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Bochner–Kodaira Formulas and the Type IIA Flow

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Abstract
A new derivation of the flow of metrics in the Type IIA flow is given. It is better adapted to the formulation of the flow as a variant of a Laplacian flow, and it uses the projected Levi–Civita connection of the metrics themselves instead of their conformal rescalings.

Keywords Type IIA flow · Type IIA structure · Type IIA equation · Bochner–Kodaira Formula

Mathematics Subject Classification 53E50 · 53E30 · 53D25 · 35K40

1 Introduction

The search for supersymmetric compactifications of string theories has revealed itself to have deep connections with special geometry. The resulting non-linear partial differential equations also turned out to be quite rich and interesting in their own right (see e.g., [4, 7, 8, 15, 18]). One feature of particular interest in these equations is invariably the presence of a cohomological constraint. In the absence of a $\partial\bar{\partial}$-lemma, the most

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natural implementation of these cohomological constraints is by a geometric flow, and this has resulted in considerable interest in the investigation of such geometric flows in recent years [1–3, 5, 12–14, 16, 17].

The present paper is mainly concerned with the Type IIA flow, which is a flow in symplectic geometry introduced in [6] and motivated by the Type IIA string. More specifically, let (M, ω) be a compact 6-dimensional symplectic manifold and ρ_A be the Poincaré dual to a finite combination of Lagrangians. Then the Type IIA flow is the flow of 3-forms φ given by

$$\partial_t \varphi = \mathrm{d}\Lambda d\left(|\varphi|^2 \star\varphi\right) - \rho_A \tag{1.1}$$

with an initial data φ_0 which is a closed, primitive, and positive 3-form on M. Here Λ is the Hodge contraction operator defined by ω, and $\star$ and $|\varphi|$ are the Hodge star operator and the norm of φ with respect to the metric g_φ which is compatible with ω and the almost-complex structure J_φ constructed by Hitchin [11] (see Sect. 2 for the precise definitions). The Type IIA flow preserves the primitiveness and closedness of φ, so that its stationary points are automatically solutions of the system investigated by Tseng and Yau [20]. This system is itself a basic case of the more general equations for supersymmetric compactifications of the Type IIA string proposed in [10, 19].

In [6], it was shown that the Type IIA flow admits at least short-time existence, and can be continued as long as $|\varphi|$ and the Riemannian curvature of g_φ remain bounded. The proof of this last assertion relied heavily on determining the flow of g_φ. This was one of the main results of [6], and it was established using the original formulation (1.1) of the Type IIA flow, and the projected Levi–Civita connection $\tilde{\mathfrak{D}}$ of a metric $\tilde{g}_\varphi$ conformal to g_φ (see (2.2 below). A key point was that, with respect to $\tilde{\mathfrak{D}}$, the manifold M has SU(3) holonomy, and the form $|\Omega|^{-1}_{\tilde{g}_\varphi}\Omega$, with $\Omega = \varphi + i\star\varphi$, is covariant constant.

The main goal of the present paper is to provide a different derivation of the flow of the metrics g_φ in the Type IIA flow. The new derivation differs from the one in [6] in two important aspects. The first aspect is that it relies on Bochner–Kodaira formulas and a different formulation of the Type IIA flow, which is closer in spirit to Bryant's G_2 flow. From this point of view, it is more easily adaptable to other Laplacian flows. The second aspect is that it relies instead on the projected Levi–Civita connection $\mathfrak{D}$ of g_φ, which is a very natural connection since it coincides with all the unitary Hermitian connections with respect to g_φ on the Gauduchon line. An important additional benefit of this second derivation is that it provides a check on the formulas obtained in [6], which is non-trivial because the calculations in both approaches are particularly long and involved.

For simplicity, we focus on the source-free case $\rho_A = 0$. Then we have

Theorem 1 *Let (M, ω) be a 6-dimensional symplectic manifold, and let $t \to \varphi(t)$ by the Type IIA flow of 3-forms defined in (1.1) with $\rho_A = 0$. If $g_{ij} = (g_\varphi)_{ij}$ is the corresponding flow of metrics, then we have*

$$\partial_t g_{ij} = -|\varphi|^2 \Big\{ 2R_{ij} - 2\nabla_i\nabla_j \log|\varphi|^2 + 4\left(N_-^2\right)_{ij} - \alpha_i\alpha_j + \alpha_{Ji}\alpha_{Jj} + 4\alpha_p\left(N_j{}^p{}_i + N_i{}^p{}_j\right)\Big\} \tag{1.2}$$

where ∇ is the Levi–Civita connection of g, R_{ij} is the Ricci curvature, N is the Nijenhuis tensor with respect to the almost-complex structure J_φ, $(N_-^2)_{ij} = N^{\lambda p}{}_i N_{p\lambda j}$, and α is the 1-form defined by $\alpha = -d\log|\varphi|^2$.

2 Background Material

We begin by providing a brief summary of the setting for the Type IIA flow, which is Type IIA geometry as introduced in [6].

2.1 Type IIA Geometry

Let M be an oriented 6-manifold. In [11], Hitchin has shown how to associate to any non-degenerate 3-form φ an almost-complex structure J_φ. Type IIA geometry arises if, in addition, M is equipped with a fixed symplectic form ω and φ is a closed form which is primitive and positive with respect to ω. The primitive condition means that $\Lambda\varphi = 0$, where $\Lambda : A^k(M) \to A^{k-2}(M)$ is the standard Hodge contraction operator with respect to ω. It is shown in [6] that ω is then preserved by J_φ, and the positivity condition means that the resulting Hermitian form $g_\varphi(X, Y) = \omega(X, J_\varphi Y)$ is positive definite and defines a metric. Thus $(J_\varphi, g_\varphi, \omega)$ is an almost-Kähler manifold. However, the condition in Type IIA geometry that this almost-Kähler structure arise from a closed 3-form results in many subtle properties which are essential for the Type IIA flow.

Explicitly, the metric g_φ is given by

$$(g_\varphi)_{ij} = -|\varphi|^{-2}\varphi_{iab}\varphi_{jkp}\omega^{ak}\omega^{bp} \tag{2.1}$$

where $|\varphi|$ is the norm of the 3-form φ with respect to J_φ, and ω^{ak} is the inverse of the symplectic form ω, $\omega^{ak}\omega_{kp} = \delta^a{}_p$. The volume form of g_φ is the same as $\omega^3/3!$. The following metric $\tilde{g}_\varphi$ conformally equivalent to g_φ also plays an important role in Type IIA geometry,

$$(\tilde{g}_\varphi)_{ij} = |\varphi|^2(g_\varphi)_{ij} = -\varphi_{iab}\varphi_{jkp}\omega^{ak}\omega^{bp}. \tag{2.2}$$

In fact, one of the defining features of Type IIA geometry is that the manifold (M, J_φ) have SU(3) holonomy with respect to the projected Levi–Civita connection $\tilde{\mathfrak{D}}$ of $\tilde{g}_\varphi$. More precisely, set

$$\hat{\varphi} = \star\varphi = J\varphi \tag{2.3}$$

and let Ω be the (3, 0)-form defined by

$$\Omega = \varphi + i\hat{\varphi}. \tag{2.4}$$

Then $|\Omega|^{-1}_{\tilde{g}_\varphi}\,\Omega$ is covariantly constant with respect to $\tilde{\mathfrak{D}}$. This was a major reason why the calculations in [6] were mostly carried out with the connection $\tilde{\mathfrak{D}}$.

In the present paper, we shall use instead the unitary connections with respect to g_φ. Since ω is closed, the Gauduchon line of Hermitian unitary connections with respect to J_φ collapses to a single connection, which can be viewed as either the Chern connection or the projected Levi–Civita connection $\mathfrak{D}$ of g_φ. Henceforth we drop the subindex φ when there is no possibility of confusion, and denote g_φ, $\tilde{g}_\varphi$, J_φ simply by g, $\tilde{g}$ and J. Then the Levi–Civita connection ∇ and the projected Levi–Civita connection $\mathfrak{D}$ of g are related by

$$\mathfrak{D}_i X^m = \nabla_i X^m - N_{ip}{}^m X^p \tag{2.5}$$

where $N_{ip}{}^m$ is the Nijenhuis tensor of J,

$$N^k{}_{ij} = \frac{1}{4}\left(J^r{}_i \nabla_r J^k{}_j + J^k{}_r \nabla_j J^r{}_i - (i \leftrightarrow j)\right). \tag{2.6}$$

In [6], we showed $\mathfrak{D}^{0,1}\Omega = 0$ and $\mathfrak{D}^{1,0}\Omega = -\alpha \otimes \Omega$ (Equation (6.50) in [6]), or equivalently,

$$\mathfrak{D}_m \varphi = \frac{1}{2}(-\alpha_m \varphi - \alpha_{Jm}\hat{\varphi}), \quad \mathfrak{D}_m \hat{\varphi} = \frac{1}{2}(-\alpha_m \hat{\varphi} + \alpha_{Jm}\varphi). \tag{2.7}$$

Here the 1-form α is defined by

$$\alpha = -d \log |\varphi|^2 \tag{2.8}$$

and we used the same notation introduced in [6] for any vector field V and any 1-form W,

$$(JV)^k = J^k{}_p V^p = V^{Jk}, \quad (JW)_k = J^p{}_k W_p = W_{Jk}. \tag{2.9}$$

In particular, $\omega_{ij} = g_{Ji,j}$, $g_{ij} = \omega_{i,Jj}$, and $\omega^{ij} = g^{Ji,j}$, $g^{ij} = \omega^{i,Jj}$.

2.2 Identities from Type IIA Geometry

We list here some identities required later. Except for (2.21), they were proved in [6].

2.2.1 Identities for φ

First, the action of J on φ is given by

$$\begin{aligned} \varphi_{ijk} &= -\varphi_{Ji,Jj,k} = -\varphi_{Ji,j,Jk} = -\varphi_{i,Jj,Jk} \\ \varphi_{Ji,j,k} &= \varphi_{j,Jj,k} = \varphi_{i,j,Jk}. \end{aligned} \tag{2.10}$$

Next, bilinears in φ with two contractions with ω^{ij} give the metric g_{ij}. But bilinears with a single contraction with either ω^{ij} or g^{ij} simplify as well,

$$\omega^{ij}\varphi_{iab}\varphi_{jcd} = \frac{|\varphi|^2}{4}\left(\omega_{ac}g_{bd} + \omega_{bd}g_{ac} - \omega_{bc}g_{ad} - \omega_{ad}g_{bc}\right)$$
$$g^{ij}\varphi_{iab}\varphi_{jcd} = \frac{|\varphi|^2}{4}\left(g_{ac}g_{bd} + \omega_{ca}\omega_{bd} - \omega_{ad}\omega_{cb} - g_{bc}g_{ad}\right). \tag{2.11}$$

As a consequence, we also have bilinear identities involving φ and $\hat{\varphi}$, for example

$$\hat{\varphi}_{\lambda kp}\varphi_{iab}\omega^{ka}\omega^{pb} = |\varphi|^2\omega_{\lambda i}. \tag{2.12}$$

This reduces to the previous identity by noting that $\hat{\varphi}_{\lambda kp} = -\varphi_{J\lambda,kp}$, so that

$$\hat{\varphi}_{\lambda kp}\varphi_{iab}\omega^{ka}\omega^{pb} = -\varphi_{J\lambda,kp}\varphi_{iab}\omega^{ka}\omega^{pb} = |\varphi|^2 g_{J\lambda,i} = |\varphi|^2\omega_{\lambda i}. \tag{2.13}$$

2.2.2 Identities for the Nijenhuis Tensor

In general, the Nijenhuis tensor satisfies the following identities of a type (0, 2)-tensor in the sense of Gauduchon [9]

$$N^k{}_{Ji,j} = -N^{Jk}{}_{ij} = N^k{}_{i,Jj}, \quad N_{Ji,j,k} = N_{i,Jj,k} = N_{i,j,Jk}. \tag{2.14}$$

Since $d\omega = 0$, we also have the Bianchi identity

$$N_{ijk} + N_{jki} + N_{kij} = 0. \tag{2.15}$$

From this it follows that there are two symmetric tensors quadratic in N, denoted by

$$\left(N_+^2\right)_{ij} = N^{pq}{}_i N_{pqj}, \quad \left(N_-^2\right)_{ij} = N^{pq}{}_i N_{qpj}. \tag{2.16}$$

The relation between the Levi–Civita connection ∇ and the projected Levi–Civita connection $\mathfrak{D}$ also implies, since $\mathfrak{D}J = 0$,

$$\nabla_i J^k{}_j = -2N_{ij}{}^{Jk}. \tag{2.17}$$

In Type IIA geometry, we also have

$$N_-^2 = 2\,N_+^2 - \frac{1}{4}|N|^2 g, \quad |N|^2 = \left(N_+^2\right)^\lambda{}_\lambda = 2\left(N_-^2\right)^\lambda{}_\lambda, \tag{2.18}$$

with $|N|^2 = N^{mkp}N_{mkp}$, and the following crucial identity between the Nijenhuis tensor and φ,

$$N^p{}_{ij}\varphi_{pkl} = -N^p{}_{kl}\varphi_{pij}, \tag{2.19}$$

which was proved in Corollary 1 [6].

2.2.3 Identities for the Curvature Tensor

We shall express the desired identities for the curvature tensor of the Levi–Civita connection in the following convention. The connection ∇ is written as $\nabla_m V^k = \partial_m V^k + \Gamma^k{}_{m\ell} V^\ell$, and the curvature tensor $R_{ij}{}^k{}_\ell$ is defined by

$$[\nabla_i, \nabla_j] V^k = R_{ij}{}^k{}_\ell V^\ell. \tag{2.20}$$

The Ricci curvature is then given by $R_{ij} = R_{ipj}{}^p$.

The first curvature identity that we require gives the action of J on Rm,

$$\begin{aligned} R_{j,i,Jk,J\ell} &= R_{jik\ell} + B_{ijk\ell} \\ B_{ijk\ell} &= -2\mathfrak{D}_i N_{jk\ell} + 2\mathfrak{D}_j N_{ik\ell} - 2N^\alpha{}_{ij} N_{\alpha k\ell}, \end{aligned} \tag{2.21}$$

This identity can also be expressed as

$$R_{ji}{}^p{}_{J\ell} = R_{ji}{}^{Jp}{}_\ell + 2\mathfrak{D}_j N_i{}^{Jp}{}_\ell - 2\mathfrak{D}_i N_j{}^{Jp}{}_\ell - 2N^\mu{}_{ji} N_{\mu\ell}{}^{Jp}. \tag{2.22}$$

To see this, we consider the action of J on a vector field V,

$$\begin{aligned} R_{jk}{}^p{}_q (JV)^q &= \nabla_j \nabla_k (JV)^p - \nabla_k \nabla_j (JV)^p \\ &= J[\nabla_j, \nabla_k] V^p + (\nabla_j \nabla_k J - \nabla_k \nabla_j J)^p{}_\lambda V^\lambda. \end{aligned} \tag{2.23}$$

It follows that

$$\begin{aligned} R_{jk}{}^p{}_q J^q{}_\lambda &= J^p{}_q R_{kj}{}^q{}_\lambda + \nabla_j \nabla_k J^p{}_\lambda - \nabla_k \nabla_j J^p{}_\lambda \\ &= J^p{}_q R_{kj}{}^q{}_\lambda - 2\nabla_j \left(J^p{}_\mu N_{k\lambda}{}^\mu \right) + 2\nabla_k \left(J^p{}_\mu N_{j\lambda}{}^\mu \right) \end{aligned} \tag{2.24}$$

or, in more succinct notation,

$$R_{jk}{}^p{}_{J\lambda} = R_{jk}{}^{Jp}{}_\lambda - 2\nabla_j (N_{k\lambda}{}^{Jp}) + 2\nabla_k (N_{j\lambda}{}^{Jp}). \tag{2.25}$$

We now convert ∇ derivatives into $\mathfrak{D}$ derivatives. First lowering indices gives

$$R_{jik,J\ell} = -R_{j,i,Jk,\ell} + 2\nabla_j (N_{i,\ell,Jk}) - 2\nabla_i (N_{j,\ell,Jk}). \tag{2.26}$$

Therefore

$$R_{j,i,Jk,J\ell} = R_{jik\ell} + 2J^p{}_k \nabla_j (N_{i,\ell,Jp}) - 2J^p{}_k \nabla_i (N_{j,\ell,Jp}). \tag{2.27}$$

We write

$$\begin{aligned} 2J^p{}_k \nabla_j (N_{i,\ell,Jp}) &= 2J^p{}_k \mathfrak{D}_j (N_{i,\ell,Jp}) - 2J^p{}_k N_{ji}{}^\mu (N_{\mu,\ell,Jp}) \\ &\quad -2J^p{}_k N_{j\ell}{}^\mu (N_{i,\mu,Jp}) - 2J^p{}_k N_{jp}{}^\mu (J^n{}_\mu N_{i\ell n}) \end{aligned} \tag{2.28}$$

Since $\mathfrak{D}J = 0$,

$$\begin{aligned} 2J^p{}_k\nabla_j(N_{i,\ell,Jp}) &= -2\mathfrak{D}_j N_{i\ell k} + 2N_{ji}{}^\mu N_{\mu\ell k} + 2N_{j\ell}{}^\mu N_{i\mu k} - 2N_{j,Jk}{}^{Jn} N_{i\ell n} \\ &= 2\mathfrak{D}_j N_{ik\ell} + 2N_{ji}{}^\mu N_{\mu\ell k} - 2\left(N_{j\ell}{}^\mu N_{ik\mu} + N_{jk}{}^\mu N_{i\ell\mu}\right) \end{aligned} \tag{2.29}$$

This last term is symmetric in (i, j). Therefore

$$2J^p{}_k\nabla_j(N_{i,\ell,Jp}) - (i \leftrightarrow j) = 2\mathfrak{D}_j N_{ik\ell} - 2\mathfrak{D}_i N_{jk\ell} + 2N_{ji}{}^\mu N_{\mu\ell k} - 2N_{ij}{}^\mu N_{\mu\ell k} \tag{2.30}$$

By the Bianchi identity

$$\begin{aligned} 2J^p{}_k\nabla_j(N_{i,\ell,Jp}) - (i \leftrightarrow j) \ &= 2\mathfrak{D}_j N_{ik\ell} - 2\mathfrak{D}_i N_{jk\ell} + 2(-N^\mu{}_{ji} - N_i{}^\mu{}_j)N_{\mu\ell k} \\ &\quad -2N_{ij}{}^\mu N_{\mu\ell k} \end{aligned} \tag{2.31}$$

from which the desired identity (2.21) follows.

Finally, we shall need the following curvature identity specific to Type IIA geometry (see (6.53) in [6]),

$$\begin{aligned} R_{ij} \ &= -\mathfrak{D}_s\left(N_i{}^s{}_j + N_j{}^s{}_i\right) - 2\left(N_-^2\right)_{ij} + \frac{1}{2}\nabla_i\nabla_j \log|\varphi|^2 \\ &\quad + \frac{1}{2}J^p{}_i J^q{}_j \nabla_p\nabla_q \log|\varphi|^2. \end{aligned} \tag{2.32}$$

3 Proof of Theorem 1

We shall establish Theorem 1 using the formulation of the Type IIA flow as a Laplacian type flow [6]

$$\partial_t\varphi = -dd^\dagger(|\varphi|^2\varphi) + 2d(|\varphi|^2 N^\dagger \cdot \varphi) \tag{3.1}$$

where $N^\dagger : \Lambda^3(M) \to \Lambda^2(M)$ is the operator defined by

$$(N^\dagger \cdot \varphi)_{kj} = N^\mu{}_j{}^\lambda \varphi_{\mu k\lambda} - N^\mu{}_k{}^\lambda \varphi_{\mu j\lambda}. \tag{3.2}$$

For our present purposes, it is convenient to rewrite the above expression as

$$\partial_t\varphi = -|\varphi|^2 dd^\dagger\varphi - d|\varphi|^2 \wedge d^\dagger\varphi + d(\iota_{\nabla|\varphi|^2}\varphi) + 2d(|\varphi|^2 N^\dagger \cdot \varphi). \tag{3.3}$$

We would like to determine $\partial_t g_{ij}$ explicitly. For this, it is convenient to determine first $\partial_t \tilde{g}_{ij}$, since $\tilde{g}_{ij}$ is a quadratic expression in φ, and we have

$$\partial_t\tilde{g}_{ij} = -\left\{(\partial_t\varphi_{iab})\varphi_{jkp}\omega^{ka}\omega^{pb} + (i \leftrightarrow j)\right\}. \tag{3.4}$$

We shall determine in turn the contribution of each expression in (3.3) to $\partial_t\tilde{g}_{ij}$.

