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Towards Ginzburg–Landau Bogomolny Approach and a Perturbative Description of Superconducting Structures

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Abstract: The Bogomolny approach to the Ginzburg–Landau equations in the context of strong and semi-strong necessary conditions is formulated for various superconducting structures in a quasi-one-dimensional description, considering both flat and curved geometries. This formulation is justified by a perturbative approach to the Ginzburg–Landau theory applied to a superconducting structure that is polarized by an electric charge moving across two neighboring quantum dots. The situation considered involves an interface between a Josephson junction and a semiconductor quantum dot system in a one-dimensional setting.

Keywords: Bogomolny equations; Ginzburg–Landau model; superconductors

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1. Motivation Behind Bogomolny Approach

The Euler–Lagrange equations of many models in physics are nonlinear partial differential equations of the second order, but sometimes one can consider, instead of them, the equations of the first order, the so-called the Bogomolny equations (or Bogomol'nyi equations). They were derived by Bogomolny for, among others, the Ginzburg–Landau (G-L) model in [1] (although independently, they were derived in [2], for another model: SU(2)), and the Yang–Mills theory, and, then, they are often called Bogomolny–Prasad–Sommerfeld equations (BPS equations), because certain classes of their solutions were found in [3] and similar results were obtained in [4] (cited in this context, only in [5]). Some presentations of the Bogomolny equations and their solutions for the G-L model were given, e.g., in [6,7].

The Bogomolny equations, for a modified G-L model, were derived in [8], but there was a coupling between the Yang–Mills term in the Lagrangian with the Higgs field through a continuous function, so it was a different model from the one investigated in this paper. The models, for which one can derive the Bogomolny equations (or BPS equations), are called BPS models, or one often says that there exists a BPS bound, e.g., [9–11]. There are two advantages of them. Owing to deriving Bogomolny equations for a given model, one can expect the existence of soliton solutions. Moreover, obtaining exact solutions is possible, too. Such solutions significantly allow for a larger understanding of the considered nonlinear models. The Bogomolny equations also guarantee the existence of a topological Bogomolny bound, and it causes a topological stability of solitons carrying a non-trivial value of the corresponding topological charge. Therefore, the property of the BPS equation is very important.

In addition, one can consider deriving the Bogomolny equations as an opportunity for a reduction in the full second-order static equation of motion to a set of first-order equations. The classical Bogomolny approach (completion to square in the action functional), was applied in many papers, and gave many important results, e.g., [12–20]. However, there is a necessity to use a completely general method, which allows one to derive BPS equations (if possible) in a more systematic way. The method is known as the concept of strong necessary conditions (CSNC). It was originally introduced and analyzed in [21–28], and was further developed in 2016 by Adam and Santamaria, in [29] (they proposed therein the so-called first-order Euler–Lagrange (FOEL) formalism). This concept was successfully applied to derive the Bogomolny equations for many nonlinear field-theoretical models, e.g., the models from Skyrme-family ones and a gauged nonlinear $\sigma O(3)$ model [30–32]. Some Bogomolny equations for some BPS Skyrme submodels, and their exact solutions (compact skyrmions), are found in [33]. The classical methods for deriving the BPS equations (i.e., including the classical trick applied by Bogomolny, independently by Belavin, Polyakov, Schwartz and Tyupkin, and also by Prasad and Sommerfield) are presented in [1–3,34].

In this paper, we apply the CSNC method (more exactly, its semi-strong version) to derive the Bogomolny equations (BPS equations) for the gauged G-L model in a curved space. This model was introduced by Ginzburg and Landau in the 1950s and it provides a description of superconductivity. Formally, the G-L model is a limit of the BCS model (which explains the phenomenon of superconductivity by the well-known notion of Cooper pairs of superconducting electrons). The G-L model has been investigated for many years, for example, in [8,35–44]. For both cases, the G-L model in a flat space and a curved space, certain exact solutions of the Bogomolny equations are derived.

The next section includes some preliminaries. In Section 2, we briefly describe the Bogomolny equations for the usual gauged G-L model. Section 3 is devoted to the gauged G-L model in the curved space and the concept of strong necessary conditions. In the next section, we derive the Bogomolny equations for the gauged G-L model in a flat space. Section 5 is devoted to a derivation of the Bogomolny equations for the G-L model in a curved space. In the next section, a perturbative approach in the description of the Josepshon junction is presented.

The last section includes some conclusions.

2. Elementary Introduction to Bogomolny Theory

Now, we present the Bogomolny approach, based on the scalar field theory model ϕ^4 with spontaneous symmetry breaking [7]

$$E = \int_{-\infty}^{\infty} \left(\frac{1}{2} \left(\frac{d\psi}{dx} \right)^2 + \frac{\lambda}{2} (\psi^2 - \psi_0^2)^2 \right) dx, \tag{1}$$

where $\psi(x) \in \mathbb{R}$, and $\lim_{x \to \pm\infty} \psi(x) = \pm\psi_0$ (what can be considered as a superconducting order parameter far from the defect of the superconducting order parameter that can take place in the Josephson junction). The Euler–Lagrange equations for this model are similar to the G-L equation and have the form of [7]

$$\frac{d^2\psi}{dx^2} = 2\lambda\psi(\psi^2 - \psi_0^2). \tag{2}$$

We can avoid solving them; namely, we write the formula for E in (1), as follows

$$E = \int_{-\infty}^{\infty} \left(\frac{1}{2} \left(\frac{d\psi}{dx} + \sqrt{\lambda}(\psi^2 - \psi_0^2) \right)^2 - \sqrt{\lambda} \ \frac{d\psi}{dx} (\psi^2 - \psi_0^2) \right) dx, \tag{3}$$

We integrate the term $\sqrt{\lambda}\ \frac{d\psi}{dx}(\psi^2 - \psi_0^2)$ in (3). We obtain [7]

$$E = \int_{-\infty}^{\infty} \frac{1}{2}\left(\frac{d\psi}{dx} + \sqrt{\lambda}(\psi^2 - \psi_0^2)\right)^2 dx + \frac{2\sqrt{\lambda}}{3}\psi_0^2 \mid Q \mid, \quad Q = \psi(\infty) - \psi(-\infty), \tag{4}$$

where Q is a topological charge. The topological charge can indicate the presence of vortices in a Josephson junction or just imprint the wave function phase difference of the Josephson junction due to an external magnetic field source or a current flow through the junction (defect of order parameter). If we now require reaching the minimum by the functional (4), then the first term needs to vanish [7]

$$\frac{d\psi}{dx} = \sqrt{\lambda}(\psi_0^2 - \psi^2). \tag{5}$$

This is just the Bogomolny equation. It is a very well-known solution to the so-called "kink" [7]

$$\psi(x) = \psi_0 \tanh\left(\psi_0\sqrt{\lambda}(x - x_0)\right). \tag{6}$$

This kink solution can correspond to a non-superconductor vs. superconductor interface in the Josephson junction, where we have two superconductors with a different asymptotic phase imprint on a macroscropic wave function. As we can see, the essence of the above derivation of the Bogomolny Equation (5) is completing the energy functional to the square term. Another approach to the derivation of the Bogomolny equations is a variational one. It is based on the concept of strong necessary conditions, and we describe it in Section 3.

3. A Short Description of the Bogomolny Equations for a Usual Gauged G-L Model

The Lagrangian for the standard gauged G-L model is well known and has the form of [7]

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} - \left(\frac{\partial}{\partial x^\mu} + 2iA_\mu\right)\psi^*\left(\frac{\partial}{\partial x_\mu} - 2iA^\mu\right)\psi - \frac{\beta}{4}(\mid \psi \mid^2 - \eta^2)^2, \tag{7}$$

where $\mu = 0, 1, 2$; $\psi \in \mathbb{C}$; $\eta^2 = \frac{1}{8e^2\lambda^2}$; $F_{\mu\nu} = \frac{\partial A_\mu}{\partial x^\nu} - \frac{\partial A_\nu}{\partial x^\mu}$ is an anti-symmetric electromagnetic tensor; and $F_{\mu\nu}F^{\mu\nu} = \frac{1}{2}(B^2 - \frac{E^2}{c^2})$. We consider a static field configuration; then, we have only a magnetic field in the z direction and we focus on the Hamiltonian

$$\mathcal{H} = \left| \left(\left(\frac{\partial}{\partial x} - 2iA_x\right) \pm i\left(\frac{\partial}{\partial y} - 2iA_y\right)\right)\psi\right|^2 + \frac{1}{2}(B_z \pm 2e(\mid \psi \mid^2 - \eta^2))^2 \pm 2e\eta^2 B_z + \frac{1}{4}(\beta - 8e^2)(\mid \psi \mid^2 - \eta^2)^2, \tag{8}$$

The most interesting case is when $\beta = 8e^2$. The Hamiltonian $\mathcal{H}$ can be now written in the following form:

$$\mathcal{H} = \left| \left(\left(\frac{\partial}{\partial x} - 2iA_x\right) \pm i\left(\frac{\partial}{\partial y} - 2iA_y\right)\right)\psi \right|^2 + \frac{1}{2}(B_z \pm 2e(\mid \psi \mid^2 - \eta^2))^2 \pm 2e\eta^2 B_z. \tag{9}$$

Then,

$$H \geq 2e\eta^2 \left| \int B_z d^2x \right|. \tag{10}$$

This inequality is saturated, i.e., when the ground state energy of a macroscopic wave function reaches the minimum, when canonical momentum is set to zero, so we have [7]

$$\left(\left(\frac{\partial}{\partial x} - 2iA_x\right) \pm i\left(\frac{\partial}{\partial y} - 2iA_y\right)\right)\psi = 0, \\ B_z = \mp 2e(\mid \psi \mid^2 - \eta^2). \tag{11}$$

The last equation is the value of the critical magnetic field above which superconducting phenomena are vanishing. We call these Equations (11) the Bogomolny equations (Bogomol'nyi equations, BPS equations).

