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Quantization in terms of symplectic groups: The harmonic oscillator as a generic example

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Abstract. The conventional quantization of the harmonic oscillator in terms of operators Q and P can be implemented with the help of irreducible unitary representations of the Heisenberg-Weyl group which acts transitively and effectively on the simply connected classical phase space $\mathcal{S}_{q,p} \cong \mathbb{R}^2$. In the description of the harmonic oscillator in terms of angle and action variables φ and I the associated phase space $\mathcal{S}_{\varphi,I}$ corresponds to the multiply connected punctured plane $\mathbb{R}^2 - \{0\}$, on which the 3-dimensional symplectic group $Sp(2,\mathbb{R})$ acts transitively, leaving the origin invariant. As this group contains the compact subgroup $U(1)$ it has infinitely many covering groups. In the here relevant irreducible unitary representations (positive discrete series) the self-adjoint generator K_0 of $U(1)$ represents the classical action variable I. It has the possible spectra $n+k,\, n=0,1,\ldots;\, k>0$, where k depends on the covering group. This implies different possible spectra for the action variable Hamiltonian $\hbar\omega K_0$ of the harmonic oscillator. On the other hand, expressing the operators Q and P (non-linearly) in terms of the three generators K_0 etc. of $Sp(2,\mathbb{R})$ leads to the usual framework. Possible physical (experimental) implications and generalizations to higher dimensions are discussed briefly.

1. Introduction: Angle and action variable for the HO

It may appear as a provocation or even a joke to present a contribution on the old-fashioned harmonic oscillator (HO in the following) to an established international conference devoted to current and relevant research in physics! But see for yourself! The basics for the present paper are described in my long article [1] which contains a wealth of references to the work of others which will not be quoted here again. The present contribution will sketch the main idea and emphasize some new important physical aspects.
The canonical Eqs. of motion

$$\dot{p} = -b\,q\,,\; p = M\,\dot{q}\,,\; b>0\,, \tag{1}$$

for the HO can be simplified by the canonical transformation

$$q(\varphi,I) = \sqrt{\frac{2\,I}{M\,\omega}}\,\cos\varphi\,,\quad p(\varphi,I) = -\sqrt{2\,M\,\omega\,I}\,\sin\varphi\,,\quad \omega=\sqrt{b/M}\,, \tag{2}$$

which is *locally* symplectic:

$$dq\wedge dp = d\varphi\wedge dI\,,\;\text{ or }\;\frac{\partial(q,p)}{\partial(\varphi,I)} = 1\,. \tag{3}$$

It is singular for $(q,p)=(0,0), I=0$ (equilibrium or critical point) corresponding to the branch point $I=0$ of $\sqrt{I}$.

The Hamiltonian is now given by

$$H(q,p) = \frac{1}{2M}\,p^2 + \frac{1}{2}\,M\,\omega^2\,q^2 = H(\varphi, I) = \omega\, I\,, \tag{4}$$

implying the (φ, I)-Eqs. of motion

$$\dot{\varphi} = \frac{\partial H}{\partial I} = \omega\,, \quad \dot{I} = -\frac{\partial H}{\partial \varphi} = 0\,. \tag{5}$$

They have the solutions

$$\varphi(t) = \omega\, t + \varphi_0\,, \quad I = \text{ const.} > 0\,. \tag{6}$$

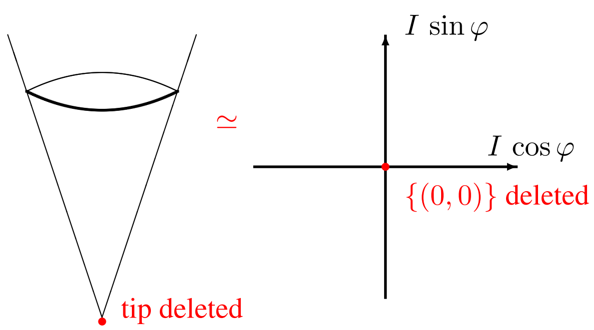
Here the angle φ is essentially the time variable t. Notice that the time t does not stop after a period $\omega\,\hat{t} = 2\pi + \varphi_0$!

