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Theory of Classical Electrodynamics with Topologically Quantized Singularities as Electric Charges

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A theory of classical electrodynamics, where the only admissible electric charges are topological singularities in the electromagnetic field, is formulated. Charge quantization is accounted by the Chern theorem, such that Dirac magnetic monopoles are not needed. The theory allows positive and negative charges of equal magnitude, where the sign of the charge corresponds to the chirality of the topological singularity. Given the trajectory w(*t*) of the singularity, one can calculate electric and magnetic fields identical to those produced by Maxwell's equations for a moving point charge, apart from a multiplicative constant factor related to electron charge and vacuum permittivity. The theory is based on the relativistic Weyl equation in frequency-wavevector space, with eigenstates comprising the position, velocity, and acceleration of the singularity, and eigenvalues defining the retarded position of the charge. From the eigenstates, one calculates the Berry connection and the Berry curvatures, and identifies the curvatures as electric and magnetic fields.

1. Introduction

Quantization of charge is one of the long-standing unresolved questions of theoretical physics. Dirac discovered that if there exists a single magnetic monopole in our universe, this would account for the quantization of charge.[1] By exploring the behavior of an electron in the presence of the magnetic monopole in the realm of quantum mechanics, Dirac found that the product of the electron and magnetic monopole charge has to be quantized.[1,2] This appealing idea attracted great interest including hypothesizing the existence of dyons, that is, elementary particles carrying both electric and magnetic charge.[3–5] The existence of magnetic monopole structures was discovered in classical non-Abelian gauge theories with spontaneously broken gauge symmetry by 't Hooft[6] and Polyakov.[7] However, magnetic monopoles and dyons were not experimentally found (for a review of literature on magnetic monopoles see[8]). Here we attempt a more conservative approach and ask the following question: Can we develop a theory that can account for the quantization of charge, which would simultaneously be consistent with currently accepted theories and experiments? Toward this goal, we formulate a theory of classical electrodynamics where the only admissible electric charges are topological singularities in the electromagnetic field. There are only two opposite - positive and negative - values of the charge, which correspond to the two values of chirality of topological singularities. Continuous distributions of charges are not possible, whereas quantization of charge is guaranteed by the Chern theorem[9] (**Figure 1**). We point out that it may be possible to account for charge quantization by the classical theory of electrodynamics. If in some future work this theory is quantized, and if the quantum theory yields identical results as conventional quantum electrodynamics, this would account for charge quantization and answer the question posed above.

Maxwell's theory of classical electrodynamics is conventionally formulated with continuous distributions of charges and currents, which are sources of electric and magnetic fields.[10,11] During the development of quantum electrodynamics, several different formulations of classical electrodynamics emerged including Lorentz-Dirac[12] and Wheeler-Feynman formulation,[13,14] e.g., see Ref. [15] and refs. therein. A nice account of the problems faced, especially that of infinities related to the self-energy of a point charge, and the line of reasoning used in developing the theory, can be found in Feynman's Nobel lecture.[16] However, these theories were not formulated to address the problem of charge quantization.

According to Maxwell's theory, the scalar and the vector potential for a point charge q moving along a trajectory $\mathbf{w}(t)$ are the Liénard–Wiechert potentials[10,11]:

$$V_M(x^\mu) = \frac{1}{4\pi\varepsilon_0}\frac{qc}{sc - \mathbf{s}\cdot\mathbf{v}}, \quad \mathbf{A}_M(x^\mu) = \frac{\mathbf{v}}{c^2}V_M. \tag{1}$$

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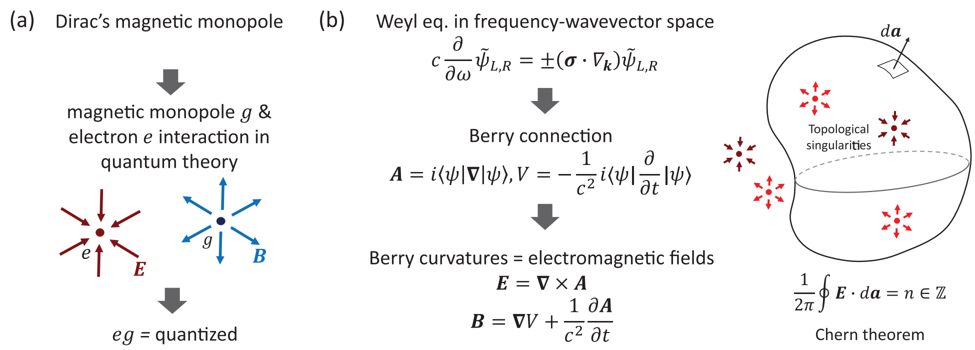

Figure 1. Comparison of Dirac's quantization condition and the present theory. a) If a single magnetic charge g (i.e., magnetic monopole) exists in our universe, then the interaction of the magnetic charge g and an electron e in quantum theory leads to quantization of the product ge.[1,8] b) In the present theory we construct electric and magnetic fields by using Berry connections and curvatures from the solution of Weyl equation in frequency-wavevector space. These solutions are identical to the solutions of Maxwell's equations for moving point charges, obtained via the Liénard–Wiechert potentials; however, Chern theorem plays the role of Gauss law and guarantees quantization of electric charges, which are here topological singularities in space.