3.1 The Bochner–Kodaira Formula for the Levi–Civita Connection

We begin with the contribution of $|\varphi|^2 dd^\dagger\varphi$ using a Bochner–Kodaira formula. In general, if M is any compact Riemannian manifold and we express any p-form in components as

$$\varphi = \frac{1}{p!}\sum_{i_1,\ldots,i_p} \varphi_{i_1\ldots i_p} \mathrm{d}x^{i_1} \wedge \cdots \wedge \mathrm{d}x^{i_p} = \frac{1}{p!}\sum_I \varphi_I dx^I \tag{3.5}$$

with antisymmetric coefficients $\varphi_{i_1\cdots i_p}$, then the adjoint $d^\dagger$ of the de Rham exterior differential with respect to a given metric g_{ij} is given by

$$(d^\dagger\varphi)_{I'} = -g^{\ell m}\nabla_m\varphi_{\ell I'}, \tag{3.6}$$

where ∇ denotes the covariant derivative with respect to the Levi–Civita connection and we have split the index I into $I = (\ell, I')$, $I' = (i_2, \ldots, i_p)$. It follows that

$$(dd^\dagger\varphi)_I = -\left(\nabla_{i_1}\left(g^{\ell m}\nabla_m\varphi_{\ell I'}\right) - \sum_{q=2}^{p}\left(i_1 \leftrightarrow i_q\right)\right). \tag{3.7}$$

Next, we have

$$(d\varphi)_{\ell I} = \nabla_\ell\varphi_I - \sum_{q=1}^{p}(\ell \leftrightarrow i_q) \tag{3.8}$$

and hence

$$(d^\dagger d\varphi)_I = -g^{\ell m}\nabla_m\left(\nabla_\ell\varphi_I - \sum_{q=1}^{p}(\ell \leftrightarrow i_q)\right). \tag{3.9}$$

Altogether, we obtain the version of the Bochner–Kodaira formula that we need,

$$((dd^\dagger + d^\dagger d)\varphi)_I = -g^{\ell m}\nabla_m\nabla_\ell\varphi_I + g^{\ell m}\sum_{q=1}^{p}[\nabla_m, \nabla_{i_q}]\varphi_{\cdots i_{q-1}\ell i_{q+1}\cdots} \tag{3.10}$$

In the case of interest, namely 3-forms φ with $d\varphi = 0$, we obtain

$$\begin{aligned} dd^\dagger\varphi_{jkp} &= -g^{\ell m}\nabla_m\nabla_\ell\varphi_{jkp} + g^{\ell m} \\ &\quad \times \left\{[\nabla_m, \nabla_j]\varphi_{kp\ell} + [\nabla_m, \nabla_k]\varphi_{pj\ell} + [\nabla_m, \nabla_p]\varphi_{jk\ell}\right\}. \end{aligned} \tag{3.11}$$

3.2 The Laplacian Term $g^{\ell m}\nabla_m\nabla_\ell\varphi_{jkp}$

Recall that the covariant derivatives of φ with respect to the projected Levi–Civita connection $\mathfrak{D}$ are given by (2.7). It follows that

$$g^{\ell m}\mathfrak{D}_\ell\mathfrak{D}_m\varphi = -\frac{1}{2}\left(\nabla_\mu\alpha^\mu\right)\varphi \tag{3.12}$$

and

$$\begin{aligned}[\mathfrak{D}_m,\mathfrak{D}_\ell]\varphi &= \frac{1}{2}\left(-\mathfrak{D}_m\alpha_\ell+\mathfrak{D}_\ell\alpha_m\right)\varphi+\frac{1}{2}\left(-\mathfrak{D}_m\alpha_{J\ell}+\mathfrak{D}_\ell\alpha_{Jm}\right)\hat{\varphi}\\ &= -\frac{1}{2}N_{m\ell}{}^j\alpha_j\,\varphi+\frac{1}{2}N_{\ell m}{}^j\alpha_j\,\varphi+\frac{1}{2}\left(-\mathfrak{D}_m\alpha_{J\ell}+\mathfrak{D}_\ell\alpha_{Jm}\right)\hat{\varphi}.\end{aligned} \tag{3.13}$$

Now the difference between ∇ and $\mathfrak{D}$ on vectors is given by (2.5). On 3-forms, it is given by

$$\begin{aligned}\nabla_\ell\varphi_{jkp} &= \mathfrak{D}_\ell\varphi_{jkp}-\varphi_{\lambda kp}N_{\ell j}{}^\lambda-\varphi_{j\lambda p}N_{\ell k}{}^\lambda-\varphi_{jk\lambda}N_{\ell p}{}^\lambda\\ &= \mathfrak{D}_\ell\varphi_{jkp}-E_{\ell;jkp},\end{aligned} \tag{3.14}$$

where

$$E_{\ell;jkp}=\varphi_{\lambda kp}N_{\ell j}{}^\lambda+\varphi_{j\lambda p}N_{\ell k}{}^\lambda+\varphi_{jk\lambda}N_{\ell p}{}^\lambda. \tag{3.15}$$

Similarly, we write

$$\begin{aligned}\nabla_m\mathfrak{D}_\ell\varphi_{jkp} &= \mathfrak{D}_m\mathfrak{D}_\ell\varphi_{jkp}-\mathfrak{D}_\mu\varphi_{jkp}N_{m\ell}{}^\mu-\mathfrak{D}_\ell\varphi_{\mu kp}N_{mj}{}^\mu-\mathfrak{D}_\ell\varphi_{j\mu p}N_{mk}{}^\mu\\ &\quad-\mathfrak{D}_\ell\varphi_{jk\mu}N_{mp}{}^\mu\\ &:= \mathfrak{D}_m\mathfrak{D}_\ell\varphi_{jkp}-E_{m;\ell jkp},\end{aligned} \tag{3.16}$$

and hence

$$g^{m\ell}\nabla_m\nabla_\ell\varphi_{jkp}=g^{m\ell}\mathfrak{D}_m\mathfrak{D}_\ell\varphi_{jkp}-g^{m\ell}E_{m;\ell jkp}-g^{m\ell}\nabla_m E_{\ell;jkp}. \tag{3.17}$$

We begin by computing the contributions of $g^{m\ell}\nabla_m E_{\ell;jkp}$,

$$\begin{aligned}g^{m\ell}\nabla_m E_{\ell;jkp} &= \left(g^{\ell m}\nabla_m\varphi_{\lambda kp}\right)N_{\ell j}{}^\lambda+\left(g^{\ell m}\nabla_m\varphi_{j\lambda p}\right)N_{\ell k}{}^\lambda+\left(g^{\ell m}\nabla_m\varphi_{jk\lambda}\right)N_{\ell p}{}^\lambda\\ &\quad+\varphi_{\lambda kp}g^{\ell m}\nabla_m N_{\ell j}{}^\lambda+\varphi_{j\lambda p}g^{\ell m}\nabla_m N_{\ell k}{}^\lambda+\varphi_{jk\lambda}g^{\ell m}\nabla_m N_{\ell p}{}^\lambda\\ &= \left(g^{\ell m}\mathfrak{D}_m\varphi_{\lambda kp}\right)N_{\ell j}{}^\lambda+\left(g^{\ell m}\mathfrak{D}_m\varphi_{j\lambda p}\right)N_{\ell k}{}^\lambda+\left(g^{\ell m}\mathfrak{D}_m\varphi_{jk\lambda}\right)N_{\ell p}{}^\lambda\\ &\quad-g^{\ell m}\left(E_{m;\lambda kp}N_{\ell j}{}^\lambda+E_{m;j\lambda p}N_{\ell k}{}^\lambda+E_{m;jk\lambda}N_{\ell p}{}^\lambda\right)\\ &\quad+\varphi_{\lambda kp}g^{\ell m}\nabla_m N_{\ell j}{}^\lambda+\varphi_{j\lambda p}g^{\ell m}\nabla_m N_{\ell k}{}^\lambda+\varphi_{jk\lambda}g^{\ell m}\nabla_m N_{\ell p}{}^\lambda.\end{aligned} \tag{3.18}$$

3.2.1 Contributions of the Terms $E_{\ell;jkp}$

Consider the contributions of the second row on the right hand side of the last equation. Paired with $\varphi_{iab}\omega^{ka}\omega^{pb}$, it gives

$$\begin{aligned} g^{\ell m}E_{m;\lambda kp}N_{\ell j}{}^{\lambda}\,\varphi_{iab}\omega^{ka}\omega^{pb} &= g^{\ell m}(\varphi_{\mu kp}N_{m\lambda}{}^{\mu} \\ &\quad +\varphi_{\lambda\mu p}N_{mk}{}^{\mu}+\varphi_{\lambda k\mu}N_{mp}{}^{\mu})N_{\ell j}{}^{\lambda}\,\varphi_{iab}\omega^{ka}\omega^{pb} \\ &= (\mathrm{I}+\mathrm{II}+\mathrm{III})\cdot\varphi_{iab}\omega^{ka}\omega^{pb} \end{aligned} \tag{3.19}$$

with

$$\begin{aligned} (\mathrm{I})\cdot\varphi_{iab}\omega^{ka}\omega^{pb} &= g^{\ell m}\varphi_{\mu kp}N_{m\lambda}{}^{\mu}N_{\ell j}{}^{\lambda}\,\varphi_{iab}\omega^{ka}\omega^{pb} \\ (\mathrm{II})\cdot\varphi_{iab}\omega^{ka}\omega^{pb} &= g^{\ell m}\varphi_{\lambda\mu p}N_{mk}{}^{\mu}N_{\ell j}{}^{\lambda}\,\varphi_{iab}\omega^{ka}\omega^{pb} \\ (\mathrm{III})\cdot\varphi_{iab}\omega^{ka}\omega^{pb} &= g^{\ell m}\varphi_{\lambda k\mu}N_{mp}{}^{\mu}N_{\ell j}{}^{\lambda}\,\varphi_{iab}\omega^{ka}\omega^{pb}. \end{aligned} \tag{3.20}$$

Next, we have

$$(\mathrm{I})\cdot\varphi_{iab}\omega^{ka}\omega^{pb} = -|\varphi|^2 g^{\ell m}g_{\mu i}N_{m\lambda}{}^{\mu}N_{\ell j}{}^{\lambda} = -|\varphi|^2 N^{\ell}{}_{\lambda i}N_{\ell j}{}^{\lambda} = |\varphi|^2\left(N_+^2\right)_{ij} \tag{3.21}$$

and, using (2.11), we compute

$$\begin{aligned} (\mathrm{II})\cdot\varphi_{iab}\omega^{ka}\omega^{pb} &= \frac{|\varphi|^2}{4}g^{\ell m}\left(\omega_{\lambda i}g_{\mu a}+\omega_{\mu a}g_{\lambda i}-\omega_{\lambda a}g_{\mu i}-\omega_{\mu i}g_{\lambda a}\right)N_{mk}{}^{\mu}N_{\ell j}{}^{\lambda}\omega^{ka} \\ &= \frac{|\varphi|^2}{4}g^{\ell m}\left(\omega_{\lambda i}J^{k}{}_{\mu}-\delta^{k}{}_{\mu}g_{\lambda i}+\delta^{k}{}_{\lambda}g_{\mu i}-\omega_{\mu i}J^{k}{}_{\lambda}\right)N_{mk}{}^{\mu}N_{\ell j}{}^{\lambda} \\ &= \frac{|\varphi|^2}{4}\left(\omega_{\lambda i}N_{mk}{}^{Jk}N^{m}{}_{j}{}^{\lambda}-N_{mk}{}^{k}N^{m}{}_{j}{}^{\lambda}+N_{mki}N^{m}{}_{j}{}^{k}-\omega_{\mu i}N_{mk}{}^{\mu}N^{m}{}_{j}{}^{Jk}\right). \end{aligned}$$

Now $N_{mk}{}^{k}=0$, and by the Nijenhuis tensor identities,

$$N_{mk}{}^{Jk} = N_{m,Jk}{}^{k} = N_{Jm,k}{}^{k} = 0. \tag{3.22}$$

Furthermore, we have by definition $N_{mki}N^{m}{}_{j}{}^{k} = -(N_+^2)_{ij}$, while

$$\begin{aligned} -\omega_{\mu i}N_{mk}{}^{\mu}N^{m}{}_{j}{}^{Jk} &= g_{\mu\nu}J^{\nu}{}_{i}N_{mk}{}^{\mu}N^{m}{}_{j}{}^{Jk} = N_{mk\,Ji}N^{m}{}_{j}{}^{Jk} = N_{m,Jk,i}N^{m}{}_{j}{}^{Jk} \\ &= -N_{mki}N^{m}{}_{j}{}^{k} = \left(N_+^2\right)_{ij} \end{aligned} \tag{3.23}$$

and hence

$$(\mathrm{II})\cdot\varphi_{iab}\omega^{ka}\omega^{pb} = 0. \tag{3.24}$$

Since (III) can be obtained from (II) by the simultaneous interchange $a \leftrightarrow b$ and $k \leftrightarrow p$, we also have

$$(\mathrm{III}) \cdot \varphi_{iab}\omega^{ka}\omega^{pb} = 0. \tag{3.25}$$

We consider next the expression

$$\begin{aligned} g^{\ell m} E_{m;j\lambda p} N_{\ell k}{}^{\lambda} \varphi_{iab}\omega^{ka}\omega^{pb} &= g^{\ell m} \left(\varphi_{\mu\lambda p} N_{mj}{}^{\mu} + \varphi_{j\mu p} N_{m\lambda}{}^{\mu} + \varphi_{j\lambda\mu} N_{mp}{}^{\mu} \right) \\ &\quad \times N_{\ell k}{}^{\lambda} \varphi_{iab}\omega^{ka}\omega^{pb} \\ &= (\mathrm{IV} + \mathrm{V} + \mathrm{VI}) \varphi_{iab}\omega^{ka}\omega^{pb}. \end{aligned} \tag{3.26}$$

The contributions of the term (IV) worked out to be 0,

$$\begin{aligned} (\mathrm{IV}) \cdot \varphi_{iab}\omega^{ka}\omega^{pb} &= \frac{|\varphi|^2}{4} \left(\omega_{\mu i} g_{\lambda a} + \omega_{\lambda a} g_{\mu i} - \omega_{\mu a} g_{\lambda i} - \omega_{\lambda i} g_{\mu a} \right) N_{mj}{}^{\mu} N^{m}{}_{k}{}^{\lambda} \omega^{ka} \\ &= \frac{|\varphi|^2}{4} \left(\omega_{\mu i} J^{k}{}_{\lambda} - \delta^{k}{}_{\lambda} g_{\mu i} + \delta^{k}{}_{\mu} g_{\lambda i} - \omega_{\lambda i} J^{k}{}_{\mu} \right) N_{mj}{}^{\mu} N^{m}{}_{k}{}^{\lambda}. \end{aligned} \tag{3.27}$$

The first two terms on the right hand side vanish individually, since

$$\begin{aligned} \omega_{\mu i} J^{k}{}_{\lambda} N_{mj}{}^{\mu} N^{m}{}_{k}{}^{\lambda} &= \omega_{\mu i} N_{mj}{}^{\mu} N^{m}{}_{J\lambda}{}^{\lambda} = 0 \\ \delta^{k}{}_{\lambda}\, g_{\mu i} N_{mj}{}^{\mu} N^{m}{}_{k}{}^{\lambda} &= N_{mj}{}^{\mu} N^{m}{}_{k}{}^{k} = 0. \end{aligned} \tag{3.28}$$

Of the remaining two terms, we have obviously

$$\delta^{k}{}_{\mu} g_{\lambda i} N_{mj}{}^{\mu} N^{m}{}_{k}{}^{\lambda} = N_{mj}{}^{k} N^{m}{}_{ki} = -N_{m}{}^{k}{}_{j} N^{m}{}_{ki} = -\left(N_{+}^{2}\right)_{ij}, \tag{3.29}$$

while

$$\begin{aligned} -\omega_{\lambda i} N_{mj}{}^{Jk} N^{m}{}_{k}{}^{\lambda} &= g_{\lambda\nu} J^{\nu}{}_{i} N_{mj}{}^{Jk} N^{m}{}_{k}{}^{\lambda} = J^{\nu}{}_{i} N_{mj}{}^{Jk} N^{m}{}_{k\nu} = N_{mj}{}^{Jk} N^{m}{}_{k,Ji} \\ &= N_{mj}{}^{Jk} N^{m}{}_{Jk,i} = -N_{mj}{}^{k} N^{m}{}_{ki} = \left(N_{+}^{2}\right)_{ij} \end{aligned} \tag{3.30}$$

so they cancel each other out and we obtain, as claimed,

$$(\mathrm{IV}) \cdot \varphi_{iab}\omega^{ka}\omega^{pb} = 0. \tag{3.31}$$

The next group of terms is given by

$$\begin{aligned} (\mathrm{V}) \cdot \varphi_{iab}\omega^{ka}\omega^{pb} &= \varphi_{j\mu p} N_{m\lambda}{}^{\mu} N^{m}{}_{k}{}^{\lambda} \varphi_{iab}\omega^{ka}\omega^{pb} = -\varphi_{j\mu p} g^{\mu\nu} \left(N_{+}^{2}\right)_{\nu k} \varphi_{iab}\omega^{ka}\omega^{pb} \\ &= -\frac{|\varphi|^2}{4} \left(\omega_{ji} g_{\mu a} + \omega_{\mu a} g_{ji} - \omega_{ja} g_{\mu i} - \omega_{\mu i} g_{ja} \right) \omega^{ka} \left(N_{+}^{2}\right)_{\nu k} g^{\mu\nu}. \end{aligned} \tag{3.32}$$

The first term on the right produces 0, since it can be computed as $\omega_{ji}\omega^{k\nu}(N_+^2)_{\nu k}$. This term vanishes due to the anti-symmetrization of k and ν. We are left with

$$\begin{aligned}(\mathrm{V})\cdot\varphi_{iab}\omega^{ka}\omega^{pb} &= -\frac{|\varphi|^2}{4}\left(-\delta^k{}_\mu g_{ji}g^{\mu\nu}+\delta^k{}_j g_{\mu i}g^{\mu\nu}-\omega_{\mu i}J^k{}_j g^{\mu\nu}\right)\left(N_+^2\right)_{\nu k}\\ &= -\frac{|\varphi|^2}{4}\left(-|N|^2 g_{ij}+\left(N_+^2\right)_{ij}+\left(N_+^2\right)_{Ji,Jj}\right)\end{aligned} \tag{3.33}$$

Since we have

$$\left(N_+^2\right)_{Ji,Jj}=N^{mk}{}_{Ji}N_{mk,Jj}=-N^{m,Jk}{}_i N_{m,Jk,j}=N^{mk}{}_i N_{mkj}=\left(N_+^2\right)_{ij} \tag{3.34}$$

we are left with

$$(\mathrm{V})\cdot\varphi_{iab}\omega^{ka}\omega^{pb}=\frac{|\varphi|^2}{4}|N|^2 g_{ij}-\frac{|\varphi|^2}{2}\left(N_+^2\right)_{ij}. \tag{3.35}$$

Finally, we observe that

$$(\mathrm{VI})\cdot\varphi_{iab}\omega^{ka}\omega^{pb}=g^{\ell m}\varphi_{j\lambda\mu}N_{mp}{}^{\mu}N_{\ell k}{}^{\lambda}\varphi_{iab}\omega^{ka}\omega^{pb}=0. \tag{3.36}$$

We can readily see this in a complex frame. Since $\varphi\in\Lambda^{3,0}\oplus\Lambda^{0,3}$, the only components of $\varphi_{j\lambda\mu}$ which are not 0 must have both barred or both unbarred indices. But the contraction with $g^{\ell m}$ implies that the indices ℓ and m must be mixed. But then for $N_{mp}{}^{\mu}N_{\ell k}{}^{\lambda}$ not to be 0, the indices λ and μ must be mixed too, contradicting the requirement that they must be both barred or both unbarred. This establishes our claim.

We still have one more contribution from the second row of (3.18), given by

$$g^{\ell m}E_{m;jk\lambda}N_{\ell p}{}^{\lambda}\varphi_{iab}\omega^{ka}\omega^{pb} \tag{3.37}$$

but which can be recognized as coinciding with the term that we just computed

$$g^{\ell m}E_{m;j\lambda p}N_{\ell k}{}^{\lambda}\varphi_{iab}\omega^{ka}\omega^{pb}=\frac{|\varphi|^2}{4}|N|^2 g_{ij}-\frac{|\varphi|^2}{2}\left(N_+^2\right)_{ij} \tag{3.38}$$

upon the renaming of indices $a\leftrightarrow b$, $p\leftrightarrow k$.

It is convenient to summarize the formula which we have obtained as a lemma:

Lemma 1 *We have*

$$g^{\ell m}\left(E_{m;\lambda kp}N_{\ell j}{}^{\lambda}+E_{m;j\lambda p}N_{\ell k}{}^{\lambda}+E_{m;jk\lambda}N_{\ell p}{}^{\lambda}\right)\varphi_{iab}\omega^{ka}\omega^{pb}=\frac{|\varphi|^2}{2}|N|^2 g_{ij}. \tag{3.39}$$

3.2.2 Contributions of the Term $E_{m;\ell jkp}$

The term $E_{m;\ell jkp}$ involves $\mathfrak{D}_\mu\varphi_{jkp}$, $\mathfrak{D}_\ell\varphi_{\mu kp}$, $\mathfrak{D}_\ell\varphi_{j\mu p}$, and $\mathfrak{D}_\ell\varphi_{jk\mu}$. We use (2.7) to evaluate the contribution of each term in turn,

$$\begin{aligned}\mathfrak{D}_\mu\varphi_{jkp}N_{m\ell}{}^\mu\varphi_{iab}\omega^{ka}\omega^{pb} &= -\frac{1}{2}(\alpha_\mu\varphi_{jkp}+\alpha_{J\mu}\hat{\varphi}_{jkp})N_{m\ell}{}^\mu\varphi_{iab}\omega^{ka}\omega^{pb}\\ &= \frac{1}{2}|\varphi|^2(\alpha_\mu g_{ji}-\alpha_{J\mu}\omega_{ji})N_{m\ell}{}^\mu \end{aligned}\tag{3.40}$$

Upon symmetrization in i and j, and contracting with $g^{\ell m}$, we obtain

$$g^{\ell m}\mathfrak{D}_\mu\varphi_{jkp}N_{m\ell}{}^\mu\varphi_{iab}\omega^{ka}\omega^{pb}+(i\leftrightarrow j)=|\varphi|^2\,g_{ij}\alpha_\mu g^{m\ell}N_{m\ell}{}^\mu=0 \tag{3.41}$$

where we have used the fact that N is of type (0, 2) to write

$$g^{m\ell}N_{m\ell}{}^\mu=g^{Jm,J\ell}N_{m\ell}{}^\mu=g^{m\ell}N_{Jm,J\ell}{}^\mu=-g^{m\ell}N_{m\ell}{}^\mu$$

and therefore

$$g^{m\ell}N_{m\ell}{}^\mu=0. \tag{3.42}$$

Next, we consider the term

$$\begin{aligned}\mathfrak{D}_\ell\varphi_{\mu kp}N_{mj}{}^\mu\varphi_{iab}\omega^{ka}\omega^{pb} &= -\frac{1}{2}(\alpha_\ell\varphi_{\mu kp}+\alpha_{J\ell}\hat{\varphi}_{\mu kp})N_{mj}{}^\mu\varphi_{iab}\omega^{ka}\omega^{pb}\\ &= \frac{1}{2}|\varphi|^2(\alpha_\ell g_{\mu i}-\alpha_{J\ell}\omega_{\mu i})N_{mj}{}^\mu\\ &= \frac{1}{2}|\varphi|^2(\alpha_\ell N_{mji}+\alpha_{J\ell}N_{mj,Ji}). \end{aligned}\tag{3.43}$$

The first term on the right symmetrizes to 0. So does the second, using the fact that N is a type (0, 2)-tensor, so that $N_{mj,Ji}=N_{Jm,j,i}$ which is antisymmetric in the last two indices.