We now have two *globally* different phase spaces for the HO:

$$\mathcal{S}_{q,p} = \{(q,p) \in \mathbb{R}^2\} \cong \mathbb{R}^2 \tag{7}$$

$$\text{and} \quad \mathcal{S}_{\varphi,I} = \{(\varphi, I), \varphi \in \mathbb{R} \bmod 2\pi\,,\ I > 0\} \cong S^1 \times \mathbb{R}^+ \cong \mathbb{R}^2 - \{0\}\,. \tag{8}$$

The phase space $\mathcal{S}_{\varphi,I}$ is topologically a simple cone with the tip deleted:



It may also be interpreted as the orbifold

$$\mathcal{S}_{\varphi,I} = \mathcal{S}_{q,p}/Z_2\,. \tag{9}$$

It is shown in Ref. [1] (the reasons will briefly be indicated in the next Sec.) that the proper global coordinates on $\mathcal{S}_{\varphi,I}$ are the following ones:

$$h_0(\varphi, I) = I > 0\,, \quad h_1(\varphi, I) = I\cos\varphi\,, \quad h_2(\varphi, I) = -I\sin\varphi\,, \tag{10}$$

which obey the Pythagorean relation

$$\vec{h}^2 \equiv h_1^2 + h_2^2 = h_0^2\,. \tag{11}$$

The invariant measure on $\mathcal{S}_{\varphi,I}$ is

$$d\mu(\varphi, I) = 2\theta(h_0)\,\delta(h_0^2 - \vec{h}^2)\,dh_0 dh_1 dh_2 = d\varphi dI = dq dp\,. \tag{12}$$

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The $h_j(\varphi, I)$ obey the Poisson Lie algebra

$$\{h_0, h_1\}_{\varphi,I} = -h_2\,,\ \{h_0, h_2\}_{\varphi,I} = h_1\,,\ \{h_1, h_2\}_{\varphi,I} = h_0\,, \tag{13}$$

where

$$\{h_j, h_k\}_{\varphi,I} \equiv \partial_\varphi h_j\,\partial_I h_k - \partial_I h_j\,\partial_\varphi h_k. \tag{14}$$

Thus, the h_j generate the isomorphic *simple* Lie algebras

$$\mathfrak{so}(1,2) = \mathfrak{sp}(2,\mathbb{R}) = \mathfrak{sl}(2,\mathbb{R}) = \mathfrak{su}(1,1)\,. \tag{15}$$

This is an analogue of the *nilpotent* Poisson Lie algebra of the Heisenberg-Weyl group:

$$\{q, p\} = 1\,,\ \ \{q, 1\} = 0\,,\ \ \{p, 1\} = 0\,. \tag{16}$$

On $\mathcal{S}_{\varphi,I}$ the basic "observables" for the construction of Hamilton functions are now the $h_j, j = 0, 1, 2$. Simple examples are:

(i) Time-dependent frequency ω :

$$H = \omega(t)\,I\,,\ \ \dot{\omega}(t) \neq 0\,, \tag{17}$$

with the Eqs. of motion

$$\dot{\varphi} = \partial_I H = \omega(t)\,,\ \ \dot{I} = -\partial_\varphi H = 0\,, \tag{18}$$

so that

$$\varphi(t) = \int_{t_0}^{t} d\tau\omega(\tau) + \varphi_0\,,\ \ I(t) = I_0 = \text{ const. }. \tag{19}$$

The energy

$$E(t) = \omega(t)\,I_0 \tag{20}$$

is *not* conserved! Conserved is the action variable $I = I_0$.

(ii) If the Hamilton function

$$H(q, p) = \frac{1}{2M}p^2 + V(q) = E = \text{ const.} \tag{21}$$

has bounded periodic motions on $\mathcal{S}_{q,p}$ then the associated action variable is defined as

$$I(E) = \oint_{C(E)} dq\,p(q; E)\,,\ p(q; E) = \pm\sqrt{2M(E - V(q))}\,, \tag{22}$$

where $C(E)$ is the closed contour in phase space, determined by the relation $H(q, p) = E$. Here $I(E)$ is the area of the phase space region with boundary $C(E)$.
E.g. for the Morse potential

$$V_{\mathrm{Mo}}(q) = V_0 \tanh(aq) \tag{23}$$

one gets

$$H(I) = \omega_0\,I(1 - \frac{\omega_0\,I}{4V_0})\,,\ \omega_0 = a\sqrt{2V_0/M}\,. \tag{24}$$