So, as we can see in both examples, for the scalar field model and the G-L model, the Bogomolny equations are the differential equations of the first order, contrary to the Euler–Lagrange equations, which are (for this model) the partial differential equations of the second order. In addition, all the solutions of the Bogomolny equations are also the solutions of the Euler–Lagrange equations [25] (but the inverse situation does not hold in general; see, e.g., [45]).

One can show that the flux of the magnetic field, as the Abrikosov vortex, is always subjected to the following quantization:

$$\Phi = \int B_z dxdy = \frac{n\pi}{e}, \tag{12}$$

then,

$$H \geq 2\pi \mid n \mid \eta^2 \tag{13}$$

(for the solutions of (11), we have the saturation of this inequality). So, we have a connection between the minimal value of the energy of the Cooper pair condensate and the topological aspect of the solution.

One applies the following ansätzes for a vector potential in two dimensions, present in the Abrikosov vortex, with a rotational symmetry given as

$$\psi(\vec{r}) = \eta e^{in\theta}(1 + f(r)), \tag{14}$$

$$A_x = -\frac{n}{e}\frac{y}{r^2}(1 + \alpha(r)), \tag{15}$$

$$A_y = \frac{n}{e}\frac{x}{r^2}(1 + \alpha(r)), \tag{16}$$

Then, the magnetic field, $B_z = \frac{n\alpha'(r)}{2er}$, and the Bogomolny Equation (11) are as follows [7]:

$$f' = -\frac{n\alpha(1 + f)}{r}, \quad \alpha' = -\frac{r}{2n\lambda^2}(2f + f^2). \tag{17}$$

We require certain asymptotic limits, which gives us the following conditions:

$$\lim_{r \to 0} \alpha(r) = -1, \lim_{r \to 0} f(r) = -1, \lim_{r \to \infty} \alpha(r) = 0, \lim_{r \to \infty} f(r) = 0. \tag{18}$$

The first two equations express superconducting order parameters in a vortex core, set to zero, and the very last two equations point towards a constant flat superconducting order parameter, far away from a single vortex core.

No exact solutions of (17) are known; however, one can apply numerical procedures to find non-trivial and physically interesting solutions to (17). For big r values, we obtain asymptotic relations [7]:

$$f(r) = -nK_0\left(\frac{r}{\lambda}\right) + O(e^{-2r/\lambda}), \tag{19}$$

$$\alpha = -\frac{r}{\lambda}K_1\left(\frac{r}{\lambda}\right) + O(e^{-2r/\lambda}), \tag{20}$$

where K_0 and K_1 are the modified Bessel functions of the second kind. In this case,

$$2e\lambda^2 B_z \to -nK_0\left(\frac{r}{\lambda}\right) + O(e^{-2r/\lambda}). \tag{21}$$

For a small r value, we have [7]

$$f(r) \approx -1 + C(n)r^n, \tag{22}$$

$$\alpha(r) \approx -1 + \frac{r^2}{4n\lambda^2}, \tag{23}$$

where $C(n) = const$ can be determined numerically for the given n that is an integer number of fluxons inside the Abrikosov vortex core, and B_z tends to $B_z(0) = \frac{1}{4e\lambda^2}$.

4. The Gauged G-L Model in a Curved Space and the Concept of Strong Necessary Conditions in the Equations of Motion

Being motivated by [46,47], we formulate the Lagrangian in a curvilinear case, referring to Figure 1.

The Lagrangian of the explored gauged nonlinear G-L model, in the case of a spiral predefined by $r(\phi), r' = \frac{dr}{d\phi}$, has the following dependence in the cylindrical coordinates:

$$\begin{aligned}\mathcal{L} = \frac{1}{2m}a_1\omega^*\left(-\frac{h}{i}\frac{d}{d\phi} - ea_2A_\phi(\phi)\right)^2\omega + a_3\omega^*\omega + V(\omega,\omega^*) + \\ c_1\left(\frac{1}{r}\right)^2\left(\frac{1}{r'}\frac{d}{d\phi}(rA_\phi) - \frac{d}{d\phi}A_r\right)^2,\end{aligned} \tag{24}$$

where $a_1(r(\phi)) = \frac{1}{\frac{1}{r(\phi)^2} + (\frac{d\phi}{dr})^2}$ or effective mass $m_{eff} = m[\frac{1}{r(\phi)^2} + (\frac{d\phi}{dr})]^2$. Here, the effective GL potential $V_{eff}(\omega,\omega^*)$ is a renormalized version of the standard GL potential $V(\omega,\omega^*)$ for a straight nanowire and due to a non-zero curvature of the nanocable $V(\omega,\omega^*) \to V_{eff}(\omega,\omega^*)$, so we have the form of

$$\begin{aligned}V_{eff}(\omega,\omega^*) = [\alpha(\phi)\omega^*\omega + \beta|\omega^*\omega|^2] + a_3\omega^*\omega = V(\omega,\omega^*) + a_3\omega^*\omega = V(\omega,\omega^*) + \\ \omega^*\frac{(\frac{1}{r'})^2 + (\frac{1}{r})^2]}{2m}\Big[-\Big[e\frac{d}{d\phi}\frac{[A_r\frac{1}{r'} + A_\phi\frac{1}{r}]}{[(\frac{1}{r'})^2 + (\frac{1}{r})^2]}\Big]^2 + \frac{e^2(A_r^2 + A_\phi^2)}{[(\frac{1}{r'})^2 + (\frac{1}{r})^2]} \\ +e\frac{\hbar}{i}\Big[\Big[\frac{d}{d\phi}\frac{[A_r\frac{1}{r'} + A_\phi\frac{1}{r}]}{[(\frac{1}{r'})^2 + (\frac{1}{r})^2]}\Big] - \frac{[\frac{1}{r'}(\frac{d}{d\phi}A_r)\Big] + \frac{1}{r}(\frac{d}{d\phi}A_\phi)]}{[(\frac{1}{r'})^2 + (\frac{1}{r})^2]}\Big]\Big]\omega\end{aligned} \tag{25}$$

and it encodes all superconducting properties incorporated in a geometry of superconducting bent wire. The coefficient a_3 brings a non-Hermicity to the Hamiltonian due to the term $e\frac{\hbar}{i}\Big[\Big[\frac{d}{d\phi}\frac{[A_r\frac{1}{r'}+A_\phi\frac{1}{r}]}{[(\frac{1}{r'})^2+(\frac{1}{r})^2]}\Big] - \frac{[\frac{1}{r'}(\frac{d}{d\phi}A_r)]+\frac{1}{r}(\frac{d}{d\phi}A_\phi)]}{[(\frac{1}{r'})^2+(\frac{1}{r})^2]}\Big]$, and thus, the described system can be categorized as a dissipative one. We observe that the two-dimensional vector potential is renormalized into the effective one-dimensional vector potential, in cases of the two-dimensional open or closed superconducting spiral, so effectively, one observes that

$$(A(\phi)_r), A(\phi)_\phi) \to \left(0, \frac{[A_r(\phi)\frac{1}{r'(\phi)} + A_\phi(\phi)\frac{1}{r(\phi)}]}{(\frac{1}{r'(\phi)})^2 + (\frac{1}{r(\phi)})^2}\right),$$
$$a_2 = \frac{1}{A_\phi(\phi)} \frac{[A_r(\phi)\frac{1}{r'(\phi)} + A_\phi(\phi)\frac{1}{r(\phi)}]}{(\frac{1}{r'(\phi)})^2 + (\frac{1}{r(\phi)})^2}. \quad (26)$$

Furthermore, bending a superconducting cable reshapes a coefficient $\alpha(\phi)$, denoting the level of superconductivity or non-superconductivity (when its value is zero or positive), so the following renormalized value is obtained.

$$\alpha(\phi) \to \alpha(\phi) + \left[\frac{[(\frac{1}{r'})^2 + (\frac{1}{r})^2]}{2m}\Big[- \Big[e\frac{d}{d\phi}\frac{[A_r\frac{1}{r'} + A_\phi\frac{1}{r}]}{[(\frac{1}{r'})^2 + (\frac{1}{r})^2]}\Big]^2 + \frac{e^2(A_r^2 + A_\phi^2)}{[(\frac{1}{r'})^2 + (\frac{1}{r})^2]}\right.$$
$$\left. + e\frac{\hbar}{i}\Big[\Big[\frac{d}{d\phi}\frac{[A_r\frac{1}{r'} + A_\phi\frac{1}{r}]}{[(\frac{1}{r'})^2 + (\frac{1}{r})^2]}\Big] - \frac{[\frac{1}{r'}(\frac{d}{d\phi}A_r)\Big] + \frac{1}{r}(\frac{d}{d\phi}A_\phi)]}{[(\frac{1}{r'})^2 + (\frac{1}{r})^2]}\Big]\right]. \quad (27)$$

In case of a superconducting cable with a constant preimposed electric flow, we have

$$A_\phi(\phi) = -\frac{Jm}{e}\frac{1}{|\omega(\phi)|^2}\frac{r(\phi)}{\sqrt{r(\phi)^2 + (\frac{dr}{d\phi})^2}}, A_r(\phi) = -\frac{Jm}{e}\frac{1}{|\omega|^2}\frac{\frac{dr}{d\phi}}{\sqrt{r(\phi)^2 + (\frac{dr}{d\phi})^2}}. \quad (28)$$

