Cauchy horizon stability and mass inflation with a cosmological constant

João L. Costa[1,2]**, Pedro M. Girão**[2]**, José Natário**[2] **and Jorge Drumond Silva**[2]

(1) ISCTE - Instituto Universitário de Lisboa, Portugal, (2) Center for Mathematical Analysis, Geometry and Dynamical Systems, Instituto Superior Técnico, Universidade de Lisboa, Portugal.

E-mail: `jlca@iscte.pt, pgirao@math.ist.utl.pt, jnatar@math.ist.utl.pt, jsilva@math.ist.utl.pt`

Abstract. Motivated by the strong cosmic censorship conjecture, we consider the Einstein-Maxwell-scalar field system with a cosmological constant Λ (of any sign), under spherical symmetry, for characteristic initial conditions, with outgoing data prescribed by a (complete) subextremal Reissner-Nordström black hole event horizon. We study the structure of the future maximal (globally hyperbolic) development, analyze the mass inflation scenarios, identifying, in particular, large choices of parameters for which the Hawking mass remains bounded, and study the existence of regular extensions. We also discuss why our results, although valid for all signs of Λ, only provide evidence for the failure of strong cosmic censorship in the case of a positive cosmological constant.

1. Introduction
The existence and stability of Cauchy horizons is intimately related to the question of global uniqueness for the Einstein equations and, in particular, to the celebrated strong cosmic censorship conjecture. To study this, we consider the following problem:

Problem. Given spherically symmetric characteristic initial data for the Einstein-Maxwell-scalar field system with a cosmological constant Λ, with the data on the outgoing part of the null initial hypersurface prescribed by a (complete) subextremal Reissner-Nordström black hole event horizon, with non-vanishing charge, and the remaining data otherwise free, verify if the corresponding maximal globally hyperbolic development is future inextendible as a suitably regular Lorentzian manifold.

This is a direct extension of the framework of [1] with the introduction of a cosmological constant in the field equations.

Here, we will review some of the results obtained in a recent series of papers [2,3,4] dedicated to the above problem, where we generalize the results of Dafermos concerning the stability of the radius function, for a cosmological constant of any sign. This has the remarkable consequence of allowing continuous extensions of the metric across the Cauchy horizon. We analyze the mass inflation scenarios and, in particular, identify large choices of parameters for which the Hawking mass remains bounded. Then we carefully unveil the consequences of this last fact concerning the existence of regular extensions of the metric beyond the Cauchy horizon. More precisely, we obtain continuous extensions of the metric across the Cauchy horizon with

square integrable Christoffel symbols and, under slightly stronger conditions, we construct (non-isometric) extensions which are classical solutions of the Einstein equations. To the best of our knowledge, these are the first results of this kind.

2. Framework

The Einstein-Maxwell-scalar field system with a cosmological constant Λ is given by

$$R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} + \Lambda g_{\mu\nu} = 2T_{\mu\nu}, \tag{1}$$

with $T_{\mu\nu}$ obtained from the sum of the stress energy tensor of a massless scalar field ϕ, satisfying the wave equation, and an electromagnetic field F solving the (source free) Maxwell equations.

Imposing the following (spherically symmetric) ansatz

$$g = -\Omega^2(u,v)du\,dv + r^2(u,v)\left(d\theta^2 + \sin^2\theta\,d\varphi^2\right), \tag{2}$$

with u the ingoing and v the outgoing null coordinates, respectively, the electromagnetic field completely decouples:

$$F = -\frac{e\Omega^2(u,v)}{2r^2(u,v)}du \wedge dv. \tag{3}$$

Introducing the *renormalized Hawking mass* ϖ through

$$1 - \frac{2\varpi}{r} + \frac{e^2}{r^2} - \frac{\Lambda}{3}r^2 = -\frac{4\partial_u r\partial_v r}{\Omega^2}, \tag{4}$$

the field equations reduce to

$$\partial_u\partial_v\phi = -\frac{\partial_u r\,\partial_v\phi}{r} - \frac{\partial_v r\,\partial_u\phi}{r}, \tag{5}$$

$$\partial_u\partial_v r = \partial_u r\,\partial_v r\,\frac{\frac{2\varpi}{r^2} - \frac{2e^2}{r^3} - \frac{2\Lambda}{3}r}{1 - \frac{2\varpi}{r} + \frac{e^2}{r^2} - \frac{\Lambda}{3}r^2}, \tag{6}$$

$$\partial_u\varpi = \left(1 - \frac{2\varpi}{r} + \frac{e^2}{r^2} - \frac{\Lambda}{3}r^2\right)\frac{(r\partial_u\phi)^2}{2\partial_u r}, \tag{7}$$

$$\partial_v\varpi = \left(1 - \frac{2\varpi}{r} + \frac{e^2}{r^2} - \frac{\Lambda}{3}r^2\right)\frac{(r\partial_v\phi)^2}{2\partial_v r}. \tag{8}$$