Here, $x^\mu = (ct, \mathbf{r})$ denotes a point in spacetime, c is the speed of light, $\mathbf{s} = \mathbf{r} - \mathbf{w}(t_r)$, $s = |\mathbf{s}|$, and $\mathbf{v} = \frac{d\mathbf{w}(t)}{dt}|_{t_r}$ is the velocity of the point charge at the retarded time t_r. The retarded time is implicitly defined by the equation

$$|\mathbf{r} - \mathbf{w}(t_r)| = c(t - t_r), \tag{2}$$

which arises from the fact that information on the position, velocity, and acceleration of the point charge travels at the speed of light. The electric and magnetic fields are given by $\mathbf{E}_M(x^\mu) = -\nabla V_M - \partial\mathbf{A}_M/\partial t$, and $\mathbf{B}_M(x^\mu) = \nabla\times\mathbf{A}_M$. The lower index M stands for Maxwell's equations.

The formulation of classical electrodynamics presented here yields identical expressions for the electric and magnetic fields of a point charge (except for a multiplicative constant), however, the scalar and the vector potentials considerably differ. A fundamental difference between Maxwell's equations and our theory is the following: Maxwell's equations allow solutions with continuous distributions of charges and currents, while the present theory allows only point charges with two opposite (positive and negative) values of the charge.

2. Results

The theory is based on the relativistic Weyl equation[17] in frequency-wavevector (that is, energy–momentum) space:

$$c\frac{\partial}{\partial\omega}\tilde{\psi}_R(k^\mu) = -(\sigma_x\frac{\partial}{\partial k_x} + \sigma_y\frac{\partial}{\partial k_y} + \sigma_z\frac{\partial}{\partial k_z})\tilde{\psi}_R(k^\mu), \tag{3}$$

where σ_i, $i = x, y, z$, are the Pauli matrices, and $k^\mu = (\omega/c, \mathbf{k})$ is a 4-vector in the frequency-wavevector space. Equation (3) is the right-handed form of the Weyl equation, hence the index R in $\tilde{\psi}_R(k^\mu)$. The left-handed form is obtained by placing a minus sign on the left-hand side of Equation (3). Wavefunction $\tilde{\psi}_R(k^\mu)$ is a two-component spinor. In this theory, we calculate the electromagnetic fields from eigenstates of the Weyl equation by using Berry connection and Berry curvature machinery,[18] where "parameters" are space and time. Thus, $\tilde{\psi}_R(k^\mu)$ should be regarded as an auxiliary mathematical field used for generating the electromagnetic field. The auxiliary field contains information on the trajectory of a moving point charge:

$$\tilde{\psi}_R(k^\mu) = \psi_R(x^\mu)\exp(i\mathbf{k}\cdot\boldsymbol{\rho} - i\frac{\omega}{c}\rho^0), \tag{4}$$

where $\rho^\mu = (\rho^0, \boldsymbol{\rho})$ is a displacement 4-vector

$$\rho^\mu = \Lambda^\mu_\nu(x^\nu - w^\nu). \tag{5}$$

Here, $w^\nu = (ct_c, \mathbf{w}(t_c))$ denotes the position of the charge at time t_c, that is, w^ν describes world-line of the moving charge, whereas $x^\nu = (ct, \mathbf{r})$ is a point in spacetime where we want to know the electric and magnetic fields. We did not specify how w^ν and x^ν are related, as this connection naturally arise from the theory. Tensor Λ^μ_ν in Equation (5) is a Lorentz transformation that depends on the velocity and acceleration of the moving charge in a manner specified below.

From Equations (3) and (4) we obtain an eigenvalue equation

$$H\psi_R(x^\mu) = \boldsymbol{\sigma}\cdot\boldsymbol{\rho}\psi_R(x^\mu) = \rho^0\psi_R(x^\mu), \tag{6}$$

which has two eigenstates $|\psi_{R,n}\rangle$ and $|\psi_{R,p}\rangle$, with opposite eigenvalues equal in magnitude: $\rho^0 = \pm|\boldsymbol{\rho}|$ (this is equivalent to $\rho^\mu\rho_\mu = 0$ in 4-vector notation). The eigenstate $|\psi_{R,p}\rangle$ corresponds to the positive eigenvalue $\rho^0 > 0$, whereas the negative eigenvalue corresponds to $|\psi_{R,n}\rangle$. Because ρ^μ is obtained from the displacement 4-vector $s^\nu = x^\nu - w^\nu$ with a Lorentz transformation, $\rho^\mu\rho_\mu = 0$ implies that $s^\mu s_\mu = 0$. Thus, the two events, x^ν and w^ν, are connected by a signal traveling at the speed of light. The Lorentz transformation cannot reverse the time-ordering of two events separated by a light-like interval, therefore, $\rho^0 > 0$ implies $s^0 > 0$ and vice versa.