This implies

$$\alpha(\phi) \to \alpha(\phi) + \left[\frac{[(\frac{d\phi}{dr})^2 + (\frac{1}{r})^2]}{2m}\Big[- \Big[Jm\frac{d}{d\phi}\frac{\frac{1}{\sqrt{r^2+(\frac{dr}{d\phi})^2}}}{[(\frac{d\phi}{dr})^2 + (\frac{1}{r})^2]}\Big]^2 + \frac{1}{|\psi|^4}\frac{(m^2J^2)}{[(\frac{d\phi}{dr})^2 + (\frac{1}{r})^2]}\right.$$
$$+ \frac{\hbar}{i}Jm\Big[- \frac{d}{d\phi}\Big[\frac{1}{|\omega|^2}\frac{1}{[(\frac{d\phi}{dr})^2 + (\frac{1}{r})^2]}\frac{1}{\sqrt{r(\phi)^2 + (\frac{dr}{d\phi})^2}}\Big] +$$
$$\left. + \frac{\Big[\frac{d\phi}{dr}(\frac{d}{d\phi}\frac{\frac{dr}{d\phi}}{\sqrt{r(\phi)^2+(\frac{dr}{d\phi})^2}})\Big] + \frac{1}{r}(\frac{d}{d\phi}\frac{r}{\sqrt{r(\phi)^2+(\frac{dr}{d\phi})^2}})]}{[(\frac{d\phi}{dr})^2 + (\frac{1}{r})^2]}\Big]\right]. \quad (29)$$

The essence of the concept of strong necessary conditions is that we replace the consideration of the Euler–Lagrange equations,

$$\mathcal{L}_{,u} - \frac{d}{dx}\mathcal{L}_{,u_{,x}} - \frac{d}{dt}\mathcal{L}_{,u_{,t}} = 0, \quad (30)$$

following from the extremum principle, applied to the functional

$$S[u] = \int_{E^2} \mathcal{L}(u, u_{,x}, u_{,t})\, dxdt, \quad (31)$$

with considering the strong necessary conditions [21–40,42–48]

$$\mathcal{L}_{,u} = 0, \quad (32)$$
$$\mathcal{L}_{,u_{,t}} = 0, \quad (33)$$
$$\mathcal{L}_{,u_{,x}} = 0, \quad (34)$$

where $\mathcal{L}_{,u} \equiv \frac{\partial \mathcal{L}}{\partial u}$, etc. Some good graphs showing the strong necessary conditions are presented in [25] (p. 2787) and [28] (p. 346).

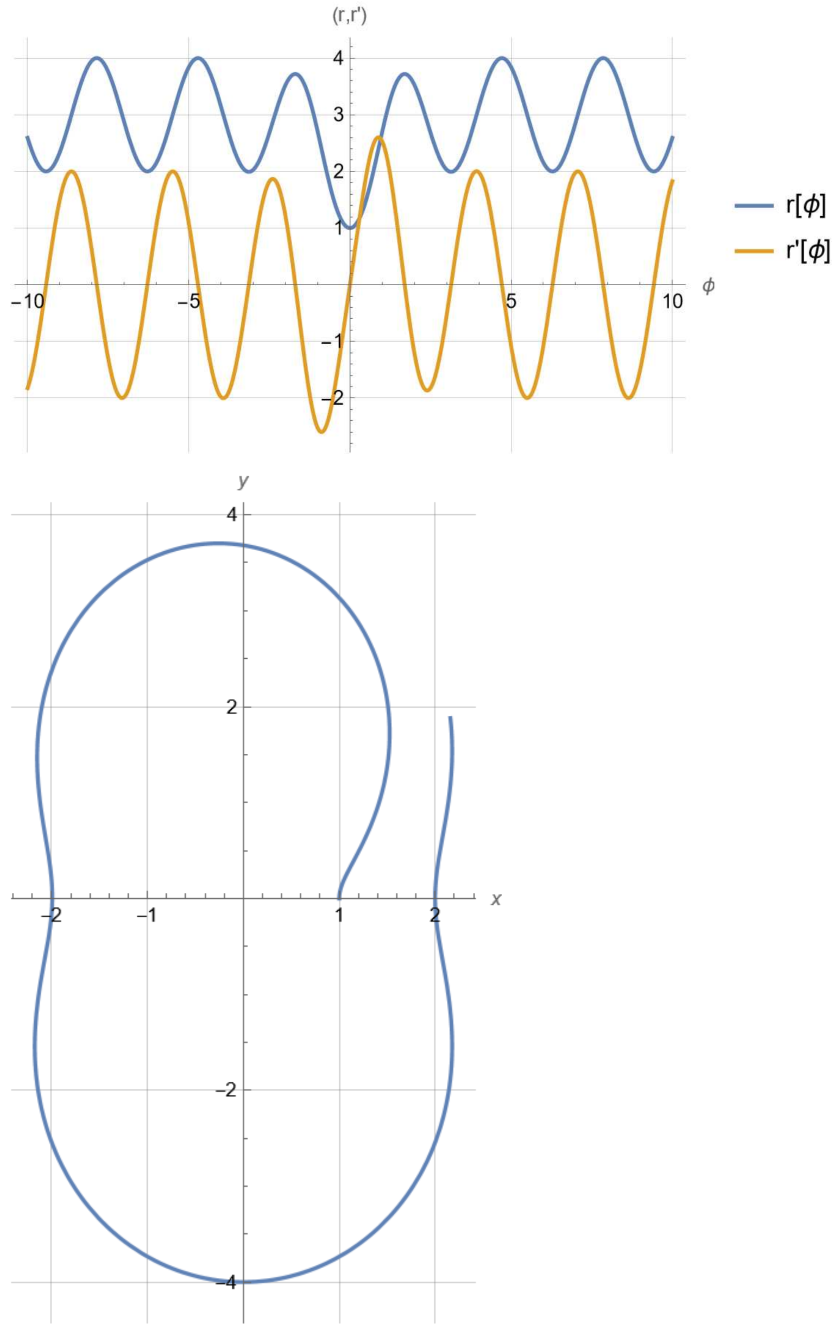

Figure 1. The case of the curved superconducting cable geometrically parameterized in polar coordinates by ϕ and $r(\phi)$. Non-zero curvature of the cable and an external magnetic field can induce the Josephson [46,47] junctions by bringing the G-L α parameter from negative to positive and again to negative values. Furthermore, additional polarization of the superconductor or the Josephson junction by a time-dependent electric and magnetic field is given in Figure 2.

However, a condition $\mathcal{L}_{,u} = 0$ might refer to a quite trivial case of a bulk uniform superconductor. In order to increase the possibilities of tracking various physical phenomena, one needs to derive the Bogomolny equations for a wider class of field-theoretical models, and one can use the so-called semi-strong necessary conditions (SSNCs). There are many versions of this approach. One of these versions (adapted by us later) assumes that the system (32)–(34) should be modified in the following way:

$$\mathcal{L}_{,u} = f_1, \tag{35}$$
$$\mathcal{L}_{,u_{,t}} = f_2, \tag{36}$$
$$\mathcal{L}_{,u_{,x}} = f_3, \tag{37}$$

where $f_j, j = 1,2,3$ are some functions chosen in such a way that the Euler–Lagrange equations are still unchanged. Function f_1 can tune "the bulk superconductor like-solution" and thus reflect various structures as a superconductor–normal material interface or a superconductor–normal material–superconductor case. One can easily see that all solutions of the system of Equations (32)–(34) are the solutions of the Euler–Lagrange Equation (30). However, these solutions, if they exist, are very often trivial. Hence, we make a gauge transformation of the functional (31):

$$S \to S + Inv, \tag{38}$$

where Inv is so functional that its local variation with respect to $u(x,t)$ vanishes: $\delta Inv \equiv 0$. The gauge transformation from (38) can correspond to the change in the system ground energy, and one example is the placement of a superconducting system inside a solenoid with a small magnetic field.

Thanks to this feature, the Euler–Lagrange Equations (30) and the Euler–Lagrange equations resulting from requiring the extremum of $S + Inv$ have the same form. On the other hand, the strong necessary conditions (32)–(34) are non-invariant with respect to the gauge transformation (38). Then, we have a chance to obtain non-trivial solutions. Obviously, the strong necessary conditions (32)–(34) constitute the system of the partial differential equations of the order lower than the order of Euler–Lagrange Equations (30).

Since we use the notion of topological charge or the topological invariant, it is useful to say more about this issue.

Let us take into account the sine-Gordon model (also known as the "sinus-Gordon model", and used for the description of the solitons in the Josepshon conjunction) [49]:

$$\mathcal{L} = -\frac{1}{2}\partial_\alpha u \partial^\alpha u - \frac{\mu^2}{\lambda^2}(1 - \cos(\lambda u)). \tag{39}$$

Its symmetry is translational ($u \to u + \frac{2\pi}{\lambda}k, k \in \mathbb{Z}$), and one can label different sectors in this model using a pair (m,n), where $m,n \in \mathbb{Z}$ in such a way that a field configuration, which belongs to the energetic sectors $E_{n,m}$, satisfies the following boundary conditions [49]:

$$\begin{aligned} \lim_{x \to -\infty} u(x,t) &= \frac{2\pi}{\lambda}n, \\ \lim_{x \to \infty} u(x,t) &= \frac{2\pi}{\lambda}m. \end{aligned} \tag{40}$$

Then, the asymptotic values of $u(x,t)$: $\lim_{x\to\pm\infty} u(x,t)$, are conserved; in other words, they do not depend on time. We denote their difference (e.g., having the meaning of the phase drop across the superconductor–non-superconductor interface) by Q_{topol1} [49]

$$Q_{topol1} = u_\infty - u_{-\infty}, \tag{41}$$

and we interpret it as a conserved charge, so its density is $\rho[u] = \frac{\partial u}{\partial x}$, and [49]

$$Q_{topol1} = \int_{-\infty}^{\infty} \frac{\partial u}{\partial x} dx \equiv \int_{-\infty}^{\infty} u_{,x} dx = u_{\infty} - u_{-\infty}. \tag{42}$$

The characteristic property of such charges is such that they are conserved independently of the dynamics. This is the simplest situation, when the so-called a homotopy group is $\pi_1(S^1)$. If the homotopy group is $\pi_2(S^2)$, then the topological charge is (cf. [50])

$$Q_{topol2} = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} (\omega_{,x}\omega^*_{,y} - \omega_{,y}\omega^*_{,x}) dxdy \tag{43}$$

which, after a generalization, has the form of

$$Q_{topol3} = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} G_1(\omega, \omega^*)(\omega_{,x}\omega^*_{,y} - \omega_{,y}\omega^*_{,x}) dxdy, \tag{44}$$

which was achieved in ([25]).

