Research Article
A Quantum Space behind Simple Quantum Mechanics

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In physics, experiments ultimately inform us about what constitutes a good theoretical model of any physical concept: physical
space should be no exception. The best picture of physical space in Newtonian physics is given by the configuration space of a free
particle (or the center of mass of a closed system of particles). This configuration space (as well as phase space) can be constructed as
a representation space for the relativity symmetry. From the corresponding quantum symmetry, we illustrate the construction of a
quantum configuration space, similar to that of quantum phase space, and recover the classical picture as an approximation through
a contraction of the (relativity) symmetry and its representations. The quantum Hilbert space reduces into a sum of one-dimensional
representations for the observable algebra, with the only admissible states given by coherent states and position eigenstates for the
phase and configuration space pictures, respectively. This analysis, founded firmly on known physics, provides a quantum picture
of physical space beyond that of a finite-dimensional manifold and provides a crucial first link for any theoretical model of quantum
space-time at levels beyond simple quantum mechanics. It also suggests looking at quantum physics from a different perspective.

1. Introduction

*Preface for the Special Issue: "Planck-Scale Deformations of
Relativistic Symmetries."* Our group has been working on a
relativity deformation scheme within the Lie group/algebra
framework. This setting has the contraction process as the
reverse of the deformation procedure, and as such it can
be applied to the full physical picture through tracing the
contraction of the relevant representation(s). Both the "quantum Galilean" and the classical Galilean symmetries arise
within the contraction limits of the full quantum relativity
symmetry. This article focuses on the simple quantum to
classical contraction and discusses a quantum model of the
physical space from this perspective.

Quantum mechanics came into physics after a few hundred years of Newtonian mechanics, as the latter failed
to describe physics at the atomic scale and beyond. In
our opinion, however, that quantum revolution has not
been completed. It was easy to accept the mathematical
formulation of the theory but a lot more difficult to adopt
a fundamental change in our basic perspective. It is not a
surprise, then, that even the great physicists who created the
theory kept trying to think and talk about it in terms of
Newtonian concepts, many of which are really not compatible
with quantum mechanics. The famous Bohr-Einstein debate,
in a way, has never ended, such has been the pursuit of a "classical" theory behind quantum mechanics. Statements about
quantum physics being counterintuitive, for example, are
commonly seen and believed by many. We tell our students
that the quantum world is impossible to make sense of and
that quantum mechanics gives only probabilistic predictions.
The thesis presented here is that some, if not all, of those
beliefs may simply be the result of our reluctance to take
the necessary quantum jump in our fundamental perspective,
as well as our indulgence in Newtonian concepts. The latter
is not really any more intuitive than the modified versions
suggested by quantum mechanics, only more familiar. A key
concept, and the main focus here, is that of space or position.
The perspective here is that quantum mechanics should be
looked at as a dynamical theory for physical entities in a

space that is really quantum instead of classical. Position, as a dynamical variable, is not real-valued because a quantum space cannot be modeled on a continuum of points as it can be in classical commutative geometry, at least not a finite-dimensional one [1].

The idea of a quantum geometry is certainly not new; however, here we are talking about a picture of that quantum space completely at the level of simple, textbook, so-called nonrelativistic quantum mechanics. Moreover, we will justify it and illustrate explicitly how the classical Newtonian picture is retrieved in the classical approximation. The formulation presented here is based on relativity symmetries and symmetry contractions.

As said above, and as can hardly be emphasized enough, every precise formulation of any physical concept is really only a model, or part of a model, of nature. Hence, all such concepts need to have their mathematical and physical content reevaluated as theories develop. Quantum mechanics as it is to date inherits, with little critical revision, many Newtonian conceptual notions, while we see that perhaps a lot more fundamental changes are called for, even down to the most basic one: that of physical space and position within it. The key question then is how we are going to look at the latter as a feature of the model instead of just as a background assumption. Instead of thinking about a theory of mechanics as to be constructed on a model of physical space, we need to see how the mechanical theory informs us as to what space is. Only then we can analyze what quantum mechanics says about physical space and how that is related to the more familiar Newtonian picture, which one must be able to retrieve as a limit or an approximation. Here, relativity symmetry, the Galilean symmetry for the case of Newtonian mechanics, is the crucial link. It is as fundamental as the assumption of the structure of the physical space itself. It is the set of admissible reference frame transformations, hence the symmetry of space itself. In fact, both physical space taken as the configuration space and as the phase space, at least for the most basic physical system of a free particle, should be seen as representations of this symmetry. Recall that within the Newtonian theory the center of mass for any closed system (of particles) behaves exactly as a free particle, which illustrates the unbiased structure of physical space. The relativity symmetry is therefore central to a theory of mechanics. Another good illustration of this point is provided by the Poincaré symmetry for Einsteinian special relativistic physics. The problem, though, is that quantum mechanics has not been exactly described as having its own relativity symmetry. We suggest it does, as illustrated below.

