Macroscopic boundaries and the wave function of the universe in the c = -2 matrix model," *Mod. Phys. Lett.* **A6** (1991) 2901–2908.

[50] N. Seiberg and D. Shih, "Branes, rings and matrix models in minimal (super)string theory," *JHEP* **02** (2004) 021, arXiv:hep-th/0312170 [hep-th].

[51] N. Seiberg and D. Shih, "Minimal string theory," *Comptes Rendus Physique* **6** (2005) 165–174, arXiv:hep-th/0409306 [hep-th].

[52] N. Seiberg and D. Stanford *Work in Progress* .

[53] A. M. Polyakov, "Quantum Geometry of Bosonic Strings," *Phys. Lett.* **103B** (1981) 207–210. [,598(1981)].

[54] K. Isler and C. A. Trugenberger, "A Gauge Theory of Two-dimensional Quantum Gravity," *Phys. Rev. Lett.* **63** (1989) 834.

[55] A. H. Chamseddine and D. Wyler, "Gauge Theory of Topological Gravity in (1+1)-Dimensions," *Phys. Lett.* **B228** (1989) 75–78.

[56] D. Cangemi and R. Jackiw, "Gauge invariant formulations of lineal gravity," *Phys. Rev. Lett.* **69** (1992) 233–236, arXiv:hep-th/9203056 [hep-th].

[57] C. G. Callan, Jr., S. B. Giddings, J. A. Harvey, and A. Strominger, "Evanescent black holes," *Phys. Rev.* **D45** no. 4, (1992) R1005, arXiv:hep-th/9111056 [hep-th].

[58] D. Grumiller, R. McNees, J. Salzer, C. Valcrcel, and D. Vassilevich, "Menagerie of AdS_2 boundary conditions," *JHEP* **10** (2017) 203, arXiv:1708.08471 [hep-th].

[59] H. A. Gonzlez, D. Grumiller, and J. Salzer, "Towards a bulk description of higher spin SYK," *JHEP* **05** (2018) 083, arXiv:1802.01562 [hep-th].

[60] P. Schaller and T. Strobl, "Diffeomorphisms versus nonAbelian gauge transformations: An Example of (1+1)-dimensional gravity," *Phys. Lett.* **B337** (1994) 266–270, arXiv:hep-th/9401110 [hep-th].

[61] D. Bagrets, A. Altland, and A. Kamenev, "SachdevYeKitaev model as Liouville quantum mechanics," *Nucl. Phys.* **B911** (2016) 191–205, arXiv:1607.00694 [cond-mat.str-el].

[62] V. V. Belokurov and E. T. Shavgulidze, "Correlation functions in the Schwarzian theory," *JHEP* **11** (2018) 036, arXiv:1804.00424 [hep-th].

[63] T. G. Mertens and G. J. Turiaci, "Defects in Jackiw-Teitelboim Quantum Gravity," *JHEP* **08** (2019) 127, arXiv:1904.05228 [hep-th].

[64] H. T. Lam, T. G. Mertens, G. J. Turiaci, and H. Verlinde, "Shockwave S-matrix from Schwarzian Quantum Mechanics," *JHEP* **11** (2018) 182, arXiv:1804.09834 [hep-th].

[65] A. Goel, H. T. Lam, G. J. Turiaci, and H. Verlinde, "Expanding the Black Hole Interior: Partially Entangled Thermal States in SYK," *JHEP* **02** (2019) 156, arXiv:1807.03916 [hep-th].

[66] P. Kraus, J. Liu, and D. Marolf, "Cutoff AdS_3 versus the $T\overline{T}$ deformation," *JHEP* **07** (2018) 027, arXiv:1801.02714 [hep-th].

[67] B. S. DeWitt, "Quantum Theory of Gravity. 1. The Canonical Theory," *Phys. Rev.* **160** (1967) 1113–1148.

[68] M. Henneaux, "Quantum Gravity In Two Dimensions: Exact Solution Of The Jackiw Model," *Phys. Rev. Lett.* **54** (1985) 959–962.

[69] D. Louis-Martinez, J. Gegenberg, and G. Kunstatter, "Exact Dirac quantization of all 2-D dilaton gravity theories," *Phys. Lett.* **B321** (1994) 193–198, arXiv:gr-qc/9309018 [gr-qc].

[70] D. J. Gross, J. Kruthoff, A. Rolph, and E. Shaghoulian, "$T\overline{T}$ in AdS_2 and Quantum Mechanics," arXiv:1907.04873 [hep-th].