(iii) Example where $H = H(\varphi, I)$ depends only on some h_j:

$$H = \omega(h_0 + g\,h_1) = \omega I\,(1 + g\,\cos\varphi)\,,\ |g| < 1\,. \tag{25}$$

Here

$$H(\varphi, I) = E = \text{ const. }. \tag{26}$$

Solutions of the associated Eqs. of motion are:

$$\tan[\varphi(\tilde{t})/2] = \sqrt{\frac{1+g}{1-g}}\,\tan[(\sqrt{1-g^2})\,(\tilde{t}/2)]\,, \tag{27}$$

$$I(\tilde{t}) = I_0\,[1 + g\,\cos\varphi(\tilde{t})\,]^{-1}\,;\ \ I_0 = E/\omega\,,\ \tilde{t} = \omega\,t\,,\ \cos\varphi = \frac{1 - \tan^2(\varphi/2)}{1 + \tan^2(\varphi/2)}\,. \tag{28}$$

2. Brief justification of the global coordinates $\mathbf{h_j, j = 1, 2, 3}$ on $\mathcal{S}_{\varphi,\mathbf{I}}$

I sketch briefly the justification for the choice (10) of the global basic coordinates h_j on the phase space $\mathcal{S}_{\varphi,I}$. The main tool is "group theoretical quantization" as developed by Isham, Sternberg and others. Details and references can be found in Ref. [1]. One has to look for an appropriate symplectic and transitive transformation group on the phase space in question. In our case the 3-parametric symplectic group

$$G \equiv Sp(2, \mathbb{R}) \tag{29}$$

(being isomorphic to the groups $SL(2, \mathbb{R}) \cong SU(1, 1)$) is defined by

$$g = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}, \; a_{jk} \in \mathbb{R}, \; \det g = 1, \tag{30}$$

$$g^T \cdot \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \cdot g = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}. \tag{31}$$

It acts on the real plane $\mathbb{R}^2$ in an obvious way by matrix multiplication of $(\tilde{q}, \tilde{p})^T \in \mathbb{R}^2$. The action of G leaves the origin $(0, 0)$ invariant and therefore is the proper "canonical" group of the punctured plane

$$\mathcal{S}_{\tilde{q},\tilde{p};\,0} \equiv \mathcal{S}_{\tilde{q},\tilde{p}} - \{(\tilde{q}, \tilde{p}) = (0, 0)\} \cong \mathbb{R}^2 - \{(0, 0)\}, \tag{32}$$

with g acting symplectically:

$$d[g(\tilde{q})] \wedge d[g(\tilde{p})] = d\tilde{q} \wedge d\tilde{p}. \tag{33}$$

N.b.: quantities with "tilde" are made dimensionless by using the intrinsic length $\lambda_0 = \sqrt{\hbar/(\omega M)}$ and $\hbar$: $\tilde{q} = q/\lambda_0$; $\tilde{p} = \lambda_0\, p/\hbar$.
In order to derive the coordinates (10) one needs the first three of the 1-dimensional subgroups

$$R: \quad r = \begin{pmatrix} \cos(\theta/2) & \sin(\theta/2) \\ -\sin(\theta/2) & \cos(\theta/2) \end{pmatrix}, \; \theta \in (-2\pi, +2\pi], \tag{34}$$

$$A: \quad a = \begin{pmatrix} e^{-\tau/2} & 0 \\ 0 & e^{\tau/2} \end{pmatrix}, \; \tau \in \mathbb{R}, \tag{35}$$

$$B: \quad b = \begin{pmatrix} \cosh(s/2) & \sinh(s/2) \\ \sinh(s/2) & \cosh(s/2) \end{pmatrix}, \; s \in \mathbb{R}, \tag{36}$$

$$N: \quad n = \begin{pmatrix} 1 & \xi \\ 0 & 1 \end{pmatrix}, \; \xi \in \mathbb{R}. \tag{37}$$