2. Quantum Kinematics from a Relativity Symmetry

Let us look at the mathematical formulation first, and leave issues with the conceptual perspective to be discussed below. With justification for the terminology being quite self-evident as the formulation develops, we consider a (partial) relativity symmetry for simple quantum mechanics

as being given by the Lie algebra with the following nonzero commutators:

$$
\begin{aligned}
\left[J_{ij}, J_{hk}\right] &= i\left(\delta_{jk}J_{ih} - \delta_{jh}J_{ik} + \delta_{ih}J_{jk} - \delta_{ik}J_{jh}\right), \\
\left[X_i, P_j\right] &= i\delta_{ij}I, \\
\left[J_{ij}, P_k\right] &= i\left(\delta_{jk}P_i - \delta_{ik}P_j\right), \\
\left[J_{ij}, X_k\right] &= i\left(\delta_{jk}X_i - \delta_{ik}X_j\right),
\end{aligned}
\tag{1}
$$

with indices going from 1 to 3. We could have included the missing generator H with only one nonzero commutator: $[X_i, H] = -iP_i$. The full algebra would then just be the nontrivial $U(1)$ central extension of the algebra for the Galilean group, for which X_i are usually denoted by K_i and interpreted as generators for the Galilean boosts. In fact, that symmetry has been used as the starting point for the quantization of Newtonian particle physics [2]. K_i, as observables, indeed give the (mass times) position, while the central extension is what allows for the Heisenberg commutation relation. The Hamiltonian H has no role to play in the kinematical descriptions here, nor is including it much of a problem. Note that without H we do have a closed subalgebra. We denote by $H_R(3)$ the symmetry generated by this subalgebra, a three-dimensional Heisenberg(-Weyl) symmetry with rotations included. As we will illustrate below, representations of this symmetry describe quantum space, that is, the quantum configuration space, as well as the phase space, for a quantum "particle" with no spin.

We start with the coset space representation obtained by factoring out the SO(3) subgroup. The explicit form of a generic infinitesimal transformation is given by

$$
\begin{pmatrix} dp^i \\ dx^i \\ d\theta \\ 0 \end{pmatrix} = \begin{pmatrix} \omega^i_j & 0 & 0 & \overline{p}^i \\ 0 & \omega^i_j & 0 & \overline{x}^i \\ -\dfrac{1}{2}\overline{x}_j & \dfrac{1}{2}\overline{p}_j & 0 & \overline{\theta} \\ 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} p^j \\ x^j \\ \theta \\ 1 \end{pmatrix}
$$

$$
= \begin{pmatrix} \omega^i_j p^j + \overline{p}^i \\ \omega^i_j x^j + \overline{x}^i \\ \dfrac{1}{2}\left(\overline{p}_j x^j - \overline{x}_j p^j\right) + \overline{\theta} \\ 0 \end{pmatrix},
\tag{2}
$$

where the real parameters ω^i_j, $\overline{p}^i$, $\overline{x}^i$, and $\overline{\theta}$ describe the algebra element $-i((1/2)\omega^{ij}J_{ij} + \overline{p}^i X_i - \overline{x}^i P_i + \overline{\theta}I)$. We will see that the coset space with coordinates (p^i, x^i, θ) is, in a way, the counterpart of the phase space for Newtonian mechanics, written as a coset space. The fact that the representation is not unitary, however, is not what we want for quantum mechanics. Nonetheless, it is closely related to the quantum phase space.

The Heisenberg subalgebra generated by $\{X_i, P_i, I\}$ is an invariant one. Note that by taking out the central charge

generator I one does not even have a subalgebra. We start with the familiar coherent state representation:

$$e^{i\theta}\left|p^i, x^i\right\rangle = U\left(p^i, x^i, \theta\right)|0\rangle, \tag{3}$$

where

$$U\left(p^i, x^i, \theta\right) \equiv e^{i(x_i p^i/2)} e^{i\theta\widehat{I}} e^{-ix^i \widehat{P}_i} e^{ip^i \widehat{X}_i} = e^{i(p^i \widehat{X}_i - x^i \widehat{P}_i + \theta\widehat{I})}, \tag{4}$$

and $|0\rangle \equiv |0,0\rangle$ is a fiducial normalized vector, $\widehat{X}_i$ and $\widehat{P}_i$ are representations of generators X_i and P_i as Hermitian operators on the Hilbert space spanned by all of the six-parameter set of vectors $|p^i, x^i\rangle$, and $\widehat{I}$ is the identity operator representing the central generator I. Here, (p^i, x^i, θ) corresponds to a generic element of the (Heisenberg-Weyl) subgroup as

$$W\left(p^i, x^i, \theta\right) = \exp i\left(p^i X_i - x^i P_i + \theta I\right), \tag{5}$$

with

$$W\left(p'^i, x'^i, \theta'\right) W\left(p^i, x^i, \theta\right)$$
$$= W\left(p'^i + p^i, x'^i + x^i, \theta' + \theta - \frac{x'_i p^i - p'_i x^i}{2}\right), \tag{6}$$

where $x'_i p^i - p'_i x^i$ is the classical mechanical symplectic form [3, 4]. This is an infinite-dimensional unitary representation [3, 4]. This Hilbert space, or rather its projective counterpart, is the phase space for quantum mechanics. The projective Hilbert space is, in fact, an infinite-dimensional symplectic manifold. Note that p^i and x^i, as labels of the coherent states, correspond to expectation values, but not eigenvalues of the $\widehat{P}_i$ and $\widehat{X}_i$ observables. The coherent states give an overcomplete basis, with overlap given by

$$\left\langle p'^i, x'^i \mid p^i, x^i\right\rangle = \exp\left[i\frac{x'_i p^i - p'_i x^i}{2}\right]$$
$$\cdot \exp\left[-\frac{\left(x'^i - x^i\right)\left(x'_i - x_i\right) + \left(p'^i - p^i\right)\left(p'_i - p_i\right)}{4}\right] \tag{7}$$
$$\xrightarrow{p' \to p, \, x' \to x} 1.$$

