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General spherically symmetric elastic stars in relativity

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Abstract The relativistic theory of elasticity is reviewed within the spherically symmetric context with a view towards the modeling of star interiors possessing elastic properties such as the ones expected in neutron stars. Emphasis is placed on generality in the main sections of the paper, and the results are then applied to specific examples. Along the way, a few general results for spacetimes admitting isometries are reviewed, and their consequences are fully exploited in the case of spherical symmetry relating them next to the case in which the material content of the spacetime is some elastic material. Specific examples are provided satisfying the dominant energy condition and admitting a constitutive equation, including a static two-layer star ‘toy model’ consisting of an elastic core surrounded by a perfect fluid corresponding to the interior Schwarzschild solution matched to the vacuum Schwarzschild solution. This paper extends and generalizes the pioneering work by Magli and Kijowski (Gen Relat Gravit 24:139, 1992), Magli (Gen Relat Gravit 25:1277, 1993; 25:441, 1993), and complements, in a sense, that by Karlovini and Samuelsson in their interesting series of papers (Karlovini and Samuelsson in Class Quantum Grav 20:363, 2003; 21:1559, 21:4531, 2004).

Keywords Elasticity, Spherical symmetry

1 Introduction

The interest of a relativistic theory of elasticity is twofold; on the one hand there is its purely theoretical interest, namely that of providing a relativistic extension of a well-known (and very fruitful) classical theory; on the other hand and on theoretical grounds, it is expected that neutron stars possess a solid crust with elastic properties which may help explaining certain observational issues (see [1] for a

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thorough account of these and other related features). Further, anisotropy in pressures is a phenomenon occurring in many situations of equilibrium which are of interest in astrophysics and whose corresponding dynamics has been thoroughly studied (see for instance [2] and references therein); the assumed point of view however has been an heuristic one, without providing mechanisms explaining how the anisotropy in pressures may arise and using instead *ad hoc* assumptions. Magli and Kijowski [3] and Magli [4] have shown that in spherical symmetry, the anisotropy in pressures arises quite naturally as the relativistic extension of the classical (non-relativistic) non-isotropic stress in elasticity theory. See also [5] for an excellent study of the static case in spherical symmetry, where existence theorems for regular solutions near the center are proven under rather mild, physically meaningful, hypotheses. Beig and Schmidt [6] have shown that, in general, the field equations for elastic matter can be cast into a first-order symmetric hyperbolic system and that as a consequence, local-in-time existence and uniqueness theorems may be obtained under various circumstances.

The aim of this paper is to extend and generalize the work presented in [3; 4] as well as to set up a set of mathematical tools and equations that may facilitate the obtention of exact solutions to Einstein's field equations (EFE) describing the interior of elastic materials and satisfying the dominant energy condition (DEC).

The paper is organized as follows: in the next section we provide a brief account of the theory of relativistic elasticity, much along the lines followed in [1; 3; 4], but we shall also include some comments on the relationship between the isometries in the material space and in the spacetime. Most of the results in that section are well known and could be found in the above references, but we are still including them in order to set up the notation which will be followed in the remainder of the paper as well as for making the present paper more self-contained. Section 3 contains a digression on spherically symmetric spacetimes and the restrictions that such an assumption imposes on the physics in these spacetimes, which we then apply it to the case of elastic materials. In Sect. 4 EFEs are obtained for the general case and some particular cases are commented upon. In Sect. 5 we analyze in detail the case of shear-free solutions, paying special attention to the fulfilment of the dominant energy condition as well as to the necessary and sufficient conditions that must be satisfied for a constitutive equation to be admitted; finally we present a few selected examples satisfying the dominant energy condition and admitting a constitutive equation, in particular that of a static two-layer star 'toy model' consisting of an elastic core surrounded by a perfect fluid corresponding to the interior Schwarzschild solution matched to the vacuum Schwarzschild solution. We do not claim that these examples are of any particular physical relevance, as much more work is needed in order to obtain realistic models, however they illustrate the points raised in the general analysis. Further, an appendix containing the analysis of the elasticity difference tensor for the non-static case has been included, most along the lines followed in a previous paper by two of the authors [7].

2 Relativistic elasticity revisited

Let (M, g) be a spacetime, M then being a 4-dimensional Hausdorff, simply connected manifold of class C^2 at least, and g a Lorentz metric of signature $(-, +, +, +)$.

The *material space* X is a 3-dimensional manifold endowed with a Riemannian metric γ , the *material metric*; points in X can then be thought of as the particles of which the material is made of. Coordinates in M will be denoted as x^a for $a = 0, 1, 2, 3$, and coordinates in X as $y^A, A = 1, 2, 3$. The material metric γ is not a dynamical quantity of the theory, but it is frozen in the material, and it roughly describes distances between neighboring particles in the relaxed state of the material.

The spacetime configuration of the material is said to be completely specified whenever a submersion $\psi : M \rightarrow X$ is given; if one chooses coordinate charts in M and X as above, the coordinate representative of ψ is given by three fields

$$y^A = y^A(x^b), \quad A = 1, 2, 3$$

and the physical laws describing the mechanical properties of the material can then be expressed in terms of a hyperbolic second order system of PDE. The differential map $\psi_* : T_p M \rightarrow T_{\psi(p)} X$ is then represented in the above charts by the rank 3 matrix

$$(y_b^A)_p, \quad y_b^A = \frac{\partial y^A}{\partial x^b} \quad A = 1, 2, 3, \quad b = 0, 1, 2, 3$$

which is sometimes called *relativistic deformation gradient*. Since ψ_* has maximal rank 3, its kernel is spanned at each point by a single timelike vector which we may take as normalized to unity, the resulting vector field, say $\mathbf{u} = u^a \partial_a$, satisfies then

$$y_b^A u^b = 0, \quad u^a u_a = -1, \quad u^0 > 0$$

the last condition stating that we choose it future oriented; \mathbf{u} is called the *velocity field of the matter*, and in the above picture in which the points in X are material points, it turns out that the spacetime manifold M (or, to be more precise, an open submanifold of it) is then made up by the worldlines of the material particles, whose tangent vector is precisely \mathbf{u} .

The material space is said to be in a *locally relaxed state* at an event $p \in M$ if, at p , it holds $k_{ab} \equiv (\psi^* \gamma)_{ab} = h_{ab}$ where $h_{ab} = g_{ab} + u_a u_b$. Otherwise, it is said to be *strained*, and a measurement of the difference between k_{ab} and h_{ab} is the *strain*, whose definition varies in the literature; thus, it can be defined simply as $S_{ab} = -\frac{1}{2}(k_{ab} - h_{ab}) = -\frac{1}{2}(k_{ab} - u_a u_b - g_{ab})$. We shall follow instead the convention in [4] and use

$$K_{ab} \equiv k_{ab} - u_a u_b \tag{1}$$

Notice that $K_b^a u^b = u^a$, and therefore one of its eigenvalues is 1. Definitions using the logarithm of the above tensor¹ also appear in the literature as that allows simple interpretations of the associated algebraic invariants, see e.g. [3; 5].

The strain tensor determines the elastic energy stored in an infinitesimal volume element of the material space (or energy per particle), hence that energy will be a scalar function of K_{ab} . This function is called *constitutive equation* of the

¹ With one index raised, thus one has a linear operator which turns out to be positive and self-adjoint, its logarithm being then well-defined.

material, and its specification amounts to the specification of the material. We shall represent it as $v = v(I_1, I_2, I_3)$, where I_1, I_2, I_3 are any suitably chosen set of scalar invariants² associated with and characterizing K_{ab} completely. Following [4] we shall choose

$$\begin{aligned} I_1 &= \frac{1}{2} (\text{Tr}K - 4) \\ I_2 &= \frac{1}{4} \left[\text{Tr}K^2 - (\text{Tr}K)^2 \right] + 3 \\ I_3 &= \frac{1}{2} (\det K - 1), \end{aligned} \quad (2)$$

Notice that for $K_{ab} = g_{ab}$ (equivalently $k_{ab} = h_{ab}$) the strain tensor S_{ab} is zero, that is: the induced metric on the rest frame of an observer moving with four-velocity \mathbf{u}, h , coincides with the material metric γ (its pull-back by ψ) describing the relaxed state of the material; thus it makes sense to have zero elastic energy stored. It is immediate to check from the above expressions that in this case one has $I_1 = I_2 = I_3 = 0$.

The energy density ρ will then be the particle number density ε times the constitutive equation, that is

$$\rho = \varepsilon v(I_1, I_2, I_3) = \varepsilon_0 \sqrt{\det K} v(I_1, I_2, I_3) \quad (3)$$

where ε_0 is the particle number density as measured in the material space, or rather, with respect to the volume form associated with $k_{ab} = (\psi^* \gamma)_{ab}$, and ε is that with respect to h_{ab} ; see [8] for a proof of the above equation. In some references (e.g. [4]), the names ρ and ε are exchanged and the density measured w.r.t. $k_{ab} = (\psi^* \gamma)_{ab}$ (ε_0 in our notation) is then called ‘‘density of the relaxed material’’ (see the above comments on the meaning of γ), whereas that measured w.r.t. h_{ab} is referred to as the ‘‘density in the rest frame’’.