Given a smooth function $f(x)$ each subgroup $\{g(u), g(u = 0) = 1\}$ induces a *global* Hamiltonian vector field

$$[Af](x) = \lim_{u\to 0} \frac{1}{u}[f(g(-u) \cdot x) - f(x)], \tag{38}$$

where A is a vector field of the type

$$A = -[\partial_{\tilde{p}} j(x)\, \partial_{\tilde{q}} - \partial_{\tilde{q}} j(x)\, \partial_{\tilde{p}}], \; x = (\tilde{q}, \tilde{p})^T. \tag{39}$$

For the above subgroups R, A and B the corresponding Hamiltonian functions $j(x)$ are

$$R: \; j_0(x) = \frac{1}{4}(\tilde{q}^2 + \tilde{p}^2) = \frac{I}{2\hbar}, \tag{40}$$

$$A: \; j_2(x) = -\frac{1}{2}\,\tilde{q}\,\tilde{p} = \frac{I}{2\hbar}\sin 2\varphi, \tag{41}$$

$$B: \; j_1(x) = \frac{1}{4}(-\tilde{q}^2 + \tilde{p}^2) = -\frac{I}{2\hbar}\cos 2\varphi, \tag{42}$$

where the relations (2) have been used. Thus, by a very simple symplectic rescaling $I/2 \to I$, $2\varphi \to \varphi$, $j_1 \to -j_1$, $j_2 \to -j_2$ one obtains the basic coordinates (10). In other words: the global coordinate functions $h_j(\varphi, I)$ on $\mathcal{S}_{\varphi,I}$ are determined by the action of the "canonical" symplectic group $Sp(2,\mathbb{R})$ on $\mathcal{S}_{q,p;0}$!

Compare this to the analogous global coordinate functions q and p on $\mathcal{S}_{q,p}$, where the canonical group consists of the translations

$$q \to q + a,\;\; p \to p\,;\;\; q \to q,\;\; p \to p - b,\;\; a, b \in \mathbb{R}\,, \tag{43}$$

which generate on $\mathcal{S}_{q,p}$ the vector fields

$$A_q = -\partial_q,\; A_p = \partial_p, \tag{44}$$

with the associated global Hamiltonian functions

$$j_q(x) = p, \quad j_p(x) = q\,. \tag{45}$$

Let me add some structural relations on the classical level: The coordinates $h_j, j = 0, 1, 2$, transform as *vectors* with respect to the group $SO^{\uparrow}(1,2) = Sp(2,\mathbb{R})/Z_2$, whereas the coordinates q, p transform as vectors with respect to the group $Sp(2,\mathbb{R})$ which is a twofold covering of $SO^{\uparrow}(1,2)$.
The phase spaces $\mathcal{S}_{\varphi,I}$ and $\mathcal{S}_{q,p}$ may be represented as homogeneous spaces as follows:

$$\mathcal{S}_{\varphi,I} = SO^{\uparrow}(1,2)/N\,,\;\; \mathcal{S}_{q,p} = Sp(2,\mathbb{R})/N\,, \tag{46}$$

which again yield the orbifold

$$\mathcal{S}_{\varphi,I} = \mathcal{S}_{q,p}/Z_2\,. \tag{47}$$

3. Quantum mechanics for angle and action variables of the HO

The quantization of the global "coordinates" h_j from Eq. (10) is implemented by promoting them to self-adjoint operators,

$$h_j \to K_j = \hbar\,\tilde{K}_j \tag{48}$$

which obey the associated Lie algebra (13):

$$[\tilde{K}_0,\, \tilde{K}_1] = i\,\tilde{K}_2\,,\;\; [\tilde{K}_0,\, \tilde{K}_2] = -i\,\tilde{K}_1\,,\;\; [\tilde{K}_1,\, \tilde{K}_2] = -i\,\tilde{K}_0\,. \tag{49}$$

The self-adjoint generators $\tilde{K}_j$ may be obtained from unitary irreducible representations of the corresponding groups $SO^{\uparrow}(1,2), Sp(2,\mathbb{R})$ $[\, = SL(2,\mathbb{R}), SU(1,1)]$ or one of their infinitely many covering groups.