We also have

$$\left\langle p'^i, x'^i \mid \widehat{X}_i \mid p^i, x^i\right\rangle$$
$$= \frac{\left(x'_i + x_i\right) - i\left(p'_i - p_i\right)}{2} \left\langle p'^i, x'^i \mid p^i, x^i\right\rangle,$$
$$\left\langle p'^i, x'^i \mid \widehat{P}_i \mid p^i, x^i\right\rangle \tag{8}$$
$$= \frac{\left(p'_i + p_i\right) + i\left(x'_i - x_i\right)}{2} \left\langle p'^i, x'^i \mid p^i, x^i\right\rangle,$$

which are important results for our analysis below.

The above coset space is modeled on the Heisenberg-Weyl subgroup. Explicitly,

$$\begin{pmatrix} 1 & 0 & 0 & p^i \\ 0 & 1 & 0 & x^i \\ -\frac{1}{2}x_i & \frac{1}{2}p_i & 1 & \theta \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} R^i_j & 0 & 0 & 0 \\ 0 & R^i_j & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$
$$= \begin{pmatrix} R^i_j & 0 & 0 & p^i \\ 0 & R^i_j & 0 & x^i \\ -\frac{1}{2}x_i R^i_j & \frac{1}{2}p_i R^i_j & 1 & \theta \\ 0 & 0 & 0 & 1 \end{pmatrix}. \tag{9}$$

For fixed (p^i, x^i, θ), the above gives a generic element of the coset with R^i_j taken as elements of the SO(3) subgroup. The fiducial vector $|0,0\rangle$ corresponds to $(0,0,0,1)^t$ which is taken by any such coset onto $(p^i, x^i, \theta, 1)^t$ corresponding to $e^{i\theta}|p^i, x^i\rangle$. This illustrates explicitly the $U(p^i, x^i, \theta)$ action of the Heisenberg-Weyl subgroup, and in fact also its extension to the full group on the Hilbert space, as depicted on the coset space. Each transformation of the unitary representation sends a coherent state to another coherent state and hence its action can be depicted in the coset space with elements of the latter mapped to the coherent states.

As inspired by the Galilean/Newtonian case, we can take a different coset space representation by factoring out an ISO(3) subgroup generated by X_i and J_{ij}. The infinitesimal, or algebra, representation is then given as

$$\begin{pmatrix} dx^i \\ d\theta \\ 0 \end{pmatrix} = \begin{pmatrix} \omega^i_j & 0 & \bar{x}^i \\ \bar{p}_j & 0 & \bar{\theta} \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} x^j \\ \theta \\ 1 \end{pmatrix} = \begin{pmatrix} \omega^i_j x^j + \bar{x}^i \\ \bar{p}_j x^j + \bar{\theta} \\ 0 \end{pmatrix}. \tag{10}$$

The (x^i, θ) space is the quantum counterpart for the coset space that describes Newtonian (configuration) space. We can also construct a unitary representation whose relation to the coset is the same as the above for the phase space. The P_i and I generators give group elements matching to the points in the coset space and also generate an invariant subalgebra, which is, however, trivial. This can also be seen in the corresponding group structure; that is, by defining

$$W'\left(x^i, \theta\right) = \exp i\left(-x^i P_i + \theta I\right), \tag{11}$$

we obtain

$$W'\left(x'^i, \theta'\right) W'\left(x^i, \theta\right) = W'\left(x'^i + x^i, \theta' + \theta\right). \tag{12}$$

We have a picture here very similar to the coherent state representation above with basis vectors labeled by the coset coordinates x^i such that

$$e^{i\theta}\left|x^i\right\rangle = U'\left(x^i, \theta\right)|0\rangle, \tag{13}$$

where

$$U'\left(x^i, \theta\right) \equiv e^{i\theta\widehat{I}} e^{-ix^i \widehat{P}_i}, \tag{14}$$

$|0\rangle$ is the fiducial normalized vector, and $\hat{P}_i$ and $\hat{I}$ Hermitian operators on the Hilbert space spanned by all of the three-parameter set of $|x^i\rangle$ vectors. Much the same as before, $\hat{P}_i$ generates translations in x^i, while $\hat{I}$ is the identity operator effectively generating only a phase rotation of a vector on the Hilbert space spanned by all $|x^i\rangle$. Following the coset action, we can see again the action of the unitary representation for the full group of $H_R(3)$. In particular, we see that

$$e^{ip^i \hat{X}_i} e^{i\theta} \left| x^i \right\rangle = e^{i(p^i x_i + \theta)} \left| x^i \right\rangle, \tag{15}$$

thus illustrating that the vectors $|x^i\rangle$ are really the usual position eigenstates. The unitary representation constructed here from the coset space describing the quantum analog of the free particle configuration space, or physical space, is the configuration analog along the lines of the phase space construction. It is however equivalent to that of the latter as a Hilbert space.

