The SCFT/VOA correspondence for twisted class $\mathcal{S}$

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Abstract

The correspondence between four-dimensional $\mathcal{N} = 2$ superconformal field theories and vertex operator algebras, when applied to theories of class $\mathcal{S}$, leads to a rich family of vertex algebras that have been given the moniker *chiral algebras of class $\mathcal{S}$*. These vertex algebras are fascinating from both a physical and mathematical point of view since they furnish novel representations of critical level affine Kac–Moody algebras. A remarkably uniform construction of these vertex operator algebras has been put forward by Tomoyuki Arakawa in [Ara18]. The construction takes as input a choice of simple Lie algebra $\mathfrak{g}$, and applies equally well regardless of whether $\mathfrak{g}$ is simply laced or not. In the non-simply laced case, however, the resulting VOAs do not correspond in any clear way to known four-dimensional theories. On the other hand, the standard realisation of class $\mathcal{S}$ theories involving non-simply laced symmetry algebras requires the inclusion of punctures that have been twisted by an outer automorphism of the Lie algebra.

In this thesis, we extend the construction of *loc. cit.* to theories of class $\mathcal{S}$ with twisted punctures. The resulting family of vertex algebras are, simultaneously, modules over two different critical level affine Kac–Moody algebras. We show that our proposal passes a number of consistency checks and establish results on gluing isomorphisms, and the action of generalised S-duality.

Statement of Authorship

Parts of this thesis are based on joint work with collaborators.

— Chapters 1 and 2 contain material that first appeared in joint work with Christopher Beem in [BN23b]

— Chapter 1 also contains examples and computations that first appeared in joint work with Christopher Beem in [BN23a]

— Chapter 2 contains ideas from an ongoing project [BN], joint with Dylan Butson

For അമ്മ *and* അച്ഛാ

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Contents

List of notation

- G is a simple, simply connected, algebraic, Lie group, with Lie algebra $\mathfrak{g}$
- G_u is a simply laced, simple, simply connected, algebraic, Lie group, with Lie algebra $\mathfrak{g}_u$
- P^+ is the set of integral dominant weights of G.
- κ_c is the critical level, equal to the negative dual Coxeter number.
- KL is the Kazhdan–Lusztig category of $\widehat{\mathfrak{g}}$, at the critical level.
- $\mathcal{O}$ is the ring of power series $\mathbb{C}[\![t]\!]$ and $\mathcal{K}$ is its field of fractions $\mathbb{C}(\!(t)\!)$.
- $\mathbb{D} = \operatorname{Spf} \mathcal{O}$ is the formal disc, and $\mathbb{D}^\times = \operatorname{Spf} \mathcal{K}$ is the formal punctured disc.

Introduction

Supersymmetric quantum field theories have proven to be a fertile ground for producing results in pure mathematics. The sector of BPS states, which enjoy enhanced supersymmetry, often collect together into rich algebraic structures. This leads to a wonderful cross-pollination of ideas from geometric representation theory and physics.

Some important examples in this regard include the quiver varieties of Nakajima [Nak94], which appear, in physics, as a moduli space of vacua of certain three-dimensional $\mathcal{N} = 4$ gauge theories. Or, indeed, the AGT correspondence of [AGT10] and its relations to instanton moduli spaces.

In certain theories, these BPS states may collect together into a vertex algebra. Vertex algebras, of course, already have a natural role in two-dimensional physics—where they originated as the holomorphic sectors of two-dimensional conformal field theories. However, there has been much work in recent years on how one may associate vertex algebras to quantum field theories in higher (greater than two) dimensions.

Within the context of four-dimensional $\mathcal{N} = 2$ superconformal field theories (hereafter abbreviated as SCFT or $\mathcal{N} = 2$ SCFT), an SCFT/VOA correspondence first appeared in work of Beem–Lemos–Liendo–Peelaers–Rastelli–van Rees [BLL$^+$15]. Given an $\mathcal{N} = 2$ SCFT, the authors detail how to extract the data of a vertex algebra from the spectrum of local operators of the SCFT. Moreover, the associated vertex algebra is invariant under the action of (generalised) S-duality on its parent SCFT. This makes it a sort of invariant of an SCFT—hinting at when two SCFTs are related by S-duality. The associated vertex algebra captures much of the intricacy of four dimensional physics—a number of observables of the parent SCFT can be fully recovered from the vertex algebra.

For example, the character of the vertex algebra recovers the Schur limit of the supersymmetric index of the SCFT. A conjecture of [BR18] (verified in infinitely many examples) also identifies the Higgs branch of the parent SCFT with the associated variety of the vertex algebra. The global symmetries of the SCFT have affine counterparts in the associated vertex algebra.

Actually computing the associated VOA, usually requires some detailed knowledge about the local operators of an SCFT. However, this requirement can be waived within the setting of the theories of class $\mathcal{S}$.

The theories of class $\mathcal{S}$, introduced in [Gai09, GMN09], constitute a highly structured, special family of four-dimensional $\mathcal{N} = 2$ SCFTs. These theories are best understood as the compactification of a six-dimensional $\mathcal{N} = (2, 0)$ SCFT, labelled by a simply laced Lie algebra $\mathfrak{g}_u$. This six-dimensional theory is compactified on a punctured algebraic curve, Σ, over $\mathbb{C}$—the UV curve.

A number of properties of the four-dimensional SCFT are characterised in terms of the data of the UV curve, *e.g.*, the marginal gauge couplings of the SCFT are identified with the complex structure moduli of the curve. Generalised S-duality for these theories can be identified with the mapping class group of the UV curve.

Each puncture on Σ gives rise to a $\mathfrak{g}_u$ global symmetry for the SCFT. Starting with two surfaces Σ and Σ', one can glue these along two punctures to produce a new surface Σ''. On the SCFT side, this procedure corresponds to gauging the diagonal action of the $\mathfrak{g}_u$ global symmetry.

The vertex algebras associated to theories of class $\mathcal{S}$ were first systematically studied in [BPRvR15] where they went under the name *chiral algebras of class* $\mathcal{S}$. In that work, a number of key properties of this family of VOAs were identified and some explicit computations performed in simple cases (with parent six dimensional theory of type $\mathfrak{g}_u = \mathfrak{a}_1, \mathfrak{a}_2$).

Since generalised S-duality acts via diffeomorphisms, the vertex algebras are labelled only by the topological data of the curve. Restricting to genus zero, this is just the number of punctures of Σ. This gives rise to a family of vertex algebras, $\mathbf{V}_{\mathfrak{g}_u,s}$, labelled by a simply laced Lie algebra[1] $\mathfrak{g}_u$ and $\mathbb{P}^1$ with s-punctures. The SCFT/VOA correspondence of [BLL+15], states that these vertex algebras must satisfy certain gluing isomorphisms, coming from the gluing of curves along punctures. Namely, we must have that

$$\mathrm{H}^{\frac{\infty}{2}+\bullet}(\widehat{\mathfrak{g}}_{u,-2h^\vee}, \mathfrak{g}_u, \mathbf{V}_{\mathfrak{g}_u,s} \otimes \mathbf{V}_{\mathfrak{g}_u,s'}) \cong \mathbf{V}_{\mathfrak{g},s+s'-2} \ ,$$

where $\mathrm{H}^{\frac{\infty}{2}+\bullet}(\widehat{\mathfrak{g}}_{u,-2h^\vee}, \mathfrak{g}_u, -)$ is the functor of relative semi-infinite cohomology, with respect to an action of the affine Kac–Moody algebra $\widehat{\mathfrak{g}}_u$ at level equal to twice the negative dual Coxeter number, $h^\vee$.

A speculative vision was also put forward, wherein the general chiral algebras of class $\mathcal{S}$ might be uniquely determined by their various duality properties. Indeed, invariance under the action of generalised S-duality turns out to be strong enough to completely fix the superconformal index of class $\mathcal{S}$ theories [GRR13].

Such speculation was answered in the affirmative by a remarkable construction of Arakawa in [Ara18]. This gives a purely mathematical construction of the chiral algebras of class $\mathcal{S}$—at genus zero. Arakawa's construction produces a family of vertex algebras $\mathbf{V}_{\mathfrak{g},s}$, parameterised by the curve $\mathbb{P}^1$ with s-marked points and a simple Lie algebra $\mathfrak{g}$. Key to this construction is a gluing operation, which we refer to as *Feigin–Frenkel* gluing, that seemingly has no physical counterpart. Pictorially, this glues UV curves along interior points as opposed to punctures.

Each maximal puncture on the UV curve gives rise to a critical level universal affine vertex algebra, $V^{\kappa_c}(\mathfrak{g})$ inside the associated VOA—all with a common Feigin–Frenkel centre. Feigin–Frenkel gluing amounts to identifying the action of this Feigin–Frenkel centre across the VOAs associated to each surface. The construction of Arakawa, using Feigin–Frenkel gluing, is a chiral analogue of the construction of Ginzburg and Kazhdan [GK] for the Moore–Tachikawa varieties [MT12]—the Higgs branches of class $\mathcal{S}$ theories.

[1] In [Ara18], this family is labelled as $\mathbf{V}_{G,s}$ with G the simply connected Lie group with Lie algebra $\mathfrak{g}$. We will adopt this convention in later sections.

Curiously, Arakawa's construction makes sense for any simple Lie algebra $\mathfrak{g}$, whether simply laced or not. This is in direct contrast to the physics of class $\mathcal{S}$, where the Lie algebra must be simply laced. In the usual lore of class $\mathcal{S}$, non-simply laced global symmetries may be introduced by adding *twisted punctures* to the UV curve [Vaf98, Tac09, Tac11]. This amounts to refining the compactification data to a UV curve with a local system of Dynkin diagrams. The twisted punctures are labelled by a non-trivial element, $\sigma \in \operatorname{Out}\mathfrak{g}_u$, of the outer-automorphism group of a simply laced Lie algebra $\mathfrak{g}_u$. Each such twisted puncture gives rise to a $\mathfrak{g}_t$ global symmetry for the SCFT, with $\mathfrak{g}_t$ the Langlands dual of the σ-invariant subalgebra $\mathfrak{g}_u$. The theories corresponding to UV curves with such twisted punctures are called the theories of twisted class $\mathcal{S}$.

In this thesis, we focus on the twisted setting—aiming to answer the following question:

What are the associated vertex algebras for the theories of twisted class $\mathcal{S}$?

We propose a novel construction for the associated vertex algebras of twisted class $\mathcal{S}$, following the techniques of Arakawa. Our proposal allows for the realisation of all vertex algebras associated to genus zero curves. We are also able to establish a number of the gluing isomorphism, though we shall see that there are homological obstructions preventing us from establishing the full scope of expected results. This obstruction is analogous to one that appears already in [Ara18] in the case of higher-genus chiral algebras.

It is natural to wonder how Arakawa's non-simply laced construction relates to the physics of twisted class $\mathcal{S}$. There seems to be no straightforward answer to this but we provide some speculative characterisation of the non-simply laced construction in terms of three-dimensional theories.

The organisation of the rest of this thesis is as follows.

Overview of Chapter 1

The first chapter will serve as a review of the physics of $\mathcal{N}$=2 SCFTs and the SCFT/VOA correspondence of [BLL+15].

We start off in Section 1.1 with an idiosyncratically selective review of four-dimensional $\mathcal{N}=2$ SCFTs. We focus, in particular, on the case of gauge theories where a number of the prototypical features of an SCFT can be made more evident.

In Section 1.2, we review the SCFT/VOA correspondence of [BLL+15]. We discuss how the characteristics of an SCFT, that we discussed in Section 1.1, have vertex algebraic counterparts. We shall also recast the gauging construction of *loc. cit.* in the language of semi-infinite cohomology.

Overview of Chapter 2

This chapter will be devoted to reviewing Arakawa's construction of the chiral algebras of class $\mathcal{S}$, as well as the physical context of this construction.

We introduce our preferred family of $\mathcal{N}=2$ SCFTs—the theories of class $\mathcal{S}$—in Section 2.1. We review their construction in terms of the parent six-dimensional $\mathcal{N}=(2,0)$ SCFT and discuss how these theories are classified by a choice of simply laced Lie algebra, and a curve with marked points. We discuss the classification of these marked punctures in Section 2.1.2 and introduce the gluing construction in 2.1.3.

Section 2.2 is a review of Arakawa's construction of the chiral algebras of class $\mathcal{S}$—the vertex algebras associated to the theories of class $\mathcal{S}$. We start of with an overview of the properties one should expect from these vertex algebras in light of the correspondence of [BLL+15]. We also introduce the key tool of Feigin–Frenkel gluing, a kind of semi-infinite cohomology with respect to the action of the Feigin–Frenkel centre.

This leads to the construction of an inverse Hamiltonian-reduction functor in Theorem 2.2.8

$$\mathrm{KL} \underset{\mathbf{W}_{G^*-}}{\overset{H^0_{\mathrm{DS}}(-)}{\rightleftarrows}} \mathcal{Z}-\mathrm{Mod} \quad .$$

which inverts principal Drinfel'd–Sokolov reduction. In particular, this gives an equivalence between the Kazhdan–Lusztig category of an affine Kac–Moody algebra $\widehat{\mathfrak{g}}$ at the critical level, and a full subcategory $H^0_{\mathrm{DS}}(\mathrm{KL}_{\kappa_c}) \equiv \mathrm{KL}_0 \subset \mathscr{Z} - \mathrm{Mod}$ of modules over the mode algebra of the Feigin–Frenkel centre.

In Section 2.2.7, we move on to fully reviewing Arakawa's construction of the genus zero chiral algebras of class $\mathcal{S}$, $\mathbf{V}_{G,s}$. We shall also reproduce a number of the results appearing in [Ara18], on the properties of the family, $\mathbf{V}_{G,s}$.

Most of the material in this section will not be original, but we include it to contextualise our construction in the following chapter.

To end this chapter, we go over a few examples of $\mathbf{V}_{G,s}$ in the case where $G = \mathrm{SL}_2$ and for $s < 4$. We give explicit presentations, in terms of strong generators, relations and the OPEs between them.

Overview of Chapter 3

In Chapter 3, we present our extension of Arakawa's construction to the setting of twisted class $\mathcal{S}$.

We introduce the eponymous theories of twisted class $\mathcal{S}$ in Section 3.1. Let $\mathfrak{g}_u$ be a simply laced Lie algebra and $\sigma \in \mathrm{Out}(\mathfrak{g}_u)$ be a non-trivial outer-automorphism of order two. The twisted theories of class $\mathcal{S}$ are classified by a choice of simply laced Lie algebra $\mathfrak{g}_u$ and a curve—now with two kinds of punctures. One type of puncture—the $\mathfrak{g}_u$ punctures—are classified as before. Additionally, we have twisted punctures giving rise to $\mathfrak{g}_t$ flavour symmetries. Once again, we restrict to genus zero and write $\mathcal{C}_{m,n}$ for $\mathbb{P}^1$ with m untwisted punctures and $2n$ twisted punctures. The associated vertex algebras will be denoted by $\mathbf{V}_{m,n}$.

Incorporating twisted punctures gives rise to new moves in the web of S-duality and we review this in Section 3.1.2. In Section 3.1.3, we give expressions for the Schur limit of the superconformal index for twisted class $\mathcal{S}$ theories, following [LPR14].

Our construction begins in Section 3.2. First, we prove a number of technical results on how the Feigin–Frenkel centres of $\widehat{\mathfrak{g}}_u$ and $\widehat{\mathfrak{g}}_t$ relate to each other. In Section 3.2.2 we prove our first important theorem, Theorem 3.2.1, which leads to the construction of mixed modules in Section 3.2.3. These mixed modules look like Weyl modules for $\widehat{\mathfrak{g}}_u$ and $\widehat{\mathfrak{g}}_t$, sewn together by identifying the action of the untwisted Feigin–Frenkel centre.

We start our construction proper in Section 3.2.4, where we give a proposal for how to construct the vertex algebra $\mathbf{V}_{1,1}$—corresponding to a $\mathbb{P}^1$ with one untwisted puncture and a pair of twisted punctures. We also state our second main theorem

$$H^0_{\mathrm{DS}}(\mathbf{V}_{1,1}) \cong \mathcal{D}^{ch}_{G_t} \equiv \mathbf{V}_{G_t,2} \ ,$$

i.e., the principal Drinfel'd Sokolov reduction of $\mathbf{V}_{1,1}$ is isomorphic to Arakawa's construction for the $\mathfrak{g}_t$ cylinder, $\mathbf{V}_{G_t,2}$. In addition, we show that $\mathbf{V}_{1,1}$ satisfies the properties that four-dimensional physics predicts.

Before extending this construction to the full family of $\mathbf{V}_{m,n}$, we make a technical interlude in Section 3.2.5. Here, we establish a number of lemmas on the interplay between the various types of gluing we have available. Namely, we will be interested in when the order of these gluings can be interchanged.

In Section 3.1.4 we extend our proposal to the full family of $\mathbf{V}_{m,n}$. The FF-gluing procedure of the previous Chapter can be straightforwardly extended to produce $\mathbf{V}_{m,1}$, but our definition of the $\mathbf{V}_{m,n}$ will be more involved. We finish by providing partial results on the gluing isomorphisms for the $\mathbf{V}_{m,n}$.

In Section 3.2.8, we construct an action of the S-duality group on the $\mathbf{V}_{m,n}$ and show that they act by automorphisms. Finally, in Section 3.2.9, we discuss the order three twists for Spin(4) and the obstructions to our construction.

To finish, we shall comment on possible physical interpretations of Arakawa's construction for the non-simply laced case in Section 3.3. In this section, we shall also provide a conjectural description of the associated varieties of the subfamily $\mathbf{V}_{m,1}$, in line with the construction of [BFN17] for the Coulomb branches of Sicilian theories.

Overview of the Appendix

There is a large amount of ancillary machinery that we require in our construction of the $\mathbf{V}_{m,n}$. For the sake of cohesion, we have relegated much of this material to this appendix.

Appendix A.1 contains some basic material on nilpotent orbits in Lie algebras. This material will primarily serve as context and to introduce the concept of Slodowy slices. Appendix A.2 is an introduction to vertex algebras with some basic definitions and concepts.

This is followed by Appendix A.3, where we review affine Kac–Moody algebras and their associated universal affine vertex algebras. We also have a brief discussion on their representation theory and the Kazhdan–Lusztig category. This appendix also defines the Feigin--Frenkel centre—the centre of the universal affine vertex algebra, at critical level.

In Appendix A.4, we start our long technical digression into opers. We follow the pedagogy of [Fre07], focusing primarily on opers on the formal disc and its punctured counterpart. We will review the Feigin–Frenkel isomorphism, relating the Feigin–Frenkel centre to the algebra of function on the moduli space of opers on the formal disc. Our main goal will be review the Miura transform and its associated screening charges. These screening charges will be crucial to the proof of Theorem 3.2.1.

Finally, in Appendix A.5 we review semi-infinite cohomology in the language of homological algebra. We shall largely follow the conventions of [Vor93, Vor97]—introducing the Feigin standard complex and associated vanishing theorems. We shall also review Drinfel'd--Sokolov reduction and how it may be described in the language of semi-infinite cohomology.

Chapter 1

The SCFT/VOA correspondence

> holy the clocks in space holy the
> fourth dimension
>
> Allen Ginsberg
> *Footnote to Howl*

1.1 A primer on four-dimensional SCFTs

The following will be a *very* quick and non-comprehensive review of four-dimensional SCFTs. We shall focus, in particular, on the ingredients appearing in the SCFT/VOA dictionary and neglect most other features. We point the reader to [Tac13] for a more pedagogical introduction to the subject.

As with most quantum field theories, a mathematically precise and general definition of four-dimensional $\mathcal{N} = 2$ SCFTs remains elusive. Instead, we shall try and paint a picture of the properties that $\mathcal{N} = 2$ SCFTs possess. To do this, we focus on a family of prototypical examples of $\mathcal{N} = 2$ SCFTs—the gauge theories—where these properties are transparent. Then we shall remark on how to extrapolate these properties to a general $\mathcal{N} = 2$ SCFT.

1.1.1 The superconformal algebra and its representations

The conformal algebra in four (Euclidean) dimensions is $\mathfrak{so}(2,4)$, which we identify, after complexification, with $\mathfrak{sl}_4 \cong \mathfrak{sl}_2 \times \mathfrak{sl}_2$. The Poincaré subalgebra is generated by

$$\begin{array}{rl} P_{\alpha\dot{\alpha}} & \text{Translations} \\ M_\alpha{}^\beta\,, M^{\dot{\alpha}}{}_{\dot{\beta}} & \text{Rotations} \end{array} \tag{1.1.1}$$

where $\alpha, \dot{\alpha}$ are $\mathfrak{sl}_2 \times \mathfrak{sl}_2$ spinorial indices. The additional conformal generators are:

$$\begin{array}{ll} K^{\dot{\alpha}\alpha} & \text{Special Conformal Transformation} \\ D & \text{Dilatations} \end{array} \tag{1.1.2}$$

The eigenvalue of D, is the *conformal weight*, E.

A conformal field theory that enjoys supersymmetry actually enjoys an enhanced symmetry, known as superconformal symmetry. For four dimensional $\mathcal{N} = 2$ superconformal field theories, the superconformal algebra is $\mathfrak{sl}(4|2)$. The bosonic part is given by $\mathfrak{sl}(4) \times \mathfrak{sl}(2)_R \times \mathfrak{u}(1)_r$, where $\mathfrak{sl}(4)$ is the conformal algebra described above. The additional symmetries are R-symmetries with generators

$$R_+\,, R_-\,, R \in \mathfrak{sl}(2)\,, \quad \text{and} \quad r \in \mathfrak{u}(1)\,, \tag{1.1.3}$$

where we have adopted the Chevalley basis for $\mathfrak{sl}(2)_R$. The fermionic part of the algebra is generated by the usual supercharges

$$\mathcal{Q}^I_\alpha\,,\ \tilde{\mathcal{Q}}_{I\dot{\alpha}}\,, \tag{1.1.4}$$

where I is an $\mathfrak{sl}(2)_R$ index and each supercharge lives in the two-dimensional representation of $\mathfrak{sl}(2)_R$. As previously stated, we have an enhanced symmetry with additional supercharges

$$S^\alpha_I\,, \tilde{S}^I_{\dot{\alpha}}\,, \tag{1.1.5}$$

called the special conformal supercharges, which also transform in the two-dimensional

Multiplet	Δ	(j_1, j_2)	R	r
$\hat{\mathcal{B}}_R$	$2R$	$(0,0)$	R	0
$\mathcal{D}_{R,(0,j_2)}$	$2R + j_2 + 1$	$(0, j_2)$	R	$j_2 + 1$
$\overline{\mathcal{D}}_{R,(j_1,0)}$	$2R + j_1 + 1$	$(j_1, 0)$	R	$-j_1 - 1$
$\hat{\mathcal{C}}_{R,(j_1,j_2)}$	$2R + j_1 + j_2 + 2$	(j_1, j_2)	R	$j_1 - j_2$

Table 1.1: Table of some short representations of the superconformal algebra $\mathfrak{sl}(4|2)$, in the notation of [DO03]. The columns detail the eigenvalues under dilatation, rotation and the R-symmetries of the highest weight state. These short representation appear as the Schur operators in 1.2.

representation of $\mathfrak{sl}(2)_R$. For the full set of relations between the generators of $\mathfrak{sl}(4|2)$ we refer the reader to [DO03], whose conventions we shall adhere to.

The representation theory of $\mathfrak{sl}(4|2)$ can be quite intricate and we avoid a full review of it. In Table 1.1.1, we introduce a few examples of the so-called *short multiplets*—a particular class of highest weight representations. We use E to denote the eigenvalue of the highest weight state. The generators $M_\alpha{}^\beta$ and $M^{\dot\alpha}{}_{\dot\beta}$ generate an $\mathfrak{sl}(2) \times \mathfrak{sl}(2)$ Lie subalgebra and we denote by j_1 and j_2 the corresponding weights. We also use R and r to denote the weights under $\mathfrak{sl}(2)_R$ and $\mathfrak{u}(1)_r$ respectively.

1.1.2 Four dimensional $\mathcal{N} = 2$ gauge theories

Gauge theories are usually formulated in terms of a Lagrangian description. To do so we first introduce two particular kinds of representations of the extended supersymmetry algebra: vector and hyper multiplets.

Fix $G = \prod_i G_i$ to be a semisimple Lie group with simple summands G_i , and let $\mathfrak{g} = \operatorname{Lie} G$. We choose $\mathcal{P} \to \mathbb{R}^4$ to be a principal G-bundle with connection A and let E_N be some associated vector bundle with fibre N, a quaternionic representation of G.

The connection A combines with two Weyl spinors, λ_L and λ_R, and a scalar $\Phi \in C^\infty(\mathbb{R}^4, \mathfrak{g})$ to form a representation of the $\mathcal{N} = 2$ algebra called the *vector multiplet*. The highest weight state of this algebra is Φ, and we summarise the representation in the diagram below.

$$\begin{array}{ccc} & \Phi & \\ \swarrow & & \searrow \\ \lambda_L & & \lambda_R \\ \searrow & & \swarrow \\ & A & \end{array} \tag{1.1.6}$$

The field-strength $F = dA + A \wedge A$ appears in the multiplet $D_\alpha V$ where D_α is the covariant derivative in superspace and V represents the vector multiplet.

Similarly, sections of E_N also live in a $\mathcal{N} = 2$ multiplet—the *hypermultiplet*. Let $Q \in \Gamma(\mathbb{R}^4, E_N)$ and $Q^\dagger$ its conjugate. A quaternionic representation has a $\mathbb{C}$-anti-linear involution $I : N \to N$ and let $\widetilde{Q} = I \circ Q$ with $\widetilde{Q}^\dagger$ its conjugate. The hypermultiplet, $\mathcal{Q}$, combines the sections $Q, \widetilde{Q}^\dagger$ with Weyl spinors ψ and $\widetilde{\psi}^\dagger$ as below

$$\begin{array}{ccc} & Q & \\ \swarrow & & \searrow \\ \psi & & \widetilde{\psi}^\dagger \\ \searrow & & \swarrow \\ & \widetilde{Q}^\dagger & \end{array} \; . \tag{1.1.7}$$

There is a second hypermultiplet $\widetilde{\mathcal{Q}}$ combining $\widetilde{Q}$ and $Q^\dagger$ with the spinors $\widetilde{\psi}$ and $\psi^\dagger$

The multiplets must satisfy the equations of motion coming from extremising an action functional. Suppose G is simple and let $\tau \in \mathbb{C}$ be the *complexified gauge coupling*, the action functional in the superspace formalism is given by

$$\begin{aligned} \mathbf{S} = \int_{\mathbb{R}^4} \mathrm{d}^4 x \int \mathrm{d}^4\theta \langle Q^\dagger, e^V Q\rangle + \int \mathrm{d}^2\theta \big(\langle \widetilde{Q}, \Phi Q\rangle + \langle \widetilde{Q}^\dagger, \Phi Q^\dagger\rangle\big) \\ + \frac{\operatorname{Im}\tau}{4\pi} \int \mathrm{d}^4\theta \, \mathrm{tr}_N \Phi^\dagger e^V \Phi - \frac{i\tau}{8\pi} \int \mathrm{d}^2\theta \, \mathrm{tr}_N (W^\alpha W_\alpha + W_{\dot\alpha} W^{\dot\alpha}) \; . \end{aligned} \tag{1.1.8}$$

where $\langle \cdot, \cdot \rangle$ is a Hermitian bilinear form on N, compatible with the unitary action of G. Here, we use e^V to denote the exponential map $\mathfrak{g} \to G \to \mathrm{GL}(N)$ (acting in a suitable representation). If G is not simple then we sum over the actions for each simple summand-

–introducing a gauge coupling τ_i, for each simple summand G_i.

The action above is for a gauge theory[1] with gauge group G, matter content valued in N and gauge coupling τ. This Lagrangian description, while useful, is somewhat cumbersome for our purpose. Indeed, the theories of class $\mathcal{S}$ do not always have such a Lagrangian description. To streamline our path toward class $\mathcal{S}$ we will provide a more abstract notation for a gauge theory.

Definition 1.1.1. A four-dimensional $\mathcal{N}=2$ gauge theory is a triple $(G, N, \{\tau_i\})$, where

- $G = \prod_i G_i$ is a semisimple Lie group, with simple summands G_i, called the *gauge group*
- N is a quaternionic representation of G called the *matter content*
- For each G_i we have a $\tau_i \in \mathbb{C}$ called the *complexified gauge coupling*

A gauge theory is *superconformal* if the following *anomaly*

$$\mathrm{Dyn}_{G_i}(N) - 2h_i^\vee \ , \tag{1.1.9}$$

vanishes for each simple summand. Here $h_i^\vee$ is the dual Coxeter number of G_i and $\mathrm{Dyn}_{\mathrm{G_i}}(N)$ is the G_i-Dynkin index of N, calculated as

$$\mathrm{Dyn}_{G_i}(N) = \frac{\dim N}{\dim G_i}\langle \lambda, \lambda + \rho \rangle \ , \tag{1.1.10}$$

where λ_i are highest weights of G_i that appear in N, counted with multiplicity.

The data $(G, N, \{\tau_i\})$, fixes the form of the action (1.1.8) and so we adopt this more compact notation.

Remark 1.1.2. Two degenerate examples of a gauge theory are when either G or N are trivial. If G is trivial, any vector space N satisfies the anomaly vanishing condition, and these are known as a theory of *free hypermultiplets*. If N is trivial, this is called a *pure*

[1] The action permits a deformation by adding a mass term $\int \mathrm{d}^2\theta \langle \widetilde{Q}, MQ \rangle + \langle \widetilde{Q}^\dagger, M^\dagger, Q^\dagger \rangle$, where M is the mass matrix. However, such a theory can never by superconformal and so we set these mass terms to zero

gauge theory. The anomaly vanishing condition for a pure gauge theory can only be met if G is abelian.

A special class of superconformal gauge theories are the ones corresponding to $\mathcal{N} = 4$ supersymmetric Yang–Mills (SYM) theories. In our notation these correspond to $(G, T^*\mathfrak{g}, \tau)$, where G is a simple Lie group with (complexified) Lie algebra $\mathfrak{g}$. The matter content is valued in the product of the coadjoint and adjoint representation of $\mathfrak{g}$. These theories have enhanced supersymmetry, endowing them with a number of nice properties.

1.1.3 Product of theories

We can define a product on the space of gauge theories by the following construction. Given two gauge theories, $(G_1, N_1, \{\tau_i\})$ and $(G_2, N_2\{\tau_j\})$, the product gauge theory $(G_1, N_1, \{\tau_i\}) \boxtimes (G_2, N_2, \{\tau_j\})$ is the gauge theory $(G_1 \times G_2, N_1 \oplus N_2, \{\tau_i\} \sqcup \{\tau_j\})$, where $\{\tau_i\} \sqcup \{\tau_j\}$ is the concatenated list of gauge couplings. At the level of actions, we have that

$$\mathbf{S}_{\mathcal{T}_1 \boxtimes \mathcal{T}_2} = \mathbf{S}_{\mathcal{T}_1} + \mathbf{S}_{\mathcal{T}_2} \ . \tag{1.1.11}$$

More generally, given two SCFTs $\mathcal{T}_1$ and $\mathcal{T}_2$, there is a notion of a product SCFT $\mathcal{T}_1 \boxtimes \mathcal{T}_2$. Physically, $\mathcal{T}_1 \boxtimes \mathcal{T}_2$ contains the field content of both $\mathcal{T}_1$ and $\mathcal{T}_2$ with no interactions between fields in $\mathcal{T}_1$ and fields in $\mathcal{T}_2$.

1.1.4 Global symmetries and gauging

Suppose that the matter in a gauge theory, $(G, N, \{\tau_i\})$, is a quaternionic representation of another semisimple Lie group G_F, *i.e.*, N is a quaternionic representation of $G \times G_F$. In this case, we say that the gauge theory possesses a *flavour* or *global* symmetry, with symmetry group G_F.

Given such a global symmetry, G_F, on (G, N), we can *gauge* the action of G_F at coupling τ to produce the gauge theory $(G \times G_F, N, \{\tau_i\} \sqcup \{\tau\})$. Note that every gauge theory arises by gauging free hypermultiplets (hence the naming).

More generally, if we have a Lie group G_F that acts on the field content of some SCFT $\mathcal{T}$

by automorphisms then we can gauge this action to produce a $\mathcal{N} = 2$ theory $\mathcal{T}_G$. Such a gauged theory might not always be superconformal, unless the anomaly condition of (1.1.9) (suitably written for a more general SCFT) vanishes.

1.1.5 S-duality

The natural notion of isomorphism between QFTs is *duality*. One says that two QFTs are dual if one can build a correspondence between the observables of both theories. For $\mathcal{N} = 2$ theories, we will be particularly interested in a kind of duality called S-duality.

For a given theory, the action of S-duality is described by the action of a group (the S-duality group) which traces out some orbit in the space of $\mathcal{N} = 2$ theories by acting on the complexified couplings. A homological definition of this S-duality group can be found in [CC18]. For the theories of class $\mathcal{S}$ we shall find a more geometric description.

As an example let us consider a theory with gauge group $G = SU(2)$, matter $N = T^*\mathfrak{su}(2)$ valued in the adjoint and coadjoint representations of $SU(2)$, and some fixed gauge coupling τ. This is $\mathcal{N} = 4$ SYM with gauge group $SU(2)$. The S-duality group is $\mathrm{SL}(2, \mathbb{Z})$ acting via Möbius transformations on τ. In particular, the S-generator that sends $\tau \mapsto -\frac{1}{\tau}$, induces a duality

$$(\mathrm{SU}(2), T^*\mathfrak{su}(2), \tau) \xleftrightarrow{\tau \mapsto -1/\tau} (\mathrm{SO}(3), T^*\mathfrak{so}(3), \frac{-1}{\tau}) \ , \qquad (1.1.12)$$

where $\mathfrak{so}(3) \cong \mathfrak{su}(2)$ since we are looking at the complexified Lie algebras. The fact that this group is the mapping class group of the torus is not an accident and hints at a more geometric description of this action.

1.1.6 Invariants of an SCFT—the index

While QFTs are hard to study in full detail, often one can extract observables which capture some shadow of the intricate structure of a full QFT. It is desirable to compute observables which are invariant under dualities. Computing these observables in various QFTs helps provide evidence in establishing a duality between them.

One such observable is the superconformal index, which is invariant under S-duality. The index is defined as a graded trace over the radially quantised Hilbert space of the theory [KMMR07, RR16],

$$\mathcal{I}(p,q,t,\mathbf{x}) = \mathrm{Tr}\,(-1)^F e^{-\beta(E-2j_2-2R+r)} p^{\frac{E+2j_1-2R-r}{2}} q^{\frac{E-2j_1-2R-r}{2}} t^{R+r} \prod_{i=1}^{\mathrm{rk}\mathfrak{g}_F} x_i^{\lambda_i} \ , \tag{1.1.13}$$

where p, q, t are called superconformal fugacities and (E, j_1, j_2, R, r) are various eigenvalues of diagonal operators in the superconformal algebra. Alternatively, the index can be thought of as the partition function of the SCFT on $S^3 \times S^1$.

If the SCFT possesses a flavour symmetry G_F, one can refine the index by including a term in the trace of the form

$$\prod_{i=1}^{\mathrm{rk}\,G_F} x_i^{\lambda_i} \ , \tag{1.1.14}$$

where the x_i are fugacities and λ_i are weights of G_F. A standard argument implies that the index can only receive a nonvanishing contribution from states obeying

$$E - 2j_2 - 2R + r = 0 \ , \tag{1.1.15}$$

and so is actually independent of β. As written, the index counts minimally supersymmetric states. There are a number of limits with enhanced supersymmetry, and we will be specifically interested in the Schur limit [GRRY13],

$$q \to t \ , \quad p \text{ arbitrary} \ . \tag{1.1.16}$$

In this limit, the index is in fact independent of p and takes the schematic form (suppressing flavour fugacities)

$$\mathcal{I}(q) = \mathrm{Tr}(-1)^F q^{E-R} \ . \tag{1.1.17}$$

Moreover, only states which satisfy

$$E + j_1 + j_2 - R = 0 \ , \tag{1.1.18}$$

contribute to this index. The short multiplets satisfying this constraint are precisely the ones in 1.1.1.

Let us compute the Schur index of a a gauge theory. First, we consider the case of free hypermultiplets valued in N an irreducible representation of some semisimple Lie group G_F. Since N is irreducible, it is of the form $N = \bigotimes_i N_i$ where the N_i are irreducible representations of the simple summands of G_F. The superconformal index of this theory is

$$\mathcal{I}_{(1,N)}(q, \mathbf{x}_i) = \mathrm{PExp}\left[\frac{\sqrt{q}}{1-q}\prod_i \chi_{N_i}(\mathbf{x}_i)\right] \ , \tag{1.1.19}$$

where PExp is the plethystic exponential[2] and χ_{N_i} is the character of the irreducible representation N_i. Gauging a simple summand $G \subset G_F$ produces a gauge theory (G, N). The index of the gauge theory is defined as an integral over a maximal torus in G,

$$\mathcal{I}_{(G,N)}(q, \mathbf{x_i}) = \oint_T [d\mathbf{z}]\mathcal{K}(q, \mathbf{z})^2 \mathcal{I}_{(1,N)}(q, \mathbf{x}_i, \mathbf{z}) \tag{1.1.20}$$

where we have singled out $\mathbf{z}$ as the fugacity for the G-symmetry, $[d\mathbf{z}]$ is the Haar-measure and where $\mathcal{R}$ are the roots of $G_{\mathbb{C}}$, the algebraic group. The $\mathcal{K}$ factors are defined by

$$\mathcal{K}(\mathbf{z}) = \frac{1}{(q;q)_\infty^{\mathrm{rk}\ \mathfrak{g}}}\prod_{\alpha \in \mathcal{R}} \frac{1}{(qe^{\alpha}(\mathbf{z});q)_\infty} \ , \tag{1.1.21}$$

where $(\cdot; q)_\infty$ are the q-Pochhammer symbols[3] .

More generally, if we know the index of an SCFT $\mathcal{T}$ with some global symmetry G, we can compute the index of $\mathcal{T}_G$ via an analogous gauging prescription

Furthermore, the index is multiplicative, in the sense that the index of a product theory $\mathcal{T}_1 \boxtimes \mathcal{T}_2$ satisfies

$$\mathcal{I}_{\mathcal{T}_1 \boxtimes \mathcal{T}_2}(q, \mathbf{a}_i, \mathbf{b}_i) = \mathcal{I}_{\mathcal{T}_1}(q, \mathbf{a}_i)\mathcal{I}_{\mathcal{T}_2}(q, \mathbf{b}_i) \ , \tag{1.1.22}$$

where $\mathbf{a}_i$ are fugacities for $\mathcal{T}_1$ and $\mathbf{b}_i$ are fugacities for $\mathcal{T}_2$.

[2]The Plethystic exponential of a power series $f(\mathbf{x})$, without a constant term is $\mathrm{Exp}\left(\sum_{k=1}^{\infty} \frac{f(\mathbf{x}^k)}{k}\right)$, where $\mathbf{x}^k$ is the tuple $(x_1^k, x_2^k, \dots)$.

[3]The q-Pochhammer symbols are defined as $(x, q) = \prod_{i=1}^{\infty}(1 - xq^k)$

1.1.7 Invariants of an SCFT—the Higgs branch

The Higgs branch of an SCFT is a geometric observable that is invariant under S-duality. Generally, the Higgs branch is a hyperkähler manifold.

Physically, the Higgs branch parameterises the space of supersymmetric vacua of an $\mathcal{N}=2$ theory. In fact the full space of supersymmetric vacua has two branches: the Higgs branch and the Coulomb branch.

For a gauge theory $(G, N, \{\tau_i\})$, the Higgs branch is defined as a hyperkähler quotient of N, by the action of G. In fact, we are only interested in the Higgs branch as a holomorphic symplectic space and so we equivalently define the Higgs branch as a holomorphic symplectic reduction of N by the complexified algebraic group $G_{\mathbb{C}}$.