As we have already stated, $w^\nu = (ct_c, \mathbf{w}(t_c))$ is the world-line of the moving charge, whereas we aim to calculate the electromagnetic fields at $x^\nu = (ct, \mathbf{r})$. If we rewrite $s^\mu s_\mu = 0$ as $c(t - t_c) = \pm|\mathbf{r} - \mathbf{w}(t_c)|$, we see that the positive (negative) sign in this equation corresponds to $t > t_c$ ($t < t_c$, respectively). This means that when we use $|\psi_{R,p}\rangle$ together with $c(t - t_c) = |\mathbf{r} - \mathbf{w}(t_c)|$ to calculate the electromagnetic fields, we obtain retarded field solutions. In contrast, from $|\psi_{R,n}\rangle$ and $c(t - t_c) = -|\mathbf{r} - \mathbf{w}(t_c)|$ we obtain advanced field solutions. By applying the same procedure for the left-handed form of the Weyl equation, we find that its eigenstates $|\psi_{L,p}\rangle$ and $|\psi_{L,n}\rangle$ yield the retarded and advanced fields, respectively. The positive eigenvalue implies $c(t-t_c) = |\mathbf{r} - \mathbf{w}(t_c)|$, which is fully equivalent to Equation (2) with $t_r = t_c$, i.e., positive eigenvalues define the retarded time t_r. Here we consider only the retarded solutions as they are consistent with causality.

The Lorentz transformation Λ is of the form $\Lambda = RB$, where $B(\mathbf{v})$ is a pure boost by velocity of a moving charge $\mathbf{v}(t_r)$ at the retarded time t_r, whereas $R(\boldsymbol{\theta})$ is a rotation $R(\boldsymbol{\theta}) = e^{i(\theta_x J_x + \theta_y J_y + \theta_z J_z)}$, which depends on the velocity and acceleration of the point charge at the retarded time t_r: $\boldsymbol{\theta}(t_r) = \hat{\mathbf{n}} \cdot \int^{t_r} (\boldsymbol{\omega}_{Th}(t') \cdot \hat{\mathbf{n}}) dt'$, where

$$\boldsymbol{\omega}_{Th}(t') = \frac{1}{c^2}\frac{\gamma^2}{\gamma+1}(\mathbf{a}(t') \times \mathbf{v}(t')) \tag{7}$$

is the Thomas precession frequency,[19] and $\hat{\mathbf{n}}$ is a fixed unit vector aligned with $\mathbf{a} \times \mathbf{v}$ at the retarded time t_r. Here, J_i are generators of rotations, and $\gamma = 1/\sqrt{1 - v^2/c^2}$ is the Lorentz contraction. Later in the text, we will demonstrate that the electromagnetic fields depend only on the derivatives of the angle variables $d\boldsymbol{\theta}(t_r)/dt_r = \boldsymbol{\omega}_{Th}(t_r)$ at t_r. For a 1D motion of the charge, $\mathbf{a} \times \mathbf{v} = 0$ and Λ is a pure boost $\Lambda = B(\mathbf{v})$. For curvilinear motion, the boost is followed by rotation. If the curvilinear motion is confined to a plane, the definition of the angle simplifies to $\boldsymbol{\theta}(t_r) = \int^{t_r} \boldsymbol{\omega}_{Th}(t') dt'$.

The vector and the scalar potentials in present theory are given by the Berry connection[18]:

$$\mathbf{A} = i\langle\psi|\nabla|\psi\rangle, \quad V = -\frac{1}{c^2} i\langle\psi|\frac{\partial}{\partial t}|\psi\rangle. \tag{8}$$

The electric and magnetic fields are given by the Berry curvature:

$$\mathbf{E} = \nabla \times \mathbf{A}, \quad \mathbf{B} = \nabla V + \frac{1}{c^2}\frac{\partial \mathbf{A}}{\partial t}. \tag{9}$$

In Equation (8), $|\psi\rangle$ is either $|\psi_{R,p}\rangle$ or $|\psi_{L,p}\rangle$. The two eigenstates $|\psi_{R,p}\rangle$ and $|\psi_{L,p}\rangle$ have a well-defined chirality. By calculating the direction of the electrostatic field (see below), we find that positive charge corresponds to the fields obtained from the eigenvector $|\psi_{L,p}\rangle$, whereas fields obtained from $|\psi_{R,p}\rangle$ correspond to the negative charge. For this reason, in the rest of the text we use the notation $|\psi_+\rangle = |\psi_{L,p}\rangle$ and $|\psi_-\rangle = |\psi_{R,p}\rangle$.

Two technical notes are in order: While evaluating the derivatives in Equations (8) and (9), one should take into account the fact that the retarded time t_r, which is implicitly defined with Equation (2), depends on the coordinates $x^\nu = (ct, \mathbf{r})$. Therefore, derivatives of $\mathbf{v}(t_r)$ and $\mathbf{a}(t_r)$ with respect to spatial coordinates x, y, and z are not zero. The connection $\mathbf{A}$ has a singularity fully equivalent to the Dirac string of the Dirac magnetic monopole.[1] This was clarified by Wu and Yang, who have shown that two connections are required to cover the entirety of parameter space, which is real space here; see **Figures** 2a,b,d,e and 3a,b,d,e for illustration. The two connections are related by a gauge transformation in a region where they overlap.[20]

Although the theory is written for a single charge, by postulating the superposition principle it is straightforward to expand it for a number of charges; for simplicity we will discuss a single charge. Note that in our theory the electric field is a curl of the vector potential $\mathbf{A}$. This means that the only admissible charged objects are singularities of the electromagnetic field in space where $\nabla \cdot (\nabla \times \mathbf{A}) = 0$ is not applicable. Moreover, by the construction of the electric field, these singularities are topological, that is, Chern theorem guarantees that

$$\oint_S \mathbf{E} \cdot d\mathbf{a} = 2\pi \times \text{integer}, \tag{10}$$

where the integer corresponds to the number and chiralities of the topological singularities inside the closed surface S, see Figure 1. In our theory, the sign of the charge corresponds to the chirality of the topological singularity in space. We emphasize that the quantization of charge [Equation (10)] is not an assumption in our theory. More specifically, the postulated equations are Equations(3)–(5) and (7)–(9), whereas Equation (10) can be analytically derived from the postulates (via the Chern theorem).