3. Newtonian Limit from a Symmetry Contraction

A naive way of interpreting the coset representations given above as quantum analogs of the classical (configuration) space and phase space is suggested by simply replacing the generator I with zero and dropping the variable θ from consideration. A symmetry contraction, however, gives a solid mathematical way to formulate the classical theory as an approximation to the quantum theory. Consider the contraction [5] of the above Lie algebra, given by the $k \to \infty$ limit under the rescaled generators $X_i^c = (1/k)X_i$ and $P_i^c = (1/k)P_i$. The J-P^c and J-X^c commutators are the same as those of J-P and J-X; however, we have

$$\left[X_i^c, P_j^c \right] = \frac{i}{k^2} \delta_{ij} I \longrightarrow 0, \tag{16}$$

giving the commuting classical position and momentum. The contracted Lie algebra gives, with the H generator included, the Galilean relativity symmetry with a trivial central extension, in which I is decoupled. The symmetry contraction applied to the above representations also gives exactly the classical phase space, as well as Newtonian space, as we will see.

The algebra element should first be written in terms of the rescaled generators as $-i((1/2)\omega^{ij} J_{ij} + \bar{p}_c^i X_i^c - \bar{x}_c^i P_i^c + \bar{\theta} I)$. It is important to note that the parameters $\bar{p}_c^i = k\bar{p}^i$ and $\bar{x}_c^i = k\bar{x}^i$ are to be taken as finite even in the $k \to \infty$ limit. They are then parameters of the contracted algebra. The coset space of (p^i, x^i, θ) should be described in terms of (p_c^i, x_c^i, θ) with the representation rewritten as

$$\begin{pmatrix} dp_c^i \\ dx_c^i \\ d\theta \\ 0 \end{pmatrix}$$

$$= \begin{pmatrix} \omega_j^i & 0 & 0 & \bar{p}_c^i \\ 0 & \omega_j^i & 0 & \bar{x}_c^i \\ -\dfrac{1}{2k^2}\bar{x}_{cj} & \dfrac{1}{2k^2}\bar{P}_{cj} & 0 & \bar{\theta} \\ 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} p_c^j \\ x_c^j \\ \theta \\ 1 \end{pmatrix}$$

$$= \begin{pmatrix} \omega_j^i p_c^j + \bar{p}_c^i \\ \omega_j^i x_c^j + \bar{x}_c^i \\ \dfrac{1}{2k^2}\left(\bar{P}_{cj} x_c^j - \bar{x}_{cj} p_c^j \right) + \bar{\theta} \\ 0 \end{pmatrix}. \tag{17}$$

This gives only $d\theta = \bar{\theta}$ in the limit; hence, θ becomes an absolute parameter not affected by the transformations, except its own translation generated by I. Note that dp_c^i and dx_c^i are also $\bar{\theta}$-independent. This reflects exactly what we mean when saying that I decouples. The $\bar{\theta}$ parameter has nothing to do with anything else any more. It may as well simply be dropped from consideration. The (p_c^i, x_c^i) space is exactly the classical phase space. We have a parallel result for the other coset; explicitly

$$\begin{pmatrix} dx_c^i \\ d\theta \\ 0 \end{pmatrix} = \begin{pmatrix} \omega_j^i & 0 & \bar{x}_c^i \\ \dfrac{1}{k^2}\bar{P}_{cj} & 0 & \bar{\theta} \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} x_c^j \\ \theta \\ 1 \end{pmatrix}$$

$$= \begin{pmatrix} \omega_j^i x_c^j + \bar{x}_c^i \\ \dfrac{1}{k^2}\bar{P}_{cj} x_c^j + \bar{\theta} \\ 0 \end{pmatrix}, \tag{18}$$

giving only $dx_c^i = \omega_j^i x_c^j + \bar{x}_c^i$ and $d\theta = \bar{\theta}$.

We can also apply the symmetry contraction to the unitary representations given on the above Hilbert space(s). We first look at the latter as a representation of the algebra of observables, based on $\hat{X}_i^c$ and $\hat{P}_i^c$ (and $\hat{I}$) at finite k. The set of $|p^i, x^i\rangle$ states should be relabeled as $|\tilde{p}_i^c, \tilde{x}_i^c\rangle$, with $\tilde{p}_i^c$ and $\tilde{x}_i^c$ characterizing the expectation values of $\hat{X}_i^c$ and $\hat{P}_i^c$. Note that $\tilde{p}_i^c$ and $\tilde{x}_i^c$ do not directly correspond to p_c^i and x_c^i above. From (7) and (8) then, we have

$$\left\langle \tilde{p}_i^{\prime c}, \tilde{x}_i^{\prime c} \right| \hat{X}_i^c \left| \tilde{p}_i^c, \tilde{x}_i^c \right\rangle$$

$$= \frac{\left(\tilde{x}_i^{\prime c} + \tilde{x}_i^c \right) - i \left(\tilde{p}_i^{\prime c} - \tilde{p}_i^c \right)}{2} \left\langle \tilde{p}_i^{\prime c}, \tilde{x}_i^{\prime c} \mid \tilde{p}_i^c, \tilde{x}_i^c \right\rangle,$$