Since N is a quaternionic representation of G, it is a holomorphic symplectic space with a Hamiltonian action of $G_{\mathbb{C}}$. This gives rise to a moment map $\mu : N \to \mathfrak{g}^*$, where $\mathfrak{g}^*$ is the linear dual of $\mathfrak{g} = \operatorname{Lie} G_{\mathbb{C}}$. The Higgs branch of (G, N, τ), as a holomorphic symplectic variety is,

$$\mathcal{M}_{(G,N,\tau)} = \mu^{-1}(0) /\!/ G_{\mathbb{C}} \tag{1.1.23}$$

where $/\!/$ is the GIT quotient (which ensures the Higgs branch is affine). Note that $\mathcal{M}_{(G,N,\tau)}$ is independent of the couplings and indeed is invariant under the action of S-duality.

The space $\mathcal{M}_{(G,N,\tau)}$ inherits a symplectic structure from the reduction but may have singularities. Indeed, $\mathcal{M}_{(G,N,\tau)}$ is expected to have symplectic singularities.

Often, the Higgs branch is a stratified space, with strata corresponding to qualitatively different vacua of the SCFT. The smallest strata, the origin, corresponds to the original SCFT, moving out to a larger strata corresponds to choosing a different vacuum and triggering the Higgs mechanism. The resulting renormalisation flow will land in a different $\mathcal{N}=2$ SCFT.

For a gauge theory $(G, N, \{\tau_i\})$, the Coulomb branch is

$$\mathcal{M}^C_{(G,N,\{\tau_i\})} \cong \mathfrak{g}^* /\!/ G \cong \mathfrak{h}^* /\!/ W \cong \mathbb{C}^{\operatorname{rk}\mathfrak{g}} \ , \tag{1.1.24}$$

as a variety. Physics endows this space with the structure of a special Kähler manifold, with the Kähler potential capturing the low energy dynamics of the theory. The special Kähler structure is described by Seiberg–Witten theory and has natural links to the study of Hitchin systems. The detailed structure of the Coulomb branch will not play a large role in the rest of this work and so we do not develop this subject further. We will note, however, that the special Kähler structure of the Coulomb branch does depend on the gauge couplings τ and S-duality does not act via automorphisms on the Coulomb branch.

While Seiberg–Witten theory is capable of computing the Coulomb branch of non-Lagrangian theories, computing the Higgs branch is difficult. In this sense, the theories of class $\mathcal{S}$ are particularly special as non-Lagrangian theories whose Higgs branches are precisely known.

Remark 1.1.3. There is an interesting trichotomy in four-dimensional theories, arising from the qualitatively different ways in which the full moduli space of vacua may branch. An SCFT may have:

- A ``pure'' Higgs branch, where the Higgs and Coulomb branch intersect transversally at the origin. This means that the generic stratum of the Higgs branch corresponds to a theory of free hypermultiplets.
- An ``enhanced'' Higgs branch, where the generic strata of the Higgs branch describes a pure abelian gauge theory. The Coulomb branch will intersect the Higgs branch in codimension zero.
- An ``interacting Higgs branch'', where the generic stratum of the Higgs branch describes an interacting SCFT

From an algebraic perspective, we can think of the presence of an enhanced Higgs branch as a shadow of an underlying *derived structure*. For gauge theories, this derived structure appears as a consequence of the failure of the moment map, $\mu : N \to \mathfrak{g}^*$, to be flat. In physics, the nonzero cohomological degrees are captured by the Hall–Littlewood chiral ring which contains fermionic operators as well as the bosonic Higgs branch operators [BBS]. The cohomology outside of degree zero corresponds to the auxiliary Coulomb branch directions one can move in starting at the generic stratum.

1.2 The SCFT/VOA correspondence

Having given a flavour of what SCFTs are and the properties they possess, let us proceed to a review of the correspondence of [BLL+15]. We introduce the notation below for the correpsondence.

$$\begin{aligned} \text{SCFT} &\xrightarrow{\chi} \text{VOA} \\ \mathcal{T} &\xmapsto{\chi} \chi[\mathcal{T}] \end{aligned} \tag{1.2.1}$$

A naive strategy for associating a vertex algebra to a four-dimensional superconformal field theory would be to restrict ourselves to a two dimensional plane and consider the algebra of operators restricted to this plane, which give rise to a two dimensional superconformal field theory. This is almost the correct strategy however the resulting two dimensional theory will not be chiral. To recover a vertex algebra we will have to pass to the cohomology of an appropriate supercharge.

We detail this cohomological construction in Sections 1.2.1 and 1.2.2. In Section 1.2.3, we describe a key element of the correspondence: global symmetries of the SCFT have affine enhancements in the VOA. In Section 1.2.4 we provide examples for the associated vertex algebras for theories of free hypermultiplets and vectormultiplets. Finally, in Section 1.2.5 we detail a gauging prescription for how to construct a gauge theory from a free one. More generally, this gauging prescription gives a vertex algebraic counterpart to the four-dimensional operation of gauging a global symmetry.

1.2.1 Twisted subalgebras

Let us fix the plane $x_1 = x_2 = 0$ in $\mathbb{R}^4$ and define complex coordinates $z = x_3 + ix_4$ and $\bar{z} = x_3 - ix_4$ on this plane. This choice of plane is completely general due to conformal invariance. Rotations within this plane are generated by

$$M := M_+{}^+ + M^{\dot{+}}{}_{\dot{+}}\,, \tag{1.2.2}$$

while orthogonal rotations are generated by

$$M^{\perp} := M_{+}{}^{+} - M^{\dot{+}}{}_{\dot{+}} \,. \tag{1.2.3}$$

There are two subalgebras of $\mathfrak{sl}(4|2)$ that stabilise our chosen plane: $\mathfrak{sl}(2|1) \times \mathfrak{sl}(2|1)$ and $\mathfrak{sl}(2) \times \mathfrak{sl}(2|2)$. We choose $\mathfrak{sl}(2) \times \mathfrak{sl}(2|2)$ since the other algebra is not chiral. The $\mathfrak{sl}(2)$ generates holomorphic translations while $\mathfrak{sl}(2|2)$ is the anti-holomorphic part.

The generators of the bosonic part are given by

$$\begin{aligned} L_{-1} = P_{+\dot{+}} \,, \quad L_1 = K^{\dot{+}+} \,, \quad L_0 = \frac{1}{2}(D + M) \\ \bar{L}_{-1} = P_{-\dot{-}} \,, \quad \bar{L}_1 = K^{\dot{-}-} \,, \quad \bar{L}_0 = D - M \,. \end{aligned} \tag{1.2.4}$$

The $\mathfrak{sl}(2)_R$ symmetry is also preserved under this construction. The holomorphic part is purely bosonic and so the supercharges will be antiholomorphic. In terms of the four-dimensional supercharges, they are given by

$$\mathcal{Q}^I = \mathcal{Q}^I_{-} \,, \quad \tilde{\mathcal{Q}}^I = \tilde{\mathcal{Q}}_{I\dot{-}} \,, \quad \mathcal{S}_I = \mathcal{S}_I^{-} \,, \quad \tilde{\mathcal{S}}^I = \tilde{\mathcal{S}}^{I\dot{-}} \,, \tag{1.2.5}$$

where the I is once again an $\mathfrak{sl}(2)_R$ index and the supercharges transform as a doublet. Finally, the plane algebra has a central element

$$Z = r + M^{\perp} \,. \tag{1.2.6}$$

A priori, one might look for vertex algebras by looking for operators transforming in trivial representations of $\mathfrak{sl}(2|2)$. However, any such operator will also transform trivially under the full four dimensional algebra $\mathfrak{sl}(4)$ — thus it must be the identity operator and the chiral algebra will be trivial. We shall evade this issue with the following strategy:

- Construct an $\widetilde{\mathfrak{sl}(2)}$ inside $\mathfrak{sl}(2|2)$, commuting with the holomorphic $\mathfrak{sl}(2)$, such that it is exact with respect to a supercharge $\mathbb{Q}$.
- Impose that anti-holomorphic Möbius transformations are generated by $\widetilde{\mathfrak{sl}(2)}$.

- Pass to the cohomology of $\mathbb{Q}$, where correlators will be meromorphic.

There is a family of choices for $\mathbb{Q}$ which give equivalent cohomologies [BLL+15] and so, for concreteness, we choose

$$\mathbb{Q} = \mathcal{Q}^i + \tilde{\mathcal{S}}^2 \,. \tag{1.2.7}$$

The $\mathbb{Q}$-exact $\widetilde{\mathfrak{sl}(2)}$ is generated by

$$\begin{aligned} \widetilde{L}_{-1} &\coloneqq \{\mathbb{Q}, \mathcal{Q}_1\} = \bar{L}_{-1} + R_- \,, \\ \widetilde{L}_1 &\coloneqq \{\mathbb{Q}, \mathcal{S}_2\} = \bar{L}_+ + R_+ \,, \\ \widetilde{L}_0 &\coloneqq \{\mathbb{Q}, \mathbb{Q}^\dagger\} = \bar{L}_0 + R \,. \end{aligned} \tag{1.2.8}$$

Notice that a spatial translation along $\bar{z}$ is a spatial translation along the plane as well as an R-symmetry transformation. Thus, the spacetime and R-symmetries have been *twisted* together to form the new conformal algebra $\mathfrak{sl}(2) \times \widetilde{\mathfrak{sl}(2)}$.

1.2.2 Schur operators and S-duality

Let us compute the spectrum of local operators in the cohomology. It is important to note that we restrict ourselves, exclusively, to *local* operators. The supercharge $\mathbb{Q}$ commutes with $\hat{L}_0$ and the central charge Z, since both are exact, and so we can decompose the cohomology into eigenspaces of the two operators. Suppose O is a representative of a cohomology class and let $A = [\mathbb{Q}, B]$ be an exact operator, then

$$[A, O] = [[\mathbb{Q}, B], O] = -\underbrace{[\mathbb{Q}, [B, O]]}_{\mathbb{Q}-\text{exact}} + [B, \underbrace{[\mathbb{Q}, O]}_{[\mathbb{Q},O]=0}] = 0 \,, \tag{1.2.9}$$

where we have used the super-Jacobi identity in the second equality. From this argument, we see that an operator O that is $\mathbb{Q}$-closed but not exact must satisfy

$$\hat{L}_0 = E + j_1 + j_2 - R = 0 \,, \quad Z = r + j_1 - j_2 = 0 \,, \tag{1.2.10}$$

where E is the (four dimensional) conformal weight and j_1, j_2 are the Lorentz spins of and R, r are the respective eigenvalues of the generators R and r. In fact, assuming our initial four dimensional SCFT is unitary, it is both necessary and sufficient to impose

$$E + j_1 + j_2 - R = 0\,. \tag{1.2.11}$$

Operators satisfying such a constraint are called Schur operators, since they are precisely the operators that the Schur limit of the index in (1.1.17) counts. The short multiplets that satisfy the Schur condition are precisely those found in Table 1.1.1. As a vector space,

$$\chi[\mathcal{T}] = \{\text{Schur operators of } \mathcal{T}\}\ . \tag{1.2.12}$$

The translation of a Schur operator away from the origin is given by

$$O(z,\bar{z}) = e^{zL_{-1}+\bar{z}\hat{L}_{-1}}O(0)e^{-zL_{-1}-\bar{z}L_{-1}}\,. \tag{1.2.13}$$

Finally, we note that the OPE of two Schur operators $O_1(z,\bar{z})$and $O_2(0)$ is also chiral i.e.

$$O_1(z,\bar{z})O_2(0) = \sum_{k \text{ Schur}} \frac{\lambda_{12k}}{z^{h_1+h_2-h_k}} O_k(0) + \mathbb{Q} - \text{exact}\,, \tag{1.2.14}$$

where h_i are the two dimensional conformal weights with respect to L_0, and λ_{ijk} are the structure constants of the four-dimensional theory. Once we pass to cohomology, the OPE becomes meromorphic. Thus, the subspace of Schur operators restricted to this plane has the structure of a vertex algebra after passing to cohomology.

The short multiplets comprising Schur operators are BPS states, and so are protected from quantum corrections. In other words, the spectrum of Schur operators is independent of the choice of gauge couplings. As a result the associated vertex algebra, $\chi[\mathcal{T}]$, is independent of any gauge couplings of $\mathcal{T}$.

The S-duality group acts on the gauge couplings of $\mathcal{T}$, tracing out an orbit in the space of theories. However, since $\chi[\mathcal{T}]$ is independent of said gauge couplings, the S-duality group must act on $\chi[\mathcal{T}]$ by automorphisms. In this sense, $\chi[\mathcal{T}]$ is an invariant of $\mathcal{T}$, just like the Higgs branch and the superconformal index.

Furthermore, the product on SCFTs descends to the tensor product over $\mathbb{C}$ for the associated vertex algebras. In other words, for SCFTs $\mathcal{T}_1$ and $\mathcal{T}_2$,

$$\chi[\mathcal{T}_1 \boxtimes \mathcal{T}_2] = \chi[\mathcal{T}_1] \otimes \chi[\mathcal{T}_2] \ . \tag{1.2.15}$$

1.2.3 Affine enhancements

We have uncovered a vertex algebra structure within a four dimensional $\mathcal{N} = 2$ superconformal field theory. One would be inclined to ask if this is a conformal vertex algebra. To find a conformal vector, we look at the four dimensional stress tensor multiplet. The Schur operator in this multiplet is $J^{11}_{+\dot{+}}$, a component of the $\mathfrak{sl}(2)_R$-symmetry current and *not* the four dimensional stress energy tensor. Its image under the correspondence is

$$T \propto \chi[J^{11}_{+\dot{+}}] \tag{1.2.16}$$

is a good guess for the conformal vector. The TT OPE is completely fixed by the OPE of the $\mathfrak{sl}(2)_R$ currents and the OPE, after normalization, is [BLL+15]

$$T(z)T(0) \sim -\frac{6c_{4d}}{z^4} + \frac{2T(0)}{z^2} + \frac{\partial T(0)}{z} \,, \tag{1.2.17}$$

where c_{4d} is the four dimensional central charge. The above has the correct form for a stress energy tensor OPE. However, we are forced to conclude that the central charge,

$$c_{2d} = -12c_{4d}, \tag{1.2.18}$$

of the two dimensional theory is negative for a unitary four-dimensional theory. Thus, for a sensible physical theory, the associated two dimensional theory must be non-unitary, which is problematic only if we wished to interpret the vertex algebras as arising from a physical two dimensional CFT.

The conformal weight of a Schur operator under this conformal vector satisfies,

$$\Delta = E - R \tag{1.2.19}$$

The character of the VOA is given by

$$\mathrm{ch}_{\chi[\mathcal{T}]} \coloneqq \mathrm{tr}_{\chi[\mathcal{T}]}(-1)^F q^{\Delta} = \mathrm{tr}_{\chi[\mathcal{T}]}(-1)^F q^{E-R} \tag{1.2.20}$$

Comparing to the expression in (1.1.17), we see that since this trace runs over the Schur operators the character of $\chi[\mathcal{T}]$ agrees with the Schur limit of the index.

The enhancement of the spacetime $\mathfrak{sl}(2)$ to the Virasoro algebra is not the only infinite enhancement. Suppose the four dimensional SCFT has some flavour symmetry G_F, then the associated conserved currents $J^F_{\alpha\dot{\alpha}}$ will be part of a multiplet. The Schur operator in this multiplet is $M^{11,A}$ a component of the moment map operator, where A is an index valued in the adjoint. We suggestively write

$$J^A \propto \chi[M^{11,A}]\,, \tag{1.2.21}$$

for the image under the correspondence. The currents J^A , after normalization, have the OPEs

$$J^A(z)J^B(0) \sim \frac{-k_{4d}/2}{z} + i\sum_C f^{ABC}\frac{J^C(0)}{z^2}\,, \tag{1.2.22}$$

which we recognise to be the current algebra OPE of the universal affine vertex algebra, $V^{k_{2d}}$, at level $k_{2d} = -k_{4d}/2$ and k_{4d} is the flavour anomaly in four dimensions[4] . For a review of some basic definitions regarding affine vertex algebras, see Appendix A.3.

Therefore, when $\mathcal{T}$ has a $\mathfrak{g}$ symmetry, $\chi[\mathcal{T}]$ possesses a chiral moment map $\mu : V^{k_{2d}}(\mathfrak{g}) \to \chi[\mathcal{T}]$. The conformal grading on $\chi[\mathcal{T}]$ is $\mathbb{Z}_{>0}$ graded and so this makes $\chi[\mathcal{T}]$ a vertex algebra object in the Kazhdan–Lusztig category, $\mathrm{KL}_{k_{2d}}$, at level k_{2d}.

[4]This flavour anomaly appears as the coefficient of a mixed $U(1)_r$-G anomaly. Alternatively, the flavour anomaly appears in the most singular term of the four-dimensional OPE between two flavour currents, see Section 2.6 of [AATM+22] for more details

1.2.4 The Vertex algebras of hypermultiplets and vectormultiplets

Since hypermultiplets and vectormultiplets are the building blocks of gauge theories, we shall discuss the vertex algebras associated to each before moving on to discussing gauge theories.

The Schur operators in a hypermultiplet are the sections $Q, \tilde{Q}$ as well as their derivatives (descendants) with respect to $\partial_{+\dot{+}}$. Along with the $SU(2)_R$ symmetry, the theory enjoys an $SU(2)_F$ symmetry under which Q and $\tilde{Q}$ transform into each other. Therefore, we shall write

$$Q_I = \begin{pmatrix} Q \\ \tilde{Q} \end{pmatrix}, \tag{1.2.23}$$

for the doublet. The corresponding element of the chiral algebra is denoted by q_I and its OPE is given by

$$q_I(z) q_J(0) \sim \frac{\epsilon_{IJ}}{z}, . \tag{1.2.24}$$

This is precisely the symplectic boson algebra, or $\mathcal{D}^{ch}(\mathbb{C})$—the chiral differential operators on $\mathbb{C}$. For hypermultiplets valued in a quaternionic (polarised) representation T^*N of a group G, the associated vertex algebra are symplectic bosons valued in T^*N or, equivalently $\mathcal{D}^{ch}(N)$.

The Schur operators of a free vectormultiplet are the gauginos $\lambda_+, \tilde{\lambda}_{\dot{+}}$ (suppressing the adjoint index) as well as their derivatives with respect to $\partial_{+\dot{+}}$. The corresponding elements in cohomology are $\lambda(z) = \chi[\lambda_+]$ and $\tilde{\lambda} = \chi[\tilde{\lambda}_{\dot{+}}]$ with OPEs

$$\tilde{\lambda}(z) \lambda(0) \sim \frac{1}{z^2}\,. \tag{1.2.25}$$

This corresponds to the vertex subalgebra of a (b, c) ghost system of weight (1,0) under the identification

$$\tilde{\lambda} \leftrightarrow b\,, \quad \lambda \leftrightarrow \partial c\,. \tag{1.2.26}$$

For free vector multiplets in the adjoint representation of a group G, we have the analogous subalgebra in the (b, c) ghosts system valued in the adjoint representation.

Note that we have introduced an asymmetry between λ and $\tilde{\lambda}$ by our choice. Furthermore, we must restrict to the subalgebra of the (b, c) system that is annihilated by b_0 to account for the fact that c_0 modes are not present. This will be important when we discuss gauging.

1.2.5 Gauge theories

Having constructed the vertex algebras of free theories, let us move onto interacting ones. We shall try to be as general as possible, so as to not exclude theories without Lagrangians. Suppose $\mathcal{T}$ is a four dimensional $\mathcal{N} = 2$ SCFT with some global symmetry G that we wish to gauge. On the four dimensional side, this is done by introducing a vector multiplet V valued in $\mathfrak{g} = \operatorname{Lie} G$ and projecting to gauge invariant states via BRST cohomology.

Analogously, the vertex algebra $\chi[\mathcal{T}]$ is has a chiral moment map $\mu : V^{k_{2d}}(\mathfrak{g}) \to \chi[\mathcal{T}]$, which gives it the structure of a module over $V^{k_{2d}}(\mathfrak{g})$. Notice that $\chi[V]$ is very similar to $\bigwedge^{\frac{\infty}{2}+\infty}(\mathfrak{g})$, the space of semi-infinite forms for $\widehat{\mathfrak{g}}$. It is natural to interpret $\chi[V]$ as the *bc* ghost system for BRST reduction.

The BRST current will be

$$J_{BRST} = \sum_A J^A c^A + \frac{1}{2} \sum_{ABC} [c^A, c^B] b^C \,, \tag{1.2.27}$$

where J^A are the images of the generators of $V^{k_{2d}}(\mathfrak{g})$ under $\mu : V^{k_{2d}}(\mathfrak{g}_F) \to \chi[\mathcal{T}]$ and normal ordering is assumed. The differential is given by

$$Q = J_{BRST,(0)} \,. \tag{1.2.28}$$

The cochain complex $(\chi[\mathcal{T}] \otimes \chi[V], Q)$ is very similar to the Feigin standard complex for semi-infinite cohomology (see Section A.5.2 for more details). Note that $\chi[V]$ is not the full ghost vertex algebra, but rather the subalgebra annihilated by b_0. This means that the BRST reduction should be interpreted as the *relative* semi-infinite cohomology of $\chi[\mathcal{T}]$-–relative to the $\mathfrak{g}$ subalgebra. Therefore, the vertex algebra of the gauged theory must satisfy

$$\chi[\mathcal{T}_G] \cong \mathrm{H}^{\frac{\infty}{2}+\bullet}(\widehat{\mathfrak{g}}_{k_{2d}}, \mathfrak{g}, \chi[\mathcal{T}]) \ . \tag{1.2.29}$$

Note that there are constraints on the level for this BRST reduction to be non-trivial. If $k_{2d} \neq -\kappa_g = -2h^\vee$, the BRST differential does not square to zero. The semi-infinite cohomology in this case is still well-defined but vanishes in all degrees. This level-matching condition is precisely the anomaly vanishing condition that ensures that the resulting gauge theory is superconformal.

In our definition of gauging, we have not restricted to zero cohomological degree. The cohomology away from degree zero is, in many interesting cases, non-vanishing and of physical interest. Recall the trichotomy (see Remark 1.1.3 of possible Higgs branches—in particular the cases of an ``enhanced'' Higgs branch which should be thought of as a derived symplectic space. Physically, these are tracked by fermionic operators in the Hall--Littlewood chiral ring. In particular, these fermionic operators are Schur operators and so contribute to $\chi[\mathcal{T}_G]$ as fermionic states in the vertex algebra. Therefore, we expect the cohomology of (1.2.29) to have support outside of degree zero.

More formally, the cohomology outside degree zero measures the failure of $\chi[\mathcal{T}]$ to be a semijective module over $\widehat{\mathfrak{g}}$—which should be understood as a chiral analogue of the failure of a moment map to be flat in symplectic reduction.

Chapter 2

The chiral algebras of class $\mathcal{S}$

,has the naughty thumb
of science prodded
thy
    beauty

E. E. Cummings
[O sweet spontaneous]

2.1 Theories of Class $\mathcal{S}$

In this section, we provide a review on the physics background of theories of class $\mathcal{S}$. This is not meant to be a comprehensive overview; more background on these theories can be found in, *e.g.*, [Gai09, GMN09, Tac11].

We start with a review of their construction as a dimensional reduction of the six-dimensional $\mathcal{N} = (2, 0)$ parent theory on a curve (the UV curve) in Section 2.1.1. The compactification data includes boundary conditions at the marked points of the curve and we discuss their classification in Section 2.1.2.

In the setting of class $\mathcal{S}$, the action of S-duality is geometric in nature, arising from the diffeomorphisms of the UV curve. We discuss this identification in Section 2.1.4, as well as giving a description of the generators of this group in terms of elementary moves swapping between pants decompositions.

In Section 2.1.5 we review the TQFT computation of the Schur limit of the superconformal index, following [GRRY13]. We shall also recast the expression for the index of *loc. cit.* in more representation theoretic terms—rewriting them in terms of characters over modules of the critical level affine Kac–Moody algebra $\widehat{\mathfrak{g}}_{\kappa_c}$.

2.1.1 A definition from six-dimensional origins

The theories of class $\mathcal{S}$ are four-dimensional $\mathcal{N}=2$ SCFTs with a six-dimensional origin. Six dimensions is special, in that superconformal theories do not exist in higher dimensions. The maximal superconformal algebra is that of $\mathcal{N}=(2,0)$ theories in six dimensions, and is isomorphic to $\mathfrak{osp}(8,4)$.

The representation theory of $\mathfrak{osp}(8,4)$, does not allow for any representation to satisfy the equations of motion of super Yang–Mills[1] and so there are no gauge theories of this type. Nevertheless, these theories are classified by the choice of a simply laced, simple Lie algebra $\mathfrak{g}_u$. One way to see the appearance of this *ADE* classification is via the string theory construction.

These theories can be geometrically engineered (see [BI97, HMV14]) as the low-energy limit of Type IIB string theory on $\mathbb{R}^6 \times \mathbb{C}^2/\Gamma$ with Γ a finite subgroup of SU(2). Such finite subgroups have an *ADE* classification following the Mckay correpsondence. The theories of A_N-type admit an M-theory uplift and can be realised as the world-volume theory of a stack of $N+1$-M5 branes in $\mathbb{R}^6$.

Starting with such a theory, characterised by a choice of $\mathfrak{g}_u$, we can compactify on a $\mathbb{C}$-curve, Σ, to produce a theory on $\mathbb{R}^4 \times \Sigma$, whose low energy limit will be a four dimensional $\mathcal{N}=2$ SCFT[2]. The resulting four-dimensional SCFTs are the eponymous *theories of class $\mathcal{S}$*. The theory depends on our choice of Σ, only up to conformal transformations. In fact

[1] There is an exception to this statement for abelian gauge groups, however we will not be interested in such theories

[2] To preserve eight supercharges one must actually perform a topological twist as part of the compactification. This involves twisting the action of the spatial symmetries by the R-symmetry. We will largely ignore this subtlety.

the complex structures of Σ correspond precisely to gauge couplings in the theory.

Our choice of curve may have boundary components, *i.e.*, punctures corresponding to the insertion of codimension-2 defects of the six-dimensional theory, which are transversal to Σ. The classification of such defects is quite involved and we shall delve into it momentarily.

For now our working definition of class $\mathcal{S}$ is an SCFT that is specified by a triple $(\mathfrak{g}_u, \Sigma_{g,s}, \{\Lambda_i\}_{i=1}^s)$ where

- $\mathfrak{g}_u$ is a simply laced simple Lie algebra
- $\Sigma_{g,s}$ is a connected Riemann surface (algebraic curve) of genus g with s many punctures (marked points). We call this the *UV curve* of the theory.
- $\{\Lambda_i\}_{i=1}^s$ are labels for each marked point

Strictly speaking, curves that are related by a conformal diffeomorphism define the ''same'' theory. As a result we should think of our gauge couplings τ_i as co-ordinates on the moduli space $\mathcal{M}_{g,s}$ of curves of genus g with s-marked points. In fact, we will also be interested in singular curves and so work over the Deligne–Mumford compactification $\overline{\mathcal{M}}_{g,s}$.

Note that if we choose a disjoint union of curves $\Sigma_{g,s} \sqcup \Sigma_{g',s'}$, the resulting theory is the product of theories defined by $\Sigma_{g,s}$ and $\Sigma_{g',s'}$.

2.1.2 Untwisted punctures

The labels of punctures in class $\mathcal{S}$ fall under a dichotomy of either *regular* or *irregular*, and in this work we shall restrict our attention entirely to the regular case. For a review of the irregular case we point the reader to [Xie13, GMN09].

Regular, untwisted punctures in a theory of type $\mathfrak{g}_u$ are labelled by a nilpotent element of $\mathfrak{g}_u$, up to conjugacy. Therefore, such punctures are labelled by a nilpotent orbit in $\mathfrak{g}_u$ or, equivalently by the Jacobson–Morozov theorem (Theorem A.1.4), by a conjugacy class of a homomorphism $\Lambda : \mathfrak{sl}_2 \to \mathfrak{g}_u$.

The defect operator corresponding to such a puncture enjoys some global symmetry that it then contributes to the total global flavour symmetry of the four dimensional SCFT. The contributed symmetry is of the form

$$\mathfrak{f}_\Lambda = \ker \mathrm{ad}_{\Lambda(\mathfrak{sl}_2)} \tag{2.1.1}$$

i.e., the commutant of the image of $\mathfrak{sl}_2$ in $\mathfrak{g}_u$.

As discussed in Proposition A.1.5, any simple Lie algebra has at least two especially important nilpotent orbits: the principal orbit (which is the unique, largest nilpotent orbit) and the trivial orbit. Their respective commutants are $\mathfrak{f} = 0$ for the principal orbit and $\mathfrak{f} = \mathfrak{g}_u$ for the trivial orbit. A puncture labelled by the trivial embedding is, therefore, called a *maximal* puncture, while a puncture labelled by the principal embedding contributes no flavour symmetry and is equivalent to having no puncture at all. With an eye towards the twisted case, we will nevertheless adopt the convention of referring to a hypothetical puncture labelled by the principal embedding as an *empty* puncture.

Given a theory where a particular puncture is maximal, the corresponding theory with that puncture replaced by a sub-maximal puncture can be realised by partially Higgsing the $\mathfrak{g}_u$ flavour symmetry associated to that puncture in the four dimensional theory. A choice of nontrivial Λ is realised by assigning an expectation value to the ``moment map'' Higgs branch operator in the conserved current multiplet that lies in the corresponding nilpotent orbit. Consequently, for many purposes it is sufficient to be able to construct theories associated to surfaces with maximal punctures, with other punctures structures being subsequently reached via partial Higgsing. Henceforth, we shall restrict ourselves to the case where $\Sigma_{g,s}$ has only maximal punctures.

We wish to point out that not all UV curves produce valid SCFTs. For example, the theories associated to $\mathbb{P}^1$ with less than three punctures are not good four-dimensional theories. This is true independent of the labelling of these punctures (so long as we only have regular punctures). Nevertheless, these objects will play an important role in the construction of the chiral algebras of class $\mathcal{S}$.

2.1.3 Trinions, gluing and residual gauge symmetry

The gauge couplings of the SCFT are precisely the complex structures τ_i of the UV curve. Therefore, the weak coupling limit of the theory corresponds to the degeneration limits of the UV curve. Consider driving the complex structures to the most singular stratum of the (compactified) moduli space—this corresponds to a limit where the UV curve decomposes into various pairs-of-pants or trinions, with thin tubes connecting maximal punctures.

The SCFTs associated to $\mathbb{P}^1$ with three maximal punctures can therefore be thought of as the indecomposable atoms of this space of theories. These are often referred to as *trinion theories* or *fixtures*. There are myriad possible (combinations of) labels that can appear on a single trinion; these have been extensively detailed by (various subsets of) Chacaltana, Distler, Tachikawa, and Trimm in [CDT13, CD10, CD13].

Given any two curves $\Sigma_{g,s}$ and $\Sigma'_{g',s'}$, we can form the disjoint union $\Sigma_{g,s} \sqcup \Sigma'_{g',s'}$, corresponding to the product on the associate SCFTs. Suppose each surface has at least one maximal puncture[3] and let us single out one such puncture on each surface. The chosen punctures give rise to a $\mathfrak{g}_u \times \mathfrak{g}_u$ global symmetry on the product theory. We can choose to gauge by the diagonal action of $\mathfrak{g}_u \times \mathfrak{g}_u$, producing a gauge theory at coupling τ. On the UV curves, this operation corresponds to gluing $\Sigma_{g,s}$ and $\Sigma'_{g',s'}$ along the singled out punctures to produce a curve $\Sigma_{g+g',s+s'-2}$, which has one additional complex structure specified by the coupling τ.

Given a single curve $\Sigma_{g,s}$ with at least two maximal punctures, one may also perform a self-gluing by gauging the diagonal $\mathfrak{g}_u \times \mathfrak{g}_u$ corresponding to two maximal punctures. This produces the curve $\Sigma_{g+1,s-2}$. The genus is not conserved under this gluing, and we shall see that this has certain implications for the underlying SCFT.

Every curve can be produced by gluing (or self-gluing) various pairs of trinions and then performing Higgsings to obtain the correct labels. Therefore the trinion theories associated to $\Sigma_{0,3}$ are the fundamental building blocks of class $\mathcal{S}$.

[3]Note that one can only perform gluings along maximal punctures. If one were to try and gauge the diagonal action of a non-maximal symmetry, the resulting theory would not be an SCFT.

Before we move on, we shall go through a brief digression on residual gauge symmetries. Recall the Higgs and Coulomb branches of Section 1.1.7 and the trichotomy of pure, enhanced and interacting Higgs branches. To the best of our knowledge, in the presence of regular punctures only pure and enhanced Higgs branches can arise, whereas interacting Higgs branches are common in Argyres–Douglas type theories that can be engineered with irregular punctures (once again see [Xie13, GMN09]).

In the untwisted class $\mathcal{S}$ setting, there is a simple characterisation of under what circumstances a theory will have an enhanced rather than a pure Higgs branch: theories of type A_n, D_n, or E_n with genus g have generic residual gauge symmetry with rank equal $n \times g$. In other words, the genus zero theories of class $\mathcal{S}$ (in the untwisted setting) all have pure Higgs branches.

2.1.4 Mapping class groups and S-duality

The parent six-dimensional theory is conformally invariant and so only sees the UV curve up to conformal transformations. This conformal invariance implies that the SCFTs associated to two UV curves, in the same conformal class, should be dual. Precisely stated, we can identify the S-duality group of a class $\mathcal{S}$ theory with the mapping class group $\mathrm{MCG}(\Sigma_{g,s})$ of its UV curve.

More pictorially, the action of S-duality swaps between the various pair-of-pants decompositions of a theory. On the SCFT side this constitutes a highly nontrivial set of quantum dualities, identifying various different inequivalent gaugings of trinion theories as being S-dual.

In the untwisted case, all pants decompositions can be reached by iterating two types of elementary move. Our presentation is similar to that of the Moore–Seiberg groupoid [MS89].

For the first type of move, consider $\mathbb{P}_1$ with four maximal punctures $\Sigma_{0,4}$ with some labelling $1, \ldots, 4$ of its punctures. We move to the singular point on the moduli space where the sphere decomposes into two trinions connected by a long tube, one of which has punctures labelled by $1, 2$ and the other by $3, 4$. In Figure 2.1, this is represented by the duality frame on the left. By moving to a different singular point, one can decompose $\Sigma_{0,4}$ into two connected trinions labelled by $1, 3$ and $2, 4$—shown on the right hand side of Figure 2.1. We call such a move, moving between the various decompositions of $\Sigma_{0,4}$, a 4-move. This can be lifted to a 4-move acting on any collection of four punctures on a general $\Sigma_{g,s}$. First, one moves to a singular locus on the moduli space where $\Sigma_{g,s}$ decomposes into a $\Sigma_{0,4}$ that contains the punctures of interest and is connected by a long tube to $\Sigma_{g,s-4}$. Then one applies a 4-move to swap between decompositions of $\Sigma_{0,4}$, before moving back out of the singular loci.

For surfaces $\Sigma_{g,s}$ with $g > 0$, one must consider another type of move. For example, consider the one punctured torus $\Sigma_{1,1}$. The S generator of the modular group acts by swapping the a and b cycles of the torus. This is a homeomorphism of the torus to itself that is not homotopic to any iterated 4-move and so must be a generator, which we call the *ab*-move. For a surface, $\Sigma_{g,s}$, there is a natural generalisation of this move. The *ab*-move acts as the S-duality $\tau \mapsto -1/\tau$ on the complex gauge coupling associated to the handle, *cf.*, the example of $SU(2)$ $\mathcal{N} = 4$ SYM in Section 1.1.5

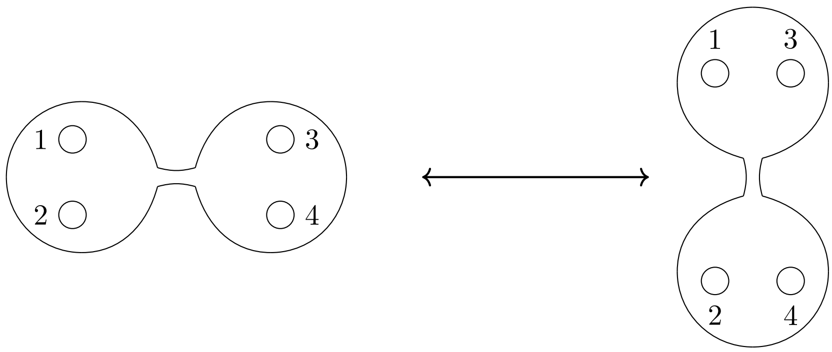

Figure 2.1: The 4-move acting on four maximal untwisted punctures.

We are primarily interested in observables that do not depend on exactly marginal deformations, such as gauge couplings. Such observables are locally constant over $\overline{\mathcal{M}}_{g,s}$. In the untwisted case, the *ab*-move is trivial for such observables and the action of the 4-moves can be phrased in terms of the action of permutations on punctures. The 4-moves, in the untwisted case, act by permuting the flavour symmetries associated to the punctures, which shows up as permutations on the flavour fugacities (see the next section) of the index or permuting the moment-maps of the Higgs branch and the associated vertex algebra. It is, therefore, useful to phrase the action of S-duality on these observables in terms of the action of a permutation group. Indeed, this corresponds to the subgroup of automorphisms $S_s \subset \mathrm{Aut}\, \mathcal{M}_{g,s}$ which permute the points in a singular stratum.

2.1.5 The superconformal index of theories of class $\mathcal{S}$

For theories of class $\mathcal{S}$, the form of the full superconformal index has been shown to follow from duality properties [GRR13], though for our purposes here we will restrict attention to the Schur limit. The index takes the form of a sum over highest weights in the set of integrable dominant weight representations P^+ (*i.e.*, a sum over finite-dimensional $\mathfrak{g}$ representations), weighted by some ``structure constants'' $C_{\lambda\lambda\lambda}$, which are functions of q. For a UV curve with all maximal punctures maximal, $\Sigma_{g,s}$, which can be realised by gluing $2g-2+s$ trinions, the index takes the form [AOSV05, GPRR10, GRRY13, ABFH13]

$$\mathcal{I}(q;\mathbf{x}_1,\mathbf{x}_2,\ldots,\mathbf{x}_s) = \sum_{\lambda\in P^+} (C_{\lambda\lambda\lambda}(q))^{2g-2+s} \prod_{i=1}^{s} \mathcal{K}(q;\mathbf{x}_i)\chi^\lambda(\mathbf{x}_i) \ , \tag{2.1.2}$$

where P^+ are the set of positive integral dominant weights of the simply connected group G_u. The $\mathbf{x}_i$ are flavour fugacities valued in a maximal torus of G_u and the $\mathcal{K}$ factors are defined as in (1.1.21). The χ^λ are Schur polynomials, *i.e.*, characters of the finite-dimensional $\mathfrak{g}_u$ representation with highest weight λ.

At the level of the index, Higgsing a maximal puncture to a non-maximal one amounts to a fugacity replacement (with a subtraction of certain divergences) that is detailed in [GRRY13, RR16, BPRvR15]. We will not use this technique but instead we wish to note the following. Since reducing to an empty puncture amounts to removing the puncture all together, one has the relation

$$C_{\lambda\lambda\lambda}(q) = \frac{1}{\mathcal{K}(\times)\chi^\lambda(\times)} \ , \tag{2.1.3}$$

where $\times$ represents the regularised fugacity replacement for the empty puncture, which is purely a function of q.

For the empty puncture, we have

$$\mathcal{K}(\times) = \prod_{i=1}^{\operatorname{rk}\mathfrak{g}} \frac{1}{(q^{d_i}, q)_\infty} \ . \tag{2.1.4}$$

Looking ahead, we can identify $\mathcal{K}(\times)$ with the character of the Feigin–Frenkel centre $\mathfrak{z}(\mathfrak{g})$. Similarly, we can identify the summand, $\mathcal{K}(q;\mathbf{x})\chi^\lambda(\mathbf{x})$ can be identified with the character of the critical level Weyl module of highest weight λ, $\mathbb{V}_\lambda$ as

$$\mathcal{K}(q;\mathbf{x})\chi^\lambda(\mathbf{x}) = \mathrm{ch}_{\mathbb{V}_\lambda} \equiv \mathrm{Tr}_{\mathbb{V}_\lambda}(q^D \mathbf{x}^T) \ , \tag{2.1.5}$$

where D is the quasiconformal weight. For the sake of clarity, we shall often suppress the flavour fugacities and use the notation $\mathrm{ch}\, V = \mathrm{Tr}_V(q^D)$.