We next turn our attention towards the energy-momentum tensor of an elastic material. Before proceeding, it will be useful to recall that any symmetric, second order covariant tensor field may be decomposed with respect to a timelike unit vector field \mathbf{v} , $v^a v_a = -1$ as follows:

$$T_{ab} = \rho v_a v_b + p h_{ab} + P_{ab} + v_a q_b + q_a v_b \quad (4)$$

where $h_{ab} = g_{ab} + v_a v_b$, $P_{ab} = h_a^m h_b^n (T_{mn} - 3p h_{mn})$, $q_a = -(T_{ab} v^b + \rho v_a)$, $\rho = T_{ab} v^a v^b$, $p = \frac{1}{3} h^{ab} T_{ab}$. From the definitions of these variables it readily follows

$$h_{ab} v^b = 0, \quad P_{ab} v^b = g^{ab} P_{ab} = 0, \quad \text{and} \quad q^a v_a = 0.$$

In the case that T_{ab} represents the energy-momentum tensor of some material distribution, ρ , p , P_{ab} , q^a are respectively the energy density, isotropic pressure, anisotropic pressure tensor and heat flow that a family of observers moving with four-velocity \mathbf{v} would measure at every point in the spacetime.

² Recall that one of the eigenvalues is 1, therefore, there exist three other scalars (in particular they could be chosen as the remaining eigenvalues) characterizing K_b^a completely along with its eigenvectors.

In the case of elastic matter, the energy-momentum tensor is obtained from the Lagrangian $\Lambda = \sqrt{-g}\rho$, which depends on y^A, y_a^A and x^a . The corresponding Euler–Lagrange equations assume the form $\frac{\partial \Lambda}{\partial y^A} - \partial_a \left(\frac{\partial \Lambda}{\partial y_a^A} \right) = 0$. Using Noether’s theorem one constructs the canonical energy-momentum tensor density

$$\mathcal{T}_b^a = \frac{\partial \Lambda}{\partial y_a^A} y_b^A - \delta_b^a \Lambda, \quad (5)$$

which satisfies the conservation law $\nabla_a \mathcal{T}^{ab} = 0$. Due to the Rosenfeld–Belinfante theorem [9; 10], the canonical energy-momentum tensor density coincides with the symmetric energy-momentum tensor density given by

$$\mathcal{T}_{ab} = -2 \frac{\partial \Lambda}{\partial g^{ab}} \quad (6)$$

up to a sign, (see for instance [11]). In fact, using the push-forward of the space-time metric, $G^{CD} = y_c^C y_d^D g^{cd}$, one can rewrite (5) as

$$- \mathcal{T}_b^a = \sqrt{-g} \left(\frac{\partial \rho}{\partial G^{CD}} \frac{\partial G^{CD}}{\partial y_a^A} y_b^A - \rho \delta_b^a \right). \quad (7)$$

and show that

$$- \mathcal{T}_b^a = \sqrt{-g} \left(2 \frac{\partial \rho}{\partial G^{AD}} g^{ad} y_b^A y_a^D - \rho \delta_b^a \right). \quad (8)$$

The right hand side of this expression is identical to the definition of the symmetric energy momentum tensor density (6), see [11]. Therefrom, it follows that the energy momentum-tensor, when decomposed with respect to \mathbf{u} , the velocity of the matter, takes the form

$$T_{ab} = \rho u_a u_b + p h_{ab} + P_{ab}, \quad (9)$$

where $p h_{ab} + P_{ab} = 2 \frac{\partial \rho}{\partial G^{AD}} y_b^A y_a^D - \rho h_{ab}$. Here, all the definitions are the same as the ones given above substituting \mathbf{u} for \mathbf{v} , i.e.: $h_{ab} = g_{ab} + u_a u_b$, $P_{ab} = h_a^m h_b^n (T_{mn} - 3p h_{mn})$, $\rho = T_{ab} u^a u^b$, $p = \frac{1}{3} h^{ab} T_{ab}$ and they satisfy $h_{ab} u^b = 0$, $P_{ab} u^b = g^{ab} P_{ab} = 0$; thus in particular one gets $q^a = 0$ and the resulting tensor is of the diagonal Segre type $\{1, 111\}$ or any of its degeneracies, \mathbf{u} being its (unit) timelike eigenvector (see [12]).

This means that an orthonormal tetrad exists $\{u_a, x_a, y_a, z_a\}$ (with $u_a u^a = -1$, $x^a x_a = y^a y_a = z^a z_a = +1$ and the mixed products zero) with respect to which T_{ab} may be written as

$$T_{ab} = \rho u_a u_b + p_1 x_a x_b + p_2 y_a y_b + p_3 z_a z_b, \quad p = \frac{1}{3} (p_1 + p_2 + p_3), \\ h_{ab} = x_a x_b + y_a y_b + z_a z_b, \text{ etc.} \quad (10)$$

It is interesting to mention that the dominant energy condition (DEC), see for instance [12], is fulfilled if and only if

$$\rho \geq 0, \quad |p_A| \leq \rho, \quad A = 1, 2, 3. \quad (11)$$

3 On symmetries and their consequences on physics

Most of the results in this section are well-known although disperse in the literature, we collect them here as a brief summary for the sake of completeness.

Let (M, g) admit a killing vector (KV) ξ , i.e.: $\mathcal{L}_\xi g_{ab} = 0$. It is then immediate to show that $\mathcal{L}_\xi R_{ab} = \mathcal{L}_\xi G_{ab} = \mathcal{L}_\xi R^a{}_{bcd} = 0$, etc. and, from EFEs it then follows that $\mathcal{L}_\xi T_{ab} = 0$ where T denotes the energy-momentum tensor describing the material in the spacetime.

Further, if \mathbf{v} is a non-degenerate unit (i.e.: $v^a v_a = \varepsilon$ with $\varepsilon = \pm 1$) eigenvector of T_{ab} with corresponding eigenvalue λ , then

$$\mathcal{L}_\xi \lambda = \mathcal{L}_\xi v^a = 0. \quad (12)$$

Under the hypothesis that ξ is a Killing vector, and on account of the above considerations the following conditions hold in the case that T_{ab} represents elastic matter and is therefore of the form (9)

$$\mathcal{L}_\xi g_{ab} = 0 \Rightarrow \mathcal{L}_\xi \rho = 0, \quad \mathcal{L}_\xi u_a = 0, \quad \mathcal{L}_\xi h_{ab} = 0, \quad \mathcal{L}_\xi P_{ab} = 0, \quad \mathcal{L}_\xi p = 0.$$

Thus, matter 4-velocity, pressure, density and anisotropic pressure tensor all stay invariant under the isometries of the space-time, together with the projection tensor³ $h_{ab} = g_{ab} + u_a u_b$.

In the case of a spherically symmetric spacetime, coordinates $x^a = t, r, \theta, \phi$ exist (and are non-unique) such that the line element can be written as

$$ds^2 = -a(r, t) dt^2 + b(r, t) dr^2 + r^2 d\theta^2 + r^2 \sin^2 \theta d\phi^2 \quad (13)$$

with a and b positive and independent of θ and ϕ . This metric possesses three Killing vectors, namely $\xi_1 = -\cos \phi \partial_\theta + \cot \theta \sin \phi \partial_\phi$, $\xi_2 = \partial_\phi$ and $\xi_3 = -\sin \phi \partial_\theta - \cot \theta \cos \phi \partial_\phi$ which generate the 3-dimensional Lie algebra $so(3)$.

Any timelike vector field \mathbf{v} such that $\mathcal{L}_{\xi_A} \mathbf{v} = 0$ is necessarily hypersurface orthogonal and of the form

$$\mathbf{v} = v^t(t, r) \partial_t + v^r(t, r) \partial_r,$$

further, a coordinate transformation taking t, r into t', r' exists such that $\mathbf{v} = v'^t(t', r') \partial_{t'}$ (i.e.: comoving coordinates) and the metric (13) reads then

$$ds^2 = -a(r', t') dt'^2 + b(r', t') dr'^2 + Y^2(r', t') (d\theta^2 + \sin^2 \theta d\phi^2) \quad (14)$$

Now, given any symmetric second order tensor P_{ab} , that is traceless: $g^{ab} P_{ab} = 0$, spherically symmetric: $\mathcal{L}_{\xi_A} P_{ab} = 0$, and orthogonal to a vector such as \mathbf{v} above: $P_{ab} v^b = 0$, it follows that P_{ab} must be proportional to the shear tensor of \mathbf{v} whenever the latter is non-zero, namely:

$$P_{ab} \propto \sigma_{ab}, \quad \sigma_{ab} = v_{(a;b)} + \dot{v}_{(a} v_{b)} - \frac{1}{3} \theta h_{ab} \quad (15)$$

³ For a general energy momentum tensor such as (4), and if one assumes that $\mathcal{L}_\xi u_a = 0$, it also follows that $\mathcal{L}_\xi q_a = 0$; but in this case that assumption has to be made, as \mathbf{u} is no longer an eigenvector of the energy-momentum tensor.

where $\theta = v_{;a}^a$ is the expansion, $v_a = v_{a;b}v^b$ is the acceleration and round brackets denote symmetrization as usual. This can be proven by direct computation using comoving coordinates; also, a more general proof is possible in the context of warped spacetimes (of which the spherically symmetric ones are special instances), see [13].