$$\left\langle \tilde{p}_i^{\prime c}, \tilde{x}_i^c \right| \hat{P}_i^c \left| \tilde{p}_i^c, \tilde{x}_i^c \right\rangle \tag{19}$$

$$= \frac{\left(\tilde{p}_i^{\prime c} + \tilde{p}_i^c \right) + i \left(\tilde{x}_i^{\prime c} - \tilde{x}_i^c \right)}{2} \left\langle \tilde{p}_i^{\prime c}, \tilde{x}_i^{\prime c} \mid \tilde{p}_i^c, \tilde{x}_i^c \right\rangle,$$

where the state overlap has the second, real, and negative exponential factor, written in terms of $\tilde{p}_i^{\prime c}$, $\tilde{x}_i^{\prime c}$, $\tilde{p}_i^c$, and $\tilde{x}_i^c$,

proportional to k^2. This therefore gives a vanishing result in the contraction limit, so long as the coherent states are not the same. Thus, we can see that $\widehat{P}^c_i$ and $\widehat{X}^c_i$ are diagonal on $|\widetilde{p}^c_i, \widetilde{x}^c_i\rangle$ with $\widetilde{p}^c_i$ and $\widetilde{x}^c_i$ as eigenvalues. The Hilbert space, as a representation for the Heisenberg-Weyl symmetry and that of the algebra of observables described as functions (or polynomials) of $\widehat{P}^c_i$ and $\widehat{X}^c_i$, is therefore reducible. It reduces to a direct sum of one-dimensional representations of the ray spaces of each $|\widetilde{p}^c_i, \widetilde{x}^c_i\rangle$. That is to say, the only admissible states are the exact coherent states, and not any linear combinations. These are really the classical states, though we are not used to describing classical mechanics in the Hilbert space language. Actually, this kind of description has been available for a long time [6]. The latter may be particularly useful in establishing the more involved dynamical picture of what we discussed here. Note that $\widehat{H}^c$ and $\widehat{J}^c_{ij}$ as classical observables would also be diagonal on $|\widetilde{p}^c_i, \widetilde{x}^c_i\rangle$. However, in getting the contracted symmetry algebra, the generators J_{ij} (and H) are not to be rescaled by k and maintain all their nonzero commutators. The transformations they generate still take one state to another, as they should in the classical picture. As they always take a coherent state to a coherent state anyway, they do not support linear combinations either. The set of coherent states essentially gives just the classical coset/phase space. Readers may find interest in an explicit expression of the generator for dynamical/time evolution on the Hilbert space of $|\widetilde{p}^c_i, \widetilde{x}^c_i\rangle$ states in terms of the classical Hamiltonian [6].

The story for the contraction of the Hilbert space as the quantum configuration space is somewhat less obvious. The basis vectors are eigenvectors of transformations in the W' group. But we know that this space serves as a representation for the Heisenberg algebra, and hence the algebra of observables, on which the action of the $\widehat{P}_i$ operators for the momentum observables have eigenstates being linear combinations of the basis $|x^i\rangle$ states. It is exactly for such considerations of momentum-dependent observables that one needs to go beyond the coset to the full Hilbert space. At the contraction limit, however, the Heisenberg algebra is trivialized. $\widehat{P}^c_i$ commuting with $\widehat{X}^c_i$ means that they have to share the same eigenvectors $|\widetilde{x}^c_i\rangle$, now labeled by the $\widehat{X}^c_i$ eigenvalues. Again the Hilbert space as a representation for the algebra of observables reduces and only the latter vectors are relevant, not linear combinations or even the phases. The result is the Newtonian three-dimensional space.

Readers should have realized that our rescaling parameter k for the implementation of the symmetry contraction corresponds to $1/\sqrt{\hbar}$, so that the contraction is really the $\hbar \to 0$ limit. The latter of course corresponds to taking the classical approximation. In fact, the quantum symmetry algebra is quite commonly written with an $\hbar$ within each commutator. Our version first refers to the natural quantum units of $\hbar = 1$, in which the J_{ij} are dimensionless. The contraction limit is obtained as described, which is the same as taking *only* the $\hbar$ in the $\widehat{X}^c$-$\widehat{P}^c$ commutator to zero. Otherwise, all commutators would be killed. If one is taking the algebra as describing relations among the classical observables, this is great; however, considering it as the relativity symmetry

algebra, this is a disaster. The algebra of observables is really not the one for the relativity symmetry, but rather the algebra of functions of $\widehat{X}^c$ and $\widehat{P}^c$, and as such a specific representation of the relativity algebra. Nevertheless, we still need to reintroduce the nonzero $\hbar$ in the rest of the relativity algebra to have J_{ij} (and H) being described in the classical units, if we want to match them to the observables $\widehat{J}^c_{ij}$ and $\widehat{H}^c$. Generators of the relativity symmetry are *not* to be identified with the operators representing the observables in the Hilbert space picture of classical mechanics [6, 7]. The contraction is not concerned with the units. In the classical picture after the contraction, it is no longer unnatural to have units for position and momentum chosen as independent, hence their product having a nontrivial unit. That unit would have fundamental significance in telling when the classical theory is a good approximation to the better quantum theory. The contraction is a mathematical procedure for getting the approximate theory characterized by a small scale [5]. The classical scale is the one which is small compared to the contraction parameter k, hence with smallness described by the $\hbar$ value. $\hbar$ serves as the fundamental unit with which we reexpress physical quantities.