The electric and magnetic fields in Equation (9) differ from the conventional Maxwell fields by a multiplicative constant

$$\mathbf{E}_M = \frac{q}{2\pi\varepsilon_0}\mathbf{E}, \ \mathbf{B}_M = \frac{q}{2\pi\varepsilon_0}\mathbf{B}, \tag{11}$$

where q is the electron charge.

First, we demonstrate that Equation (11) indeed holds. We start with the simplest example of a stationary point charge at a position $\mathbf{w}$, $\mathbf{v} = 0$, where Hamiltonian Equation (6) takes the form $H = \boldsymbol{\sigma} \cdot (\mathbf{r} - \mathbf{w})$. This is a well-known Hamiltonian that yields a Berry monopole at $\mathbf{w}$, that is,

$$\mathbf{E} = \pm\frac{1}{2}\frac{\mathbf{r} - \mathbf{w}}{|\mathbf{r} - \mathbf{w}|^3}, \ \mathbf{B} = 0; \tag{12}$$

positive (negative) sign corresponds to fields obtained with $|\psi_+\rangle$ ($|\psi_-\rangle$, respectively). The electric field in Equation (12) and the corresponding connections $\mathbf{A}$ are illustrated in Figure 2a–c.

Next, we consider a point charge moving with the constant velocity along the z-axis, $\mathbf{w} = v_z t\hat{z}$, where Hamiltonian in Equation (6) takes the form

$$H = x\sigma_x + y\sigma_y + \gamma(z - v_z t)\sigma_z. \tag{13}$$

By calculating the Berry connections and curvatures (Section SI, Supporting Information), we obtain:

$$\mathbf{E} = \pm\frac{1}{2}\frac{\gamma(\mathbf{r} - v_z t\hat{z})}{\left[x^2 + y^2 + \gamma^2(z - v_z t)2\right]^{3/2}}, \ \mathbf{B} = \frac{1}{c^2} v_z \hat{z} \times \mathbf{E}, \tag{14}$$

which are exactly the electric and magnetic fields of the point charge moving at a constant velocity. This result is perhaps not surprising because we have essentially Lorentz boosted the stationary Hamiltonian $\boldsymbol{\sigma} \cdot (\mathbf{r} - \mathbf{w})$ to obtain Equation (13). The electric field in Equation (14) and the corresponding connections $\mathbf{A}$ are illustrated in Figure 2d–f.

However, what we find surprising is that this approach yields correct Maxwell expressions for the field of a charge that accelerates and radiates. First, we address motion of a charge along a straight line, where velocity and acceleration are collinear (as in Bremsstrahlung). Because $\mathbf{a} \times \mathbf{v} = 0$, we have $R = 1_4$, and Lorentz transformation in Equation (5) is a pure boost, $\Lambda = B(\boldsymbol{v}(t_r))$. Without losing generality, we set the motion of the