In the comoving coordinate system above, one can see by direct computation that the shear σ_{ab} is

$$\sigma_{ab} = \text{diag} \left(0, \frac{1}{3} \frac{(b_t Y - 2Y_t b)}{\sqrt{a} Y}, -\frac{1}{6} \frac{Y(b_t Y - 2Y_t b)}{\sqrt{a} Y}, -\frac{1}{6} \frac{Y(b_t Y - 2Y_t b)}{\sqrt{a} Y} \sin^2 \theta \right) \quad (16)$$

and therefore this field is shearfree if and only if

$$b(r, t) = F^2(r) Y^2(r, t),$$

in which case it is always possible, by means of an obvious redefinition of the coordinate r , bring the metric to the form

$$ds^2 = -a(r, t) dt^2 + Y^2(r, t) (dr^2 + d\theta^2 + \sin^2 \theta d\phi^2). \quad (17)$$

For the class of spacetimes we shall be interested in, namely elastic, spherically symmetric, all the above apply for \mathbf{u} (the velocity of matter), as it is indeed the unit timelike eigenvector of T_{ab} given by (9), and P_{ab} the anisotropic pressure tensor since, as discussed previously, it is invariant under the KVs the spacetime possesses and is also traceless and orthogonal to \mathbf{u} , therefore we have that, whenever the shear of \mathbf{u} is non-zero

$$P_{ab} = 2\lambda \sigma_{ab}, \quad \sigma_{ab} = u_{(a;b)} + \dot{u}_{(a} u_{bd)} - \frac{1}{3} \theta h_{ab} \quad (18)$$

where $\theta = u_{;a}^a$ and $\dot{u}_a = u_{a;b} u^b$, and $\lambda = \lambda(t, r)$ is some function, therefore, for the generic (non shearfree) case, it is always possible to treat, at least formally,⁴ the elastic material as a viscous fluid with zero heat flow. This interpretation would indeed break down in the case in which \mathbf{u} is shearfree.

4 Elasticity in spherical symmetry

Let us now consider in more detail the problem of elasticity in a spherically symmetric spacetime (M, \bar{g}) with associated material space $(X, \bar{\gamma})$.

The results given in this section generalize those in [4] in the sense that here we consider a non-flat material metric $\bar{\gamma}$, while, when referring to quantities and results in [4], we shall use non-barred quantities (hence the bars on the spacetime metric and the material metric in our notation).

Recalling the notation and results in Sect. 2, we shall demand that the submersion $\psi : M \rightarrow X$ preserves the KVs, that is: $\psi_*(\xi_A) = \eta_A$ are also KVs on X .

⁴ There is a further requirement for a fully physically meaningful interpretation as a viscous fluid, namely that $\lambda < 0$ in which case the kinematical viscosity would be $\eta = -\lambda > 0$.

This implies that the metric $\bar{\gamma}$ is also spherically symmetric and therefore coordinates $y^A = (y, \bar{\theta}, \bar{\phi})$ exist with $y = y(t, r)$, $\bar{\theta} = \theta$ and $\bar{\phi} = \phi$, and are such that $\eta_A = \xi_A$ are KVs of the metric $\bar{\gamma}$. Thus, the line elements of \bar{g} and $\bar{\gamma}$ may be written as:

$$d\bar{s}^2 = -\bar{a}(t, r)dt^2 + \bar{b}(t, r)dr^2 + r^2d\theta^2 + r^2\sin^2\theta d\phi^2 \quad (19)$$

$$d\bar{\Sigma}^2 = f^2(y)(dy^2 + y^2d\theta^2 + y^2\sin^2\theta d\phi^2), \quad (20)$$

Notice that this last expression is completely general, as any 3-dimensional spherically symmetric metric is necessarily conformally flat, as it is immediate to show.

The results in [4] correspond to $f(y) = 1$, and the relation between $\bar{\gamma}$ and the flat material metric γ used in [4] is given by

$$\bar{\gamma}_{AB} = f^2(y)\gamma_{AB}. \quad (21)$$

Next, attention should be paid to the canonical definition of the energy-momentum tensor used by [4]:

$$T_b^a = \frac{1}{\sqrt{-g}} \left(\frac{\partial \Lambda}{\partial y_a^A} y_b^A - \delta_b^a \Lambda \right). \quad (22)$$

Denoting by \bar{k} the pull-back by ψ of the material metric $\bar{\gamma}$, that is: $\bar{k} = \psi^*(\bar{\gamma})$, one has:

$$\begin{aligned} \bar{k}_b^a &= \bar{g}^{ac} \bar{k}_{cb} = \bar{g}^{ac} \bar{\gamma}_{BC} y_c^C y_b^B = f^2(y) \bar{g}^{ac} \gamma_{CB} y_c^C y_b^B = f^2(y) \bar{g}^{ac} k_{cb} \\ &= f^2(y) \bar{g}^{ac} [\dot{y}^2 \delta_c^0 \delta_b^0 + \dot{y} y' (\delta_c^0 \delta_b^1 + \delta_b^0 \delta_c^1) + y'^2 \delta_b^1 \delta_c^1 + y^2 \delta_b^2 \delta_c^2 + y^2 \sin^2 \theta \delta_b^3 \delta_c^3], \end{aligned}$$

or

$$\bar{k}_b^a = \begin{pmatrix} -f^2(y)(\dot{y}^2/\bar{a}) & -f^2(y)(\dot{y}y'/\bar{a}) & 0 & 0 \\ f^2(y)(\dot{y}y'/\bar{b}) & f^2(y)(y'^2/\bar{b}) & 0 & 0 \\ 0 & 0 & f^2(y)y^2/r^2 & 0 \\ 0 & 0 & 0 & f^2(y)y^2/r^2 \end{pmatrix}, \quad (23)$$

where a dot indicates a derivative with respect to t and a prime a derivative with respect to r .

The velocity field of the matter, defined by the conditions $\bar{u}^a y_a^A = 0$, $\bar{g}_{ab} \bar{u}^a \bar{u}^b = -1$ and $\bar{u}^0 > 0$, can be expressed as

$$\bar{u}^a = \frac{\bar{\Gamma}}{\sqrt{\bar{a}}} \left(1, -\frac{\dot{y}}{y'}, 0, 0 \right), \quad (24)$$

where

$$\bar{\Gamma} \equiv \left(1 - \frac{\bar{b}}{\bar{a}} \left(\frac{\dot{y}}{y'} \right)^2 \right)^{-\frac{1}{2}}. \quad (25)$$

Therefore the projection tensor is

$$\bar{h}_b^a = \delta_b^a + \bar{u}^a \bar{u}_b = \begin{pmatrix} 1 - \bar{\Gamma}^2 & -\bar{\Gamma}^2(\bar{b}\dot{y}/\bar{a}y') & 0 & 0 \\ \bar{\Gamma}^2(\dot{y}/y') & 1 + \bar{\Gamma}^2(\bar{b}/\bar{a})(\dot{y}/y')^2 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}. \quad (26)$$

We will use an orthonormal tetrad and write the metric as $\bar{g}_{ab} = -\bar{u}_a\bar{u}_b + \bar{x}_a\bar{x}_b + \bar{y}_a\bar{y}_b + \bar{z}_a\bar{z}_b$, such that:

$$\begin{aligned}\bar{u}^a &= \left(\frac{\bar{\Gamma}}{\sqrt{\bar{a}}}, -\frac{\dot{y}}{y'} \frac{\bar{\Gamma}}{\sqrt{\bar{a}}}, 0, 0 \right) & \bar{u}_a &= \left(-\sqrt{\bar{a}}\bar{\Gamma}, -\frac{\bar{b}}{\sqrt{\bar{a}}} \frac{\dot{y}}{y'} \bar{\Gamma}, 0, 0 \right) \\ \bar{x}^a &= \left(-\frac{\sqrt{\bar{b}}}{\bar{a}} \frac{\dot{y}}{y'} \bar{\Gamma}, \frac{\bar{\Gamma}}{\sqrt{\bar{b}}}, 0, 0 \right) & \bar{x}_a &= \left(\frac{\dot{y}}{y'} \sqrt{\bar{b}}\bar{\Gamma}, \sqrt{\bar{b}}\bar{\Gamma}, 0, 0 \right) \\ \bar{y}^a &= \left(0, 0, \frac{1}{r}, 0 \right) & \bar{y}_a &= (0, 0, r, 0) \\ \bar{z}^a &= \left(0, 0, 0, \frac{1}{r \sin \theta} \right) & \bar{z}_a &= (0, 0, 0, r \sin \theta),\end{aligned}$$

where $\bar{\Gamma}$ is the auxiliary quantity given in (25). Here, \bar{u}^a is the matter velocity and \bar{x}^a, \bar{y}^a and \bar{z}^a are spacelike eigenvectors of the pulled-back material metric \bar{k}_b^a . From our developments in Sect. 2, it is immediate to see that the pressure tensor has the same eigenvectors as \bar{k}_{ab} and can be written, for the space-time under consideration as $\bar{P}_{ab} = \bar{p}_1 \bar{x}_a \bar{x}_b + \bar{p}_2 (\bar{y}_a \bar{y}_b + \bar{z}_a \bar{z}_b)$. Therefore, (10) yields

$$\bar{T}_{ab} = \bar{\rho} \bar{u}_a \bar{u}_b + \bar{p}_1 \bar{x}_a \bar{x}_b + \bar{p}_2 (\bar{y}_a \bar{y}_b + \bar{z}_a \bar{z}_b), \quad (27)$$

where $\bar{\rho}$ is the energy density, \bar{p}_1 , the radial pressure and \bar{p}_2 , the tangential pressure.

The results in [4] can be easily recovered by setting $f(y) = 1$ above.