At the critical level, the Weyl modules are not irreducible—instead they have a unique simple quotient $\mathbb{L}_\lambda$. The characters of the Weyl module and their simple quotients are related by

$$\mathrm{ch}\ \mathbb{L}_\lambda = \frac{\mathrm{ch}\ \mathbb{V}_\lambda}{\mathrm{ch}\ \mathfrak{z}_\lambda} = \frac{\mathcal{K}(q;\mathbf{x})\chi^\lambda(\mathbf{x})}{\mathcal{K}(\times)\chi^\lambda(\times)} \ , \tag{2.1.6}$$

where $\mathfrak{z}_\lambda$ is the Drinfel'd–Sokolov reduction of $\mathbb{V}_\lambda$, as in Proposition A.3.6.

The index of $\Sigma_{g,s}$ can then be completely rewritten in terms of characters of Weyl modules, or their simple quotients, as

$$\mathcal{I}(q;\mathbf{x_1}\mathbf{x_2},\ldots,\mathbf{x}_s) = \sum_{\lambda\in P^+} (\mathrm{ch}\ \mathfrak{z}_\lambda)^{2-2g} \prod_{i=1}^{s} \mathrm{ch}\ \mathbb{L}_\lambda = \sum_{\lambda\in P^+} (\mathrm{ch}\ \mathfrak{z}_\lambda)^{2-2g} \prod_{i=1}^{s} \frac{\mathrm{ch}\ \mathbb{V}_\lambda}{\mathrm{ch}\ \mathfrak{z}_\lambda} \ . \tag{2.1.7}$$

2.2 Arakawa's construction of the chiral algebras of $\mathcal{S}$

The theories of class $\mathcal{S}$ define a family of SCFTs for each simply laced $\mathfrak{g}_u$, parameterised by a choice of $\Sigma_{g,s}$. Applying the SCFT/VOA correspondence of [BLL+15] gives rise to a family of vertex algebras also parameterised by a choice of $\Sigma_{g,s}$. However, the vertex algebras are independent of exactly marginal deformation, *i.e.*, they are independent of the complexified gauge couplings of the theory. The gauge couplings of the SCFT are precisely the complex structures of $\Sigma_{g,s}$. Therefore, the resulting family of vertex algebras only depends on the topological data of $\Sigma_{g,s}$.

From here on, we shall restrict our attention to the genus zero case, since these are the focus of Arakawa's construction. We shall also restrict to the case where all punctures are maximal. The vertex algebras corresponding to non-maximal punctures can then be obtained through Drinfel'd–Sokolov reduction—the vertex algebraic counterpart of Higgsing a puncture.

Following Arakawa's lead, we shall also relax the condition that $\mathfrak{g}$ is simply laced. For the rest of this chapter $\mathfrak{g}$ will refer to a simple Lie algebra. While this is unphysical, we wish to review the construction in its full generality. We will make a comment about the physical interpretation of vertex algebras with non-simply laced $\mathfrak{g}$ in Section 3.3.

We start this section with a summary of the properties that the genus zero chiral algebras should possess in Section 2.2.1. We introduce glued modules in Section 2.2.2. These are, roughly speaking, a product of Weyl modules of $\widehat{\mathfrak{g}}$, where we identify the action of the Feigin–Frenkel centre on each module. The chiral algebras of class $\mathcal{S}$ will be limits of these glued modules.

In Section 2.2.3, we introduce the technology of Feigin–Frenkel gluing, a kind of semi-infinite cohomology with respect to the action of the Feigin–Frenkel centre. With this technique in hand, we can begin our review of Arakawa's construction. We start off with defining the vertex algebras of the cylinder and the cap in Sections 2.2.4 and 2.2.5. These are strongly constrained by their properties under gluing—gluing the cylinder is an identity operation and gluing the cap closed the puncture.

Having defined the cap, allows us to introduce inverse Hamiltonian reduction in Section 2.2.6. At the level of pictures, inverse Hamiltonian reduction introduces a puncture to the UV curve by gluing on a cap *via the interior points*. More formally, it defines a (partial) inverse to the functor of Drinfel'd–Sokolov reduction.

The construction proper will be in Section 2.2.7. Here we review Arakawa's construction as well as reproducing their results on gluing and the various other properties detailed in

Section 2.2.1.

2.2.1 Chiral algebras of class $\mathcal{S}$ at genus zero

Before we begin with the construction, let us review the properties we expect from the vertex algebras associated to the genus zero SCFTS in class $\mathcal{S}$. The following will be a high level overview of what we should expect from the construction of [Ara18], based on the SCFT/VOA correspondence of Section 1.2—more details can be found in [BPRvR15]

Fix a simple Lie algebra $\mathfrak{g}$. Let Σ_s denote $\mathbb{P}^1$ with s maximal punctures. To such a curve we associate a vertex algebra object[4] $\mathbf{V}_{G,s}$ internal to KL, the Kazhdan–Lusztig category for $\widehat{\mathfrak{g}}_{\kappa_c}$. Recall from Remark A.3.3 that a vertex algebra object, V, in KL is a vertex algebra V equipped with a vertex algebra morphism $V^{\kappa_c}(\mathfrak{g}) \to V$, such that its structure as a $V^{\kappa_c}(\mathfrak{g})$-module is a limit of positive energy representations.

The vertex algebra is independent of the complex structure of Σ_s, in other words it is locally constant over $\overline{\mathcal{M}}_{0,s}$. In other words the braid group action on $\mathbf{V}_{G,s}$, coming from the mapping class group, factors through the action of the symmetric group that permutes punctures.

Each puncture on Σ_s gives rise to a vertex algebra morphism

$$\mu_i : V^{\kappa_c}(\mathfrak{g}) \to \mathbf{V}_{G,s} \ , \quad \text{for } i = 1, \ldots, s \ , \tag{2.2.1}$$

From the universal affine vertex algebra of $\mathfrak{g}$ at the critical level κ_c. We call these morphisms the chiral moment maps and their images are commuting $V^{\kappa_c}(\mathfrak{g})$ subalgebras inside $\mathbf{V}_{G,s}$. The natural symmetric group automorphism of $\overline{\mathcal{M}}_{0,s}$ means that all punctures are on a equivalent footing.

Pick such a moment map, we can perform principal Drinfel'd–Sokolov reduction with respect to this moment map to close the corresponding puncture. Therefore, the vertex

[4] Following Arakawa, we elect to label these vertex algebras by the simply connected group G with Lie algebra $\mathfrak{g}$.

algebras $\mathbf{V}_{G,s}$ and $\mathbf{V}_{s-1}$ are related via principal Drinfel'd–Sokolov reduction,

$$\mathrm{H}^{i}_{\mathrm{DS}}(\mathbf{V}_{G,s}) \cong \delta_{0i}\mathbf{V}_{s-1} \ . \tag{2.2.2}$$

Given two surfaces, $\mathcal{C}_s$ and $\mathcal{C}_{s'}$, we can glue them along two specified punctures to produce $\mathcal{C}_{s+s'-2}$. On the SCFT side this corresponds to gauging the diagonal $\mathfrak{g}$ symmetry. In Section 1.2.5 we argued that such a gauging on the SCFT side corresponds to BRST reduction or (relative) semi-infinite cohomology on the vertex algebra side. Therefore, the family $\mathbf{V}_{G,s}$ must satisfy the following gluing relations

$$\mathrm{H}^{\frac{\infty}{2}+\bullet}(\widehat{\mathfrak{g}}_{-\kappa_g}, \mathfrak{g}, \mathbf{V}_{G,s} \otimes \mathbf{V}_{G,s'}) \cong \mathbf{V}_{G,s+s'-2} \ . \tag{2.2.3}$$

Moreover, we expect these gluings to be concentrated in degree zero, since the corresponding genus zero SCFTs all have pure Higgs branches (see the discussion in Section 1.2.5). Therefore, we should expect that

$$\mathrm{H}^{\frac{\infty}{2}+i}(\widehat{\mathfrak{g}}_{-\kappa_g}, \mathfrak{g}, \mathbf{V}_{G,s} \otimes \mathbf{V}_{s'}) \cong \delta_{i,0}\mathbf{V}_{s+s'-2} \ . \tag{2.2.4}$$

For compactness, we introduce the notation $\circ$, where for any $M, N \in \mathrm{KL}$,

$$M \circ N \equiv \mathrm{H}^{\frac{\infty}{2}+\bullet}(\widehat{\mathfrak{g}}_{-\kappa_g}, \mathfrak{g}, M \otimes N) \ . \tag{2.2.5}$$

Finally, four-dimensional physics also imposes some constraints on the algebraic structure of $\mathbf{V}_{G,s}$. We know that $\mathbf{V}_{G,s}$ must be conformal, and from the expressions for the four-dimensional central charge [CDT13], we see that

$$c_{\mathbf{V}_{G,s}} = (s-(s-2)h^{\vee})\dim\mathfrak{g} - (s-2)\mathrm{rk}\,\mathfrak{g} \ . \tag{2.2.6}$$

Unitarity demands that $c_{\mathbf{V}_{G,s}}$ is negative and this is true as long as $s > 2$. Furthermore, unitarity demands that $\mathbf{V}_{G,s}$ is non-negatively $\frac{1}{2}\mathbb{Z}$-graded by conformal weight, with the weight zero component being spanned by the vacuum vector. These constraints imply that

$\mathbf{V}_{G,s}$ is of CFT type for $s > 2$.

2.2.2 Glued modules of the Feigin–Frenkel centre

In this subsection, we shall define a certain class of modules in KL, which shall appear in the decompositions of the chiral algebras of class $\mathcal{S}$, as $V^{\kappa_c}(\mathfrak{g})$ modules.

Let $\mathfrak{z}(\mathfrak{g}) \subset V^{\kappa_c}(\mathfrak{g})$ be the Feigin–Frenkel centre of the universal affine vertex algebra. Recall that the Feigin–Frenkel centre is a commutative vertex subalgebra of $v^{\kappa_c}(\mathfrak{g})$ which has non-singular OPEs with all fields, *i.e.*, it is central. It is strongly generated, as a commutative vertex algebra, by generators P_{d_i}, where d_i are the exponents of $\mathfrak{g}$. We denote by $\mathcal{Z}$ the algebra of Fourier modes of $\mathfrak{z}(\mathfrak{g})$. See Appendix A.3.3 for more details.

As a commutative algebra,

$$\mathcal{Z} \cong \mathbb{C}[P_{d_i,(n)}, \,|\, d_i = 1, \dots, \operatorname{rk}\mathfrak{g} \; n \in \mathbb{Z}] \; . \tag{2.2.7}$$

Furthermore, we denote $\mathcal{Z}_{<0}$ to be the subalgebra,

$$\mathcal{Z}_{<0} \coloneqq \mathbb{C}[P_{d_i,n}, \,|\, d_i = 1, \dots, \operatorname{rk}\mathfrak{g} \; n \in \mathbb{Z}_{<0}] \; , \tag{2.2.8}$$

where it should be noted that we have used the physicist's gradings on the mode number with $P_{d_i,n} = P_{d_i,(n)-d_i-1}$.

First we recall the quotient, $\mathfrak{z}_\lambda$ of $\mathcal{Z}_{<0}$ from Definition A.3.5. Let $\mathbb{V}_\lambda$ be a Weyl module over $\widehat{\mathfrak{g}}_{\kappa_c}$ with $\lambda \in P^+$. Let $\mathcal{I}_\lambda$ be the annihilator ideal of $\mathbb{V}_\lambda$ inside $\mathcal{Z}_{<0}$, then

$$\mathfrak{z}_\lambda = \mathcal{Z}_{<0}/\mathcal{I}_\lambda \tag{2.2.9}$$

From the definition, $\mathfrak{z}_\lambda$ acts on $\mathbb{V}_\lambda$ freely by the projection $\mathcal{Z}_{<0} \twoheadrightarrow \mathfrak{z}_\lambda$.

Suppose $M, N \in \mathcal{Z}$-Mod, we endow $M \otimes N$ with the structure of a $\mathcal{Z}$-module in the following way. Let $\tau = -w_0$ be the Cartan involution, where w_0 is the longest word in the Weyl group, $W(\mathfrak{g})$. On $\mathfrak{g}$-modules, τ sends the highest weight representation V_λ to its contragredient dual V_{λ^*}. This lifts to an automorphism, τ of the vertex algebra $\mathfrak{z}(\mathfrak{g})$, and so its mode algebra $\mathcal{Z}$, see the proof of [Ara18, Lemma 5.4] or [FG04, Theorem 5.4].

The product, $M \otimes N$, has a $\mathcal{Z}$-Mod structure with $P \in \mathcal{Z}$ acting as

$$(P \otimes 1 - 1 \otimes \tau(P)) \ . \tag{2.2.10}$$

The twist by τ is a matter of convention, but will prove convenient in later constructions when we wish to glue together modules with respect to the Feigin–Frenkel centre.

Now consider $\mathbb{V}_\lambda \otimes \mathbb{V}_{\lambda^*}$, which has two commuting actions of $V^{\kappa_c}(\mathfrak{g})$, and so two actions of $\mathcal{Z}$. We can pass to the quotient

$$\mathbb{V}_{\lambda,2} \coloneqq \mathbb{V}_\lambda \underset{\mathcal{Z}}{\otimes} \mathbb{V}_{\lambda^*} \cong \mathbb{V}_\lambda \underset{\mathfrak{z}_\lambda}{\otimes} \mathbb{V}_{\lambda^*} \ , \tag{2.2.11}$$

by identifying the action of the Feigin–Frenkel centre on each Weyl module. The resulting ``glued'' bimodule still retains the two commuting actions of $V^{\kappa_c}(\mathfrak{g})$ coming from each factor. In fact, $\mathbb{V}_{\lambda,2}$ is naturally a module over

$$\mathbb{V}_{0,2} \cong V^{\kappa_c}(\mathfrak{g}) \underset{\mathfrak{z}(\mathfrak{g})}{\otimes} V^{\kappa_c}(\mathfrak{g}) \ , \tag{2.2.12}$$

the glued current algebra. Let us generalise this construction.

Definition 2.2.1. Let $\lambda \in P^+$ and $s \in \mathbb{N}$, we define the glued module $\mathbb{V}_{\lambda,s}$ as

$$\mathbb{V}_{\lambda,s} \coloneqq \underbrace{\mathbb{V}_\lambda \underset{\mathcal{Z}}{\otimes} \mathbb{V}_{\lambda^*} \underset{\mathcal{Z}}{\otimes} \dots \underset{\mathcal{Z}}{\otimes} \mathbb{V}_{\tilde{\lambda}}}_{s \text{ many}} \cong \underbrace{\mathbb{V}_\lambda \underset{\mathfrak{z}_\lambda}{\otimes} \mathbb{V}_{\lambda^*} \underset{\mathfrak{z}_\lambda}{\otimes} \dots \underset{\mathfrak{z}_\lambda}{\otimes} \mathbb{V}_{\tilde{\lambda}}}_{s \text{ many}} \ , \tag{2.2.13}$$

where $\tilde{\lambda}$ is equal to λ^* for even s, or λ for odd s. The resulting module is an object in KL with respect to the action of $V^{\kappa_c}(\mathfrak{g})$ on each factor. We denote their contragredient duals by $D(\mathbb{V}_{\lambda,s})$.

Note that $\mathbb{V}_{0,s}$ are vertex algebra objects for all values of s. In particular, we have vertex algebra morphisms $\mu_i : V^{\kappa_c}(\mathfrak{g})$ for $i = 1, \dots, s$, embedding $V^{\kappa_c}(\mathfrak{g})$ into the ith factor of

$\mathbb{V}_{0,s}$. Suppose M is a module over $\mathbb{V}_{0,s}$, then the embeddings μ_i give it the structure of a $V^{\kappa_c}(\mathfrak{g})$-module, by restriction.

We denote by $\mathrm{KL}^s_{\otimes\mathfrak{z}(\mathfrak{g})}$ be the category of $\mathbb{V}_{0,s}$ modules such that the action of $V^{\kappa_c}(\mathfrak{g})$ via μ_i gives the module a structure of an object in KL. By construction, the $\mathbb{V}_{\lambda,s}$ are objects of $\mathrm{KL}^s_{\otimes\mathfrak{z}(\mathfrak{g})}$.

Proposition 2.2.2 ([Ara18, Prop 8.6]). *We have the following isomorphisms*

- $H^0_{\mathrm{DS},1}(\mathbb{V}_{\lambda,s}) \cong \mathbb{V}_{\lambda^*,s-1}$ *and* $H^0_{\mathrm{DS,s}}(\mathbb{V}_{\lambda,s}) \cong \mathbb{V}_{\lambda,s-1}$ *where the DS-reduction is performed with respect to the action of* $V^{\kappa_c}(\mathfrak{g})$ *on the first, and last factor respectively.*
- $H^0_{\mathrm{DS},1}(D(\mathbb{V}_{\lambda,s})) \cong D(\mathbb{V}_{\lambda^*,s-1})$ *and* $H^0_{\mathrm{DS,s}}(D(\mathbb{V}_{\lambda,s})) \cong D(\mathbb{V}_{\lambda,s-1})$ *where the DS-reduction is performed with respect to the action of* $V^{\kappa_c}(\mathfrak{g})$ *on the first, and last factor respectively.*

The Weyl modules have the nice homological property of being projective as a $U(t^{-1}[\mathfrak{g}[t^{-1}]])$ module and their contragredient duals are injective over $U(t[\mathfrak{g}[t]])$. The glued modules retain this property, since $\mathfrak{z}_\lambda$ acts freely, with respect to each action of $V^{\kappa_c}(\mathfrak{g})$ on their factors.

Let $\mathrm{KL}^{s,\Delta}$ be the subcategory of $\mathrm{KL}^s_{\otimes\mathfrak{z}(\mathfrak{g})}$ objects, M, with an increasing filtration,

$$0 = M_0 \subset M_1 \subset M_2 \subset \cdots \subset M \ , \tag{2.2.14}$$

such that each successive quotient satisfies

$$M_i/M_{i-1} \cong \mathbb{V}_{\lambda,s} \ , \tag{2.2.15}$$

for some $\lambda \in P^+$. Analogously, we define $\mathrm{KL}^{s,\nabla}$ to be the subcategory of $\mathrm{KL}^s_{\otimes\mathfrak{z}(\mathfrak{g})}$ objects, N, equipped with a descending filtration

$$N = N_0 \supset N_1 \supset N_2 \supset \cdots \supset 0 \ , \tag{2.2.16}$$

with successive quotients

$$N_i/N_{i+1} \cong D(\mathbb{V}_{\lambda,s}) \ , \tag{2.2.17}$$

for some $\lambda \in P^+$.

Proposition 2.2.3. *Let* $\mathrm{KL}^{s,\diamondsuit}$ *denote the subcategory of objects of* $\mathrm{KL}^s_{\otimes \mathfrak{z}(\mathfrak{g})}$ *that are simultaneously objects of* $\mathrm{KL}^{s,\Delta}$ *and* $\mathrm{KL}^{s,\nabla}$*. Any object* M *in* $\mathrm{KL}^{s,\diamondsuit}$ *is semijective over* $\widehat{\mathfrak{g}}_{\kappa_c}$ *with respect to each action, i.e., projective over* $U(t^{-1}\mathfrak{g}[t^{-1}])$ *and injective over* $U(t\mathfrak{g}[t])$*.*

As a result, for any $N \in \mathrm{KL}$*,*

$$\mathrm{H}^{\frac{\infty}{2}+i}(\widehat{\mathfrak{g}}_{-\kappa_g}, \mathfrak{g}, M \otimes N) \cong \delta_{i,0}\mathrm{H}^{\frac{\infty}{2}+0}(\widehat{\mathfrak{g}}_{-\kappa_g}, \mathfrak{g}, M \otimes N) \ , \tag{2.2.18}$$

where we may use any of the s *many,* $V^{\kappa_c}(\mathfrak{g})$ *actions.*

Proof. If $M \in \mathrm{KL}^{s,\diamondsuit}$, then it is a colimit of $U(t^{-1}\mathfrak{g}[t^{-1}])$-projective objects and a limit of $U(t\mathfrak{g}[t])$-injective objects. The second statement follows from Theorem A.5.5. □

2.2.3 Feigin–Frenkel gluing

The current subalgebras of $\mathbf{V}_{G,s}$ are all at the critical level, so there could theoretically be several copies of the Feigin–Frenkel centre present. This turns out to not be the case; in fact the current subalgebras all *share a common Feigin–Frenkel centre.* This can be seen from writing the index in terms of Weyl modules as in (2.1.7)

This phenomenon of a shared FF centre is a chiral analogue of certain well-known Higgs branch relations for theories of class $\mathcal{S}$. We recall that for theory of type A_n associated to a UV curve with s punctures, there are s moment map operators μ_s subject to the relation

$$\operatorname{Tr}\mu_1^k = \operatorname{Tr}\mu_2^k = \cdots = \operatorname{Tr}\mu_s^k \ , \qquad k = 2, \ldots, n+1 \ . \tag{2.2.19}$$

More generally, for the D_n and E_n theories there are analogous relations corresponding to the respective fundamental invariants of those algebras. These play a central role in the construction of the Moore–Tachikawa varieties (Higgs branches of class $\mathcal{S}$) by Ginzburg–Kazhdan [GK].

It is not immediate that these Higgs branch relations lead to the identification of Feigin–Frenkel centres. This is because the Feigin–Frenkel generators are related to, but not equal to, the corresponding Higgs branch operators under the SCFT/VOA correspondence.[5] To illustrate, consider the case $k = 2$ of (2.2.19) The Higgs branch operators associated to the quadratic fundamental invariant are related to the Segal–Sugawara operators, $P_{1,i}$ but also receive a nonzero contribution from the VOA stress-energy tensor T [Bee19],

$$P_{1,i} = \operatorname{Tr} \mu_i^2 + \alpha T \ , \tag{2.2.20}$$

where α is a fixed (nonzero) constant computed in [Bee19] and $P_{1,i}$ is the quadratic Feigin--Frenkel generator associated to the i'th puncture. The Higgs branch relations force the $\operatorname{Tr} \mu_i^2$ to be equal and, importantly, there is a *unique* $\hat{\mathcal{C}}_{0(0,0)}$ multiplet (the four-dimensional stress tensor multiplet)—so the operator T is the same for each i. As a result, the quadratic generators of the Feigin–Frenkel centre are identified across different punctures, *i.e.*, $P_{1,1} = P_{1,2} = \ldots P_{1,s}$.

For higher order invariants, more information is required about the structure of the vertex algebra. In the case of the cubic invariant of A_n there will be mixing between Higgs branch operators ($\widehat{\mathcal{B}}_3$ multiplets) and $\mathcal{C}_{1(0,0)}$ multiplets, but the uniqueness of the latter is not apparent. Nevertheless, precisely for the class $\mathcal{S}$ theories, the known expression for the Schur index indicates that the identification of the higher Feigin–Frenkel generators should indeed hold.

As an abelian Lie algebra, $\mathcal{Z}$ has a semi-infinite structure and so can be used to define semi-infinite cohomology with coefficients in some object of $\mathcal{Z}$-Mod. Specifically, we will have coefficients of the form $M \otimes N$ for $M, N \in \mathcal{Z}$-Mod. The Feigin standard complex is then given by

$$C^\bullet(\mathcal{Z}, M \otimes N) \coloneqq M \otimes N \otimes \bigwedge{}^{\frac{\infty}{2}+\bullet} \mathfrak{z}(\mathfrak{g}) \ , \tag{2.2.21}$$

with BRST current

$$Q(z) = \sum_{i=1}^{\operatorname{rk}\mathfrak{g}} (P_{d_i} c^{d_i})(z) \ , \tag{2.2.22}$$

whose zero mode $Q_{(0)}$ acts as a differential for the cochain complex.

[5] To be precise, the Feigin–Frenkel operators are identified with the corresponding Higgs branch operators upon passing to the associated graded of the R filtration [BR18].

Definition 2.2.4. Given two modules V_1 and V_2 in $\mathcal{Z}$-Mod, we define Feigin–Frenkel (FF) gluing as the semi-infinite cohomology

$$V_1 * V_2 := \mathrm{H}^{\frac{\infty}{2}+0}(\mathcal{Z}, V_1 \otimes V_2) \ . \tag{2.2.23}$$

In the case when V_1 and V_2 are vertex algebras, $V_1 * V_2$ is also a vertex algebra (see Remark A.2.12).

As was the case with the gauge theory BRST problem, we have a vanishing theorem here.

Theorem 2.2.5 (Theorem 9.10 of [Ara18])**.** *Let $M \in \mathcal{Z}$-Mod be free as a $\mathcal{Z}_{(<0)}$ module, then*

$$\mathrm{H}^{\frac{\infty}{2}+i}(\mathcal{Z}, M) = 0 \quad \textit{for} \ \ i < 0 \ .$$

This is a weaker conclusion than in the vanishing theorem A.5.5, as the cohomology is not necessarily concentrated in degree zero. Nevertheless for many purposes it is sufficient.[6]

We see that the BRST procedure enforces that the action of the Feigin–Frenkel centre on V_1 and V_2 are identified; in some loose sense, this enforces the Higgs branch relations on the Schur operators. This is very similar to a chiral version of the Hamiltonian reduction procedure described in [GK].

2.2.4 Chiral differential operators and the cylinder

The starting point of the construction of [Ara18] is the cylinder VOA; from here one can define the cap chiral algebra by Drinfel'd–Sokolov reduction and, as it turns out, construct all genus zero VOAs by FF-gluing. The form of the cylinder algebra for A_n theories was identified concretely in [BPRvR15], but it was subsequently recognised in [Ara15] that this reproduced a more general, and purely algebraic, construction that makes no explicit reference to four dimensional physics. The construction is universal and depends only on

[6]From a derived perspective, truncating the cohomology at degree zero is somewhat unnatural. We will return to this point in a later section.

a choice of algebraic group G with Lie $G = \mathfrak{g}$. Here we take G to be the simply connected Lie group.

Starting from such a G, the arc space $G(\mathcal{O})$ is a scheme whose $\mathbb{C}$-points are

$$G(\mathcal{O}) \coloneqq Hom_{\mathrm{Sch}/\mathbb{C}}(\mathbb{D}, G) \ , \tag{2.2.24}$$

where $\mathbb{D}$ is the formal disc $\mathbb{D} = \mathrm{Spf}\, \mathbb{C}[\![t]\!]$. More abstractly, $G(\mathcal{O})$ represents the functor

$$\begin{aligned} \mathrm{Sch}_{/\mathbb{C}} &\to \mathrm{Set} \\ S &\mapsto \mathrm{Hom}_{\mathrm{Sch}_{/\mathbb{C}}}(S \times \mathbb{D}, G) \end{aligned} \tag{2.2.25}$$

For a nice discussion of arc spaces, we point the reader to [Ara17].

The Lie algebra $\mathfrak{g} = Lie(G)$ acts on the co-ordinate ring, $\mathbb{C}[G]$, via derivations, and, by functoriality, this action lifts to one of $\mathfrak{g}(\mathcal{O}) \cong \mathfrak{g}[\![t]\!]$ on $\mathbb{C}[G(\mathcal{O})]$. Therefore, $\mathbb{C}[G(\mathcal{O})]$ has the structure of a $\mathfrak{g}[\![t]\!] \oplus \mathbb{C}K$ module where K acts as the level $\kappa \in \mathbb{C}$. One can further produce a $\hat{\mathfrak{g}}_\kappa$ module via induction, which defines the *chiral differential operators* (cdos) on the simply connected, algebraic Lie group G [AG02] (see also [GMS99, GMS01, GMS04, MSV99]),

$$\mathcal{D}^{ch}_{G,\kappa} = U(\hat{\mathfrak{g}}_\kappa) \otimes_{U(\mathfrak{g}[\![t]\!] \oplus \mathbb{C}K)} \mathbb{C}[G(\mathcal{O})] \ . \tag{2.2.26}$$

We shall be interested in the chiral differential operators at the critical level, $\kappa = \kappa_c$, and we write

$$\mathcal{D}^{ch}_{G} \equiv \mathcal{D}^{ch}_{G,\kappa_c} \ . \tag{2.2.27}$$

At any level, cdos on G has the structure of a conformal vertex algebra [GMS01] with central charge

$$c_{2d} = 2 \dim G \ , \tag{2.2.28}$$

which matches with the central charge for a twice punctured sphere (see, *e.g.*, [CDT13]). Note that this central charge is positive, corresponding to a negative central charge for the putative four-dimensional SCFT—which will be non-unitary. Again, this is expected since the cylinder class $\mathcal{S}$ theory is not a good four-dimensional theory.

By construction, $\mathcal{D}^{ch}_G$ is a vertex algebra object in KL and there is an embedding of the universal affine vertex algebra $\pi_L : V^{\kappa_c}(\mathfrak{g}) \hookrightarrow \mathcal{D}^{ch}_G$. This vertex algebra homomorphism is induced by the embedding of $\mathfrak{g}$ as left invariant vector fields of G. The Lie algebra $\mathfrak{g}$ is also isomorphic to the right invariant vector fields of G and this embedding is also lifted to a vertex algebra homomorphism $\pi_R : V^{\kappa_c}(\mathfrak{g}) \hookrightarrow \mathcal{D}^{ch}_G$, such that the images of π_L and π_R commute [AG02, Theorem 3.7].

The left and right embeddings of $V^{\kappa_c}(\mathfrak{g})$ restrict to embeddings of the Feigin–Frenkel centre $\mathfrak{z}(\mathfrak{g})$, and the two embeddings of the Feigin–Frenkel centre coincide [FG04],

$$\pi_L(\mathfrak{z}(\mathfrak{g})) \cong \pi_R(\mathfrak{z}(\mathfrak{g})) \cong (\mathcal{D}^{ch}_G)^{\mathfrak{g}[t]\times\mathfrak{g}[t]} \ . \tag{2.2.29}$$

The vertex algebra $\mathcal{D}^{ch}_G$ is free as a module over $U(t^{-1}\mathfrak{g}[t^{-}1])$ and cofree over $U(t\mathfrak{g}[t])$ [Ara18]; thus the conditions of Theorem A.5.5 are met and for any $M \in \mathrm{KL}$, the cohomology $\mathcal{D}^{ch}_G \circ M$ is concentrated in degree zero, and furthermore we have the following result.

Theorem 2.2.6 ([AG02, Theorem 5.5])**.** *Let $M \in \mathrm{KL}$, then we have that*

$$\mathrm{H}^{\frac{\infty}{2}+i}(\hat{\mathfrak{g}}_{-\kappa_g}, \mathfrak{g}, \mathcal{D}^{ch}_G \otimes M) \cong \mathrm{H}^{\frac{\infty}{2}+i}(\hat{\mathfrak{g}}_{-\kappa_g}, \mathfrak{g}, M \otimes \mathcal{D}^{ch}_G) \cong \delta_{i,0} M \ . \tag{2.2.30}$$

Pictorially, gluing the cylinder to any surface must be the identity operation. The above theorem confirms that $\mathcal{D}^{ch}_G$ satisfies this condition. By abstract nonsense, $\mathcal{D}^{ch}_G$ must be the unique object in KL that satisfies such a property.

2.2.5 Equivariant affine $\mathcal{W}$-algebras and the cap

Starting from the cylinder VOA, the cap algebra is recovered by completely reducing one maximal puncture. Arakawa has named the resulting VOA the (principal) *equivariant affine $\mathcal{W}$-algebra* $\mathbf{W}_G$ (it is an affine analogue of the equivariant $\mathcal{W}$-algebra of [Los07]),

$$\mathbf{W}_G \coloneqq H^0_{DS}(\mathcal{D}^{ch}_G) \ . \tag{2.2.31}$$

In the usual way, this vertex algebra inherits a conformal structure from the cylinder, with central charge

$$c_{\mathbf{W}_G} = \dim \mathfrak{g} + \operatorname{rk} \mathfrak{g} + 24\kappa_g(\rho, \rho^\vee) \ , \tag{2.2.32}$$

where ρ is half the sum of all positive roots and $\rho^\vee$ is the half sum of all positive co-roots.

By Propositions 6.4 and 6.5 of [Ara18], $\mathbf{W}_G$ is free over $U(t^{-1}\mathfrak{g}[t^{-1}])$ and cofree over $U(t\mathfrak{g}[t])$, and so is the $\hat{\mathfrak{g}}_{-\kappa_g}$ module $\mathbf{W}_G \otimes M$. The cohomology when gauging is therefore concentrated in degree zero.

Theorem 2.2.7 ([Ara18, Theorem 6.8])**.** *Let* $M \in \mathrm{KL}$*,*

$$\mathrm{H}^{\frac{\infty}{2}+\bullet}(\hat{\mathfrak{g}}_{-\kappa_g}, \mathfrak{g}, \mathbf{W}_G \otimes M) = H^0_{\mathrm{DS}}(M) \ . \tag{2.2.33}$$

When M is a chiral algebra of class $\mathcal{S}$, this corresponds, pictorially, to the fact that gluing a cap and a surface together along a maximal puncture has the effect of closing the puncture, *i.e.*, performing principal DS reduction.

2.2.6 Inverse Hamiltonian reduction

Feigin–Frenkel gluing a cap onto another vertex algebra provides a sort of inverse to the principal DS reduction functor; by FF gluing a cap onto a vertex algebra $V \in \mathcal{Z}$-Mod we provide it with a $V^{\kappa_c}(\mathfrak{g})$ action, and it becomes a vertex algebra object in KL. The cap is free over $\mathcal{Z}_{(<0)}$ [Ara18] and so $\mathbf{W}_G * - : \mathcal{Z}\text{-Mod} \to \mathrm{KL}$ is a left-exact functor that acts, almost, as an inverse to H^0_{DS}.

Theorem 2.2.8 ([Ara18, Theorem 9.11])**.** *Let* $M \in \mathrm{KL}$*, then*

$$M \cong \mathrm{H}^{\frac{\infty}{2}+0}(\mathcal{Z}, \mathbf{W}_G \otimes H^0_{\mathrm{DS}}(M)) \ , \tag{2.2.34}$$

i.e., the composition $\mathrm{H}^{\frac{\infty}{2}+0}(\mathcal{Z}, \mathbf{W}_G \otimes H^0_{\mathrm{DS}}(-))$ *is the identity functor on* KL*. Define the* *subcategory* $\mathrm{KL}_0 \subset \mathcal{Z}$*-Mod as the image,* $H^0_{\mathrm{DS}}(\mathrm{KL})$*, then we have an equivalence of categories*

$$\mathrm{KL} \underset{\mathbf{W}_G * -}{\overset{H^0_{\mathrm{DS}}(-)}{\rightleftarrows}} \mathrm{KL}_0 \ . \tag{2.2.35}$$

Remark 2.2.9. This almost-equivalence is reminiscent of a result of Riche [Ric17, Proposition 3.3.11]. Let $\mathfrak{g}^*_{\mathrm{reg}}$ be the regular locus inside $\mathfrak{g}^*$, *i.e.*, the locus whose G-stabiliser is of dimension $\mathrm{rk}\,\mathfrak{g}$. Then

$$\kappa : \mathrm{QCoh}^G(\mathfrak{g}^*_{\mathrm{reg}}) \xrightarrow{\sim} \mathit{Rep}(\mathfrak{Z}_G) \ . \tag{2.2.36}$$

where $\mathfrak{Z}_G$ is the group scheme of G-stabilisers over the principal Slodowy slice S_{prin}. The functor κ is Kostant–Whittaker reduction, the finite-dimensional Poisson counterpart of Drinfel'd–Sokolov reduction. The inverse functor is provided by $\mathfrak{Z}_G$ symplectic reduction,$((G\times S_{\mathrm{prin}}) \times -)///\mathfrak{Z}_G$, as made precise in [GK]. The equivariant Slodowy-slice $G \times S_{\mathrm{prin}}$ is the associated variety of $\mathbf{W}_G$.

Instead of reducing by the action of the group scheme, one can reduce by the action of the Lie algebroid. Roughly speaking FF-gluing a cap, $\mathbf{W}_G * -$ is a chiralisation of this latter construction.

As a ``corollary''[7] of Theorem 2.2.8, the cylinder VOA can be recovered from the equivariant affine W-algebra by FF gluing two caps together,

$$\mathbf{W}_G * \mathbf{W}_G \equiv \mathrm{H}^{\frac{\infty}{2}+0}(\mathcal{Z}, \mathbf{W}_G \otimes \mathbf{W}_G) \cong \mathcal{D}^{ch}_G \ . \tag{2.2.37}$$

Indeed, there is an obvious generalisation to produce all $\mathbf{V}_{G,s}$ by repeatedly gluing caps.

2.2.7 Constructing genus zero VOAs and their properties

From here there is a conceptually straightforward construction of all genus zero chiral algebras of class $\mathcal{S}$: one starts with $\mathcal{D}^{ch}_G$ and repeatedly applies $\mathbf{W} * -$ to add more maximal

[7] This is not quite a corollary since Arakawa's proof of Theorem 2.2.8 requires establishing $\mathbf{W}_G * \mathbf{W}_G \cong \mathcal{D}^{ch}_G$, independently as Theorem 9.9 of [Ara18].

punctures. The construction of [Ara18] takes a slightly different approach. Instead of performing iterative FF-gluing of caps, one glues all caps together simultaneously—producing $\mathbf{V}_{G,s}$ in one step.

Definition 2.2.10. Take the chain complex

$$C^\bullet\left(\bigoplus_{i=1}^{s-1}\mathcal{Z}^{i,i+1},\mathbf{W}^s\right) \coloneqq \mathbf{W}^s\otimes\left(\bigwedge{}^{\frac{\infty}{2}+\bullet}(\mathfrak{z}(\mathfrak{g}))\right)^{s-1}, \tag{2.2.38}$$

for $s>1$, with differential equal to $Q_{(0)}$, for

$$\begin{aligned} Q(z) &= \sum_{i=1}^{s-1} Q^{i,i+1}(z)\ , \\ Q^{i,i+1}(z) &= \sum_{j=1}^{\operatorname{rk}\mathfrak{g}}(\rho_i(P_j)-\rho_{i+1}(\tau(P_j)))\rho_{gh,i}(c^j)(z)\ , \end{aligned} \tag{2.2.39}$$

where ρ_i represents the action of $\mathfrak{z}(\mathfrak{g})$ on the i-th factor of $\mathbf{W}$ and $\rho_{gh,i}(c^j)$ acts on the i-th factor of the ghost system $\bigwedge^{\frac{\infty}{2}+\bullet}(\mathfrak{z}(\mathfrak{g}))$. The vertex algebra of a sphere of type $\mathfrak{g}$ with s maximal punctures is then defined to be

$$\begin{aligned} \mathbf{V}_{G,1} &\coloneqq \mathbf{W}\ , \\ \mathbf{V}_{G,s} &\coloneqq \mathrm{H}^{\frac{\infty}{2}+0}\left(\bigoplus_{i=1}^{s-1}\mathcal{Z}^{i,i+1},\mathbf{W}^s\right)\ . \end{aligned} \tag{2.2.40}$$

Having reviewed their construction, let us continue on to establishing the various expected properties of the $\mathbf{V}_{G,s}$.

Each cap $\mathbf{W}_G$ has a morphism $\mu_i : V^{\kappa_c}(\mathfrak{g})$, and $\mathbf{V}_{G,s}$ inherits these actions making it a vertex algebra object in KL for each action μ_i. These are the chiral moment maps coming from the maximal punctures on the UV curve. By construction, the Feigin–Frenkel centres of each $\mu_i(V^{\kappa_c}(\mathfrak{g}))$ are identified and so $\mathbf{V}_{G,s}$ is a module over $\mathbb{V}_{0,s}$ and an object in $\mathrm{KL}^s_{\otimes\mathfrak{z}(\mathfrak{g})}$.

Proposition 2.2.11 ([Ara18, Proposition 10.10])**.** *The vertex algebras* $\mathbf{V}_{G,s}$ *are objects in*

$\mathrm{KL}^{s,\diamondsuit}$. *As a result, the gluings,*

$$\mathbf{V}_{G,s} \circ \mathbf{V}_{G,s'} \cong \mathrm{H}^{\frac{\infty}{2}+0}(\widehat{\mathfrak{g}}_{-\kappa_g}, \mathfrak{g}, \mathbf{V}_{G,s} \otimes \mathbf{V}_{s'}) \tag{2.2.41}$$

are concentrated in degree zero. Here, the cohomology can be taken with respect to any of the actions, μ_i for $i = 1, \ldots s$.

By way of a spectral sequence argument, one can then establish the following result.