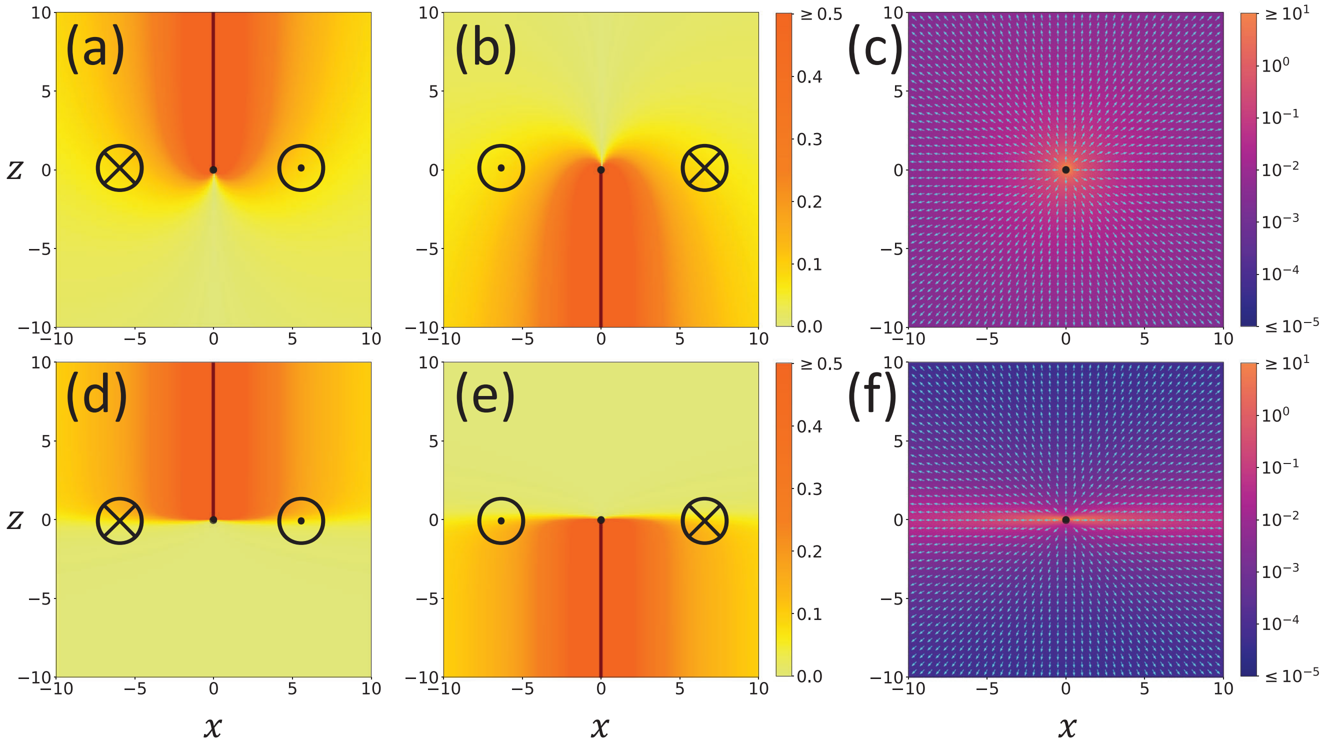

Figure 2. The Berry connection $\boldsymbol{A}$, and the corresponding electric field $\boldsymbol{E} = \nabla \times \boldsymbol{A}$ for a stationary topological singularity (a–c), and a topological singularity moving at velocity $v = 0.995c$ along the z-axis (d-f). The unit convention for illustrations is $c = 1$. a,b) The two connections $\boldsymbol{A}$ for a stationary point charge. Two connections are needed to cover the real space due to the presence of the Dirac string singularities, which are indicated with black solid lines. The amplitude of the connections is indicated by using the color map, whereas the direction is asimuthal, i.e., perpendicular o the plane with the direction indicated on the plot. The connections in (a) and (b) are connected by a gauge transformation in a region where they overlap. c) The electric field calculated from the connections in (a,b). The magnitude of the electric field is indicated by using the color map, whereas the direction is indicated with arrows (the length of the arrow does not correspond to the magnitude of the field). d–f) The same layout as in (a–c) for a moving topological singularity.

charge along the z-axis. The Hamiltonian Equation (6) is given by

$$H = x\sigma_x + y\sigma_y + \gamma(t_r)(z - w_z(t_r) - v_z(t_r)(t - t_r))\sigma_z. \tag{15}$$

Although it is essentially the same Hamiltonian as in the previous example, the derivative $dv_z(t)/dt|_{t_r}$ is now not necessarily zero, which gives rise to the radiation field. By applying the machinery of connections and curvatures (Section SII, Supporting Information), we find the electric and magnetic fields to be:

$$\mathbf{E} = \pm\frac{1}{2}\frac{s}{(s\cdot u)^3}\left[(c^2 - v_z^2)\boldsymbol{u} + \boldsymbol{s}\times(c\hat{\boldsymbol{s}}\times(a_z\hat{\boldsymbol{z}}))\right], \quad \mathbf{B} = \frac{1}{c}\hat{\boldsymbol{s}}\times\mathbf{E}, \tag{16}$$

coinciding with Maxwell's theory. Here, we have used $\boldsymbol{u} \equiv c\hat{\boldsymbol{s}} - \boldsymbol{v}$ following.[10] In Figure 3 we illustrate the electric field from Equation (16), and pertinent connections $\mathbf{A}$, for a topological singularity oscillating along the z-axis. Figure 3a–c corresponds to the topological singularity at the amplitude of the oscillating motion where $\mathbf{v} = 0$ and the magnitude of the acceleration is maximal. In Figure 3d–f we show $\mathbf{A}$ and $\mathbf{E}$ when the singularity passes the origin, $\mathbf{a} = 0$ and the magnitude of the velocity is maximal.

Now we turn to curvilinear motion. For simplicity, let us first address motion of a charge in a plane, which we choose to be the xy plane without loss of generality. In this case, the rotation component of the Lorentz transformation $\Lambda = RB$ is of the form $R(\theta_z(t_r)) = e^{i\theta_z(t_r)J_z}$, where $\theta_z(t_r) = \int^{t_r}\omega_{Th}(t')dt'$, whereas the boost is $B(v_x(t_r)\hat{\mathbf{x}} + v_y(t_r)\hat{\mathbf{y}})$. The Thomas precession frequency is at any point of motion orthogonal to the xy plane: $\boldsymbol{\omega}_{Th}(t') = \omega_{Th}(t')\hat{\mathbf{z}}$. By using $\Lambda = RB$, we construct the Hamiltonian Equation (6) and apply Equations (8) and (9) to calculate the fields. A detailed expression for $\Lambda = RB$ and calculation of the electric field component E_z is written in Section SIII (Supporting Information), while the other field components can be obtained in an analogous manner yielding

$$\mathbf{E} = \pm\frac{1}{2}\frac{s}{(s\cdot u)^3}\left[(c^2 - v^2)\boldsymbol{u} + \boldsymbol{s}\times(\boldsymbol{u}\times\boldsymbol{a})\right], \quad \mathbf{B} = \frac{1}{c}\hat{\boldsymbol{s}}\times\mathbf{E}. \tag{17}$$