As $\tilde{K}_0$ is the generator of a maximal compact abelian subgroup, its eigenstates may used as a Hilbert space basis:

$$\tilde{K}_0|k,\, n\rangle = (n + k)\,|k,\, n\rangle\,, \tag{50}$$

where k is some real number ("Bargmann index"). With

$$\tilde{K}_{\pm} = \tilde{K}_1 \pm i\,\tilde{K}_2 \tag{51}$$

it follows from the relations (49) that

$$\tilde{K}_+|k,\, n\rangle = [(2k+n)(n+1)]^{1/2}\,|k,\, n+1\rangle\,,\;\; \tilde{K}_-|k,\, n\rangle = [(2k+n-1)n]^{1/2}\,|k,\, n-1\rangle\,. \tag{52}$$

If there exists a $|k, 0\rangle$ such that

$$\tilde{K}_0|k, 0\rangle = k\,|k, 0\rangle\,, \quad \tilde{K}_-|k, 0\rangle = 0\,, \tag{53}$$

then

$$\tilde{K}_0\,|k,n\rangle = (n+k)\,|k,n\rangle\,,\; n=0,1,\ldots;\;\; k>0\,. \tag{54}$$

This is the so-called "positive discrete series" $D_k^{(+)}$ among the different types of possible irreducible unitary representations of $Sp(2,\mathbb{R})$ (see Ref. [1] for details). The Bargmann index k characterizes an irred. repr. $D_k^{(+)}$.

The Group $SO^{\uparrow}(1,2)$ has *infinitely many covering groups* because its compact subgroup $SO(2)\cong S^1$ is not simply connected! Let us denote the m-fold covering by

$$SO^{\uparrow}_{[m]}(1,2)\,,\; m=1,2,\ldots\,. \tag{55}$$

Its irreducible unitary representations $D_k^{(+)}$ have the indices

$$k=\frac{\mu}{m}\,,\; \mu\in\mathbb{N}=\{1,2,\ldots\}\,. \tag{56}$$

This shows that $k_{min}=1/m$ can be arbitrarily small >0 if m is large enough!
The 2-fold coverings

$$Sp(2,\mathbb{R})=SL(2,\mathbb{R})\;\cong SU(1,1) \tag{57}$$

have $k=1/2$. The related (q,p)-Hamiltonian

$$H(q,p)\to H_{osc}(Q,P)=\frac{1}{2\,M}\,P^2+\frac{1}{2}\,M\,\omega^2\,Q^2 \tag{58}$$

$$=-\frac{\hbar^2}{2\,M}\,\frac{d^2}{d\,q^2}+\frac{1}{2}\,M\,\omega^2\,q^2 \tag{59}$$

has eigenvalues

$$E_n=\hbar\,\omega\,(n+\frac{1}{2})\,,\;\; n=0,1,\ldots\,. \tag{60}$$

However, the (φ,I)-Hamiltonian

$$H(\varphi,I)\to H_{osc}(\vec{K})=\omega\,K_0\,,\;\; \vec{K}=\hbar\,(\tilde{K}_0,\tilde{K}_1,\tilde{K}_2) \tag{61}$$

can have spectra

$$E_{k,n}(\varphi,I)=\hbar\,\omega\,(n+k)\,,\; n=0,1,\ldots\,;\;\; k\in\mathbb{R}^+\,! \tag{62}$$

How to reconcile the two results if $k\neq 1/2$?
Interesting enough there exists an operator version of the mapping $(\varphi,I)\to(q,p)$ from Eq. (2), namely

$$Q=Q(\vec{K})=\frac{\lambda_0}{\sqrt{2}}\,(A^\dagger+A)(\vec{K})\,,\;\; P=P(\vec{K})=\frac{i\,\hbar}{\sqrt{2}\,\lambda_0}\,(A^\dagger-A)(\vec{K})\,, \tag{63}$$

$$\lambda_0=\sqrt{\hbar/(M\,\omega)}\,, \tag{64}$$

where the operators

$$A(\vec{K})=(\tilde{K}_0+k)^{-1/2}\tilde{K}_-\,,\;\; A^\dagger(\vec{K})=\tilde{K}_+(\tilde{K}_0+k)^{-1/2} \tag{65}$$

act as usual:

$$A^\dagger\,|k,\,n\rangle=\sqrt{n+1}\,|k,\,n+1\rangle\,,\;\; A\,|k,\,n\rangle=\sqrt{n}\,|k,\,n-1\rangle\,. \tag{66}$$