The coset space pictures at least illustrate well that quantum space is different from Newtonian space in much the same way as the quantum phase space is different from the classical one. The analysis of the (equivalent) infinite-dimensional unitary representations and their reductions upon the symmetry contraction gives the full, solid results.

4. Concluding Remarks

We physicists should not endow a vague common sense concept like physical space with any particular mathematical model as a given. We are supposed to learn from experiments what constitutes a good/correct theoretical/mathematical model of any physical concept, and physical space should not be an exception. We have by now roughly a century of experimental results saying that the classical/Newtonian model of physical space does not serve this purpose so well, especially not as the configuration space of quantum particle motion. We should not be reluctant to modify it. What could the notion of (classical/Newtonian) space, described in any inertial frame, be other than the configuration space of (free) particle motion under arbitrary initial conditions? What kind of coordinates would be more natural for space besides the q^i variables acting as the angle coordinates, with p^i as action coordinates, for free particle motion as described on the phase space? Looking at physical space as it can possibly be understood from practical physics, the space of all q^i values as the configuration variables is essentially the only picture we should have, so long as nonrelativistic "particle" mechanics, classical or quantum, are concerned.

It is known that the projective Hilbert space, as the true quantum phase space, is an infinite-dimensional symplectic manifold. An expansion of a state in terms of an orthonormal basis in the form $|\phi\rangle = \sum (q_n + ip_n)|n\rangle$ gives q_n and p_n as a set of real homogeneous coordinates of the projective space on which the Schrödinger equation is equivalent to the set

of Hamilton equations of motion for q_n and p_n as pairs of configuration and momentum variables with Hamiltonian function $H(p_n, q_n) = (2/\hbar)\langle\phi|\widehat{H}|\phi\rangle$. It suggests thinking about a Lagrangian submanifold, like the space of q_n, as the quantum configuration space. One can also take the real and imaginary part of the values of a wavefunction at the various points (of the classical space model) as a similar set of symplectic coordinates. However, our perspective of the quantum relativity symmetry has a complex phase rotation of the state generated by the X-P commutator which mixes the configuration and momentum coordinates. Hence, unlike the classical case, the position/configuration space and the momentum space are no longer irreducible components of the relativity symmetry. The quantum phase space is an irreducible representation, though the classical one is reducible. We get a quantum (position) space model that is equivalent to the phase space model. The projective Hilbert space is also to be a Kählër manifold [8, 9] and hence has a natural metric, though the latter notion may not be feasible on a generic symplectic manifold.

The analysis in this article is simple and straightforward, with results hardly totally new or unexpected for the phase space picture. What is new and important is the way they are pieced together consistently to illustrate the basic perspective and that the application of the latter suggests looking at familiar notions in quantum physics in a very different way. In particular, it gives a picture of the not quite discussed notion of the configuration space in quantum mechanics as a model of physical space beyond the usual one, which is nothing but the Newtonian model. This is the first step in justifying a new perspective regarding (quantum) physical space, the adoption of which may also help clarify some issues in quantum physics and beyond.

Symmetry is the single most important organizing principle in the theory of modern physics. What we performed in the above analysis is an attempt to see how the fundamental symmetry of something like free particle motion informs us about the nature of the phase space, configuration space, and hence our physical space. These types of symmetries are relativity symmetries. Different fundamental theories have different relativity symmetries, which correspond to different pictures of physical space and time, just like the fact that Einsteinian (special) relativity gives Minkowski space-time. In fact, the mathematical relation of the latter to the Newtonian one can be described exactly using the corresponding coset space picture as representations of the relativity symmetries through the symmetry contraction with c as the parameter [10] (see also [11] for a detailed pedagogical introduction). The above symmetry contraction is really the necessary, proper, and quite subtle, mathematical way to describe the Newtonian limit as an approximation to the better Einsteinian or quantum theory. We give the analogous mathematical description of the quantum to classical case here and use it to illustrate a picture of quantum space. In this case $\hbar$, or rather $1/\sqrt{\hbar}$, takes the place of c. Neither $\hbar$ nor $1/c$ is really zero: nonzero values of both are key fundamental constants. The symmetry contraction limit provides the necessary subtle approach to successfully describe the Newtonian approximation.