Proposition 2.2.12 ([Ara18, Proposition 10.11])**.** *We have the following isomorphism of vertex algebra objects in* KL*,*

$$\mathbf{V}_{G,s} \circ \mathbf{V}_{G,s'} \cong \mathbf{V}_{G,s+s'-2} \ , \tag{2.2.42}$$

where the gluing can be done with respect to any of the actions μ_i on each vertex algebra.

Remark 2.2.13. Proposition 2.2.12 also implies that gauging is associative, *i.e.*,

$$\mathbf{V}_{G,s_1} \circ (\mathbf{V}_{G,s_2} \circ \mathbf{V}_{G,s_3}) \cong (\mathbf{V}_{G,s_1} \circ \mathbf{V}_{G,s_2}) \circ \mathbf{V}_{G,s_3} \cong \mathbf{V}_{G,s_1+s_2+s_3-4} \ . \tag{2.2.43}$$

Furthermore, all cohomologies being concentrated in degree zero is compatible with the expectation that in genus zero there is no residual gauge symmetry on the Higgs branch, and so no Hall–Littlewood chiral ring beyond the Higgs chiral ring.

These genus zero vertex algebras also play nicely under FF-gluing.

Proposition 2.2.14 ([Ara18, Proposition 10.2])**.** *For any $s \geq 1$, $\mathbf{V}_{G,s}$ is free over $\mathcal{Z}_{(<0)}$. Therefore, for any $s, s' \geq 1$,*

$$\mathbf{V}_{G,s} * \mathbf{V}_{G,s'} \equiv \mathrm{H}^{\frac{\infty}{2}+0}(\mathcal{Z}, \mathbf{V}_{G,s} \otimes \mathbf{V}_{s'}) \cong \mathbf{V}_{s+s'} \ . \tag{2.2.44}$$

As a special case, the above proposition implies that the simultaneous construction of the

$\mathbf{V}_{G,s}$ agrees with the recursive definition, *i.e.*,

$$\mathbf{V}_{G,s+1} \cong \mathrm{H}^{\frac{\infty}{2}+0}(\mathcal{Z}, \mathbf{W}_G \otimes \mathbf{V}_{G,s}) \ . \tag{2.2.45}$$

Additionally, [Ara18, Proposition 10.3] grants the following isomorphism

$$H^0_{\mathrm{DS}}(\mathbf{V}_{G,s}) \cong \mathbf{V}_{G,s-1} \ . \tag{2.2.46}$$

This establishes the various gluing properties that physics predicts. Next, let us look at some of the structural results. We collect these into one proposition.

Proposition 2.2.15. *The vertex algebras, $\mathbf{V}_{G,s}$, are simple and conformal with central charge*

$$c_s = \big(b - 2(b-2)h^\vee\big) \dim \mathfrak{g} - (b-2) \operatorname{rk} \mathfrak{g} \ . \tag{2.2.47}$$

Moreover, $\mathbf{V}_{G,s}$ are of CFT type for $s \geq 2$ with characters

$$\mathrm{ch}_{\mathbf{V}_{G,s}} = \mathcal{I}(q, \mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x_s}) = \sum_{\lambda \in \ P^+} (\mathrm{ch}\, \mathfrak{z}^2_\lambda) \prod_{i=1}^{s} \frac{\mathrm{ch}\, \mathbb{V}_\lambda}{\mathrm{ch}\, \mathfrak{z}_\lambda} \ , \tag{2.2.48}$$

i.e., the characters agree with the Schur limit of the index.

An interesting feature of this construction is that it trivialises S-duality. The 4-move acts by permuting the various chiral moment maps which amounts to permuting the caps involved in the construction of $\mathbf{V}_{G,s}$. The caps involved in the construction are identical and so permuting these caps is an automorphism of the vertex algebra.

2.3 Examples of chiral algebras of class $\mathcal{S}$

The preceding construction has been quite abstract, in this section we give examples of $\mathbf{V}_{G,s}$ for the case $G = \mathrm{SL}_2$ and $s = 0, 1, 2, 3$.

A number of these vertex algebras have appeared earlier in [BPRvR15, Ara18]. The expression for $\mathbf{V}_{\mathrm{SL}_2,0}$ is based on the free-field realisation technique of [BN23a].

2.3.1 The sphere

The vertex algebra $\mathbf{V}_{G,0}$ was named the *chiral universal centraliser* in [Ara18]. As a vertex algebra, $\mathbf{V}_{\mathrm{SL}_2,0}$, is strongly generated by the fields S, X, Y, satisfying the null relation

$$SXX - YY - \tfrac{3}{2}\partial YX + \tfrac{3}{2}Y\partial X - \tfrac{3}{4}\partial X\partial X - \tfrac{1}{8}X\partial^2 X \overset{!}{=} \mathbb{1} \ . \tag{2.3.1}$$

The OPEs between strong generators are given by

$$\begin{aligned} S(z)X(w) &\sim \frac{\frac{3}{4}X(w)}{(z-w)^2} + \frac{Y(w)}{z-w} \ , \\ S(z)Y(w) &\sim \frac{\frac{3}{4}Y(w)}{(z-w)^2} + \frac{(SX)(w)}{z-w} \ , \\ Y(z)X(w) &\sim \frac{\frac{1}{2}(XX)(w)}{z-w} \ , \\ Y(z)Y(w) &\sim \frac{-\frac{1}{4}(XX)(w)}{(z-w)^2} - \frac{\frac{1}{4}(X\partial X)(w)}{z-w} \ . \end{aligned} \tag{2.3.2}$$

Here, S generates a commutative vertex subalgebra, which should be identified with $\mathfrak{z}(\mathfrak{sl}_2)$-–with S the degree two generator.

The conformal vector is

$$\begin{aligned} T_0 = 2S(\partial YX - Y\partial X) + 3S\partial X\partial X + \tfrac{1}{2}Y\partial^3 X - \partial SYX + \tfrac{5}{2}\partial S\partial XX - 2\partial Y\partial Y \\ + \tfrac{3}{2}\partial Y\partial^2 X - \tfrac{3}{4}\partial^2 X\partial^2 X - \tfrac{3}{2}\partial^2 Y\partial X - \tfrac{7}{12}\partial^3 X\partial X - \tfrac{1}{2}\partial^3 YX - \tfrac{1}{24}\partial^4 XX \ . \end{aligned} \tag{2.3.3}$$

with central charge $c = 26$, which agrees with (2.2.6). The conformal weights of S, X, Y under this choice of conformal vector are

$$\Delta_S = 2 \ , \quad \Delta_X = -1 \ , \quad \Delta_Y = 0 \ . \tag{2.3.4}$$

Note that $\mathbf{V}_{\mathrm{SL}_2,0}$ is not positively graded by conformal weight, nor is it conical since the $\Delta = 0$ subspace is infinite dimensional.

2.3.2 The cap

The cap VOA, $\mathbf{V}_{\mathrm{SL}_2,1} \equiv \mathbf{W}_{\mathrm{SL}_2}$ is strongly generated by S, X^a, Y^a, for $a = +, -$, subject to the relation

$$X^a Y_a + \tfrac{1}{2}\partial X^a X_a \stackrel{!}{=} \mathbb{1} \ . \tag{2.3.5}$$

The OPEs between strong generators is given by

$$\begin{aligned}
S(z)X^a(w) &= \frac{\frac{3}{4}X^a}{(z-w)^2} + \frac{Y^a}{z-w} \ , \\
S(z)Y^a(w) &= \frac{\frac{3}{4}Y^a}{(z-w)^2} + \frac{SX^a}{z-w} \ , \\
Y^a(z)X^b(w) &= \frac{\frac{1}{2}(X^aX^b)}{z-w} \ , \\
Y^a(z)Y^b(w) &= -\frac{\frac{1}{4}(X^aX^b)}{(z-w)^2} + \frac{\epsilon^{ab}\frac{1}{2}\big(Y^cX_d - \frac{1}{2}\partial X^c X_d\big) - \frac{1}{4}\partial X^a X^b}{z-w} \ .
\end{aligned} \tag{2.3.6}$$

The cap has a single chiral moment map $\mu_1 : V^{-2}(\mathfrak{sl}_2) \to \mathbf{W}_{\mathrm{SL}_2}$. The image of the strong generators e, h, f of $V^{-2}(\mathfrak{sl}_2)$ under this map is given by

$$\begin{aligned}
e_1 =& SX^+X^+ - Y^+Y^+ - \frac{3}{2}\big(X^+\partial Y^+ - \partial X^+ Y^+\big) - \frac{3}{4}\partial X^+\partial X^+ - \frac{1}{8}X^+\partial^2 X^+ \ , \\
h_1 =& -2SX^-X^+ + 2Y^-Y^+ + \frac{3}{2}(X^-\partial Y^+ + X^+\partial Y^- - \partial X^- Y^+ - \partial X^+ Y^-) \\
&+ \frac{3}{2}\partial X^+\partial X^- + \frac{1}{4}X^+\partial^2 X^- \ , \\
f_1 =& -SX^-X^- + Y^-Y^- + \frac{3}{2}\big(X^-\partial Y^- - \partial X^- Y^-\big) + \frac{3}{4}\partial X^-\partial X^- + \frac{1}{8}X^-\partial^2 X^- \ .
\end{aligned} \tag{2.3.7}$$

Here $e_1 \equiv \mu_1(e)$, $h_1 \equiv \mu_1(h)$, and $f_1 \equiv \mu_1(f)$. The X^a and Y^a should be thought of as highest weight states in the $\mathbb{V}_{\lambda=1}$ Weyl module.

Similarly, the stress tensor T_1 is composite and takes the form

$$T_1 = -S\big(\partial X^a X_a\big) + \partial Y^a Y_a + \frac{3}{2}\big(\partial X^a \partial Y_a\big) + \frac{3}{8}\partial^2 X^a \partial X_a + \frac{1}{6}\partial^3 X^a X_a \ , \tag{2.3.8}$$

with central charge $c_1 = 16$. The conformal weights are

$$\Delta_S = 2 \ , \quad \Delta_{X^a} = -\tfrac{1}{2} \ , \quad \Delta_{Y^a} = \tfrac{1}{2} \ . \tag{2.3.9}$$

Just like the sphere, $\mathbf{W}_G$ is not conical, nor is it positively graded by weight—again confirming the expectation that the class $\mathcal{S}$-theory on Σ_1 is not a good SCFT.

2.3.3 The cylinder

The cylinder VOA, $\mathbf{V}_{\mathrm{SL}_2,2}$ already has a presentation in terms of strong generators and relations since it is isomorphic to $\mathcal{D}^{ch}_{\mathrm{SL}_2}$. Expressions for the strong generators of $\mathcal{D}^{ch}_G$ for any affine algebraic group can be found via results of [GMS01]. For SL_2, the cylinder vertex algebra is strongly generated by X_{ab}, for $a, b = +, -$, and e_L, h_L, f_L with the null relation

$$\begin{vmatrix} X^{++} & X^{+-} \\ X^{-+} & X^{--} \end{vmatrix} = X^{++}X^{--} - X^{+-}X^{-+} \stackrel{!}{=} \mathbb{1} \ . \tag{2.3.10}$$

The e_L, h_L, f_L are strong generators of a $V^{-2}(\mathfrak{sl}_2)$ current subalgebra and the X_{ab} are strong generators of a commutative subalgebra, with the X_{ab} being highest weight states of $\mathbb{V}_2 \otimes \mathbb{V}_2$ acted on by e_L, h_L, f_L.

It is useful to collect the e_L, h_L, f_L into a matrix $J_{L\ b}^{a}$ with

$$J_L := \begin{pmatrix} e_L & -\frac{1}{2}h_L \\ -\frac{1}{2}h_L & -f_L \end{pmatrix} \ , \tag{2.3.11}$$

This vertex algebra has an obvious chiral moment map $\mu_L : V^{-2}(\mathfrak{sl}_2) \to \mathcal{D}^{ch}_{\mathrm{SL}_2}$ whose image is the $V^{-2}(\mathfrak{sl}_2)$ subalgebra generated by the e_L, h_L, f_L. Morally, one should think of this moment map as arising from the embedding of $\mathfrak{sl}_2$ as left invariant vector fields.

The embedding of $\mathfrak{sl}_2$ as right invariant vector fields gives rise to another, chiral moment map, $\mu_R : V^{-2}(\mathfrak{sl}_2) \to \mathcal{D}^{ch}_{\mathrm{SL}_2}$ with image generated by

$$J_{R\ b}^{a} = J_{L\ d}^{c} X^{da} X_{cb} + 2X^{ac}\partial X_{cb} - \delta^a_b X^{cd}\partial X_{cd} \ . \tag{2.3.12}$$

The images of μ_L and μ_R commute and one can verify that the Feigin–Frenkel centres agree.

The conformal vector is again composite, now being given by

$$T_2 = J^{ab}\partial X_a{}^c X_{cb} + \partial^2 X^{ab} X_{ab} + 2\partial X^{ab}\partial X_{ab} \ . \tag{2.3.13}$$

The conformal weights are

$$\Delta_{e_1} = \Delta_{f_1} = \Delta_{h_1} = 1 \ , \quad \Delta_{X_{ab}} = 0 \ . \tag{2.3.14}$$

While $\mathcal{D}^{ch}_{\mathrm{SL}_2}$ is not conical it is $\mathbb{Z}_{\geq 0}$-graded by conformal weight.

2.3.4 The trinion

The class $\mathcal{S}$ theory corresponding to Σ_3 is a theory of free-hypermultiplets valued in the representation $N = T^*(V_1 \otimes V_1) \cong V_1 \otimes V_1 \otimes V_1$ of $\mathfrak{sl}_2$. The corresponding vertex algebra $\mathbf{V}_{\mathrm{SL}_2,3}$ should therefore be the $\beta\gamma$ system on $T^*V_1 \otimes V_1$.

Instead, we give an equivalent presentation of $\mathbf{V}_{\mathrm{SL}_2,3}$ as a vertex algebra strongly generated by X_{abc} with $a, b, c = +, -$, subject to no relations. The singular OPEs are

$$X_{abc}X_{def} \sim \frac{\epsilon_{ad}\epsilon_{be}\epsilon_{cf}\mathbb{1}}{(z-w)} \ , \tag{2.3.15}$$

where the ϵ_{ab} are Levi-Civita symbols.

The three chiral moment maps μ_1, μ_2 and μ_3 have images generated by

$$\begin{aligned} J_1{}^a{}_b &= \epsilon^{cc'}\epsilon^{dd'}X_{acd}X_{bc'd'} \ , \\ J_2{}^a{}_b &= \epsilon^{cc'}\epsilon^{dd'}X_{cad}X_{c'bd'} \ , \\ J_3{}^a{}_b &= \epsilon^{cc'}\epsilon^{dd'}X_{cda}X_{c'd'b} \ , \end{aligned} \tag{2.3.16}$$

and one can verify that these all commute and have a shared Feigin–Frenkel centre.

The conformal vector is

$$T_3 = X_{abc}\partial X^{abc} \tag{2.3.17}$$

with $\Delta_{X_{abc}} = 1/2$. Therefore, $\mathbf{V}_{\mathrm{SL}_2,3}$ is conical and positively graded and so is of CFT type—reflecting the fact that the class $\mathcal{S}$ theory on Σ_3 is a good SCFT.

Chapter 3

The chiral algebras of twisted class $\mathcal{S}$

It's the black wind through the maples,
and the difficulty of getting tenure...

Hera Lindsay Bird
Lost Scrolls

3.1 Theories of twisted class $\mathcal{S}$

We extend our review of the theories of class $\mathcal{S}$ to incorporate twisted punctures with non-simply laced flavour symmetries.

We start by introducing twisted punctures in Section 3.1.1. Including these punctures allows for new moves in the web of generalised S-duality and we discuss these in Section 3.1.2. In Section 3.1.3, we review the computation of the Schur indices of theories with twisted punctures, following [LPR14].

In Section 3.1.4, we lay out our expectations for the associated vertex algebras of the SCFTs that feature twisted punctures. This will also serve as a sort of overview of Section 3.2, where set out to construct these vertex algebras.

Before, moving on to the construction, we divert our attention briefly, in Section 3.1.5 to

Lie algebra ($\mathfrak{g}_u$)	Order of twist	Twisted algebra ($\mathfrak{g}_t$)
$\mathfrak{a}_{2n}$	$\mathbb{Z}/2\mathbb{Z}$	$\mathfrak{c}_n$
$\mathfrak{a}_{2n-1}$	$\mathbb{Z}/2\mathbb{Z}$	$\mathfrak{b}_n$
$\mathfrak{d}_n$	$\mathbb{Z}/2\mathbb{Z}$	$\mathfrak{c}_{n-1}$
$\mathfrak{d}_4$	$\mathbb{Z}/3\mathbb{Z}$	$\mathfrak{g}_2$
$\mathfrak{e}_6$	$\mathbb{Z}/2\mathbb{Z}$	$\mathfrak{f}_4$

Table 3.1: Simply laced Lie algebras $\mathfrak{g}_u$ and their corresponding twisted algebras $\mathfrak{g}_t$ for different choices of outer automorphism twist. The Lie algebras $\mathfrak{a}_{2n}$ and $\mathfrak{d}_n$ both give rise to Lie algebras of type $\mathfrak{c}$ after a twist. The corresponding theories nevertheless have subtle differences—see, *e.g.*, [CDT14, BP20]. The algebra $\mathfrak{d}_4$ has a non-abelian outer automorphism group isomorphic to S_3, the symmetric group on three elements.

comment on residual gauge symmetries. Unlike in the untwisted case, the genus of the curve is no longer the sole indicator of whether the Higgs branch is enhanced or not. We shall see that certain genus zero theories with twisted punctures have residual gauge symmetry at a generic point of their Higgs branch.

Throughout this chapter, we shall use the subscripts u and t to distinguish between various objects related to the untwisted simply laced Lie algebra $\mathfrak{g}_u$ and its non-simply laced, twisted counterpart $\mathfrak{g}_t$. For example, KL_u will denote the Kazhdan–Lusztig category for $\widehat{\mathfrak{g}}_{u,\kappa_c}$, while KL_t will be the Kazhdan–Lusztig category for $\widehat{\mathfrak{g}}_{t,\kappa_c}$.

3.1.1 Twisted punctures

Definition 3.1.1. Let $\sigma \in \mathrm{Out}(\mathfrak{g}_{\mathrm{u}})$, be a non-trivial element. Then we define

$$\mathfrak{g}_t \coloneqq {}^L(\mathfrak{g}_u^\sigma) \ , \tag{3.1.1}$$

where ^{L} denotes the Langlands dual.

Note that σ is always a graph automorphism of the Dynkin diagram of $\mathfrak{g}_u$. The associated twisted algebra, $\mathfrak{g}_t$ will always be non-simply laced. The pairs of untwisted algebras and their twisted counterparts can be found in Table 3.1.

The setting of twisted class $\mathcal{S}$ is an extension of the usual class $\mathcal{S}$ formalism to incorporate non-simply laced flavour and gauge symmetries. We refine the compactification data to a punctured Riemann surface *with* a local system of Dynkin diagrams. The resulting four-dimensional theory is an $\mathcal{N} = 2$ SCFT.

Any such local system is specified by giving a homomorphism.

$$\pi_1(\Sigma) \to \mathrm{Out}(\mathfrak{g}_u) \ , \tag{3.1.2}$$

from the fundamental group of the UV curve, Σ, to the outer automorphisms of $\mathfrak{g}_u$. Concretely, this gives rise to two types of regular punctures: *untwisted* punctures labelled by $1 \in \mathrm{Out}(\mathfrak{g}_\mathrm{u})$ and *twisted* punctures labelled by some $\sigma \in \mathrm{Out}(\mathfrak{g}_\mathrm{u})$. It will be helpful to think of twisted punctures as appearing in pairs connected by twist-lines.

The untwisted punctures are precisely those described in Section 2.1.2. As with the untwisted case, there are a myriad of possible punctures but (in the terminology of [CDT15a]) we restrict our attention to the case of regular, typical, twisted punctures. Such twisted punctures are labelled by nilpotent orbits in $\mathfrak{g}_t$. There are, again, two special orbits in $\mathfrak{g}_t$: the maximal puncture labelled by 0, and the empty puncture labelled by the principal orbit of $\mathfrak{g}_t$.

The maximal twisted punctures carries $\mathfrak{g}_t$ flavour symmetry, while submaximal twisted punctures proceed analogously to the untwisted case. Starting from a maximal twisted puncture, one can reduce the flavour symmetry by performing nilpotent Higgsing by giving a nilpotent vacuum expectation value to the moment map.

Importantly, unlike in the untwisted case, the empty twisted puncture (labelled by the principal nilpotent orbit in $\mathfrak{g}_t$) remains a nontrivial puncture (as it still carries monodromy on the UV curve; in terms of twist lines there is still a point where the relevant twist line ends, which distinguishes the point from a generic point on the UV curve).

We restrict to genus zero, so as not to worry about twist lines that can wrap cycles. We denote a curve with m maximal punctures and n *pairs* of maximal twisted punctures by $\mathcal{C}_{m,n}$. Any such curve is uniquely fixed (up to choice of complex structures) by specifying m and n.

In the presence of twist lines, curves can be glued together along either twisted or untwisted maximal punctures. Surfaces with only maximal punctures—like $\mathcal{C}_{m,n}$—can be built from gluing copies of $\mathcal{C}_{1,1}$, *the mixed trinion*, and Σ_3 the untwisted trinion. A full classification of trinions with (not necessarily maximal) twisted punctures[1] have also been classified by Chacaltana, Distler, Tachikawa and Trimm [CDT15a, CDT15b].

3.1.2 Dualities with twisted punctures

The complex structures of $\mathcal{C}_{m,n}$ once again correspond to gauge couplings—though this time we have two types of gauge symmetries. However, there is some ambiguity as to which gauge couplings correspond to which groups. To illustrate this, let us consider the curve $\mathcal{C}_{2,1}$. This curve has only one complex structure: τ.

Looking at weak coupling limits gives two types of pants decompositions (see Figure 3.1). In one frame, $\mathcal{C}_{2,1}$ is built by gluing $\mathcal{C}_{1,1}$ and a untwisted three punctured sphere Σ_3, along untwisted punctures. In the other frame, $\mathcal{C}_{2,1}$ decomposes into two copies of $\mathcal{C}_{1,1}$ glued along a twisted puncture. The action of S-duality relates these two frames, *i.e.*, it relates a weakly coupled $\mathfrak{g}_t$ gauge theory to a strongly coupled $\mathfrak{g}_u$ gauge theory and vice versa. The complex structure τ, therefore, can be thought of as a $\mathfrak{g}_t$ or $\mathfrak{g}_u$ gauge coupling depending on which frame we are working in. Whether to think of τ as a $\mathfrak{g}_u$ or $\mathfrak{g}_t$ gauge coupling depends on which open chart of $\overline{\mathcal{M}}_{0,4}$ we work in.

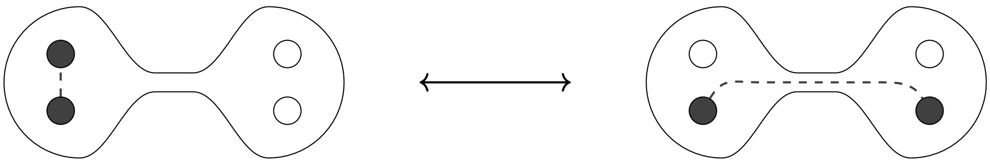

Figure 3.1: Two degeneration limits of $\mathcal{C}_{2,1}$. Note that the gluing on the left is untwisted but twisted on the right. We mark untwisted punctures by unfilled circles and twisted punctures by filled circles. We connect the twisted punctures by dashed twist-lines for clarity.

The S-duality move that swaps between these two frames is a variant of the 4-move we described in Section 2.1.4 . We call this variant the *ut*-move, since the gauge group changes across the frames. The 4-move still acts on a curve $\mathcal{C}_{m,n}$, permuting any four identical punctures—all untwisted or twisted.

Though we shall not look at higher genus curves, the *ab*-move is particular interesting in

[1]The case of $\mathfrak{d}_4$ trinions with non-abelian twists has recently been explored in [DES21], but we restrict our attention to the abelian case

the presence of twist lines. Take, for instance, the untwisted trinion Σ_3. We can gauge the diagonal action of G_u by self-gluing two punctures on the sphere together. However, we may also gauge with respect to a diagonal action of G_u that has been twisted by an outer automorphism, *i.e.*, G_u acts as $g \otimes \sigma(g)$ for (a lift of) an outer automorphism σ. Pictorially, we represent this by a cylinder, with a twist line around it, connecting the punctures. This results in a genus one curve with a twist line running around the a-cycle of the torus.

Now, consider, the curve $\mathcal{C}_{1,1}$. Again, we can construct a gauge theory, by self-gluing the two twisted punctures together. This time, we gauge with respect to the diagonal action of G_t. This results in a genus one curve with a twist line running along the b-cycle. These two theories are known to be S-dual, and the associated UV curves are related by the action of the ab-move. The two degeneration limits are shown in Figure 3.2. Unlike the untwisted case, the ab-move changes the rank of the gauge group, as well as moving from strong to weak coupling.

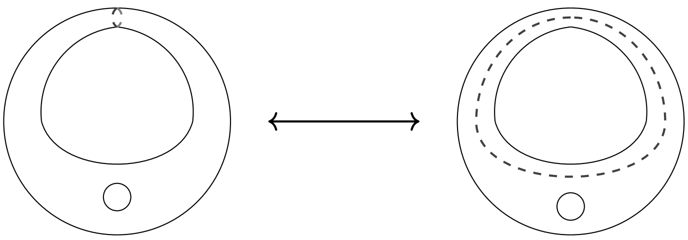

Figure 3.2: The ab-move swapping between two decompositions of the once punctured torus with a twist line.

3.1.3 The superconformal index for twisted class $\mathcal{S}$

The Macdonald limit (and so Schur limit by further specialisation) of the superconformal index in the twisted setting was studied for type D_N theories in [LPR14]. According to the analysis there, the presence of an single twisted puncture restricts the sum over P^+ to representations that are invariant under the action of the outer automorphism twist, which is equivalent to summing over the set of highest weight representations of the twisted algebra $\mathfrak{g}_t$ (which we denote by P_t^+). The overall structure constants are also modified, though they are expressed in terms of the same building blocks. For a surface of genus g with m untwisted punctures and $2n$, $\mathbb{Z}_2$-twisted punctures such that no twist lines wrap any cycles, the index then takes the form

$$\mathcal{I}(q;\mathbf{x}_1,\ldots,\mathbf{x}_m,\mathbf{y}_1,\ldots,\mathbf{y}_{2n})=\sum_{\lambda\in P_t^+}\frac{\prod_{i=1}^m\mathcal{K}_u(q;\mathbf{x}_i)\chi_u^\lambda(\mathbf{x}_i)\prod_{j=1}^{2n}\mathcal{K}_t(q;\mathbf{y}_j)\chi_t^\lambda(\mathbf{y}_j)}{(\mathcal{K}_u(\times)\chi_u^\lambda(\times))^{2g-2+m+2n}}\ . \tag{3.1.3}$$

Here $\mathbf{x}_i$ are fugacities for untwisted punctures and $\mathbf{y}_j$ are fugacities for the twisted ones. We have adopted notation where λ denotes an integral dominant weight in P_t^+ and its image $\iota(\lambda)\in P_u^+$ under the embedding (3.2.5). The characters of G_u that appear are at weights which are invariant under the action of σ.

For the non-abelian twist of D_4, a TQFT form of the index was proposed in [DES21]. This agrees with the heuristics we have so far observed—namely the sum is restricted to the integral dominant weights of D_4 that are invariant under the action of the twists that are present.

One can also rewrite the twisted index in terms of Weyl modules as

$$\mathcal{I}(q;\mathbf{x}_1,\ldots,\mathbf{x}_m,\mathbf{y}_1,\ldots,\mathbf{y}_{2n})=\sum_{\lambda\in P_t^+}(\mathrm{ch}\ \mathfrak{z}_\lambda^u)^{2-2g-2n}\left(\prod_{i=1}^m\mathrm{ch}\ \mathbb{L}_\lambda^u\right)\left(\prod_{j=1}^{2n}\mathrm{ch}\ \mathbb{V}_\lambda^t\right)\ , \tag{3.1.4}$$

where we use the superscripts u,t to distinguish between the (simple quotients of) Weyl modules over each algebra.

It may be worth remarking that all structure constants are of ``untwisted type'' in spite of the $\mathcal{K}$ factors and Schur functions of twisted type. This means that when closing a twisted puncture, the specialised puncture factor in the numerator won't cancel against a corresponding factor in the denominator. This is the index-level incarnation of the fact that empty twisted punctures are nontrivial and cannot be completely erased from the UV curve.

3.1.4 Chiral algebras of twisted class $\mathcal{S}$ at genus zero

Applying the dictionary of Section 1.2 to twisted class $\mathcal{S}$, produces a new family of vertex algebras, labelled by a simply laced Lie algebra $\mathfrak{g}_u$ and a punctured sphere with twisted punctures. We shall call these vertex algebras, *mixed* vertex algebras as they mix together the action of $\widehat{\mathfrak{g}}_u$ and $\widehat{\mathfrak{g}}_t$ via the Feigin–Frenkel centre.

Given $\mathcal{C}_{m,n}$, *i.e.*, $\mathbb{P}^1$ with m maximal untwisted punctures and n pairs of maximal twisted punctures, we wish to construct a vertex algebra $\mathbf{V}_{m,n}$ that is independent of the complex structures of $\mathcal{C}_{m,n}$. For each untwisted puncture, we expect a chiral moment map (vertex algebra morphism) $\mu_{u,i} : V^{\kappa_c}(\mathfrak{g}_u) \to \mathbf{V}_{m,n}$, making $\mathbf{V}_{m,n}$ a vertex algebra object in KL_u, with respect to $\mu_{u,i}$ for $i = 1, \dots, m$. Similarly, for each twisted puncture, we have a chiral moment map $\mu_{t,j} : V^{\kappa_c}(\mathfrak{g}_t) \to \mathbf{V}_{m,n}$ making $\mathbf{V}_{m,n}$ a vertex algebra object in KL_t for each $j = 1, \dots, 2n$. The images of all moment maps commute.

The action of the 4-move implies that all moment maps (of the same type) are on an equal footing, *i.e.*, there is an $S_m \times S_n$ permutation symmetry that acts on $\mathbf{V}_{m,n}$. We may still Higgs a maximal untwisted puncture to remove it from the UV curve, and so

$$\mathbf{V}_{m-1,n} \cong H^0_{\mathrm{DS}}(u, \mathbf{V}_{m,n}) \text{ for } m > 2 \tag{3.1.5}$$

where the reduction is done with respect to any of the untwisted moment maps. The case $m = 1$ will require some care, as we shall see in later sections.

Given two curves $\mathcal{C}_{m,n}$ and $\mathcal{C}_{p,q}$, we can glue them together via untwisted or twisted punctures. At the level of vertex algebras, this corresponds to semi-infinite cohomology and so we expect the following isomorphisms

$$\begin{aligned} \mathbf{V}_{m,n} \circ_u \mathbf{V}_{p,q} &\equiv \mathrm{H}^{\frac{\infty}{2}+\bullet}(\widehat{\mathfrak{g}}_{u,-\kappa_g}, \mathfrak{g}_u, \mathbf{V}_{m,n} \otimes \mathbf{V}_{p,q}) \cong \mathbf{V}_{m+p-2,q+n} \ , \\ \mathbf{V}_{m,n} \circ_t \mathbf{V}_{p,q} &\equiv \mathrm{H}^{\frac{\infty}{2}+\bullet}(\widehat{\mathfrak{g}}_{t,-\kappa_g}, \mathfrak{g}_t, \mathbf{V}_{m,n} \otimes \mathbf{V}_{p,q}) \cong \mathbf{V}_{m+p,q+n-1} \ . \end{aligned} \tag{3.1.6}$$

Since all punctures (of the same type) are on an equal footing, the gluing can be performed with respect to any diagonal chiral moment map of $\widehat{\mathfrak{g}}_{u,-\kappa_g}$ or $\widehat{\mathfrak{g}}_{t,-\kappa_g}$ on the two surfaces.

Furthermore, we expect that $\mathbf{V}_{m,n}$ are simple and conformal, with central charge [CDT13]

$$c_{m,n} = 2n \dim \mathfrak{g}_t + (m - 2h_u^\vee(m + 2n - 2)) \dim \mathfrak{g}_u - (m + 2n - 2)\mathrm{rk}\, \mathfrak{g}_u \ . \tag{3.1.7}$$

3.1.5 Residual gauge symmetry for twisted class $\mathcal{S}$

In the case of untwisted class $\mathcal{S}$, all genus zero theories had pure Higgs branches and so the vertex algebra $\mathbf{V}_{G,s}$ were purely bosonic and all gluings were concentrated in degree zero.

For twisted theories, the situation is a bit more complicated. To illustrate, we consider the D_2 theory. Due to the accidental isomorphism $\mathfrak{d}_2 \cong \mathfrak{sl}_2 \times \mathfrak{sl}_2$, we can recast theories of type D_2 into $\mathfrak{sl}_2$ theories. An untwisted puncture in the D_2 theory becomes a pair of untwisted punctures in the A_1 theory. The twisted subalgebra is just $\mathfrak{sl}_2$, and a full twisted puncture becomes a conventional (untwisted) puncture of the $\mathfrak{a}_1$ theory. In particular, the Spin(4) gauge theory with $N_f = 4$ flavours can be engineered via compactification of the D_2 theory on a sphere two maximal and two minimal twisted punctures. Equivalently, it can be described as the A_1 theory compactified on a genus-one surface with two punctures. Thus, the A_1 surface is a double cover of the D_2 surface, treating the twist lines as branch cuts. The D_2 theory has residual gauge symmetry at a generic point of the Higgs branch—despite being superficially of genus zero. From the A_1 perspective, this residual gauge symmetry is straightforward—since this surface has genus one. We will see that this phenomenon is characteristic of the twisted theories.

Somewhat more generally, for SO($2n$) superconformal QCD—realised in type D_n using a sphere with four twisted punctures (two minimal and two maximal)—a generic point of the Higgs branch has precisely a residual $U(1)$ gauge symmetry [APS97]. For $n \geqslant 3$, we have no more accidental isomorphisms and so there is not an immediate relation to a higher genus class $\mathcal{S}$ theory like the D_2 case. We can, nevertheless, observe that the presence of outer automorphism twist lines means that there is a natural covering space for the UV curve (thought of as the space where the corresponding local system of Dynkin diagrams has no monodromy), and in this example the covering space has genus one. We believe that in the general case of twisted class $\mathcal{S}$ theories, it is precisely the value of the genus (zero or nonzero) of this covering space that controls whether a theory in question has a pure or an enhanced Higgs branch.

More generally, we take away the lesson that in the presence of twisted punctures, genus zero theories should perhaps nevertheless be thought of as more analogous to higher-genus untwisted theories than genus-zero untwisted theories. The genus of this covering space, the Riemann–Hurwitz genus, is zero only if the number of twisted punctures does not exceed two.

The corresponding vertex algebras, $\mathbf{V}_{m,n}$ must therefore contain fermionic states for $n > 1$. Therefore, any gluing of punctures that introduces additional twisted punctures must not be concentrated in degree zero—so as to account for the fermionic states.

3.2 Constructing mixed vertex algebras

In this section we shall provide a construction of the $\mathbf{V}_{m,n}$ and provide partial results on gluing isomorphisms. We shall restrict our construction to the case where σ has order two though the results of Sections 3.2.1 and 3.2.2 hold for any non-trivial σ.

To establish properties regarding untwisted and twisted gluing, we need to examine the decomposition of the mixed vertex algebras into semijective objects in both twisted and untwisted KL categories. This will depend, in a crucial way, on the structure of critical-level Weyl modules over $\hat{\mathfrak{g}}_{t,\kappa_c}$, as modules over the untwisted Feigin–Frenkel centre $\mathfrak{z}(\mathfrak{g}_u)$. We discuss how these two centres are related in Section 3.2.1.

In Section 3.2.2 we will prove the following technical result, the proof of which will require a technical digression involving opers. Let $\lambda \in P_t^+$ be an integral dominant weight of $\mathfrak{g}_t$, and let $\iota(\lambda) \in P_u^+$ be defined as in (3.2.5). Let $\mathrm{Op}^{\lambda}_{{}^LG_t}$ denote the ${}^LG_t$ opers on $\mathbb{D}$ of coweight λ with regular singularity and trivial monodromy.

Theorem. *The restriction of the closed immersion* $\mathrm{Op}_{{}^LG_t}(\mathbb{D}^\times) \hookrightarrow \mathrm{Op}_{{}^LG_u}(\mathbb{D}^\times)$ *to the subscheme* $\mathrm{Op}^{\lambda}_{{}^LG_t}$ *factors as*

$$\mathrm{Op}^{\lambda}_{{}^LG_t} \hookrightarrow \mathrm{Op}^{\lambda}_{{}^LG_u} \hookrightarrow \mathrm{Op}_{{}^LG_u}(\mathbb{D}^\times) \ ,$$

with each map a closed immersion. Equivalently, the natural surjection $\mathrm{Fun}\,\mathrm{Op}_{{}^LG_u}(\mathbb{D}^\times) \twoheadrightarrow$

$(\mathrm{Fun}\,\mathrm{Op}_{^LG_u}(\mathbb{D}^\times))_\sigma$, *restricts to a surjection*

$$\mathrm{Fun}\,\mathrm{Op}^\lambda_{^LG_u} \twoheadrightarrow (\mathrm{Fun}\,\mathrm{Op}^\lambda_{^LG_u})_\sigma \ ,$$

on the quotient algebras.

A review of opers on curves can be found in Appendix A.4.

In Section 3.2.3, we introduce a class of $(\widehat{\mathfrak{g}}_u, \widehat{\mathfrak{g}}_t)$ bimodules, which look like Weyl modules of each algebra sewn together by identifying the action of $\mathfrak{z}(\mathfrak{g}_u)$. We shall also establish some homological properties of these modules under semi-infinite cohomology.

Our construction of the mixed trinion $\mathbf{V}_{1,1}$ can be found in Section 3.2.4. We also prove a number of the expected properties from Section 3.1.4. In particular, we establish that closing the untwisted puncture via DS-reduction recovers $\mathcal{D}^{ch}_{G_t}$.

Theorem. *We have the following isomorphism:*

$$H^0_{DS}(u, \mathbf{V}_{1,1}) \cong \mathcal{D}^{ch}_t \ , \tag{3.2.1}$$

so $\mathcal{D}^{ch}_t \in \mathrm{KL}_{u,0}$.

The proof of this theorem will be delayed to Section 3.2.10, since it requires some additional machinery. This result will be key in establishing our uniqueness result of Proposition 3.2.12. A number of the results in this subsection shall serve as the base case for inductive arguments establishing properties for the $\mathbf{V}_{m,n}$.

Before extending our construction to the full family of $\mathbf{V}_{m,n}$ we shall find it useful to prove a number of technical lemmas regarding the commutativity of the various homological operations we have introduced. In Section 3.2.5 we establish the conditions, under which, we can swap the orders of the various cohomologies.

With this result in hand we can extend our construction to the $\mathbf{V}_{m,n}$ in Section 3.2.6. There is an obvious analogue of the construction of [Ara18] for the $\mathbf{V}_{m,1}$, but such a construction fails if we wish to introduce more punctures. For $n > 1$, we will be forced to define the

$\mathbf{V}_{m,n}$ recursively, by picking a particular pants decomposition of the surface $\mathcal{C}_{m,n}$. To finish this section we shall provide partial results on the gluing isomorphisms.

In Section 3.2.8, we shall discuss the action of generalised S-duality on the $\mathbf{V}_{m,n}$ and show that the 4-moves of Section 3.1.2 act as automorphisms. This justifies that our recursive definition of the $\mathbf{V}_{m,n}$, is well-defined.

To end, we shall consider the case when σ is not of order two in Section 3.2.9 and discuss the obstructions that arise in this case.

3.2.1 The (un)twisted Feigin–Frenkel centre

The $\mathbf{V}_{m,n}$ should simultaneously be vertex algebra objects in KL_u and in KL_t, so they will admit actions of both Feigin–Frenkel centres. The construction of $\mathbf{V}_{G,s}$ suggests that the action of these Feigin–Frenkel centres should be identified, but of course the twisted and untwisted centres are not isomorphic. It will be useful, therefore, to first examine how the actions of these two Feigin–Frenkel centres interact with each other.