[71] D. J. Gross, J. Kruthoff, A. Rolph, and E. Shaghoulian, "Hamiltonian deformations in quantum mechanics, $T\overline{T}$, and SYK," arXiv:1912.06132 [hep-th].

[72] J. Preskill, P. Schwarz, A. D. Shapere, S. Trivedi, and F. Wilczek, "Limitations on the statistical description of black holes," *Mod. Phys. Lett.* **A6** (1991) 2353–2362.

[73] J. M. Maldacena, J. Michelson, and A. Strominger, "Anti-de Sitter fragmentation," *JHEP* **02** (1999) 011, arXiv:hep-th/9812073 [hep-th].

[74] D. N. Page, "Thermodynamics of near extreme black holes," arXiv:hep-th/0012020 [hep-th].

[75] C. G. Callan and J. M. Maldacena, "D-brane approach to black hole quantum mechanics," *Nucl. Phys.* **B472** (1996) 591–610, arXiv:hep-th/9602043 [hep-th].

[76] J. M. Maldacena and L. Susskind, "D-branes and fat black holes," *Nucl. Phys.* **B475** (1996) 679–690, arXiv:hep-th/9604042 [hep-th].

[77] J. M. Maldacena and A. Strominger, "Universal low-energy dynamics for rotating black holes," *Phys. Rev.* **D56** (1997) 4975–4983, arXiv:hep-th/9702015 [hep-th].

[78] L. Iliesiu, M. Heydeman, G. J. Turiaci, and W. Zhao *work in progress* .

[79] S. Banerjee, R. K. Gupta, and A. Sen, "Logarithmic Corrections to Extremal Black Hole Entropy from Quantum Entropy Function," *JHEP* **03** (2011) 147, arXiv:1005.3044 [hep-th].

[80] S. Banerjee, R. K. Gupta, I. Mandal, and A. Sen, "Logarithmic Corrections to N=4 and N=8 Black Hole Entropy: A One Loop Test of Quantum Gravity," *JHEP* **11** (2011) 143, arXiv:1106.0080 [hep-th].

[81] A. Sen, "Logarithmic Corrections to N=2 Black Hole Entropy: An Infrared Window into the Microstates," *Gen. Rel. Grav.* **44** no. 5, (2012) 1207–1266, arXiv:1108.3842 [hep-th].

[82] A. Sen, "Logarithmic Corrections to Rotating Extremal Black Hole Entropy in Four and Five Dimensions," *Gen. Rel. Grav.* **44** (2012) 1947–1991, arXiv:1109.3706 [hep-th].

[83] S. W. Hawking, G. T. Horowitz, and S. F. Ross, "Entropy, Area, and black hole pairs," *Phys. Rev.* **D51** (1995) 4302–4314, arXiv:gr-qc/9409013 [gr-qc].

[84] A. Ghosh, H. Maxfield, and G. J. Turiaci, "A universal Schwarzian sector in two-dimensional conformal field theories," arXiv:1912.07654 [hep-th].

[85] C. Teitelboim, "Gravitation and Hamiltonian Structure in Two Space-Time Dimensions," *Phys. Lett.* **126B** (1983) 41–45.

[86] D. Kapec, R. Mahajan, and D. Stanford, "Matrix ensembles with global symmetries and 't Hooft anomalies from 2d gauge theory," arXiv:1912.12285 [hep-th].

[87] A. Almheiri and J. Polchinski, "Models of AdS_2 backreaction and holography," *JHEP* **11** (2015) 014, arXiv:1402.6334 [hep-th].

[88] A. Almheiri and B. Kang, "Conformal Symmetry Breaking and Thermodynamics of Near-Extremal Black Holes," *JHEP* **10** (2016) 052, arXiv:1606.04108 [hep-th].

[89] D. Anninos, T. Anous, and R. T. D'Agnolo, "Marginal deformations & rotating horizons," *JHEP* **12** (2017) 095, arXiv:1707.03380 [hep-th].

[90] G. Srosi, "AdS_2 holography and the SYK model," *PoS* **Modave2017** (2018) 001, arXiv:1711.08482 [hep-th].

[91] P. Nayak, A. Shukla, R. M. Soni, S. P. Trivedi, and V. Vishal, "On the Dynamics of Near-Extremal Black Holes," *JHEP* **09** (2018) 048, arXiv:1802.09547 [hep-th].