The resulting fields in Equation (17) coincide with those given by Maxwell's theory.[10]

Next, consider a curvilinear motion in three-dimensions (3D). First, we note that Equation (17) coincides with solutions of Maxwell's theory for a 3D curvilinear motion of the charge.[10] The reason behind this is that the electromagnetic fields at a point x^μ depend on the instantaneous velocity and acceleration at the retarded position of the charge. The retarded velocity and acceleration vectors are in a plane that is perpendicular to the unit vector $\hat{\mathbf{n}} = \mathbf{a}(t_r)\times\mathbf{v}(t_r)/|\mathbf{a}(t_r)\times\mathbf{v}(t_r)|$ (see **Figure 4** for illustration). Thus, we can always invent a 2D motion in the plane perpendicular

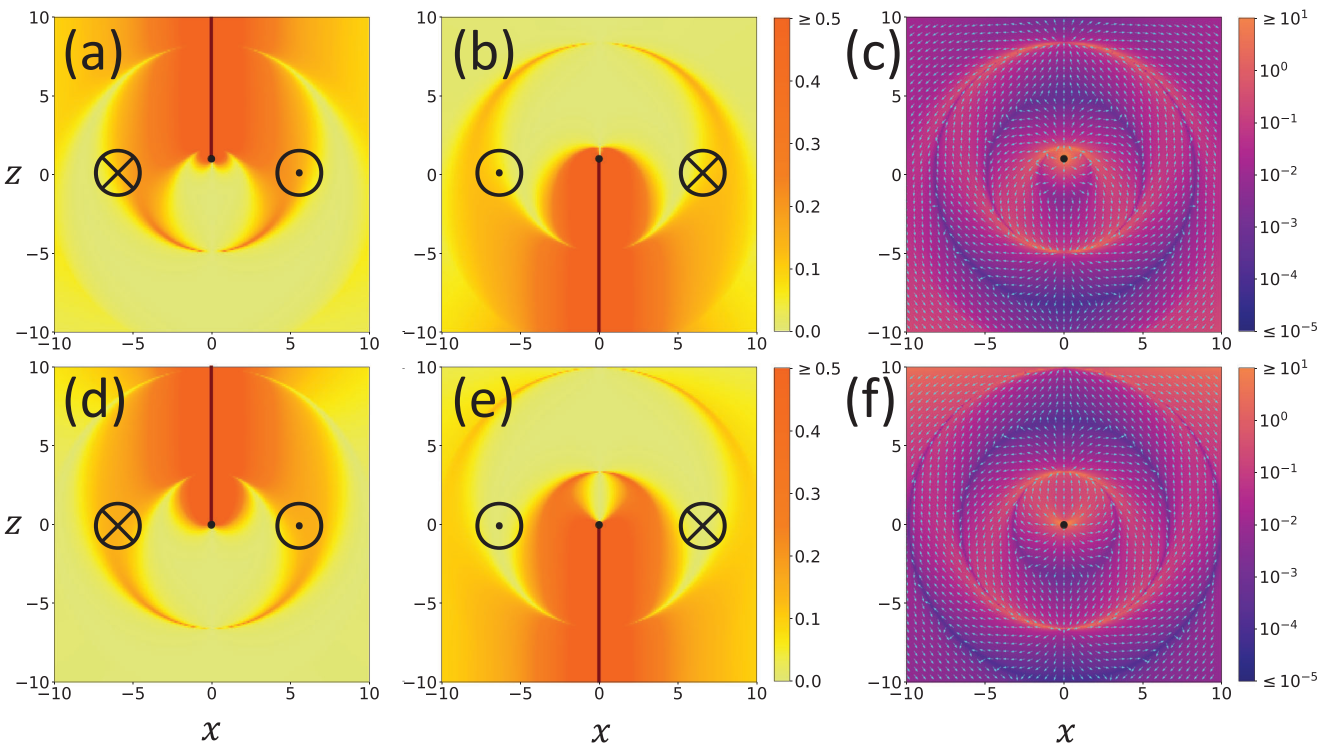

Figure 3. The Berry connection $\boldsymbol{A}$, and the corresponding electric field $\boldsymbol{E} = \nabla \times \boldsymbol{A}$ for an oscillating topological singularity. The motion is described by $z = A\cos(\omega t)$, where $A = 1$ and $A\omega = 0.95$ (using $c = 1$ unit convention). a–c) $\boldsymbol{A}$ and $\boldsymbol{E}$ at $t = 0$ when the topological singularity is at the amplitude of the oscillating motion, where $\boldsymbol{v} = 0$ and the magnitude of the acceleration is maximal. d–f) $\boldsymbol{A}$ and $\boldsymbol{E}$ at $\omega t = \frac{\pi}{2}$ when the singularity passes the origin, $\boldsymbol{a} = 0$ and the magnitude of the velocity is maximal. The direction and magnitude of the vector fields is indicated as in Figure 2.