Given the basic perspective of looking for a picture of quantum space as described by the symmetry structure of the theory instead of the corresponding classical notion, the considerations and analysis presented here are necessarily simple and somewhat naive. As such, it is definitely not the "final" answer in the general setting of quantum physics. Invariance under Einsteinian special relativity, for example, has not been incorporated. Our key point of interest here is exactly in showing how this basic perspective provides us with a notion of quantum space(time) beyond classical space(time), yet giving rise to the latter when the proper limit is taken, even for the simplest, ordinary and conventional theory of quantum mechanics without any extra assumptions. Hence, we are not interested here in putting in extra notions beyond the bare minimum, no matter how natural one may argue for them to have a part in quantum physics. In fact, the basic perspective, we believe, can take us much beyond the simple results in this article. Our study on quantum space-time, given by the work presented here, is therefore necessarily incomplete. Moreover, our discussion has been entirely restricted to kinematics; analysis of the full dynamical picture will be given in a separate publication [12]. There are two main reasons for separation of the two. Conceptually, as seen in the Newtonian example, the constructions of the notion of particle configuration space and phase space, as well as that of physical space, require only kinematical considerations. Besides this, as to be reported in [12], the dynamical picture should firstly be considered as one on the algebra of observables rather than the configuration space or phase space. Otherwise, the Schrödinger equation applied to the set of coherent states is known to be equivalent to the classical dynamics on the states taken as classical ones. That is all that is relevant so long as the dynamics of the pure states of the quantum Hilbert space is concerned. A further source of incompleteness lies in the fact that field theory issues are not discussed here either. Note that practical field theories are either quantum or at least (Einstein) relativistic. It goes without saying that we have the big task at hand of extending this framework to the fully deformed/stabilized fundamental quantum relativity. We hope that the simple analysis here can help make our basic perspective more accessible to general readers, beyond those who have more experience with space-time physics and the foundations of quantum mechanics, as well as new developments in these areas.

Our group has worked on a notion of a quantum relativity for deep microscopic quantum space-time [13], from much the same theoretical perspective as that which lies behind the current analysis. The basic starting point there is the old idea of relativity deformations [14–18] to which contraction of the relativity symmetries is the reverse process, so long as one stays within the Lie group/algebra framework. While we have presented some picture of the physics from a sequence of contractions [10], we are currently working on the details of the descriptions of an alternative contraction scheme, via an approach that naturally incorporates symmetries like $H_R(3)$ and $\widetilde{G}(3)$. The results here are really part of that work. The "final" symmetry is considered to have noncommuting X_i and P_i [13], to which no real number picture of space-time is expected to work. Within the domain of simple quantum

mechanics investigated here, the physical space picture still looks like a real manifold, albeit of infinite dimension. The results here may also serve as the crucial first link from the bottom-up to any theoretical model of space-time beyond the level of simple quantum mechanics.

A fair question is if it is too conservative to stay within the Lie group/algebra framework. While we sure encourage other alternative bold approaches within the deformed relativity picture, what we want to emphasize is that our chosen framework is a very powerful one. The $H_R(3)$, or $\widetilde{G}(3)$, group obviously corresponds to an observable algebra which is quantum/noncommutative. In fact, the latter is more or less just the group C^* algebra [12], which is a completion of the group algebra [19]. The quantum Hilbert space is naturally a cyclic irreducible representation of the algebra corresponding to its space of pure states [12]. The theory of noncommutative geometry [20] says any (noncommutative) algebra has a matching topological/geometric space which we see as essentially the projective Hilbert space in our case. It is then indeed quite plausible that the picture of relativity symmetries as Lie groups is a good enough starting point to formulate the noncommutative geometries of quantum space-time. Again, the representation contraction picture gives the setting to build kinematic and dynamic models which can be systematically traced back to those of well-known physics.

To look at the dynamical picture at the quantum level under a formulation completely in line with our approach here is mathematically involved. The Weyl-Wigner-Groenewold-Moyal formalism has to first be rewritten with the coherent state basis or wavefunctions $\langle p^i, x^i \mid \phi \rangle$ as the starting point and fully matches to a representation picture of the group C^* algebra, though restriction of the latter to that of the Heisenberg-Weyl subgroup is good enough. Thanks to the semidirect product structure, a representation of the subgroup and its C^* algebra serves as a representation the full group (C^* algebra) in which elements beyond the subgroup act as inner automorphisms [12]. The observable algebra is the representation of the group C^* algebra. Naively summarized, so long as the contraction to the classical limit is concerned, it is just the reverse of the standard deformation quantization in the $\hbar \rightarrow 0$ limit. A generator of the full relativity symmetry group G_s is represented by a function $G_s(\widehat{P}_i, \widehat{X}_i)$ with $G_s(\widehat{P}_i, \widehat{X}_i)\star = G_s(p_i\star, x_i\star)$, as an operator acting on the Hilbert space of wavefunctions and the observable algebra itself, in which $\star$ is the standard Moyal star product. The latter action is the left regular representation of the algebra on itself, and there is a corresponding right action. However, the corresponding automorphisms of the observable algebra which match with the unitary transformations on the Hilbert space are really generated by the difference of the left and the right action. This can be written as $\{G_s(p_i, x_i), \cdot\}_\star$, that is, in terms of the Moyal bracket. In the $\hbar \rightarrow 0$ limit, formulated here as the $k \rightarrow \infty$ limit as described above, the $G_s(p_i\star, x_i\star)$ action reduces to the classical multiplicative action of $G_s(p_i, x_i)$, as all classical observables commute. The generators for the automorphisms as symmetry transformations in the Heisenberg picture, however, reduce to the

classical Liouville operator, hence giving the Poisson algebra structure. Time evolution is just the symmetry transformation generated by the Hamiltonian operator/function. Hence, one retrieves classical dynamics. The separate notion of a function as a multiplicative operator and its corresponding Liouville operator have been studied in the Koopman-von Neumann formalism [7], which is really a Hilbert space picture for the mixed states. All of this can be retrieved as the contraction limit [12], except the naive Schrödinger picture of dynamics. We have seen above that the quantum Hilbert space of pure states reduces to essentially that of the classical phase space. In the Hilbert space picture, the classical pure states are essentially disconnected vectors/rays. It is then no surprise at all that one does not have a Schrödinger dynamics for the classical pure states as the contraction limit. Classical dynamics is really one of the Heisenberg picture. For details, readers are referred to [12].