Now, much clarity is gained by making use of the comoving coordinates adapted to \mathbf{u} , the timelike eigenvector of the energy-momentum tensor, which were introduced in the above section. The form of the metric is given by

$$ds^2 = -\bar{a}(r,t) dt^2 + \bar{b}(r,t) dr^2 + \bar{Y}^2(r,t) (d\theta^2 + \sin^2 \theta d\phi^2), \quad (28)$$

\mathbf{u} , being then

$$u^a = \left(\frac{1}{\sqrt{\bar{a}}}, 0, 0, 0 \right), \quad u_a = \left(-\sqrt{\bar{a}}, 0, 0, 0 \right), \quad (29)$$

hence, we have for the material space $(M, \bar{\gamma})$ that coordinates $y^A = (y, \tilde{\theta}, \tilde{\phi})$ exist with $y = y(r)$, $\tilde{\theta} = \theta$ and $\tilde{\phi} = \phi$, as follows from the condition $y_a^A u^a = 0$ and the requirement that $\psi_*(\xi_A) = \eta_A$ are KVs of the metric $\bar{\gamma}$.

Further, and since the line element of the material space is

$$d\bar{\sigma}^2 = f^2(y) [dy^2 + y^2 (d\theta^2 + \sin^2 \theta d\phi^2)],$$

with $y = y(r)$, no generality is lost by setting $y = r$, as this amounts to a redefinition of the r coordinate in spacetime, and leaves unchanged the form of the metric (28) as well as that of the velocity field of the matter (29). We shall do that in the sequel.

Thus, the pulled-back material metric \bar{k} (23) is

$$\begin{aligned}\bar{k}_b^a &= \bar{g}^{ac} \bar{k}_{cb} = \bar{g}^{ac} \bar{\gamma}_{CB} y_c^C y_b^B = f^2(y) \bar{g}^{ac} \gamma_{CB} y_c^C y_b^B = f^2(y) \bar{g}^{ac} k_{cb} \\ &= f^2(y) \bar{g}^{ac} [y'^2 \delta_b^1 \delta_c^1 + y^2 \delta_b^2 \delta_c^2 + y^2 \sin^2 \theta \delta_b^3 \delta_c^3],\end{aligned}$$

where a prime indicates a derivative with respect to r , which upon setting $y = r$ as discussed above it simplifies further to:

$$\bar{k}_b^a = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & f^2(r)(1/\bar{b}) & 0 & 0 \\ 0 & 0 & f^2(r)r^2/\bar{Y}^2 & 0 \\ 0 & 0 & 0 & f^2(r)r^2/\bar{Y}^2 \end{pmatrix}. \quad (30)$$

The operator $\bar{K}_b^a = \bar{g}^{ac}\bar{k}_{cb} - \bar{u}^a\bar{u}_b$, introduced in Sect. 2 and used to measure the state of strain of the material has one eigenvalue equal to 1 (corresponding to the eigenvector \mathbf{u}), while the other eigenvalues are

$$\begin{aligned} \bar{s} &= f^2(y) \frac{y^2}{\bar{Y}^2} = f^2(r) \frac{r^2}{\bar{Y}^2} \\ \bar{\eta} &= f^2(y) \frac{y^2}{\bar{b}} = \frac{f^2(r)}{\bar{b}}, \end{aligned} \quad (31)$$

and \bar{s} has algebraic multiplicity two.

The three invariants I_1, I_2, I_3 of \bar{K} introduced in (2) have the following expressions

$$\begin{aligned} \bar{I}_1 &= \frac{1}{2} (\text{Tr}\bar{K} - 4) = \frac{1}{2} (\bar{\eta} + 2\bar{s} - 3) \\ \bar{I}_2 &= \frac{1}{4} [\text{Tr}\bar{K}^2 - (\text{Tr}\bar{K})^2] + 3 = -\frac{1}{2} (\bar{s}^2 + 2\bar{\eta}\bar{s} + \bar{\eta} + 2\bar{s}) - 3 \\ \bar{I}_3 &= \frac{1}{2} (\det\bar{K} - 1) = \frac{1}{2} (\bar{\eta}\bar{s}^2 - 1) \end{aligned} \quad (32)$$

In [4], the energy-momentum tensor was calculated from these invariants for a flat material metric. A similar calculation shows that, for the non-flat material metrics under consideration, the same expression holds so that

$$\bar{T}_b^a = \bar{\rho} \delta_b^a - \frac{\partial \bar{\rho}}{\partial \bar{I}_3} \det\bar{K} \bar{h}_b^a + \left(\text{Tr}\bar{K} \frac{\partial \bar{\rho}}{\partial \bar{I}_2} - \frac{\partial \bar{\rho}}{\partial \bar{I}_1} \right) \bar{k}_b^a - \frac{\partial \bar{\rho}}{\partial \bar{I}_2} \bar{k}_c^a \bar{k}_b^c. \quad (33)$$

Therefore, the non-zero components are

$$\begin{aligned} \bar{T}_0^0 &= \bar{\rho}, \\ \bar{T}_1^1 &= \bar{\rho} - \frac{y^2}{\bar{b}} \bar{\Sigma}, \\ \bar{T}_2^2 = \bar{T}_3^3 &= \bar{\rho} - \frac{y^2}{\bar{Y}^2} \left[\bar{\Sigma} + \left(\frac{\partial \bar{\rho}}{\partial \bar{I}_2} - f^2(y) \frac{y^2}{\bar{Y}^2} \frac{\partial \bar{\rho}}{\partial \bar{I}_3} \right) \left(f^4(y) \frac{y^2}{\bar{Y}^2} - f^4(y) \frac{y^2}{\bar{b}} \right) \right], \end{aligned} \quad (34)$$

where

$$\bar{\Sigma} = f^2(y) \left[\frac{\partial \bar{\rho}}{\partial \bar{I}_1} - \frac{\partial \bar{\rho}}{\partial \bar{I}_2} \left(1 + 2f^2(y) \frac{y^2}{\bar{Y}^2} \right) + \frac{\partial \bar{\rho}}{\partial \bar{I}_3} f^4(y) \frac{y^4}{\bar{Y}^4} \right]. \quad (35)$$

The rest frame energy per unit volume,⁵ $\bar{\rho}$, is defined by

$$\bar{\rho} = \bar{\varepsilon} \bar{v} = \varepsilon_0 \bar{s} \sqrt{\bar{\eta}} \bar{v}(\bar{s}, \bar{\eta}), \quad (36)$$

where, as discussed in Sect. 2, $\bar{v} = \bar{v}(\bar{I}_1, \bar{I}_2, \bar{I}_3) = \bar{v}(\bar{s}, \bar{\eta})$ represents the constitutive equation, ε_0 , the density of the relaxed material (density w.r.t. the pulled-back material metric \bar{k}) and

$$\bar{\varepsilon} = \varepsilon_0 \sqrt{\det \bar{K}} = \varepsilon_0 \bar{s} \sqrt{\bar{\eta}}, \quad (37)$$

the density calculated in the rest frame (that is, w.r.t. h).

Then, using (32), one can prove the following relations:

$$\frac{\partial \bar{\rho}}{\partial \bar{\eta}} = \frac{1}{2f^2} \bar{\Sigma}, \quad (38)$$

$$\frac{\partial \bar{\rho}}{\partial \bar{s}} = \frac{1}{f^2} \bar{\Sigma} + \left(f^2 \frac{\partial \bar{\rho}}{\partial \bar{I}_2} - f^4 \frac{y^2}{\bar{Y}^2} \frac{\partial \bar{\rho}}{\partial \bar{I}_3} \right) \left(\frac{y^2}{\bar{Y}^2} - \frac{y'^2}{\bar{b}} \right). \quad (39)$$

Alternatively, one can express the components of the energy-momentum tensor in terms of the eigenvalues \bar{s} and $\bar{\eta}$ by substituting the last results in (34):

$$\begin{aligned} \bar{T}_0^0 &= \bar{\varepsilon} \bar{v}, \\ \bar{T}_1^1 &= -\bar{\varepsilon} 2 \bar{\eta} \frac{\partial \bar{v}}{\partial \bar{\eta}}, \\ \bar{T}_2^2 &= -\bar{\varepsilon} \bar{s} \frac{\partial \bar{v}}{\partial \bar{s}}. \end{aligned} \quad (40)$$