Let $\sigma \in \mathrm{Out}(\mathfrak{g}_u)$ be an outer automorphism (not necessarily of order two), $\mathfrak{g}_u^\sigma$ be the σ-invariant subalgebra of $\mathfrak{g}_u$, and $\mathfrak{g}_t = (\mathfrak{g}_u^\sigma)^\vee$. There exists a projection [FSS96],

$$\pi_\sigma : \mathfrak{h}_u \twoheadrightarrow \mathfrak{h}_t \ , \tag{3.2.2}$$

from the Cartan subalgebra of $\mathfrak{g}_u$ to that of $\mathfrak{g}_t$ that projects to elements that are invariant under σ. The outer automorphism lifts to an automorphism of $U(\mathfrak{g}_u)$ and we have a surjection

$$Z(U(\mathfrak{g}_u)) \twoheadrightarrow Z(U(\mathfrak{g}_t)) \ , \tag{3.2.3}$$

which is just the projection of the centre of $U(\mathfrak{g}_u)$ to its $\langle\sigma\rangle$-coinvariants, *i.e.*, we set the σ-non-invariant generators of $Z(\mathfrak{g}_u)$ to zero. The action of σ can be lifted to $\hat{\mathfrak{g}}_{u,\kappa_c}$ according to $\sigma(xt^n) = \sigma(x)t^n$. This gives a projection

$$\mathcal{Z}_u \twoheadrightarrow \mathcal{Z}_t \cong (\mathcal{Z}_u)_\sigma \ , \tag{3.2.4}$$

$\mathfrak{g}_u$	$\mathfrak{z}(\mathfrak{g}_u)$	$\mathfrak{g}_t$	$\mathfrak{z}(\mathfrak{g}_t)$
$\mathfrak{a}_{2n-1}$	$P_2, P_3, \ldots, P_{2n}$	$\mathfrak{b}_n$	$P_2, P_4, \ldots, P_{2n}$
$\mathfrak{a}_{2n}$	$P_2, P_3, \ldots, P_{2n+1}$	$\mathfrak{c}_n$	$P_2, P_4, \ldots, P_{2n}$
$\mathfrak{d}_n$	$\tilde{P}_n; P_2, P_4, \ldots, P_n, \ldots, P_{2n-2}$	$\mathfrak{c}_{n-1}$	$P_2, P_4, \ldots, P_{2n-2}$
$\mathfrak{e}_6$	$P_2, P_5, P_6, P_8, P_9, P_{12}$	$\mathfrak{f}_4$	P_2, P_6, P_8, P_{12}
$\mathfrak{d}_4$	$\tilde{P}_4, P_2, P_4, P_6$	$\mathfrak{g}_2$	P_2, P_6

Table 3.2: The monomial generators of the Feigin–Frenkel centres of the untwisted algebra $\mathfrak{g}_u$ and its associated twisted algebra $\mathfrak{g}_t$. Note that algebras of type $\mathfrak{d}_n$ have two generators of degree n, only one of which is invariant under the outer automorphism. The last row shows the $\mathbb{Z}/3\mathbb{Z}$ twist for $\mathfrak{d}_4$; neither generator of degree four is invariant under this outer automorphism.

where $(\mathcal{Z}_u)_\sigma$ is the space of $\langle\sigma\rangle$-coinvariants of the untwisted Feigin–Frenkel centre. The projection π_σ also induces an embedding of weight spaces

$$\iota : P_t^+ \hookrightarrow P_u^+ \ , \tag{3.2.5}$$

with image $\iota(P_t^+)$ equal to the subset of elements in P_u^+ that are invariant under the action of σ. For example if $\mathfrak{g}_u = \mathfrak{d}_n$ and $\mathfrak{g}_t = \mathfrak{c}_{n-1}$ (so σ is the $\mathbb{Z}/2\mathbb{Z}$ outer automorphism), we have

$$\iota(\lambda_1, \lambda_2, \ldots, \lambda_{n-1}) = (\lambda_1, \lambda_2, \ldots, \lambda_{n-1}, \lambda_{n-1}) \ . \tag{3.2.6}$$

In [LPR14], this was indicated with the notation $\lambda' = \lambda$. We will abuse notation and use λ for both the weight in P_t^+ and its image under $\iota : P_t^+ \hookrightarrow P_u^+$. For example, V_λ^t denotes the finite dimensional irreducible representation of $\mathfrak{g}_t$ with highest weight λ, and V_λ^u is the finite dimensional irreducible representation of $\mathfrak{g}_u$ with highest weight $\iota(\lambda)$.

Given an object $M \in \mathcal{Z}_t$-Mod, we can lift it to a module in $\mathcal{Z}_u$-Mod via the restriction of scalars associated to $\mathcal{Z}_u \twoheadrightarrow \mathcal{Z}_t$, giving a functor $\mathcal{Z}_t$-Mod $\to \mathcal{Z}_u$-Mod. Even in the case where there are no untwisted punctures, the algebra must still have a $\mathcal{Z}_u$-Mod structure. This is because any such algebra should be obtained via DS reduction of a vertex algebra object in KL_u, and the DS reduction functor lands in $\mathcal{Z}_u$-Mod.

Similarly to the FF glued modules of Section 2.2.2, we would like to define mixed modules that are glued via the action of $\mathcal{Z}_u$ on $\mathbb{V}_{\lambda^t}^t$ and $\mathbb{V}_{\lambda'}^u$ for some $\lambda \in P_u^+$ and $\lambda' \in P_t^+$. Such a glued module should have the form

$$\mathbb{V}^u_{\lambda'} \underset{\mathfrak{z}^u_\lambda}{\otimes} \mathbb{V}^t_\lambda \tag{3.2.7}$$

While it is obvious that $\mathcal{Z}_t$ modules can be lifted to $\mathcal{Z}_u$ modules by way of the projection $\mathcal{Z}_u \twoheadrightarrow \mathcal{Z}_t$, it is not so obvious that this should hold for the quotient modules $\mathfrak{z}^u_{\lambda'}$, which are more complicated. There is also a question of which values of λ and λ' result in non-trivial modules.

Altogether, there is room for doubt over whether the suggested tensor product over $\mathfrak{z}^u_{\lambda'}$ is non-trivial. To show that the restriction of scalars descends compatibly to the quotients, we will have to use the machinery of opers.

3.2.2 Opers with monodromy

Much of the machinery below is introduced in a more pedagogical manner in Section A.4.2. For the sake of brevity we shall refer to results and definitions in the appendix, rather than reproducing them in full.

By the Feigin–Frenkel isomorphism, the Feigin–Frenkel centre and its mode algebra are related to ${}^LG_u$-opers on the disc and punctured disc:

$$\mathfrak{z}(\mathfrak{g}_u) \cong \mathrm{Op}_{{}^LG_u}(\mathbb{D}) \ , \quad \mathcal{Z}_u \cong \mathrm{Op}_{{}^LG_u}(\mathbb{D}^\times) \ , \tag{3.2.8}$$

and analogously for $\mathfrak{z}(\mathfrak{g}_t)$ and $\mathcal{Z}_t$. An ${}^LG_u$-oper has a representative,

$$\nabla = \partial_t + p_{-1} + \sum_{i=1}^{\mathrm{rk}\,\mathfrak{g}_u} v_{d_j}(t) p_{d_j} \ , \tag{3.2.9}$$

where $v_{d_j} \in \mathbb{C}[\![t]\!]$ for $\mathbb{D}$ and $v_{d_j} \in \mathbb{C}(\!(t)\!)$ for $\mathbb{D}^\times$. Here p_{d_i} are the basis from Remark A.1.13. The outer automorphism acts naturally on the p_{d_j}, leaving p_{d_1} invariant. We can identify the fixed points of this action with

$$(\mathrm{Op}_{{}^LG_u})^\sigma \cong \mathrm{Op}_{{}^LG_t} \ , \tag{3.2.10}$$

for $\mathbb{D}$ and $\mathbb{D}^\times$. The closed immersion $\mathrm{Op}_{^LG_t}(\mathbb{D}^\times) \cong (\mathrm{Op}_{^LG_u})^\sigma \hookrightarrow \mathrm{Op}_{^LG_u}$ is precisely the projection to coinvariants in (3.2.4).

To proceed, we want to show that we can lift $\mathfrak{z}^t_\lambda$ modules, for $\lambda \in P^+_t$ t $\mathfrak{z}^u_{\lambda'}$ modules for some $\lambda' \in P^+_u$. Recall, from Appendix A.4.4 the subspace, $\mathrm{Op}^\lambda_{^LG}$ of opers on $\mathbb{D}$ with regular singularity and no monodromy—specified by a choice of coweight. The main theorem of [FG10], reproduced in the appendix as Theorem A.4.9, tells us that

$$\mathfrak{z}^t_\lambda \cong \mathrm{Fun}\,\mathrm{Op}^\lambda_{^LG_t}\ , \text{ and } \mathfrak{z}^u_{\lambda'} \cong \mathrm{Fun}\,\mathrm{Op}^{\lambda'}_{^LG_u}\ . \tag{3.2.11}$$

Setting all weights to zero recovers the usual spaces $\mathrm{Op}_{^LG_u}(\mathbb{D})$ and $\mathrm{Op}_{^LG_t}(\mathbb{D})$ in the Feigin--Frenkel isomorphism.

We would like to construct a morphism $\mathrm{Fun}\,\mathrm{Op}^{\lambda'}_{^LG_u} \to \mathrm{Fun}\,\mathrm{Op}^\lambda_{^LG_t}$ along which we can restrict scalars. Equivalently, we would like to find a morphism, $\mathrm{Op}^\lambda_{^LG_t} \to \mathrm{Op}^{\lambda'}_{^LG_u}$, on the spaces.

We also have natural closed immersions $\mathrm{Op}^\lambda_{^LG_t} \hookrightarrow \mathrm{Op}_{^LG_t}(\mathbb{D}^\times)$ and $\mathrm{Op}^{\lambda'}_{^LG_u} \hookrightarrow \mathrm{Op}_{^LG_u}(\mathbb{D}^\times)$, along with the inclusion of fixed points, $\mathrm{Op}_{^LG_t}(\mathbb{D}^\times) \hookrightarrow \mathrm{Op}_{^LG_u}(\mathbb{D}^\times)$. We would like to show that these morphisms are all compatible, *i.e.*, the composition $\mathrm{Op}^\lambda_{^LG_t} \hookrightarrow \mathrm{Op}_{^LG_t}(\mathbb{D}^\times) \hookrightarrow \mathrm{Op}_{^LG_u}(\mathbb{D}^\times)$ factors via $\mathrm{Op}^{\lambda'}_{^LG}$.

Theorem 3.2.1. *Let $\lambda \in P^+_t$ and also denote its image under the inclusion $\iota : P^+_t \hookrightarrow P^+_u$ by $\lambda \in P^+_u$.*

The restriction of the inclusion of fixed points, $\mathrm{Op}_{^LG_t}(\mathbb{D}^\times) \hookrightarrow \mathrm{Op}_{^LG_u}(\mathbb{D}^\times)$ *to the subscheme* $\mathrm{Op}^\lambda_{^LG_t}$ *factors as*

$$\mathrm{Op}^\lambda_{^LG_t} \hookrightarrow \mathrm{Op}^\lambda_{^LG_u} \hookrightarrow \mathrm{Op}_{^LG_u}(\mathbb{D}^\times)\ ,$$

with each map a closed immersion. Equivalently, the natural surjection $\mathrm{Fun}\,\mathrm{Op}_{^LG_u}(\mathbb{D}^\times) \twoheadrightarrow (\mathrm{Fun}\,\mathrm{Op}_{^LG_u}(\mathbb{D}^\times))_\sigma$*, restricts to a surjection*

$$\mathrm{Fun}\,\mathrm{Op}^\lambda_{^LG_u} \twoheadrightarrow (\mathrm{Fun}\,\mathrm{Op}^\lambda_{^LG_u})_\sigma\ ,$$

on the quotient algebras.

Note that the theorem requires $\lambda' = \iota(\lambda)$, *i.e.*, this restriction of scalars is only non-trivial when the untwisted weight $\lambda' \in P_u^+$ is outer-automorphism invariant.

Corollary 3.2.2. *The restriction of scalars* $\mathfrak{z}_\lambda^t - \mathrm{mod} \to \mathfrak{z}_\lambda^u - \mathrm{mod}$ *is fully faithful.*

The rest of this subsection will be devoted to the proof of this theorem.

To start with, we recall the Miura transforms (A.4.26)

$$\begin{aligned} \mu_{u,\mathrm{Miura}} &: \mathrm{Conn}(\Omega^{\rho_u})_{{}^L H_u, \mathbb{D}^\times} \twoheadrightarrow \mathrm{Op}_{{}^L G_u}(\mathbb{D}^\times) \ , \\ \mu_{t,\mathrm{Miura}} &: \mathrm{Conn}(\Omega^{\rho_t})_{{}^L H_t, \mathbb{D}^\times} \twoheadrightarrow \mathrm{Op}_{{}^L G_t}(\mathbb{D}^\times) \ , \end{aligned} \tag{3.2.12}$$

where $\mathrm{Conn}(\Omega^{\rho_u})_{{}^L H_t, \mathbb{D}^\times}$ is the space of Cartan connections. Fixing dominant integral weights $\lambda \in P_t^+$ and $\lambda' \in P_u^+$, we denote by, $\mathrm{Conn}(\Omega^{\rho_t})^{\lambda'}_{{}^L H_t}$ and $\mathrm{Conn}(\Omega^{\rho_u})^{\lambda}_{{}^L H_u}$, the space of Cartan connections with residue at zero equal to λ and λ', respectively. The Miura transform, restricted to these subspaces, gives a surjection (see Proposition A.4.14)

$$\begin{aligned} \mu^{\lambda'}_{u,\mathrm{Miura}} &: \mathrm{Conn}(\Omega^{\rho_u})_{{}^L H_u} \twoheadrightarrow \mathrm{Op}^{\lambda'}_{{}^L G_u} \cong \mathrm{Spec}\, \mathfrak{z}^u_{\lambda'} \ , \\ \mu^{\lambda}_{t,\mathrm{Miura}} &: \mathrm{Conn}(\Omega^{\rho_t})^{\lambda}_{{}^L H_t} \twoheadrightarrow \mathrm{Op}^{\lambda}_{{}^L G_t} \cong \mathrm{Spec}\, \mathfrak{z}^t_{\lambda} \ . \end{aligned} \tag{3.2.13}$$

The functions on these spaces are easy to describe:

$$\begin{aligned} \mathrm{Fun}\,\mathrm{Conn}(\Omega^{\rho_u})^{\lambda'}_{{}^L H_u} &\cong \mathbb{C}[u_{i,m} \,|\, i = 1, \ldots, \mathrm{rk}\, \mathfrak{g}_u \ , \ m \in \mathbb{Z}_{<0}] \ , \\ \mathrm{Fun}\,\mathrm{Conn}(\Omega^{\rho_t})^{\lambda}_{{}^L H_t} &\cong \mathbb{C}[u_{i,m} \,|\, i = 1, \ldots, \mathrm{rk}\, \mathfrak{g}_t \ , \ m \in \mathbb{Z}_{<0}] \ . \end{aligned} \tag{3.2.14}$$

The $\mathbb{C}$-points of $\mathrm{Conn}(\Omega^{\rho_u})^{\lambda'}_{{}^L H_u}$ are connections of the form

$$\nabla_u = \partial_t + \frac{\lambda'}{t} + \sum_{m<0} u_m t^{-m-1} \ , \tag{3.2.15}$$

with $u_m \in {}^L\mathfrak{h}_u$ and $u_{i,m} = (\alpha_i^\vee, u_m)$ for simple coroots $\alpha_i^\vee$. There is a natural action of σ

on this subspace, given by

$$\sigma \cdot \nabla_u \coloneqq \partial_t + \frac{\sigma(\lambda')}{t} + \sum_{m<0} \sigma(u_m) t^{-m-1} \ . \tag{3.2.16}$$

From the above expression, it is clear to see that this action is free unless λ' is σ-invariant. Suppose λ' is σ-invariant, then it is equal to $\iota(\lambda)$ for some $\lambda \in P_t^+$. We continue to abuse notation and use λ to denote $\iota(\lambda) \in P_u^+$. Now, a connection in $\mathrm{Conn}(\Omega^{\rho_u})^{\lambda'}_{^L H_u}$ is σ-invariant if the $u_m \in ({}^L\mathfrak{h}_u)^\sigma \cong (\mathfrak{h}_u)^\sigma \cong {}^L\mathfrak{h}_t$. Therefore, we have an isomorphism

$$\mathrm{Conn}(\Omega^{\rho_t})^{\lambda}_{^L H_t} \xrightarrow{\sim} \left(\mathrm{Conn}(\Omega^{\rho_u})^{\lambda}_{^L H_u}\right)^\sigma \ , \tag{3.2.17}$$

and an inclusion $\mathrm{Conn}(\Omega^{\rho_t})^{\lambda}_{^L H_t} \hookrightarrow \mathrm{Conn}(\Omega^{\rho_u})^{\lambda}_{^L H_u}$. On functions, we can realise the space of coinvariants as

$$\left(\mathrm{Fun}\,\mathrm{Conn}(\Omega^{\rho_u})^{\lambda}_{^L H_u}\right)_\sigma \cong \mathbb{C}\left[\tilde{u}_{i,n} | i = 1, \ldots, \mathrm{rk}\,\mathfrak{g}_t;\ n \in \mathbb{Z}\right] \ , \tag{3.2.18}$$

where

$$\tilde{u}_{i,n} = \frac{1}{|\langle\sigma\rangle|} \sum_{\sigma' \in \langle\sigma\rangle} \sigma'(u_{i,n}) \ . \tag{3.2.19}$$

These are precisely the linear combinations of the generators of $\mathrm{Fun}\,\mathrm{Conn}(\Omega^\rho)^\lambda_{^L G_u}$ that are invariant under $\langle\sigma\rangle$.

The fibres of the Miura transforms from (3.2.13), $\mu^\lambda_{u,\mathrm{Miura}}$ and $\mu^\lambda_{t,\mathrm{Miura}}$, are principal ${}^L N_u$ and ${}^L N_t$ torsors over $\mathrm{Op}^\lambda_{^L G_u}$ and $\mathrm{Op}^\lambda_{^L G_t}$ respectively. Therefore, we can identify $\mathrm{Op}^\lambda_{^L G_u}$ and $\mathrm{Op}^\lambda_{^L G_t}$ with the ${}^L N_u$ orbit space of $\cong \mathrm{Conn}(\Omega^\rho)^\lambda_{^L G_u}$ and the ${}^L N_t$ orbit space of $\mathrm{Conn}(\Omega^\rho)^\lambda_{^L G_t}$. On functions, the infinitesimal action of the unipotent groups is given by the action of the vector fields (A.4.39).

Assembling, we have the diagram,

$$\begin{array}{ccc} \mathrm{Fun}\,\mathrm{Op}^\lambda_{G_u} & \xhookrightarrow{\iota} & \mathrm{Fun}\,\mathrm{Conn}(\Omega^\rho)^\lambda \\ \big\downarrow{\scriptstyle \pi} & & \big\downarrow{\scriptstyle \pi_\sigma} \\ \mathrm{Fun}\,\mathrm{Op}^\lambda_{^L G_t} & \xhookrightarrow{\iota} & (\mathrm{Fun}\,\mathrm{Conn}(\Omega^\rho)^\lambda)_\sigma \end{array} \tag{3.2.20}$$

where the horizontal morphisms are the natural inclusions of ${}^L\mathfrak{n}_u \cong \mathfrak{n}_u$ and ${}^L\mathfrak{n}_t = (\mathfrak{n}_u)^\sigma$ invariants and the vertical arrow is the natural projection to the σ-coinvariants. We can define a map π that, we claim, makes the diagram commute via $\pi = \pi_\sigma \circ \iota$. The map π is precisely the projection of $\operatorname{Fun}\operatorname{Op}^\lambda_{G_u}$ to its $\langle\sigma\rangle$ coinvariants.

Our Theorem 3.2.1, is therefore equivalent to the claim that π is surjective. Indeed, surjectivity of π is precisely the statement that $\mathfrak{z}^u_\lambda \twoheadrightarrow \mathfrak{z}^t_\lambda$ is a surjection.

Proof of Theorem 3.2.1. Recall that in Section A.4.6, we described the action of ${}^L\mathfrak{n}_u \cong \mathfrak{n}_u$ on $\operatorname{Fun}\operatorname{Conn}(\Omega^{\rho_u})^\lambda_{{}^L H_u}$ in terms of the vector fields, or screening charges:

$$V_i[\lambda_i+1] = -\sum_{j=1}^{\operatorname{rk}\mathfrak{g}} a_{ji} \sum_{n\geq\lambda_i} x_{i,n-\lambda_i}\frac{\partial}{\partial u_{j,-n-1}} \ , \tag{3.2.21}$$

for $i = 1,\ldots,\operatorname{rk}\mathfrak{g}_u$, a_{ji} the Cartan matrix of $\mathfrak{g}_u$, and $x_{i,n}$ determined by

$$\sum_{n\leq 0} x_{i,n}t^{-n} = \operatorname{Exp}\left(-\sum_{m>0}\frac{u_{i,-m}}{m}t^m\right) \ . \tag{3.2.22}$$

The action of ${}^L\mathfrak{n}_t = (\mathfrak{n}_u)^\sigma$ on the space of coinvariants can be realised through the symmetrised screening charges

$$\tilde{V}_i[\lambda_i+1] = \frac{1}{|\langle\sigma\rangle|}\sum_{\sigma'\in\langle\sigma\rangle}\sigma'(V_i[\sigma'(\lambda)_i+1]) \ , \tag{3.2.23}$$

for $i = 1,\ldots,\operatorname{rk}\mathfrak{g}_t$.

A polynomial $P \in \operatorname{Fun}\operatorname{Conn}(\Omega^\rho)^\lambda_{{}^L H_u}$ is in the space of invariants, $(\operatorname{Fun}\operatorname{Conn}(\Omega^\rho)^\lambda)^{\mathfrak{n}_u}$, if and only if it is in the intersection of the kernels of the screening charges, *i.e.*

$$V_i[\lambda_i+1]P \overset{!}{=} 0\,, \qquad \text{for } i = 1,\ldots,\operatorname{rk}\mathfrak{g}_u \ . \tag{3.2.24}$$

Similarly, a polynomial $P \in (\operatorname{Fun}\operatorname{Conn}(\Omega^\rho)^\lambda_{{}^L H_u})_\sigma$ in the space of coinvariants is an $\mathfrak{n}_u^\sigma = {}^L\mathfrak{n}_t$ invariant, if and only if it lies in the intersection of the kernels of the symmetrised screening charges $\tilde{V}_i[\lambda_i+1]$.

It is, therefore, sufficient to show that if P is a representative of a σ-coinvariant and lies in the intersection of the kernels of the symmetrised screening charges, then it must lie in the intersection of the kernels of the $\mathfrak{n}_u$ screening charges. We make the following observations. Suppose P lies in the space of coinvariants, then we can realise it as $P \in \mathbb{C}[\tilde{u}_{i,n} | i = 1, \ldots, \operatorname{rk}\mathfrak{g}_t;\ n < 0]$. Therefore,

$$\frac{\partial P}{\partial u_{j,n}} = \sigma' \left(\frac{\partial P}{\partial u_{j,n}} \right) \ , \tag{3.2.25}$$

for any $\sigma' \in \langle \sigma \rangle$. As a result, we must have that

$$\sigma'(V_i[\lambda_i + 1]P) = \sigma'(V_i[\lambda_i + 1])P \ . \tag{3.2.26}$$

Suppose P lies in the intersection of kernels of the symmetrised screening charges. First, we consider the case where σ has order two. Now, we have that

$$\tilde{V}_i[\lambda_1 + 1]P = \tfrac{1}{2}\big(V_i[\lambda_i + 1] + \sigma(V_i[\lambda_i + 1])\big)P = 0 \ , \tag{3.2.27}$$

but from (3.2.26), this means that

$$\sigma\left(V_i[\lambda_i + 1]P\right) = -V_i[\lambda_i + 1]P \ . \tag{3.2.28}$$

We shall now show that for any P in the space of coinvariants, the polynomial $V_i[\lambda_i + 1]P$, cannot have eigenvalue -1 under σ.

The image of P under the ith screening charge is

$$V_i[\lambda_i + 1]P = \sum_{n \geq \lambda_i} x_{i,n-\lambda_i} \cdot \left(a_{ji} \frac{\partial P}{\partial u_{j,-n-1}} \right) \ , \tag{3.2.29}$$

and we have

$$\sigma\left(V_i[\lambda_i + 1]P\right) = \sum_{n \geq \lambda_i} \sigma(x_i)_{,n-\lambda_i} \cdot \left(a_{ji} \frac{\partial P}{\partial u_{j,-n-1}} \right) \ , \tag{3.2.30}$$

where we have made use of the fact that the derivatives of P and the highest weight, λ, are invariant under σ. The outer automorphism does not act with eigenvalue -1 on any simple root and so we must have that.

$$\sigma(V_i[\lambda_i+1]P) \neq -V_i[\lambda_i+1]P \ , \tag{3.2.31}$$

unless both are identically zero, as desired.

Now we address the $\mathbb{Z}/3\mathbb{Z}$ case of $\mathfrak{g}_u = \mathfrak{d}_4$ and $\mathfrak{g}_t = \mathfrak{g}_2$. A $\mathfrak{n}_u^\sigma$ invariant P must satisfy the following:

$$(V_1[\lambda_1+1] + V_3[\lambda_1+1] + V_4[\lambda_1+1])P = 0 \ , \quad V_2[\lambda_2+1]P = 0 \ . \tag{3.2.32}$$

We have, once more, made use of the fact that the derivatives of P and the weight λ are invariant under the action of $\mathbb{Z}_3$. Expanding the first requirement, we have that

$$\sum_{n\geq\lambda_1} (x_{1,n-\lambda_1} + x_{3,n-\lambda_1} + x_{4,n-\lambda_i}) \left(-\frac{\partial P}{\partial u_{2,-n-1}} + 2\frac{\partial P}{\partial u_{i,-n-1}} \right) = 0 \ . \tag{3.2.33}$$

Once again, this can only hold if each screening charge individually acts as zero. □

3.2.3 Mixed modules over the (un)twisted Feigin–Frenkel centre

Having proven our main technical result, let us move to defining and establishing various properties of the putative mixed modules.

In what follows we will make use of the following technical proposition.

Proposition 3.2.3. *Let N be a $\mathfrak{z}^t_\lambda$-module. Then N is an object of $KL_{u,0}$, i.e., there exists some $M \in \mathrm{KL}_u$ such that*

$$N = H^0_{DS}(u, M) \ . \tag{3.2.34}$$

Proof. We prove this by explicitly constructing an object in KL_u whose DS reduction is isomorphic to N as an object of $\mathfrak{z}^t_\lambda$-mod. From Theorem 3.2.1, the $\mathfrak{z}^t_\lambda$ action can be lifted to an action of $\mathfrak{z}^u_\lambda$.

Let $\mathbb{V}^u_\lambda$ be the Weyl module of $\hat{\mathfrak{g}}_{u,\kappa_c}$ with highest weight $\iota(\lambda)$. The tensor product,

$$\mathbb{V}^u_\lambda \otimes_{\mathfrak{z}^u_\lambda} N \ , \tag{3.2.35}$$

is well-defined, where the $\mathfrak{z}^u_\lambda$ action on N is from the lift. By construction, this is an object in KL_u with respect to the $\hat{\mathfrak{g}}_{u,\kappa_c}$ action on the untwisted Weyl module. Let us consider its DS reduction,

$$H^0_{\mathrm{DS}}(u, \mathbb{V}^u_\lambda \otimes_{\mathfrak{z}^u_\lambda} N) \ .$$

Note that by Proposition A.3.6 ([FG10, Theorem 2]) $H^0_{\mathrm{DS}}(\mathbb{V}_\lambda) \cong \mathfrak{z}_\lambda$ is manifestly free over $\mathfrak{z}_\lambda$. Combining this with the Künneth theorem, we have that

$$H^0_{DS}(u, \mathbb{V}^u_\lambda \otimes_{\mathfrak{z}^u_\lambda} N) \cong \mathfrak{z}^u_\lambda \otimes_{\mathfrak{z}^u_\lambda} N \cong N \ , \tag{3.2.36}$$

as desired. □

Definition 3.2.4. Let $\lambda \in P^+_t$ be an integral dominant weight and let $\mathbb{V}^t_\lambda$ be the associated Weyl module of $\widehat{\mathfrak{g}}_{t,\kappa_c}$. We also use $\lambda \in P^+_u$ to denote the image of the embedding $\iota : P^+_t \hookrightarrow P^+_u$—let $\mathbb{V}^u_\lambda$ be the associated Weyl module of $\widehat{\mathfrak{g}}_{u,\kappa_c}$. For $m \in \mathbb{N}$, we define,

$$\mathbb{V}^{ut}_{\lambda,m} \coloneqq \mathbb{V}^u_{\lambda,m} \underset{\mathfrak{z}^u_\lambda}{\otimes} \mathbb{V}^t_{\lambda,2} \equiv \underbrace{\big(\mathbb{V}^u_\lambda \underset{\mathfrak{z}^u_\lambda}{\otimes} \mathbb{V}^u_{\lambda^*} \underset{\mathfrak{z}^u_\lambda}{\otimes} \dots \underset{\mathfrak{z}^u_\lambda}{\otimes} \mathbb{V}^u_{\overline{\lambda}}\big)}_{m \text{ copies}} \underset{\mathfrak{z}^u_\lambda}{\otimes} \big(\mathbb{V}^t_\lambda \underset{\mathfrak{z}^t_\lambda}{\otimes} \mathbb{V}^t_{\lambda *}\big) \tag{3.2.37}$$

where $\overline{\lambda}$ is equal to λ for odd m and λ^* otherwise. We denote their contragredient duals by $D(\mathbb{V}^{ut}_{\lambda,m})$.

The mixed modules $\mathbb{V}^{ut}_{\lambda,m}$ are projective over $U(t^{-1}\mathfrak{g}_t[t^{-1}]$ and their contragredient duals are injective over $U(t\mathfrak{g}_t[t])$. The analogous statement for $\mathfrak{g}_u$ does not hold. To see this, note that the modes $P_{i,-n}$, which are not invariant under σ, must act as zero on the lowest degree subspace. Hence, $\mathbb{V}^{ut}_{\lambda,m}$ are not torsion-free—and so cannot be projective—over $U(t^{-1}\mathfrak{g}_u[t^{-1}])$.

An immediate application of Proposition 3.2.3 gives the following isomorphisms

$$\begin{aligned} H^0_{\mathrm{DS}}(u, \mathbb{V}^{ut}_{\lambda,m}) &\cong \mathbb{V}^{ut}_{\lambda,m-1} \ , \ \text{for } m \geq 1 \ , \\ H^0_{\mathrm{DS}}(u, D(\mathbb{V}^{ut}_{\lambda,m})) &\cong D(\mathbb{V}^{ut}_{\lambda,m-1}) \ , \ \text{for } m \geq 1 \ , \end{aligned} \tag{3.2.38}$$

Proposition 3.2.5. *Suppose $N \in \mathcal{Z}_u{-}Mod$ has an increasing filtration $0 = N_0 \subset N_1 \subset \cdots \subset N$ with successive quotients*

$$N_i/N_{i-1} \cong \mathbb{V}^{ut}_{\lambda,m} \tag{3.2.39}$$

for some $\lambda \in P^+_t$ and a fixed m for all quotients. Then $M = \mathrm{H}^{\frac{\infty}{2}+0}(\mathcal{Z}_u, \mathbf{W}_u \otimes N) \in \mathrm{KL}_u$ has an increasing filtration, $0 \subset M_0 \subset M_1 \subset \cdots \subset M$ whose successive quotients satisfy

$$M_i/M_{i-1} \cong \mathbb{V}^{ut}_{\lambda,m+1} \ . \tag{3.2.40}$$

Proof. Note that since $\mathbb{V}^{ut}_{\lambda,m} \in \mathrm{KL}_{u,0}$ by (3.2.38), and so each quotient N_i/N_{i-1} is an object of $\mathrm{KL}_{u,0}$. Applying [Ara18, Theorem 9.14], we have that M has an increasing filtration with successive quotients,

$$M_i/M_{i-1} \cong \mathrm{H}^{\frac{\infty}{2}+0}(\mathcal{Z}_u, \mathbf{W}_u \otimes N_i/N_{i-1}) \cong \mathrm{H}^{\frac{\infty}{2}+0}(\mathcal{Z}_u, \mathbf{W}_u \otimes \mathbb{V}^{ut}_{\lambda,m}) \ , \tag{3.2.41}$$

for some $\lambda \in P^+_t$. Now by Theorem 2.2.8, $\mathrm{H}^{\frac{\infty}{2}+0}(\mathcal{Z}_u, \mathbf{W}_u \otimes \mathbb{V}^{ut}_{\lambda,m}) \cong \mathbb{V}^{ut}_{\lambda,m+1}$ and we have the desired result. □

Remark 3.2.6. Dualising the statement and proof of Proposition 3.2.5 implies the following. Suppose that, $N \in \mathcal{Z}_u{-}\mathrm{Mod}$ has a decreasing filtration $N = N_0 \supset N_1 \supset \cdots \supset 0$ with successive quotients, $N_i/N_{i+1} \cong D(\mathbb{V}^{ut}_{\lambda,m})$. Then $M = \mathrm{H}^{\frac{\infty}{2}+0}(\mathcal{Z}_u, \mathbf{W}_u \otimes N) \in \mathrm{KL}_u$ has a decreasing filtration, with successive quotients,

$$M_i/M_{i+1} \cong D(\mathbb{V}^{ut}_{\lambda,m}) \ . \tag{3.2.42}$$

3.2.4 The mixed trinion

In this subsection, we construct the first member of the $\mathbf{V}_{m,n}$ family, $\mathbf{V}_{1,1}$. We shall prove that our candidate for $\mathbf{V}_{1,1}$ possesses a number of desirable properties and also establish a uniqueness result.

The vertex algebra $\mathbf{V}_{1,1}$ corresponds to $\mathbb{P}^1$ with two $\mathfrak{g}_t$ punctures and one $\mathfrak{g}_u$ puncture. The Schur index of $\mathbf{V}_{0,1}$, computed via [LPR14] matches the character of $\mathcal{D}_t^{ch}$, the chiral differential operators on G_t[2]. Thinking back to our cartoon where FF-gluing $\mathbf{W}_u$ adds an untwisted puncture, there is a reasonably natural guess for $\mathbf{V}_{1,1}$: FF-glue an untwisted cap ($\mathbf{W}_u$) to a twisted cylinder ($\mathcal{D}_t^{ch}$). As such, we propose

$$\mathbf{V}_{1,1} \coloneqq \mathbf{W}_u *_u \mathcal{D}_t^{ch} \equiv \mathrm{H}^{\frac{\infty}{2}+0}(\mathcal{Z}_u, \mathbf{W}_u \otimes \mathcal{D}_t^{ch}) \ . \tag{3.2.43}$$

Theorem 3.2.7. *We have the following isomorphism:*

$$H^0_{DS}(u, \mathbf{V}_{1,1}) \cong \mathcal{D}_t^{ch} \ , \tag{3.2.44}$$

so $\mathcal{D}_t^{ch} \in \mathrm{KL}_{u,0}$.

This is the statement that the mixed vertex algebra we have constructed can indeed be identified with the UV curve $\mathcal{C}_{0,1,1}$ insofar as closing the maximal untwisted puncture results in the cylinder of type $\mathfrak{g}_t$. The proof of the above theorem is not entirely straightforward because $H^0_{DS}(u, \mathrm{H}^{\frac{\infty}{2}+0}(\mathcal{Z}_u, \mathbf{W}_u \otimes -))$ is not necessarily the identity on a generic object in $\mathcal{Z}_u$-Mod. The full proof of the the theorem is relegated to Subsection 3.2.10; here we provide a sketch.

The proof proceeds by first establishing that at the level of formal characters,

$$\mathrm{ch}\ H^0_{DS}(u, \mathbf{V}_{1,1})_\lambda \leqslant \mathrm{ch}\ \mathcal{D}^{ch}_{t,[\lambda]} \ . \tag{3.2.45}$$

[2]Strictly speaking, the characters of the cylinder and cap VOAs don't exist due to infinite-dimensional weight spaces. However, one can proceed formally by working term-by-term in the sum over integral dominant weights; this can be understood from a vertex algebra perspective as considering the decomposition into blocks belonging to KL^λ for each $\lambda \in P^+$.

In words, each weight space (with fixed generalised eigenvalue under the action of the Feigin–Frenkel zero modes) of $H^0_{DS}(u, \mathbf{V}_{1,1})$ is of dimension less than or equal to that of the corresponding weight space of $\mathcal{D}^{ch}_t$. This is argued by leveraging the fact that $\mathcal{D}^{ch}_t$ has an increasing filtration with subquotients $\mathbb{V}^t_{\lambda,2}$, which are in $\mathrm{KL}_{u,0}$ by Proposition 3.2.3. We show that passing (in a careful sense) to the associated graded of this filtration can only increase the dimensions of the weight spaces, and on the associated graded the composition of FF gluing and DS reduction acts as the identity; this leads to (3.2.45). Since $\mathcal{D}^{ch}_t$ is simple, we need only construct a non-zero homomorphism $\mathcal{D}^{ch}_t \to H^0_{DS}(u, \mathbf{V}_{1,1})$ to establish the isomorphism. The construction of such a homomorphism follows an adaptation of the proof of Theorem 9.9 of [Ara18] to this twisted setting.

Theorem 3.2.7 will serve as the foundation which lets us build up a number of other important properties of the genus zero mixed vertex algebras.

Proposition 3.2.8. *The mixed trinion* $\mathbf{V}_{1,1}$ *has an ascending filtration* $0 \subset N_0 \subset N_1 \subset \cdots \subset \mathbf{V}_{1,1}$ *with successive quotients*

$$N_i/N_{i-1} \cong \mathbb{V}^{ut}_{\lambda,1} \ . \tag{3.2.46}$$

Similarly, $\mathbf{V}_{1,1}$ *has a descending filtration* $\mathbf{V}_{1,1} \supset M_0 \supset M_1 \supset \cdots \supset 0$ *with successive quotients,*

$$M_i/M_{i+1} \cong D(\mathbb{V}^{ut}_{\lambda,1}) \ . \tag{3.2.47}$$

Therefore, $\mathbf{V}_{1,1}$ *is semijective in* KL_t*, with respect to the* $\widehat{\mathfrak{g}}_{\kappa_c}$ *actions of either twisted puncture.*

Proof. The cylinder $\mathcal{D}^{ch}_t$ has an increasing filtration [FG04],

$$0 = N_0 \subset N_1 \subset N_2 \subset \dots \ , \quad N = \bigcup N_i \cong \mathcal{D}^{ch}_t \ , \tag{3.2.48}$$

whose successive quotients take the form

$$N_i/N_{i-1} \cong \mathbb{V}^t_\lambda \otimes_{\mathfrak{z}^t_\lambda} \mathbb{V}^t_{\lambda^*} \equiv \mathbb{V}^{ut}_{\lambda,0} \ , \tag{3.2.49}$$

for some $\lambda \in P^+_t$ and λ^* the dual representation. Therefore Proposition 3.2.5 applies and $\mathbf{V}_{1,1}$ has an increasing filtration

$$0 = M_0 \subset M_1 \subset M_2 \subset \dots \ , \quad M = \bigcup M_i \cong \mathbf{V}_{1,1} \ , \tag{3.2.50}$$

with successive quotients $M_i/M_{i-1} \cong \mathbb{V}^{ut}_{\lambda,1}$, for some $\lambda \in P^+_t$. We have already argued that the mixed modules are projective over $U(t^{-1}\mathfrak{g}_t[t^{-1}])$, so $\mathbf{V}_{1,1}$ is projective over $U(t^{-1}\mathfrak{g}_t[t^{-1}])$.