Now, we explain why such a generalization is always important when we apply strong necessary conditions. As we have written, the aim of this paper is to derive the Bogomolny decomposition (the Bogomolny equations), using strong necessary conditions, for the G-L model with $U(1)$ gauge. In the order to derive the Bogomolny decomposition, it is necessary to make dual equations (following from strong necessary conditions) self-consistent. The generalizations of the topological charges (topological invariants) allow us to properly choose functions like $G_1(\omega, \omega^*)$, in order to make the dual equations self-consistent. Thus, we have an opportunity to achieve the following:

- Make a certain part of the dual equations linearly dependent—the remaining equations are just the Bogomolny equations;
- Obtain a condition for the potential of the given field-theoretical model. The Bogomolny decomposition (the Bogomolny equations) exists only for this model, whose potential satisfies such a condition.

An important issue is how to construct the general form of the density of the topological invariant for the case of the topology of this model. In general, such an invariant should be also a gauge-invariant term (with respect to $U(1)$ gauge). A general construction of this density was given (in the cases of the gauged models: nonlinear σ $O(3)$, restricted BPS baby Skyrme, and full BPS baby Skyrme), in [32] (proposed there for the first time). At least one of the gauge invariance densities has the following form:

$$\begin{aligned} I_1 = \lambda_3 \cdot \Big\{ R_1' \cdot [i \cdot (\omega_{,\phi}\omega^*_{,z} - \omega_{,z}\omega^*_{,\phi}) - A_\phi \cdot (\omega_{,z}\omega^* + \omega\omega^*_{,z}) + \\ A_z \cdot (\omega_{,\phi}\omega^* + \omega\omega^*_{,\phi})] \Big\} + R_1 \cdot (A_{\phi,z} - A_{z,\phi}), \end{aligned} \tag{45}$$

where we assume that $R_1 = R_1(\omega\omega^*)$ is dependent on the order parameter magnitude, and this is a function which is usually to be determined later, when one applies the concept of strong necessary conditions. $R_1' = \frac{dR_1}{d(\omega\omega^*)}$, so R_1' denotes the derivative of the function R_1 with respect to its argument: $\omega\omega^*$. However, for the sake of simplicity, in the case of deriving the Bogomolny equations for the gauged G-L model in a flat space, we use the usual set of topological invariants, i.e.,

$$\begin{aligned} G_1(\omega, \omega^*)(\omega_{,x}\omega^*_{,y} - \omega_{,y}\omega^*_{,x}) + G_2(\omega, \omega^*)(A_{x,y} - A_{y,x}) + \\ \frac{d}{dx} G_3(\omega, \omega^*) + \frac{d}{dy} G_4(\omega, \omega^*), \end{aligned} \tag{46}$$

where the expressions $\frac{d}{dx}G_3(\omega,\omega^*)$, $\frac{d}{dy}G_4(\omega,\omega^*)$ are the so-called divergent invariants [25,26]. The functions $G_k, k = 1, ..., 4$ are to be determined later. As it turns out, in the case of deriving the Bogomolny equations for the gauged G-L model in a curved space, only divregent invariants will be needed. Let us notice here that by using the Lagrangian gauged with the focus on divergent invariants, we can obtain the same Euler–Lagrange equations, as these ones, obtained by using the Lagrangian, are ungauged on these invariants, even in cases when the divergent invariants are not invariant under the gauge transformations of the field A_k, $k = 1, 2$.

5. Derivation of the Bogomolny Equations for the Gauged G-L Model in a Flat Space

Now, in order to effectively apply the strong necessary conditions for a flat space, we need to gauge this Lagrangian on the so-called invariants, and we obtain

$$\begin{gathered}\tilde{\mathcal{L}} = \frac{1}{2m}\left(\frac{h}{i}\omega_{,x} - eA_x\omega\right)\left(-\frac{h}{i}\omega^*_{,x} - eA_x\omega^*\right)+\\ \frac{1}{2m}\left(\frac{h}{i}\omega_{,y} - eA_y\omega\right)\left(-\frac{h}{i}\omega^*_{,y} - eA_y\omega^*\right) + V(\omega,\omega^*,A_x,A_y)+\\ c_1(A_{y,x} - A_{x,y})^2 + G_1(\omega,\omega^*)(\omega_{,x}\omega^*_{,y} - \omega_{,y}\omega^*_{,x})+\\ G_2(\omega,\omega^*)(A_{x,y} - A_{y,x}) + \frac{d}{dx}G_3(\omega,\omega^*) + \frac{d}{dy}G_4(\omega,\omega^*),\end{gathered} \tag{47}$$

where $c_1 = \frac{1}{2}\frac{1}{\mu\mu_0}$ and the functions $G_k(k = 1, 2, 3,4)$ will be determined later.

According to the strong necessary conditions, instead of considering the Euler–Lagrange equations

$$\frac{d}{dx}\frac{\partial\mathcal{L}}{\partial\omega_{,x}} + \frac{d}{dy}\frac{\partial\mathcal{L}}{\partial\omega_{,y}} - \frac{\partial\mathcal{L}}{\partial\omega} = 0, \tag{48}$$

$$\frac{d}{dx}\frac{\partial\mathcal{L}}{\partial\omega^*_{,x}} + \frac{d}{dy}\frac{\partial\mathcal{L}}{\partial\omega^*_{,y}} - \frac{\partial\mathcal{L}}{\partial\omega^*} = 0, \tag{49}$$

$$\frac{d}{dx}\frac{\partial\mathcal{L}}{\partial A_{x,x}} + \frac{d}{dy}\frac{\partial\mathcal{L}}{\partial A_{x,y}} - \frac{\partial\mathcal{L}}{\partial A_x} = 0, \tag{50}$$

$$\frac{d}{dx}\frac{\partial\mathcal{L}}{\partial A_{y,x}} + \frac{d}{dy}\frac{\partial\mathcal{L}}{\partial A_{y,y}} - \frac{\partial\mathcal{L}}{\partial A_y} = 0, \tag{51}$$

we can consider simplified equations as given below.

The equations in the case of a flat space (Equation (47)), generated by the Lagrangian renormalized with the invariants, have the form of

$$\begin{gathered}\tilde{\mathcal{L}}_{,\omega} : -\frac{(-ih\omega^*_{,x} - eA_x\omega^*)eA_x}{2m} - \frac{(-ih\omega^*_{,y} - eA_y\omega^*)eA_y}{2m} + V_{,\omega}+\\ iG_{,\omega}(\omega_{,x}\omega^*_{,y} - \omega_{,y}\omega^*_{,x}) + G_{2,\omega}(A_{y,x} - A_{x,y}) + D_xG_{3,\omega} + D_yG_{4,\omega} = 0\end{gathered} \tag{52}$$

$$\begin{gathered}\tilde{\mathcal{L}}_{,\omega^*} : -\frac{(ih\omega^*_{,x} - eA_x\omega^*)eA_x}{2m} - \frac{(ih\omega^*_{,y} - eA_y\omega^*)eA_y}{2m} + V_{,\omega^*}+\\ iG_{,\omega^*}(\omega_{,x}\omega^*_{,y} - \omega_{,y}\omega^*_{,x}) + G_{2,\omega^*}(A_{y,x} - A_{x,y}) + D_xG_{3,\omega^*} + D_yG_{4,\omega^*} = 0\end{gathered} \tag{53}$$

$$\begin{gathered}\tilde{\mathcal{L}}_{,A_x} : -\frac{(ih\omega_{,x} - eA_x\omega)e\omega^*}{2m} - \frac{(-ih\omega^*_{,x} - eA_x\omega^*)e\omega}{2m}+\\ V_{,A_x} + D_xG_{3,A_x} + D_yG_{4,A_x} = 0\end{gathered} \tag{54}$$

$$\begin{gathered}\tilde{\mathcal{L}}_{,A_y} : -\frac{(ih\omega_{,y} - eA_y\omega)e\omega^*}{2m} - \frac{(-ih\omega^*_{,y} - eA_y\omega^*)e\omega}{2m}+\\ V_{,A_y} + D_xG_{3,A_y} + D_yG_{4,A_y} = 0\end{gathered} \tag{55}$$

$$\tilde{\mathcal{L}}_{,\omega_{,x}} : \frac{i(-ih\omega^*_{,x} - eA_x\omega^*)}{2m} + iG_1\omega^*_{,y} + G_{3,\omega} = 0 \tag{56}$$

$$\tilde{\mathcal{L}}_{,\omega^*_{,x}} : -\frac{i(ih\omega_{,x} - eA_x\omega)}{2m} - iG_1\omega_{,y} + G_{3,\omega^*} = 0 \tag{57}$$

$$\tilde{\mathcal{L}}_{,\omega_{,y}} : \frac{i(-ih\omega^*_{,y} - eA_y\omega^*)}{2m} - iG_1\omega^*_{,x} + G_{4,\omega} = 0 \tag{58}$$

$$\tilde{\mathcal{L}}_{,\omega^*_{,y}} : -\frac{i(ih\omega_{,y} - eA_y\omega)}{2m} + iG_1\omega_{,x} + G_{4,\omega^*} = 0, \tag{59}$$

$$\tilde{\mathcal{L}}_{,A_{x,y}} : -2c_1(A_{x,y} - A_{y,x}) - G_2 + G_{4,A_x} = 0, \tag{60}$$

$$\tilde{\mathcal{L}}_{,A_{y,x}} : 2c_1(A_{x,y} - A_{y,x}) + G_2 + G_{3,A_y} = 0. \tag{61}$$