[92] U. Moitra, S. P. Trivedi, and V. Vishal, "Extremal and near-extremal black holes and near-CFT_1," *JHEP* **07** (2019) 055, arXiv:1808.08239 [hep-th].

[93] S. Hadar, "Near-extremal black holes at late times, backreacted," *JHEP* **01** (2019) 214, arXiv:1811.01022 [hep-th].

[94] A. Castro, F. Larsen, and I. Papadimitriou, "5D rotating black holes and the $nAdS_2/nCFT_1$ correspondence," *JHEP* **10** (2018) 042, arXiv:1807.06988 [hep-th].

[95] F. Larsen and Y. Zeng, "Black hole spectroscopy and AdS_2 holography," *JHEP* **04** (2019) 164, arXiv:1811.01288 [hep-th].

[96] U. Moitra, S. K. Sake, S. P. Trivedi, and V. Vishal, "Jackiw-Teitelboim Gravity and Rotating Black Holes," arXiv:1905.10378 [hep-th].

[97] S. Sachdev, "Universal low temperature theory of charged black holes with AdS_2 horizons," *J. Math. Phys.* **60** no. 5, (2019) 052303, arXiv:1902.04078 [hep-th].

[98] J. Hong, F. Larsen, and J. T. Liu, "The scales of black holes with $nAdS_2$ geometry," *JHEP* **10** (2019) 260, arXiv:1907.08862 [hep-th].

[99] A. Castro and V. Godet, "Breaking away from the near horizon of extreme Kerr," arXiv:1906.09083 [hep-th].

[100] A. M. Charles and F. Larsen, "A One-Loop Test of the near-AdS_2/near-CFT_1 Correspondence," arXiv:1908.03575 [hep-th].

[101] R. M. Wald, "General relativity, chicago, usa: Univ," *Pr. 491p* (1984) .

[102] M. Blau and G. Thompson, "Quantum Yang-Mills theory on arbitrary surfaces," *Int. J. Mod. Phys.* **A7** (1992) 3781–3806. [,28(1991)].

[103] E. Witten, "On quantum gauge theories in two-dimensions," *Commun. Math. Phys.* **141** (1991) 153–209.

[104] S. Cordes, G. W. Moore, and S. Ramgoolam, "Lectures on 2-d Yang-Mills theory, equivariant cohomology and topological field theories," *Nucl. Phys. Proc. Suppl.* **41** (1995) 184–244, arXiv:hep-th/9411210 [hep-th].

[105] G. Moore and N. Seiberg, "Taming the conformal zoo," *Physics Letters B* **220** no. 3, (1989) 422–430.

[106] C. P. Constantinidis, O. Piguet, and A. Perez, "Quantization of the jackiw-teitelboim model," *Physical Review D* **79** no. 8, (2009) 084007.

[107] A. A. Tseytlin, "On gauge theories for nonsemisimple groups," *Nucl. Phys.* **B450** (1995) 231–250, arXiv:hep-th/9505129 [hep-th].

[108] A. Migdal, "Phase transitions in gauge and spin-lattice systems," *Zh. Eksp. Teor. Fiz* **69** (1975) 1457–1465.

[109] A. Migdal, "Loop Equations and 1/N Expansion," *Phys. Rept.* **102** (1983) 199–290.

[110] D. S. Fine, "Quantum Yang-Mills on a Riemann surface," *Commun. Math. Phys.* **140** (1991) 321–338.

[111] E. Witten, "Two-dimensional gauge theories revisited," *J. Geom. Phys.* **9** (1992) 303–368, arXiv:hep-th/9204083 [hep-th].

[112] O. Ganor, J. Sonnenschein, and S. Yankielowicz, "The string theory approach to generalized 2d yang-mills theory," *Nuclear Physics B* **434** no. 1-2, (1995) 139–178.

[113] D. Stanford and E. Witten, "JT Gravity and the Ensembles of Random Matrix Theory," arXiv:1907.03363 [hep-th].

[114] Z. Yang, "The Quantum Gravity Dynamics of Near Extremal Black Holes," *JHEP* **05** (2019) 205, `arXiv:1809.08647 [hep-th]`.

[115] A. Kitaev, "Notes on $\widetilde{\mathrm{SL}}(2,\mathbb{R})$ representations," `arXiv:1711.08169 [hep-th]`.

[116] D. Gaiotto, A. Kapustin, N. Seiberg, and B. Willett, "Generalized Global Symmetries," *JHEP* **02** (2015) 172, `arXiv:1412.5148 [hep-th]`.