We consider now

$$\mathfrak{D}_\ell\varphi_{j\mu p}N_{mk}{}^\mu\varphi_{iab}\omega^{ka}\omega^{pb}=-\frac{1}{2}(\alpha_\ell\varphi_{j\mu p}+\alpha_{J\ell}\hat{\varphi}_{j\mu p})N_{mk}{}^\mu\varphi_{iab}\omega^{ka}\omega^{pb}. \tag{3.44}$$

We work out separately the contributions of the two terms $\varphi_{j\mu p}$ and $\hat{\varphi}_{j\mu p}$ on the right hand side. First, we have

$$\begin{aligned}\alpha_\ell\varphi_{j\mu p}N_{mk}{}^\mu\varphi_{iab}\omega^{ka}\omega^{pb} &= \frac{|\varphi|^2}{4}\alpha_\ell\left(\omega_{ji}g_{\mu a}+\omega_{\mu a}g_{ji}-\omega_{ja}g_{\mu i}-\omega_{\mu i}g_{ja}\right)N_{mk}{}^\mu\omega^{ka}\\ &= \frac{|\varphi|^2}{4}\alpha_\ell\left(\omega_{ji}J^k{}_\mu-\delta^k{}_\mu g_{ji}+\delta^k{}_j g_{\mu i}-\omega_{\mu i}J^k{}_j\right)N_{mk}{}^\mu. \end{aligned}\tag{3.45}$$

We claim that, upon symmetrization in i and j, the net result is 0. This is obviously true of the term ω_{ji}, while $N_{mk}{}^{k}=0$ and N_{mji} also symmetrizes to 0. The fourth term can be rewritten as

$$\omega_{\mu i} J^{k}{}_{j} N_{mk}{}^{\mu} = -g_{\mu\nu} J^{\nu}{}_{i} N_{m,Jj}{}^{\mu} = -J^{\nu}{}_{i} N_{m,Jj,\nu} = N_{m,Jj,Ji} \tag{3.46}$$

which symmetrizes to 0. We come to the contribution of the term involving $\hat{\varphi}$,

$$\begin{aligned}
\hat{\varphi}_{j\mu p}\varphi_{iab}\omega^{ka}\omega^{pb} &= -\varphi_{Jj,\mu p}\varphi_{iab}\omega^{ka}\omega^{pb} \\
&= -\frac{|\varphi|^2}{4}(\omega_{Jj,i}g_{\mu a} + \omega_{\mu a}g_{Jj,i} - \omega_{Jj,a}g_{\mu i} - \omega_{\mu i}g_{Jj,a})\omega^{ka} \\
&= -\frac{|\varphi|^2}{4}(-g_{ij}g_{\mu a} + \omega_{\mu a}\omega_{ji} + g_{aj}g_{\mu i} - \omega_{\mu i}\omega_{ja})\omega^{ka} \\
&= -\frac{|\varphi|^2}{4}(-g_{ij}J^{k}{}_{\mu} - \delta^{k}{}_{\mu}\omega_{ji} + J^{k}{}_{j}g_{\mu i} + \delta^{k}{}_{j}\omega_{\mu i}).
\end{aligned} \tag{3.47}$$

Dropping the term ω_{ji} since it symmetrizes to 0, we arrive at

$$\begin{aligned}
\hat{\varphi}_{j\mu p}\varphi_{iab}\omega^{ka}\omega^{pb}N_{mk}{}^{\mu} + (i \leftrightarrow j) &= -\frac{|\varphi|^2}{4}(-g_{ij}N_{m,J\mu}{}^{\mu} + J^{k}{}_{j}N_{mki} + N_{mj}{}^{\mu}\omega_{\mu i}) \\
&\quad + (i \leftrightarrow j) \\
&= -\frac{|\varphi|^2}{4}(N_{m,Jj,i} - N_{mj,Ji}) + (i \leftrightarrow j) \\
&= -\frac{|\varphi|^2}{4}(N_{Jm,j,i} - N_{Jm,j,i}) + (i \leftrightarrow j) = 0.
\end{aligned} \tag{3.48}$$

The last term $\mathfrak{D}_{\ell}\varphi_{jk\mu}$ makes an identical contribution as $\mathfrak{D}_{\ell}\varphi_{j\mu p}$, upon renaming the summation indices $a \leftrightarrow b$, $k \leftrightarrow p$. Thus its contribution is also 0. In summary, we have established

Lemma 2 *We have*

$$g^{m\ell}E_{m;\ell jkp}\varphi_{iab}\omega^{ka}\omega^{pb} + (i \leftarrow j) = 0. \tag{3.49}$$

3.2.3 Completion of the Calculations for $\nabla^{\ell}E_{\ell;jkp}$

The terms from $\nabla^{\ell}E_{\ell;jkp}$ in (3.18) whose contributions we have not worked out as yet are the following

$$\begin{aligned}
&\left(g^{\ell m}\mathfrak{D}_{m}\varphi_{\lambda kp}\right)N_{\ell j}{}^{\lambda} + \left(g^{\ell m}\mathfrak{D}_{m}\varphi_{j\lambda p}\right)N_{\ell k}{}^{\lambda} + \left(g^{\ell m}\mathfrak{D}_{m}\varphi_{jk\lambda}\right)N_{\ell p}{}^{\lambda} \\
&\quad + \varphi_{\lambda kp}g^{\ell m}\nabla_{m}N_{\ell j}{}^{\lambda} + \varphi_{j\lambda p}g^{\ell m}\nabla_{m}N_{\ell k}{}^{\lambda} + \varphi_{jk\lambda}g^{\ell m}\nabla_{m}N_{\ell p}{}^{\lambda}. \\
&= \varphi_{\lambda kp}\left(-\frac{1}{2}\alpha^{\ell}N_{\ell j}{}^{\lambda} + \nabla^{\ell}N_{\ell j}{}^{\lambda}\right) - \frac{1}{2}\hat{\varphi}_{\lambda kp}\,\alpha_{Jm}g^{\ell m}N_{\ell j}{}^{\lambda}
\end{aligned}$$

$$
\begin{aligned}
&+\varphi_{j\lambda p}\left(-\frac{1}{2}\alpha^{\ell}N_{\ell k}{}^{\lambda}+\nabla^{\ell}N_{\ell k}{}^{\lambda}\right)-\frac{1}{2}\hat{\varphi}_{j\lambda p}\,\alpha_{Jm}g^{\ell m}N_{\ell k}{}^{\lambda}\\
&+\varphi_{jk\lambda}\left(-\frac{1}{2}\alpha^{\ell}N_{\ell p}{}^{\lambda}+\nabla^{\ell}N_{\ell p}{}^{\lambda}\right)-\frac{1}{2}\hat{\varphi}_{jk\lambda}\,\alpha_{Jm}g^{\ell m}N_{\ell p}{}^{\lambda}\\
=&\ \mathrm{VII}+\hat{\mathrm{VII}}+\mathrm{VIII}+\hat{\mathrm{VIII}}+\mathrm{IX}+\hat{\mathrm{IX}}. \qquad (3.50)
\end{aligned}
$$

Again, we evaluate each contribution in turn. We have

$$
\begin{aligned}
(\mathrm{VII})\cdot\varphi_{iab}\omega^{ka}\omega^{pb}&=-|\varphi|^{2}g_{\lambda i}\left(-\frac{1}{2}\alpha^{\ell}N_{\ell j}{}^{\lambda}+\nabla^{\ell}N_{\ell j}{}^{\lambda}\right)\\
&=|\varphi|^{2}\left(\frac{1}{2}\alpha^{\ell}N_{\ell ji}-\nabla^{\ell}N_{\ell ji}\right)=0 \qquad (3.51)
\end{aligned}
$$

upon symmetrization in $i\leftrightarrow j$. Next,

$$
\begin{aligned}
(\hat{\mathrm{VII}})\cdot\varphi_{iab}\omega^{ka}\omega^{pb}&=-\frac{1}{2}\alpha_{Jm}\varphi_{J\lambda,k,p}N^{m}{}_{j}{}^{\lambda}\varphi_{iab}\omega^{ka}\omega^{pb}\\
&=\frac{1}{2}|\varphi|^{2}\alpha_{Jm}\omega_{\lambda i}N^{m}{}_{j}{}^{\lambda}=-\frac{1}{2}|\varphi|^{2}\alpha_{Jm}N^{m}{}_{j,Ji}\\
&=-\frac{1}{2}|\varphi|^{2}\alpha_{Jm}g^{m\ell}N_{\ell,j,Ji}=-\frac{1}{2}|\varphi|^{2}\alpha_{Jm}g^{m\ell}N_{J\ell,j,i} \qquad (3.52)
\end{aligned}
$$

which produces 0 upon symmetrization in j and i. Next,

$$
\begin{aligned}
(\mathrm{VIII})\cdot\varphi_{iab}\omega^{ka}\omega^{pb}&=\varphi_{j\lambda p}\left(-\frac{1}{2}\alpha^{\ell}N_{\ell k}{}^{\lambda}+\nabla^{\ell}N_{\ell k}{}^{\lambda}\right)\varphi_{iab}\omega^{ka}\omega^{pb}\\
&=\frac{|\varphi|^{2}}{4}\left(\omega_{ji}g_{\lambda a}+\omega_{\lambda a}g_{ji}-\omega_{ja}g_{\lambda i}-\omega_{\lambda i}g_{ja}\right)\omega^{ka}\left(-\frac{1}{2}\alpha^{\ell}N_{\ell k}{}^{\lambda}+\nabla^{\ell}N_{\ell k}{}^{\lambda}\right)\\
&=\frac{|\varphi|^{2}}{4}\left(\omega_{ji}J^{k}{}_{\lambda}-\delta^{k}{}_{\lambda}g_{ji}+\delta^{k}{}_{j}g_{\lambda i}-\omega_{\lambda i}J^{k}{}_{j}\right)\left(-\frac{1}{2}\alpha^{\ell}N_{\ell k}{}^{\lambda}+\nabla^{\ell}N_{\ell k}{}^{\lambda}\right)\\
&=\frac{|\varphi|^{2}}{4}\Big\{g_{ji}\left(\frac{1}{2}\alpha^{\ell}N_{\ell\lambda}{}^{\lambda}-\nabla^{\ell}N_{\ell\lambda}{}^{\lambda}\right)-\frac{1}{2}\alpha^{\ell}N_{\ell ji}+\nabla^{\ell}N_{\ell ji}-\frac{1}{2}\alpha^{\ell}N_{\ell,Jj,Ji}\\
&\quad-\omega_{\lambda i}J^{k}{}_{j}\nabla^{\ell}N_{\ell k}{}^{\lambda}\Big\} \qquad (3.53)
\end{aligned}
$$

Note that, the first two terms are zero because $N_{\ell\lambda}{}^{\lambda}=0$; the next three terms also adds up to 0 upon symmetrization in i and j. Indeed, the last term is also zero upon symmetrization in i and j because it is antisymmetric about i and j as

$$
\begin{aligned}
\omega_{\lambda i}J^{k}{}_{j}\nabla^{\ell}N_{\ell k}{}^{\lambda}&=\omega_{\lambda i}g^{kp}J_{pj}\nabla^{\ell}N_{\ell k}{}^{\lambda}\\
&=\omega_{\lambda i}\omega_{pj}\nabla^{\ell}\left(N_{\ell k}{}^{\lambda}g^{kp}\right)\\
&=\omega_{\lambda i}\omega_{pj}\nabla^{\ell}N_{\ell}{}^{p\lambda}\\
&=-\omega_{\lambda j}\omega_{pi}\nabla^{\ell}N_{\ell}{}^{p\lambda}
\end{aligned}
$$

The last identity is seen by switching indices $p \leftrightarrow \lambda$ and using the antisymmetry of N.

The next term to be considered is

$$
\begin{aligned}
(\hat{\text{VIII}}) \cdot \varphi_{iab}\omega^{ka}\omega^{pb} &= -\frac{1}{2}\alpha_{Jm}\hat{\varphi}_{j\lambda p}g^{\ell m}N_{\ell k}{}^{\lambda}\varphi_{iab}\omega^{ka}\omega^{pb} \\
&= \frac{1}{2}\alpha_{Jm}\varphi_{Jj,\lambda,p}g^{\ell m}N_{\ell k}{}^{\lambda}\varphi_{iab}\omega^{ka}\omega^{pb} \\
&= \frac{|\varphi|^2}{8}\left(\omega_{Jj,i}g_{\lambda a} + \omega_{\lambda a}g_{Jj,i} - \omega_{Jj,a}g_{\lambda i} - \omega_{\lambda i}g_{Jj,a}\right)\omega^{ka}\alpha_{Jm}N^m{}_k{}^{\lambda} \\
&= \frac{|\varphi|^2}{8}\left(-g_{ij}J^k{}_{\lambda} - \delta^k{}_{\lambda}\omega_{ji} + J^k{}_j g_{\lambda i} + \omega_{\lambda i}\delta^k{}_j\right)\alpha_{Jm}N^m{}_k{}^{\lambda} \\
&= \frac{|\varphi|^2}{8}\left(-g_{ij}\alpha_{Jm}N^m{}_{J\lambda}{}^{\lambda} - \alpha_{Jm}N^m{}_{\lambda}{}^{\lambda} + \alpha_{Jm}N^m{}_{Jj,i} - \alpha_{Jm}N^m{}_{j,Ji}\right)
\end{aligned}
$$

Using the fact that N is a tensor of type (0, 2), we readily see that each of these terms reduces to 0.

In summary, the contribution of the Laplacian term is given by

Lemma 3 *We have*

$$
\left(g^{\ell m}\nabla_m\nabla_\ell\varphi_{jkp}\right)\varphi_{iab}\omega^{ka}\omega^{pb} + (i \leftrightarrow j) = |\varphi|^2\left\{\nabla_\mu\alpha^\mu + |N|^2\right\}g_{ij}.
$$

3.2.4 Contributions of the Curvature Terms

Turning next to the curvature contributions, we write

$$
\begin{aligned}
g^{\ell m}[\nabla_m, \nabla_j]\varphi_{kp\ell} &= -g^{\ell m}\left(R_{mj}{}^{\lambda}{}_k\varphi_{\lambda p\ell} + R_{mj}{}^{\lambda}{}_p\varphi_{k\lambda\ell} + R_{mj}{}^{\lambda}{}_\ell\varphi_{kp\lambda}\right) \\
&= -R^\ell{}_j{}^\lambda{}_k\varphi_{\lambda p\ell} - R^\ell{}_j{}^\lambda{}_p\varphi_{k\lambda\ell} + R_j{}^\lambda\varphi_{kp\lambda} \\
&= -R^\ell{}_j{}^\lambda{}_k\varphi_{\lambda p\ell} + R^\ell{}_j{}^\lambda{}_p\varphi_{\lambda k\ell} + R_j{}^\lambda\varphi_{kp\lambda}
\end{aligned} \tag{3.54}
$$

We consider for the moment only the contribution of the last term.

$$
R_j{}^\lambda\,\varphi_{kp\lambda}\,\varphi_{iab}\omega^{ka}\omega^{pb} = -|\varphi|^2R_j{}^\lambda\,g_{\lambda i} = -|\varphi|^2R_{ji}. \tag{3.55}
$$

The next curvature contribution is similar

$$
\begin{aligned}
g^{\ell m}[\nabla_m, \nabla_p]\varphi_{jk\ell} &= -g^{\ell m}\left(R_{mp}{}^\lambda{}_j\varphi_{\lambda k\ell} + R_{mp}{}^\lambda{}_k\varphi_{j\lambda\ell} + R_{mp}{}^\lambda{}_\ell\varphi_{jk\lambda}\right) \\
&= -R^\ell{}_p{}^\lambda{}_j\varphi_{\lambda k\ell} + R^\ell{}_p{}^\lambda{}_k\varphi_{\lambda j\ell} + R_p{}^\lambda\varphi_{jk\lambda}
\end{aligned} \tag{3.56}
$$

and the corresponding last term gives

$$
\begin{aligned}
R_p{}^\lambda\varphi_{jk\lambda}\,\varphi_{iab}\omega^{ka}\omega^{pb} &= \frac{|\varphi|^2}{4}R_p{}^\lambda(\omega_{ji}g_{\lambda b} + \omega_{\lambda b}g_{ji} - \omega_{jb}g_{\lambda i} - \omega_{\lambda i}g_{jb})\omega^{pb} \\
&= \frac{|\varphi|^2}{4}R_p{}^\lambda(\omega_{ji}J^p{}_\lambda - \delta^p{}_\lambda g_{ji} + \delta^p{}_j g_{\lambda i} - \omega_{\lambda i}J^p{}_j)
\end{aligned}
$$

$$= \frac{|\varphi|^2}{4}(-R\, g_{ji} + R_{ji} + R_{Jj,Ji}) \tag{3.57}$$

where we have dropped the term proportional to ω_{ji} since it symmetrizes to 0. The remaining terms gives an identical contribution. Indeed,

$$\begin{aligned} g^{\ell m}[\nabla_m, \nabla_k]\varphi_{pj\ell} &= -g^{\ell m}(R_{mk}{}^{\lambda}{}_{p}\varphi_{\lambda j\ell} + R_{mk}{}^{\lambda}{}_{j}\varphi_{p\lambda\ell} + R_{mk}{}^{\lambda}{}_{\ell}\varphi_{pj\lambda}) \\ &= -R^{\ell}{}_{k}{}^{\lambda}{}_{p}\varphi_{\lambda j\ell} + R^{\ell}{}_{k}{}^{\lambda}{}_{j}\varphi_{\lambda p\ell} + R_k{}^{\lambda}\,\varphi_{pj\lambda}. \end{aligned} \tag{3.58}$$

Considering for the moment only the contribution of the last term, we can write

$$\begin{aligned} R_k{}^{\lambda}\,\varphi_{pj\lambda}\varphi_{iab}\omega^{ka}\omega^{pb} &= R_k{}^{\lambda}\left(\omega^{pb}\varphi_{pj\lambda}\varphi_{bia}\right)\omega^{ka} \\ &= \frac{|\varphi|^2}{4}R_k{}^{\lambda}\left(\omega_{ji}g_{\lambda a} + \omega_{\lambda a}g_{ji} - \omega_{ja}g_{\lambda i} - \omega_{\lambda i}g_{ja}\right)\omega^{ka} \\ &= \frac{|\varphi|^2}{4}R_k{}^{\lambda}\left(\omega_{ji}J^k{}_{\lambda} - \delta^k{}_{\lambda}g_{ji} + \delta^k{}_{j}g_{\lambda i} - \omega_{\lambda i}J^k{}_{j}\right) \\ &= \frac{|\varphi|^2}{4}\left(-R\,g_{ji} + R_{ji} + R_{Jj,Ji}\right) \end{aligned} \tag{3.59}$$

where we have dropped the antisymmetric term ω_{ji} just as before. Assembling all the terms, we have proved the following lemma

Lemma 4 *We have the following formula*

$$\begin{aligned} &g^{\ell m}([\nabla_m, \nabla_j]\varphi_{kp\ell} + [\nabla_m, \nabla_k]\varphi_{pj\ell} + [\nabla_m, \nabla_p]\varphi_{jk\ell})\varphi_{iab}\omega^{ka}\omega^{pb} + (i \leftrightarrow j) \\ &= -2|\varphi|^2\, R_{ji} - |\varphi|^2 R\, g_{ij} + |\varphi|^2 R_{ij} + |\varphi|^2 R_{Jj,Ji} + F \end{aligned} \tag{3.60}$$

where the term F is given by

$$\begin{aligned} F = &\Big\{ \left(R^{\ell}{}_{j}{}^{\lambda}{}_{p} - R^{\ell}{}_{p}{}^{\lambda}{}_{j}\right)\varphi_{\lambda k\ell} + \left(-R^{\ell}{}_{j}{}^{\lambda}{}_{k} + R^{\ell}{}_{k}{}^{\lambda}{}_{j}\right)\varphi_{\lambda p\ell} \\ &+ (R^{\ell}{}_{p}{}^{\lambda}{}_{k} - R^{\ell}{}_{k}{}^{\lambda}{}_{p})\varphi_{\lambda j\ell}\Big\}\varphi_{iab}\omega^{ka}\omega^{pb} + (i \leftrightarrow j) \end{aligned} \tag{3.61}$$

3.2.5 Evaluation of the Term *F*

We begin with

$$\begin{aligned} &\left(R^{\ell}{}_{j}{}^{\lambda}{}_{p} - R^{\ell}{}_{p}{}^{\lambda}{}_{j}\right)\varphi_{\lambda k\ell}\varphi_{iab}\omega^{ka}\omega^{pb} \\ &= \frac{|\varphi|^2}{4}\left(R^{\ell}{}_{j}{}^{\lambda}{}_{p} - R^{\ell}{}_{p}{}^{\lambda}{}_{j}\right)(\omega_{\lambda i}g_{\ell b} + \omega_{\ell b}g_{\lambda i} - \omega_{\lambda b}g_{\ell i} - \omega_{\ell i}g_{\lambda b})\,\omega^{pb} \\ &= \frac{|\varphi|^2}{4}\left(R^{\ell}{}_{j}{}^{\lambda}{}_{p} - R^{\ell}{}_{p}{}^{\lambda}{}_{j}\right)\left(\omega_{\lambda i}J^p{}_{\ell} - \delta^p{}_{\ell}g_{\lambda i} + \delta^p{}_{\lambda}g_{\ell i} - \omega_{\ell i}J^p{}_{\lambda}\right) \end{aligned}$$

$$
\begin{aligned}
&= \frac{|\varphi|^2}{4}\left(-R^{Jp}{}_{j,Ji,p} + R^{\ell}{}_{J\ell Ji,j}\right) - \frac{|\varphi|^2}{4}\left(-R_{ji} + R^{\ell}{}_{\ell ij}\right) \\
&\quad + \frac{|\varphi|^2}{4}(R_{ij}{}^{\lambda}{}_{\lambda} - R_{i\lambda}{}^{\lambda}{}_{j}) + \frac{|\varphi|^2}{4}(R_{Ji,j}{}^{\lambda}{}_{J\lambda} - R_{Ji,J\lambda}{}^{\lambda}{}_{j}) \qquad (3.62)
\end{aligned}
$$