This means

$$[A,\,A^\dagger]=\mathbf{1}\;\;\forall\,D_k^{(+)}\,, \tag{67}$$

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independent of the value of k!!
So again

$$H_{osc}(Q,P)|k,n\rangle = \hbar\,\omega\,(n+1/2)\,|k,n\rangle\,, \tag{68}$$

where now

$$H_{osc}(Q,P) = \hbar\,\omega\,(A^\dagger A + 1/2)\,. \tag{69}$$

Thus, the (φ, I)-variable QM of the HO is more subtle than its conventional (q,p)-variable one, due to the topologically non-trivial classical phase space $\mathcal{S}_{\varphi,I}$! As the transformation (65) "erases" the k-dependence, the usual Stone-von Neumann uniqueness theorem for self-adjoint Q and P still holds!
Eqs. (65) are the operator version of the classical relations

$$q(\varphi,I) = \sqrt{\frac{2}{M\,\omega}}\,\frac{h_1(\varphi,I)}{\sqrt{h_0(\varphi,I)}}\,,\quad p(\varphi,I) = \sqrt{2\,M\,\omega}\,\frac{h_2(\varphi,I)}{\sqrt{h_0(\varphi,I)}}\,. \tag{70}$$

The most important physical question is, of course, whether one can detect the k-dependence experimentally. More about that below.

3.1. Simple consequences

The Casimir operator of a representation $D_k^{(+)}$ is

$$\mathfrak{C} = \tilde{K}_1^2 + \tilde{K}_2^2 - \tilde{K}_0^2 = k(1-k)\,. \tag{71}$$

This means that the "classical Pythagoras" (11) is violated quantum mechanically for $k \neq 1$, e.g. for the HO with $k = 1/2$!
The unitary time evolution is given by

$$U(\tilde{t}) = e^{-i\,\tilde{H}\,\tilde{t}}\,,\;\; \tilde{H} = \tilde{K}_0 = N + k\,,\;\; \omega\,t = \tilde{t} = \theta\,, \tag{72}$$

i.e. time is an angle variable here!
The unitary operator (72) implies the usual Heisenberg Eqs. of motion:

$$U(-\tilde{t})\,\tilde{K}_1\,U(\tilde{t}) = \cos\tilde{t}\,\tilde{K}_1 + \sin\tilde{t}\,\tilde{K}_2\,,\quad U(-\tilde{t})\,\tilde{K}_2\,U(\tilde{t}) = -\sin\tilde{t}\,\tilde{K}_1 + \cos\tilde{t}\,\tilde{K}_2\,; \tag{73}$$

$$U(-\tilde{t})\,\tilde{Q}\,U(\tilde{t}) = \cos\tilde{t}\,\tilde{Q} + \sin\tilde{t}\,\tilde{P}\,,\quad U(-\tilde{t})\,\tilde{P}\,U(\tilde{t}) = -\sin\tilde{t}\,\tilde{Q} + \cos\tilde{t}\,\tilde{P}\,. \tag{74}$$

3.1.1. Covering groups For $\tilde{t} = 2\pi$ the operator (72) becomes

$$U(\tilde{t} = 2\pi) = e^{-2\pi i k}\mathbf{1}\,. \tag{75}$$

If

$$k = n/m\,,\; n,m \in \mathbb{N}\,, \tag{76}$$

this implies for $SO^\uparrow_{[m]}(1,2)$:

$$U(\tilde{t} = m\,2\pi) = \mathbf{1}\,. \tag{77}$$

The ground state $|k,0\rangle$ has the time evolution

$$U(\tilde{t})\,|k,0\rangle = e^{-i\,k\,\tilde{t}}\,|k,0\rangle\,, \tag{78}$$

with the associated time period

$$T_{2\pi,k} = \frac{2\pi}{\omega_k}\,,\;\; \omega_k \equiv k\,\omega\,, \tag{79}$$

which can become arbitrarily large for $k = 1/m$, $m \to \infty$.