These are a wave equation for the scalar field (5), a wave equation for the radial function (6), and equations prescribing the gradient of the mass (7) and (8); from these we can easily derive the remaining unknowns: the metric component Ω from (4) and the Faraday tensor F from (3).

3. The Reissner-Nordström solution and strong cosmic censorship

The first reason to consider the previous Einstein-matter system is that it admits the full Reissner-Nordström (RN) family of solutions, where we include all the anti-de Sitter ($\Lambda < 0$) the asymptotically flat ($\Lambda = 0$) and the de Sitter ($\Lambda > 0$) subfamilies, as particular solutions. This family is obtained by setting $\phi \equiv 0$. Then, the usual mass and charge parameters are given by $M = \varpi$ and $Q = 4\pi e$.

In this context, it is natural to try to derive the black hole solutions of this family from a characteristic initial value problem for the system (5)-(8). To do that in a unified way for all signs of Λ, we can prescribe their corresponding characteristic data on the event horizon: the future maximal globally hyperbolic developments obtained will then coincide with the shaded region in figure 1. But such developments can be continued to the future in a highly non unique
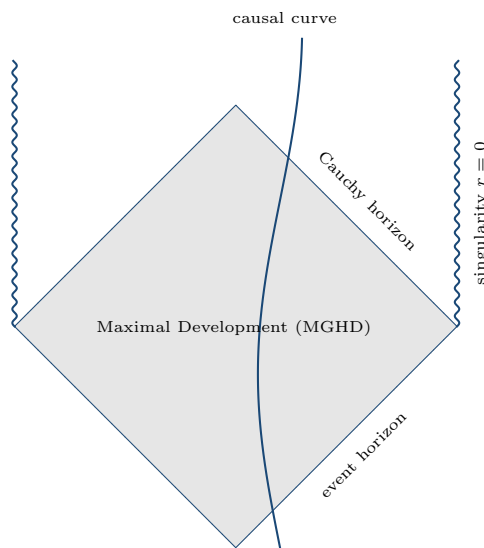
Figure 1. Part of the future of the event horizon of RN black holes.

way, with the metric extending smoothly. This well known fact is quite disturbing since what happens in such extensions is not determined, in any way, by the prescribed initial data. In such extensions, a Cauchy horizon arises as the boundary of the maximal development, signaling the failure of global uniqueness.

This puts into question the deterministic character of general relativity. A heuristic argument devised by Simpson and Penrose [5] suggests that small perturbations might turn the Cauchy horizon into a singularity, via a blue-shift instability, beyond which spacetime cannot be continued in any meaningful way. As a consequence, global uniqueness might still hold as a generic feature of initial value problems for the Einstein equations. These are the core ideas behind the *strong cosmic censorship conjecture* (SCC). Later, Israel and Poisson [6] identified the blow up of the Hawking mass as a source for the expected Cauchy horizon instability, a process known as *mass inflation*.

The previous discussion motivates the problem in the Introduction. Regarding the specific choices made there, the self-gravitating real massless scalar field is the simplest non-pathological matter model with dynamical degrees of freedom in spherical symmetry; moreover it exhibits a wavelike behavior reminiscent of the general Einstein vacuum equations. A non-vanishing charge parameter is needed to exclude the Schwarzschild subfamily, whose solutions do not contain a Cauchy horizon to start with.

We introduce a cosmological constant because it plays a fundamental role in modern physics and, from a purely mathematical point of view, provides rich geometrical structures and dynamical behaviors. But it is not clear why it should matter in the context of the SCC. Note, for instance, that our problem concerns what happens in the interior of a black hole region, whereas the effects of the cosmological constant are specially relevant at large scales. Nonetheless, during the late 90s, there was a considerable amount of activity concerning the SCC with $\Lambda > 0$. The first impressions where that the SCC should fail. But, later on, such claims were dismissed by arguing that the conclusions were based on over-simplified models

that did not take into proper account the effects of backscattering (see Section 15.2 of [7] for a detailed discussion).