[117] D. Gaiotto, A. Kapustin, Z. Komargodski, and N. Seiberg, "Theta, Time Reversal, and Temperature," *JHEP* **05** (2017) 091, `arXiv:1703.00501 [hep-th]`.

[118] G. M. Tuynman and W. A. J. J. Wiegerinck, "Central extensions and physics," *Journal of Geometry and Physics* **4** no. 2, (1987) 207–258.

[119] E. Witten, "Analytic Continuation Of Chern-Simons Theory," *AMS/IP Stud. Adv. Math.* **50** (2011) 347–446, `arXiv:1001.2933 [hep-th]`.

[120] A. Blommaert, T. G. Mertens, and H. Verschelde, "Edge dynamics from the path integral Maxwell and Yang-Mills," *JHEP* **11** (2018) 080, `arXiv:1804.07585 [hep-th]`.

[121] A. Blommaert, T. G. Mertens, and H. Verschelde, "The Schwarzian Theory - A Wilson Line Perspective," `arXiv:1806.07765 [hep-th]`.

[122] A. Blommaert, T. G. Mertens, and H. Verschelde, "Fine Structure of Jackiw-Teitelboim Quantum Gravity," `arXiv:1812.00918 [hep-th]`.

[123] S. Carlip, "Exact quantum scattering in 2+ 1 dimensional gravity," *Nuclear Physics B* **324** no. 1, (1989) 106–122.

[124] C. Vaz and L. Witten, "Wilson loops and black holes in 2+ 1 dimensions," *arXiv preprint gr-qc/9401017* (1994) .

[125] P. de Sousa Gerbert, "On spin and (quantum) gravity in 2+ 1 dimensions," *Nuclear Physics B* **346** no. 2-3, (1990) 440–472.

[126] B.-S. Skagerstam and A. Stern, "Topological quantum mechanics in 2+ 1 dimensions," *International Journal of Modern Physics A* **5** no. 08, (1990) 1575–1595.

[127] M. Ammon, A. Castro, and N. Iqbal, "Wilson Lines and Entanglement Entropy in Higher Spin Gravity," *JHEP* **10** (2013) 110, `arXiv:1306.4338 [hep-th]`.

[128] A. L. Fitzpatrick, J. Kaplan, D. Li, and J. Wang, "Exact Virasoro Blocks from Wilson Lines and Background-Independent Operators," *JHEP* **07** (2017) 092, `arXiv:1612.06385 [hep-th]`.

[129] J. Cotler and K. Jensen, "A theory of reparameterizations for AdS_3 gravity," *JHEP* **02** (2019) 079, `arXiv:1808.03263 [hep-th]`.

[130] W. Groenevelt, "The wilson function transform," *International Mathematics Research Notices* **2003** no. 52, (2003) 2779–2817.

[131] W. Groenevelt, "Wilson function transforms related to racah coefficients," *Acta Applicandae Mathematica* **91** no. 2, (2006) 133–191.

[132] J. Lin, "Entanglement entropy in Jackiw-Teitelboim Gravity," arXiv:1807.06575 [hep-th].

[133] M. Aganagic, H. Ooguri, N. Saulina, and C. Vafa, "Black holes, q-deformed 2d Yang-Mills, and non-perturbative topological strings," *Nucl. Phys.* **B715** (2005) 304–348, arXiv:hep-th/0411280 [hep-th].

[134] M. Berkooz, M. Isachenkov, V. Narovlansky, and G. Torrents, "Towards a full solution of the large N double-scaled SYK model," arXiv:1811.02584 [hep-th].

[135] L. Iliesiu, S. Pufu, and Y. Wang, "Work in progress,".

[136] J. Maldacena, G. J. Turiaci, and Z. Yang, "Two dimensional Nearly de Sitter gravity," arXiv:1904.01911 [hep-th].

[137] H. W. Lin, J. Maldacena, and Y. Zhao, "Symmetries Near the Horizon," arXiv:1904.12820 [hep-th].

[138] S. Dubovsky, V. Gorbenko, and G. Hernández-Chifflet, "$T\overline{T}$ partition function from topological gravity," *Journal of High Energy Physics* **2018** no. 9, (Sep, 2018) .

[139] J. Maldacena, G. J. Turiaci, and Z. Yang, "Two dimensional Nearly de Sitter gravity," arXiv:1904.01911 [hep-th].