3.1.2. Some matrix elements It follows from the relations (51)-(53) that

$$\langle k,n|\tilde{K}_j|k,n\rangle = 0\,,\ j=1,2, \tag{80}$$

$$(\Delta\tilde{K}_j)^2_{k,n} = \frac{1}{2}(n^2+2nk+k)\,,\ j=1,2, \tag{81}$$

so that

$$(\Delta\tilde{K}_1)_{k,n}\,(\Delta\tilde{K}_2)_{k,n} = \frac{1}{2}(n^2+2kn+k)\,, \tag{82}$$

$$(\Delta\tilde{K}_1)_{k,n=0}\,(\Delta\tilde{K}_2)_{k,n=0} = \frac{k}{2}\,. \tag{83}$$

The last relation shows that $k\to 0$ is a kind of classical limit in the angle-action framework!
The composite $\tilde{Q}$ and $\tilde{P}$ have the usual k-independent properties

$$\langle k,n|\tilde{Q}|k,n\rangle = 0\,,\ \ \langle k,n|\tilde{P}|k,n\rangle = 0\,, \tag{84}$$

$$(\Delta\tilde{Q})^2_{k,n} = (\Delta\tilde{P})^2_{k,n} = n+1/2\,, \tag{85}$$

$$(\Delta\tilde{Q})_{k,n}\,(\Delta\tilde{P})_{k,n} = n+1/2\,. \tag{86}$$

4. Possible applications

For applications of the angle-action framework it is important to use the variables h_j and K_j as primary observables, e.g. as building blocks for the construction of Hamiltonians! An additional experimental problem is how to avoid the dominance of the q- and p-degrees of freedom!
For $k=1/2$ a possible Hilbert space for the HO is the Hardy space characterized by

$$(f_2,f_1)_+ = \int_0^{2\pi}\frac{d\vartheta}{2\pi}\,f_2^*(\vartheta)f_1(\vartheta)\,,\ \text{basis}: e_n(\vartheta)=e^{i\,n\,\vartheta}\,,\ n=0,1,\ldots \tag{87}$$

Now

$$\tilde{K}_0 = \frac{1}{i}\partial_\vartheta + 1/2\,. \tag{88}$$

For $k\neq 1/2$ one can use the associated Hilbert space with the scalar product (see Ref. [1])

$$(f_2,f_1)_{+,k} = (f_2,A_kf_1)_+\,,\ (A_k\,e_n)(\vartheta) = \frac{n!}{(2k)_n}\,e_n(\vartheta)\,. \tag{89}$$

It has the orthonormal basis

$$\hat{e}_{k,n}(\vartheta) = \sqrt{\frac{(2k)_n}{n!}}\,e_n(\vartheta)\,,\ \ (2k)_n = \frac{\Gamma(2k+n)}{\Gamma(2k)}\,. \tag{90}$$

Now one has

$$\tilde{K}_0 = \frac{1}{i}\partial_\vartheta + k\,, \tag{91}$$

$$\tilde{K}_+ = e^{i\vartheta}(\frac{1}{i}\partial_\vartheta + 2k)\,,\ \ \tilde{K}_- = e^{-i\vartheta}\,\frac{1}{i}\partial_\vartheta\,. \tag{92}$$

Examples:

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(i) For experimental tests a HO with a time-dependent frequency $\omega(t)$ may be of considerable interest (see examples (iii)). The associated Schrödinger equation ($\hbar = 1$)

$$H = \omega(t)\tilde{K}_0\,,\ i\partial_t\psi(t,\vartheta) = H\psi(t,\vartheta) \tag{93}$$

may easily be solved by a separation of variables:

$$\psi(t,\vartheta) = \sigma(t)\,f(\vartheta)\,,\ \tilde{K}_0 f(\vartheta) = I_0\,f(\vartheta)\,, \tag{94}$$

where

$$\sigma(t) = \sigma(t_0)\,e^{-iI_0\int_{t_0}^{t}d\tau\,\omega(\tau)}\,. \tag{95}$$

If $f(\vartheta)$ is an eigenfunction (90), then

$$I_0 = n + k\,. \tag{96}$$

Here I_0 is constant, not the energy $E = \omega(t)\,I_0$!