The Einstein field equations $\bar{G}_b^a = 8\pi \bar{T}_b^a$ can be written as follows:

$$\bar{G}_0^0 = 8\pi \bar{T}_0^0:$$

$$-\frac{\dot{\bar{Y}}}{\bar{Y}^2 \bar{a}} - \frac{\dot{\bar{Y}}}{\bar{Y}} \frac{\dot{\bar{b}}}{\bar{a} \bar{b}} + \frac{2\bar{Y}''}{\bar{Y} \bar{b}} + \frac{\bar{Y}'^2}{\bar{Y}^2 \bar{b}} - \frac{\bar{Y}' \bar{b}'}{\bar{Y} \bar{b}^2} - \frac{1}{\bar{Y}^2} = \bar{\varepsilon} \bar{v} 8\pi \quad (41)$$

$$\bar{G}_0^1 = 8\pi \bar{T}_0^1:$$

$$2\dot{\bar{Y}}' - \frac{\bar{a}'}{\bar{a}} \dot{\bar{Y}} - \frac{\dot{\bar{b}}}{\bar{b}} \bar{Y}' = 0, \quad (42)$$

$$\bar{G}_1^1 = 8\pi \bar{T}_1^1:$$

$$-\frac{\dot{\bar{Y}}^2}{\bar{Y}^2 \bar{a}} + \frac{\dot{\bar{Y}}}{\bar{Y}} \frac{\dot{\bar{a}}}{\bar{a}^2} + \frac{\bar{Y}' \bar{a}'}{\bar{Y} \bar{a} \bar{b}} + \frac{\bar{Y}'^2}{\bar{Y}^2 \bar{b}} - \frac{2\ddot{\bar{Y}}}{\bar{Y} \bar{a}} - \frac{1}{\bar{Y}^2} = -\bar{\varepsilon} 2 \bar{\eta} \frac{\partial \bar{v}}{\partial \bar{\eta}} 8\pi, \quad (43)$$

$$\bar{G}_2^2 = 8\pi \bar{T}_2^2:$$

$$\begin{aligned} &\frac{1}{2} \frac{\dot{\bar{Y}} \dot{\bar{a}}}{\bar{Y} \bar{a}^2} - \frac{1}{2} \frac{\dot{\bar{Y}} \dot{\bar{b}}}{\bar{Y} \bar{a} \bar{b}} - \frac{1}{4} \frac{\bar{a}''}{\bar{a}^2 \bar{b}} + \frac{1}{2} \frac{\bar{Y}' \bar{a}'}{\bar{Y} \bar{a} \bar{b}} - \frac{1}{4} \frac{\bar{a}' \bar{b}'}{\bar{a} \bar{b}^2} + \frac{\bar{Y}''}{\bar{Y} \bar{b}} - \frac{1}{2} \frac{\bar{Y}' \bar{b}'}{\bar{Y} \bar{b}^2} \\ &+ \frac{1}{2} \frac{\bar{a}''}{\bar{a} \bar{b}} - \frac{1}{2} \frac{\ddot{\bar{b}}}{\bar{a} \bar{b}} + \frac{1}{4} \frac{\dot{\bar{a}} \dot{\bar{b}}}{\bar{a}^2 \bar{b}} + \frac{1}{4} \frac{\dot{\bar{b}}^2}{\bar{a} \bar{b}^2} - \frac{\ddot{\bar{Y}}}{\bar{Y} \bar{a}} = -\bar{\varepsilon} \bar{s} \frac{\partial \bar{v}}{\partial \bar{s}} 8\pi. \end{aligned} \quad (44)$$

⁵ In [4] the quantities $\bar{\rho}$ and $\bar{\varepsilon}$ are ε and ρ , respectively.

It is interesting to express the contracted Bianchi identities for \bar{T}_b^a in terms of \bar{v} and its derivatives w.r.t the quantities $\bar{\eta}$ and \bar{s} . Thus, from $\bar{T}_{b;a}^a = 0$ one has:

$$\bar{T}_{b,a}^a + \partial_n(\ln \sqrt{-\bar{g}})\bar{T}_b^n - \bar{\Gamma}_{ba}^n \bar{T}_n^a = 0 \quad (45)$$

and specifying this equation to $b = 0, 1$ one gets respectively (for non-stationary solutions):

$$\partial_t(\bar{\varepsilon}\bar{v}) + \frac{\dot{\bar{b}}}{\bar{b}} \left(\frac{1}{2} \bar{\varepsilon}\bar{v} + \bar{\varepsilon}\bar{\eta} \frac{\partial \bar{v}}{\partial \bar{\eta}} \right) + \frac{\dot{\bar{Y}}}{\bar{Y}} (2\varepsilon\bar{v} + 2\bar{\varepsilon}\bar{s} \frac{\partial \bar{v}}{\partial \bar{s}}) = 0, \quad (46)$$

$$-2 \left(\bar{\varepsilon}\bar{\eta} \frac{\partial \bar{v}}{\partial \bar{\eta}} \right)_{,r} - \frac{1}{2} \bar{\varepsilon}\bar{v} \frac{\bar{a}'}{\bar{a}} - \bar{\varepsilon}\bar{\eta} \frac{\partial \bar{v}}{\partial \bar{\eta}} \left(\frac{\bar{a}'}{\bar{a}} + 4 \frac{\bar{Y}'}{\bar{Y}} \right) + 2\bar{\varepsilon}\bar{s} \frac{\partial \bar{v}}{\partial \bar{s}} \frac{\bar{Y}'}{\bar{Y}} = 0. \quad (47)$$

The remaining equations for $b = 2, 3$ which can be obtained from (45) are identically satisfied.

Equation (46) for non-stationary solutions, implies readily

$$\bar{\varepsilon} = \frac{1}{\sqrt{\bar{b}}\bar{Y}^2} \varepsilon_0(r), \quad (48)$$

which can then be substituted into (47) to get a slightly simplified equation.

From this point onwards, we shall drop the bars, as no confusion may arise with the results in [4].

5 Shear-free solutions: examples

In this section we shall consider in detail the case of spacetimes with a material content that may be represented by some elastic material such that the velocity of the matter is shearfree, in which case coordinates exist such that the metric can be written in the form (17). For this case, the interpretation as a viscous fluid with kinematical viscosity is not possible, and therefore the anisotropy in the pressures must be a consequence of the elastic properties of the material. The study of solutions with non-vanishing shear tensor and their possible interpretations as viscous fluids, will be carried out elsewhere as this would render the present paper too lengthy.

We will study separately the cases of static and non-static solutions, presenting examples of each instance which are regular at the origin, possess a constitutive equation and satisfy the dominant energy condition (at least in some open submanifold of the spacetime).

Consider the metric (17) which we rewrite here for convenience:

$$ds^2 = -a(r,t)dt^2 + Y^2(r,t) (dr^2 + d\theta^2 + \sin^2 \theta d\phi^2) \quad (49)$$

From the field equations it follows that $G_r^r = 0$ which in turn implies that

$$a = L(t) \frac{\dot{Y}}{Y} \quad (50)$$

whenever $\dot{Y} \neq 0$, $L(t)$ being a function of time.

If $\dot{Y} = 0$ then $G_r^r = 0$ is identically satisfied, and from (31) follows that s and η , and therefore $v(\eta, s)$ are functions of r alone, then (46) implies that $\varepsilon = \varepsilon(r)$; further from the field equation $G_r^r = -8\pi\varepsilon 2\eta \frac{\partial v}{\partial \eta}$ it follows that G_r^r can only depend on r as well, which in turn implies that $a(t, r) = a_0(t)a_1(r)$, the solution being then static, as a trivial redefinition of the coordinate t coordinate shows.

It is interesting now to see that in the shear-free case, if one sets either η or s equal to 1, so that matter is strained in tangential directions (but not in the radial direction $\eta = 1$), or it is strained only in the radial direction ($s = 1$), from the definition of these quantities it follows that $Y = Y(r)$, and according to the statements in the above paragraph, it follows that the solution must be static, and therefore the results in [5] apply. Thus, we have proven that: *if the velocity field of the matter is shear-free and the matter is stressed either in the radial direction only or in the tangential directions only, the spacetime is necessarily static.*

5.1 Static shear-free solutions

In the static, shear-free case (metric (49) with no dependence on t), the field equations yield

$$\varepsilon v 8\pi = 2 \frac{Y''}{Y^3} - \frac{Y'^2}{Y^4} - \frac{1}{Y^2}, \quad (51)$$

$$-2\varepsilon\eta \frac{\partial v}{\partial \eta} 8\pi = \frac{a' Y'}{a Y^3} + \frac{Y'^2}{Y^4} - \frac{1}{Y^2}, \quad (52)$$

$$-\varepsilon s \frac{\partial v}{\partial s} 8\pi = \frac{Y''}{Y^3} + \frac{1}{2} \frac{a''}{a Y^2} - \frac{1}{4} \frac{a'^2}{a^2 Y^2} - \frac{Y'^2}{Y^4}, \quad (53)$$

solving (51) for ε and substituting it in (52) and (53) one gets two equations which depend only on r and elementary considerations show that for given $a(r)$ and $Y(r)$, functions $y(r)$, $f(y)$ and v can be found so that the two equations are satisfied. It remains to be seen, though, that the DEC are satisfied and therefore the solution is physically acceptable.

The following simple example shows that solutions with these characteristics do indeed exist.

Example 1 Consider the line element

$$ds^2 = -Y^{-2}(r)dt^2 + Y^2(r) (dr^2 + d\theta^2 + \sin^2\theta d\phi^2). \quad (54)$$

A direct calculation yields

$$8\pi\varepsilon v = 2 \frac{Y''}{Y^3} - \frac{Y'^2}{Y^4} - \frac{1}{Y^2}, \quad -16\pi\varepsilon\eta \frac{\partial v}{\partial \eta} = -\frac{Y'^2}{Y^4} - \frac{1}{Y^2}, \quad -8\pi\varepsilon s \frac{\partial v}{\partial s} = \frac{Y'^2}{Y^4}. \quad (55)$$

Fig. 1 The density, radial pressure and tangential pressure

The dominant energy condition (11) implies:

$$8\pi\rho = 2\frac{Y''}{Y^3} - \frac{Y'^2}{Y^4} - \frac{1}{Y^2} \geq 0, \quad (56)$$

$$8\pi(\rho - p_1) = 2\frac{Y''}{Y^3} \geq 0, \quad (57)$$

$$8\pi(\rho + p_1) = 2\left(\frac{Y''}{Y^3} - \frac{Y'^2}{Y^4} - \frac{1}{Y^2}\right) \geq 0 \quad (58)$$

$$8\pi(\rho - p_2) = 2\frac{Y''}{Y^3} - 2\frac{Y'^2}{Y^4} - \frac{1}{Y^2} \geq 0, \quad (59)$$

$$8\pi(\rho + p_2) = 2\frac{Y''}{Y^3} - \frac{1}{Y^2} \geq 0, \quad (60)$$

where we put $\rho = \varepsilon v$, $p_1 = -2\varepsilon\eta \frac{\partial v}{\partial \eta}$ and $p_2 = -\varepsilon s \frac{\partial v}{\partial s}$.