[140] T. Thiemann and H. A. Kastrup, "Canonical quantization of spherically symmetric gravity in Ashtekar's selfdual representation," *Nucl. Phys.* **B399** (1993) 211–258, arXiv:gr-qc/9310012 [gr-qc].

[141] L. Freidel, "Reconstructing AdS/CFT," arXiv:0804.0632 [hep-th].

[142] J. B. Hartle and S. W. Hawking, "Wave function of the universe," *Phys. Rev. D* **28** (Dec, 1983) 2960–2975.

[143] O. Aharony, S. Datta, A. Giveon, Y. Jiang, and D. Kutasov, "Modular invariance and uniqueness of $T\overline{T}$ deformed CFT," *JHEP* **01** (2019) 086, arXiv:1808.02492 [hep-th].

[144] A. Alekseev and S. L. Shatashvili, "Path Integral Quantization of the Coadjoint Orbits of the Virasoro Group and 2D Gravity," *Nucl. Phys. B* **323** (1989) 719–733.

[145] F. A. Smirnov and A. B. Zamolodchikov, "On space of integrable quantum field theories," *Nucl. Phys.* **B915** (2017) 363–383, arXiv:1608.05499 [hep-th].

[146] S. B. Giddings and A. Strominger, "Baby Universes, Third Quantization and the Cosmological Constant," *Nucl. Phys. B* **321** (1989) 481–508.

[147] J. Cotler, K. Jensen, and A. Maloney, "Low-dimensional de Sitter quantum gravity," arXiv:1905.03780 [hep-th].

[148] J. M. Maldacena, "Non-Gaussian features of primordial fluctuations in single field inflationary models," *JHEP* **05** (2003) 013, arXiv:astro-ph/0210603 [astro-ph].

[149] A. Castro and A. Maloney, "The Wave Function of Quantum de Sitter," *JHEP* **11** (2012) 096, arXiv:1209.5757 [hep-th].

[150] A. Goel, L. Iliesiu, and Z. Yang, "work in progress,".

[151] A. Hosoya and K.-i. Nakao, "(2+1)-Dimensional Quantum Gravity: Case of Torus Universe," *Progress of Theoretical Physics* **84** no. 4, (10, 1990) 739–748.

[152] A. Hosoya and K.-i. Nakao, "(2+1)-dimensional Pure Gravity for an Arbitrary Closed Initial Surface," *Class. Quant. Grav.* **7** (1990) 163.

[153] E. A. Mazenc, V. Shyam, and R. M. Soni, "A $T\overline{T}$ Deformation for Curved Spacetimes from 3d Gravity," arXiv:1912.09179 [hep-th].

[154] N. Callebaut, J. Kruthoff, and H. Verlinde, "$T\overline{T}$ deformed CFT as a non-critical string," arXiv:1910.13578 [hep-th].

[155] A. Giveon, N. Itzhaki, and D. Kutasov, "$\mathrm{T}\overline{\mathrm{T}}$ and LST," *JHEP* **07** (2017) 122, arXiv:1701.05576 [hep-th].

[156] A. Bzowski and M. Guica, "The holographic interpretation of $J\overline{T}$-deformed CFTs," *JHEP* **01** (2019) 198, arXiv:1803.09753 [hep-th].

[157] G. Turiaci and H. Verlinde, "Towards a 2d QFT Analog of the SYK Model," *JHEP* **10** (2017) 167, arXiv:1701.00528 [hep-th].

[158] M. Guica and R. Monten, "$T\overline{T}$ and the mirage of a bulk cutoff," arXiv:1906.11251 [hep-th].

[159] S. Sachdev and J. Ye, "Gapless spin fluid ground state in a random, quantum Heisenberg magnet," *Phys. Rev. Lett.* **70** (1993) 3339, arXiv:cond-mat/9212030 [cond-mat].

[160] A. Georges, O. Parcollet, and S. Sachdev, "Quantum fluctuations of a nearly critical heisenberg spin glass," *Physical Review B* **63** no. 13, (2001) 134406.

[161] R. Jackiw, "Gauge theories for gravity on a line," *Theor. Math. Phys.* **92** (1992) 979–987, arXiv:hep-th/9206093 [hep-th]. [,197(1992)].

[162] N. Ikeda, "Two-dimensional gravity and nonlinear gauge theory," *Annals Phys.* **235** (1994) 435–464, arXiv:hep-th/9312059 [hep-th].