The standard procedure of deriving the Bogomolny equations, using a strong necessary condition method, is such that one has to make Equations (52)–(61) self-consistent. Thus, the reduction in the number of independent equations by an appropriate choice of the functions $G_k, (k = 1, 2, 3\ 4)$ is necessary. Usually, such ansatzes exist only for a special $V(\omega, \omega^*)$. Hence, in most cases of $V(\omega, \omega^*)$ for many nonlinear field models, the reduction of the system of the corresponding dual equations to the Bogomolny equations is impossible. In such a case, we have an overdetermined system of the first-order partial differential equations for the unknown $\omega, \omega^*, A_x, A_y$. However, even if deriving the Bogomolny equations for a given model is impossible, we still have a system of first-order equations, and we can try to solve them. Just now, we consider the situation with the overdetermined system of the first-order PDEs, where the unknown functions are $\omega, \omega^*, A_x, A_y$.

At the beginning, we eliminate all terms including the derivatives $\omega_{,x}, \ldots, A_{y,x}$ from Equations (52)–(55), using Equations (56)–(59).

We obtain a system of partially differential equations for the unknown functions $G_k, V(k = 1, \ldots, 4)$. It is highly nonlinear, so we have to choose some ansatzes intuitively, given as the following functions: $G_3 = A_y, G_4 = A_x$ (it additionally eliminates the terms including the derivatives $A_{x,x}, A_{y,y}$ from Equations (52)–(55)). We obtain, among others, the following solutions of this system:

$$G_1 = 0, G_2 = -1 - \sqrt{1 + 4c_1c_2 + 4c_1f_1}, V = f_1, \tag{62}$$

where $f_1 = f_1(\omega, \omega^*) \in \mathcal{C}^2$ is an arbitrary function, which we can fix, e.g., as the G-L potential ($\frac{\alpha}{2} \mid \omega\omega^* \mid + \frac{\beta}{4} \mid \omega\omega^* \mid^2$, where $\alpha < 0, \beta > 0$). If we insert the forms of the functions G_1, G_2 and $G_3 = A_y, G_4 = -A_x$ into Equations (56)–(61), we obtain

$$\frac{i(-ih\omega^*_{,x} - eA_x\omega^*)}{2m} = 0, \tag{63}$$

$$-\frac{i(ih\omega_{,x} - eA_x\omega)}{2m} = 0, \tag{64}$$

$$\frac{i(-ih\omega^*_{,y} - eA_y\omega^*)}{2m} = 0, \tag{65}$$

$$-\frac{i(ih\omega_{,y} - eA_y\omega)}{2m} = 0, \tag{66}$$

$$-2c_1(A_{x,y} - A_{y,x}) + \sqrt{1 + 4c_1c_2 + 4c_2f_1} = 0, \tag{67}$$

$$2c_1(A_{x,y} - A_{y,x}) - \sqrt{1 + 4c_1c_2 + 4c_2f_1} = 0. \tag{68}$$

As we can see, the first two equations point the canonical momentum being zero, which is the feature of the superconductor "defending" its ground state against excitation, while the last two equations are linearly dependent, so we can reuse one of them immediately.

Then, we have to solve the system (63)–(67).

We obtain the following results for $f_1 = \alpha \mid \omega\omega^* \mid + \beta \mid \omega\omega^* \mid^2$, where $\alpha < 0, \beta > 0$,

$$A_x = \int A_{y,x} dy + \frac{d}{dx} f_2(x), \tag{69}$$

$$\omega(x,y) = c_3 e^{\frac{-ief_2}{h}} e^{\int \frac{-ieA_y}{h} dy}, \tag{70}$$

and the complex conjugation of ω has to satisfy the following conditions:

$$\begin{gathered}
4c_1c_2 + \beta c_1 c_3^2 e^{\frac{-2ief_2}{h}} e^{2\int \frac{-ieA_y}{h} dy} (\omega^*)^2 + 2c_1c_3\alpha e^{\frac{-ief_2}{h}} e^{\int \frac{-ieA_y}{h} dy} \omega^* = 0, \\
-2c_2 e^{\frac{-ief_2}{h}} e^{\int \frac{-ieA_y}{h} dy} \left(ie \frac{df_2}{dx} \omega^* - \left(\int \frac{-ieA_{y,x}}{h} dy \right) h\omega^* - h\omega^*_{,x} \right) \times \\
\times (e^{\frac{-ief_2}{h}} e^{\int \frac{-ieA_y}{h} dy} c_3 \beta \omega^* + \alpha) c_3 = 0, \\
-2c_1 e^{\int \frac{-ieA_y}{h} dy} (e^{\frac{-ief_2}{h}} e^{\int \frac{-ieA_y}{h} dy} c_2 \beta \omega^* + \alpha) c_3 e^{\frac{-ief_2}{h}} (ieA_y \omega^* - h\omega^*_{,y}) = 0, \\
\omega^*_{,x} = \frac{ie}{h} \left(\int A_{y,x} dy + \frac{df_2}{dx} \right) \omega^*, \omega^*_{,y} = \frac{ieA_y \omega^*}{h}.
\end{gathered} \tag{71}$$

where $f_2 = f_2(x)$ can be the complex-valued function. The very last equations point out bulk solutions in the y direction, while the complex value properties of f_2 account for the magnitude of the superconducting order parameter change in the x direction as well as a phase imprint (70). It is naturally obtained from the Aharonov–Bohm effect, present in superconductors. Equation (66) points out the unusual co-dependence of A_x and A_y fields, as depicted in Figure 2. The obtained solutions validate the Bogomolny approach to the G-L model. The next step is to describe the bend of superconducting cables.

6. Deriving the Bogomolny Equations of the G-L Model in a Curved Space

Now, we consider a bent superconducting cable of small thickness with the preimposed electric current flow J (described by the London relation that associates superconducting order parameters and a vector potential with the electric current intensity) in two-dimensional Cartesian coordinates with $(x, y) = (x, g(x))$, which gives the following version in a curved space:

$$\begin{gathered}
\tilde{\mathcal{L}} = \frac{1}{2m} \left(\frac{h}{i} \omega_{,x} + \frac{\frac{J}{\sqrt{1+g'2(x)}} 2mc\omega}{\mid \omega \mid^2} \right) \left(\frac{h}{-i} \omega^*_{,x} + \frac{\frac{J}{\sqrt{1+g'^2(x)}} 2mc\omega^*}{\mid \omega \mid^2} \right) + \\
\frac{1}{2m} \left(\frac{h}{ig'} \omega_{,x} + \frac{\frac{Jg'}{\sqrt{1+g'^2(x)}} 2mc\omega}{\mid \omega \mid^2} \right) \left(\frac{h}{-ig'} \omega^*_{,x} + \frac{\frac{Jg'}{\sqrt{1+g'^2(x)}} 2mc\omega^*}{\mid \psi \mid^2} \omega^* \right) + \\
+ \alpha(x) \mid \omega \mid^2 + \beta(x) \mid \omega \mid^4 + JG_{1,\omega} \omega_{,x} + JG_{1,\omega^*} \omega^*_{,x} + \frac{J}{g'} G_{1,\omega} \omega_{,x} + \frac{J}{g'} G_{1,\omega^*} \omega^*_{,x},
\end{gathered} \tag{72}$$

where $\alpha(x) < 0$, as in the case of a superconducting state. We have assumed a London relation between electric current density and superconducting order parameter density ω.

It is convenient to write down this Lagrangian density by using real and imaginary parts of ω: $u = \Re(\omega), v = \Im(\omega)$. Then, for the simplicity of mathematical structure and perturbative J, we put $J = 0$ in the kinetic term, so J appears only in divergent invariants:

$$\begin{gathered}
\tilde{\mathcal{L}} = h^2((u_{,x})^2 + (v_{,x})^2)) + \frac{h^2}{g'^2}((u_{,x})^2 + (v_{,x})^2)) + \alpha(x)(u^2 + v^2) + \beta(x)(u^2 + v^2)^2 + \\
JG_{2,u} u_{,x} + JG_{2,v} v_{,x} + \frac{J}{g'} G_{2,u} u_{,x} + \frac{J}{g'} G_{2,v} v_{,x},
\end{gathered} \tag{73}$$

For certain simplicity of mathematical structure and perturbative J, we also put $J = 0$ in the kinetic term, so J appears only in divergent invariants.

Now, we apply the semi-strong necessary conditions expressing the decay of a superconducting order parameter and thus describe the superconductor–non-superconductor interface,

$$\frac{\partial \tilde{\mathcal{L}}}{\partial u} = ke^{-\lambda x}, \ \frac{\partial \tilde{\mathcal{L}}}{\partial v} = ke^{-\lambda x}, \tag{74}$$

$$\frac{\partial \tilde{\mathcal{L}}}{\partial u_{,x}} = -ke^{-\lambda x}, \frac{\partial \tilde{\mathcal{L}}}{\partial v_{,x}} = -ke^{-\lambda x}, \tag{75}$$

in reference to the Lagrangian density (73).