This reduces to

$$
\begin{aligned}
&\left(R^{\ell}{}_{j}{}^{\lambda}{}_{p} - R^{\ell}{}_{p}{}^{\lambda}{}_{j}\right)\varphi_{\lambda k\ell}\varphi_{iab}\omega^{ka}\omega^{pb} \\
&= \frac{|\varphi|^2}{4}\left(-R^{Jp}{}_{j,Ji,p} + R^{\ell}{}_{J\ell,Ji,j} + R_{Ji,j}{}^{\lambda}{}_{J\lambda} - R_{Ji,J\lambda}{}^{\lambda}{}_{j}\right) + \frac{|\varphi|^2}{2}R_{ij}.
\end{aligned}
$$

Using the symmetries of the Riemann curvature tensor, we simplify this to

$$
\left(R^{\ell}{}_{j}{}^{\lambda}{}_{p} - R^{\ell}{}_{p}{}^{\lambda}{}_{j}\right)\varphi_{\lambda k\ell}\varphi_{iab}\omega^{ka}\omega^{pb} = \frac{|\varphi|^2}{2}\left(-R_{Ji,J\lambda}{}^{\lambda}{}_{j} + R_{Ji,j}{}^{\lambda}{}_{J\lambda}\right) + \frac{|\varphi|^2}{2}R_{ij}.
$$

We work out the next term, which after relabeling is

$$
\begin{aligned}
&\left(-R^{\ell}{}_{j}{}^{\lambda}{}_{k} + R^{\ell}{}_{k}{}^{\lambda}{}_{j}\right)\varphi_{\lambda p\ell}\varphi_{iab}\omega^{ka}\omega^{pb} \\
&= \left(R^{\ell}{}_{j}{}^{\lambda}{}_{k} - R^{\ell}{}_{k}{}^{\lambda}{}_{j}\right)\varphi_{\lambda\ell p}\varphi_{iab}\omega^{ka}\omega^{pb} \\
&= \left(R^{\ell}{}_{j}{}^{\lambda}{}_{p} - R^{\ell}{}_{p}{}^{\lambda}{}_{j}\right)\varphi_{\lambda\ell k}\varphi_{iba}\omega^{pb}\omega^{ka}, \qquad (3.63)
\end{aligned}
$$

and is therefore identical to the previous term,

$$
\begin{aligned}
&\left(-R^{\ell}{}_{j}{}^{\lambda}{}_{k} + R^{\ell}{}_{k}{}^{\lambda}{}_{j}\right)\varphi_{\lambda p\ell}\varphi_{iab}\omega^{ka}\omega^{pb} \\
&= \frac{|\varphi|^2}{2}\left(-R_{Ji,J\lambda}{}^{\lambda}{}_{j} + R_{Ji,j}{}^{\lambda}{}_{J\lambda}\right) + \frac{|\varphi|^2}{2}R_{ij}. \qquad (3.64)
\end{aligned}
$$

We work out the final term. We start with

$$
\left(R^{\ell}{}_{p}{}^{\lambda}{}_{k} - R^{\ell}{}_{k}{}^{\lambda}{}_{p}\right)\varphi_{\lambda j\ell}\varphi_{iab}\omega^{ka}\omega^{pb} = -R_{pk}{}^{\lambda\ell}\varphi_{\lambda j\ell}\varphi_{iab}\omega^{ka}\omega^{pb} \qquad (3.65)
$$

by the Bianchi identity $R^{\ell}{}_{p}{}^{\lambda}{}_{k} + R_{pk}{}^{\lambda\ell} + R_{k}{}^{\ell\lambda}{}_{p} = 0$. Applying the identity (2.21) gives

$$
\begin{aligned}
-R_{pk}{}^{\lambda\ell}\varphi_{\lambda j\ell}\varphi_{iab}\omega^{ka}\omega^{pb} &= \left(-R_{pk}{}^{J\lambda,J\ell} + B_{kp}{}^{\lambda\ell}\right)\varphi_{\lambda j\ell}\varphi_{iab}\omega^{ka}\omega^{pb} \\
&= -R_{pk}{}^{\lambda\ell}\varphi_{J\lambda,j,J\ell}\varphi_{iab}\omega^{ka}\omega^{pb} + B_{kp}{}^{\lambda\ell}\varphi_{\lambda j\ell}\varphi_{iab}\omega^{ka}\omega^{pb} \\
&= R_{pk}{}^{\lambda\ell}\varphi_{\lambda,j,\ell}\varphi_{iab}\omega^{ka}\omega^{pb} + B_{kp}{}^{\lambda\ell}\varphi_{\lambda j\ell}\varphi_{iab}\omega^{ka}\omega^{pb}
\end{aligned} \qquad (3.66)
$$

Therefore

$$- R_{pk}{}^{\lambda\ell}\varphi_{\lambda j\ell}\varphi_{iab}\omega^{ka}\omega^{pb} = \frac{1}{2}B_{kp}{}^{\lambda\ell}\varphi_{\lambda j\ell}\varphi_{iab}\omega^{ka}\omega^{pb} \tag{3.67}$$

and hence

$$(R^{\ell}{}_{p}{}^{\lambda}{}_{k} - R^{\ell}{}_{k}{}^{\lambda}{}_{p})\varphi_{\lambda j\ell}\varphi_{iab}\omega^{ka}\omega^{pb} = \frac{1}{2}B_{kp}{}^{\lambda\ell}\varphi_{\lambda j\ell}\varphi_{iab}\omega^{ka}\omega^{pb} \tag{3.68}$$

By definition of B,

$$B_{kp}{}^{\lambda\ell} = -2\mathfrak{D}_k N_p{}^{\lambda\ell} + 2\mathfrak{D}_p N_k{}^{\lambda\ell} - 2N^{\alpha}{}_{kp}N_{\alpha}{}^{\lambda\ell}. \tag{3.69}$$

Therefore

$$\begin{aligned}\left(R^{\ell}{}_{p}{}^{\lambda}{}_{k} - R^{\ell}{}_{k}{}^{\lambda}{}_{p}\right)\varphi_{\lambda j\ell}\varphi_{iab}\omega^{ka}\omega^{pb} &= -\mathfrak{D}_k N_p{}^{\lambda\ell}\varphi_{\lambda j\ell}\varphi_{iab}\omega^{ka}\omega^{pb} \\ &\quad +\mathfrak{D}_p N_k{}^{\lambda\ell}\varphi_{\lambda j\ell}\varphi_{iab}\omega^{ka}\omega^{pb} \\ &\quad -N^{\alpha}{}_{kp}N_{\alpha}{}^{\lambda\ell}\varphi_{\lambda j\ell}\varphi_{iab}\omega^{ka}\omega^{pb} \\ &= -2\mathfrak{D}_k N_p{}^{\lambda\ell}\varphi_{\lambda j\ell}\varphi_{iab}\omega^{ka}\omega^{pb} \\ &\quad -N^{\alpha}{}_{kp}N_{\alpha}{}^{\lambda\ell}\varphi_{\lambda j\ell}\varphi_{iab}\omega^{ka}\omega^{pb}.\end{aligned} \tag{3.70}$$

We start with the last term. By the Bianchi identity $N_{ijk} + N_{kij} + N_{jki} = 0$,

$$N^{\alpha}{}_{kp}N_{\alpha}{}^{\lambda\ell}\varphi_{\lambda j\ell}\varphi_{iab}\omega^{ka}\omega^{pb} = -N^{\alpha}{}_{kp}\left[N^{\ell}{}_{\alpha}{}^{\lambda}\varphi_{\lambda j\ell} + N^{\lambda\ell}{}_{\alpha}\varphi_{\lambda j\ell}\right]\varphi_{iab}\omega^{ka}\omega^{pb} \tag{3.71}$$

Recall the identity (2.19) for switching indices on contractions of N and φ. Therefore

$$\begin{aligned}N^{\alpha}{}_{kp}N_{\alpha}{}^{\lambda\ell}\varphi_{\lambda j\ell}\varphi_{iab}\omega^{ka}\omega^{pb} &= N^{\alpha}{}_{kp}\left[N^{\ell}{}_{\lambda j}\varphi_{\ell\alpha}{}^{\lambda} + N^{\lambda}{}_{j\ell}\varphi_{\lambda}{}^{\ell}{}_{\alpha}\right]\varphi_{iab}\omega^{ka}\omega^{pb} \\ &= -2N^{\alpha}{}_{kp}N^{\ell\lambda}{}_{j}\varphi_{\alpha\ell\lambda}\varphi_{iab}\omega^{ka}\omega^{pb}.\end{aligned} \tag{3.72}$$

Applying the identity (2.19) again,

$$N^{\alpha}{}_{kp}N_{\alpha}{}^{\lambda\ell}\varphi_{\lambda j\ell}\varphi_{iab}\omega^{ka}\omega^{pb} = 2N^{\alpha}{}_{\ell\lambda}N^{\ell\lambda}{}_{j}\varphi_{\alpha kp}\varphi_{iab}\omega^{ka}\omega^{pb}. \tag{3.73}$$

We can now apply the bilinear identities (2.11), so that

$$\begin{aligned}N^{\alpha}{}_{kp}N_{\alpha}{}^{\lambda\ell}\varphi_{\lambda j\ell}\varphi_{iab}\omega^{ka}\omega^{pb} &= -2|\varphi|^2 N^{\alpha}{}_{\ell\lambda}N^{\ell\lambda}{}_{j}g_{\alpha i} \\ &= -2|\varphi|^2 N_{i\ell\lambda}N^{\ell\lambda}{}_{j}.\end{aligned} \tag{3.74}$$

Next, we need to handle the $\mathfrak{D}N$ terms in (3.70). By the Bianchi identity $N_{ijk}+N_{kij}+N_{jki}=0$, we have

$$\begin{aligned} -2\mathfrak{D}_k N_p{}^{\lambda\ell}\varphi_{\lambda j\ell}\varphi_{iab}\omega^{ka}\omega^{pb} &= 2\mathfrak{D}_k N^{\ell}{}_p{}^{\lambda}\varphi_{\lambda j\ell}\varphi_{iab}\omega^{ka}\omega^{pb} \\ &\quad +2\mathfrak{D}_k N^{\lambda\ell}{}_p\varphi_{\lambda j\ell}\varphi_{iab}\omega^{ka}\omega^{pb} \end{aligned} \tag{3.75}$$

This is

$$-2\mathfrak{D}_k N_p{}^{\lambda\ell}\varphi_{\lambda j\ell}\varphi_{iab}\omega^{ka}\omega^{pb} = -4\mathfrak{D}_k N^{\ell\lambda}{}_p\varphi_{\ell\lambda j}\varphi_{iab}\omega^{ka}\omega^{pb}. \tag{3.76}$$

To apply the bilinear identities (2.11), we will need to switch some indices.

Lemma 5

$$\mathfrak{D}_k N^p{}_{ij}\varphi_{p\lambda l} = -\mathfrak{D}_k N^p{}_{\lambda l}\varphi_{pij} + N^p{}_{\lambda,Jl}\alpha_{Jk}\varphi_{pij}. \tag{3.77}$$

Proof Differentiating identity (2.19) gives

$$\mathfrak{D}_k N^p{}_{ij}\varphi_{p\lambda l} + N^p{}_{ij}\mathfrak{D}_k\varphi_{p\lambda l} = -\mathfrak{D}_k N^p{}_{\lambda l}\varphi_{pij} - N^p{}_{\lambda l}\mathfrak{D}_k\varphi_{pij}. \tag{3.78}$$

Using the formula (2.7), we obtain

$$\begin{aligned} \mathfrak{D}_k N^p{}_{ij}\varphi_{p\lambda l} &= -\mathfrak{D}_k N^p{}_{\lambda l}\varphi_{pij} + \frac{1}{2}N^p{}_{ij}\alpha_k\varphi_{p\lambda l} + \frac{1}{2}N^p{}_{ij}\alpha_{Jk}\hat{\varphi}_{p\lambda l} \\ &\quad +\frac{1}{2}N^p{}_{\lambda l}\alpha_k\varphi_{pij} + \frac{1}{2}N^p{}_{\lambda l}\alpha_{Jk}\hat{\varphi}_{pij}. \end{aligned} \tag{3.79}$$

Using (2.19) and $\hat{\varphi}_{p\lambda l} = -\varphi_{pJ\lambda,l} = -\varphi_{Jp,\lambda l}$, we simplify this to

$$\mathfrak{D}_k N^p{}_{ij}\varphi_{p\lambda l} = -\mathfrak{D}_k N^p{}_{\lambda l}\varphi_{pij} - \frac{1}{2}N^p{}_{ij}\alpha_{Jk}\varphi_{p,J\lambda,l} - \frac{1}{2}N^p{}_{\lambda l}\alpha_{Jk}\varphi_{Jp,ij}. \tag{3.80}$$

Using (2.19) again and $N^{Jp}{}_{\lambda l} = -N^p{}_{\lambda,Jl}$, we obtain the desired identity. □

Applying now this lemma to (3.76), we find

$$-2\mathfrak{D}_k N_p{}^{\lambda\ell}\varphi_{\lambda j\ell}\varphi_{iab}\omega^{ka}\omega^{pb} = \left(4\mathfrak{D}_k N^{\ell\lambda}{}_j - 4N^{\ell\lambda}{}_{Jj}\alpha_{Jk}\right)\varphi_{\ell\lambda p}\varphi_{iab}\omega^{ka}\omega^{pb} \tag{3.81}$$

We can now use the bilinear identities

$$\begin{aligned} &-2\mathfrak{D}_k N_p{}^{\lambda\ell}\varphi_{\lambda j\ell}\varphi_{iab}\omega^{ka}\omega^{pb} \\ &\quad = |\varphi|^2(\mathfrak{D}_k N^{\ell\lambda}{}_j - N^{\ell\lambda}{}_{Jj}\alpha_{Jk})(\omega_{\ell i}g_{\lambda a} - \omega_{\lambda i}g_{\ell a} - \omega_{\ell a}g_{\lambda i} + \omega_{\lambda a}g_{\ell i})\omega^{ka} \\ &\quad = |\varphi|^2(\mathfrak{D}_k N^{\ell\lambda}{}_j - N^{\ell\lambda}{}_{Jj}\alpha_{Jk})(\omega_{\ell i}J^k{}_\lambda - \omega_{\lambda i}J^k{}_\ell + \delta^k{}_\ell g_{\lambda i} - \delta^k{}_\lambda g_{\ell i}) \end{aligned}$$

$$= |\varphi|^2(-\mathfrak{D}_k N_{Ji}{}^{Jk}{}_j + N_{Ji}{}^{Jk}{}_{Jj}\alpha_{Jk}) + |\varphi|^2(\mathfrak{D}_k N^{Jk}{}_{Ji,j} - N^{Jk}{}_{Ji,Jj}\alpha_{Jk})$$
$$+|\varphi|^2(\mathfrak{D}_k N^k{}_{ij} - N^k{}_{i,Jj}\alpha_{Jk}) + |\varphi|^2(-\mathfrak{D}_k N_i{}^k{}_j + N_i{}^k{}_{Jj}\alpha_{Jk}).$$

This simplifies to

$$\begin{aligned} &-2\mathfrak{D}_k N_p{}^{\lambda\ell}\varphi_{\lambda j\ell}\varphi_{iab}\omega^{ka}\omega^{pb} \\ &\quad = |\varphi|^2(-2\mathfrak{D}_k N_i{}^k{}_j + 2\mathfrak{D}_k N^k{}_{ij} + 2N_i{}^k{}_j\alpha_k - 2N^k{}_{ij}\alpha_k). \end{aligned} \tag{3.82}$$

Substituting (3.74) and (3.82) into (3.70),

$$\begin{aligned} &\left(R^\ell{}_p{}^\lambda{}_k - R^\ell{}_k{}^\lambda{}_p\right)\varphi_{\lambda j\ell}\varphi_{iab}\omega^{ka}\omega^{pb} \\ &= |\varphi|^2(-2\mathfrak{D}_k N_i{}^k{}_j + 2\mathfrak{D}_k N^k{}_{ij} + 2N_i{}^k{}_j\alpha_k - 2N^k{}_{ij}\alpha_k) + 2|\varphi|^2 N_{i\ell\lambda}N^{\ell\lambda}{}_j \end{aligned} \tag{3.83}$$

By the Bianchi identity,

$$\begin{aligned} 2|\varphi|^2 N_{i\ell\lambda}N^{\ell\lambda}{}_j &= 2|\varphi|^2(-N_{\lambda i\ell} - N_{\ell\lambda i})N^{\ell\lambda}{}_j \\ &= 2|\varphi|^2 N_{\lambda\ell i}N^{\ell\lambda}{}_j - 2|\varphi|^2 N_{\ell\lambda i}N^{\ell\lambda}{}_j \end{aligned} \tag{3.84}$$

and hence

$$\begin{aligned} \left(R^\ell{}_p{}^\lambda{}_k - R^\ell{}_k{}^\lambda{}_p\right)\varphi_{\lambda j\ell}\varphi_{iab}\omega^{ka}\omega^{pb} &= |\varphi|^2(-2\mathfrak{D}_k N_i{}^k{}_j + 2\mathfrak{D}_k N^k{}_{ij} + 2N_i{}^k{}_j\alpha_k - 2N^k{}_{ij}\alpha_k) \\ &\quad +2|\varphi|^2(N_-^2)_{ij} - 2|\varphi|^2(N_+^2)_{ij} \end{aligned}$$

The result is

$$\begin{aligned} F &= |\varphi|^2\Big\{(-R_{Ji,J\lambda}{}^\lambda{}_j - R_{Jj,J\lambda}{}^\lambda{}_i) + (R_{Ji,j}{}^\lambda{}_{J\lambda} + R_{Jj,i}{}^\lambda{}_{J\lambda}) + 2R_{ij} \\ &\quad -2(\mathfrak{D}_k N_i{}^k{}_j + \mathfrak{D}_k N_j{}^k{}_i) + 2(N_i{}^k{}_j + N_j{}^k{}_i)\alpha_k + 4(N_-^2)_{ij} - 4(N_+^2)_{ij}\Big\} \end{aligned} \tag{3.85}$$

Lemma 6 *We have the following formula*

$$\begin{aligned} &g^{\ell m}([\nabla_m,\nabla_j]\varphi_{kp\ell} + [\nabla_m,\nabla_k]\varphi_{pj\ell} + [\nabla_m,\nabla_p]\varphi_{jk\ell})\varphi_{iab}\omega^{ka}\omega^{pb} + (i\leftrightarrow j) \\ &= |\varphi|^2\Big\{-2R_{ji} - Rg_{ij} + R_{ij} + R_{Jj,Ji} \\ &\quad -(R_{Ji,J\lambda}{}^\lambda{}_j + R_{Jj,J\lambda}{}^\lambda{}_i) + (R_{i,Jj}{}^\lambda{}_{J\lambda} + R_{j,Ji}{}^\lambda{}_{J\lambda}) + 2R_{ij} \\ &\quad -2(\mathfrak{D}_k N_i{}^k{}_j + \mathfrak{D}_k N_j{}^k{}_i) + 2(N_i{}^k{}_j + N_j{}^k{}_i)\alpha_k + 4(N_-^2)_{ij} - 4(N_+^2)_{ij}\Big\} \end{aligned} \tag{3.86}$$

3.2.6 Contributions of the Curvature Terms, Continued

We now simplify Lemma 6 by applying identities for the action of J on the Riemann curvature tensor. We start with the terms

$$- R_{Ji,J\lambda}{}^{\lambda}{}_{j} - R_{Jj,J\lambda}{}^{\lambda}{}_{i} \tag{3.87}$$

which can be manipulated using the relation (2.21) into

$$\begin{aligned} - R_{Ji,J\lambda}{}^{\lambda}{}_{j} - R_{Jj,J\lambda}{}^{\lambda}{}_{i} &= -R_{j}{}^{\lambda}{}_{J\lambda,Ji} - R_{i}{}^{\lambda}{}_{J\lambda,Jj} \\ &= -R_{j}{}^{\lambda}{}_{\lambda i} - R_{i}{}^{\lambda}{}_{\lambda j} - B^{\lambda}{}_{j\lambda i} - B^{\lambda}{}_{i\lambda j} \\ &= 2R_{ij} - B^{\lambda}{}_{j\lambda i} - B^{\lambda}{}_{i\lambda j}. \end{aligned} \tag{3.88}$$

Next, we have the terms

$$R_{i,Jj}{}^{\lambda}{}_{J\lambda} + R_{j,Ji}{}^{\lambda}{}_{J\lambda}. \tag{3.89}$$

By the Bianchi identity,

$$\begin{aligned} R_{i,Jj}{}^{\lambda}{}_{J\lambda} + (i \leftrightarrow j) &= -R_{j,J\lambda}{}^{\lambda}{}_{Ji} - R_{J\lambda,Ji}{}^{\lambda}{}_{j} + (i \leftrightarrow j) \\ &= -R_{j\lambda}{}^{J\lambda}{}_{Ji} - R_{j}{}^{\lambda}{}_{Ji,J\lambda} + (i \leftrightarrow j) \\ &= g^{\lambda\mu} R_{j,\lambda,J\mu,Ji} - R_{j}{}^{\lambda}{}_{Ji,J\lambda} + (i \leftrightarrow j) \end{aligned} \tag{3.90}$$

Using the relation (2.21),

$$\begin{aligned} R_{i,Jj}{}^{\lambda}{}_{J\lambda} + (i \leftrightarrow j) &= g^{\lambda\mu} R_{j,\lambda,\mu,i} - R_{j}{}^{\lambda}{}_{i,\lambda} + g^{\lambda\mu} B_{\lambda,j,\mu,i} - B^{\lambda}{}_{ji\lambda} + (i \leftrightarrow j) \\ &= -2R_{ij} - 2R_{ij} + \{B^{\lambda}{}_{j\lambda i} - B^{\lambda}{}_{ji\lambda} + B^{\lambda}{}_{i\lambda j} - B^{\lambda}{}_{ij\lambda}\} \end{aligned} \tag{3.91}$$