To establish that $\mathbf{V}_{1,1}$ has an ascending filtration and so is injective over $U(t\mathfrak{g}_t[t])$, we can repeat the same argument after taking $(\mathcal{D}^{ch}_t)^{op}$ and using the identification

$$(\mathcal{D}^{ch}_t)^{op} \cong \mathcal{D}^{ch}_t \ .$$

□

We observe that $\mathbf{V}_{1,1}$ is not semijective in KL_u. Intuitively, this is because the extra generators of the Feigin–Frenkel centre must be set to zero when glued to the twisted cylinder, and these relations spoil projectivity. More precisely, we consider the vacuum vector $|0\rangle$. Any state element $P_{i,-n}|0\rangle$ can be written as $\phi|0\rangle$ for some $\phi \in U(t^{-1}\mathfrak{g}_u[t^{-1}])$ a regular element. However, the modes $P_{i,-n}$, which are not invariant under σ, must act as zero. As there are regular elements in $U(t^{-1}\mathfrak{g}_u[t^{-1}])$ which act as zero, $\mathbf{V}_{1,1}$ cannot be torsion free—so cannot be projective—over $U(t^{-1}\mathfrak{g}_u[t^{-1}])$.

The semijectivity of $\mathbf{V}_{1,1}$ in KL_t is in accordance with our expectations regarding enhanced Higgs branches/residual gauge symmetries. The twisted class $\mathcal{S}$ theories for surfaces $\mathcal{C}_{m,1}$ formed by gluing $\mathcal{C}_{1,1}$ along twisted punctures have no residual gauge symmetry, so the gauge theory gluing $\mathbf{V}_{1,1} \circ_t -$ should be concentrated in cohomological degree zero. We have just established this for our $\mathbf{V}_{1,1}$ algebra by showing semijectivity in KL_t. On the other hand, gluing along the untwisted puncture may lead to a higher-genus local system covering space (*cf.* Section 3.1.5), which falls in line with our observation that $\mathbf{V}_{1,1}$ is not semijective in KL_u.

Having shown that $\mathbf{V}_{1,1}$ has the expected properties under gluing, we move on to some more intrinsic properties of $\mathbf{V}_{1,1}$. Though our construction ensures that $\mathbf{V}_{1,1}$ is a vertex algebra, it is not at all clear that it has the properties expected from four-dimensional unitarity. Namely, $\mathbf{V}_{1,1}$ must be a conical, conformal vertex algebra with negative central charge. Let us first address the issue of the character.

Proposition 3.2.9. *The character of the vertex algebra* $\mathbf{V}_{1,1}$ *is given by*

$$\mathrm{ch}\,\mathbf{V}_{1,1} = \sum_{\lambda \in P_t^+} \frac{\mathcal{K}_u(\mathbf{a})\chi_u^\lambda(\mathbf{a})\mathcal{K}_t(\mathbf{b}_1)\chi_t^\lambda(\mathbf{b}_1)\mathcal{K}_t(\mathbf{b}_2)\chi_t^\lambda(\mathbf{b}_2)}{\mathcal{K}_u(\times)\chi_u^\lambda(\times)} \ ,$$

where $\mathbf{a}$ *is a* G_u *fugacity and the* $\mathbf{b}_i$ *are* G_t *fugacities. Furthermore,* $\mathbf{V}_{1,1}$ *is conical.*

Proof. By Theorem 3.2.7, we have that

$$H^0_{DS}(u, \mathbf{V}_{1,1}) \cong \mathcal{D}^{ch}_t \ .$$

As a graded vector space $\mathbf{V}_{1,1} \cong \bigoplus_{\lambda \in P_t^+} \mathbf{V}_{1,1,\lambda}$, since it is a colimit of objects in KL_t. By Proposition 8.4 of [Ara18].

$$\mathrm{ch}\mathbf{V}_{1,1,\lambda} = q^{\lambda(\rho^\vee)}\mathrm{ch}\,\mathbb{L}_\lambda \mathrm{ch} H^0_{DS}(u, \mathbf{V}_{1,1,\lambda}) \ .$$

From the structure of the cylinder, we know that $H^0_{DS}(u, \mathbf{V}_{1,1,\lambda})$ is zero unless it is in the image of $\iota : P_t^+ \hookrightarrow P_u^+$. Therefore, we have

$$\mathrm{ch}\,\mathbf{V}_{1,1,\lambda} = q^{\lambda(\rho^\vee)}\mathrm{ch}\,\mathbb{L}^u_\lambda \mathrm{ch}\mathbb{V}^t_\lambda \otimes_{\mathfrak{z}^t_\lambda} \mathbb{V}^t_{\lambda^*} \ .$$

Recalling Section 3.1.3 and using the appendix of [LP15], we can rewrite this in the notation of $\mathcal{K}$-factors, giving the desired result.

To show that $\mathbf{V}_{1,1}$ is conical, note that the cylinder, $\mathcal{D}^{ch}_t$, is non-negatively graded and

$\lambda(\rho^\vee) \geqslant 0$ since it is integral dominant, with equality only for $\lambda = 0$. This establishes that $\mathbf{V}_{1,1}$ is non-negatively graded by weight. The character ch $\mathbf{V}_{1,1,0} = 1 + \dots$ since $\mathbb{L}^u_{\lambda=0}$ and $\mathcal{D}^{ch}_{t,\lambda=0}$ are both conical. Thus the mixed trinion is conical. $\square$

Proposition 3.2.10. *The vertex algebra* $\mathbf{V}_{1,1}$ *is conformal with central charge*

$$c_{\mathbf{V}_{1,1}} = 2\dim \mathfrak{g}_t + \dim\mathfrak{g}_u - \mathrm{rk}\mathfrak{g}_u - 24\rho_u \cdot \rho_u^\vee \ .$$

Proof. This proof relies on ideas from the proof of Proposition 10.7 of [Ara18], but with modifications. The vertex algebras $\mathbf{W}_u$, $\mathcal{D}^{ch}_t$, and the ghost system $\bigwedge^{\frac{\infty}{2}+\bullet}(\mathfrak{z}(\mathfrak{g}_u))$ are all conformal, and we denote their respective conformal vectors by $\omega_{\mathbf{W}}$, $\omega_{\mathcal{D}^{ch}}$ and ω_{gh}. Clearly, $\omega = \omega_{\mathbf{W}} + \omega_{\mathcal{D}^{ch}} + \omega_{gh}$ is a conformal vector for the complex, $\mathbf{W}_u \otimes \mathcal{D}^{ch}_t \otimes \bigwedge^{\frac{\infty}{2}+\bullet}(\mathfrak{z}(\mathfrak{g}_u))$. We write

$$\omega(z) = \sum_{m\in\mathbb{Z}} L_m z^{-m-1} \ ,$$

for the associated field.

By Lemma 9.4 of [Ara18], the Feigin–Frenkel centre of $\mathbf{W}_u$ is preserved by the action of L_m for $m \geqslant -1$. For a generator $P_i \in \mathfrak{z}(\mathfrak{g}_u)$,

$$\omega(z)P_i(w) \sim \frac{\partial P_i}{z-w} + \frac{(d_i+1)P_i}{(z-w)^2} + \sum_{j=2}^{d_i+2} \frac{(-1)^j j!}{(z-w)^{j+1}} q_j^{(i)}(w) \ ,$$

where $q_j^{(i)}$ is some homogeneous state in $\mathfrak{z}(\mathfrak{g}_u)$ with weight $d_i - j + 2$. Let us denote by $\widetilde{P}_i$ the image of P_i under the projection $\mathfrak{z}(\mathfrak{g}_u) \twoheadrightarrow \mathfrak{z}(\mathfrak{g}_t)$. One then has

$$\omega(z)\widetilde{P}_i(w) \sim \frac{\partial \widetilde{P}_i}{z-w} + \frac{(d_i+1)\widetilde{P}_i}{(z-w)^2} + \sum_{j=2}^{d_i+2} \frac{(-1)^j j!}{(z-w)^{j+1}} \widetilde{q_j}^{(i)}(w) \ ,$$

where we think of $\widetilde{P}_i$ as a state in $\mathfrak{z}(\mathfrak{g}_t) \subset \mathcal{D}^{ch}_t$. Let $Q(z)$ be the BRST differential for Feigin–Frenkel gluing. We have that

$$Q_{(0)}(z)\omega(w) = \sum_{i=1}^{\mathrm{rk}\,\mathfrak{g}} \sum_{j=2}^{d_i+1} \partial^j \big(\rho_{\mathbf{W}}(q_j^{(i)}) - \rho_{\mathcal{D}^{ch}_t}(\tau(q_j^{(i)}))\big) c_i \ ,$$

where $\rho_W : \mathfrak{z}(\mathfrak{g}_u) \hookrightarrow \mathbf{W}_u$ and $\rho_{\mathcal{D}^{ch}_t} : \mathfrak{z}(\mathfrak{g}_u) \hookrightarrow \mathcal{D}^{ch}_t$ denote the action of the untwisted Feigin-Frenkel centres on $\mathbf{W}_u$ and on $\mathcal{D}^{ch}_t$ via the projection to $\mathfrak{z}(\mathfrak{g}_t)$. Unfortunately, ω does not descend directly to cohomology, so correction terms must be introduced to construct a putative conformal vector in cohomology.

If the right hand side of the above equation equals $Q_{(0)}\chi$ for some state χ, then $\widetilde{\omega} = \omega + \chi$ is Q-closed and defines a vector in $\mathbf{V}_{1,1}$. to show that such a χ exists, it is sufficient to show that $\widetilde{q_j}^{(i)} = \pi(q_j^{(i)})$.

The action of L_m for $m \geqslant -1$ on $\mathfrak{z}(\mathfrak{g}_u)$ is given by the action of $\mathrm{Der}(\mathcal{O})$ on $\mathrm{Op}_{^L G_u}(\mathbb{D})$, which correspond to infinitesimal coordinate changes on the formal disc [Fre07]. The action of the group $\mathrm{Aut}(\mathcal{O})$ on opers is given in (A.4.13) and we note that it intertwines the action the action of σ. Therefore, $\widetilde{q_j}^{(i)} = \pi(q_j^{(i)})$. Thus $\widetilde{\omega} \in \mathbf{V}_{1,1}$.

Now, we wish to show that $\widetilde{\omega}$ is a conformal vector. The vector χ can be written as

$$\chi = \sum_{i=1}^{\mathrm{rk}\,\mathfrak{g}_u} \sum_{j=2}^{d_i+2} \partial^j (\rho_{\mathbf{W}} \otimes \rho_{\mathcal{D}^{ch}_t} \otimes \rho_{gh})(z_{ij}) \ ,$$

for some $z_{ij} \in \mathfrak{z}(\mathfrak{g}_u) \otimes \mathfrak{z}(\mathfrak{g}_u) \otimes \bigwedge^{\frac{\infty}{2}+0}(\mathfrak{z}(\mathfrak{g}_u))$. Therefore, $\widetilde{\omega}_{(i)} = \omega_{(i)}$ for $i = 0, 1$, so the OPEs agree up to the quadratic pole. Since $\mathbf{V}_{1,1}$ is non-negatively graded by Proposition 3.2.9, Lemma 3.1.2 of [Fre07] says that all we need to check is that $\widetilde{\omega}_{(3)}\widetilde{\omega} = c/2|0\rangle$ for some $c \in \mathbb{C}$., *i.e.*, the quartic pole in the $Vir \times Vir$ OPE is a multiple of the identity. However, as $\mathbf{V}_{1,1}$ is conical, the only operator of dimension zero that can appear in the OPE is the identity. Thus, $\widetilde{\omega}$ is a conformal vector of $\mathbf{V}_{1,1}$.

Finally, we wish to show that $\widetilde{\omega}$ and ω have the same central charge in cohomology. Note that $\mathbf{V}_{1,1} = \sum_{\Delta \in \mathbb{N}} \mathbf{V}^{\Delta}_{1,1}$ with $\dim \mathbf{V}^{\Delta}_{1,1} < \infty$ and is conical—so $\mathbf{V}_{1,1}$ is of CFT type. As a result, Lemma 4.1 of [Mor20] applies. Namely, for any $x \in \mathbf{V}_{1,1}$, if $\Delta(x) \geqslant 2$ and $x_{(0)}\widetilde{\omega} = 0$ then $x \in \mathrm{im}_{\widetilde{\omega}_{(-1)}}(\mathbf{V}_{1,1})$.

Now suppose ω' was some other conformal vector in $\mathbf{V}_{1,1}$, such that $\omega'_{(1)}$ agrees with $\widetilde{\omega}_{(1)}$. Then, we have that $(\omega' - \widetilde{\omega})_{(0)}\widetilde{\omega} = 0$, so $\omega' - \widetilde{\omega} = \partial x$ for some $x \in \mathbf{V}_{1,1}$ with $\Delta = 1$. Since $\widetilde{\omega}_{(i)} = \omega'_{(i)}$ for $i = 0, 1$, we must have that $\partial x_{(i=0,1)} = 0$ so x is central in $\mathbf{V}^{\Delta=1}_{1,1}$. Repeating the argument with $(\omega - \widetilde{\omega})_{(1)}$ and using the Borcherds identities, leads us to conclude that $\partial x = 0$, so $\widetilde{\omega}$ is unique.

Under DS reduction, $H^0_{DS}(u, \mathbf{V}_{1,1}) \cong \mathcal{D}^{ch}_t$ and the image of $\widetilde{\omega}$ gives rise to a conformal vector in $\mathcal{D}^{ch}_t$. By a similar argument as the preceding, one can show that this is the unique conformal vector which agrees with the grading on $\mathcal{D}^{ch}_t$ (see the proof of Proposition 10.7 in [Ara18]). The central charge of $\mathcal{D}^{ch}_t$ is $c_{\mathcal{D}^{ch}_t} = 2\dim \mathfrak{g}_t$ and the central charge of the image of $\widetilde{\omega}$ is related to $c_{\mathbf{V}_{1,1}}$ by

$$c_{\mathcal{D}^{ch}_t} = c_{\mathbf{V}_{1,1}} + \mathrm{rk}\,\mathfrak{g}_u - \mathrm{dim}\mathfrak{g}_u + 24\rho_u \cdot \rho_u^\vee \ ,$$

under DS reduction, see Remark A.5.9. □

Proposition 3.2.11. *The vertex algebra* $\mathbf{V}_{1,1}$ *is simple.*

Proof. We proceed by contradiction. Suppose $\mathbf{V}_{1,1}$ contained some proper $\mathbf{V}_{1,1}$-submodule $V \not\subseteq \mathbf{V}_{1,1}$. From Theorem 3.2.7, we have that the DS reduction $0_{\mathrm{DS}}(u, V) \not\subseteq H^0_{\mathrm{DS}}(u, \mathbf{V}_{1,1}) \cong \mathcal{D}^{ch}_t$ must also be a submodule by functoriality.

However, $\mathcal{D}^{ch}_t$ is simple [AM21, Corollary 9.3] and so $H^0_{\mathrm{DS}}(u, V) = 0$ or $H^0_{\mathrm{DS}}(u, V) = \mathcal{D}^{ch}_t$. Now from Theorem 2.2.8, we can invert this DS reduction by FF-gluing a cap. Therefore, we should have that $V \cong \mathrm{H}^{\frac{\infty}{2}+0}(\mathcal{Z}_u, \mathbf{W}_u \otimes 0) \cong 0$ or $V \cong \mathrm{H}^{\frac{\infty}{2}+0}(\mathcal{Z}_u, \mathbf{W}_u \otimes \mathcal{D}^{ch}_t) \cong \mathbf{V}_{1,1}$—which is a contradiction. □

With these propositions established, we know that $\mathbf{V}_{1,1}$ obeys many of the desirable properties one would expect from the chiral algebra associated to $\mathcal{C}_{1,1}$. However, we have not explicitly tied this object to the construction of [BLL+15]. One might reasonably wonder whether the object we have constructed is necessarily the mixed trinion of class $\mathcal{S}$. We have the following uniqueness result.

Proposition 3.2.12. *The mixed trinion* $\mathbf{V}_{1,1}$ *is the unique vertex algebra object in* KL_u *such that*

$$H^0_{DS}(u, \mathbf{V}_{1,1}) \cong \mathcal{D}^{ch}_t \ .$$

Proof. This follows easily from the fact that $\mathcal{D}_t^{ch}$ is an object in $\mathrm{KL}_{u,0}$. Suppose $V \in \mathrm{KL}_u$ is a vertex algebra object such that $H^0_{DS}(u, V) \cong \mathcal{D}_t^{ch}$. Then it must be the case that $V \cong \mathbf{W}_u *_u \mathcal{D}_t^{ch}$, since $\mathbf{W}_u *_u (H^0_{DS}(u, -))$ is the identity in KL_u. However, $\mathbf{V}_{1,1} \coloneqq \mathbf{W}_u *_u \mathcal{D}_t^{ch}$, so indeed $V \cong \mathbf{V}_{1,1}$. □

Suppose now that $\widetilde{\mathbf{V}}_{1,1}$ is the vertex algebra canonically associated to $\mathcal{C}_{1,1}$ via the four-dimensional construction of [BLL+15]. This must be a vertex algebra object in KL_u, since it has an action of $V^{\kappa_c}(\mathfrak{g}_u)$ coming from the untwisted puncture and inherits a suitable grading from the physical grading of superconformal quantum numbers. Performing untwisted DS reduction on $\tilde{\mathbf{V}}_{1,1}$ must produce the cylinder $\mathcal{C}_{0,1}$, which has corresponding vertex algebra $\mathcal{D}_t^{ch}$.[3] Proposition 3.2.12 then applies, so we have

$$\widetilde{\mathbf{V}}_{1,1} \cong \mathbf{W}_u *_u \mathcal{D}_t^{ch} \cong \mathbf{V}_{1,1} \ . \tag{3.2.51}$$

Thus far, this has been fairly abstract. Let us provide some concrete observations and predictions. We have argued that the mixed trinion $\mathbf{V}_{1,1}$, as we have constructed it, is the unique vertex algebra that could be associated to $\mathcal{C}_{1,1}$. Let us consider the vertex algebra associated to $\mathcal{C}_f$, which is a genus zero surface with one maximal untwisted puncture, one maximal twisted puncture and an empty twisted puncture. The corresponding vertex algebra is $\mathbf{V}_f = H^0_{DS}(t, \mathbf{V}_{1,1})$.

The surface $\mathcal{C}_f$ does correspond to a physical SCFT, and in particular, when $\mathfrak{g}_u = \mathfrak{d}_n$ and $\mathfrak{g}_t = \mathfrak{c}_{n-1}$ the corresponding SCFT is a free hypermultiplet theory (hence the subscript f). In this case, $\mathbf{V}_f$ should be a symplectic boson vertex algebra with a commuting $\hat{\mathfrak{g}}_{u,\kappa_c} \times \hat{\mathfrak{g}}_{t,\kappa_c}$ subalgebra. Unfortunately, this does not hold for the other choices of $\mathfrak{g}_u$ (at generic rank).

There is, however, another example that has appeared in recent literature. The even rank A-type Lie algebras $\mathfrak{g}_u = \mathfrak{a}_{2n}$ have, as their twisted algebras, $\mathfrak{g}_t = \mathfrak{c}_n$—unlike the $\mathfrak{d}_n$ theories,

[3] To see that one must recover the cylinder upon DS reduction, even though this doesn't correspond to a physical four-dimensional theory, one may proceed as follows. The physical vertex algebra $\widetilde{\mathbf{V}}_{1,1}$ must satisfy $\widetilde{\mathbf{V}}_{1,1} \circ_t \widetilde{\mathbf{V}}_{1,1}$. Performing DS reduction, we must have $H^0_{DS}(u, \widetilde{\mathbf{V}}_{1,1} \circ_t \widetilde{\mathbf{V}}_{1,1}) \cong \widetilde{\mathbf{V}}_{1,1}$. By means of a spectral sequence argument (we delay this until the next subsection) one can rearrange the order of cohomologies to show $H^0_{DS}(u, \widetilde{\mathbf{V}}_{1,1}) \circ_t \widetilde{\mathbf{V}}_{1,1} \cong \widetilde{\mathbf{V}}_{1,1}$, which implies that $H^0_{DS}(u, \widetilde{\mathbf{V}}_{1,1}) = \mathcal{D}_t^{ch}$.

these SCFTs have global Witten anomalies. For $\mathfrak{g}_u = \mathfrak{a}_2$ and $\mathfrak{g}_t = \mathfrak{c}_1 = \mathfrak{a}_1$, the SCFT is the $\mathcal{T}_X$ theory of [BLN17]. This was identified as the rank-two H_2 F-theory SCFT [BMPR20], and the class $\mathcal{S}$ realisation was given in [BP20]. In the E_6 and A_{2n+1} cases, these vertex algebras remain unstudied to the best of our knowledge.

It is perhaps of technical interest to note that $\mathbf{V}_f$ should be conical, despite the fact that it is obtained from the Feigin–Frenkel gluing of just two caps. In the cases where the two caps are of the same type, one obtains the cylinder $\mathcal{D}^{ch}_u$ or $\mathcal{D}^{ch}_t$—neither of which are conical.

The identification of $\mathbf{V}_f$ with a symplectic boson system for $\mathfrak{g}_u = \mathfrak{d}_n$ leads to a curious observation.

Conjecture 3.2.13. *Let* $\mathrm{SB}(\mathfrak{d}_n)$ *be the symplectic boson system associated to the ``bifundamental'' representation of* $\mathfrak{d}_n \times \mathfrak{c}_{n-1}$*. In other words,* $\mathrm{SB}(\mathfrak{d}_n) = \mathcal{D}^{ch}(V_{\mathfrak{d}_n} \times V_{\mathfrak{c}_{n-1}})$*, where* $V_{\mathfrak{d}_n}$ *and* $V_{\mathfrak{c}_{n-1}}$ *are defining representations of* $\mathfrak{d}_n$ *and* $\mathfrak{c}_{n-1}$*. Then,*

$$\mathbf{W}_{\mathrm{Sp}(2n-2)} \cong H^0_{\mathrm{DS}}(u, \mathrm{SB}(\mathfrak{d}_n)) \ . \tag{3.2.52}$$

Indeed, this presents an alternate hypothetical construction for the equivariant affine $\mathcal{W}$-algebra of C_n type.

3.2.5 Rearrangement lemmas

Having established many key properties of the mixed trinion, which is the building block of the twisted chiral algebras of class $\mathcal{S}$, we would like to extend our results to other genus zero, mixed vertex algebras. As we increase the number of twisted and untwisted punctures, we are naturally required to consider how the various types of gluing interact with each other. It will be useful in this endeavour to have a collection of rearrangement lemmas that establish the extent to which the various gluings associate.

In this subsection we establish a series of technical rearrangement lemmas concerning the interplay between the various cohomological operations we have defined thus far. The proofs of these lemmas, which are modifications of proofs of [Ara18], rely heavily on the machinery of spectral sequences. The reader who is uninterested in highly technical details may wish to skip this section and pick up in the following, where we extend our construction to the vertex algebras associated to $\mathcal{C}_{m,n}$.

Since we are interested in the interplay of gluings, we will need to consider objects that have multiple, commuting $\hat{\mathfrak{g}}_{\kappa_c}$ actions. To be completely precise, one should decorate each $\circ$ in this section with subscripts to indicate the diagonal action that is being gauged. This would be somewhat cumbersome, so we will overload the $\circ$ notation and rely on context to make the relevant actions clear. Our lemmas only ever concern two such actions, for the sake of argument we call them the left and right actions. Suppose U, V, W are in KL, such that V has two actions of $\hat{g}_{\kappa_c}$ and V is in KL with respect to both actions. One should then interpret the symbol $U \circ (V \circ W)$ as the semi-infinite cohomology $\mathrm{H}^{\frac{\infty}{2}+\bullet}(\hat{\mathfrak{g}}_{-\kappa_g}, \mathfrak{g}, U \otimes \mathrm{H}^{\frac{\infty}{2}+\bullet}(\hat{\mathfrak{g}}_{-\kappa_g}, \mathfrak{g}, V \otimes W))$, where the diagonal action of $\hat{\mathfrak{g}}_{-\kappa_g}$ is with respect to the right $\hat{\mathfrak{g}}_{\kappa_c}$ action on V and the sole $\hat{\mathfrak{g}}_{\kappa_c}$ action on W. Similarly, the diagonal action on $U \otimes \mathrm{H}^{\frac{\infty}{2}+\bullet}(\hat{\mathfrak{g}}_{-\kappa_g}, \mathfrak{g}, V \otimes W)$ is with respect to the sole $\hat{\mathfrak{g}}_{\kappa_c}$ action on U and the $\hat{\mathfrak{g}}_{\kappa_c}$ action on $V \circ W$ induced by the left action on V.

Similarly, for objects $U, V, W \in \mathcal{Z}$-Mod, we will denote the iterated Feigin–Frenkel gluing, $\mathrm{H}^{\frac{\infty}{2}+0}(\mathcal{Z}, U \otimes \mathrm{H}^{\frac{\infty}{2}+0}(\mathcal{Z}, V \otimes W))$ by $U * (V * W)$. Here one should take the action on $V \otimes W$ for the first cohomology and on $U \otimes V$ for the second—recall that the $\mathcal{Z}$-action on V descends to the cohomology $V * W$.

Hereafter, it should be understood that when there are many KL_u or KL_t actions present, we choose two such actions for the purposes of the rearrangement lemmas. For the chiral algebras of class $\mathcal{S}$ (both untwisted and twisted), all the moment maps (from the same algebra) are related by discrete automorphisms, and one can make such a choice without loss of generality.

First, we recast some of the results of [Ara18] as rearrangement lemmas. The following results hold for any simple Lie algebra $\mathfrak{g}$.

Lemma 3.2.14. *Let U, V, W be vertex algebra objects in* KL *such that U is semijective in*

KL *and* V *has two* KL *actions, then*

$$U \circ (V \circ W) \cong (U \circ V) \circ W \ .$$

Proof. This proof is similar to the proof of Theorem 10.11 of [Ara18]. Consider the bicomplex

$$C^{\bullet\bullet} = U \otimes V \otimes W \otimes \bigwedge\nolimits^{\frac{\infty}{2}+\bullet}(\mathfrak{g}) \otimes \bigwedge\nolimits^{\frac{\infty}{2}+\bullet}(\mathfrak{g}) \ ,$$

with the differential d_1 acting on $U \otimes V$ and the first $\bigwedge^{\frac{\infty}{2}+\bullet}(\mathfrak{g})$ and d_2 acting on $V \otimes W$ and the second $\bigwedge^{\frac{\infty}{2}+\bullet}(\mathfrak{g})$. It is easy to see that $d_1 d_2 + d_2 d_1 = 0$, so the total complex $C^p_{tot} = \bigoplus_{m+n=p} C^{m,n}$ is a cochain complex with the differential $d = d_1 + (-1)^m d_2$ [Wei94]. There are two spectral sequences converging to the total cohomology of (C_{tot}, d)

$$\begin{aligned} {}_{I}E_2^{p,q} &= \mathrm{H}^{\frac{\infty}{2}+p}(\hat{\mathfrak{g}}_{-\kappa_g}, \mathfrak{g}, U \otimes \mathrm{H}^{\frac{\infty}{2}+q}(\hat{\mathfrak{g}}_{-\kappa_g}, \mathfrak{g}, V \otimes W)) \ , \\ {}_{II}E_2^{p,q} &= \mathrm{H}^{\frac{\infty}{2}+p}(\hat{\mathfrak{g}}_{-\kappa_g}, \mathfrak{g}, W \otimes \mathrm{H}^{\frac{\infty}{2}+q}(\hat{\mathfrak{g}}_{-\kappa_g}, \mathfrak{g}, U \otimes V)) \ . \end{aligned}$$

By Theorem A.5.5, the cohomologies $\mathrm{H}^{\frac{\infty}{2}+p}(\hat{\mathfrak{g}}_{-\kappa_g}, \mathfrak{g}, U \otimes -)$ are concentrated in degree zero, so both spectral sequences collapse at the second page. The only nonzero entries are ${}_{I}E_2^{0,q}$ and ${}_{II}E_2^{p,0}$. Thus, we have

$${}_{I}E_2^{0,p} \cong \ {}_{II}E_2^{p,0} \cong H^p_{tot}(C_{tot}, d) \ ,$$

which gives the desired isomorphism. □

The vertex algebra objects $\mathbf{V}_{G,s}$ are semijective in KL so the composition of vertex algebras is associative—in the untwisted setting. Furthermore, gauging at genus zero is always concentrated in degree zero. In the twisted setting, we will have to work harder.

We have a similar result for Feigin–Frenkel gluing.

Lemma 3.2.15. *Suppose* $M_1, M_2, M_3 \in \mathcal{Z}$*-Mod and suppose that* M_1 *and* M_3 *are free over* $\mathcal{Z}_{(<0)}$*. Then*

$$M_1 * (M_2 * M_3) \cong (M_1 * M_2) * M_3 \ .$$

Proof. This proof is similar to the proof of Proposition 10.2 of [Ara18]. Consider the bicomplex

$$C^{\bullet\bullet} = M_1 \otimes M_2 \otimes M_3 \otimes \bigwedge{}^{\frac{\infty}{2}+\bullet}(\mathfrak{z}(\mathfrak{g})) \otimes \bigwedge{}^{\frac{\infty}{2}+\bullet}(\mathfrak{z}(g)) \ ,$$

with differentials d_{12} acting on $M_1 \otimes M_2 \otimes \bigwedge^{\frac{\infty}{2}+\bullet}(\mathfrak{z}(\mathfrak{g}))$ and d_{23} acting on $M_2 \otimes M_3 \otimes \bigwedge^{\frac{\infty}{2}+\bullet}(\mathfrak{z}(\mathfrak{g}))$. The two differentials anticommute, so we can form the total complex $C^n_{tot} = \bigoplus_{p+q=n} C^{p,q}$ with total differential $d_{tot} = d_{12} + (-1)^q d_{23}$. There are two spectral sequences converging to the total cohomology H_{tot} of C_{tot}, whose second pages are given by

$$
\begin{aligned}
{}_I E_2^{p,q} &:= \mathrm{H}^{\frac{\infty}{2}+p}(\mathcal{Z}, M_1 \otimes \mathrm{H}^{\frac{\infty}{2}+q}(\mathcal{Z}, M_2 \otimes M_3)) \ , \\
{}_{II} E_2^{p,q} &:= \mathrm{H}^{\frac{\infty}{2}+p}(\mathcal{Z}, \mathrm{H}^{\frac{\infty}{2}+q}(\mathcal{Z}, M_1 \otimes M_2) \otimes M_3) \ .
\end{aligned}
$$

By Theorem 2.2.5, the entries $E_2^{p,q}$ in either spectral sequence vanish if $p < 0$ or $q < 0$. Thus, we have $E_2^{00} = E_\infty^{00}$, which gives the isomorphism

$$H^0_{tot}(C_{tot}, d_{tot}) \cong \mathrm{H}^{\frac{\infty}{2}+0}(\mathcal{Z}, M_1 \otimes \mathrm{H}^{\frac{\infty}{2}+0}(\mathcal{Z}, M_2 \otimes M_3)) \cong \mathrm{H}^{\frac{\infty}{2}+0}(\mathcal{Z}, \mathrm{H}^{\frac{\infty}{2}+0}(\mathcal{Z}, M_1 \otimes M_2) \otimes M_3) \ ,$$

as desired. □

Since the algebras $\mathbf{V}_{G,s}$ are free over $\mathcal{Z}_{<0}$ (Proposition 10.2 of [Ara18]), the above lemma applies, and Feigin–Frenkel gluing $*_u$ is associative. We hold off on analysing associativity of $*$ for the twisted algebras, since it is a challenge to understand the twisted FF gluing between two mixed vertex algebras.

We next consider the combination of the two gluing operations, $*$ and $\circ$.

Lemma 3.2.16. *Suppose $U \in \mathcal{Z}$-Mod, $V \in$ KL and $\mathbf{W} \in$ KL. Additionally, suppose U is free over $\mathcal{Z}_{(<0)}$ and W is semijective in* KL*. Then we have the isomorphism*

$$U * (V \circ W) \cong (U * V) \circ W \ .$$

Proof. Consider the bicomplex

$$C^{\bullet\bullet} = U \otimes V \otimes W \bigwedge{}^{\frac{\infty}{2}+\bullet}(\mathfrak{z}(\mathfrak{g})) \otimes \bigwedge{}^{\frac{\infty}{2}+\bullet}(\mathfrak{g}) \ ,$$

with differentials $d_{\mathfrak{g}}$ acting on $V \otimes W \otimes \otimes \bigwedge^{\frac{\infty}{2}+\bullet}(\mathfrak{g})$ and $d_{\mathfrak{z}}$ acting on $U \otimes V \otimes \bigwedge^{\frac{\infty}{2}+\bullet}(\mathfrak{z}(\mathfrak{g}))$. The differentials anticommute, so we can form the total complex $C^p_{tot} = \bigoplus_{m+n=p} C^{p,q}$ with differential $d_{tot} = d_{\mathfrak{g}} + (-1)^q d_{\mathfrak{z}}$. There are two spectral sequences converging to the total cohomology:

$$
\begin{aligned}
{}_I E_2^{p,q} &= \mathrm{H}^{\frac{\infty}{2}+p}(\mathcal{Z}, U \otimes \mathrm{H}^{\frac{\infty}{2}+q}(\hat{\mathfrak{g}}_{-\kappa_g}, \mathfrak{g}, V \otimes W)) \ , \\
{}_{II} E_2^{p,q} &= \mathrm{H}^{\frac{\infty}{2}+p}(\hat{\mathfrak{g}}_{-\kappa_g}, \mathfrak{g}, \mathrm{H}^{\frac{\infty}{2}+q}(\mathcal{Z}, U \otimes V) \otimes W) \ .
\end{aligned}
\tag{3.2.53}
$$

The cohomology $\mathrm{H}^{\frac{\infty}{2}+p}(\mathcal{Z}, U \otimes -)$ vanishes for $p < 0$ and the cohomology $\mathrm{H}^{\frac{\infty}{2}+p}(\hat{\mathfrak{g}}_{-\kappa_g}, \mathfrak{g}, - \otimes W)$ is concentrated in degree zero. Thus both spectral sequences will collapse at the second page and we have

$$
{}_I E_2^{0,0} \cong {}_{II} E_2^{0,0} \cong H^0_{tot}(C_{tot}, d_{tot}) \ , \tag{3.2.54}
$$

as desired. □

Now we come to the rearrangement of twisted and untwisted gluing. We again adopt our conventions of using subscripts u and t to denote objects associated to $\mathfrak{g}_u$ or $\mathfrak{g}_t$.

The results here are more limited; the obvious generalisations of the above spectral sequence arguments often fail in the twisted setting, since the mixed vertex algebras are not semijective in KL_u. Nevertheless, we will manage to demonstrate that some properties of the gluing of twisted algebras are as we expect. To start with, we have a result for the interchange of twisted and untwisted gauging.

Lemma 3.2.17. *Suppose V_1 is in KL_u, V_2 is in KL_u and in KL_t, and V_3 is in KL_t. Furthermore, suppose that V_3 is semijective for $\mathfrak{g}_t$. Then, we have the isomorphism*

$$
V_1 \circ_u (V_2 \circ_t V_3) \cong (V_1 \circ_u V_2) \circ_t V_3 \ .
$$

Proof. As usual, the proof is via a spectral sequence, but with the new feature that cohomology is not necessarily concentrated in degree zero. We define the bicomplex

$$
C^{p,q} = V_1 \otimes V_2 \otimes V_3 \otimes \bigwedge^{\frac{\infty}{2}+p}(\hat{\mathfrak{g}}_u) \otimes \bigwedge^{\frac{\infty}{2}+q}(\hat{\mathfrak{g}}_t) \ ,
$$

with the differential d_u and d_t acting on $V_1 \otimes V_2 \otimes \bigwedge^{\frac{\infty}{2}+\bullet}(\hat{\mathfrak{g}}_u)$ and $V_2 \otimes V_2 \otimes \bigwedge^{\frac{\infty}{2}+\bullet}(\hat{\mathfrak{g}}_t)$ respectively. The differentials anticommute, so we can form the total complex $C^m_{tot} = \bigoplus_{p+q=m} C^{p,q}$ with the differential $d_{tot} = d_u + (-1)^p d_t$. There are two spectral sequences converging to the total cohomology, given by

$$
\begin{aligned}
{}_I E_2^{p,q} &= \mathrm{H}^{\frac{\infty}{2}+p}(\hat{\mathfrak{g}}_{u,-\kappa_g}, \mathfrak{g}_u, V_1 \otimes \mathrm{H}^{\frac{\infty}{2}+p}(\hat{\mathfrak{g}}_{t,-\kappa_g}, \mathfrak{g}_t, V_2 \otimes V_3)) \ , \\
{}_{II} E_2^{p,q} &= \mathrm{H}^{\frac{\infty}{2}+p}(\hat{\mathfrak{g}}_{t,-\kappa_g}, \mathfrak{g}_t, \mathrm{H}^{\frac{\infty}{2}+q}(\hat{\mathfrak{g}}_{u,-\kappa_g}, \mathfrak{g}_u, V_1 \otimes V_2) \otimes V_3) \ .
\end{aligned}
$$

Since V_3 satisfies the conditions of Theorem A.5.5, the spectral sequences will collapse on the second page and ${}_I E_2^{p,q} = 0$ for $q \neq 0$ and ${}_{II} E_2^{p,q} = 0$ for $p \neq 0$. Thus we have the isomorphism

$$H^m_{tot} = {}_I E_2^{m,0} = {}_{II} E_2^{0,m} \ .$$

□

We can also show that the twisted gauging and untwisted Feigin–Frenkel gluing are nicely compatible,

Lemma 3.2.18. *Let M be an object of $\mathcal{Z}_u$-Mod such that it is free over $\mathcal{Z}_{u,<0}$ and let W be semijective in* KL_t. *Suppose, V is a vertex algebra object in* KL_t, *then we have the following isomorphism.*

$$M *_u (V \circ_t W) \cong (M *_u V) \circ_t W \ .$$

Proof. Let $C^{p,q}$ be the bicomplex

$$C^{p,q} = M \otimes V \otimes W \otimes \bigwedge{}^{\frac{\infty}{2}+p}(\mathfrak{z}(\mathfrak{g}_u)) \otimes \bigwedge{}^{\frac{\infty}{2}+q}(\hat{\mathfrak{g}}_t) \ ,$$

with the differentials $d_{\mathfrak{z}}$ acting on $M \otimes V \otimes \bigwedge^{\frac{\infty}{2}+\bullet}(\mathfrak{z}(\mathfrak{g}_u))$ and $d_{\mathfrak{g}}$ acting on $V \otimes W \bigwedge^{\frac{\infty}{2}+\bullet}(\hat{\mathfrak{g}}_t)$. The differentials $d_{\mathfrak{z}}$ and $d_{\mathfrak{g}}$ anticommute, so we can form the total complex $C^m_{tot} = \bigoplus_{p+q=m} C^{p,q}$ with the total differential $d_{tot} = d_{\mathfrak{z}} + (-1)^p d_{\mathfrak{g}}$. There are two spectral sequences converging to the total cohomology, given by

$$
\begin{aligned}
{}_{I}E_2^{p,q} &= \mathrm{H}^{\frac{\infty}{2}+p}(\mathcal{Z}_u, M \otimes \mathrm{H}^{\frac{\infty}{2}+q}(\hat{\mathfrak{g}}_{t,-\kappa_g}, \mathfrak{g}_t, V \otimes W)) \ , \\
{}_{II}E_2^{p,q} &= \mathrm{H}^{\frac{\infty}{2}+p}(\hat{\mathfrak{g}}_{t,-\kappa_g}, \mathfrak{g}_t, \mathrm{H}^{\frac{\infty}{2}+q}(\mathcal{Z}_u, M \otimes V) \otimes W) \ .
\end{aligned}
$$

The cohomology $\mathrm{H}^{\frac{\infty}{2}+q}(\hat{\mathfrak{g}}_{t,-\kappa_g}, \mathfrak{g}_t, - \otimes W)$ is concentrated in degree zero, so both spectral sequence will collapse in the second page. Thus, we have the isomorphism

$$
H^0_{tot}(C_{tot}, d_{tot}) \cong {}_{I}E_2^{0,0} \cong {}_{II}E_2^{0,0} \ .
$$

□

3.2.6 Mixed vertex algebras at genus zero

We first give a construction of the vertex algebras associated to spheres with only one pair of twisted punctures: $\mathcal{C}_{m,1}$ before considering the more general case. As in the untwisted case, one could provide a recursive definition of $\mathbf{V}_{m,1}$ by repeatedly gluing untwisted caps. We elect, instead, to perform a simultaneous gluing and will show the equivalence between the two definitions later on.