We find a corresponding superconducting parameter $\alpha(x)$ and the geometry of the superconducting cable $y = g(x)$, keeping $\beta = 1$ with the electric current intensity J and Planck constant $\hbar = h$:

$$\begin{aligned} \alpha(x) = &\frac{1}{h^2 \cos(x)(3J\cos(x)e^{\lambda x} + 4h^2 + N)}\Big(-h^2 J e^{-\lambda x} - 2\cos(x)((u^2+v^2)h^2 + \frac{J^2}{8})N - \\ &8\cos(x)(\frac{h^2J^2}{8} + \frac{3\cos(x)J((u^2+v^2)h^2 - \frac{J^2}{24})e^{\lambda x}}{4} + ((u^2+v^2)h^2 + \frac{J^2}{8}))) \Big), \\ g(x) = &\int (-\frac{J\cos(x)e^{\lambda x} + N}{2(J\cos(x)e^{\lambda x} + 2h^2)}dx + c_1 = \\ = &\int (-\frac{J\cos(x)e^{\lambda x} + [\sqrt{J^2(\cos(x))^2 e^{2\lambda x} - 8Je^{\lambda x}\cos(x)h^2 - 16h^4}]}{2(J\cos(x)e^{\lambda x} + 2h^2)}dx + c_1, \end{aligned} \tag{76}$$

where $N = \sqrt{J^2(\cos(x))^2 e^{2\lambda x} - 8Je^{\lambda x}\cos(x)h^2 - 16h^4}$.

We also find the form of $u(x)$ and $v(x)$ with the use of $sec(x) = \frac{1}{cos(x)} = \frac{2}{(e^{+ix}+e^{-ix})}$ as

$$u(x) = c_1 \sinh\Big(\int(-\sec(x)e^{-\lambda x}dx)\Big) + c_2\cosh\Big(\int(-\sec(x)e^{-\lambda x}dx)\Big), \tag{77}$$

$$v(x) = -c_1\cosh\Big(\int(-\sec(x)e^{-\lambda x}dx)) - c_2\sinh(\int(-\sec(x)e^{-\lambda x}dx)\Big). \tag{78}$$

Quite similar considerations can be obtained for preassumed superconductor–non-superconductor–superconductor interface encoding properties of the Josephson junction with $\frac{\partial \tilde{\mathcal{L}}}{\partial u} = ke^{-\lambda x} + ke^{+\lambda x}, \frac{\partial \tilde{\mathcal{L}}}{\partial v} = ke^{-\lambda x} + ke^{\lambda x}$ that result in very complicated formulas for $u(x), v(x), g(x), \alpha(x)$, and thus, it is beyond the scope of this work. So far, we have considered stationary situations. They can be modified by the subjection of the Josephson junction to various perturbing factors as a time-dependent electric field, naturally occurring in the Josephson junction, interfaced with a semiconductor quantum dot system. It is natural to start with a perturbative G-L model and later propose its validation by the Bogomolny approach.

7. Perturbative Approach in the Description of the Josephson Junction

A thin superconducting nanowire can be described as having external perturbations that are of magnetic or electric origin. We might preimpose superconducting current flowing via a superconductor that is always associated with the existence of the non-zero vector potential, accompanied with a time-dependent component. Due to the importance of the interface between semiconductor quantum dots, interfaced with the Josephson junction, we will consider the situations depicted in Figure 2. In such a case, we encounter a simplified form of an AC vector potential component, given as

$$A_{x,p}(x,t) = \frac{1}{4\pi\epsilon}\frac{qv(t)}{\sqrt{d^2+(x-(\frac{L}{2})sin(\omega t))^2}} = \frac{1}{4\pi\epsilon}\frac{q\omega(\frac{L}{2})cos(\omega t))}{\sqrt{d^2+(x-(\frac{L}{2})sin(\omega t))^2}} \approx \lambda \to 0 \qquad (79)$$

that implies an electric field component as

$$E_x = -\frac{d}{dt}A_{x,p}(x,t) = -\frac{q}{4\pi\epsilon}[\frac{-\omega(\frac{L}{2})sin(\omega t))}{\sqrt{d^2+(x-(\frac{L}{2})sin(\omega t))^2}} + \frac{1}{2}\frac{\omega^2(\frac{L}{2})^2cos(\omega t)^2)}{(d^2+(x-(\frac{L}{2})sin(\omega t))^2)^{\frac{3}{2}}}], \qquad (80)$$

$$V_p(x) = -\int_{-\infty}^{+x} dx_1 E_x(x_1) = \int_{-\infty}^{+x} dx_1 \frac{d}{dt}A_{x_1,p}(x,t) = \int_{-\infty}^{+x} \frac{q dx_1}{4\pi\epsilon}[\frac{-\omega(\frac{L}{2})sin(\omega t))}{\sqrt{d^2+(x_1-(\frac{L}{2})sin(\omega t))^2}} + \frac{\frac{1}{2}\omega^2(\frac{L}{2})^2cos(\omega t)^2)}{(d^2+(x_1-(\frac{L}{2})sin(\omega t))^2)^{\frac{3}{2}}}]. \qquad (81)$$

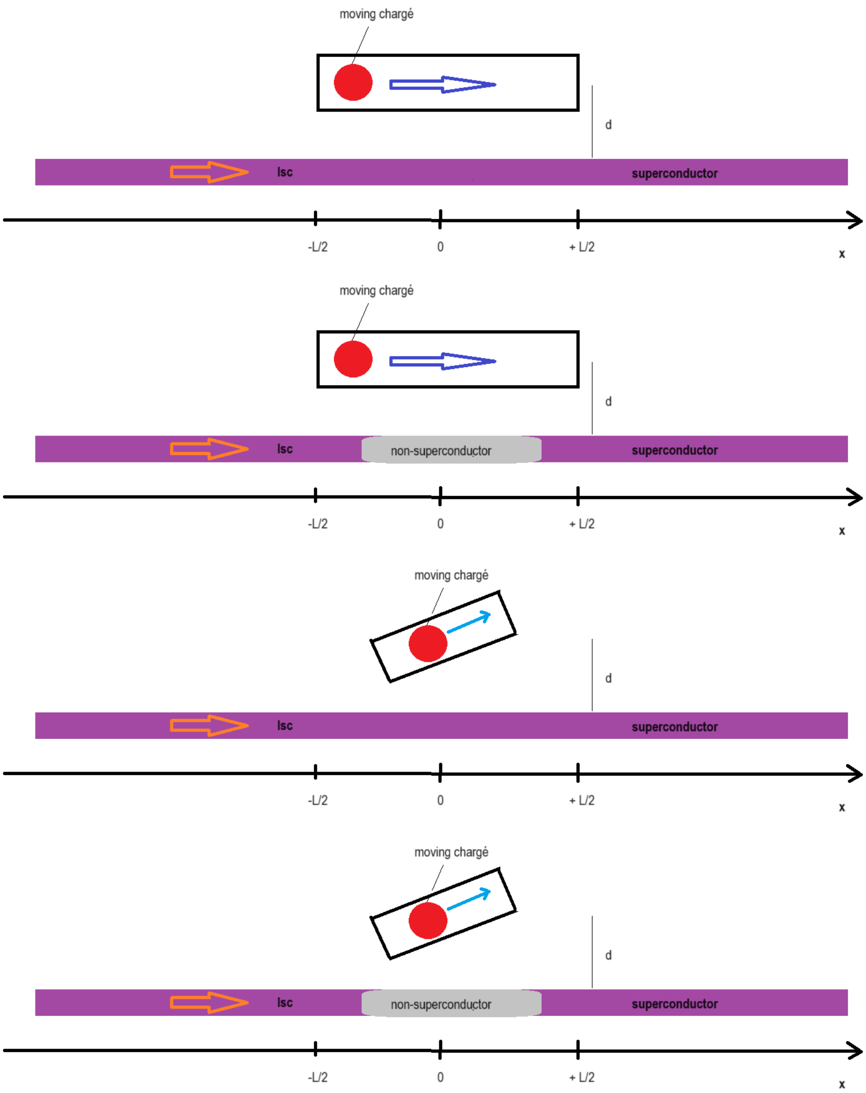

Figure 2. The case of a thin superconducting nanowire and the Josephson junction interfaced to the system of quantum dots with a moving electric charge [51].