Now, it is immediate to see that the above conditions are all satisfied if and only if (58) is, which in turn can be written as

$$\frac{1}{Y^2} \left(\frac{Y''}{Y} - \frac{Y'^2}{Y^2} - 1 \right) \geq 0 \Leftrightarrow (\ln Y)'' - 1 \geq 0, \quad (61)$$

which is equivalent to

$$Y = \exp(r^2/2) f^2(r) \quad \text{such that} \quad (\ln f)'' \geq 0. \quad (62)$$

Take, for instance,

$$Y = e^{\frac{5}{2}r^2}, \quad (63)$$

one then has

$$\begin{aligned} \rho = \varepsilon v &= \frac{1}{8\pi} e^{-5r^2} (25r^2 + 9), \\ p_1 = -2\varepsilon\eta \frac{\partial v}{\partial \eta} &= -\frac{1}{8\pi} e^{-5r^2} (25r^2 + 1), \quad p_2 = -\varepsilon s \frac{\partial v}{\partial s} = \frac{1}{8\pi} 25r^2 e^{-5r^2} \end{aligned} \quad (64)$$

which is obviously well behaved: satisfies the dominant energy condition and is non-singular at the origin. Notice that the radial pressure is negative (compressed material) and the tangential pressures are zero at the center, as one would expect. See Fig. 1 for details.

The field equations in this case read:

$$\varepsilon v = \frac{1}{8\pi} e^{-5r^2} (25r^2 + 9) \quad (65)$$

$$-2\varepsilon\eta \frac{\partial v}{\partial \eta} = -\frac{1}{8\pi} e^{-5r^2} (25r^2 + 1) \quad (66)$$

$$-\varepsilon s \frac{\partial v}{\partial s} = \frac{1}{8\pi} e^{-5r^2} 25r^2 \quad (67)$$

and one has that

$$\eta = f^2(r)e^{-5r^2}, \quad s = r^2 f^2(r)e^{-5r^2}. \quad (68)$$

Now, dividing (66) and (67) through by (65), and setting $E \equiv \ln \eta$, $\Sigma \equiv \ln s$, one gets

$$\frac{\partial \ln v}{\partial E} = \frac{1}{2} \frac{25r^2 + 1}{25r^2 + 9}, \quad \frac{\partial \ln v}{\partial \Sigma} = -\frac{25r^2}{25r^2 + 9}. \quad (69)$$

From the expressions for η and s one has that

$$E = 2 \ln f(r) - 5r^2, \quad \Sigma = E + 2 \ln r, \quad (70)$$

hence one can express r as a function of E , and Σ as a function of E as well, thus

$$\frac{\partial \ln v}{\partial E} = \frac{\partial \Sigma}{\partial E} \frac{\partial \ln v}{\partial \Sigma}$$

from where it follows that

$$\frac{\partial r}{\partial E} = -\frac{3}{4}r - \frac{1}{100r}, \quad \text{or else} \quad E = -\frac{2}{3} \ln(75r^2 + 1),$$

that is

$$r = \sqrt{\frac{1}{75} \left(e^{-\frac{3}{2}E} - 1 \right)}. \quad (71)$$

Plugging the expression of E in terms of r into (70) one gets that

$$f(r) = \frac{e^{\frac{5}{2}r^2}}{(75r^2 + 1)^{\frac{1}{3}}}, \quad (72)$$

whence expressions for η , s and $\varepsilon = \varepsilon_0 s \sqrt{\eta}$ can be easily derived.

Next, from (71) and the first equation in (69), one can easily find an expression for the equation of state, namely:

$$v = F(\Sigma) \left(\frac{e^{\frac{3}{2}E}}{\left(e^{-\frac{3}{2}E} + 26 \right)^{12}} \right)^{\frac{1}{39}}, \quad (73)$$

where $F(\Sigma)$ must satisfy

$$\frac{\partial \ln F}{\partial \Sigma} = -\frac{25r^2}{25r^2 + 9},$$

where r in the right hand side of the equation has to be expressed in terms of Σ . From the second equation in (70) it follows that r must be the only real solution of

$$r^3 - 75e^{\frac{3}{2}\Sigma} r^2 - e^{\frac{3}{2}\Sigma} = 0,$$

which has a rather complicated form. In any case, one gets

$$F(\Sigma(r)) = (25r^2 + 9)^{-\frac{12}{39}} (75r^2 + 1)^{-\frac{1}{39}},$$

and thus we have proven that a solution exists, which is regular at the origin $r = 0$, satisfies the dominant energy condition and possesses a constitutive equation which can be given in a closed form.

5.1.1 A toy star model

We next use the solution found above to build up a toy model for a star. Ideally, one would like to match the solution obtained to Schwarzschild vacuum solution. A detailed analysis of the matching (or junction) conditions, i.e.: continuity of the first and second fundamental forms on a certain matching surface Σ , shows that this is not possible over any spherically symmetric surface (not necessarily static), which is consistent with the physical intuition which seems to indicate that, for this to be possible, the radial pressure p_1 should vanish on the matching surface. The reader is referred to [14] for a thorough and rigorous study of the matching of two spherically symmetric spacetimes.

The idea is to build up a ‘two-layer’ model in which the central core consists of elastic matter corresponding to the spacetime metric (54) with Y given by (63), i.e.

$$ds_1^2 = -e^{-5r^2} dt^2 + e^{5r^2} (dr^2 + d\theta^2 + \sin^2 \theta d\phi^2) \quad (74)$$

match it to the Schwarzschild interior solution across a static surface $r = r_0$, and match this to the Schwarzschild vacuum solution. This double matching turns out to be possible as we shall show subsequently. While we do not claim at all that the model thus obtained is of any particular physical significance, it does illustrate the possibility of obtaining, in a relatively simple way, a star model consisting of an elastic core surrounded by a ‘crust’ of some other material behaving as a perfect fluid of constant density.

As it is well known [12] the interior Schwarzschild solution may be written as

$$ds_2^2 = - \left(\frac{3}{2} \sqrt{1 - \frac{2M}{R_1^3}} - \frac{1}{2} \sqrt{1 - \frac{2M}{R_1^3} R^2} \right)^2 dt^2 + \frac{dR^2}{\left(1 - \frac{2M}{R_1^3} R^2\right)} + R^2 (d\theta^2 + \sin^2 \theta d\phi^2) \quad (75)$$

where M is the corresponding Schwarzschild mass of the vacuum solution it is matched to, and R_1 the radius of the matching surface in curvature (Schwarzschild) coordinates t, R, θ, ϕ . Note that we use the same names for the angular and time coordinates for all the solutions involved here (elastic, Schwarzschild interior and Schwarzschild) since they can be associated to the Killing vectors the three spacetimes have in common. The radial coordinate though is not the same and we thus

distinguish it by using R . The metric above satisfies the EFEs for a perfect fluid source with comoving velocity and density and pressure given by

$$8\pi\rho_2 = 6\frac{M}{R_1^3}, \quad 8\pi p = 2M \frac{3\sqrt{1 - \frac{2M}{R_1^3}R^2} - \sqrt{1 - \frac{2M}{R_1^3}}}{R_1^3 \left(3\sqrt{1 - \frac{2M}{R_1^3}} - \sqrt{1 - \frac{2M}{R_1^3}R^2}\right)} \quad (76)$$

Let us next show that the spacetimes with metrics given by (75) and (74) can be matched across a spherically symmetric static surface Σ . Henceforth we shall designate geometric quantities (metric, fundamental forms, normals, etc.) corresponding to the above spacetimes by means of the subindices 1 and 2, respectively, thus $\Sigma = \{r = r_0\}$ and $\Sigma = \{R = R_0\}$ and its unit normal will read $\mathbf{n}_1 = \exp(-\frac{5}{2}r^2)\partial_r$ and $\mathbf{n}_2 = (1 - \frac{2M}{R_1^3}R^2)^{\frac{1}{2}}\partial_R$. The tangent space to Σ is then spanned, in an obvious notation, by

$$T_1\Sigma = \text{span} \left\{ \mathbf{u}_1 = e^{\frac{5}{2}r^2} \partial_t, \partial_\theta, \partial_\phi \right\},$$

$$T_2\Sigma = \text{span} \left\{ \mathbf{u}_2 = \left(\frac{3}{2} \sqrt{1 - \frac{2M}{R_1^3}} - \frac{1}{2} \sqrt{1 - \frac{2M}{R_1^3}R^2} \right)^{-1} \partial_t, \partial_\theta, \partial_\phi \right\}$$

Now, a ready calculation of the first and second fundamental forms on Σ yields for the non-vanishing components (see for instance the appendix in [15]):

$$d\sigma_1^2 = -e^{-5r_0^2} dt^2 + e^{5r_0^2} (d\theta^2 + \sin^2\theta d\phi^2) \quad (77)$$

$$d\sigma_2^2 = - \left(\frac{3}{2} \sqrt{1 - \frac{2M}{R_1^3}} - \frac{1}{2} \sqrt{1 - \frac{2M}{R_1^3}R_0^2} \right)^2 dt^2 + R_0^2 (d\theta^2 + \sin^2\theta d\phi^2) \quad (78)$$

$$K_{1tt} = 5r_0 e^{-\frac{15}{2}r_0^2}, \quad K_{1\theta\theta} = 5r_0 e^{\frac{5}{2}r_0^2} \quad (79)$$

$$K_{2tt} = \frac{M}{R_1^3} R_0, \quad K_{2\theta\theta} = \frac{R_0}{\frac{3}{2} \sqrt{1 - \frac{2M}{R_1^3}} - \frac{1}{2} \sqrt{1 - \frac{2M}{R_1^3}R_0^2}} \left(1 - \frac{2M}{R_1^3} R_0^2 \right) \quad (80)$$

and equating componentwise one gets:

$$r_0^2 = \ln R_0^{\frac{2}{3}}, \quad R_1^3 = MR_0^2(R_0 + 2) \quad \text{and} \quad R_0^4 - 10(R_0 + 2)^2 \ln R_0 = 0 \quad (81)$$

where the last equation can be solved numerically giving a value $R_0 \simeq 1.01161$.