[163] P. Schaller and T. Strobl, "Poisson structure induced (topological) field theories," *Mod. Phys. Lett.* **A9** (1994) 3129–3136, arXiv:hep-th/9405110 [hep-th].

[164] A. S. Cattaneo and G. Felder, "On the AKSZ formulation of the Poisson sigma model," *Lett. Math. Phys.* **56** (2001) 163–179, arXiv:math/0102108 [math].

[165] L. Iliesiu and H. Verlinde, "Moving the Schwarzian deep into the bulk," *Work in Progress* .

[166] S. Sachdev, "Bekenstein-Hawking Entropy and Strange Metals," *Phys. Rev.* **X5** no. 4, (2015) 041025, `arXiv:1506.05111 [hep-th]`.

[167] R. A. Davison, W. Fu, A. Georges, Y. Gu, K. Jensen, and S. Sachdev, "Thermoelectric transport in disordered metals without quasiparticles: The Sachdev-Ye-Kitaev models and holography," *Phys. Rev.* **B95** no. 15, (2017) 155131, `arXiv:1612.00849 [cond-mat.str-el]`.

[168] D. J. Gross and V. Rosenhaus, "A Generalization of Sachdev-Ye-Kitaev," *JHEP* **02** (2017) 093, `arXiv:1610.01569 [hep-th]`.

[169] W. Fu, D. Gaiotto, J. Maldacena, and S. Sachdev, "Supersymmetric Sachdev-Ye-Kitaev models," *Phys. Rev.* **D95** no. 2, (2017) 026009, `arXiv:1610.08917 [hep-th]`. [Addendum: Phys. Rev.D95,no.6,069904(2017)].

[170] P. Narayan and J. Yoon, "SYK-like Tensor Models on the Lattice," *JHEP* **08** (2017) 083, `arXiv:1705.01554 [hep-th]`.

[171] J. Yoon, "SYK Models and SYK-like Tensor Models with Global Symmetry," *JHEP* **10** (2017) 183, `arXiv:1707.01740 [hep-th]`.

[172] P. Narayan and J. Yoon, "Supersymmetric SYK Model with Global Symmetry," *JHEP* **08** (2018) 159, `arXiv:1712.02647 [hep-th]`.

[173] I. R. Klebanov, A. Milekhin, F. Popov, and G. Tarnopolsky, "Spectra of eigenstates in fermionic tensor quantum mechanics," *Phys. Rev.* **D97** no. 10, (2018) 106023, `arXiv:1802.10263 [hep-th]`.

[174] J. Liu and Y. Zhou, "Note on global symmetry and SYK model," *JHEP* **05** (2019) 099, `arXiv:1901.05666 [hep-th]`.

[175] A. Solovyov, "Open/Closed String Duality on Orbifolds: a Toy Model," *pre-thesis work at Princeton University advised by H. Verlinde* .

[176] M. Mulase and J. T. Yu, "A generating function of the number of homomorphisms from a surface group into a finite group," *arXiv preprint math/0209008* (2002) .

[177] M. Mulase and J. T Yu, "Non-commutative matrix integrals and representation varieties of surface groups in a finite group," in *Annales de l'institut Fourier*, vol. 55, pp. 2161–2196. 2005.

[178] M. Mulase, "Geometry of character varieties of surface groups," *arXiv preprint arXiv:0710.5263* (2007) .

[179] M. Marinov and M. Terentyev, "Dynamics on the group manifold and path integral," *Fortschritte der Physik* **27** no. 11-12, (1979) 511–545.

[180] R. Picken, "The propagator for quantum mechanics on a group manifold from an infinite-dimensional analogue of the duistermaat-heckman integration formula," *Journal of Physics A: Mathematical and General* **22** no. 13, (1989) 2285.

[181] M. Chu and P. Goddard, "Quantisation of a particle moving on a group manifold," *Physics Letters B* **337** no. 3-4, (1994) 285–293.

[182] D. Kapec, R. Mahajan, and D. Stanford *Work in Progress* .

[183] A. Castro, D. Grumiller, F. Larsen, and R. McNees, "Holographic Description of AdS(2) Black Holes," *JHEP* **11** (2008) 052, `arXiv:0809.4264 [hep-th]`.

[184] D. Grumiller, R. McNees, and J. Salzer, "Cosmological constant as confining U(1) charge in two-dimensional dilaton gravity," *Phys. Rev.* **D90** no. 4, (2014) 044032, `arXiv:1406.7007 [hep-th]`.