Perturbative G-L Approach

The Bogomolny approach can also be redefined by the G-L approach (time-independent or time-dependent) with the treatment of the vector potential as perturbation $\lambda(t)$, which justifies the usage of perturbation calculus [52]. We start from the case of the zero vector potential and the solution of the GL equation given in the form of

$$\begin{gathered}\frac{1}{2m}(-\hbar^2\frac{d^2}{dx^2})\psi_s(x,t)+\alpha(x,t)\psi_s(x,t)+\beta(x,t)|\psi_s(x,t)|^2\psi_s(x,t)=0,\\ \frac{1}{2m}(\frac{\hbar}{i}\frac{d}{dx}-\frac{2e}{c}A_x(t))^2\psi_s(x,t)+\alpha(x,t)\psi_s(x,t)+\beta(x,t)|\psi_s(x,t)|^2\psi_s(x,t)=\\ \eta\frac{d}{dt}\psi(x,t)\end{gathered} \tag{82}$$

and we preassume the existence of the perturbative solution $\psi(x)_p=e^f e^{\frac{2e}{\hbar}\int_{x_0}^{x}A_x(x_1)dx_1}\psi(x)_s$ that justifies the following sequence of steps given below. After multiplying the time-independent GL equation by perturbative terms

$$\begin{gathered}e^f e^{i\frac{2e}{\hbar}\int A_x dx}\frac{1}{2m}(-\hbar^2\frac{d^2}{dx^2})\psi_s(x,t)+\alpha(x,t)e^f e^{i\frac{2e}{\hbar}\int A_x dx}\psi_s(x,t)+\\ \beta(x,t)|\psi_s(x,t)|^2\psi_s(x,t)e^f e^{i\frac{2e}{\hbar}\int A_x dx}=0,\end{gathered} \tag{83}$$

and rearranging the expression around a nonlinear SCOP term

$$\begin{gathered}e^f e^{i\frac{2e}{\hbar}\int A_x dx}\frac{1}{2m}(-\hbar^2\frac{d^2}{dx^2})\psi_s(x,t)+\alpha(x,t)e^f e^{i\frac{2e}{\hbar}\int A_x dx}\psi_s(x,t)+\\ e^{-2f}\beta(x,t)|e^{2f}\psi_s(x,t)|^2\psi_s(x,t)e^f e^{i\frac{2e}{\hbar}\int A_x dx}=0,\end{gathered} \tag{84}$$

we obtain the equivalent form

$$\begin{gathered}e^f e^{i\frac{2e}{\hbar}\int A_x dx}\frac{1}{2m}(-\hbar^2\frac{d^2}{dx^2})\psi_s(x,t)+\alpha(x,t)\psi_p(x,t)+\\ (1-1+e^{-2f})\beta(x,t)|\psi_p(x,t)|^2\psi_p(x,t)=0,\end{gathered} \tag{85}$$

that can be rearranged by the vector potential perturbation A_x^2.

$$\begin{gathered}e^f e^{i\frac{2e}{\hbar}\int A_x dx}\frac{1}{2m}(-\hbar^2\frac{d^2}{dx^2})\psi_s(x,t)+\alpha(x,t)\psi_p(x,t)+\\ \beta(x,t))|\psi_p(x,t)|^2\psi_p(x,t)+\frac{1}{2m}(4e^2A_x^2)\psi_p(x,t)-\frac{\hbar^2}{2m}(\frac{d^2}{dx^2}e^f e^{i\frac{2e}{\hbar}\int A_x dx})\psi_s(x,t)=\\ (1-e^{-2f})\beta(x,t))|\psi_p(x,t)|^2\psi_p(x,t)+\frac{1}{2m}(e^2A_x^2)\psi_p(x,t)-\\ \frac{\hbar^2}{2m}(\frac{d^2}{dx^2}e^f e^{i\frac{2e}{\hbar}\int A_x dx})\psi_s(x,t).\end{gathered} \tag{86}$$

We have two other vector potential perturbation terms leading to a time-dependent G-L equation on the left side of the equation, so we end up with

$$
\begin{aligned}
-e^{f}e^{i\frac{2e}{\hbar}\int A_x dx}\frac{\hbar^2}{2m}(\frac{d^2}{dx^2})\psi_s(x,t)+(\alpha(x,t)+\beta(x,t)|\psi_p(x,t)|^2)\psi_p(x,t)+\frac{4e^2A_x^2}{2m}\psi_p(x,t)+\\
-\frac{\hbar}{i}\frac{1}{2m}(2e\frac{d}{dx}A_x)\psi_p(x,t)-2\frac{\hbar}{i}\frac{1}{2m}(2eA_x)\frac{d}{dx}\psi_p(x,t)+\\
-\frac{\hbar^2}{2m}(\frac{d^2}{dx^2}e^{f}e^{i\frac{2e}{\hbar}\int A_x dx})\psi_s(x,t)-\frac{\hbar^2}{2m}[\frac{df}{dx}+i\frac{2e}{\hbar}A_x(x)]\psi_p(x,t)-\eta V_p(t)\psi_p(x,t)=0=\\
=(1-e^{-2f})\beta(x,t))|\psi_p(x,t)|^2\psi_p(x,t)+\frac{1}{2m}(e^2A_x^2)\psi_p(x,t)+\\
-\frac{\hbar^2}{2m}(\frac{d^2}{dx^2}e^{f}e^{\int A_x dx})\psi_s(x,t)-\frac{\hbar^2}{2m}[\frac{df}{dx}+i\frac{e}{\hbar}A_x(x)]\psi_p(x,t)-\eta V_p(t)\psi_p(x,t)+\\
-\frac{\hbar}{i}\frac{1}{2m}(2e\frac{d}{dx}A_x)\psi_p(x,t)-2\frac{\hbar}{i}\frac{1}{2m}(2eA_x)\frac{d}{dx}\psi_p(x,t).
\end{aligned}
\tag{87}
$$

Inevitably, it results in the situation that the right side, expressing a function (f), will describe dynamics of the perturbation (f). We recognize that both A_x and f are proportional to $\lambda \to 0$ as the perturbation, so we have

$$
\begin{aligned}
0=(1-e^{-2f})\beta(x,t))|\psi_p(x,t)|^2\psi_p(x,t)+\frac{1}{2m}(e^2A_x^2)\psi_p(x,t)+\\
-\frac{\hbar^2}{2m}(\frac{d^2}{dx^2}e^{f}e^{\int A_x dx})\psi_s(x,t)+e^{f}e^{\int A_x dx}\frac{-\hbar^2}{2m}[\frac{df}{dx}+i\frac{e}{\hbar}A_x(x)]\frac{d}{dx}\psi_S(x,t)-\\
\eta V_p(t)\psi_p(x,t)+-\frac{\hbar}{i}\frac{1}{2m}(2e\frac{d}{dx}A_x)\psi_p(x,t)-2\frac{\hbar}{i}\frac{1}{2m}(2eA_x)\frac{d}{dx}\psi_p(x,t).
\end{aligned}
\tag{88}
$$

Finally, we end up with the reduced equation

$$
\begin{aligned}
0=(e^{2f}-1)\beta(x,t))|\psi_s(x,t)|^2+\frac{1}{2m}(4e^2A_x^2)+\\
+\frac{1}{2m\psi_p(x,t)}(-\hbar^2\frac{d^2}{dx^2}e^{f}e^{\frac{2e}{\hbar}\int A_x dx})\psi_s(x,t)+\frac{-\hbar^2}{2m}[\frac{df}{dx}+i\frac{2e}{\hbar}A_x(x)]-\\
\eta V_p(t)+-\frac{\hbar}{i}\frac{1}{2m}(2e\frac{d}{dx}A_x)-2\frac{\hbar}{i}\frac{1}{2m}(2eA_x)\frac{\frac{d}{dx}\psi_p(x,t)}{\psi_p(x,t)}.
\end{aligned}
\tag{89}
$$

In the meantime, we use the identity

$$
\begin{aligned}
(-\hbar^2\frac{d^2}{dx^2}e^{f}e^{i\frac{2e}{\hbar}\int A_x dx})=\\
=-\hbar^2 i\frac{d}{dx}[(\frac{df}{dx}+\frac{2e}{\hbar}A_x)e^{i\frac{2e}{\hbar}\int A_x dx}]=\\
=-\hbar^2[-(\frac{df}{dx}+\frac{2e}{\hbar}A_x)^2+i(\frac{d^2f}{dx^2}+\frac{2e}{\hbar}\frac{dA_x}{x})]e^{i\frac{2e}{\hbar}\int A_x dx}
\end{aligned}
\tag{90}
$$

that allows us to simplify the equation for GL perturbation f, so we obtain

$$
\begin{aligned}
0=(e^{2f}-1)\beta(x,t))|\psi_s(x,t)|^2+\frac{1}{2m}(4e^2A_x^2)+\\
-\hbar^2[-(\frac{df}{dx}+\frac{2e}{\hbar}A_x)^2+i(\frac{d^2f}{dx^2}+\frac{2e}{\hbar}\frac{dA_x}{dx})]\\
+\frac{-\hbar^2}{2m}[\frac{df}{dx}+i\frac{2e}{\hbar}A_x(x)]\frac{\frac{d}{dx}\psi_S(x,t)}{\psi_S(x,t)}-\eta V_p(t)+\\
-\frac{\hbar}{i}\frac{1}{2m}(2e\frac{d}{dx}A_x)-2\frac{\hbar}{i}\frac{1}{2m}(2eA_x)[\frac{\frac{d}{dx}\psi_S(x,t)}{\psi_S(x,t)}+i(\frac{df}{dx}+\frac{2e}{\hbar}A_x)].
\end{aligned}
\tag{91}
$$

Real values of the previous equation bring the following equation:

$$
\begin{aligned}
0 = (e^{2f} - 1)\beta(x,t))|\psi_s(x,t)|^2 + \frac{1}{2m}(4e^2A_x^2) + \\
-\hbar^2[-(\frac{df}{dx} + \frac{2e}{\hbar}A_x)^2] \\
+ \frac{-\hbar^2}{2m}[\frac{df}{dx}]\frac{\frac{d}{dx}\psi_S(x,t)}{\psi_S(x,t)} - \eta V_p(t) + \\
-\frac{\hbar}{1}\frac{1}{2m}(2e\frac{d}{dx}A_x) - 2\frac{\hbar}{1}\frac{1}{2m}(2eA_x)[(\frac{df}{dx} + \frac{2e}{\hbar}A_x)],
\end{aligned}
\tag{92}
$$

and imaginary values result in another equation

$$
\begin{aligned}
0 = -\hbar^2[(\frac{d^2f}{dx^2} + \frac{2e}{\hbar}\frac{dA_x}{dx})] \\
+ \frac{-\hbar^2}{2m}[\frac{2e}{\hbar}A_x(x)]\frac{\frac{d}{dx}\psi_S(x,t)}{\psi_S(x,t)} - \frac{\hbar}{i}\frac{1}{2m}(2e\frac{d}{dx}A_x) - 2\frac{\hbar}{1}\frac{1}{2m}(2eA_x)[\frac{\frac{d}{dx}\psi_S(x,t)}{\psi_S(x,t)}],
\end{aligned}
\tag{93}
$$