Therefore

$$R_{Ji,j}{}^{\lambda}{}_{J\lambda} + R_{Jj,i}{}^{\lambda}{}_{J\lambda} = -4R_{ij} + \{B^{\lambda}{}_{j\lambda i} - B^{\lambda}{}_{ji\lambda} + B^{\lambda}{}_{i\lambda j} - B^{\lambda}{}_{ij\lambda}\}. \tag{3.92}$$

The next term in Lemma 6 that we consider is $R_{Jj,Ji}$. This term becomes

$$\begin{aligned} R_{Jj,Ji} = g^{\lambda\mu} R_{\lambda,Jj,\mu,Ji} &= -g^{\lambda\mu} R_{\lambda,Jj,J\mu,i} - g^{\lambda\mu} B_{Jj,\lambda,J\mu,i} \\ &= -g^{\lambda\mu} R_{i,J\mu,Jj,\lambda} - g^{\lambda\mu} B_{Jj,\lambda,J\mu,i} \\ &= g^{\lambda\mu} R_{i,J\mu,j,J\lambda} + g^{\lambda\mu} B_{J\mu,i,j,J\lambda} - g^{\lambda\mu} B_{Jj,\lambda,J\mu,i} \end{aligned} \tag{3.93}$$

and thus

$$R_{Jj,Ji} = R_{ij} + B^{\lambda}{}_{ij\lambda} - B_{Jj}{}^{\lambda}{}_{J\lambda,i}. \tag{3.94}$$

Substituting (3.88), (3.92) and (3.94) into Lemma 6, we obtain

$$
\begin{aligned}
&g^{\ell m}([\nabla_m,\nabla_j]\varphi_{kp\ell}+[\nabla_m,\nabla_k]\varphi_{pj\ell}+[\nabla_m,\nabla_p]\varphi_{jk\ell})\varphi_{iab}\omega^{ka}\omega^{pb}+(i\leftrightarrow j)\\
&\quad=-|\varphi|^2R\,g_{ij}-2(\mathfrak{D}_kN_i{}^k{}_j+\mathfrak{D}_kN_j{}^k{}_i)+2(N_i{}^k{}_j+N_j{}^k{}_i)\alpha_k+4(N_-^2)_{ij}-4(N_+^2)_{ij}\\
&\qquad-B^\lambda{}_{ji\lambda}-B_{Jj}{}^\lambda{}_{J\lambda,i}
\end{aligned}\tag{3.95}
$$

Using the definition of B,

$$
\begin{aligned}
-B^\lambda{}_{ji\lambda}-B_{Jj}{}^\lambda{}_{J\lambda,i}&=-[-2D^\lambda N_{ji\lambda}+2D_jN^\lambda{}_{i\lambda}-2N^{\alpha\lambda}{}_jN_{\alpha i\lambda}]\\
&\quad-[-2D_{Jj}N^\lambda{}_{J\lambda,i}+2D^\lambda N_{Jj,J\lambda,i}-2N^\alpha{}_{Jj}{}^\lambda N_{\alpha,J\lambda,i}]\\
&=-4(N_+^2)_{ij}
\end{aligned}\tag{3.96}
$$

where we use the symmetries of N to get the last equality. Therefore

Lemma 7 *We have the following formula*

$$
\begin{aligned}
&g^{\ell m}([\nabla_m,\nabla_j]\varphi_{kp\ell}+[\nabla_m,\nabla_k]\varphi_{pj\ell}+[\nabla_m,\nabla_p]\varphi_{jk\ell})\varphi_{iab}\omega^{ka}\omega^{pb}+(i\leftrightarrow j)\\
&\quad=|\varphi|^2\left\{-Rg_{ij}+2\mathfrak{D}_kN_{ij}{}^k+2\mathfrak{D}_kN_{ji}{}^k+2(N_i{}^k{}_j+N_j{}^k{}_i)\alpha_k+4(N_-^2)_{ij}-8(N_+^2)_{ij}\right\}
\end{aligned}\tag{3.97}
$$

3.2.7 Bochner–Kodaira Contributions

By (3.11), we have

$$
\begin{aligned}
(-|\varphi|^2dd^\dagger\varphi)_{jkp}\varphi_{iab}\omega^{ka}\omega^{pb}&=(|\varphi|^2g^{\ell m}\nabla_m\nabla_\ell\varphi_{jkp})\varphi_{iab}\omega^{ka}\omega^{pb}\\
&\quad-|\varphi|^2(g^{\ell m}\{[\nabla_m,\nabla_j]\varphi_{kp\ell}\\
&\qquad+[\nabla_m,\nabla_k]\varphi_{pj\ell}+[\nabla_m,\nabla_p]\varphi_{jk\ell}\})\varphi_{iab}\omega^{ka}\omega^{pb}
\end{aligned}
$$

By Lemmas 3 and 7, we obtain

$$
\begin{aligned}
&(-|\varphi|^2dd^\dagger\varphi)_{jkp}\varphi_{iab}\omega^{ka}\omega^{pb}+(i\leftrightarrow j)=|\varphi|^4\{\nabla_\mu\alpha^\mu+|N|^2\}g_{ij}\\
&\quad+|\varphi|^4\left\{Rg_{ij}+2(-\mathfrak{D}_kN_{ij}{}^k-\mathfrak{D}_kN_{ji}{}^k)-2(N_i{}^k{}_j+N_j{}^k{}_i)\alpha_k-4(N_-^2)_{ij}+8(N_+^2)_{ij}\right\}
\end{aligned}
$$

Altogether,

Lemma 8 *We have the following formula*

$$
\begin{aligned}
&(-|\varphi|^2dd^\dagger\varphi)_{jkp}\varphi_{iab}\omega^{ka}\omega^{pb}+(i\leftrightarrow j)\\
&\quad=|\varphi|^4\Big\{Rg_{ij}-2\left(\mathfrak{D}_kN_{ij}{}^k+\mathfrak{D}_kN_{ji}{}^k\right)+\left(\nabla_\mu\alpha^\mu+|N|^2\right)g_{ij}-2\left(N_i{}^k{}_j+N_j{}^k{}_i\right)\alpha_k\\
&\qquad-4\left(N_-^2\right)_{ij}+8\left(N_+^2\right)_{ij}\Big\}.
\end{aligned}\tag{3.98}
$$

3.3 Other Contributions

3.3.1 Gradient Dagger

Returning to (3.3), we study the contributions of the second term $-d|\varphi|^2 \wedge d^\dagger\varphi$. We let $\alpha = -d\log|\varphi|^2$ as before, and write

$$-d|\varphi|^2 = |\varphi|^2\alpha, \quad (d^\dagger\varphi)_{kp} = -g^{\mu\beta}\nabla_\beta\varphi_{\mu kp}. \tag{3.99}$$

Since

$$\begin{aligned}(-d|\varphi|^2 \wedge d^\dagger\varphi)_{jkp} &= (-d|\varphi|^2)_j(d^\dagger\varphi)_{kp} + (-d|\varphi|^2)_p(d^\dagger\varphi)_{jk} \\ &\quad + (-d|\varphi|^2)_k(d^\dagger\varphi)_{pj}\end{aligned} \tag{3.100}$$

we have

$$\begin{aligned}&(-d|\varphi|^2 \wedge d^\dagger\varphi)_{jkp} \\ &\quad = |\varphi|^2\left(-\alpha_j g^{\mu\beta}\nabla_\beta\varphi_{\mu kp} - \alpha_p g^{\mu\beta}\nabla_\beta\varphi_{\mu jk} - \alpha_k g^{\mu\beta}\nabla_\beta\varphi_{\mu pj}\right)\end{aligned} \tag{3.101}$$

Using previous notation,

$$\nabla_\beta\varphi_{\mu kp} = \mathfrak{D}_\beta\varphi_{\mu kp} - E_{\beta;\mu kp}. \tag{3.102}$$

By the formula (2.7), we conclude

$$\nabla_\beta\varphi_{\mu kp} = -\frac{1}{2}\alpha_\beta\varphi_{\mu kp} + \frac{1}{2}\alpha_{J\beta}\varphi_{J\mu,kp} - E_{\beta;\mu kp}. \tag{3.103}$$

Therefore

$$\begin{aligned}(-d|\varphi|^2 \wedge d^\dagger\varphi)_{jkp} &= |\varphi|^2\Big(\frac{1}{2}\alpha_j g^{\mu\beta}\alpha_\beta\varphi_{\mu kp} - \frac{1}{2}\alpha_j g^{\mu\beta}\alpha_{J\beta}\varphi_{J\mu,kp} + \alpha_j g^{\mu\beta}E_{\beta;\mu kp} \\ &\quad + \frac{1}{2}\alpha_p g^{\mu\beta}\alpha_\beta\varphi_{\mu jk} - \frac{1}{2}\alpha_p g^{\mu\beta}\alpha_{J\beta}\varphi_{J\mu,jk} + \alpha_p g^{\mu\beta}E_{\beta;\mu jk} \\ &\quad + \frac{1}{2}\alpha_k g^{\mu\beta}\alpha_\beta\varphi_{\mu pj} - \frac{1}{2}\alpha_k g^{\mu\beta}\alpha_{J\beta}\varphi_{J\mu,pj} + \alpha_k g^{\mu\beta}E_{\beta;\mu pj}\Big)\end{aligned} \tag{3.104}$$

which simplifies to

$$\begin{aligned}(-d|\varphi|^2 \wedge d^\dagger\varphi)_{jkp} &= |\varphi|^2\left(\alpha_j g^{\mu\beta}E_{\beta;\mu kp} + \alpha_p g^{\mu\beta}E_{\beta;\mu jk} + \alpha_k g^{\mu\beta}E_{\beta;\mu pj}\right) \\ &:= \text{(I)} + \text{(II)} + \text{(III)}.\end{aligned} \tag{3.105}$$

We now work out the bilinears.

$$(\mathrm{I})\cdot\varphi_{iab}\omega^{ka}\omega^{pb}=|\varphi|^2\alpha_j g^{\mu\beta}(\varphi_{\lambda kp}N_{\beta\mu}{}^{\lambda}+\varphi_{\mu\lambda p}N_{\beta k}{}^{\lambda}+\varphi_{\mu k\lambda}N_{\beta p}{}^{\lambda})\varphi_{iab}\omega^{ka}\omega^{pb} \tag{3.106}$$

Since $N^{\mu}{}_{\mu}{}^{\lambda}=0$ and we can relabel $p\leftrightarrow k$ and $a\leftrightarrow b$,

$$(\mathrm{I})\cdot\varphi_{iab}\omega^{ka}\omega^{pb}=2|\varphi|^2\alpha_j g^{\mu\beta}(\varphi_{\mu\lambda p}N_{\beta k}{}^{\lambda})\varphi_{iab}\omega^{ka}\omega^{pb}. \tag{3.107}$$

By the bilinear identities

$$\begin{aligned}(\mathrm{I})\cdot\varphi_{iab}\omega^{ka}\omega^{pb}&=\frac{|\varphi|^4}{2}\alpha_j g^{\mu\beta}N_{\beta k}{}^{\lambda}(g_{\mu i}\omega_{\lambda a}-g_{\lambda i}\omega_{\mu a}-g_{\mu a}\omega_{\lambda i}+g_{\lambda a}\omega_{\mu i})\omega^{ka}\\&=\frac{|\varphi|^4}{2}\alpha_j g^{\mu\beta}N_{\beta k}{}^{\lambda}(-g_{\mu i}\delta^{k}{}_{\lambda}+g_{\lambda i}\delta^{k}{}_{\mu}-J^{k}{}_{\mu}\omega_{\lambda i}+J^{k}{}_{\lambda}\omega_{\mu i})\\&=\frac{|\varphi|^4}{2}(0+0+\alpha_j N^{Jk}{}_{k}{}^{Ji}-\alpha_j N_{Ji,J\lambda}{}^{\lambda})=0\end{aligned} \tag{3.108}$$

using the type (0, 2) and trace-free property of N. Next,

$$\begin{aligned}(\mathrm{II}+\mathrm{III})\cdot\varphi_{iab}\omega^{ka}\omega^{pb}&=2|\varphi|^2\alpha_p g^{\mu\beta}(\varphi_{\lambda jk}N_{\beta\mu}{}^{\lambda}+\varphi_{\mu\lambda k}N_{\beta j}{}^{\lambda}+\varphi_{\mu j\lambda}N_{\beta k}{}^{\lambda})\varphi_{iab}\omega^{ka}\omega^{pb}\\&=2|\varphi|^2\alpha_p(0+\varphi_{\mu\lambda k}N^{\mu}{}_{j}{}^{\lambda}+\varphi_{\mu j\lambda}N^{\mu}{}_{k}{}^{\lambda})\varphi_{iab}\omega^{ka}\omega^{pb}\end{aligned} \tag{3.109}$$

The first term is

$$\begin{aligned}2|\varphi|^2\alpha_p(\varphi_{\mu\lambda k}N^{\mu}{}_{j}{}^{\lambda})\varphi_{iab}\omega^{ka}\omega^{pb}&=-2|\varphi|^2\alpha_p N^{\mu}{}_{j}{}^{\lambda}(\varphi_{\mu\lambda k}\varphi_{iba}\omega^{ka})\omega^{pb}\\&=-\frac{|\varphi|^4}{2}\alpha_p N^{\mu}{}_{j}{}^{\lambda}(g_{\mu i}\omega_{\lambda b}-g_{\lambda i}\omega_{\mu b}-g_{\mu b}\omega_{\lambda i}+g_{\lambda b}\omega_{\mu i})\omega^{pb}\\&=-\frac{|\varphi|^4}{2}\alpha_p N^{\mu}{}_{j}{}^{\lambda}(-g_{\mu i}\delta^{p}{}_{\lambda}+g_{\lambda i}\delta^{p}{}_{\mu}-J^{p}{}_{\mu}\omega_{\lambda i}+J^{p}{}_{\lambda}\omega_{\mu i})\\&=-\frac{|\varphi|^4}{2}(-\alpha_p N_{ij}{}^{p}+\alpha_p N^{p}{}_{ji}+\alpha_p N^{Jp}{}_{j,Ji}-\alpha_p N_{Ji,j}{}^{Jp})\\&=|\varphi|^4(\alpha_p N_{ij}{}^{p}-\alpha_p N^{p}{}_{ji})\end{aligned} \tag{3.110}$$

For the second term, we use the identity (2.19) to obtain

$$\begin{aligned}2|\varphi|^2\alpha_p(\varphi_{\mu j\lambda}N^{\mu}{}_{k}{}^{\lambda})\varphi_{iab}\omega^{ka}\omega^{pb}&=2|\varphi|^2\alpha_p(-\varphi_{\mu k\lambda}N^{\mu}{}_{j}{}^{\lambda})\varphi_{iab}\omega^{ka}\omega^{pb}\\&=-2|\varphi|^2\alpha_p N^{\mu}{}_{j}{}^{\lambda}\varphi_{\mu\lambda k}\varphi_{iba}\omega^{ka}\omega^{pb}.\end{aligned} \tag{3.111}$$

This term is identical to the one above. Therefore

$$(\mathrm{II}+\mathrm{III})\cdot\varphi_{iab}\omega^{ka}\omega^{pb}=2|\varphi|^4(\alpha_p N_{ij}{}^{p}-\alpha_p N^{p}{}_{ji}) \tag{3.112}$$

Altogether,

$$(-d|\varphi|^2 \wedge d^\dagger \varphi)_{jkp}\varphi_{iab}\omega^{ka}\omega^{pb} = 2|\varphi|^4(\alpha_p N_{ij}{}^p - \alpha_p N^p{}_{ji}). \tag{3.113}$$

Therefore

$$(-d|\varphi|^2 \wedge d^\dagger \varphi)_{jkp}\varphi_{iab}\omega^{ka}\omega^{pb} + (i \leftrightarrow j) = 2|\varphi|^4(\alpha_p N_{ij}{}^p - \alpha_p N^p{}_{ji} + \alpha_p N_{ji}{}^p - \alpha_p N^p{}_{ij}). \tag{3.114}$$

By the Bianchi identity $N^p{}_{ij} + N_j{}^p{}_i + N_{ij}{}^p = 0$, and hence

$$\begin{aligned}
&(-d|\varphi|^2 \wedge d^\dagger \varphi)_{jkp}\varphi_{iab}\omega^{ka}\omega^{pb} + (i \leftrightarrow j)\\
&= 2|\varphi|^4 \left(\alpha_p N_{ij}{}^p - \alpha_p(-N_i{}^p{}_j - N_{ji}{}^p) + \alpha_p N_{ji}{}^p - \alpha_p(-N_j{}^p{}_i - N_{ij}{}^p)\right)\\
&= 2|\varphi|^4\alpha_p(N_{ij}{}^p + N_i{}^p{}_j + N_{ji}{}^p + N_{ji}{}^p + N_j{}^p{}_i + N_{ij}{}^p)
\end{aligned} \tag{3.115}$$

Thus we have established the following lemma:

Lemma 9

$$(-d|\varphi|^2 \wedge d^\dagger \varphi)_{jkp}\varphi_{iab}\omega^{ka}\omega^{pb} + (i \leftrightarrow j) = -2|\varphi|^4\alpha_p(N_j{}^p{}_i + N_i{}^p{}_j). \tag{3.116}$$

3.3.2 Interior Product

Returning to (3.3), we study the contributions of the third term $d(\iota_{\nabla|\varphi|^2}\varphi)$. We can write

$$(\iota_{\nabla|\varphi|^2}\varphi)_{kp} = g^{\mu\nu}(\partial_\nu|\varphi|^2)\varphi_{\mu kp} = -|\varphi|^2 g^{\mu\nu}\alpha_\nu\varphi_{\mu kp} \tag{3.117}$$

since $\alpha_i = -\partial_i \log|\varphi|^2$, or $\partial_j|\varphi|^2 = -|\varphi|^2\alpha_j$. Next,

$$d(\iota_{\nabla|\varphi|^2}\varphi)_{jkp} = \nabla_j(\iota_{\nabla|\varphi|^2}\varphi)_{kp} + \nabla_p(\iota_{\nabla|\varphi|^2}\varphi)_{jk} + \nabla_k(\iota_{\nabla|\varphi|^2}\varphi)_{pj}. \tag{3.118}$$

We start with

$$\nabla_j(\iota_{\nabla|\varphi|^2}\varphi)_{kp} = |\varphi|^2\alpha_j\alpha^\mu\varphi_{\mu kp} - |\varphi|^2\nabla_j\alpha^\mu\varphi_{\mu kp} - |\varphi|^2\alpha^\mu\nabla_j\varphi_{\mu kp} \tag{3.119}$$

Since

$$\nabla_j\varphi_{\mu kp} = -\frac{1}{2}\alpha_j\varphi_{\mu kp} + \frac{1}{2}\alpha_{Jj}\varphi_{J\mu,k,p} - E_{j;\mu kp}, \tag{3.120}$$

we have

$$\begin{aligned}\nabla_j(\iota_{\nabla|\varphi|^2}\varphi)_{kp} &= \frac{3}{2}|\varphi|^2\alpha_j\alpha^\mu\varphi_{\mu kp} - |\varphi|^2\nabla_j\alpha^\mu\varphi_{\mu kp}\\ &\quad -\frac{1}{2}|\varphi|^2\alpha^\mu\alpha_{Jj}\varphi_{J\mu,k,p} + |\varphi|^2\alpha^\mu E_{j;\mu kp}.\end{aligned} \tag{3.121}$$

We now work out the bilinears.

$$\left(\frac{3}{2}|\varphi|^2\alpha_j\alpha^\mu\varphi_{\mu kp}\right)\varphi_{iab}\omega^{ka}\omega^{pb} = -\frac{3}{2}|\varphi|^4\alpha_j\alpha^\mu g_{\mu i} = -\frac{3}{2}|\varphi|^4\alpha_i\alpha_j, \tag{3.122}$$

$$\left(-|\varphi|^2\nabla_j\alpha^\mu\varphi_{\mu kp}\right)\varphi_{iab}\omega^{ka}\omega^{pb} = |\varphi|^4\nabla_j\alpha_i, \tag{3.123}$$

$$\left(-\frac{1}{2}|\varphi|^2\alpha^\mu\alpha_{Jj}\varphi_{J\mu,k,p}\right)\varphi_{iab}\omega^{ka}\omega^{pb} = \frac{1}{2}|\varphi|^4\alpha^\mu\alpha_{Jj}g_{J\mu,i} = -\frac{1}{2}|\varphi|^4\alpha_{Ji}\alpha_{Jj}. \tag{3.124}$$

Therefore

$$\begin{aligned}(\nabla_j(\iota_{\nabla|\varphi|^2}\varphi)_{kp})\varphi_{iab}\omega^{ka}\omega^{pb} &= -\frac{3}{2}|\varphi|^4\alpha_i\alpha_j + |\varphi|^4\nabla_j\alpha_i\\ &\quad -\frac{1}{2}|\varphi|^4\alpha_{Ji}\alpha_{Jj}\\ &\quad +|\varphi|^2\alpha^\mu E_{j;\mu kp}\varphi_{iab}\omega^{ka}\omega^{pb}.\end{aligned} \tag{3.125}$$

Next, we work out the two next contributions of this term with the indices (jkp) cyclically permuted. After forming bilinears, these two extra terms are identical.

$$\begin{aligned}&(\nabla_p(\iota_{\nabla|\varphi|^2}\varphi)_{jk})\varphi_{iab}\omega^{ka}\omega^{pb} + (\nabla_k(\iota_{\nabla|\varphi|^2}\varphi)_{pj})\varphi_{iab}\omega^{ka}\omega^{pb}\\ &= 2(\nabla_p(\iota_{\nabla|\varphi|^2}\varphi)_{jk})\varphi_{iab}\omega^{ka}\omega^{pb}\end{aligned} \tag{3.126}$$

As before, we have

$$\begin{aligned}\nabla_p(\iota_{\nabla|\varphi|^2}\varphi)_{jk} &= \frac{3}{2}|\varphi|^2\alpha_p\alpha^\mu\varphi_{\mu jk} - |\varphi|^2\nabla_p\alpha^\mu\varphi_{\mu jk}\\ &\quad -\frac{1}{2}|\varphi|^2\alpha^\mu\alpha_{Jp}\varphi_{J\mu,j,k} + |\varphi|^2\alpha^\mu E_{p;\mu jk}\end{aligned} \tag{3.127}$$