Somewhat after the posting of the first version of this paper, another study of the notion of model for the physical space behind quantum mechanics [21] came up. The approach there has nothing to do with the theme of relativity symmetry contraction/deformation here. Nevertheless, it may be in the interest of the readers for us to give a comparison between their approach and ours in this paper and beyond. As stated with emphasis in their introductory section, [21] is focused on *"quantum systems with a built-in length scale."* We sure share the idea that some fundamental scale(s) being built into the basic formulation would indeed be an important part of any theory of deep microscopic quantum space-time. We have the relativity symmetries for simple quantum mechanics and classical Newtonian mechanics as retrieved from the proper (contraction) limits of the such a quantum relativity symmetry [13]. The limits provide the setting within which the fundamental scales can be neglected. No matter how natural the idea of having fundamental quantum scales may sound to many of us, saying that it is a part of the ordinary (formulation of) quantum mechanics may really be pushing it too much. Our analysis here is particularly interested in developing a notion of quantum space without putting such kinds of extra theoretical structure into ordinary quantum mechanics. Reference [21] illustrates how their notion of modular space-time is arguably a natural part of quantum mechanics with a fundamental (length) scale, which is certainly of great interest. It is, however, beyond the setting of ordinary quantum mechanics. There is however an important difference between our perspectives on quantum space-time in general. The *"point of view that any choice of a maximally commutative ∗-subalgebra of the Heisenberg algebra can be thought of as defining our concept of quantum Euclidean space"* [21] is to be contrasted against our point of view that the full quantum noncommutative algebra of observables can be thought of as defining a concept of quantum space(time), which is generally noncommutative [20]. As discussed in [13], fundamental scales are supposed to characterize noncommutativity of the classical notion of space-time coordinates as well as momentum coordinates. This perspective is the key that gives, even in the current (limited) setting without fundamental scales, a notion of quantum space beyond the classical one. The notion of "quantum Euclidean space" in

[21] will likely be retrievable from proper limits of our idea of noncommutative quantum space-time from the full relativity symmetry with fundamental scales incorporated, which is still to be constructed (It is interesting to note the following: the fundamental quantum relativity symmetry of [13] can be written as

$$
\begin{aligned}
\left[X_\mu, X_\nu\right] &= iM_{\mu\nu}, \\
\left[P_\mu, P_\nu\right] &= -iM_{\mu\nu}, \\
\left[X_\mu, P_\nu\right] &= i\eta_{\mu\nu}F, \\
\left[X_\mu, F\right] &= -iP_\mu, \\
\left[P_\mu, F\right] &= -iX_\mu,
\end{aligned}
\tag{20}
$$

$\eta_{\mu\nu} = diag\ \{-1, 1, 1, 1\}$, with all fundamental scales taken as unity. On an eigenspace of $M_{\mu\nu}$ and F of integral eigenvalues, as a representation space, the set of $e^{2\pi iX_\mu}$ and e^{iP_μ} behaves like the commuting set of U and V of the "Heisenberg group" discussed in the modular picture of [17].).

We have not touched on the measurement problem so far. A couple of comments on this issue are in order. To the extent that we do not have any dynamical theory to describe a measurement process, leaving such issues on the sideline is justified (the theory of decoherence should be noted on the subject matter; see, for example, [22–25]). We sure do not see the quantum space picture here as, in any sense, "final," and we do not aim at describing measurements. We want to note, however, that most if not all, discussions about measurements are really about classical measurements, as Bohr did a good job in elaborating. They are about extracting pieces of classical information, as represented by numbers, from a quantum system. It is not surprising that the nature of the information/physical attributes of the system being quantum does not fit in well with such measurements. If the quantum position is to be described by infinitely many real numbers, our decision to "get" one or three real numbers reads to the so-called probabilistic results. Only statistics from many such measurements can give a better approximation of those infinite coordinate values. Actually, we essentially only obtain values of any measurements by comparison. For example, position or distance between two positions is measured by comparing it to a length standard, admitting some uncertainty. The nature of that "ratio" being a piece of classical information, a real number, is never more than a mathematical model or an assumption. With development of quantum information theory, physicists in the future may be proficient in handling quantum information and true quantum measurements may then be the rule, rather than the exception. We would like to advance the notion of measurements as possibly extracting quantum, non-real-number, information from a system which describes some of its properties. Even the idea of a "definite" position in physical quantum space may plausibly be useful for that kind of position information. However, we are certainly not defending the classical notion of being able to extract full information about a dynamical state without disturbing it at all. It is not our intent either to take a stand in that kind of philosophical debate about realism here, which we see as beyond, and not at all necessary to, the study of physics.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

Acknowledgments

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