A full non-linear analysis, for instance as provided by our problem, was clearly in order.

4. Regularity of extensions and Cauchy horizon stability

In the formulation of our problem, a fundamental role is played by the regularity that we allow the extensions to have, beyond the maximal globally hyperbolic development. It turns out that this regularity may be viewed as a measure of the stability of the Cauchy horizon: the more regular the extensions are, the more stable the Cauchy horizon can be thought of. We will now briefly discuss some regularity requirements on the metric.

Inextendibility of the metric in C^2: This is, first of all, motivated by the fact that the Einstein equations are of second order. Also, the blow up of the Kretschmann scalar (a measure of tidal forces) implies this criterion. It is clear that its failure would provide overwhelming evidence against the SCC. Nonetheless, its success would not provide such a compelling argument in favor of the conjecture, since there are plenty of relevant solutions of the Einstein equations whose regularity is well below this threshold. In fact, we have constructed classical solutions which are not C^2.

Inextendibility of the metric in C^0: This corresponds to Christodoulou's choice in his original formulation of the SCC. Its validity would provide overwhelming evidence in favor of the deterministic nature of general relativity.

It turns out that neither of the two previous criteria are enough to capture the full richness of the subject. In fact, in [8], Dafermos proved the existence of (spherically symmetric) extension, with metric in $C^0 \setminus C^1$, for the $\Lambda = 0$ case.

We are thus lead to consider a third type of requirement, based on the possibility of trying to exclude extensions satisfying the field equations.

Inextendibility as a Lorentzian manifold with Christoffel symbols in L^2_{loc}: This is enough to ensure inextendibility of the metric, even as a weak solution of the Einstein equations, and it was proposed by Christodoulou in response to Dafermos' results. Chruściel had already proposed a similar criterion with the following conditions imposed directly at the level of the metric: inextendibility with $g \in H^1_{\mathrm{loc}}$ and $g^{-1} \in L^\infty_{\mathrm{loc}}$. It turns out that, under the genericity conditions alluded to above, this criterion holds for the ($\Lambda = 0$) solutions of Dafermos.

5. Main results

Given a reference subextremal RN solution, from whose event horizon we take the outgoing initial data (recall that the remaining data is fixed freely), we first show that it has a unique maximal development $(\mathcal{M}, g, \phi)$ with the metric (2) and scalar field ϕ defined on $\mathcal{M} = \mathcal{P} \times_r S^2$, where $\mathcal{P}$ is a past set $\mathcal{P} \subset [0, U] \times [0, \infty[$. In our coordinates, the Cauchy horizon of the the reference RN solution is given by $v = +\infty$.

Stability of radius function at the Cauchy horizon. We show that there exists $U > 0$ and $r_0 > 0$ for which

$$r(u, v) > r_0, \text{ for all } (u, v) \in [0, U] \times [0, \infty[\subset \mathcal{P}.$$

So, we see that the symmetry orbits do not collapse to points on $[0, U] \times [0, \infty[$. This is a stability result for the Cauchy horizon. We have that $(\mathcal{M}, g, \phi)$ extends, across the Cauchy horizon $\{v = \infty\}$, to $(\hat{\mathcal{M}}, \hat{g}, \hat{\phi})$, with $\hat{g}$ and $\hat{\phi}$ in C^0.

We now need to introduce the quantity ρ defined as the ratio between the surface gravity of the Cauchy horizon and the surface gravity of the event horizon of the reference solution. Note that $\rho > 1$ and that it provides a measure of how close the initial data is from extremality; the case $\rho = 1$, which we have excluded from beginning, corresponds to the extremal case where the radius of the event horizon and the radius of the Cauchy horizon coincide.

After controlling r, the fundamental quantities that will determine the regularity of extensions are the mass ϖ and the radial derivative of the scalar field along lines of fixed u, $\partial_r(\phi|_{u=c})$. As we will see, our control over these quantities will depend on the decay rate imposed on the free data $\phi(\,\cdot\,, v = 0)$ and on how close we are to extremality.