Definition 3.2.19. We define the family of mixed vertex algebras, $\mathbf{V}_{m,1}$, by

$$
\begin{aligned}
\mathbf{V}_{0,1} &\coloneqq \mathcal{D}^{ch}_t \ , \\
\mathbf{V}_{m,1} &\coloneqq H^0(C_{m,1}, Q^m) \ ,
\end{aligned}
\tag{3.2.55}
$$

where $C_{m,1}$ is the chain complex

$$
C^\bullet_{m,1} = \mathbf{W}_u^{\otimes m} \otimes \mathcal{D}^{ch}_t \otimes \left(\bigwedge{}^{\frac{\infty}{2}+\bullet}(\mathfrak{z}(\mathfrak{g}_u)) \right)^m \ , \tag{3.2.56}
$$

with differential

$$\begin{aligned} Q^m(z) &= \sum_{i=1}^{m} Q^{i,i+1} \ , \\ Q^{i,i+1}(z) &= \sum_{j=1}^{\operatorname{rk}\mathfrak{g}_u} :(\rho_i(P_j) - \rho_{i+1}(\tau(P_j)))\rho_{gh_i}(c^j):(z) \ . \end{aligned} \tag{3.2.57}$$

Here ρ_i, for $i \leqslant m$ denotes the action of $\mathfrak{z}(\mathfrak{g}_u)$ on the ith factor of $\mathbf{W}_u$ and ρ_{m+1} denotes the action of $\mathfrak{z}(\mathfrak{g}_u)$ on $\mathcal{D}_t^{ch}$ along the projection $\mathcal{Z}_u \twoheadrightarrow \mathcal{Z}_t$.

Pictorially, this is the simultaneous FF-gluing of m caps to the twisted cylinder. The vertex algebras $\mathbf{V}_{m,1}$ live entirely in cohomological degree zero (no fermions) since we manually restrict to the zeroth cohomology—this is compatible with our expectation on the basis of residual gauge symmetries (see Section 3.1.5. We will later reinforce this by showing that the cohomology of $\mathbf{V}_{m,1} \circ_t \mathbf{V}_{n,1}$ is concentrated in degree zero.

The naive generalisation of our previous construction to the full family $\mathbf{V}_{m,n}$ would be to take the zeroth cohomology of

$$C^{\bullet}_{m,n} = \mathbf{W}_u^m \otimes \mathbf{V}_{G_t,2n} \otimes \left(\bigwedge{}^{\frac{\infty}{2}+\bullet}(\mathfrak{z}(\mathfrak{g}_u))\right)^m \ . \tag{3.2.58}$$

However, this cannot be quite right. Indeed, the vertex algebras associated to $\mathcal{C}_{m,n}$ for $n > 1$ should be supported outside of cohomological degree zero in order to express the presence of enhanced Higgs branches for the corresponding SCFTs. On the other hand, the vertex algebras, $\mathbf{W}_u$ and $\mathbf{V}_{G_t,2n}$, lie in degree zero, and the truncation to zeroth cohomology means this will persist. One should also expect to see a $\mathbb{Z}/2\mathbb{Z}$ symmetry exchanging positive and negative cohomological degree (a shadow of CPT in four dimensions). However, the untwisted caps $\mathbf{W}_u$ are projective over $\mathcal{Z}_{u,(<0)}$ and as a consequence the cohomology vanishes in negative degree. Therefore, even if we do not truncate to degree zero, the resulting vertex algebra would not have the right form.

Instead, we will define the vertex algebras $\mathbf{V}_{m,n}$, by going to some (non-canonically chosen) duality frame, *i.e.*, pants decomposition. We will choose to recursively consider the decomposition of $\mathcal{C}_{m+1,n-1}$ as $\mathcal{C}_{m+1,n-1}$ and $\mathcal{C}_{1,1}$ connected by an untwisted cylinder. This gives us our definition:

$$\mathbf{V}_{m,n} \coloneqq \mathbf{V}_{1,1} \circ_u \mathbf{V}_{m+1,n-1} \ . \tag{3.2.59}$$

As defined, it is not clear whether the $\mathbf{V}_{m,n}$ are independent of our choice of duality frame in which to define them. For example, we could have also obtained this from a twisted gluing of $\mathbf{V}_{1,1}$ to $\mathbf{V}_{m-1,n}$. This is just the vertex algebra version of generalised S-duality, which is now not made manifest by our definition (3.2.59). In the following sections, we shall work to establish how our definitions fit in with the duality web of class $\mathcal{S}$.

3.2.7 Properties of the genus zero mixed vertex algebras

First, we will check that our definition of the mixed vertex algebras with $n = 1$ agrees with the recursive definition, $\mathbf{V}_{m,1} \overset{?}{\cong} \mathbf{W}_u *_u \mathbf{V}_{m-1,1}$. We have an extension of Lemma 10.1 of Arakawa to the twisted case.

Lemma 3.2.20. *For $m \geqslant 1$, we have that*

(i) $H^n(C_{m,1}, Q_{(0)}^{m-1}) \cong 0$ *for* $n < 0$,

(ii) $\mathbf{W}_u *_u \mathbf{V}_{m-1,1} \cong \mathbf{V}_{m,1}$.

Proof. This proof is largely adapted from the proof of Lemma 10.1 in [Ara18]. We proceed by induction on m. For the base case, $m = 1$, (i) is true, since $\mathbf{W}_u$ is projective over $\mathcal{Z}_{u,(<0)}$. The second statement is true by definition, since $\mathbf{V}_{1,1} \coloneqq \mathbf{W}_u *_u \mathcal{D}_t^{ch}$. Next, suppose $m > 1$ and consider the bicomplex

$$C^{\bullet\bullet} = \mathbf{W}_u \otimes C_{m-1,1}^{\bullet} \otimes \bigwedge\nolimits^{\frac{\infty}{2}+\bullet}(\mathfrak{z}(\mathfrak{g}_u)) \ , \tag{3.2.60}$$

with differentials $Q_{(0)}^{m-1}$ acting on $C_{m-1,1}$ and d acting on $\mathbf{W}_u \otimes C_{m-1,1} \otimes \bigwedge^{\frac{\infty}{2}+\bullet}(\mathfrak{z}(\mathfrak{g}_u))$. The two differentials anticommute and the corresponding total complex is just $C_{m,1}$ with differential $Q_{(0)}^{m}$. There is a spectral sequence, with second page

$$E_2^{p,q} = \mathrm{H}^{\frac{\infty}{2}+p}(\mathcal{Z}_u, \mathbf{W}_u \otimes H^q(C_{m-1,1}, Q_{(0)}^{m-1})) \ , \tag{3.2.61}$$

which converges to the total cohomology. By the inductive assumption, $H^q(C_{m,1}, Q_{(0)}^{m-1})$ vanishes for $q < 0$ and $\mathbf{W}_u *_u -$ is left exact. Therefore, $E_2^{p,q} = 0$ for $p, q < 0$, so $H^n(C_{m,1}, Q_{(0)}^m)$ vanishes for $n < 0$. Moreover, the entry $E_2^{0,0}$ is stable and we have

$$E_2^{0,0} = \mathbf{W}_u *_u \mathbf{V}_{m-1,1} \cong H^0(C_{m,1}, Q_{(0)}^m) \cong \mathbf{V}_{m,1} \ , \tag{3.2.62}$$

as desired. □

Next, we examine the case of untwisted DS reduction.

Proposition 3.2.21. *The mixed vertex algebras* $\mathbf{V}_{m,n}$ *are objects in* $\mathrm{KL}_{u,0}$. *In particular, for* $m \geqslant 0$ *and* $n \geqslant 1$,

$$H^0_{DS}(u, \mathbf{V}_{m+1,n}) \cong \mathbf{V}_{m,n} \ . \tag{3.2.63}$$

Proof. We proceed by double induction on m and n, first examining the base case of $n = 1$ and arbitrary m. Note that for $m = 0$, $H^0_{DS}(u, \mathbf{V}_{1,1}) = \mathcal{D}_t^{ch}$ from Theorem 3.2.7. Next, suppose $m > 0$; for any object in KL_u, $H^0_{DS}(u, -)$ and $\mathbf{W}_u \circ_u -$ are isomorphic. Therefore,

$$H^0_{DS}(u, \mathbf{V}_{m+1,1}) \cong \mathbf{W}_u \circ_u \mathbf{V}_{m+1,1} \cong \mathbf{W}_u \circ_u (\mathbf{W}_u *_u \mathbf{V}_{m,1}) \ , \tag{3.2.64}$$

where we have used Lemma 3.2.20. Consider the bicomplex

$$C^{\bullet\bullet} = \mathbf{W}_u \otimes \mathbf{V}_{m,1} \otimes \mathbf{W}_u \otimes \bigwedge{}^{\frac{\infty}{2}+\bullet}(\mathfrak{z}(\mathfrak{g}_u)) \otimes \bigwedge{}^{\frac{\infty}{2}+\bullet}(\hat{\mathfrak{g}}_{u,-\kappa_g}) \ , \tag{3.2.65}$$

with differential $d_{\mathfrak{g}}$ acting on $\mathbf{W}_u \otimes \mathbf{V}_{m,1} \otimes \bigwedge^{\frac{\infty}{2}+\bullet}(\hat{\mathfrak{g}}_{u,-\kappa_g})$ and $d_{\mathfrak{z}}$ acting on $\mathbf{V}_{m,1} \otimes \mathbf{W}_u \otimes \bigwedge^{\frac{\infty}{2}+\bullet}(\mathfrak{z}(\mathfrak{g}_u))$. The two differentials anticommute, so we form the total complex $C^n_{tot} = \bigoplus_{p+q=n} C^{pq}$, with total differential $d_{tot} = d_{\mathfrak{g}} + (-1)^q d_{\mathfrak{z}}$. There are two spectral sequences converging to the total cohomology, given by

$$
\begin{aligned}
{}_I E_2^{pq} &= \mathrm{H}^{\frac{\infty}{2}+p}(\mathcal{Z}_u, \mathbf{W}_u \otimes \mathrm{H}^{\frac{\infty}{2}+q}(\hat{\mathfrak{g}}_{u,-\kappa_g}, \mathfrak{g}_u, \mathbf{V}_{m,1} \otimes \mathbf{W}_u)) \ , \\
{}_{II} E_2^{pq} &= \mathrm{H}^{\frac{\infty}{2}+p}(\hat{\mathfrak{g}}_{u,-\kappa_g}, \mathfrak{g}_u, \mathrm{H}^{\frac{\infty}{2}+q}(\mathcal{Z}_u, \mathbf{W}_u \otimes \mathbf{V}_{m,1}) \otimes W_u) \ .
\end{aligned}
$$

The cohomology, $\mathrm{H}^{\frac{\infty}{2}+\bullet}(\hat{\mathfrak{g}}_{u,-\kappa_g}, \mathfrak{g}_u, -\otimes \mathbf{W}_u)$ is concentrated in degree zero, and $\mathrm{H}^{\frac{\infty}{2}+i}(\mathcal{Z}_u \mathbf{W}_u \otimes$

$-$) vanishes for $i < 0$. Therefore, both spectral sequences collapse at the second page and we have

$$\mathbf{W}_u *_u (\mathbf{W}_u \circ_u \mathbf{V}_{m,1}) = \ _IE_2^{00} \cong \ _{II}E_2^{00} = \mathbf{W}_u \circ_u (\mathbf{W}_u *_u \mathbf{V}_{m,1}) \ .$$

Therefore,

$$H^0_{DS}(u, \mathbf{V}_{m+1,1}) \cong \mathbf{W}_u *_u (\mathbf{W}_u \circ_u \mathbf{V}_{m,1}) \cong \mathbf{W}_u *_u (H^0_{DS}(u, \mathbf{V}_{m,1}) \ , \tag{3.2.66}$$

but $\mathbf{W}_u *_u (H^0_{DS}(u, \mathbf{V}_{m,1})) \cong \mathbf{V}_{m,1}$ by Theorem 9.11 of [Ara18].

Now suppose $n > 1$. Then, we have that

$$H^0_{DS}(u, \mathbf{V}_{m+1,n}) \cong \mathbf{W}_u \circ_u (\mathbf{V}_{1,1} \circ_u \mathbf{V}_{m+2,n-1}) \ . \tag{3.2.67}$$

Since, $\mathbf{W}_u$ is semijective in KL_u, Lemma 3.2.14 applies and we have that

$$\begin{aligned} H^0_{DS}(u, \mathbf{V}_{m+1,n}) &\cong \mathbf{V}_{1,1} \circ_u (\mathbf{W}_u \circ_u \mathbf{V}_{m+2,n-1}) \ , \\ &\cong \mathbf{V}_{1,1} \circ_u (H^0_{DS}(u, \mathbf{V}_{m+2,n-1})) \ , \\ &\cong \mathbf{V}_{1,1} \circ_u \mathbf{V}_{m+1,n-1} \ , \end{aligned}$$

where we have used the inductive assumption. Of course, $\mathbf{V}_{1,1} \circ_u \mathbf{V}_{m+1,n-1}$ is $\mathbf{V}_{m,n}$ by definition and we are done. □

Remark 3.2.22. The uniqueness result for $\mathbf{V}_{1,1}$ can be readily extended to the $\mathbf{V}_{m,n}$. To reiterate, $\mathbf{V}_{m,n}$ is the unique vertex algebra object in $\mathrm{KL}_u^{\otimes m}$ such that

$$H^0_{DS}(u, \mathbf{V}_{m,n}) \cong \mathbf{V}_{m-1,n} \ . \tag{3.2.68}$$

Using Proposition 3.2.5, we can present our version of Proposition 10.10 of [Ara18].

Proposition 3.2.23. *The vertex algebras* $\mathbf{V}_{m,1}$ *have an ascending filtration whose successive quotients are isomorphic to* $\mathbb{V}^{ut}_{\lambda,m}$ *for some* $\lambda \in P^+_t$.

Additionally, the vertex algebras $\mathbf{V}_{m,1}$ *have a descending filtration whose successive quotients* *are isomorphic to* $D(\mathbb{V}^{ut}_{\lambda,m})$ *for some* $\lambda \in P_t^+$*.*

Therefore, the vertex algebras $\mathbf{V}_{m,1}$ *are semijective in* KL_t*, i.e.we have:*

$$\mathbf{V}_{m,1} \circ_t V \cong \mathrm{H}^{\frac{\infty}{2}+0}(\hat{\mathfrak{g}}_{t,\kappa_g}, \mathfrak{g}, \mathbf{V}_{m,1} \otimes V) \ , \tag{3.2.69}$$

for any $V \in \mathrm{KL}_t$*, and where we perform the reduction with respect to the moment maps coming from either twisted puncture on* $\mathbf{V}_{m,1}$*.*

Proof. We proceed by induction, noting that the base case for $\mathbf{V}_{1,1}$ has been established in Proposition 3.2.8. Now assume, $\mathbf{V}_{m,1}$ has such filtrations for some m, then from Proposition 3.2.5 the vertex algebra $\mathbf{V}_{m+1,1}$ does too since, by Lemma, 3.2.20 $\mathbf{V}_{m+1,1} \cong W_u *_u \mathbf{V}_{m,1}$. □

One could hope to strengthen this to the statement that all $\mathbf{V}_{m,n}$ are semijective in KL_t but such a result is beyond what is easily realised by our technologies. With the current arguments, we would have to independently establish that the vertex algebras $\mathbf{V}_{0,n}$ are semijective in KL_t. Given that our definition of $\mathbf{V}_{0,n}$ involves an unbounded cohomology, it seems difficult to verify such a property.

From a similar argument as with $\mathbf{V}_{1,1}$, the mixed vertex algebra, $\mathbf{V}_{m,1}$ cannot be semijective in KL_u, so the derived functor $\mathbf{V}_{1,1} \circ_u -$, which increases the number of pairs of twisted punctures by one, is not exact. By construction, the $\mathbf{V}_{m,n}$ cannot be concentrated in cohomological degree zero for $n > 1$. This is, in one sense, a good thing—going back to our discussion of residual gauge symmetry, these vertex algebras should have fermionic states lying in non-zero cohomological degree. On the other hand, this is a rather large roadblock to our spectral sequence powered proofs of associativity. We will only be able to provide partial results for how $\mathbf{V}_{m,n}$ fit in the duality web.

Similar to the mixed trinion, we can derive the characters and central charges of the mixed vertex algebras.

Proposition 3.2.24. *The character of the vertex algebra $\mathbf{V}_{m,1}$ is given by*

$$\mathrm{Tr}_{\mathbf{V}_{m,1}}\left(q^D \prod_{i=1}^{m} \mathbf{a}_i\ \mathbf{b}_1\mathbf{b}_2\right) = \sum_{\lambda\in P_t^+} \frac{\prod_{i=1}^m \mathcal{K}_u(\mathbf{a}_i)\chi_u^\lambda(\mathbf{a}_i)\mathcal{K}_t(\mathbf{b}_1)\chi_t^\lambda(\mathbf{b}_1)\mathcal{K}_t(\mathbf{b}_2)\chi_t^\lambda(\mathbf{b}_2)}{\left(\mathcal{K}_u(\times)\chi_u^\lambda(\times)\right)^m} .$$

so the vertex algebra, $\mathbf{V}_{m,1}$ is conical for all $m \in \mathbb{N}$.

Proof. For $m = 1$ this is just Proposition 3.2.9, so we take $m > 1$. We have that $H^0_{DS}(u, \mathbf{V}_{m,1}) \cong \mathbf{V}_{m-1,1}$. As graded vector spaces, we have the decomposition

$$\mathbf{V}_{m-1,1} \cong \sum_{\lambda\in P_t^+} \mathbb{V}^u_{\lambda,m-1} \otimes_{\mathfrak{z}^u_\lambda} (\mathbb{V}^t_\lambda \otimes_{\mathfrak{z}^t_\lambda} \mathbb{V}^t_{\lambda*}) .$$

Applying Proposition 8.4 of [Ara18] gives us the desired result. The fact that $\mathbf{V}_{m,1}$ are conical follows from the same argument as Proposition 3.2.9. ◻

Remark 3.2.25. The vertex algebra $\mathbf{V}_{m,n}$ is constructed by repeatedly gauging $\mathbf{V}_{m+n-1,1}$ with copies of $\mathbf{V}_{1,1}$. We know how the character behaves under gauging, so this result can be extended to

$$\mathrm{Tr}_{\mathbf{V}_{m,n}}\left(q^D \prod_{i=1}^{m} \mathbf{a}_i \prod_{j=1}^{2n} \mathbf{b}_j\right) = \frac{\prod_{i=1}^m \mathcal{K}_u(\mathbf{a}_i)\chi_u^\lambda(\mathbf{a}_i)\prod_{j=1}^{2n}\mathcal{K}_t(\mathbf{b}_j)\chi_t^\lambda(\mathbf{b}_j)}{\left(\mathcal{K}_u(\times)\chi_u^\lambda(\times)\right)^{m+2n-2}} , \tag{3.2.70}$$

which agrees with the expression in [LPR14].

Proposition 3.2.26. *The vertex algebras $\mathbf{V}_{m,1}$ are simple for all $m \in \mathbb{N}$.*

Proof. We proceed by induction on m, noting that $\mathbf{V}_{1,1}$ is simple by Proposition 3.2.11. Now suppose $\mathbf{V}_{m,1}$ is simple for some $m \geq 1$, and consider $\mathbf{V}_{m+1,1}$. If $V \subset \mathbf{V}_{m+1,1}$ is a submodule then $H^0_{\mathrm{DS}}(u, V)$ is a submodule of $\mathbf{V}_{m,1}$ by Proposition 3.2.21. However, by the inductive assumption $\mathbf{V}_{m,1}$ is simple and so $H^0_{\mathrm{DS}}(V)$ must be 0 or $\mathbf{V}_{m,1}$. From Theorem 2.2.8, we see that $V \cong W_u *_u H^0_{\mathrm{DS}}(V)$ and so $V \cong 0$ or $V \cong \mathbf{V}_{m+1,1}$. Therefore $\mathbf{V}_{m+1,1}$ is simple. ◻

Proposition 3.2.27. *The vertex algebras* $\mathbf{V}_{m,n}$ *are conformal with central charge*

$$c_{\mathbf{V}_{m,n}} = 2n \dim \mathfrak{g}_t + m \dim \mathfrak{g}_u - (m + 2n - 2)\mathrm{rk}\, \mathfrak{g}_u - 24(m + 2n - 2)\, \rho_u \cdot \rho_u^\vee \ .$$

Proof. First, we prove this statement for the family $\mathbf{V}_{m,1}$ before moving on to the full family of $\mathbf{V}_{m,n}$.

From Proposition 3.2.10, we know that $\mathbf{V}_{m,1}$ is conformal with central charge $c_{\mathbf{V}_{1,1}} = 2\dim \mathfrak{g}_t + \dim \mathfrak{g}_u - \mathrm{rk}\, \mathfrak{g}_u - 24\rho_u \cdot \rho_u^\vee$ so the statement is true of $m = 1$.

We shall first show that $\mathbf{V}_{m,1}$ has a conformal vector, then show that it is the unique conformal vector whose grading agrees with the character and finally show that this results in the correct central charge.

Consider the vector $\omega_m \in C_{m,1}$, defined by

$$\omega_m = \omega_{\mathcal{D}_t^{ch}} + \sum_{i=1}^{m} \omega_{\mathbf{W}_i} + \sum_{i=1}^{m} \omega_{gh,i} \ ,$$

where $\omega_{\mathcal{D}_t^{ch}}$ is the conformal vector of $\mathcal{D}_t^{ch}$, $\omega_{\mathbf{W}_i}$ is the conformal vector of the ith factor of $\mathbf{W}_u$ and $\omega_{gh,i}$ is the conformal vector of the ith ghost system, $\bigwedge^{\frac{\infty}{2}+\bullet}(\mathfrak{z}(\mathfrak{g}_u))$. Clearly, ω_m defines a conformal vector on the complex, $C_{m,1}$. Like we did for $\mathbf{V}_{1,1}$ we shall argue that this descends to a conformal vector in cohomology.

For an element $P_i \in \mathfrak{z}(\mathfrak{g}_u)$, where we think of $\mathfrak{z}(\mathfrak{g}_u)$ as a subalgebra of one of the $\mathbf{W}_u$ factors, ω_m acts as

$$\omega_m(z)P_i(w) \sim \frac{\partial P_i}{z - w} + \frac{(d_i + 1)P_i}{(z - w)^2} + \sum_{j=2}^{d_i+2} \frac{(-1)^j j!}{(z - w)^{j+1}} q_j^{(i)}(w) \ ,$$

where $q_j(i)$ is some homogeneous state in $\mathfrak{z}(\mathfrak{g}_u)$ with weight $d_i - j + 2$. Let us denote by $\widetilde{P}_i$ the image of P_i under the projection $\mathfrak{z}(\mathfrak{g}_u) \twoheadrightarrow \mathfrak{z}(\mathfrak{g}_t)$. One then has

$$\omega(z)\widetilde{P}_i(w) \sim \frac{\partial \widetilde{P}_i}{z - w} + \frac{(d_i + 1)\widetilde{P}_i}{(z - w)^2} + \sum_{j=2}^{d_i+2} \frac{(-1)^j j!}{(z - w)^{j+1}} \widetilde{q_j}^{(i)}(w) \ ,$$

where we think of $\widetilde{P}_i$ as a state in $\mathfrak{z}(\mathfrak{g}_t) \subset \mathcal{D}_t^{ch}$. Let Q^m be the differential of $C_{m,1}$; the action of Q^m on the vector ω_m is

$$Q^m_{(0)}(z)\omega(w) = \sum_{l=1}^{m} \sum_{i=1}^{\mathrm{rk}\,\mathfrak{g}_u} \sum_{j=2}^{d_i+1} \partial^j \big(\rho_l(q_j^{(i)}) - \rho_{l+1}(\tau(q^{(i)}))\big) c_i \ ,$$

where ρ_i for $i \leqslant m$ denotes the action of the Feigin-Frenkel centre on the ith $\mathbf{W}_u$ factor and ρ_{m+1} once again denotes the action of $\mathfrak{z}(\mathfrak{g}_u)$ on $\mathcal{D}_t^{ch}$ along the projection to $\mathfrak{z}(\mathfrak{g}_t)$.

If the right hand side of the above equation equals $Q^m_{(0)}\chi$ for some state χ, then $\widetilde{\omega}_m = \omega_m + \chi$ is Q-closed and defines a vector in $\mathbf{V}_{m,1}$. For $l \neq m$ it is clear that $\rho_l(q_j^{(i)}) - \rho_{l+1}(q_j^{(i)})$ is a coboundary, and we have addressed the $l = m$ case in the proof of Proposition 3.2.10. Therefore, such a χ exists and may be written as

$$\chi = \sum_{l=1}^{m} \sum_{i=1}^{\mathrm{rk}\,\mathfrak{g}_u} \sum_{j=2}^{d_i+2} \partial^j (\rho_l \otimes \rho_{l+1} \otimes \rho_{gh,l})(z_{ij})$$

for some $z_{ij} \in \mathfrak{z}(\mathfrak{g}_u) \otimes \mathfrak{z}(\mathfrak{g}_u) \otimes \bigwedge^{\frac{\infty}{2}+0}(\mathfrak{z}(\mathfrak{g}_u))$. Therefore, $\widetilde{\omega}_{m,(i)} = \omega_{m,(i)}$ for $i = 0, 1$, so the OPEs agree up to the quadratic pole. Since $\mathbf{V}_{m,1}$ is conical by Proposition 3.2.24, Lemma 3.1.2 of [Fre07] applies once more and we can conclude that $\widetilde{\omega}_m$ is a conformal vector in $\mathbf{V}_{m,1}$.

Now we wish to show that $\widetilde{\omega}_m$ is the unique conformal vector whose L_0-grading agrees with ω_m. The argument from Proposition 3.2.10 using Lemma 4.1 of [Mor20] still works, with minor alteration, since $\mathbf{V}_{m,1}$ are conical.

The DS reduction of $\widetilde{\omega}_m$ gives a conformal vector in $\mathbf{V}_{m-1,1}$ with central charge

$$c_{\mathbf{V}_{m-1,1}} = c_{\mathbf{V}_{1,1}} + \mathrm{rk}\,\mathfrak{g}_u - \mathrm{dim}\mathfrak{g}_u + 24\rho_u \cdot \rho_u^\vee \ ,$$

and which agrees with the grading by ω_{m-1}. But, by the inductive assumption, such a conformal vector on $\mathbf{V}_{m-1,1}$ is unique.

Unwrapping the induction from the base case of $\mathbf{V}_{0,1} = \mathcal{D}_t^{ch}$, we get that

$$c_{\mathbf{V}_{m,1}} = m\,\mathrm{dim}\mathfrak{g}_u + 2\mathrm{dim}\mathfrak{g}_t - 24m\,\rho_u \cdot \rho_u^\vee - m\,\mathrm{rk}\,\mathfrak{g}_u\ .$$

To extend this to $\mathbf{V}_{m,n}$, we shall once more make use of the fact the $\mathbf{V}_{m,n}$ are constructed by repeated twisted gaugings of the $\mathbf{V}_{m,1}$ with copies of $\mathbf{V}_{1,1}$. It is well known that the conformal vector

$$T = \widetilde{\omega}_{\mathbf{V}_{k,l}} + \widetilde{\omega}_{\mathbf{V}_{1,1}} + \omega_{gh}\ ,$$

where ω_{gh} is the conformal vector of the b, c ghost system, descends to a conformal vector in the BRST cohomology with central charge equal to the central charge of T. Therefore the vertex algebras $\mathbf{V}_{m,n}$ are conformal, with central charge

$$c_{\mathbf{V}_{m,n}} = 2n\dim\mathfrak{g}_t + m\dim\mathfrak{g}_u - (m+2n-2)\mathrm{rk}\,\mathfrak{g}_u - 24(m+2n-2)\,\rho_u\cdot\rho_u^\vee\ ,$$

as desired. □

Having established a number of intrinsic properties of the genus zero, mixed vertex algebras, let us examine how they interact with each other under gluing. This will shed some light on how the $\mathbf{V}_{m,n}$ fit into the class $\mathcal{S}$ duality web.

Given our rearrangement lemmas, we can show that the mixed vertex algebras of the previous section have the expected behaviour under $\circ_u$ and $\circ_t$. First, we establish the partial result

Lemma 3.2.28. *We have the isomorphism*

$$\mathbf{V}_{1,1} \circ_t \mathbf{V}_{m,n} \cong \mathbf{V}_{m+1,n}\ .$$

Proof. First, let us treat the base case of $n = 1$. We have that

$$\mathbf{V}_{1,1} \circ_t \mathbf{V}_{m,1} \cong (\mathbf{W}_u *_u \mathcal{D}_t^{ch}) \circ_t \mathbf{V}_{m,1}\ ,$$

so $\mathbf{V}_{1,1} \circ_t \mathbf{V}_{m,1} \cong \mathbf{W}_u *_u \mathbf{V}_{m,1} \cong \mathbf{V}_{m+1,1}$. By Lemma 3.2.18, we have that

$$(\mathbf{W}_u *_u \mathcal{D}_t^{ch}) \circ_t \mathbf{V}_{m,1} \cong \mathbf{W}_u *_u (\mathcal{D}_t^{ch} \circ_t \mathbf{V}_{m,1}) \cong \mathbf{W}_u *_u \mathbf{V}_{m,1} \cong \mathbf{V}_{m+1,1} \ ,$$

where we have used Lemma 3.2.20. Now, we proceed by induction on n. We have just established the base case for $n = 1$, so suppose $n > 1$. Then,

$$\begin{aligned} \mathbf{V}_{1,1} \circ_t \mathbf{V}_{m,n} &\cong \mathbf{V}_{1,1} \circ_t (\mathbf{V}_{m+1,n-1} \circ_u \mathbf{V}_{1,1}) \ , \\ &\cong (\mathbf{V}_{1,1} \circ_t \mathbf{V}_{m+1,n-1}) \circ_u \mathbf{V}_{1,1} \ , \\ &\cong \mathbf{V}_{m+2,n-1} \circ_u \mathbf{V}_{1,1} \ , \\ &\cong \mathbf{V}_{m+1,n} \ , \end{aligned}$$

where in the second line we have used Lemma 3.2.17. □

Proposition 3.2.29. *Under gauging, the vertex algebras $\mathbf{V}_{m,1}$ behave as expected, namely,*

$$\begin{aligned} \mathbf{V}_{m,1} \circ_t \mathbf{V}_{p,q} &\cong \mathbf{V}_{m+p,q} \ , \\ \mathbf{V}_{m,1} \circ_u \mathbf{V}_{p,q} &\cong \mathbf{V}_{p+m-2,q+1} \ . \end{aligned}$$

Proof. We proceed via induction for each type of gluing, noting that the base case $m = 1$ is true, either by Lemma 3.2.28 or by definition.

Suppose $m > 1$. Then,

$$\begin{aligned} \mathbf{V}_{m,1} \circ_t \mathbf{V}_{p,q} &\cong (\mathbf{V}_{1,1} \circ_t \mathbf{V}_{m-1,1}) \circ_t \mathbf{V}_{p,q} \ , \\ &\cong \mathbf{V}_{1,1} \circ_t (\mathbf{V}_{m-1,1} \circ_t \mathbf{V}_{p,q}) \ , \\ &\cong \mathbf{V}_{p+m,q} \ , \end{aligned}$$

where in the second line, we have used Lemma 3.2.14 to arrive at the desired result.

Next we treat the $\circ_u$ case. Again, suppose $m > 1$. Then,

$$\begin{aligned}\mathbf{V}_{m,1} \circ_u \mathbf{V}_{p,q} &\cong (\mathbf{V}_{1,1} \circ_t \mathbf{V}_{m-1,1}) \circ_u \mathbf{V}_{p,q} \ , \\ &\cong \mathbf{V}_{1,1} \circ_t (\mathbf{V}_{m-1,1} \circ_u \mathbf{V}_{p,q}) \ , \\ &\cong \mathbf{V}_{p+m-2,q+1} \ ,\end{aligned}$$

where, in the second line we have used a slight modification of Lemma 3.2.17—which applies, since the cohomology $\mathbf{V}_{1,1} \circ_t -$ is concentrated in degree zero, so the spectral sequences will collapse at the second page. □

Of course, we expect that these results should extend to the general case,

$$\mathbf{V}_{m,n} \circ_t \mathbf{V}_{p,q} \cong \mathbf{V}_{m+p,n+q-1} \ , \quad \mathbf{V}_{m,n} \circ_u \mathbf{V}_{p,q} \cong \mathbf{V}_{m+p-2,q+n} \ . \tag{3.2.71}$$

The obstructions to proving this are as follows. In the case of $\circ_t$, the inductive step is $\mathbf{V}_{m,n} \circ_t (\mathbf{V}_{p+1,q-1} \circ_u \mathbf{V}_{1,1})$ and the corresponding spectral sequence is unbounded and does not collapse at the second page. Similarly, for the untwisted gluing we have not been able to establish the putative isomorphism

$$(\mathbf{V}_{1,1} \circ_u \mathbf{V}_{m+1,n-1}) \circ_u \mathbf{V}_{p,q} \cong \mathbf{V}_{1,1} \circ_u (\mathbf{V}_{m+1,n-1}) \circ_u \mathbf{V}_{p,q} \ , \tag{3.2.72}$$

for the inductive step. Neither cohomology is concentrated in degree zero, so the second page of the spectral sequence is unbounded. To make progress we require more sophisticated machinery or a different strategy.

Proposition 3.2.30. *We have the isomorphism*

$$\mathbf{V}_{G_u,s} *_u \mathbf{V}_{m,n} \cong \mathbf{V}_{m+s,n} \ .$$

Proof. We proceed by induction on s, noting that, for $s = 1$, the statement is true since $V_{m,n}$ are in $\mathrm{KL}_{u,0}$. Now suppose $s > 1$, and consider $H^0_{DS}(u, \mathbf{V}_{G_u,s} *_u \mathbf{V}_{m,n})$. Of course,

$H^0_{DS}(u,-)$ and $\mathbf{W}_u \circ_u -$ are isomorphic. We form the bicomplex

$$C^{\bullet,\bullet} = \mathbf{W}_u \otimes \mathbf{V}_{G_u,s} \otimes \mathbf{V}_{m,n} \otimes \bigwedge{}^{\frac{\infty}{2}+\bullet}(\mathfrak{z}(\mathfrak{g}_u))) \otimes \bigwedge{}^{\frac{\infty}{2}+\bullet}(\mathfrak{g}_u) \ ,$$

with differentials $d_{\mathfrak{z}}$ acting on $\mathbf{V}_{G_u,s} \otimes \mathbf{V}_{m,n} \otimes \bigwedge^{\frac{\infty}{2}+\bullet}(\mathfrak{z}(\mathfrak{g}_u))$ and $d_{\mathfrak{g}}$ acting on $\mathbf{W}_u \otimes \mathbf{V}_{G_u,s} \otimes \bigwedge^{\frac{\infty}{2}+\bullet}(\mathfrak{g}_u)$. The differentials anticommute and we can form the total complex as usual. The two relevant spectral sequences are

$$\begin{aligned} {}_IE_2^{p,q} &= \mathrm{H}^{\frac{\infty}{2}+p}(\hat{\mathfrak{g}}_{u,-\kappa_g}, \mathfrak{g}_u, \mathbf{W}_u \otimes \mathrm{H}^{\frac{\infty}{2}+q}(\mathcal{Z}_u, \mathbf{V}_{G_u,s} \otimes \mathbf{V}_{m,n})) \ , \\ {}_{II}E_2^{p,q} &= \mathrm{H}^{\frac{\infty}{2}+p}(\mathcal{Z}_u, \mathrm{H}^{\frac{\infty}{2}+q}(\hat{\mathfrak{g}}_{u,-\kappa_g}, \mathfrak{g}_u, \mathbf{W}_u \otimes \mathbf{V}_{G_u,s}) \otimes \mathbf{V}_{m,n}) \ . \end{aligned}$$

The functor of DS-reduction is exact, so both sequences collapse at page two with ${}_IE_2^{0,q}$ and ${}_{II}E_2^{p,0}$ being the only non-zero entries. This gives the isomorphism,

$$H^0_{DS}(u, \mathbf{V}_{G_u,s} *_u \mathbf{V}_{m,n}) \cong \mathbf{V}_{G_u,s-1} *_u \mathbf{V}_{m,n} \cong \mathbf{V}_{m+s-1,n} \ ,$$

where we used the inductive hypothesis. Acting by $\mathbf{W}_u *_u -$, we have that

$$\mathbf{V}_{G_u,s} *_u \mathbf{V}_{m,n} \cong \mathbf{V}_{m+s,n} \ , \tag{3.2.73}$$

as desired. □

Finally, let us consider gauging the untwisted and mixed vertex algebras together.

Proposition 3.2.31. *Under untwisted gauging of untwisted vertex algebras, the mixed vertex algebras behave as expected, i.e.,*

$$\mathbf{V}_{G_u,s} \circ_u \mathbf{V}_{m,n} \cong \mathbf{V}_{m+s-2,n}$$

Proof. We have the following chain of isomorphisms,

$$\begin{aligned}\mathbf{V}_{G_u,s} \circ_u \mathbf{V}_{m,n} &\cong (\mathbf{V}_{G_u,s-1} *_u \mathbf{W}_u) \circ_u \mathbf{V}_{m,n} \ , \\ &\cong \mathbf{V}_{G_u,s-1} *_u (\mathbf{W}_u \circ_u \mathbf{V}_{m,n}) \ , \\ &\cong \mathbf{V}_{G_u,s-1} *_u \mathbf{V}_{m-1,n} \ , \\ &\cong \mathbf{V}_{m+s-2,n} \ ,\end{aligned}$$

where in the second line we have used Lemma 3.2.16 and in the third we have used Proposition 3.2.30. □

To conclude this subsection, we should comment on the general issue of associativity. In [Ara18], the cohomology $\mathbf{V}_{G,s} \circ \mathbf{V}_{G,s'}$ was concentrated in degree zero, so ``gauging is associative'' as a consequence of a by-now-standard spectral sequence argument. In our case, the argument is not so simple—we have repeatedly remarked that zero genus is no longer sufficient for a gluing to be concentrated in degree zero. The obvious spectral sequence no longer collapses on the second page, so the proofs of rearrangement lemmas no longer hold. Nevertheless, it is our belief that associativity must still hold in general.

Yanagida [Yan20] has defined a derived version of the construction of [Ara18] in a suitably defined category of dgVOAs. In addition, that work also imported the Moore–Tachikawa TQFT to the derived setting. In the derived analysis, associativity of gauging follows from general properties of the derived pushforward—even for nonzero genus. However, the notion of associativity in the derived setting is a slightly weaker result than the notion of associativity in this work.

In short, our prescription is normally to pass cohomology before the second gauging—unlike the derived case, where one does not pass to cohomology but instead holds on to the full data of the chain complex (as an object in an appropriate derived category). A sufficient condition for the derived associativity to imply our version of associativity is to show that the relevant chain complexes are formal, *i.e.*, are isomorphic to their cohomology (thought of as a complex with zero differential) in that derived category. This is an interesting problem in its own right but is far beyond the scope of this current work.

3.2.8 Generalised S-duality and 4-moves

Recall that in our construction of the $\mathbf{V}_{m,n}$, we were forced to non-canonically pick a particular pants decomposition. In the following, we wish to justify that our construction is in fact independent of such a choice, by establishing the invariance of our construction under the various moves of S-duality. To do so, we shall construct an action of the various 4-moves of Sections 1.1.5 and 3.1.2.

Let $\mathbf{V}$ be some mixed vertex algebra, for now we assume genus zero with only maximal punctures. There are three different types of 4-move that can act on $\mathbf{V}$.

- The first acts on a collection of four untwisted punctures.
- The second acts on two pairs of twisted punctures.
- The third kind (the *ut*-move) acts on a pair of twisted punctures and two untwisted punctures.

We examine each case in turn to establish invariance.

In the purely untwisted case, invariance under the 4-move is baked into Arakawa's construction–it permutes the s-many chiral moment maps of $\mathbf{V}_{G,s}$. Each moment map is inherited from the s-many caps that are FF-glued together to build $\mathbf{V}_{G,s}$. These caps are all identical and the gluing happens simultaneously—hence the permutation group symmetry is manifest. Let us present an alternate description of this action, which is generalisable to the twisted case.