to $\hat{\mathbf{n}}$ that has identical instantaneous velocity and acceleration at the retarded position of the charge as a given 3D curvilinear motion, and the electromagnetic fields at x^μ must be, according to Maxwell, identical for the two different motions. In other words, if we have two world-lines, one describing 2D and the other 3D curvilinear motion, such that at the retarded position $w^\nu = (ct_r, \mathbf{w}(t_r))$ the velocity and acceleration of the two motions coincide, these two motions will yield identical electromagnetic fields at points $x^\mu = (ct, \mathbf{r})$, which are connected with w^ν by a signal traveling at the speed of light: $s^\mu s_\mu = 0$, $s^\mu = x^\mu - w^\mu$. Therefore, as we wish our theory to yield identical solutions as Maxwell's theory, we define the unit vector $\hat{\mathbf{n}}$ to be fixed perpendicular to the plane spanned by the retarded velocity and acceleration, and define the angle of rotation as $\theta(t_r) = \hat{\mathbf{n}} \cdot \int^{t_c} (\boldsymbol{\omega}_{Th}(t') \cdot \hat{\mathbf{n}}) dt'$. With this definition, we have economically constructed the Lorentz transformation $\Lambda = R(\theta(t_r))B(\mathbf{v}(t_r))$ that yields correct expressions for the electromagnetic field of a moving charge. Besides the general analytical construction described above, we have verified this result numerically as well.

Let us address the Lorentz covariance of the theory. Suppose that we observe the motion of a charge in an inertial frame S. We insert the world-line of that charge w^μ, which contains the velocity $\mathbf{v}$ and acceleration $\mathbf{a}$ of the charge in frame S into Equations (3)–(5) and (7)–(9) that constitute our theory, and obtain the fields $\mathbf{E}$ and $\mathbf{B}$ at x^μ in frame S. If we move to another inertial frame S', where the world-line w'^μ, the velocity $\mathbf{v}'$, and acceleration $\mathbf{a}'$ can be obtained via Lorentz transformations, and insert these quantities in the same Equations (3)–(9), but now with primed quantities, we obtain the fields $\mathbf{E}'$ and $\mathbf{B}'$ at x'^μ in frame S'. We know that the fields $\mathbf{E}$ and $\mathbf{B}$ transform into $\mathbf{E}'$ and $\mathbf{B}'$ as a 2nd rank tensor under Lorentz transformations, simply because they are solutions of Maxwell's equations. Therefore, the Lorentz covariance of our theory is connected to the fact that we reproduce Maxwell's theory (for point charges). The Weyl Equation (3) is manifestly covariant.

3. Discussion

The idea for formulating electrodynamics in terms of the Weyl equation in frequency-momentum space arose from studies of the Weyl semimetals in condensed-matter physics, photonics, and ultracold quantum gases (e.g., see Refs. [21–25]). Under specific circumstances, Weyl points may occur in momentum space of crystalline, photonic, or optical lattices. These momentum space topological singularities are located somewhere in the Brillouin zone(s) of these materials. The equivalent of Gauss law for these Weyl points in momentum space is the Chern theorem.[21–25] The idea for this study was simply to exchange the real and momentum space to obtain quantized topological charges as electric charges in real space, such that the Gauss law in real space would be equivalent to the Chern theorem.

It has been previously shown that Maxwell's equations can be derived from the massless Dirac equation in spinor form (e.g., see Refs. [26, 27] and refs. therein). These approaches are equivalent to Maxwell's electrodynamics that allows for continuous distributions of charges, and therefore they considerably differ from our theory. The Berry phase effects have been ad-

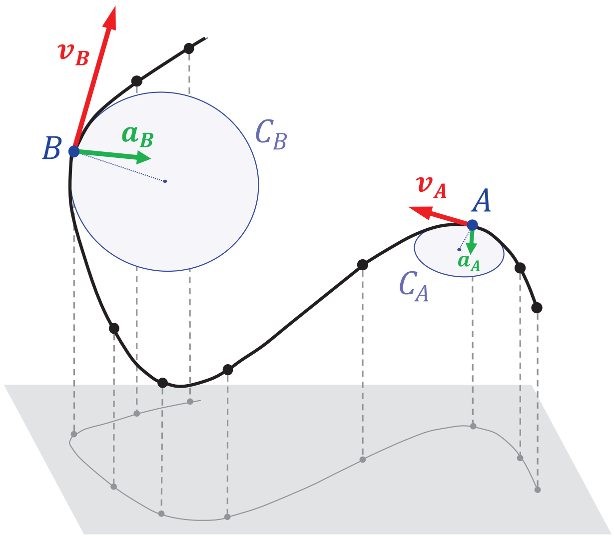

Figure 4. A topological singularity (i.e., electric charge) moving on an arbitrary, non-planar trajectory in 3D space, with its velocity and acceleration shown at two arbitrary points of the trajectory: A and B. Circles C_A and C_B describe the local curvature of the trajectory, and lie in the same planes as the velocity and acceleration vectors evaluated at the corresponding points (every 3D trajectory is locally identical to a 2D circular curve).

dressed in momentum space of the massive,[28] and more recently the massless Dirac (i.e., Weyl) equation,[29] with Dirac monopoles in momentum space. Ref. [29] also addressed the monopole in momentum space of Maxwell equations. These calculations are in sharp contrast to this theory, which addresses electric monopoles in real space arising from the Weyl equation in frequency-wavevector space, and establishes the connection to the fields obtained from Liénard–Wiechert potentials.[10,11] Nevertheless, by drawing upon the analogy between the solutions of the Maxwell equations for moving point charges in real space, which are fully equivalent to solutions of our theory, one could get inspiration for potential observations of analogous phenomena in momentum space of Weyl semimetals in condensed matter, photonic or optical lattices. By changing the parameters of these lattices, Weyl points can be moved in the Brillouin zone (i.e., in momentum space), which is analogous to the motion of charges in real space.