Forming bilinears,

$$\begin{aligned}\left(\frac{3}{2}|\varphi|^2\alpha_p\alpha^\mu\varphi_{\mu jk}\right)\varphi_{iab}\omega^{ka}\omega^{pb} &= -\frac{3}{8}|\varphi|^4\alpha_p\alpha^\mu(\omega_{\mu i}g_{jb} - \omega_{ji}g_{\mu b} - \omega_{\mu b}g_{ji} + \omega_{jb}g_{\mu i})\omega^{pb}\\ &= \frac{3}{8}|\varphi|^4\alpha_p\alpha^\mu(-\omega_{\mu i}J^p{}_j + \omega_{ji}J^p{}_\mu - \delta^p{}_\mu g_{ji} + \delta^p{}_j g_{\mu i})\\ &= \frac{3}{8}|\varphi|^4(\alpha_{Jj}\alpha_{Ji} + \alpha_{J\mu}\alpha^\mu\omega_{ij} - \alpha_\mu\alpha^\mu g_{ij} + \alpha_j\alpha_i),\end{aligned} \tag{3.128}$$

and

$$\left(-|\varphi|^2\nabla_p\alpha^\mu\varphi_{\mu jk}\right)\varphi_{iab}\omega^{ka}\omega^{pb} = \frac{1}{4}|\varphi|^4\nabla_p\alpha^\mu(\omega_{\mu i}g_{jb} - \omega_{ji}g_{\mu b} - \omega_{\mu b}g_{ji} + \omega_{jb}g_{\mu i})\omega^{pb}$$
$$= \frac{1}{4}|\varphi|^4\nabla_p\alpha^\mu(\omega_{\mu i}J^p{}_j - \omega_{ji}J^p{}_\mu + \delta^p{}_\mu g_{ji} - \delta^p{}_j g_{\mu i})$$
$$= \frac{|\varphi|^4}{4}(-J^n{}_j\nabla_n\alpha_q J^q{}_i - J^p{}_\mu\nabla_p\alpha^\mu\omega_{ji} + \nabla_\mu\alpha^\mu g_{ij} - \nabla_j\alpha_i),$$

and

$$\left(-\frac{1}{2}|\varphi|^2\alpha^\mu\alpha_{Jp}\varphi_{J\mu,j,k}\right)\varphi_{iab}\omega^{ka}\omega^{pb}$$
$$= \frac{1}{8}|\varphi|^4\alpha^\mu\alpha_{Jp}(\omega_{J\mu,i}g_{jb} - \omega_{ji}g_{J\mu,b}$$
$$-\omega_{J\mu,b}g_{ji} + \omega_{jb}g_{J\mu,i})\omega^{pb}$$
$$= \frac{1}{8}|\varphi|^4\alpha^\mu\alpha_{Jp}(-g_{\mu i}J^p{}_j + \omega_{ji}\delta^p{}_\mu$$
$$+J^p{}_\mu g_{ji} - \delta^p{}_j g_{J\mu,i})$$
$$= \frac{|\varphi|^4}{8}(\alpha_i\alpha_j + \alpha^p\alpha_{Jp}\omega_{ji} - \alpha^\mu\alpha_\mu g_{ij} + \alpha_{Ji}\alpha_{Jj}). \quad (3.129)$$

Altogether,

$$(\nabla_p(\iota_{\nabla|\varphi|^2}\varphi)_{jk})\varphi_{iab}\omega^{ka}\omega^{pb} = \frac{|\varphi|^4}{8}\Big(3\alpha_{Jj}\alpha_{Ji} + 3\alpha_{J\mu}\alpha^\mu\omega_{ij} - 3\alpha_\mu\alpha^\mu g_{ij} + 3\alpha_j\alpha_i$$
$$-2\nabla_{Jj}\alpha_{Ji} - 2\nabla_{J\mu}\alpha^\mu\omega_{ji} + 2\nabla_\mu\alpha^\mu g_{ij} - 2\nabla_j\alpha_i$$
$$+\alpha_i\alpha_j + \alpha^p\alpha_{Jp}\omega_{ji} - \alpha^\mu\alpha_\mu g_{ij} + \alpha_{Ji}\alpha_{Jj}\Big)$$
$$+|\varphi|^2\alpha^\mu E_{p;\mu jk}\varphi_{iab}\omega^{ka}\omega^{pb} \quad (3.130)$$

It follows that

$$2(\nabla_p(\iota_{\nabla|\varphi|^2}\varphi)_{jk})\varphi_{iab}\omega^{ka}\omega^{pb} = \frac{|\varphi|^4}{4}\Big(4\alpha_{Jj}\alpha_{Ji} + 2\alpha_{J\mu}\alpha^\mu\omega_{ij} - 4\alpha_\mu\alpha^\mu g_{ij} + 4\alpha_j\alpha_i$$
$$-2J^n{}_j\nabla_n\alpha_q J^q{}_i - 2J^p{}_\mu\nabla_p\alpha^\mu\omega_{ji} + 2\nabla_\mu\alpha^\mu g_{ij} - 2\nabla_j\alpha_i\Big)$$
$$+2|\varphi|^2\alpha^\mu E_{p;\mu jk}\varphi_{iab}\omega^{ka}\omega^{pb} \quad (3.131)$$

We can now combine all of our calculations. By (3.118), (3.125), (3.131),

Lemma 10

$$(d\iota_{\nabla|\varphi|^2}\varphi)_{jkp}\varphi_{iab}\omega^{ka}\omega^{pb} + (i \leftrightarrow j)$$

$$
\begin{aligned}
&= |\varphi|^4 \bigg\{ \frac{1}{2}(\nabla_j \alpha_i + \nabla_i \alpha_j) - \alpha_i \alpha_j + \alpha_{Ji} \alpha_{Jj} - 2\alpha_\mu \alpha^\mu g_{ij} \\
&\quad - \frac{1}{2}(J^p{}_j J^q{}_i \nabla_p \alpha_q + J^p{}_i J^q{}_j \nabla_p \alpha_q) \\
&\quad + \nabla_\mu \alpha^\mu g_{ij} \bigg\} + |\varphi|^2 \alpha^\mu E_{j;\mu kp} \varphi_{iab} \omega^{ka} \omega^{pb} + |\varphi|^2 \alpha^\mu E_{i;\mu kp} \varphi_{jab} \omega^{ka} \omega^{pb} \\
&\quad + 2|\varphi|^2 \alpha^\mu E_{p;\mu jk} \varphi_{iab} \omega^{ka} \omega^{pb} + 2|\varphi|^2 \alpha^\mu E_{p;\mu ik} \varphi_{jab} \omega^{ka} \omega^{pb}. \qquad (3.132)
\end{aligned}
$$

It remains to evaluate the E terms.

$$
|\varphi|^2 \alpha^\mu E_{j;\mu kp} \varphi_{iab} \omega^{ka} \omega^{pb} + 2|\varphi|^2 \alpha^\mu E_{p;\mu jk} \varphi_{iab} \omega^{ka} \omega^{pb} + (i \leftrightarrow j) \qquad (3.133)
$$

We start with

$$
\begin{aligned}
|\varphi|^2 \alpha^\mu E_{j;\mu kp} \varphi_{iab} \omega^{ka} \omega^{pb} &= |\varphi|^2 \alpha^\mu \Big(\varphi_{\lambda kp} N_{j\mu}{}^\lambda + \varphi_{\mu\lambda p} N_{jk}{}^\lambda \\
&\quad + \varphi_{\mu k\lambda} N_{jp}{}^\lambda \Big) \varphi_{iab} \omega^{ka} \omega^{pb} \qquad (3.134)
\end{aligned}
$$

which by symmetry is

$$
\begin{aligned}
|\varphi|^2 \alpha^\mu E_{j;\mu kp} \varphi_{iab} \omega^{ka} \omega^{pb} &= |\varphi|^2 \alpha^\mu (\varphi_{\lambda kp} N_{j\mu}{}^\lambda) \varphi_{iab} \omega^{ka} \omega^{pb} \\
&\quad + 2|\varphi|^2 \alpha^\mu (\varphi_{\mu\lambda p} N_{jk}{}^\lambda) \varphi_{iab} \omega^{ka} \omega^{pb} \qquad (3.135)
\end{aligned}
$$

The first term is

$$
|\varphi|^2 \alpha^\mu (\varphi_{\lambda kp} N_{j\mu}{}^\lambda) \varphi_{iab} \omega^{ka} \omega^{pb} = -|\varphi|^4 \alpha^\mu N_{j\mu}{}^\lambda g_{\lambda i} = -|\varphi|^4 \alpha^\mu N_{j\mu i}. \qquad (3.136)
$$

The second term is

$$
\begin{aligned}
2|\varphi|^2 \alpha^\mu (\varphi_{\mu\lambda p} N_{jk}{}^\lambda) \varphi_{iab} \omega^{ka} \omega^{pb} &= \frac{|\varphi|^2}{4} \alpha^\mu N_{jk}{}^\lambda (g_{\mu i} \omega_{\lambda a} - g_{\lambda i} \omega_{\mu a} - g_{\mu a} \omega_{\lambda i} + g_{\lambda a} \omega_{\mu i}) \omega^{ka} \\
&= \frac{|\varphi|^2}{4} \alpha^\mu N_{jk}{}^\lambda (-g_{\mu i} \delta^k{}_\lambda + g_{\lambda i} \delta^k{}_\mu - J^k{}_\mu \omega_{\lambda i} + J^k{}_\mu \omega_{\mu i}) \\
&= \frac{|\varphi|^2}{4} \alpha^\mu N_{jk}{}^\lambda (-g_{\mu i} \delta^k{}_\lambda + g_{\lambda i} \delta^k{}_\mu - J^k{}_\mu \omega_{\lambda i} + J^k{}_\lambda \omega_{\mu i}) \\
&= \frac{|\varphi|^2}{4} (-\alpha_i N_{j\lambda}{}^\lambda + \alpha^\mu N_{j\mu i} + \alpha^\mu N_{j,J\mu,Ji} - \alpha_{Ji} N_{j,J\lambda}{}^\lambda) \\
&= \frac{|\varphi|^2}{4} (0 + \alpha^\mu N_{j\mu i} - \alpha^\mu N_{j\mu i} + 0) = 0. \qquad (3.137)
\end{aligned}
$$

Therefore

$$
|\varphi|^2 \alpha^\mu E_{j;\mu kp} \varphi_{iab} \omega^{ka} \omega^{pb} = -|\varphi|^4 \alpha^\mu N_{j\mu i} = |\varphi|^4 \alpha^\mu N_{ji\mu}. \qquad (3.138)
$$

Next, we consider

$$2|\varphi|^2\alpha^\mu E_{p;\mu jk}\varphi_{iab}\omega^{ka}\omega^{pb} = 2|\varphi|^2\alpha^\mu(\varphi_{\lambda jk}N_{p\mu}{}^\lambda + \varphi_{\mu\lambda k}N_{pj}{}^\lambda + \varphi_{\mu j\lambda}N_{pk}{}^\lambda)\varphi_{iab}\omega^{ka}\omega^{pb}$$
$$:= (\tilde{\mathrm{I}} + \tilde{\mathrm{II}} + \tilde{\mathrm{III}}) \cdot \varphi_{iab}\omega^{ka}\omega^{pb}. \tag{3.139}$$

We start with

$$\begin{aligned}(\tilde{\mathrm{I}}) \cdot \varphi_{iab}\omega^{ka}\omega^{pb} &= -2|\varphi|^2\alpha^\mu N_{p\mu}{}^\lambda(\varphi_{\lambda kj}\varphi_{iab}\omega^{ka})\omega^{pb} \\ &= -\frac{|\varphi|^4}{2}\alpha^\mu N_{p\mu}{}^\lambda(\omega_{\lambda i}g_{jb} - \omega_{ji}g_{\lambda b} - \omega_{\lambda b}g_{ji} + \omega_{jb}g_{\lambda i})\omega^{pb} \\ &= -\frac{|\varphi|^4}{2}\alpha^\mu N_{p\mu}{}^\lambda(\omega_{\lambda i}J^p{}_j - \omega_{ji}J^p{}_\lambda + \delta^p{}_\lambda g_{ji} - \delta^p{}_j g_{\lambda i}) \\ &= \frac{|\varphi|^4}{2}(\alpha^\mu N_{Jj,\mu,Ji} + \omega_{ji}\alpha^\mu N_{J\lambda,\mu}{}^\lambda - g_{ji}\alpha^\mu N_{\lambda\mu}{}^\lambda + \alpha^\mu N_{j\mu i}) \\ &= \frac{|\varphi|^4}{2}(-\alpha^\mu N_{j\mu i} + 0 - 0 + \alpha^\mu N_{j\mu i}) = 0. \end{aligned} \tag{3.140}$$

Similarly, we can also compute

$$(\tilde{\mathrm{II}}) \cdot \varphi_{iab}\omega^{ka}\omega^{pb} = 0 \tag{3.141}$$

The third term is

$$(\tilde{\mathrm{III}}) \cdot \varphi_{iab}\omega^{ka}\omega^{pb} = 2|\varphi|^2\alpha^\mu(\varphi_{\mu j\lambda}N_{pk}{}^\lambda)\varphi_{iab}\omega^{ka}\omega^{pb} \tag{3.142}$$

It can be rearranged using the symmetry $p \leftrightarrow k$, $a \leftrightarrow b$

$$\begin{aligned}(\tilde{\mathrm{III}}) \cdot \varphi_{iab}\omega^{ka}\omega^{pb} &= 2|\varphi|^2\alpha^\mu\varphi_{\mu j\lambda}\frac{(N_{pk}{}^\lambda - N_{kp}{}^\lambda)}{2}\varphi_{iab}\omega^{ka}\omega^{pb} \\ &= -|\varphi|^2\alpha^\mu\varphi_{\lambda\mu j}N^\lambda{}_{pk}\varphi_{iab}\omega^{ka}\omega^{pb} \end{aligned} \tag{3.143}$$

and then using the Bianchi identity. By the identity $N^\lambda{}_{pk}\varphi_{\lambda\mu j} = -N^\lambda{}_{\mu j}\varphi_{\lambda pk}$,

$$(\tilde{\mathrm{III}}) \cdot \varphi_{iab}\omega^{ka}\omega^{pb} = -|\varphi|^2\alpha^\mu N^\lambda{}_{\mu j}\varphi_{\lambda pk}\varphi_{iba}\omega^{ka}\omega^{pb}. \tag{3.144}$$

We can now use the bilinear identity.

$$(\tilde{\mathrm{III}}) \cdot \varphi_{iab}\omega^{ka}\omega^{pb} = |\varphi|^4\alpha^\mu N^\lambda{}_{\mu j}g_{\lambda i} = |\varphi|^4\alpha^\mu N_{i\mu j} = -|\varphi|^4\alpha^\mu N_{ij\mu}. \tag{3.145}$$

Substituting our results into (3.139), we obtain

$$2|\varphi|^2\alpha^\mu E_{p;\mu jk}\varphi_{iab}\omega^{ka}\omega^{pb} = -|\varphi|^4\alpha^\mu N_{ij\mu}. \tag{3.146}$$

Combining the above equation with (3.138),

$$|\varphi|^2\alpha^\mu E_{j;\mu kp}\varphi_{iab}\omega^{ka}\omega^{pb} + 2|\varphi|^2\alpha^\mu E_{p;\mu jk}\varphi_{iab}\omega^{ka}\omega^{pb} + (i \leftrightarrow j)$$
$$= |\varphi|^4\alpha^\mu N_{ji\mu} - |\varphi|^4\alpha^\mu N_{ij\mu} + (i \leftrightarrow j) = 0. \quad (3.147)$$

Therefore the E terms do not contribute, and we are left with:

Lemma 11

$$(d\iota_{\nabla|\varphi|^2}\varphi)_{jkp}\varphi_{iab}\omega^{ka}\omega^{pb} + (i \leftrightarrow j) = |\varphi|^4\bigg\{\frac{1}{2}(\nabla_j\alpha_i + \nabla_i\alpha_j) - \alpha_i\alpha_j + \alpha_{Ji}\alpha_{Jj} - 2\alpha_\mu\alpha^\mu g_{ij}$$
$$-\frac{1}{2}(J^p{}_jJ^q{}_i\nabla_p\alpha_q + J^p{}_iJ^q{}_j\nabla_p\alpha_q) + \nabla_\mu\alpha^\mu g_{ij}\bigg\}$$

3.4 $N^\dagger$ Term: $d(|\varphi|^2N^\dagger\cdot\varphi)$

Recall from the definition of the operator $N^\dagger$ that $(N^\dagger\varphi)_{kj} = 2N^\mu{}_j{}^\lambda\varphi_{\mu k\lambda}$, and thus

$$d(|\varphi|^2N^\dagger\cdot\varphi)_{jkp} = \nabla_j(|\varphi|^2(N^\dagger\cdot\varphi)_{kp}) + \nabla_p(|\varphi|^2(N^\dagger\cdot\varphi)_{jk}) + \nabla_k(|\varphi|^2(N^\dagger\cdot\varphi)_{pj})$$
$$:= \mathrm{I} + \mathrm{II} + \mathrm{III}. \quad (3.148)$$

3.4.1 Computation for (I)

We start with the first term

$$\nabla_j(|\varphi|^2(N^\dagger\cdot\varphi)_{kp}) = -2|\varphi|^2\,\alpha_jN^\mu{}_p{}^\lambda\varphi_{\mu k\lambda} + 2|\varphi|^2\nabla_j(N^\mu{}_p{}^\lambda\varphi_{\mu k\lambda})$$
$$= -2|\varphi|^2\,\alpha_jN^\mu{}_p{}^\lambda\varphi_{\mu k\lambda} + 2|\varphi|^2\nabla_jN^\mu{}_p{}^\lambda\varphi_{\mu k\lambda} + 2|\varphi|^2N^\mu{}_p{}^\lambda\nabla_j\varphi_{\mu k\lambda} \quad (3.149)$$

We now work out the bilinears term by term

$$-2|\varphi|^2\,\alpha_jN^\mu{}_k{}^\lambda\varphi_{\mu p\lambda}\varphi_{iab}\omega^{ka}\omega^{pb} = \frac{1}{2}|\varphi|^4\,\alpha_jN^\mu{}_k{}^\lambda\,(\omega_{\mu i}g_{\lambda a} + \omega_{\lambda a}g_{\mu i} - \omega_{\mu a}g_{\lambda i} - \omega_{\lambda i}g_{\mu a})\omega^{ka}$$
$$= \frac{1}{2}|\varphi|^4\,\alpha_jN^\mu{}_k{}^\lambda\,(\omega_{\mu i}J^k{}_\lambda - \delta^k{}_\lambda g_{\mu i} + \delta^k{}_\mu g_{\lambda i} - \omega_{\lambda i}J^k{}_\mu)$$
$$= \frac{1}{2}|\varphi|^4\,\alpha_j(-N_{Ji,k}{}^{Jk} - N_{ik}{}^k + N^k{}_{ki} + N^{Jk}{}_{k,Ji}) = 0.$$

The first two terms are zero due to antisymmetry of N in the second and third indices. The third and fourth terms are also zero since $g^{ml}N_{mlj} = 0$ and $N^{Jk}{}_{k,Ji} = -N^k{}_{Jk,Ji} = N^k{}_{ki}$.