5.2 Non-static shear-free solutions

Assume now that $\dot{Y} \neq 0$, so that $a(t, r)$ takes the form (50), substituting this into (49) and redefining the coordinate t so as to absorb the arbitrary function $L(t)$ one has

$$ds^2 = -\frac{\dot{Y}}{Y} dt^2 + Y^2(r, t) (dr^2 + d\theta^2 + \sin^2\theta d\phi^2) \quad (82)$$

From $G^t_r = 0$ it follows now that $Y(r, t) = A(t)B(r)$, which substituted above, yields, after a trivial redefinition of the coordinate t :

$$ds^2 = -dt^2 + A^2(t)B^2(r) (dr^2 + d\theta^2 + \sin^2\theta d\phi^2) \quad (83)$$

A direct computation of the EFEs for the above metric gives

$$8\pi\varepsilon v = -\frac{B'^2 - 2B''B + 3\dot{A}^2B^4 + B^2}{A^2B^4}, \quad (84)$$

$$-16\pi\varepsilon\eta \frac{\partial v}{\partial \eta} = -\frac{2AB^4\ddot{A} - B'^2 + \dot{A}^2B^4 + B^2}{A^2B^4}, \quad (85)$$

$$-8\pi\varepsilon s \frac{\partial v}{\partial s} = -\frac{2AB^4\ddot{A} - B''B + \dot{A}^2B^4 + B'^2}{A^2B^4}. \quad (86)$$

where the energy density, radial and tangential pressures are

$$\rho = \varepsilon v, \quad p_1 = -2\varepsilon\eta \frac{\partial v}{\partial \eta}, \quad p_2 = -\varepsilon s \frac{\partial v}{\partial s}.$$

At this point, it is worth mentioning that general (i.e. not restricted by the shear-free condition) non-static solutions with vanishing radial pressure have been studied in detail by Magli [16] using Ori's mass-area coordinates, although in a rather different context (the cosmic censorship conjecture), solving the problem up to a quadrature.

In our case (i.e.: non-static and shear-free solutions), vanishing radial pressure implies, from (85) above, that

$$2A\ddot{A} - \frac{B'^2}{B^4} + \dot{A}^2 + \frac{1}{B^2} = 0$$

which is separable in the coordinates t and r , thus we have

$$B(r) = \frac{2Cqe^{-r}}{C^2e^{-2r} + \varepsilon}, \quad 2A\ddot{A} + \dot{A}^2 = \varepsilon q^{-2}$$

where⁶ $\varepsilon = \pm 1$ and q and C are constants, and therefore a well characterized subset of the solutions considered here are included in those studied in [16].

Back to the general non-static shear-free case, the dominant energy condition $\rho \geq 0, \rho \pm p_1 \geq 0$ and $\rho \pm p_2 \geq 0$ implies

$$-(B'^2 - 2B''B + 3\dot{A}^2B^4 + B^2) \geq 0, \quad (87)$$

$$-B'^2 + B''B - \dot{A}^2B^4 + AB^4\ddot{A} \geq 0, \quad (88)$$

$$-(AB^4\ddot{A} - BB'' + 2\dot{A}^2B^4 + B^2) \geq 0, \quad (89)$$

$$BB'' - 2\dot{A}^2B^4 - B^2 + 2AB^4\ddot{A} \geq 0, \quad (90)$$

$$-(2B'^2 - 3B''B + 4\dot{A}^2B^4 + B^2 + 2AB^4\ddot{A}) \geq 0. \quad (91)$$

⁶ Note that the equation for A can be simplified to $\ddot{a} = \frac{3}{4}\varepsilon q^{-2}a^{-\frac{1}{2}}$.

We next address the question of the existence of a constitutive equation $v = v(\eta, s)$ where in this case, η and s are given by [see (31)]

$$\eta = \frac{f^2(r)}{A^2(t)B^2(r)}, \quad s = \frac{r^2 f^2(r)}{A^2(t)B^2(r)}. \quad (92)$$

Dividing (85) and (86) by (84), and defining as before $E = \ln \eta, \Sigma = \ln s$, we get

$$\frac{\partial \ln v}{\partial E} = -\frac{1}{2} \frac{2AB^4\ddot{A} - B'^2 + \dot{A}^2 B^4 + B^2}{B'^2 - 2B''B + 3\dot{A}^2 B^4 + B^2} \equiv V_E, \quad (93)$$

$$\frac{\partial \ln v}{\partial \Sigma} = -\frac{2AB^4\ddot{A} - B''B + \dot{A}^2 B^4 + B'^2}{B'^2 - 2B''B + 3\dot{A}^2 B^4 + B^2} \equiv V_S. \quad (94)$$

In order for a constitutive equation $v = v(\eta, s)$ (or equivalently $v = v(E, \Sigma)$) to exist, it must be that

$$\frac{\partial^2 \ln v}{\partial \Sigma \partial E} = \frac{\partial^2 \ln v}{\partial E \partial \Sigma} \Leftrightarrow \frac{\partial V_E}{\partial \Sigma} = \frac{\partial V_S}{\partial E}. \quad (95)$$

Notice that

$$\partial_E = \frac{\partial t}{\partial E} \partial_t + \frac{\partial r}{\partial E} \partial_r, \quad \partial_\Sigma = \frac{\partial t}{\partial \Sigma} \partial_t + \frac{\partial r}{\partial \Sigma} \partial_r. \quad (96)$$

Now, from the expression (92) and the corresponding one for E and Σ , it follows that

$$r = e^{\frac{1}{2}(\Sigma - E)}, \quad A(t) = \frac{f}{B} e^{-\frac{1}{2}E}, \quad (97)$$

and differentiating them with respect to E and Σ and applying the chain rule, we get

$$\frac{\partial r}{\partial E} = -\frac{1}{2}r, \quad \frac{\partial r}{\partial \Sigma} = \frac{1}{2}r, \quad (98)$$

$$\frac{\partial t}{\partial E} = -\frac{1}{2\dot{A}} \left[r \left(\frac{f}{B} \right)' + \frac{f}{B} \right] \frac{AB}{f}, \quad \frac{\partial t}{\partial \Sigma} = \frac{1}{2\dot{A}} r \left(\frac{f}{B} \right)' \frac{AB}{f}. \quad (99)$$

Substituting the above expressions into (96), equation (95) reads, after some manipulations

$$\frac{A}{\dot{A}} r \left(\frac{f'}{f} - \frac{B'}{B} \right) \partial_t (V_E + V_S) + r \partial_r (V_E + V_S) + \frac{A}{\dot{A}} \partial_t V_S = 0. \quad (100)$$

Solving for $\frac{f'}{f}$ we get, after some algebra,

$$3rB^2 \frac{f'}{f} = \frac{-K [(S + 3B^4 \dot{A}^2) A_{III} - 6B^4 \dot{A} \ddot{A}^2 + 3B^4 A^{-1} \dot{A}^3] + MA^{-1} \dot{A} \ddot{A}}{[(S + 3B^4 \dot{A}^2) A_{III} - 6B^4 \dot{A} \ddot{A}^2 + 3B^4 A^{-1} \dot{A}^3] + SA^{-1} \dot{A} \ddot{A}}, \quad (101)$$

where, K, S and M are functions of r alone given by:

$$K = 2B - 3rB', \quad S = B'^2 - 2BB'' + B^2, \\ M = -6rB^2 B_{rrr} + 2B(9rB' + B)B'' + (2BB' - 9rB'^2 - 3rB^2)B' - 4B^3.$$

Since the left hand side of (101) depends only on r this implies that the time derivative of the right hand side must vanish. A careful but otherwise trivial analysis, reveals that there are only three possibilities, assuming the metric is non-static, namely

1. $M + KS = 0$, in which case $B(r)$ is determined by the resulting ordinary third order differential equation. $A(t)$ is in principle arbitrary, and $f(r)$ is fixed by (101) once the solution for $B(r)$ to the equation $M + KS = 0$ is given. We have not been able to find an integral for $B(r)$ in closed form, but in this case one has for $f(r)$:

$$\frac{f'}{f} = \frac{B'}{B^2} - \frac{2}{3rB}.$$

2. $\ddot{A} = 0$, which in turn implies $A = t$ without loss of generality, since the two constants of integration may be absorbed by suitable redefinitions of t and B . In this case, $B(r)$ is free, constrained only by the requirements imposed by the DEC, and once it is chosen, $f(r)$ is determined through (101), which implies as in the previous case

$$\frac{f'}{f} = \frac{B'}{B^2} - \frac{2}{3rB}.$$

3. In this case, both $A(t)$ and $B(r)$ are determined as the solutions of the following two third order differential equations:

$$\begin{aligned} -2BB'' + B'^2 + B^2 - kB^4 &= 0, \quad k = \text{constant}, \\ AA_{ttt} \left(\frac{k}{3} + \dot{A}\ddot{A} \right) - 2A\dot{A}\ddot{A}^2 + \dot{A}^3 - q\dot{A}\ddot{A} &= 0, \quad k, q = \text{constant}. \end{aligned}$$

Since $A(t)$ and $B(r)$ are fixed, so is $f(r)$, and (101) implies in this case

$$3rB^2 \frac{f'}{f} = \frac{M + (k - 3q)KB^4}{3qB^4}.$$

As in the first case above, we have not been able to find integrals for $A(t)$ or $B(r)$ in closed form.