[185] D. Grumiller, J. Salzer, and D. Vassilevich, "AdS_2 holography is (non-)trivial for (non-)constant dilaton," *JHEP* **12** (2015) 015, `arXiv:1509.08486 [hep-th]`.

[186] M. Cveti and I. Papadimitriou, "AdS_2 holographic dictionary," *JHEP* **12** (2016) 008, `arXiv:1608.07018 [hep-th]`. [Erratum: JHEP01,120(2017)].

[187] F. J. Dyson, "Statistical theory of the energy levels of complex systems. I," *J. Math. Phys.* **3** (1962) 140–156.

[188] P. Norbury, "Lengths of geodesics on non-orientable hyperbolic surfaces," *Geometriae Dedicata* **134** no. 1, (2008) 153–176.

[189] M. Gendulphe, "What's wrong with the growth of simple closed geodesics on nonorientable hyperbolic surfaces," *arXiv preprint arXiv:1706.08798* (2017) .

[190] M. Mulase and A. Waldron, "Duality of orthogonal and symplectic matrix integrals and quaternionic feynman graphs," *Communications in mathematical physics* **240** no. 3, (2003) 553–586.

[191] T. G. Mertens, "The Schwarzian Theory - Origins," `arXiv:1801.09605 [hep-th]`.

[192] A. Blommaert, T. G. Mertens, H. Verschelde, and V. I. Zakharov, "Edge State Quantization: Vector Fields in Rindler," *JHEP* **08** (2018) 196, `arXiv:1801.09910 [hep-th]`.

[193] D. J. Gross and W. Taylor, "Two-dimensional QCD is a string theory," *Nucl. Phys.* **B400** (1993) 181–208, `arXiv:hep-th/9301068 [hep-th]`.

[194] D. J. Gross and W. Taylor, "Twists and Wilson loops in the string theory of two-dimensional QCD," *Nucl. Phys.* **B403** (1993) 395–452, `arXiv:hep-th/9303046 [hep-th]`.

[195] M. G. Laidlaw and C. M. DeWitt, "Feynman functional integrals for systems of indistinguishable particles," *Physical Review D* **3** no. 6, (1971) 1375.

[196] D. de Laat, "Contractibility and self-intersections of curves on surfaces,".

[197] H. W. Braden, J. D. Brown, B. F. Whiting, and J. W. York, Jr., "Charged black hole in a grand canonical ensemble," *Phys. Rev.* **D42** (1990) 3376–3385.

[198] S. W. Hawking and S. F. Ross, "Duality between electric and magnetic black holes," *Phys. Rev.* **D52** (1995) 5865–5876, `arXiv:hep-th/9504019 [hep-th]`.

[199] M. Henningson and K. Skenderis, "Holography and the Weyl anomaly," *Fortsch. Phys.* **48** (2000) 125–128, `arXiv:hep-th/9812032 [hep-th]`.

[200] V. Balasubramanian and P. Kraus, "A Stress tensor for Anti-de Sitter gravity," *Commun. Math. Phys.* **208** (1999) 413–428, `arXiv:hep-th/9902121 [hep-th]`.

[201] A. Chamblin, R. Emparan, C. V. Johnson, and R. C. Myers, "Charged AdS black holes and catastrophic holography," *Phys. Rev.* **D60** (1999) 064018, `arXiv:hep-th/9902170 [hep-th]`.

[202] D. Jafferis, B. Mukhametzhanov, and A. Zhiboedov, "Conformal Bootstrap At Large Charge," *JHEP* **05** (2018) 043, `arXiv:1710.11161 [hep-th]`.

[203] A. Salam and E. Sezgin, "Chiral Compactification on Minkowski$\times S^2$ of $N = 2$ Einstein-Maxwell Supergravity in Six-Dimensions," *Phys. Lett.* **147B** (1984) 47. [,47(1984)].

[204] J. Michelson and M. Spradlin, "Supergravity spectrum on AdS(2)$\times S^2$," *JHEP* **09** (1999) 029, `arXiv:hep-th/9906056 [hep-th]`.

[205] G. W. Gibbons and C. N. Pope, "Consistent S^2 Pauli reduction of six-dimensional chiral gauged Einstein-Maxwell supergravity," *Nucl. Phys.* **B697** (2004) 225–242, `arXiv:hep-th/0307052 [hep-th]`.

[206] D. Grumiller and R. McNees, "Thermodynamics of black holes in two (and higher) dimensions," *JHEP* **04** (2007) 074, `arXiv:hep-th/0703230 [HEP-TH]`.