Next, we work with the second group of terms in (3.149):

$$2|\varphi|^2\nabla_jN^\mu{}_p{}^\lambda\varphi_{\mu k\lambda}\varphi_{iab}\omega^{ka}\omega^{pb} = \frac{1}{2}|\varphi|^4\nabla_jN^\mu{}_p{}^\lambda(\omega_{\mu i}g_{\lambda b} + \omega_{\lambda b}g_{\mu i} - \omega_{\mu b}g_{\lambda i} - \omega_{\lambda i}g_{\mu b})\omega^{pb}$$
$$= \frac{1}{2}|\varphi|^4\nabla_jN^\mu{}_p{}^\lambda(\omega_{\mu i}J^p{}_\lambda - \delta^p{}_\lambda g_{\mu i} + \delta^p{}_\mu g_{\lambda i} - \omega_{\lambda i}J^p{}_\mu)$$

$$= \frac{1}{2}|\varphi|^4(\nabla_j N^\mu{}_p{}^\lambda \omega_{\mu i} J^p{}_\lambda - \nabla_j N_{ip}{}^p + \nabla_j N^p{}_{pi} - \nabla_j N^\mu{}_p{}^\lambda \omega_{\lambda i} J^p{}_\mu)$$
$$= \frac{1}{2}|\varphi|^4(\nabla_j N^\mu{}_p{}^\lambda \omega_{\mu i} J^p{}_\lambda - \nabla_j N^\mu{}_p{}^\lambda \omega_{\lambda i} J^p{}_\mu) \quad (3.150)$$

The last two terms require extra work since J may not be covariantly constant under ∇.

$$\begin{aligned}\omega_{\mu i}\nabla_j N^\mu{}_p{}^\lambda J^p{}_\lambda &= \omega_{\mu i}(\nabla_j(N^\mu{}_p{}^\lambda J^p{}_\lambda) - N^\mu{}_p{}^\lambda \nabla_j J^p{}_\lambda)\\ &= \omega_{\mu i}(\nabla_j N^\mu{}_p{}^{Jp} + 2N^\mu{}_p{}^\lambda N_{j\lambda}{}^{Jp})\\ &= 2\omega_{\mu i} N^\mu{}_p{}^\lambda N_{j\lambda}{}^{Jp}\\ &= -2N_{Ji,p}{}^\lambda N_{j\lambda}{}^{Jp}\\ &= -2N_{i,Jp}{}^\lambda N_{j\lambda}{}^{Jp}\\ &= 2N_{ip}{}^\lambda N_{j\lambda}{}^p.\end{aligned}$$

Similarly, we can compute

$$-\nabla_j N^\mu{}_p{}^\lambda \omega_{\lambda i} J^p{}_\mu = 2N^\mu{}_{Jp,i} N_{j\mu}{}^{Jp} = -2N^\mu{}_{pi} N_{j\mu}{}^p. \quad (3.151)$$

Altogether, we have

$$2|\varphi|^2 \nabla_j N^\mu{}_p{}^\lambda \varphi_{\mu k\lambda}\varphi_{iab}\omega^{ka}\omega^{pb} = |\varphi|^4(N_{ip}{}^\lambda N_{j\lambda}{}^p - N^\lambda{}_{pi} N_{j\lambda}{}^p). \quad (3.152)$$

Next, we consider the last group of terms in (3.149).

$$2|\varphi|^2 N^\mu{}_p{}^\lambda \nabla_j \varphi_{\mu k\lambda}\varphi_{iab}\omega^{ka}\omega^{pb} \quad (3.153)$$

Since

$$\nabla_j \varphi_{\mu k\lambda} = -\frac{1}{2}\alpha_j \varphi_{\mu k\lambda} + \frac{1}{2}\alpha_{Jj}\varphi_{J\mu,k\lambda} - E_{j;\mu k\lambda}, \quad (3.154)$$

then

$$\begin{aligned}&2|\varphi|^2 N^\mu{}_p{}^\lambda \nabla_j \varphi_{\mu k\lambda}\varphi_{iab}\omega^{ka}\omega^{pb}\\ &= 2|\varphi|^2 N^\mu{}_p{}^\lambda \left(-\frac{1}{2}\alpha_j \varphi_{\mu k\lambda} + \frac{1}{2}\alpha_{Jj}\varphi_{J\mu,k\lambda} - E_{j;\mu k\lambda}\right)\varphi_{iab}\omega^{ka}\omega^{pb}.\end{aligned}$$

We work out the bilinears term by term

$$\begin{aligned}2|\varphi|^2 N^\mu{}_p{}^\lambda \left(-\frac{1}{2}\alpha_j \varphi_{\mu k\lambda}\right)\varphi_{iab}\omega^{ka}\omega^{pb} &= -|\varphi|^2 \alpha_j N^\mu{}_p{}^\lambda \varphi_{\mu k\lambda}\varphi_{iab}\omega^{ka}\omega^{pb}\\ &= -\frac{1}{4}|\varphi|^4 \alpha_j N^\mu{}_p{}^\lambda(\omega_{\mu i} g_{\lambda b} + \omega_{\lambda b} g_{\mu i}\end{aligned}$$

$$
\begin{aligned}
&\quad -\omega_{\mu b} g_{\lambda i} - \omega_{\lambda i} g_{\mu b})\omega^{pb} \\
&= -\frac{1}{4}|\varphi|^4 \alpha_j N^\mu{}_p{}^\lambda (\omega_{\mu i} J^p{}_\lambda \\
&\quad -\delta^p{}_\lambda g_{\mu i} + \delta^p{}_\mu g_{\lambda i} - \omega_{\lambda i} J^p{}_\mu) \\
&= -\frac{1}{4}|\varphi|^4 \alpha_j (-N_{Ji,p}{}^{Jp} \\
&\quad -N_{ip}{}^p + N^p{}_{pi} + N^p{}_{p,Ji}) = 0. \\
2|\varphi|^2 N^\mu{}_p{}^\lambda (\frac{1}{2}\alpha_{Jj}\varphi_{J\mu,k\lambda})\varphi_{iab}\omega^{ka}\omega^{pb} &= |\varphi|^2 \alpha_{Jj} N^\mu{}_p{}^\lambda \varphi_{J\mu,k\lambda}\varphi_{iab}\omega^{ka}\omega^{pb} \\
&= -\frac{1}{4}|\varphi|^4 \alpha_{Jj} N^\mu{}_p{}^\lambda (\omega_{J\mu,i} g_{\lambda b} + \omega_{\lambda b} g_{J\mu,i} \\
&\quad -\omega_{J\mu,b} g_{\lambda i} - \omega_{\lambda i} g_{J\mu,b})\omega^{pb} \\
&= -\frac{1}{4}|\varphi|^4 \alpha_{Jj} N^\mu{}_p{}^\lambda (-g_{\mu i} J^p{}_\lambda - \delta^p{}_\lambda \omega_{\mu i} + J^p{}_\mu g_{\lambda i} \\
&\quad +\delta^p{}_\mu \omega_{\lambda i}) \\
&= -\frac{1}{4}|\varphi|^4 \alpha_{Jj} (-N_{ip}{}^{Jp} \\
&\quad +N_{Ji,p}{}^p + N^{Jp}{}_{pi} - N^p{}_{p,Ji}) = 0. \\
2|\varphi|^2 N^\mu{}_p{}^\lambda (-E_{j;\mu k\lambda})\varphi_{iab}\omega^{ka}\omega^{pb} &= -2|\varphi|^2 N^\mu{}_p{}^\lambda E_{j;\mu k\lambda}\varphi_{iab}\omega^{ka}\omega^{pb} \\
&= -2|\varphi|^2 N^\mu{}_p{}^\lambda (N_{j\mu}{}^\ell \varphi_{\ell k\lambda} + N_{jk}{}^\ell \varphi_{\mu\ell\lambda} \\
&\quad +N_{j\lambda}{}^\ell \varphi_{\mu k\ell})\varphi_{iab}\omega^{ka}\omega^{pb} \qquad (3.155)
\end{aligned}
$$

The first term in the above last line is easy to handle

$$
\begin{aligned}
2|\varphi|^2 N^\mu{}_p{}^\lambda N_{j\mu}{}^\ell \varphi_{\ell k\lambda}\varphi_{iab}\omega^{ka}\omega^{pb} &= \frac{1}{2}|\varphi|^4 N^\mu{}_p{}^\lambda N_{j\mu}{}^\ell (\omega_{\ell i} g_{\lambda b} + \omega_{\lambda b} g_{\ell i} - \omega_{\ell b} g_{\lambda i} - \omega_{\lambda i} g_{\ell b})\omega^{pb} \\
&= \frac{1}{2}|\varphi|^4 N^\mu{}_p{}^\lambda N_{j\mu}{}^\ell (\omega_{\ell i} J^p{}_\lambda - \delta^p{}_\lambda g_{\ell i} + \delta^p{}_\ell g_{\lambda i} - \omega_{\lambda i} J^p{}_\ell) \\
&= \frac{1}{2}|\varphi|^4 (-N^\mu{}_{J\lambda}{}^\lambda N_{j\mu,Ji} - N^\mu{}_p{}^p N_{j\mu i} \\
&\quad +N^\mu{}_{pi} N_{j\mu}{}^p + N^\mu{}_{J\ell,Ji} N_{j\mu}{}^\ell \\
&= \frac{1}{2}|\varphi|^4 (N^\mu{}_{pi} N_{j\mu}{}^p - N^\mu{}_{pi} N_{j\mu}{}^p) = 0. \qquad (3.156)
\end{aligned}
$$

The third term can also be handled in the similar way

$$
\begin{aligned}
2|\varphi|^2 N^\mu{}_p{}^\lambda N_{j\lambda}{}^\ell \varphi_{\mu k\ell}\varphi_{iab}\omega^{ka}\omega^{pb} &= \frac{1}{2}|\varphi|^4 N^\mu{}_p{}^\lambda N_{j\lambda}{}^\ell (\omega_{\mu i} g_{\ell b} + \omega_{\ell b} g_{\mu i} - \omega_{\mu b} g_{\ell i} - \omega_{\ell i} g_{\mu b})\omega^{pb} \\
&= \frac{1}{2}|\varphi|^4 N^\mu{}_p{}^\lambda N_{j\lambda}{}^\ell (\omega_{\mu i} J^p{}_\ell - \delta^p{}_\ell g_{\mu i} + \delta^p{}_\mu g_{\ell i} - \omega_{\ell i} J^p{}_\mu) \\
&= \frac{1}{2}|\varphi|^4 (-N_{Ji,p}{}^\lambda N_{j\lambda}{}^{Jp} - N_{ip}{}^\lambda N_{j\lambda}{}^p \\
&\quad +N^p{}_p{}^\lambda N_{j\lambda i} + N^{Jp}{}_p{}^\lambda N_{j\lambda,Ji}) \\
&= \frac{1}{2}|\varphi|^4 (N_{ip}{}^\lambda N_{j\lambda}{}^p - N_{ip}{}^\lambda N_{j\lambda}{}^p) = 0. \qquad (3.157)
\end{aligned}
$$

For the second term in (3.155), we will use the Bianchi identity and switch the indices as before, $N^p{}_{ij}\varphi_{pkl} = -N^p{}_{kl}\varphi_{pij}$, obtaining

$$
\begin{aligned}
N^{\mu}{}_{p}{}^{\lambda} N_{jk}{}^{\ell} \varphi_{\mu\ell\lambda} &= -N^{\mu}{}_{p}{}^{\lambda}(N^{\ell}{}_{jk} + N_{k}{}^{\ell}{}_{j})\varphi_{\mu\ell\lambda} \\
&= N^{\mu}{}_{p}{}^{\lambda} N^{\ell}{}_{jk}\varphi_{\ell\mu\lambda} - N_{k}{}^{\ell}{}_{j} N^{\mu}{}_{p}{}^{\lambda}\varphi_{\mu\ell\lambda} \\
&= -N^{\mu}{}_{p}{}^{\lambda} N^{\ell}{}_{\mu\lambda}\varphi_{\ell jk} + N_{k}{}^{\ell}{}_{j} N^{\mu}{}_{\ell}{}^{\lambda}\varphi_{\mu p\lambda}
\end{aligned}
\tag{3.158}
$$

Therefore,

$$
\begin{aligned}
-2|\varphi|^2 N^{\mu}{}_{p}{}^{\lambda} N_{jk}{}^{\ell}\varphi_{\mu\ell\lambda}\varphi_{iab}\omega^{ka}\omega^{pb} &= 2|\varphi|^2(N^{\mu}{}_{p}{}^{\lambda} N^{\ell}{}_{\mu\lambda}\varphi_{\ell jk} - N_{k}{}^{\ell}{}_{j} N^{\mu}{}_{\ell}{}^{\lambda}\varphi_{\mu p\lambda}) \\
&\quad \times\varphi_{iab}\omega^{ka}\omega^{pb} \\
&= -\frac{1}{2}|\varphi|^4 N^{\mu}{}_{p}{}^{\lambda} N^{\ell}{}_{\mu\lambda}(\omega_{\ell i} g_{jb} + \omega_{jb} g_{\ell i} \\
&\quad -\omega_{\ell b} g_{ji} - \omega_{ji} g_{\ell b})\omega^{pb} \\
&\quad +\frac{1}{2}|\varphi|^4 N_{k}{}^{\ell}{}_{j} N^{\mu}{}_{\ell}{}^{\lambda}(\omega_{\mu i} g_{\lambda a} + \omega_{\lambda a}\omega_{\mu i} \\
&\quad -\omega_{\mu a} g_{\lambda i} - \omega_{\lambda i} g_{\mu a})\omega^{ka} \\
&= -\frac{1}{2}|\varphi|^4 N^{\mu}{}_{p}{}^{\lambda} N^{\ell}{}_{\mu\lambda}(\omega_{\ell i} J^{p}{}_{j} \\
&\quad -\delta^{p}{}_{j} g_{\ell i} + \delta^{p}{}_{\ell} g_{ji} - \omega_{ji} J^{p}{}_{\ell}) \\
&\quad +\frac{1}{2}|\varphi|^4 N_{k}{}^{\ell}{}_{j} N^{\mu}{}_{\ell}{}^{\lambda}(\omega_{\mu i} J^{k}{}_{\lambda} \\
&\quad -\delta^{k}{}_{\lambda} g_{\mu i} + \delta^{k}{}_{\mu} g_{\lambda i} - \omega_{\lambda i} J^{k}{}_{\mu}) \\
&= -\frac{1}{2}|\varphi|^4(-N^{\mu}{}_{Jj}{}^{\lambda} N_{Ji,\mu\lambda} \\
&\quad -N^{\mu}{}_{j}{}^{\lambda} N_{i\mu\lambda} + N^{\mu}{}_{p}{}^{\lambda} N^{p}{}_{\mu\lambda} g_{ji}) \\
&\quad +\frac{1}{2}|\varphi|^4(-N_{J\lambda}{}^{\ell}{}_{j} N_{Ji,\ell}{}^{\lambda} - N_{\lambda}{}^{\ell}{}_{j} N_{i\ell}{}^{\lambda} \\
&\quad +N_{\mu}{}^{\ell}{}_{j} N^{\mu}{}_{\ell i} + N_{J\mu}{}^{\ell}{}_{j} N^{\mu}{}_{\ell,Ji})
\end{aligned}
$$

The right hand side can be readily simplified as follows,

$$
\begin{aligned}
&-\frac{1}{2}|\varphi|^4(-2N^{\mu}{}_{j}{}^{\lambda} N_{i\mu\lambda} + N^{\mu}{}_{p}{}^{\lambda} N^{p}{}_{\mu\lambda} g_{ji}) \\
&+\frac{1}{2}|\varphi|^4(-2N_{\lambda}{}^{\ell}{}_{j} N_{i\ell}{}^{\lambda} + N_{\mu}{}^{\ell}{}_{j} N^{\mu}{}_{\ell i} - N_{J\mu}{}^{\ell}{}_{j} N^{J\mu}{}_{\ell i}) \\
&= -\frac{1}{2}|\varphi|^4(-2N^{\mu}{}_{j}{}^{\lambda} N_{i\mu\lambda} + N^{\mu}{}_{p}{}^{\lambda} N^{p}{}_{\mu\lambda} g_{ji} + 2N_{\lambda}{}^{\ell}{}_{j} N_{i\ell}{}^{\lambda} - 2N_{\mu}{}^{\ell}{}_{j} N^{\mu}{}_{\ell i}) \\
&= -\frac{1}{2}|\varphi|^4(2N^{\mu\lambda}{}_{j} N_{i\mu\lambda} - 2N^{\lambda\ell}{}_{j} N_{i\lambda\ell} + (N_{-}^{2})^{\lambda}{}_{\lambda} g_{ji} - 2N_{\mu}{}^{\ell}{}_{j} N^{\mu}{}_{\ell i}) \\
&= -|\varphi|^4(\frac{1}{2}(N_{-}^{2})^{\lambda}{}_{\lambda} g_{ij} - N_{\mu}{}^{\ell}{}_{j} N^{\mu}{}_{\ell i}).
\end{aligned}
\tag{3.159}
$$

Putting the above computations into (3.153), we obtain

$$2|\varphi|^2 N^{\mu}{}_{p}{}^{\lambda}\nabla_j\varphi_{\mu k\lambda}\varphi_{iab}\omega^{ka}\omega^{pb} = -|\varphi|^4\left(\frac{1}{2}(N_-^2)^{\lambda}{}_{\lambda}g_{ij} - N_{\mu}{}^{\ell}{}_{j}\,N^{\mu}{}_{\ell i}\right) \tag{3.160}$$

Therefore, we obtain the first term (I) in (3.148):

$$\begin{aligned}(\mathrm{I})\cdot\varphi_{iab}\omega^{ka}\omega^{pb} &= |\varphi|^4\left(N_{ip}{}^{\lambda}N_{j\lambda}{}^{p} - N^{\lambda}{}_{pi}\,N_{j\lambda}{}^{p}\right)\\ &\quad -|\varphi|^4\left(\frac{1}{2}\left(N_-^2\right)^{\lambda}{}_{\lambda}g_{ij} - N_{p}{}^{\lambda}{}_{j}\,N^{p}{}_{\lambda i}\right)\\ &= -\frac{1}{2}|\varphi|^4(N_-^2)^{\lambda}{}_{\lambda}g_{ij} + |\varphi|^4(N_{ip}{}^{\lambda}N_{j\lambda}{}^{p} - N^{\lambda}{}_{pi}\,N_{j\lambda}{}^{p} + N_{p}{}^{\lambda}{}_{j}\,N^{p}{}_{\lambda i})\end{aligned}$$

Using the Bianchi identity, we readily find

$$N_{ip}{}^{\lambda}N_{j\lambda}{}^{p} - N^{\lambda}{}_{pi}\,N_{j\lambda}{}^{p} + N_{p}{}^{\lambda}{}_{j}\,N^{p}{}_{\lambda i} = -N_{\lambda j}{}^{p}N_{p}{}^{\lambda}{}_{i} \tag{3.161}$$

Therefore,

$$(\mathrm{I})\cdot\varphi_{iab}\omega^{ka}\omega^{pb} = -\frac{1}{2}|\varphi|^4(N_-^2)^{\lambda}{}_{\lambda}g_{ij} + |\varphi|^4N_{\lambda pj}N^{p\lambda}{}_{i}. \tag{3.162}$$

3.4.2 Computation for (II)

Next we work out the contributions of (II) in (3.148). The contributions from (III) will turn out to be similar.

$$\begin{aligned}\frac{1}{2}\mathrm{II} &= \frac{1}{2}\nabla_p(|\varphi|^2(N^{\dagger}\cdot\varphi)_{jk}) = -\frac{1}{2}\nabla_p(|\varphi|^2(N^{\dagger}\cdot\varphi)_{kj})\\ &= |\varphi|^2\alpha_pN^{\mu}{}_{j}{}^{\lambda}\varphi_{\mu k\lambda} - |\varphi|^2\nabla_pN^{\mu}{}_{j}{}^{\lambda}\varphi_{\mu k\lambda} - |\varphi|^2N^{\mu}{}_{j}{}^{\lambda}\nabla_p\varphi_{\mu k\lambda}\end{aligned} \tag{3.163}$$

Again, we will work out the bilinears term by term.

$$\begin{aligned}|\varphi|^2\alpha_pN^{\mu}{}_{j}{}^{\lambda}\varphi_{\mu k\lambda}\varphi_{iab}\omega^{ka}\omega^{pb} &= \frac{1}{4}|\varphi|^4\alpha_pN^{\mu}{}_{j}{}^{\lambda}(\omega_{\mu i}g_{\lambda b} + \omega_{\lambda b}g_{\mu i} - \omega_{\mu b}g_{\lambda i} - \omega_{\lambda i}g_{\mu b})\omega^{pb}\\ &= \frac{1}{4}|\varphi|^4\alpha_pN^{\mu}{}_{j}{}^{\lambda}(\omega_{\mu i}J^{p}{}_{\lambda} - \delta^{p}{}_{\lambda}g_{\mu i} + \delta^{p}{}_{\mu}g_{\lambda i} - \omega_{\lambda i}J^{p}{}_{\mu})\\ &= \frac{1}{4}|\varphi|^4\alpha_p(-N_{Ji,j}{}^{Jp} - N_{ij}{}^{p} + N^{p}{}_{ji} + N^{Jp}{}_{j,Ji})\\ &= \frac{1}{4}|\varphi|^4\alpha_p(-N_{ij}{}^{p} - N_{ij}{}^{p} + N^{p}{}_{ji} + N^{p}{}_{ji})\\ &= \frac{1}{2}|\varphi|^4\alpha_p(-N_{ij}{}^{p} + N^{p}{}_{ji})\end{aligned} \tag{3.164}$$

Next, we deal with the second term in (3.163)

$$
\begin{aligned}
-|\varphi|^2\nabla_p N^\mu{}_j{}^\lambda\varphi_{\mu k\lambda}\varphi_{iab}\omega^{ka}\omega^{pb} &= -\frac{1}{4}|\varphi|^4\nabla_p N^\mu{}_j{}^\lambda(\omega_{\mu i}g_{\lambda b}\\
&\quad+\omega_{\lambda b}g_{\mu i}-\omega_{\mu b}g_{\lambda i}-\omega_{\lambda i}g_{\mu b})\omega^{pb}\\
&= -\frac{1}{4}|\varphi|^4\nabla_p N^\mu{}_j{}^\lambda(\omega_{\mu i}J^p{}_\lambda\\
&\quad-\delta^p{}_\lambda g_{\mu i}+\delta^p{}_\mu g_{\lambda i}-\omega_{\lambda i}J^p{}_\mu)\\
&= \frac{1}{4}|\varphi|^4(\nabla_p N_{ij}{}^p-\nabla_p N^p{}_{ji})\\
&\quad-\frac{1}{4}|\varphi|^4(\omega_{\mu i}\nabla_p N^\mu{}_j{}^\lambda J^p{}_\lambda\\
&\quad-\omega_{\lambda i}\nabla_p N^\mu{}_j{}^\lambda J^p{}_\mu)
\end{aligned}
\tag{3.165}
$$

For the second group of terms in (3.165) , we need to take care of ∇J,

$$
\omega_{\mu i}\nabla_p N^\mu{}_j{}^\lambda J^p{}_\lambda-\omega_{\lambda i}\nabla_p N^\mu{}_j{}^\lambda J^p{}_\mu \tag{3.166}
$$

$$
\begin{aligned}
&= \omega_{\mu i}\nabla_p N^\mu{}_j{}^\lambda J^p{}_\lambda-\omega_{\mu i}\nabla_p N^\lambda{}_j{}^\mu J^p{}_\lambda\\
&= \omega_{\mu i}\nabla_p(N^\mu{}_j{}^\lambda+N^{\lambda\mu}{}_j)J^p{}_\lambda\\
&= -\omega_{\mu i}\nabla_p N_j{}^{\lambda\mu}J^p{}_\lambda\\
&= -\omega_{\mu i}(\nabla_p(N_j{}^{\lambda\mu}J^p{}_\lambda)-N_j{}^{\lambda\mu}\nabla_p J^p{}_\lambda)\\
&= \omega_{\mu i}\nabla_p N_{Jj}{}^{p\mu}-2\omega_{\mu i}N_j{}^{\lambda\mu}N_{p\lambda}{}^{Jp}\\
&= \omega_{\mu i}\nabla_p N_{Jj}{}^{p\mu}\\
&= \nabla_p N_{Jj}{}^{p\mu}J^\ell{}_\mu g_{\ell i}\\
&= \nabla_p(N_{Jj}{}^{p\mu}J^\ell{}_\mu g_{\ell i})-N_{Jj}{}^{p\mu}\nabla_p J^\ell{}_\mu g_{\ell i}\\
&= \nabla_p N_j{}^p{}_i-2N_j{}^{p\mu}N_{p\mu i}
\end{aligned}
\tag{3.167}
$$

Putting this back into the calculation,

$$
\begin{aligned}
-|\varphi|^2\nabla_p N^\mu{}_j{}^\lambda\varphi_{\mu k\lambda}\varphi_{iab}\omega^{ka}\omega^{pb} &= \frac{1}{4}|\varphi|^4(\nabla_p N_{ij}{}^p-\nabla_p N^p{}_{ji}-\nabla_p N_j{}^p{}_i)\\
&\quad+\frac{1}{2}|\varphi|^4N_j{}^{p\mu}N_{p\mu i}.\\
&= \frac{1}{2}|\varphi|^4(\nabla_p N_{ij}{}^p-\nabla_p N^p{}_{ji})+\frac{1}{2}|\varphi|^4N_j{}^{p\mu}N_{p\mu i}
\end{aligned}
\tag{3.168}
$$

where we used the Bianchi identity $-N_j{}^p{}_i = N_{ij}{}^p+N^p{}_{ij}$ to obtain the last equality above.