Example 2 Let us next investigate in some detail the second case above, that is $A = t$. Substituting this into the EFEs we get

$$\rho = \frac{1}{8\pi t^2} \left(\frac{2B''}{B^3} - \frac{B'^2}{B^4} - \frac{1}{B^2} - 3 \right), \quad p_1 = \frac{1}{8\pi t^2} \left(\frac{B'^2}{B^4} - \frac{1}{B^2} - 1 \right),$$

$$p_2 = \frac{1}{8\pi t^2} \left(\frac{B''}{B^3} - \frac{B'^2}{B^4} - 1 \right). \quad (102)$$

On the other hand, the DEC (87–91) are all satisfied if and only if the following three inequalities hold:

$$3BB'' - 2B'^2 - B^2 - 4B^4 \geq 0, \quad BB'' - B'^2 - B^4 \geq 0, \quad BB'' - B'^2 - 2B^4 \geq 0 \quad (103)$$

which, upon setting $B \equiv e^b$ are equivalent to

$$3b'' + b'^2 - 1 - 4e^{2b} \geq 0, \quad b'' - e^{2b} \geq 0, \quad b'' + b'^2 - 1 - 2e^{2b} \geq 0. \quad (104)$$

It is easy to see that these conditions can be satisfied, at least for certain ranges of the radial coordinate $r \in [0, R)$, for suitably chosen functions $b(r)$, such as

$$b(r) = \frac{3}{2} \ln \left[\frac{2}{3} + \sinh^2 \left(\frac{r-r_0}{c} \right) \right]$$

$$-\ln c \Leftrightarrow B(r) = \frac{\sqrt{3}}{9c} \ln \left[-1 + 3 \cosh^2 \left(\frac{r-r_0}{c} \right) \right]^{\frac{3}{2}} \quad (105)$$

The form of $f(r)$ can be given explicitly up to a quadrature.

As in previous cases, we do not claim that it has any particular significance, but it provides a relatively simple instance of solution with the desired properties.

Example 3 Another simple example with similar characteristics is provided by the following choice of $B(r)$:

$$b(r) = \frac{3}{2} \ln \left(\frac{2}{3} + r^2 \right) \Leftrightarrow B(r) = \frac{\sqrt{3}}{9} (2 + 3r^2)^{\frac{3}{2}}, \quad (106)$$

in which case $f(r)$ can be integrated out yielding:

$$f(r) = \exp \left\{ \frac{-15 - 9r^2 + \frac{3}{2}(2 + 3r^2)\sqrt{4 + 6r^2} \tanh^{-1} \left(\frac{2}{\sqrt{4 + 6r^2}} \right)}{\sqrt{6 + 9r^2} (2 + 3r^2)} \right\}$$

Similar remarks to the ones above regarding its physical significance, apply also here.

6 Conclusions

In this paper we have considered spherically symmetric spacetimes with elastic material content. We started considering the symmetries these spacetimes possess in order to exploit their consequences on physics for elastic spacetime configurations. By doing this, we have generalized previous work done by [3; 4] for non-static spherically symmetric configurations, where only flat material metrics were considered. In fact, we have shown that all material metrics compatible with a given spacetime are conformally related and, moreover, are conformally flat. Next we have used comoving coordinates to relate the EFEs with quantities characterizing elasticity properties (constitutive equation, material and energy density, eigenvalues of the pulled back material metric) as well as the conformal factor referred to above. The case in which the velocity of the matter is shearfree has been considered in detail, giving the necessary and sufficient condition for a constitutive equation to be admitted; further, we have provided three examples of static and non-static shear-free solutions. For non-static spherically symmetric spacetimes, the elasticity difference tensor has been studied, thus extending some previous work for the static case (see Appendix).

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Appendix: The elasticity difference tensor for non-static solutions

Here we obtain the elasticity difference tensor, defined in [1], for non-static spherically symmetric spacetimes and analyze this tensor following the procedure developed in [7], where the static, spherically symmetric case was presented as an example.

This third order tensor, symmetric on the two covariant indices, is completely flow-line orthogonal and is related with the (pulled back) material metric according to

$$S_{bc}^a = \frac{1}{2}k^{-am}(D_b k_{mc} + D_c k_{mb} - D_m k_{bc}). \quad (107)$$

Here k^{-am} is such that $k^{-am}k_{mb} = h_b^a$ and D represents the spatially projected connection obtained from the spacetime connection ∇ associated with g by

$$D_a t_{c\dots}^{b\dots} = h_a^d h_e^b \dots h_c^f \dots \nabla_d t_{f\dots}^{e\dots}, \quad (108)$$

with the property $D_a h_{bc} = 0$.

The non-zero components of S_{bc}^a for non-static, spherically symmetric spacetimes, using the space-time metric (28) and the pulled-back material metric (30)

Table 1 Eigenvectors and eigenvalues for M_1

Eigenvectors	Eigenvalues
x	$\mu_1 = \frac{f'}{\sqrt{b}f} - \frac{\bar{b}'}{2b\sqrt{\bar{b}}}$
y	$\mu_2 = \frac{\bar{Y}'}{\sqrt{\bar{b}}\bar{Y}} - \frac{r^2 f' \sqrt{\bar{b}}}{\bar{Y}^2 f} - \frac{r\sqrt{\bar{b}}}{\bar{Y}^2}$
z	$\mu_3 = \mu_2$

Table 2 Eigenvectors and eigenvalues for M_2

Eigenvectors	Eigenvalues
$x + y$	$\mu_4 = \frac{f'}{\sqrt{b}f} + \frac{1}{\sqrt{b}r} - \frac{\bar{Y}'}{\bar{Y}\sqrt{\bar{b}}}$
$x - y$	$\mu_5 = -\mu_4$
z	$\mu_6 = 0$

can be written as:

$$\begin{aligned}
S_{rr}^r &= \frac{f'}{f} - \frac{\bar{b}'}{2\bar{b}} \\
S_{\theta r}^\theta &= \frac{f'}{f} + \frac{1}{r} - \frac{\bar{Y}'}{\bar{Y}} \\
S_{\phi r}^\phi &= \frac{f'}{f} + \frac{1}{r} - \frac{\bar{Y}'}{\bar{Y}} \\
S_{\theta\theta}^r &= -\frac{f'r^2}{f} - \frac{1}{r} + \frac{\bar{Y}'\bar{Y}}{\bar{b}} \\
S_{\phi\phi}^r &= -\frac{r^2 f' \sin^2 \theta}{f} - \frac{r \sin^2 \theta}{1} + \frac{\bar{Y}'\bar{Y} \sin^2 \theta}{\bar{b}}.
\end{aligned}$$

In this case, the pulled-back material metric k_{ab} is

$$k_{ab} = n_1^2 x_a x_b + n_2^2 (y_a y_b + z_a z_b). \quad (109)$$

Here, x, y, z are eigenvectors of k_b^a with eigenvalues n_1^2 and $n_2^2 = n_3^2$ which depend on t and r according to

$$n_1^2 = f^2 \frac{1}{b}, \quad n_2^2 = n_3^2 = f^2 \frac{r^2}{\bar{Y}^2}. \quad (110)$$

The elasticity difference tensor can be decomposed along the directions determined by the eigenvectors of k_b^a as follows

$$S_{bc}^a = M_{bc}^a x^a + M_{bc}^a y^a + M_{bc}^a z^a, \quad (111)$$

where $M_i^a, i = 1, 2, 3$ are second order, symmetric tensors (see [7]).

Here we determine the eigenvectors and eigenvalues of these tensors, complementing the results obtained in [7] for the static case, the result being summarized in Tables 1, 2 and 3:

Table 3 Eigenvectors and eigenvalues for M_3

Eigenvectors	Eigenvalues
$x+z$	$\mu_7 = \mu_4$
$x-z$	$\mu_8 = -\mu_4$
y	$\mu_9 = 0$

Therefore, the canonical forms for the three tensors M_i are:

$$\begin{aligned}
 M_{bc}^1 &= \mu_1 x_b x_c + \mu_2 (y_b y_c + z_b z_c) \\
 M_{bc}^2 &= 2\mu_4 (x_b y_c + y_b x_c) \\
 M_{bc}^3 &= 2\mu_4 (x_b z_c + z_b x_c).
 \end{aligned} \tag{112}$$

Although the eigenvalues are different from the ones obtained in the static case, the eigenvectors of the above tensors are the same for the static and non-static case.

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