(ii) The Hamiltonian

$$H = \omega(\tilde{K}_0 + g\tilde{K}_1)\,,\ |g| < 1\,, \tag{97}$$

is the quantized version of the classical one (25). It has the eigenvalues

$$E_n = \omega(n+k)\sqrt{1-g^2} \tag{98}$$

and the eigenfunctions

$$f_{k,n}(\vartheta) = C_k\,(1+g\,\cos\vartheta)^{-k}\,e^{i\,\{(n+k)2\arctan[\sqrt{\frac{1-g}{1+g}}\tan(\vartheta/2)]-k\vartheta\}}\,,\ C_k = \text{ const.}. \tag{99}$$

First-order perturbation theory yields

$$\langle k,m|H|k,n\rangle = \omega[\langle k,m|\tilde{K}_0|k,n\rangle + g\langle k,m|\tilde{K}_1|k,n\rangle]\,; \tag{100}$$

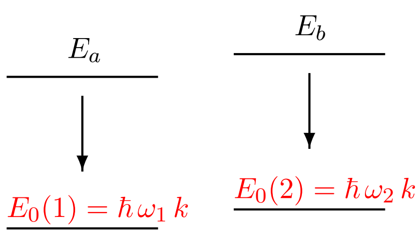
so that

$$\langle k,m|H|k,m\rangle = \omega(n+k); \tag{101}$$

notice that the exact eigenvalues (98) are of second order in g! Furthermore

$$\langle k,m|H|k,m-1\rangle = \frac{g\omega}{2}[m(2k+m-1)]^{1/2}\,,\ \langle k,1|H|k,0\rangle = g\,\omega\sqrt{k/2}\,. \tag{102}$$

(iii) Possible experiments should make use of the essential property that the action I or K_0 is the basic observable, not the Hamiltonian $\omega\,K_0$! This is an old idea which was already exploited in the framework of the old quantum mechanics (e.g. by Mulliken in 1924) even before the "correct" modern QM was found in 1925. The main idea is as follows: The frequency $\omega = \sqrt{b/M}$ of a given HO type may be changed, either by modifying the mass M or (and) the oscillator strength b!
Consider two different oscillator frequencies ω_1 and ω_2 and the transitions from some "exterior" levels as indicated in the following figure:



The transitions are associated with the frequencies

$$\omega_{a,1} = [E_a - E_0(1)]/\hbar\,, \quad \omega_{b,2} = [E_b - E_0(2)]/\hbar\,, \tag{103}$$

which lead to the k-dependent differences

$$\omega_{a,1} - \omega_{b,2} = (E_a - E_b)/\hbar - k\,(\omega_1 - \omega_2)\,, \tag{104}$$

which - in principle - can serve to determine k!
Examples:

- The relation (104) has been used by Mulliken and others in order to determine the vibrational levels (infrared bands) of diatomic molecules with isotopic atoms which lead to different (reduced) masses $M_i\,, i = 1, 2$, for the oscillator: the isotopes were $B^{10}O$ and $B^{11}O$; $AgCl^{35}$ and $AgCl^{37}$.
  The conclusion then was that $k \approx 1/2$. Nowadays it should be possible to make corresponding experiments with much higher precision!
- Presently one can build sophisticated 1-dimensional harmonic traps for ultra-cold ions, atoms and BE-condensates for which the trap-frequency $\omega(t)$ can be tuned! Determining the ratio $E(t)/\omega(t)$ could - in principle - yield information about k.

5. Problems and generalizations

The previous discussion raises a large number of questions and suggests possible generalizations. Only a few will be mentioned here:

- Most important above all are ideas for appropriate experiments!
- What about fermions?
- How are essential features of the usual quantum field theories affected: locality; Casimir effect; etc. etc.?
- Generalizations: The symplectic group $Sp(2n, \mathbb{R})$ of a $2n$-dimensional symplectic space $\mathcal{S}^{2n}$ has dimension dimension of $2n^2 + n$ and rank n .
  Its maximal compact subgroup is $U(n) = SU(n) \times U(1)$, where $SU(n)$ is simply connected; the rank of $U(n)$ is n; thus, rank [$Sp(2n, \mathbb{R})$] = rank [$U(n)$]; according to a famous theorem by Harish-Chandra this implies the existence of a positive discrete series among the irreducible unitary representations of $Sp(2n, \mathbb{R})$.

Conclusion: HO has an interesting "hidden side" which appears worthwhile to be investigated further!

6. References

[1] Kastrup H A 2007 A new look at the quantum mechanics of the harmonic oscillator *Ann. Physik (Leipzig)* **16** 437-580; [arXiv:quant-ph/0612032]