Now, we deal with the ∇N terms using the projected Levi–Civita connection

$$
\nabla_p N_{ij}{}^p-\nabla_p N^p{}_{ji} = \mathfrak{D}_p N_{ij}{}^p-N_{\alpha j}{}^p N_{pi}{}^\alpha-N_{i\alpha}{}^p N_{pj}{}^\alpha-N_{ij\alpha}N_p{}^{p\alpha}
$$

$$
\begin{aligned}
&-\mathfrak{D}_p N^p{}_{ji} + N_{\alpha ji} N_p{}^{p\alpha} + N^p{}_{\alpha i} N_{pj}{}^\alpha + N^p{}_{j\alpha} N_{pi}{}^\alpha \\
&= \mathfrak{D}_p N_{ij}{}^p - \mathfrak{D}_p N^p{}_{ji} - (N_{\alpha j}{}^p - N^p{}_{j\alpha}) N_{pi}{}^\alpha \\
&-(N_{i\alpha}{}^p - N^p{}_{\alpha i}) N_{pj}{}^\alpha
\end{aligned} \tag{3.169}
$$

since $N_p{}^{p\alpha} = 0$. Next, apply the Bianchi identity of N to the last two terms, and get

$$
\nabla_p N_{ij}{}^p - \nabla_p N^p{}_{ji} = \mathfrak{D}_p N_{ij}{}^p - \mathfrak{D}_p N^p{}_{ji} + N_j{}^p{}_\alpha N_{pi}{}^\alpha + N_\alpha{}^p{}_i N_{pj}{}^\alpha. \tag{3.170}
$$

So, we have

$$
\begin{aligned}
-|\varphi|^2 \nabla_p N^\mu{}_j{}^\lambda \varphi_{\mu k\lambda} \varphi_{iab} \omega^{ka} \omega^{pb} &= \frac{1}{2}|\varphi|^4 (\mathfrak{D}_p N_{ij}{}^p - \mathfrak{D}_p N^p{}_{ji}) \\
&+\frac{1}{2}|\varphi|^4 (N_j{}^p{}_\alpha N_{pi}{}^\alpha + N_\alpha{}^p{}_i N_{pj}{}^\alpha + N_j{}^{p\alpha} N_{p\alpha i}) \\
&= \frac{1}{2}|\varphi|^4 (\mathfrak{D}_p N_{ij}{}^p - \mathfrak{D}_p N^p{}_{ji}) + \frac{1}{2}|\varphi|^4 N_\alpha{}^p{}_i N_{pj}{}^\alpha
\end{aligned} \tag{3.171}
$$

Next, we deal with the last term in (3.163). Since $\nabla_p \varphi_{\mu k\lambda} = -\frac{1}{2}\alpha_p \varphi_{\mu k\lambda} + \frac{1}{2}\alpha_{Jp} \varphi_{J\mu,k\lambda} - E_{p;\mu k\lambda}$, we have

$$
-|\varphi|^2 N^\mu{}_j{}^\lambda \nabla_p \varphi_{\mu k\lambda} = |\varphi|^2 N^\mu{}_j{}^\lambda \left(\frac{1}{2}\alpha_p \varphi_{\mu k\lambda} - \frac{1}{2}\alpha_{Jp} \varphi_{J\mu,k\lambda} + E_{p;\mu k\lambda} \right). \tag{3.172}
$$

We work out the bilinears term by term.

$$
\begin{aligned}
\frac{1}{2}|\varphi|^2 \alpha_p N^\mu{}_j{}^\lambda \varphi_{\mu k\lambda} \varphi_{iab} \omega^{ka} \omega^{pb} &= \frac{1}{8}|\varphi|^4 \alpha_p N^\mu{}_j{}^\lambda (\omega_{\mu i} J^p{}_\lambda \\
&\quad -\delta^p{}_\lambda g_{\mu i} + \delta^p{}_\mu g_{\lambda i} - \omega_{\lambda i} J^p{}_\mu) \\
&= \frac{1}{8}|\varphi|^4 \alpha_p (-N_{Ji,j}{}^{Jp} - N_{ij}{}^p + N^p{}_{ji} + N^{Jp}{}_{j,Ji}) \\
&= \frac{1}{4}|\varphi|^4 \alpha_p (-N_{ij}{}^p + N^p{}_{ji})
\end{aligned} \tag{3.173}
$$

$$
\begin{aligned}
-\frac{1}{2}|\varphi|^2 \alpha_{Jp} N^\mu{}_j{}^\lambda \varphi_{J\mu,k\lambda} \varphi_{iab} \omega^{ka} \omega^{pb} &= -\frac{1}{8}|\varphi|^4 \alpha_{Jp} N^\mu{}_j{}^\lambda (-g_{\mu i} J^p{}_\lambda \\
&\quad -\delta^p{}_\lambda \omega_{\mu i} + J^p{}_\mu g_{\lambda i} + \delta^p{}_\mu \omega_{\lambda i}) \\
&= -\frac{1}{8}|\varphi|^4 \alpha_{Jp} (-N_{ij}{}^{Jp} + N_{Ji,j}{}^p + N^{Jp}{}_{ji} - N^p{}_{j,Ji}) \\
&= -\frac{1}{8}|\varphi|^4 \alpha_{Jp} (-2N_{ij}{}^{Jp} + 2N^{Jp}{}_{ji}) = \frac{1}{4}|\varphi|^4 \alpha_p N_j{}^p{}_i
\end{aligned} \tag{3.174}
$$

by the Bianchi identity satisfied by N. The terms E lead to

$$|\varphi|^2 N^{\mu}{}_{j}{}^{\lambda} E_{p;\mu k\lambda}\varphi_{iab}\omega^{ka}\omega^{pb} = |\varphi|^2 N^{\mu}{}_{j}{}^{\lambda}(N_{p\mu}{}^{\ell}\varphi_{\ell k\lambda} + N_{pk}{}^{\ell}\varphi_{\mu\ell\lambda} + N_{p\lambda}{}^{\ell}\varphi_{\mu k\ell})\varphi_{iab}\omega^{ka}\omega^{pb}$$

We compute the three terms

$$\begin{aligned}
|\varphi|^2 N^{\mu}{}_{j}{}^{\lambda} N_{p\mu}{}^{\ell}\varphi_{\ell k\lambda}\varphi_{iab}\omega^{ka}\omega^{pb} &= \frac{1}{2}|\varphi|^4 N^{\mu}{}_{j}{}^{\lambda} N_{p\mu}{}^{\ell}(\omega_{\ell i}g_{\lambda b} \\
&\quad + \omega_{\lambda b}g_{\ell i} - \omega_{\ell b}g_{\lambda i} - \omega_{\lambda i}g_{\ell b})\omega^{pb} \\
&= \frac{1}{2}|\varphi|^4 N^{\mu}{}_{j}{}^{\lambda} N_{p\mu}{}^{\ell}(\omega_{\ell i}J^{p}{}_{\lambda} - \delta^{p}{}_{\lambda}g_{\ell i} \\
&\quad + \delta^{p}{}_{\ell}g_{\lambda i} - \omega_{\lambda i}J^{p}{}_{\ell}) \\
&= \frac{1}{2}|\varphi|^4(-N^{\mu}{}_{j}{}^{Jp}N_{p\mu,Ji} - N^{\mu}{}_{j}{}^{p}N_{p\mu i} \\
&\quad + N^{\mu}{}_{ji}N_{p\mu}{}^{p} + N^{\mu}{}_{j,Ji}N_{p\mu}{}^{Jp}) \\
&= \frac{1}{2}|\varphi|^4(N^{\mu}{}_{j}{}^{p}N_{p\mu i} - N^{\mu}{}_{j}{}^{p}N_{p\mu i}) = 0
\end{aligned} \tag{3.175}$$

$$\begin{aligned}
|\varphi|^2 N^{\mu}{}_{j}{}^{\lambda} N_{p\lambda}{}^{\ell}\varphi_{\mu k\ell}\varphi_{iab}\omega^{ka}\omega^{pb} &= \frac{1}{2}|\varphi|^4 N^{\mu}{}_{j}{}^{\lambda} N_{p\lambda}{}^{\ell}(\omega_{\mu i}g_{\ell b} \\
&\quad + \omega_{\ell b}g_{\mu i} - \omega_{\mu b}g_{\ell i} - \omega_{\ell i}g_{\mu b})\omega^{pb} \\
&= \frac{1}{2}|\varphi|^4 N^{\mu}{}_{j}{}^{\lambda} N_{p\lambda}{}^{\ell}(\omega_{\mu i}J^{p}{}_{\ell} - \delta^{p}{}_{\ell}g_{\mu i} \\
&\quad + \delta^{p}{}_{\mu}g_{\ell i} - \omega_{\ell i}J^{p}{}_{\mu}) \\
&= \frac{1}{2}|\varphi|^4(-N_{Ji,j}{}^{\lambda}N_{p\lambda}{}^{Jp} - N_{ij}{}^{\lambda}N_{p\lambda}{}^{p} \\
&\quad + N^{p}{}_{j}{}^{\lambda}N_{p\lambda i} + N^{Jp}{}_{j}{}^{\lambda}N_{p\lambda,Ji}) \\
&= \frac{1}{2}|\varphi|^4(N^{p}{}_{j}{}^{\lambda}N_{p\lambda i} - N^{Jp}{}_{j}{}^{\lambda}N_{p\lambda,Ji}) = 0
\end{aligned} \tag{3.176}$$

The second term in (3.175) is more complicated, we first note that, by interchanging indices $k \leftrightarrow p$ and $a \leftrightarrow b$,

$$N^{\mu}{}_{j}{}^{\lambda} N_{pk}{}^{\ell}\varphi_{\mu\ell\lambda}\varphi_{iab}\omega^{ka}\omega^{pb} = -N^{\mu}{}_{j}{}^{\lambda} N_{kp}{}^{\ell}\varphi_{\mu\ell\lambda}\varphi_{iab}\omega^{ka}\omega^{pb} \tag{3.177}$$

It follows that

$$\begin{aligned}
N^{\mu}{}_{j}{}^{\lambda} N_{pk}{}^{\ell}\varphi_{\mu\ell\lambda}\varphi_{iab}\omega^{ka}\omega^{pb} &= \frac{1}{2}N^{\mu}{}_{j}{}^{\lambda}(N_{pk}{}^{\ell} - N_{kp}{}^{\ell})\varphi_{\mu\ell\lambda}\varphi_{iab}\omega^{ka}\omega^{pb} \\
&= \frac{1}{2}N^{\mu}{}_{j}{}^{\lambda}(N_{pk}{}^{\ell} + N_{k}{}^{\ell}{}_{p})\varphi_{\mu\ell\lambda}\varphi_{iab}\omega^{ka}\omega^{pb}
\end{aligned}$$

$$= -\frac{1}{2} N^{\mu}{}_{j}{}^{\lambda} N^{\ell}{}_{pk} \varphi_{\mu\ell\lambda} \varphi_{iab} \omega^{ka} \omega^{pb} \tag{3.178}$$

where we use Bianchi identity to get the last equality. Now, we can ready to use the identity $N^{p}{}_{k\ell}\varphi_{pij} = -N^{p}{}_{ij}\varphi_{pk\ell}$ to handle the second term in (3.175)

$$\begin{aligned} |\varphi|^2 N^{\mu}{}_{j}{}^{\lambda} N_{pk}{}^{\ell} \varphi_{\mu\ell\lambda}\varphi_{iab}\omega^{ka}\omega^{pb} &= -\frac{1}{2}|\varphi|^2 N^{\mu}{}_{j}{}^{\lambda} N^{\ell}{}_{pk}\varphi_{\mu\ell\lambda}\varphi_{iab}\omega^{ka}\omega^{pb} \\ &= \frac{1}{2}|\varphi|^2 N^{\mu}{}_{j}{}^{\lambda} N^{\ell}{}_{\mu\lambda}\varphi_{\ell kp}\varphi_{iab}\omega^{ka}\omega^{pb} \\ &= -\frac{1}{2}|\varphi|^4 N^{\mu}{}_{j}{}^{\lambda} N^{\ell}{}_{\mu\lambda} g_{\ell i} \\ &= \frac{1}{2}|\varphi|^4 N^{\mu\lambda}{}_{j} N_{i\mu\lambda}. \end{aligned} \tag{3.179}$$

So,

$$|\varphi|^2 N^{\mu}{}_{j}{}^{\lambda} E_{p;\mu k\lambda}\varphi_{iab}\omega^{ka}\omega^{pb} = \frac{1}{2}|\varphi|^4 N^{\mu\lambda}{}_{j} N_{i\mu\lambda}. \tag{3.180}$$

Putting the above calculation together, we obtain

$$\begin{aligned} -|\varphi|^2 N^{\mu}{}_{j}{}^{\lambda}\nabla_p\varphi_{\mu k\lambda}\varphi_{iab}\omega^{ka}\omega^{pb} &= \frac{1}{4}|\varphi|^4\alpha_p(-N_{ij}{}^{p} + N^{p}{}_{ji}) \\ &\quad + \frac{1}{4}\|\varphi\|^4\alpha_p N_{j}{}^{p}{}_{i} + \frac{1}{2}\|\varphi\|^4 N^{\mu\lambda}{}_{j} N_{i\mu\lambda} \\ &= \frac{1}{2}|\varphi|^4 N^{\mu\lambda}{}_{j} N_{i\mu\lambda} + \frac{1}{2}|\varphi|^4\alpha_p N_{j}{}^{p}{}_{i} \end{aligned} \tag{3.181}$$

using Bianchi identity $N^{p}{}_{ji} + N_{i}{}^{p}{}_{j} + N_{ji}{}^{p} = 0$.

Back to (3.163), using (3.164) (3.171) and (3.181), we complete the calculation for (II):

$$\begin{aligned} \text{(II)} \cdot \varphi_{iab}\omega^{ka}\omega^{pb} &= |\varphi|^4\alpha_p(-N_{ij}{}^{p} + N^{p}{}_{ji}) + |\varphi|^4(\mathfrak{D}_p N_{ij}{}^{p} - \mathfrak{D}_p N^{p}{}_{ji}) \\ &\quad + |\varphi|^4 N_{\lambda}{}^{p}{}_{i} N_{pj}{}^{\lambda} + |\varphi|^4 N^{\mu\lambda}{}_{j} N_{i\mu\lambda} + |\varphi|^4\alpha_p N_{j}{}^{p}{}_{i} \\ &= |\varphi|^4\mathfrak{D}_p N_{ji}{}^{p} + 2|\varphi|^4\alpha_p N_{j}{}^{p}{}_{i} - |\varphi|^4 N^{p\lambda}{}_{i} N_{p\lambda j} \end{aligned} \tag{3.182}$$

Note that $N^{p}{}_{ji} = 0$ up to the symmetrization for $(i \leftrightarrow j)$. So, terms involving $N^{p}{}_{ij}$ vanish up to the symmetrization for $(i \leftrightarrow j)$. For the two quadratic terms about N, we use Bianchi identity to obtain the last line. Thus

$$\text{(II)} \cdot \varphi_{iab}\omega^{ka}\omega^{pb} = |\varphi|^4\left\{\mathfrak{D}_p N_{ji}{}^{p} + 2\alpha_p N_{j}{}^{p}{}_{i} - N^{p\lambda}{}_{i} N_{p\lambda j}\right\} \tag{3.183}$$

3.4.3 Computation for (III)

Next, we consider (III) in (3.148). We simply observe that by switching the indices $k \leftrightarrow p$ and $a \leftrightarrow b$ and exploiting the antisymmetry of $(N^{\dagger}\varphi)_{kj}$ in j and k, we may write

$$
\begin{aligned}
(\mathrm{III}) \cdot \varphi_{iab}\omega^{ka}\omega^{pb} &= \nabla_k(|\varphi|^2(N^\dagger \cdot \varphi)_{pj})\varphi_{iab}\omega^{ka}\omega^{pb} \\
&= -\nabla_p(|\varphi|^2(N^\dagger \cdot \varphi)_{kj})\varphi_{iab}\omega^{ka}\omega^{pb} \\
&= \nabla_p(|\varphi|^2(N^\dagger \cdot \varphi)_{jk})\varphi_{iab}\omega^{ka}\omega^{pb} \\
&= (\mathrm{II}) \cdot \varphi_{iab}\omega^{ka}\omega^{pb}.
\end{aligned} \tag{3.184}
$$

We can now put (I), (II) and (III) all together,

$$
\begin{aligned}
&(d(|\varphi|^2 N^\dagger \cdot \varphi))_{jkp} \cdot \varphi_{iab}\omega^{ka}\omega^{pb} + (i \leftrightarrow j) \\
&\quad = -|\varphi|^4 (N_-^2)^\lambda{}_\lambda g_{ij} + |\varphi|^4 \left\{ N_{\lambda pj} N^{p\lambda}{}_i + (i \leftrightarrow j) \right\} \\
&\qquad + 2|\varphi|^4 \left\{ \mathfrak{D}_p N_{ji}{}^p + 2\alpha_p N_j{}^p{}_i - N^{p\lambda}{}_i N_{p\lambda j} + (i \leftrightarrow j) \right\}
\end{aligned}
$$

Lemma 12 *In conclusion, we have*

$$
\begin{aligned}
&d(|\varphi|^2 N^\dagger \cdot \varphi)_{jkp} \cdot \varphi_{iab}\omega^{ka}\omega^{pb} + (i \leftrightarrow j) \\
&\quad = |\varphi|^4 \big\{ 2(\mathfrak{D}_p N_{ji}{}^p + \mathfrak{D}_p N_{ij}{}^p) + 4\alpha_p (N_j{}^p{}_i + N_i{}^p{}_j) \\
&\qquad - (N_-^2)^\lambda{}_\lambda g_{ij} - 4(N_+^2)_{ij} + 2(N_-^2)_{ij} \big\}
\end{aligned} \tag{3.185}
$$

3.5 The Flow of g_{ij}

Assembling all the terms in (3.3) and putting them in (3.4), we obtain the flow of $\tilde{g}_{ij}$,

$$
\begin{aligned}
\partial_t \tilde{g}_{ij} = -\Big\{ &(-|\varphi|^2 d d^\dagger \varphi)_{jkp}\varphi_{iab}\omega^{ka}\omega^{pb} - (d|\varphi|^2 \wedge d^\dagger \varphi)_{jkp}\varphi_{iab}\omega^{ka}\omega^{pb} \\
&+ (d\iota_{\nabla|\varphi|^2}\varphi)_{jkp}\varphi_{iab}\omega^{ka}\omega^{pb} \\
&+ (2d(|\varphi|^2 N^\dagger \cdot \varphi))_{jkp}\varphi_{iab}\omega^{ka}\omega^{pb} + (i \leftrightarrow j) \Big\}
\end{aligned} \tag{3.186}
$$

By (3.98), (3.116), Lemmas 11 and 12, and the identity (2.18),

$$
\begin{aligned}
\partial_t \tilde{g}_{ij} &= -|\varphi|^4 \Big\{ 2(\mathfrak{D}_k N_{ij}{}^k + \mathfrak{D}_k N_{ji}{}^k) + R g_{ij} + 2\nabla_\mu \alpha^\mu g_{ij} + \frac{1}{2}(\nabla_j \alpha_i + \nabla_i \alpha_j) \\
&\quad - \frac{1}{2}(J^p{}_j J^q{}_i \nabla_p \alpha_q + J^p{}_i J^q{}_j \nabla_p \alpha_q) - \alpha_i \alpha_j + \alpha_{Ji}\alpha_{Jj} \\
&\quad - 2\alpha_\mu \alpha^\mu g_{ij} + 4\alpha_p (N_j{}^p{}_i + N_i{}^p{}_j) \Big\}
\end{aligned} \tag{3.187}
$$

Recall that $\tilde{g}_{ij} = |\varphi|^2 g_{ij}$. Therefore

$$
\partial_t \log \det \tilde{g} = |\varphi|^{-2} g^{ij} \partial_t \tilde{g}_{ij} = |\varphi|^2 \big\{ -12\nabla_\mu \alpha^\mu - 6R + 12|\alpha|^2 \big\}. \tag{3.188}
$$

Since $\det\tilde{g} = |\varphi|^{12}\det g$ and $\partial_t \det g = 0$ as the volume form of g equals to $\omega^3/3!$ and ω is fixed, we have

$$\partial_t \log|\varphi|^2 = \frac{1}{6}\log\det\tilde{g} \tag{3.189}$$

Then, we conclude

$$\partial_t \log|\varphi|^2 = |\varphi|^2\Big\{-2\nabla_\mu\alpha^\mu - R + 2|\alpha|^2\Big\} \tag{3.190}$$

The flow of $g_{ij} = |\varphi|^{-2}\tilde{g}_{ij}$ is

$$\partial_t g_{ij} = |\varphi|^{-2}\{\partial_t\tilde{g}_{ij} - (\partial_t \log|\varphi|^2)g_{ij}\}. \tag{3.191}$$

Substituting the equations derived above,

$$\begin{aligned}\partial_t g_{ij} = -|\varphi|^2\Big\{&2(\mathfrak{D}_p N_{ij}{}^p + \mathfrak{D}_p N_{ji}{}^p) - \nabla_i\nabla_j \log|\varphi|^2 + J^p{}_i J^q{}_j \nabla_p\nabla_q \log|\varphi|^2\\ &-\alpha_i\alpha_j + \alpha_{Ji}\alpha_{Jj} + 4\alpha_p(N_j{}^p{}_i + N_i{}^p{}_j)\Big\}\end{aligned} \tag{3.192}$$

using $\alpha_i = -\partial_i \log|\varphi|^2$. The Ricci curvature of g_{ij} is given by (2.32). Substituting this into (3.192), we obtain the flow of g_{ij} as stated in Theorem 1. □

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