Mass inflation. If $\rho > 2$ and

$$\partial_u \phi(u, 0) \geq c u^s, \; for \; some \; 0 < s < \frac{\rho}{2} - 1,$$

then

$$\lim_{v \to \infty} \varpi(u, v) = \infty, \; for \; each \; 0 < u \leq U.$$

In particular, no C^1 (spherically symmetric) extensions across the Cauchy horizon exist, since ϖ is at the level of first derivatives of the metric. In fact, the Christodoulou-Chruściel inextendibility criterion holds. We stress the fact that we are only considering the existence of spherically symmetric extensions: for instance, when considering general extensions, the blow up of the mass, a priori, only excludes the existence of C^2 extensions, since it leads to the blow up of the Kretschmann scalar. The existence, under these circumstances, of non-spherically symmetric extensions with metric in C^1 is an open problem, to our knowledge.

No mass inflation. If

$$|\partial_u \phi(u, 0)| \leq c u^s, \; for \; some \; s > \frac{7\rho}{9} - 1,$$

then there exists $C > 0$ such that,

$$|\varpi(u, v)| < C, \; for \; all \; (u, v) \in [0, U] \times [0, \infty[.$$

Moreover, the Christodoulou-Chruściel inextendibility criterion fails, i.e. $(\mathcal{M}, g, \phi)$ extends, across the Cauchy horizon, to $(\hat{\mathcal{M}}, \hat{g}, \hat{\phi})$, with $\hat{g}$ and $\hat{\phi}$ in C^0, Christoffel symbols $\hat{\Gamma}$ in L^2_{loc}, and $\hat{\phi}$ in H^1_{loc}.

Bounding $\partial_r(\phi|_{u=c})$***.*** If

$$|\partial_u \phi(u, 0)| \leq c u^s, \; for \; some \; s > \frac{13\rho}{9} - 1,$$

then, there exists $C > 0$ such that,

$$|\partial_r(\phi|_{u=c})| < C, \; for \; all \; (u, v) \in [0, U] \times [0, \infty[.$$

Consequently, $(\mathcal{M}, g, \phi)$ extends (in a non-unique way), across the Cauchy horizon, to a spherically symmetric (classical) solution $(\hat{\mathcal{M}}, \hat{g}, \hat{\phi})$ of the Einstein-Maxwell-scalar field system with cosmological constant Λ, with $\hat{g}$ and $\hat{\phi}$ in C^1. Moreover, the Kretschmann scalar is uniformly bounded. To the best of our knowledge, these are the first results where the generic existence of extensions as solutions of the Einstein equations is established.

6. Remarks about the strong cosmic censorship conjecture

Strictly speaking, our results, which hold for all signs of Λ, do not apply directly to the SCC. The reason for this is that we are considering exact RN data on the initial outgoing hypersurface, with our dynamical degrees of freedom provided by the free data on the ingoing hypersurface. So our results only cover the case where the scalar field has compact support on the event horizon. This scenario is not expected to arise from the gravitational collapse of generic (and

appropriate) Cauchy (for $\Lambda \geq 0$) or Cauchy and boundary (for $\Lambda < 0$) initial data, to which a proper formulation of the SCC should refer to.

In fact, for $\Lambda = 0$, the expected generic behavior along the event horizon is described by a Price's law of the form

$$|\partial_v \phi(0, v)| \sim v^{-p},$$

for some $p > 0$ and v a Eddington-Finkelstein coordinate as above. For initial data satisfying this law, in [8], Dafermos has shown that, for all $1 < \rho < \infty$, mass inflation occurs and the Christodoulou-Chruściel criterion holds (see also [7]).

For $\Lambda < 0$, in the general non-symmetric case, the decay along the horizon is expected to be even slower, and the process of gravitational collapse is far less understood. Therefore we excuse ourselves from extrapolating from our results (which, recall, hold when $\Lambda < 0$ as well).

For $\Lambda > 0$ the expected Price law provides "fast" decay

$$|\partial_v \phi(0, v)| \sim e^{-\Delta v}, \tag{9}$$

for some positive Δ, which leads to an exponential approach to the stationary data considered here. So, in the $\Lambda > 0$ case, the *no mass inflation* scenario and some of its remarkable consequences are expected to be true for dynamical data close to extremality $\rho \approx 1$ and satisfying (9). This puts the validity of the SCC, in the $\Lambda > 0$ case, into question.

From the previous discussion it is clear that the stationary data case at the event horizon, discussed here, is the most natural setting to study, in a unified framework, the influence of the sign of Λ in the stability of Cauchy horizons. Our results and techniques show that *differences concerning Cauchy horizon stability and mass inflation when Λ changes sign are due solely to the differences in the decay rates of ϕ along the event horizon, as described by a Price's law.*

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