In treating f and A_x as perturbations proportional to λ, we dropped the terms proportional to the square of perturbation and thus obtained the simplified equation

$$
0 = (e^{2f} - 1)\beta(x,t))|\psi_s(x,t)|^2 - \frac{\hbar^2}{2m}[\frac{df}{dx}]\frac{\frac{d}{dx}\psi_S(x,t)}{\psi_S(x,t)} - \eta V_p(t) - \frac{\hbar}{1}\frac{1}{2m}(2e\frac{d}{dx}A_x) \tag{94}
$$

that, after linearization ($e^{2f} \approx 1 + 2f$), brings the linear ODE of the form

$$
0 = (2f)\beta(x,t))|\psi_s(x,t)|^2 - \frac{\hbar^2}{2m}[\frac{df}{dx}]\frac{\frac{d}{dx}\psi_S(x,t)}{\psi_S(x,t)} - \eta V_p(t) - \frac{\hbar}{1}\frac{1}{2m}(2e\frac{d}{dx}A_x) \tag{95}
$$

and thus its reduced form is given as

$$
\frac{\hbar^2}{2m}[\frac{df}{dx}]\frac{\frac{d}{dx}\psi_S(x,t)}{\psi_S(x,t)} - (2)\beta(x,t))|\psi_s(x,t)|^2 f = -[\hbar\frac{1}{2m}(2e\frac{d}{dx}A_x) + \eta V_p(t)]. \tag{96}
$$

The derivative of perturbation f with respect to the position brings the following equation:

$$
\begin{aligned}
\frac{df}{dx} = \\
\frac{4m}{\hbar^2}\beta(x,t))|\psi_s(x,t)|^2\frac{\psi_S(x,t)}{\frac{d}{dx}\psi_S(x,t)}f - [\frac{\hbar}{2m}(2e\frac{d}{dx}A_x) + \eta V_p(t)]\frac{\psi_S(x,t)}{\frac{d}{dx}\psi_S(x,t)}\frac{2m}{\hbar^2}
\end{aligned}
\tag{97}
$$

and by the introduction of e^u, we have the equation of the form

$$
\begin{aligned}
\frac{df}{dx}e^u = \\
\frac{4m}{\hbar^2}\beta(x,t))|\psi_s(x,t)|^2\frac{\psi_S(x,t)}{\frac{d}{dx}\psi_S(x,t)}e^u f - [\frac{\hbar}{2m}(2e\frac{d}{dx}A_x) + \eta V_p(t)]\frac{\psi_S(x,t)}{\frac{d}{dx}\psi_S(x,t)}\frac{2m}{\hbar^2}e^u
\end{aligned}
\tag{98}
$$

that can be rearranged after using the identity $\frac{d}{dx}(fe^u) = (\frac{d}{dx}f + f\frac{du}{dx})e^u$ into a form given below:

$$\begin{aligned}
&\frac{d}{dx}(fe^u) = (\frac{d}{dx}f + f\frac{du}{dx})e^u = \\
&= \frac{d}{dx}fe^u - \frac{2m}{\hbar^2}(2)\beta(x,t))|\psi_s(x,t)|^2\frac{\psi_S(x,t)}{\frac{d}{dx}\psi_S(x,t)}e^u f = \\
&\frac{d}{dx}(fe^{-\int_{x_0}^x dx_1((\frac{2m}{\hbar^2}(2)\beta(x_1,t))|\psi_s(x_1,t)|^2\frac{\psi_S(x_1,t)}{\frac{d}{dx_1}\psi_S(x_1,t)})}) = \\
&-[\hbar\frac{1}{2m}(2e\frac{d}{dx}A_x) + \eta V_p(t)]\frac{\psi_S(x,t)}{\frac{d}{dx}\psi_S(x,t)}\frac{2m}{\hbar^2}e^{-\int_{x_0}^x dx_1((\frac{2m}{\hbar^2}(2)\beta(x_1,t))|\psi_s(x_1,t)|^2\frac{\psi_S(x_1,t)}{\frac{d}{dx_1}\psi_S(x_1,t)})} = \\
&-[\hbar\frac{1}{2m}(2e\frac{d}{dx}A_x) + \eta V_p(t)]\frac{\psi_S(x,t)}{\frac{d}{dx}\psi_S(x,t)}\frac{2m}{\hbar^2}e^u.
\end{aligned} \tag{99}$$

Consequently, we obtain the perturbation $f(x)$ solution as

$$f(x) = e^R\Lambda, e^{f(x)} = \exp{(e^R\Lambda)}, \tag{100}$$

where $R = \int_{x_0}^x dx_1((\frac{4m}{\hbar^2}\beta(x_1,t))|\psi_s(x_1,t)|^2\frac{\psi_S(x_1,t)}{\frac{d}{dx_1}\psi_S(x_1,t)})$, $\Lambda = C_1 + \int_{x_0}^x dx_3\Bigg[-[\frac{2e\hbar}{2m}(\frac{d}{dx_3}A_x(x_3)) + \eta V_p(x_3,t)]\frac{\psi_S(x_3,t)}{\frac{d}{dx_3}\psi_S(x_3,t)}\frac{2m}{\hbar^2}e^{-R}\Bigg]$ with the identification of constant $C_1 = const$. The final perturbative solution of the G-L equation is of the form

$$\begin{aligned}
&\psi_p(x) = \psi_S(x)e^{f(x)}e^{\frac{i2e}{\hbar}\int_{x_0}^x dx_4 A_x(x_4)} = \psi_S(x)e^{\frac{i2e}{\hbar}\int_{x_0}^x dx_4 A_x(x_4)}Exp\Bigg[e^R \\
&\Bigg[C_1 + \int_{x_0}^x dx_3\Bigg[-[\frac{2e\hbar}{2m}(\frac{d}{dx_3}A_x(x_3)) + \eta V_p(x_3,t)]\frac{\psi_S(x_3,t)}{\frac{d}{dx_3}\psi_S(x_3,t)}\frac{2m}{\hbar^2}e^{-R}\Bigg]\Bigg]\Bigg],
\end{aligned} \tag{101}$$

where $R = \int_{x_0}^x dx_1((\frac{4m}{\hbar^2}\beta(x_1,t))|\psi_s(x_1,t)|^2\frac{\psi_S(x_1,t)}{\frac{d}{dx_1}\psi_S(x_1,t)})$.

In the meantime, we can use certain strong constraints due to the electric charge conservation I_{sc} and the fact of a quasi-one-dimensional situation giving $const = I_{sc} = -c_1 A_x(x)|\psi(x)_s|^2 e^{2f(x)}$ that implies $\frac{1}{2}Ln[-\frac{I_{sc}}{c_1 A_x(x)|\psi(x)_s|^2}] = f(x)$. The very last expression allows us to extract the value of the C_1 constant given by the formula

$$\begin{aligned}
C_1 = &\frac{1}{2}Ln[-\frac{I_{sc}}{c_1 A_x(x)|\psi(x)_s|^2}]\Big[e^{-\int_{x_0}^x dx_1((\frac{2m}{\hbar^2}(2)\beta(x_1,t))|\psi_s(x_1,t)|^2\frac{\psi_S(x_1,t)}{\frac{d}{dx_1}\psi_S(x_1,t)})}\Big] + \\
&\Big[\int_{x_0}^x dx_3\Bigg[[\frac{2e\hbar}{2m}(\frac{d}{dx_3}A_x(x_3)) + \\
&\eta V_p(x_3,t)]\frac{\psi_S(x_3,t)}{\frac{d}{dx_3}\psi_S(x_3,t)}\frac{2m}{\hbar^2}e^{-\int_{x_0}^{x_3} dx_1((\frac{2m}{\hbar^2}(2)\beta(x_1,t))|\psi_s(x_1,t)|^2\frac{\psi_S(x_1,t)}{\frac{d}{dx_1}\psi_S(x_1,t)})}\Bigg]\Big].
\end{aligned} \tag{102}$$

We can conclude, obtaining the perturbative solution of the G-L equation, that it is complementary to the previously obtained Bogomolny approach, since $f_2 = f$, as given in (71). Furthermore, the Bogomolny approach presented in (72) will be applied to a $p(x)\psi_p(x)$ wave function, so the $p(x)$ is determined.

8. Conclusions

The first achieved result is a specification of the existence of the solutions of the Bogomolny equations for the case of the G-L model in a flat (71) and curved superconducting wire (76) with the presence of a preimposed magnetic field (as by the preimposed vector potential) or with a preimposed electric current flow. It opens applications of the G-L Bogomolny scheme for various interfaces such as Josephson junctions and superconductor–non-superconductor interfaces.

The obtained solutions expressed in (70) and (76) correspond to physical intuitions and are obtained with the use of strong and semi-strong necessary conditions.

Perturbative solutions of the GL equation, with the first level of perturbation, bring the results consistent with the Bogomolny solutions. One of the domains of the application of the obtained results and the used methodology is an interface between the superconducting Josephson junction and the system of coupled semiconductor quantum dots [51], characterized by the movement of the electric charge between neighboring semiconductor dots under various electromagnetic conditions. One can also apply the specified methodology for a system of superconductors biased by an external magnetic field inducing or modifying the Josephson effect [47]. Furthermore, we can also use the Bogomolny invariant as encoders of physically preserved quantities.

In future, the Bogomolny approach will be attempted with the background of the other formalism (as in the case of the Bogoliubov–de Gennes or Eilenberger formalisms given by [53–55]) describing a superconducting state. It is, however, beyond the scope of the G-L picture.

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