[207] R. Emparan, C. V. Johnson, and R. C. Myers, "Surface terms as counterterms in the AdS / CFT correspondence," *Phys. Rev.* **D60** (1999) 104001, `arXiv:hep-th/9903238 [hep-th]`.

[208] A. Kitaev and S. J. Suh, "The soft mode in the Sachdev-Ye-Kitaev model and its gravity dual," *JHEP* **05** (2018) 183, `arXiv:1711.08467 [hep-th]`.

[209] A. Sen, "Black hole entropy function and the attractor mechanism in higher derivative gravity," *JHEP* **09** (2005) 038, `arXiv:hep-th/0506177 [hep-th]`.

[210] F. Larsen and P. Lisbao, "Quantum Corrections to Supergravity on $\mathrm{AdS}_2 \times S^2$," *Phys. Rev.* **D91** no. 8, (2015) 084056, `arXiv:1411.7423 [hep-th]`.

[211] I. M. Gel'fand and A. M. Yaglom, "Integration in functional spaces and its applications in quantum physics," *Journal of Mathematical Physics* **1** no. 1, (1960) 48–69.

[212] S. S. Gubser, "Breaking an Abelian gauge symmetry near a black hole horizon," *Phys. Rev.* **D78** (2008) 065034, `arXiv:0801.2977 [hep-th]`.

[213] S. S. Gubser and S. S. Pufu, "The Gravity dual of a p-wave superconductor," *JHEP* **11** (2008) 033, `arXiv:0805.2960 [hep-th]`.

[214] H. Liu, J. McGreevy, and D. Vegh, "Non-Fermi liquids from holography," *Phys. Rev.* **D83** (2011) 065029, `arXiv:0903.2477 [hep-th]`.

[215] T. Faulkner, H. Liu, J. McGreevy, and D. Vegh, "Emergent quantum criticality, Fermi surfaces, and AdS(2)," *Phys. Rev.* **D83** (2011) 125002, `arXiv:0907.2694 [hep-th]`.

[216] M. Berkooz, P. Narayan, M. Rozali, and J. Simon, "Higher Dimensional Generalizations of the SYK Model," *JHEP* **01** (2017) 138, `arXiv:1610.02422 [hep-th]`.

[217] J. Murugan, D. Stanford, and E. Witten, "More on Supersymmetric and 2d Analogs of the SYK Model," *JHEP* **08** (2017) 146, `arXiv:1706.05362 [hep-th]`.

[218] H. Maxfield and G. J. Turiaci *work in progress* .

[219] A. Almheiri, T. Hartman, J. Maldacena, E. Shaghoulian, and A. Tajdini, "Replica Wormholes and the Entropy of Hawking Radiation," `arXiv:1911.12333 [hep-th]`.

[220] G. Penington, S. H. Shenker, D. Stanford, and Z. Yang, "Replica wormholes and the black hole interior," `arXiv:1911.11977 [hep-th]`.

[221] A. Comtet and P. J. Houston, "Effective Action on the Hyperbolic Plane in a Constant External Field," *J. Math. Phys.* **26** (1985) 185.

[222] A. Comtet, "On the Landau Levels on the Hyperbolic Plane," *Annals Phys.* **173** (1987) 185.

[223] O. Matsushita, "The Plancherel Formula for the Universal Covering Group of $SL(2,\mathbb{R})$,".

[224] J. Repka, "Tensor products of unitary representations of $sl(2,\setminus)$," *American Journal of Mathematics* **100** no. 82, (1976) 930–932.

[225] J. Repka, "Tensor products of unitary representations of $sl(2,\setminus)$," *American Journal of Mathematics* **100** no. 4, (1978) 747–774.

[226] E. Witten, "Quantum Field Theory and the Jones Polynomial," *Commun. Math. Phys.* **121** (1989) 351–399. [,233(1988)].

[227] C. Beasley, "Localization for Wilson Loops in Chern-Simons Theory," *Adv. Theor. Math. Phys.* **17** no. 1, (2013) 1–240, `arXiv:0911.2687 [hep-th]`.

[228] W. Israel, "Event horizons in static vacuum space-times," *Phys. Rev.* **164** (1967) 1776–1779.

[229] B. Carter, "Axisymmetric Black Hole Has Only Two Degrees of Freedom," *Phys. Rev. Lett.* **26** (1971) 331–333.