Although the physics of Weyl semi-metals is most closely related to this work, we point out there are numerous studies of topological quantization in condensed matter physics, photonics, ultracold atomic gases, acoustic crystals and other systems, for example, see Refs. [30–39]. Our focus here is solely on the quantization of electric charge within the realm of classical electrodynamics.

Up to this point, we have considered the field $\tilde{\psi}_{R,L}(k^\mu)$ simply as an auxiliary field that enforces quantization of charge via Chern theorem, which very conveniently yields, via Berry connection-curvature machinery, electromagnetic fields of moving point charges. One may ask, are these Weyl fields more than a mathematical convenience? Can they be interpreted as particles that interact with electric charges and how? The hypothetical particles described by fields $\tilde{\psi}_{R,L}(k^\mu)$ are solutions of the Weyl equation in frequency-wavevector space, that is, in energy–momentum space. Therefore, they live in a space dual to Minkowski spacetime. However, the dynamics in the two spaces are not independent. When a particle moves through spacetime, under the influence of interactions, its energy and momentum can change. Inversely, if a hypothetical particle described by the field $\tilde{\psi}_{R,L}(k^\mu)$ moves through energy–momentum space, its spatial and temporal coordinates can change. For dynamics in such a space, one should find the laws of "conservation of space" and "conservation of time," which are analogous (or perhaps dual) to conventional laws of conservation of momentum and energy, respectively. Equation (8) would represent interactions of these hypothetical $\tilde{\psi}_{R,L}(k^\mu)$ particles with electric charges. Thus, when such a particle moves through spacetime, its wavefunction $\tilde{\psi}_{R,L}(k^\mu)$ acquires a phase fully analogous to the (geometric) Berry phase;[18] in this case spacetime coordinates can be thought of as parameters, such that a change of parameters imprints the geometric phase on the particle's wavefunction. This discussion implies that to think of $\tilde{\psi}_{R,L}(k^\mu)$ as a physical field, we should change the standard paradigms of physics such as conservation of energy and momentum. Therefore, in this paper we regard $\tilde{\psi}_{R,L}(k^\mu)$ as auxiliary mathematical fields, and leave potential manifestation of their physical reality for future studies.

In this theory, we establish the electromagnetic fields from the world-lines of topological singularities (sources). Up to this point, we have not discussed the force on the singularities, that is, the Lorentz force on point charges. In conventional classical electrodynamics, the Lorentz force is postulated (e.g., see Refs. [10, 11]):

$$\mathbf{F} = q(\mathbf{E}_M + \mathbf{v} \times \mathbf{B}_M). \tag{18}$$

In classical mechanics, it is used to derive the motion of charges in electromagnetic fields, while Maxwell's equations are used to calculate the fields from these charges. In a fully equivalent manner, we can postulate an appropriate version of the Lorentz force for our theory,

$$\mathbf{F} = \frac{q^2}{2\pi\epsilon_0}(\mathbf{E} + \mathbf{v} \times \mathbf{B}), \tag{19}$$

which is identical to the force in Equation (18). The coupling constant $q^2/2\pi\epsilon_0$ is not fixed by our theory, however, the coupling constant is also not fixed in conventional electrodynamics; it rather follows from experimental observations. The key distinction is that our theory allows only two opposite values of the charge, and they are necessarily point charges, which are constraints on the fields not contained in Maxwell's theory.

4. Conclusion

In conclusion, we have formulated a theory of classical electrodynamics where electric charges are topological singularities in the electromagnetic field, and their sign corresponds to the chirality of the singularity. Charge quantization is thus accounted by the Chern theorem. The electric and magnetic fields are identical to those produced by the Maxwell's equations, apart from the multiplicative constant $q/2\pi\epsilon_0$. Quantization of charge conventionally relies on the prediction of magnetic monopoles.[1] However, these theories addressing quantization of charge are inherently quantum,[8] whereas our theory is entirely classical. The theory is based on the relativistic Weyl equation in frequency-wavevector space, with eigenstates depending on the spacetime coordinates. From the eigenstates, we calculate the Berry connection and curvatures, by using spacetime coordinates as "parameters", and then identify the curvatures as electric and magnetic fields. In outlook, we foresee efforts to develop the quantum version of this theory to test whether it will yield identical results as conventional quantum electrodynamics. One of the greatest challenges of these efforts will be to resolve the issue of infinities related to self-energy of a point charge; it is not clear at this point whether conventional quantum electrodynamics techniques could be used to address this challenge in that new theory.

Supporting Information

Supporting Information is available from the Wiley Online Library or from the author.

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Conflict of Interest

The authors declare no conflict of interest.

Author Contributions

B. G. and D. J. contributed equally to this work. All authors participated in this work and made substantial contributions. H.B. conceived and supervised the project.

Data Availability Statement

The data that support the findings of this study are available from the corresponding author upon reasonable request.

Keywords

charge quantization, Chern theorem, topological singularity, Weyl equation in energy-momentum space

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