We may endow $\mathbf{V}_{G,s}$ with the action[4] of a permutation group as follows. Fix a labelling of the punctures, equivalently, a labelling of the chiral moment maps. We shall describe the action of the transposition (23). First, we close off the punctures labelled 2 and 3 in sequence, *i.e.*, we perform DS reduction with respect to the moment maps ι_2, associated to puncture 2, and then with respect to ι_3, associated to puncture 3. From Theorem 2.2.7, we

[4] As written, the action of the 4-move is not an automorphism but an isomorphism to some vertex algebra object in KL. To correct this to an automorphism, we appeal to the uniqueness property of $\mathbf{V}_{G,s}$ (see Remark 10.13 of [Ara18]) and fix an isomorphism from this vertex algebra back to $\mathbf{V}_{G,s}$. One can perform a similar trick with $\mathbf{V}_{m,n}$ using the uniqueness statements of the previous section. To avoid having to make such a choice one can work with the Moore–Seiberg groupoid [MS89], but we will not do so here.

can think of this DS reduction as gluing a cap to the chosen puncture.

We can invert this procedure by Feigin–Frenkel gluing a cap with moment map ι_3 and then Feigin–Frenkel gluing a cap with moment map ι_2. Instead, we reverse the order of inversions, that is to say we first glue a cap with moment map ι_2 and then a cap with moment map ι_3. This is an isomorphism since

$$\mathbf{W}_3 *_u (\mathbf{W}_2 *_u (\mathbf{W}_3 \circ_u (\mathbf{W}_2 \circ_u \mathbf{V}_s))) \cong \mathbf{W}_2 *_u (\mathbf{W}_3 *_u (\mathbf{W}_3 \circ_u (\mathbf{W}_2 \circ_u \mathbf{V}_s))) \cong \mathbf{V}_s \ , \quad (3.2.74)$$

where we have used Lemma 3.2.15 to swap the order of Feigin–Frenkel gluings. The subscripts on the caps keep track of the labelling of the moment maps and we have suppressed the G-subscript for clarity. One can realise the actions of the other transpositions in the same way, and thus generate the action of the full symmetric group on s punctures. The action of the symmetric group should be understood as swapping the labellings of the moment maps associated to each puncture. This argument also establishes invariance under the four move of the first type for the mixed vertex algebras $\mathbf{V}_{m,n}$.

Now let us consider the case $\mathbf{V} = \mathbf{V}_{m,n}$, the mixed vertex algebra with n pairs of twisted punctures and m untwisted punctures. We can define the action of transpositions on twisted punctures, as in the untwisted case, by closing pairs of punctures and gluing caps. Again, let us pick two twisted punctures, labelled 2 and 3, with moment maps j_2 and j_3 respectively. We perform DS reduction once more, closing the punctures labelled 2 and then 3, in order. Once again, we can think of this DS reduction as gluing a twisted cap to the chosen puncture.

We restore the punctures by Feigin–Frenkel gluing, via the twisted Feigin–Frenkel centre, two twisted caps with moment maps j_2 and j_3. We have

$$\mathbf{W}_{t,2} *_t (\mathbf{W}_{t,3} *_t (\mathbf{W}_{t,3} \circ_t (\mathbf{W}_{t,2} \circ_t \mathbf{V}))) \cong \mathbf{W}_{t,3} *_t (\mathbf{W}_{t,3} *_t (\mathbf{W}_{t,3} \circ_t (\mathbf{W}_{t,2} \circ_t \mathbf{V}))) \cong \mathbf{V} \ . \quad (3.2.75)$$

Once more, by Lemma 3.2.15, this results in a vertex algebra that is isomorphic to $\mathbf{V}_{m,n}$. All transpositions of twisted punctures can be arrived at using this method and we can generate the full symmetric group on $2n$ twisted punctures. It should be noted that the automorphism group allows swaps between twisted punctures regardless of whether one has connected them by twist lines (the twist lines in this abelian setting are a fiction anyways; they just record monodromies of the punctures). The preceding argument establishes the action of S_m on the untwisted punctures and so the $\mathbf{V}_{m,n}$ has an action of $S_m \times S_{2n}$ by automorphisms.

For the 4-move of the third kind—the *ut*-move. Our analysis in terms of permutations fails. This move swaps between the degeneration limits shown in Figure 3.1—unlike the other cases the decompositions are no longer related by a simple permutation on the punctures. Instead, we appeal to Proposition 3.2.30, which states that the two BRST gluings $\mathbf{V}_{m-1,n} \circ_u \mathbf{V}_{G_u,3}$ and $\mathbf{V}_{1,1} \circ_t \mathbf{V}_{m-1,n}$ are isomorphic.

This construction endows the vertex algebras $\mathbf{V}_{m,n}$ with an action of the generalised S-duality group. Therefore, our recursive definition for $\mathbf{V}_{m,n}$ is well-defined and independent of the choice of pants decomposition.

A proof of invariance under the *ab*-move eludes us but we pose this as a conjecture in the language of semi-infinite cohomology.

Conjecture 3.2.32. *Let $\mathbf{V}_{G_u,3}$ be the trinion vertex algebra with maximal untwisted punctures and $\mathbf{V}_{1,1}$ be the mixed trinion vertex algebra as before. Let i_1, i_2, i_3 denote the three actions of $V^{\kappa_c}(\mathfrak{g}_u)$ on $\mathbf{V}_{G_u,3}$. Similarly, let j_2, j_3 be the actions of $V^{\kappa_c}(\mathfrak{g}_t)$ on $\mathbf{V}_{1,1}$. Then the following isomorphisms hold,*

$$\mathrm{H}^{\frac{\infty}{2}+\bullet}(\hat{g}_{u,-\kappa_g}, \mathfrak{g}_u, \mathbf{V}_{G_u,3}) \cong \mathrm{H}^{\frac{\infty}{2}+\bullet}(\hat{\mathfrak{g}}_{t,-\kappa_g}, \mathfrak{g}_t, \mathbf{V}_{1,1}) \ ,$$

where $\hat{\mathfrak{g}}_{u,-\kappa_g}$ acts on $\mathbf{V}_{G_u,3}$ via $i_2 \otimes (i_3 \circ \sigma)$, with σ the $\mathbb{Z}_2$ outer-automorphism, and $\hat{\mathfrak{g}}_{t,-\kappa_g}$ acts on $\mathbf{V}_{1,1}$ via $j_2 \otimes j_3$.

Establishing invariance under the *ab*-move for the one punctured torus is sufficient to ensure invariance for the vertex algebras of all other surfaces. This construction is only relevant at nonzero genus, so states of higher cohomological degrees will be present. Of course, replacing the maximal puncture with a minimal puncture in this duality just recovers S-duality for non-simply-laced $\mathcal{N} = 4$ super Yang-Mills theory; this also remains an open conjecture at the level of associated vertex algebras. A computation to show the matching of indices can be found in [AS14].

3.2.9 The $\mathbb{Z}/3\mathbb{Z}$ twist of D_4

Much of the preceding discussion goes through for case of $\mathbb{Z}_3$-twisted punctures in the D_4 theory, but there are some new features that are worth mentioning. First, let us lay out the details of the construction.

The chiral differential operators over G_2 are well defined. Here we take G_2 to be the exponentiated form of the $\mathfrak{g}_2$ Lie algebra, *i.e.*, G_2 is a simply connected, simple, algebraic Lie group. The superconformal index assigned to the twisted cylinder agrees (summand by summand) with the character of $\mathbf{V}_{G_2,2}$, and this once more motivates our construction.

Pictorially, one imagines the G_2 cylinder as having a puncture twisted by ω and the other by ω^2. One might, *a priori*, expect that there are two possible $\mathfrak{g}_2$ caps, $\mathbf{W}_\omega$ and $\mathbf{W}_{\omega^2}$, depending on which puncture is closed. Yet, from the uniqueness argument of [Ara18], the two putative caps are isomorphic. The outer automorphism, σ, that exchanges ω with ω^2 should therefore lift to an isomorphism of vertex-algebras $\mathbf{W}_\omega \xrightarrow{\sigma} \mathbf{W}_{\omega^2}$.

The total monodromy around all punctures must be trivial. This can be satisfied in a number of ways, but for now we will restrict our attention to the case where punctures labelled by ω and ω^2, respectively, come in pairs (one might think of them as having twist lines connecting them pairwise). We denote a genus zero surface with m untwisted punctures and n ω, ω^2 pairs of punctures by $\mathcal{C}_{0,m,n}$. We define the mixed trinion, $\mathbf{V}_{1,1}$ as

$$\mathbf{V}_{1,1} \coloneqq \mathbf{W}_{\mathrm{Spin}(8)} *_u \mathcal{D}^{ch}_{\mathrm{G}_2} \ , \tag{3.2.76}$$

and the construction of the $\mathbf{V}_{m,n}$ proceeds analogously. The ambiguity in the two versions of the cap is present here again, and we can ask whether this is physical. Namely, is there an S-duality move that swaps two punctures with ω and ω^2 labels? We will show that the vertex algebras $\mathbf{V}_{m,n}$ are indeed invariant under such a move, though there is no expectation that the underlying SCFTs will enjoy the same symmetry.

Let us first establish some rearrangement lemmas for the $\mathbb{Z}/3\mathbb{Z}$ case. The isomorphism of vertex algebras $\mathbf{W}_\omega \cong \mathbf{W}_{\omega^2}$ lifts to a natural transformation of functors,

$$\mathrm{H}^{\frac{\infty}{2}+0}(\mathscr{Z}_t, \mathbf{W}_\omega \otimes -) \simeq \mathrm{H}^{\frac{\infty}{2}+0}(\mathscr{Z}_t, \mathbf{W}_{\omega^2} \otimes -) \ . \tag{3.2.77}$$

In other words, the twisted cylinders whose endpoints are labelled by any combination of ω, ω^2 have isomorphic vertex algebras—all isomorphic to chiral differential operators on G_2. Though unphysical, one can consider the vertex algebra $\widetilde{\mathbf{V}}_{1,1} \coloneqq \mathbf{W}_{\mathrm{Spin}(8)} *_u (\mathbf{W}_\omega *_t \mathbf{W}_\omega)$. This would, naively, correspond to the trinion with two maximal punctures twisted by ω and one untwisted puncture. From the natural isomorphism, however, we have that

$$\widetilde{\mathbf{V}}_{1,1} \cong \mathbf{V}_{1,1} \ . \tag{3.2.78}$$

We can, therefore, use the trinion $\widetilde{\mathbf{V}}_{1,1}$ as the building block for an equivalent construction of genus zero vertex algebras, which are isomorphic to the $\mathbf{V}_{1,1}$ construction. This construction, however, makes manifest the enhanced symmetry of the vertex algebras, *i.e.*, the labelling by ω *versus* ω^2 is redundant.

Let us reiterate, as this strikes us as a surprising result. At the level of the associated vertex algebra, there is are additional automorphisms that swap ω and ω^2 punctures which, as far as we know, do not arise from an underlying S-duality of the four-dimensional physics. For example, the naive S-duality group of a surface with a pair of ω punctures and a pair of ω^2 punctures is $\mathbb{Z}_2 \times \mathbb{Z}_2$, which swaps punctures with the same label. This is enhanced to S_4—swapping between all four punctures—as if these were all identical punctures!

In addition to the mixed trinion that we described above, one can compactify on a curve with a three pronged twist line—connecting three punctures each with ω (or ω^2) monodromy. The trinion with three ω punctures does indeed correspond to a physical SCFT, and one might expect that this trinion is $\mathbf{V}_{G_2,3}$. Comparing the superconformal index of this trinion theory with the character of $\mathbf{V}_{G_2,3}$, however, exposes this as wishful thinking. Indeed, this trinion is something of a mystery, and we currently have no way of constructing it with the machinery of [Ara18]. For now, we shall refer to the surface with the three pronged twist line by $\mathcal{C}_{0,0,3\omega}$ and its corresponding vertex algebra by $\mathbf{V}_{0,3\omega}$. We also introduce their σ-conjugates $\mathcal{C}_{0,0,3\omega^2}$ and $\mathbf{V}_{0,3\omega^2}$, which correspond to the surface with a three pronged twist line between three ω^2 twisted punctures and its associated vertex algebra.

Unlike all of the other trinions we have considered, we expect $\mathbf{V}_{0,3\omega}$ to be have support outside cohomological degree zero, *i.e.*, contain fermions. This prediction comes from our proposed diagnostic concerning the covering space of the UV curve, and is compatible with some speculative analyses of the superconformal index. Any construction involving Feigin--Frenkel gluing is forced to be in degree zero, since we manually truncate to the zeroth cohomology, so this expectation implies the necessity of other tools to get at this vertex algebra.

3.2.10 The proof of Theorem 3.2.7

In this section we present our proof of Theorem 3.2.7, reproduced below.

Theorem. *We have the following isomorphism:*

$$H^0_{DS}(u, \mathbf{V}_{1,1}) \cong \mathcal{D}^{ch}_t \ ,$$

so $\mathcal{D}^{ch}_t \in \mathrm{KL}_{u,0}$.

Proof. First, we establish notation. Let $\mathcal{F}$ denote the composition $H^0_{DS}(u, \mathrm{H}^{\frac{\infty}{2}+0}(\mathcal{Z}, \mathbf{W}_u \otimes -))$, which is an endofunctor on $\mathcal{Z}_u-\mathrm{Mod}$. Notably, on the subcategory $\mathrm{KL}_{u,0}$, $\mathcal{F}(M) \cong M$ for any $M \in \mathrm{KL}_{u,0}$ (see Proposition 9.12 of [Ara18]), and in general $\mathcal{F}$ is left-exact. We wish to show that $\mathcal{F}(\mathcal{D}^{ch}_t) \cong \mathcal{D}^{ch}_t$.

Let $\chi_\lambda : \mathcal{Z}_t \to \mathbb{C}$ be a character defined by $\chi_\lambda(P_{i,n}) = 0$ for $n \neq 0$ and $P_{i,0}\mathbb{V}^t_\lambda = \chi_\lambda(P_{i,0})\mathbb{V}^t_\lambda$. The Kazhdan–Lusztig category decomposes into blocks $\mathrm{KL}_t \cong \bigoplus \mathrm{KL}^{[\lambda]}_t$ where $\mathrm{KL}^{[\lambda]}_t$ is the subcategory of objects on which $P_{i,0}$ acts as the *generalised* eigenvalue $\chi_\lambda(0)$, *i.e.*, these are objects which are supported on the formal completion of the ideal $(P_{i,0} - \chi_\lambda(P_{i,0}))$ in $\mathrm{Spec}\, \mathcal{Z}_t$.

As $\mathcal{D}_t^{ch}$ is a vertex algebra object in KL_t, it decomposes as

$$\mathcal{D}_t^{ch} \cong \bigoplus_{\lambda \in P_t^+} \mathcal{D}_{t,[\lambda]}^{ch} \ ,$$

where $\mathcal{D}_{t,[\lambda]}^{ch}$ are objects in $\mathrm{KL}_t^{[\lambda]}$. The increasing filtration on $\mathcal{D}_t^{ch}$ induces a filtration on each $\mathcal{D}_{t,[\lambda]}^{ch}$,

$$0 = N_0 \subset N_1 \subset \cdots \subset N = \bigcup_i N_i = \mathcal{D}_{t,[\lambda]}^{ch} \ ,$$

such that successive quotients are isomorphic to $\mathbb{V}_{\lambda,2}^t$. While the character of $\mathcal{D}_t^{ch}$ is ill-defined (since each weight space is infinite dimensional), the character of each $\mathcal{D}_{t,[\lambda]}^{ch}$ is well-defined. We have that $\mathcal{F}(\mathcal{D}_t^{ch})_{[\lambda]} \cong \mathcal{F}(\mathcal{D}_{t,[\lambda]}^{ch})$ since $\mathcal{F}$ is left exact. Therefore,

$$\mathrm{ch}\ \mathcal{F}(\mathcal{D}_t^{ch})_{[\lambda]} \cong \mathrm{ch}\ \mathcal{F}(\mathcal{D}_{t,[\lambda]}^{ch}) \ .$$

The filtration on $\mathcal{D}_{t,[\lambda]}^{ch}$ induces a filtration on $\mathcal{F}(\mathcal{D}_{t,[\lambda]}^{ch})$ and since $\mathcal{F}$ is left exact, we have that $\mathcal{F}(N_i)/\mathcal{F}(N_{i-1}) \subseteq \mathcal{F}(N_i/N_{i-1})$, and so

$$\mathrm{ch}\ \mathcal{F}(\mathcal{D}_{t,[\lambda]}^{ch}) \leqslant \sum_i \mathrm{ch}\ \mathcal{F}(N_i/N_{i-1}) \ .$$

Each subquotient, N_i/N_{i-1} is an object in $\mathrm{KL}_{u,0}$ by Proposition 3.2.3. Consequently,

$$\mathrm{ch}\ \mathcal{F}(\mathcal{D}_{t,[\lambda]}^{ch}) \leqslant \sum_i \mathrm{ch}\ (N_i/N_{i-1}) = \mathrm{ch}\mathcal{D}_{t,[\lambda]}^{ch} \ .$$

From this we conclude that $\mathcal{F}(\mathcal{D}_t^{ch})$ must be $\mathbb{Z}_{\geqslant 0}$ graded. Now, since $\mathcal{D}_t^{ch}$ is simple, if we can show that there is a non-trivial vertex algebra morphism $\mathcal{D}_t^{ch} \to \mathcal{F}(\mathcal{D}_t^{ch})$, then we will have $\mathcal{F}(\mathcal{D}_t^{ch}) \cong \mathcal{D}_t^{ch}$. From here we are in a very similar situation as the proof of Theorem 9.9 of [Ara18], so we adapt that proof to our setting. Before we do so, however, we will need some subsidiary lemmas. Let $\mathfrak{g}_t^+ \subset \hat{\mathfrak{g}}_{t,-\kappa_c} = t\mathfrak{g}_t[t]$. Then we have:

Lemma 3.2.33. $\mathbf{V}_{1,1}^{\mathfrak{g}_t^+} = \mathrm{H}^{\frac{\infty}{2}+0}(\mathcal{Z}_u, \mathbf{W}_u \otimes \mathcal{D}_t^{ch})^{\mathfrak{g}_t^+} \cong \mathrm{H}^{\frac{\infty}{2}+0}(\mathcal{Z}_u, \mathbf{W}_u \otimes (\mathcal{D}_t^{ch})^{\mathfrak{g}_t^+})$.

Proof. Let $C^{\bullet,\bullet}$ be defined by

$$C^{p,q} = \mathbf{W}_u \otimes \mathcal{D}_t^{ch} \otimes \bigwedge{}^{\frac{\infty}{2}+p}(\mathfrak{z}(\mathfrak{g}_u)) \otimes \bigwedge{}^q((\mathfrak{g}_t^+)^*) ,$$

where $\mathbf{W}_u \otimes \mathcal{D}_t^{ch} \otimes \bigwedge^{\frac{\infty}{2}+\bullet}(\mathfrak{z}(\mathfrak{g}_u)$ is the Feigin standard complex for computing $\mathrm{H}^{\frac{\infty}{2}+0}(\mathcal{Z}_u, \mathbf{W}_u \otimes \mathcal{D}_t^{ch})$ and $\mathcal{D}_t^{ch} \otimes \bigwedge^q((\mathfrak{g}_t^+)^*$ is the Chevalley–Eilenberg complex for computing the *ordinary* Lie algebra cohomology of $\mathfrak{g}_t^+$. We denote the differentials of each complex by $d_{\mathfrak{z}}$ and $d_{\mathfrak{g}}$, respectively, and extend them to $C^{\bullet\bullet}$ by letting $d_{\mathfrak{z}}$ act trivially on $\bigwedge^\bullet((\mathfrak{g}_t^+)^*$ and letting $d_{\mathfrak{g}}$ act trivially on $\mathbf{W}_u \otimes \bigwedge^{\frac{\infty}{2}+\bullet}(\mathfrak{z}(\mathfrak{g}_u))$. The two differentials anticommute and so $C^{\bullet\bullet}$ is a bicomplex from which we form the total complex $C^i_{tot} = \bigoplus_{p+q=i} C^{p,q}$ with differential $d = d_{\mathfrak{z}} + (-1)^q d_{\mathfrak{g}}$.

There are two spectral sequences converging to the total cohomology $H^\bullet_{tot}(C)$, with second pages given by

$$_IE_2^{p,q} = \mathrm{H}^{\frac{\infty}{2}+p}(\mathcal{Z}_u, \mathbf{W}_u \otimes H^q(\mathfrak{g}_t^+, \mathcal{D}_t^{ch})) ,$$
$$_{II}E_2^{p,q} = H^p(\mathfrak{g}_t^+, \mathrm{H}^{\frac{\infty}{2}+q}(\mathcal{Z}_u, \mathbf{W}_u \otimes \mathcal{D}_t^{ch})) .$$

Note that $\mathcal{D}_t^{ch}$ is injective over $U(t\mathfrak{g}_t[t])$, so $H^q(\mathfrak{g}_t^+, \mathcal{D}_t^{ch})$ is concentrated in degree zero. Furthermore, both $H^i(\mathfrak{g}_t^+, -)$ and $\mathrm{H}^{\frac{\infty}{2}+i}(\mathcal{Z}_u, \mathbf{W}_u \otimes -)$ vanish in negative degrees because $\mathbf{W}_u$ is free over $\mathcal{Z}_{u,(<0)}$ and $(-)^{\mathfrak{g}_t^+}$ is left exact. Therefore, $_IE_r^{p,q}$ collapses at the second page and $_{II}E_2^{0,0}$ is stable. This gives the isomorphism

$$_IE_2^{0,0} \cong H^0_{tot}(C) \cong {}_{II}E_2^{0,0} ,$$

as desired. ☐

Lemma 3.2.34. $\mathcal{F}(\mathcal{D}_t^{ch})^{\mathfrak{g}_t^+} = H^0_{DS}(u, \mathbf{V}_{1,1})^{\mathfrak{g}_t^+} \cong H^0_{DS}(u, \mathbf{V}_{1,1}^{\mathfrak{g}_t^+})$.

Proof. By Theorem 6.8 of [Ara18], the functors $H^0_{DS}(u, -)$ and $\mathrm{H}^{\frac{\infty}{2}+0}(\hat{\mathfrak{g}}_{u,-\kappa_g}, \mathfrak{g}_u, \mathbf{W}_u \otimes -)$

are isomorphic. Let $C^{\bullet\bullet}$ be defined by

$$C^{p,q} = \mathbf{W}_u \otimes \mathbf{V}_{1,1} \otimes \bigwedge{}^{\frac{\infty}{2}+p}(\mathfrak{g}_u) \otimes \bigwedge{}^q((\mathfrak{g}_t^+)^*) \ .$$

Let d_u be the differential on $\mathbf{W}_u \otimes \mathbf{V}_{1,1} \otimes \bigwedge{}^{\frac{\infty}{2}+\bullet}(\mathfrak{g}_u)$, which computes the relative $\hat{\mathfrak{g}}_{u,-\kappa_g}$ semi-infinite cohomology, and we extend d_u to $C^{\bullet\bullet}$ by letting it act trivially on $\bigwedge{}^q((\mathfrak{g}_t^+)^*)$. Similarly, let d_t be the differential on $\mathbf{V}_{1,1} \otimes \bigwedge{}^q((\mathfrak{g}_t^+)^*)$, which computes the ordinary Lie algebra cohomology of $\mathfrak{g}_t^+$, and we can extend this to $C^{\bullet\bullet}$ by letting it act trivially on $\mathbf{W}_u \otimes \bigwedge{}^{\frac{\infty}{2}+p}(\mathfrak{g}_u)$. The two differentials anticommute and so $C^{\bullet\bullet}$ is a bicomplex. We can form the total complex $C^i = \bigoplus_{p+q=i} C^{p,q}$ with total differential $d_{tot} = d_u + (-1)^q d_t$.

There are two spectral sequences converging to the total cohomology $H_{tot}(C)$, with second pages given by

$$\begin{aligned} {}_I E_2^{p,q} &= \mathrm{H}^{\frac{\infty}{2}+p}(\hat{\mathfrak{g}}_{u,-\kappa_g}, \mathfrak{g}_u, \mathbf{W}_u \otimes H^q(\mathfrak{g}_t^+, \mathbf{V}_{1,1})) \ , \\ {}_{II} E_2^{p,q} &= H^p(\mathfrak{g}_t^+, \mathrm{H}^{\frac{\infty}{2}+q}(\hat{\mathfrak{g}}_{u,-\kappa_g}, \mathfrak{g}_u, \mathbf{W}_u \otimes \mathbf{V}_{1,1})) \ . \end{aligned}$$

Since $\mathbf{W}_u$ is semijective in KL_u, $\mathrm{H}^{\frac{\infty}{2}+\bullet}(\hat{\mathfrak{g}}_{u,-\kappa_g}, \mathfrak{g}_u, \mathbf{W}_u \otimes -)$ is concentrated in degree zero. Therefore, both spectral sequences collapse on the second page, and we have

$${}_I E_2^{0,0} \cong H^0_{tot}(C) \cong_{II} E_2^{0,0} \ ,$$

as desired. □

Combining both of the above lemmas, we have

$$\mathcal{F}(\mathcal{D}_t^{ch})^{\mathfrak{g}_t^+} \cong \mathcal{F}((\mathcal{D}_t^{ch})^{\mathfrak{g}_t^+}) \cong \bigoplus_{\lambda \in P_t^+} \mathcal{F}(\mathbb{V}_\lambda^t \otimes V_{\lambda^*}^t) \ ,$$

where we have used the fact that

$$(\mathcal{D}_t^{ch})^{\mathfrak{g}_t^+} \cong U(\hat{\mathfrak{g}}_{t,\kappa_c}) \otimes_{U(\mathfrak{g}_t[t]) \oplus \mathbb{C}K} \mathcal{O}(G_t) \cong \bigoplus_{\lambda \in P_t^+} \mathbb{V}_\lambda^t \otimes V_{\lambda^*}^t \ .$$

The $\mathbb{V}_\lambda^t \otimes V_{\lambda^*}^t$ are naturally modules over $\mathfrak{z}_\lambda^t$ and are therefore objects in $\mathrm{KL}_{u,0}$ by Proposition 3.2.3. As a result, we have that

$$\mathcal{F}(\mathcal{D}_t^{ch})^{\mathfrak{g}_t^+} \cong \bigoplus_{\lambda \in P_t^+} \mathbb{V}_\lambda \otimes V_{\lambda^*} \ .$$

Looking at the $\Delta = 0$ subspace, $\mathcal{F}(\mathcal{D}_t^{ch})_0^{\mathfrak{g}_t^+}$, and comparing with our constraint on the character, we have that

$$\mathcal{F}(\mathcal{D}_t^{ch})_0 \cong \bigoplus_{\lambda \in P_t} V_\lambda \otimes V_{\lambda^*} \cong \mathcal{O}(G_t) \ ,$$

as $\mathfrak{g}_t \otimes \mathfrak{g}_t$ modules. Since, $\mathcal{F}(\mathcal{D}_t^{ch})$ is $\mathbb{Z}_{\geqslant 0}$-graded, the zero weight subspace $\mathcal{F}(\mathcal{D}_t^{ch})_0$ is a unital commutative, associative algebra under the normal product. The quadratic Casimir provides a $\mathbb{Q}_{\geqslant 0}$-grading,

$$\mathcal{F}(\mathcal{D}_t^{ch})_0 = \bigoplus_{d \in \mathbb{Q}_{\geqslant 0}} \mathcal{F}(\mathcal{D}_t^{ch})_0(d) \ ,$$

where $\mathcal{F}(\mathcal{D}_t^{ch})_0(d)$ has eigenvalue d with respect to the quadratic Casimir. The natural projection $\mathcal{F}(\mathcal{D}_t^{ch})_0 \twoheadrightarrow \mathcal{F}(\mathcal{D}_t^{ch})_0(0)$ is an algebra homomorphism.

These observations mean that we are exactly in the situation of the proof of Theorem 9.9 in [Ara18] and so we can apply Lemma 9.10 of *loc. cit.* to conclude that $\mathcal{F}(\mathcal{D}_t^{ch})_0$ is isomorphic to $\mathcal{O}(G_t)$ as a commutative $G_t \times G_t$ algebra. Additionally, $\mathcal{F}(\mathcal{D}_t^{ch})$ is a KL_t object and so we have an action of $V^{\kappa_c}(\mathfrak{g}_t)$. All together, this gives a nonzero homomorphism $\mathcal{D}_t^{ch} \to \mathcal{F}(\mathcal{D}_t^{ch})$, as desired.

□

3.3 Observations and future directions

Having identified the appropriate vertex algebras to associate with the twisted theories of type $\mathcal{C}_{m,n}$, there remains a question of how to understand Arakawa's $\mathbf{V}_{G_t,s}$ for non-simply laced G. We wish to speculatively suggest a physical interpretation of these vertex algebras.

Our suggestion will require a digression to three dimensions. In Section 3.3.1, we describe the mirrors of the three-dimensional circle reductions of the theories of class $\mathcal{S}$.

These mirrors are star-shaped quiver gauge theories and their Coulomb branches have been described (in the A_n case) by [BFN17]. We review the construction of these coulomb branches, *à la* Braverman–Finkelberg–Nakajima, in Section 3.3.2. After this review, we suggest a physical interpretation for the $\mathbf{V}_{G,s}$ in terms of certain quiver gauge theories.

In Section 3.3.3, we discuss the associated varieties of the mixed vertex algebras $\mathbf{V}_{m,n}$. In particular, we present a conjectural description of the associated varieties of the subfamily, with one pair of twisted punctures, $\mathbf{V}_{m,1}$, in terms of the geometric Satake correspondence.

3.3.1 Three dimensional mirrors

The Higgs branches of class $\mathcal{S}$ theories, which in the mathematical literature have come to be known as *Moore–Tachikawa varieties* following the work of [MT12], are at present most uniformly understood in terms of circle reduction to three-dimensions. Reducing a four-dimensional $\mathcal{N} = 2$ theory on a circle results in a three dimensional $\mathcal{N} = 4$ theory.

Three dimensional $\mathcal{N} = 4$ theories share the same branching structure of their moduli space of vacua as four dimensional $\mathcal{N} = 2$ theories. The two branches are also called the Higgs and Coulomb branch and both are hyperkähler (holomorphic symplectic) spaces. When a three-dimensional $\mathcal{N} = 4$ theory arises as a result of S^1-compactification of a four dimensional theory, the Higgs branches of the three-dimensional theory agrees with that of its parent four-dimensional theory.

Three dimensional physics has its own incarnation of mirror symmetry: two theories that are mirror dual are characterised[5] by the fact that the Higgs branch of one is the Coulomb branch of the other (and vice versa). Reducing a class $\mathcal{S}$ theory on the circle and then taking its mirror dual results in a three-dimensional $\mathcal{N} = 4$ quiver gauge theory. These theories are

[5] Strictly speaking, this only holds true for so-called *good* theories. These theories have Coulomb branches whose zero dimensional strata consist of a singular point. The quivers we consider will be good theories so we ignore this nuance.

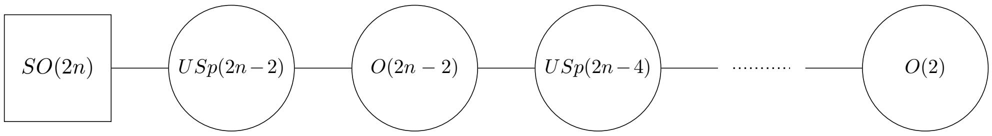

Figure 3.3: The Lagrangian for the $T[SO(2n)]$ theory that is used to introduce a maximal untwisted puncture in a theory of type $\mathfrak{d}_n$.

called the *Sicilian theories*. The three dimensional mirrors of the A_n and D_n series of class $\mathcal{S}$ theories were found by Benini–Tachikawa–Xie in [BTX10] using brane web technology. In that same work, mirror theories for D_n theories with twisted punctures were also presented. The twisted A_{2n} case has also been explored more recently, in [BGMS20].

In all of these cases, the three dimensional mirrors are quiver gauge theories—they are *star shaped quivers* with a central node from which tails radiate outwards, one for each puncture. Each type of puncture gives rise to a different tail. For example, the maximal untwisted punctures in type D_n give rise to tails matching the Lagrangian description for the theory $T[\mathrm{SO}(2n)]$ seen in Figure 3.3 [GW09]. For a genus zero theory with maximal punctures, the quiver is a central $SO(2n)$ node with $T[SO(2n)]$ tails radiating off. For genus $g > 0$, the quiver is the same but with the addition of g many hypermultiplets valued in the adjoint of SO(2n), which look like g loops starting and ending on the central $\mathrm{SO}(2n)$ node.

When there is a mixture of twisted and untwisted punctures, the central node is then replaced by O(2n − 1), and the tails for maximal twisted punctures correspond to the Lagrangian for the T[SO(2n − 1)] theory, which is displayed in Figure 3.4. Finally, in addition to the extra adjoint matter arising from positive genus, there are an extra $2s_t+2g-2$ fundamental hypermultiplets of O(2n − 1), where $2s_t$ is the number of twisted punctures. Note that at genus zero, these additional fundamental hypermultiplets only appear in the presence of *four or more twisted punctures*. In light of our discussions on residual gauge symmetries, these extra fundamental hypers might be seen as indicative of the residual gauge symmetry/derived structure that arises in the twisted setting. (An interesting special case is for the D_2 theory, where the extra fundamentals can be directly identified with extra adjoints of the twisted algebra $\mathfrak{sl}_2$.)

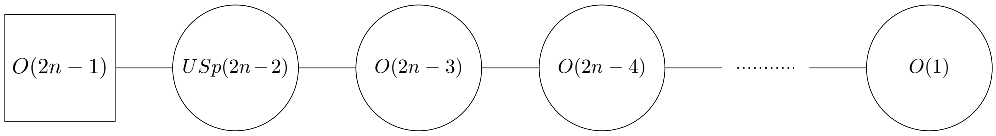

Figure 3.4: The quiver for the $T[SO(2n-1)]$ theory that is used to introduce a maximal $\mathbb{Z}_2$-twisted puncture in a theory of type $\mathfrak{d}_n$.

3.3.2 The non simply laced construction

In a series of papers [Nak16, BFN18, BFN19, BFN17], (subsets of) Braverman, Finkelberg, and Nakajima (BFN) have introduced a mathematical construction of the Coulomb and Higgs branches of three dimensional $\mathcal{N}=4$ gauge theories. Note that the construction of the Coulomb branch requires the matter to be valued in a representation of the gauge group that is of cotangent type—T^*N for some $\mathbb{C}$-representation, N, of the gauge group G. This technical assumption can be relaxed and this has been done so in [BDF$^+$22], at the cost of more complicated machinery.

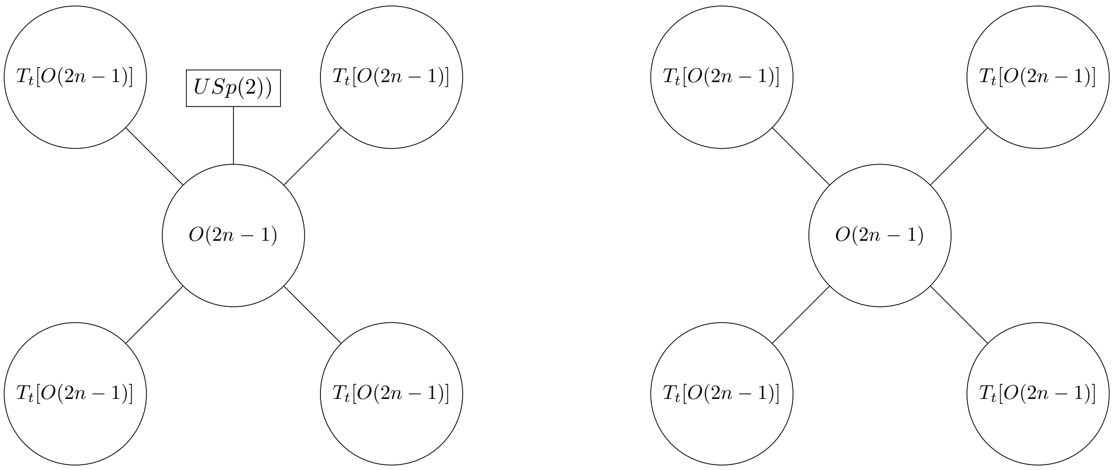

Figure 3.5: The three dimensional mirrors of the D_n theory on $\mathcal{C}_{0,0,2}$ (left) and of Arakawa's $\mathbf{V}_{\mathfrak{c}_{n-1},4}$ (right). We claim that the quiver variety of this mirror corresponds to the associated variety of $\mathbf{V}_{\mathfrak{c}_{n-1},4}$.

For a general quiver with matter in representations of cotangent type, the construction of BFN involves a vector bundle over the affine Grassmannian. For star shaped quivers in the A_n case, one has a much slicker construction via the geometric Satake correspondence [BFN17]. Let G be a simply connected simple algebraic Lie group, and let $\mathrm{Gr}_{^LG}$ be the affine Grassmannian, modelled by the quotient

$$\mathrm{Gr}_{{}^L G} = {}^L G(\mathcal{K}) / {}^L G(\mathcal{O}) \ . \tag{3.3.1}$$

The geometric Satake is an equivalence

$$\mathrm{Rep}(G) \xrightarrow{\mathbb{S}} \mathrm{Perv}_{{}^L G(\mathcal{O})}(\mathrm{Gr}_{{}^L G}) \tag{3.3.2}$$

between representations of G and ${}^L G(\mathcal{O})$-equivariant perverse sheaves on the affine Grassmannian. In particular, consider the ind-representation of G, $\mathcal{R}_G = \sum_{\lambda \in P^+} V_\lambda \otimes V_{\lambda^*}$, which by the Peter-Weyl theorem is isomorphic to the co-ordinate ring $\mathbb{C}[G]$. This corresponds to some ind-object in the category of perverse sheaves $\mathcal{A}_G \equiv \mathbb{S}(\mathcal{R}_G)$. Now, let Δ_b denote the diagonal embedding

$$\Delta_b : \mathrm{Gr}_{{}^L G} \hookrightarrow \underbrace{\mathrm{Gr}_{{}^L G} \times \cdots \times \mathrm{Gr}_{{}^L G}}_{b \text{ copies}} \ . \tag{3.3.3}$$

Define,

$$X_b \coloneqq \mathrm{Spec}\, \mathrm{H}^{\bullet}_{{}^L G(\mathcal{O})}(\mathrm{Gr}_{{}^L G(\mathcal{O})}, \Delta_b^!(\boxtimes_{k=1}^b \mathcal{A}_G)) \ . \tag{3.3.4}$$

where $\mathrm{H}^{\bullet}_{{}^L G(\mathcal{O})}$ denotes the ${}^L G(\mathcal{O})$-equivariant Borel–Moore cohomology. For $G = \mathrm{SL}_n$, [BFN17, Theorem 2.11] states that the X_b are isomorphic to the Coulomb branches of the $\mathfrak{a}_n$ Sicilian theories, *i.e.*, they are isomorphic to the Higgs branches of Σ_b. Indeed, these varieties first appeared in unpublished work of Ginzburg and Kazhdan [GK].

This construction is well-defined for any G, and so one might hope that by choosing the simply connected Lie group, $\mathrm{Exp}(\mathfrak{g})$, one can construct the corresponding Higgs branches of the Σ_b theories for any simply laced Lie algebra. This, however, remains conjectural—such an isomorphism between the Coulomb branches of star shaped quivers in non A_n-type and the varieties of form (3.3.4) has yet to be established.

Nevertheless, [Ara18, Theorem 10.14] shows that the associated variety of $\mathbf{V}_{G,b}$ is isomorphic to the X_b for any simple $\mathfrak{g}$, *i.e.*,

$$X_{V_{G,b}} \cong X_b \ . \tag{3.3.5}$$

With this context, there is a reasonably natural guess for the quivers for $X_{V_{G,b}}$ for non-simply laced G. Namely, we should think of these as the mirror quivers of the $\mathcal{C}_{0,b}$ theories *without the extra fundamental matter*—see Figure 3.5 for an illustration.

How about the vertex algebras, $\mathbf{V}_{G,b}$, themselves? In three-dimensions there is a construction that produces a vertex algebra associated to a boundary condition for an $\mathcal{N} = 4$ theory [Gai19, CCG19, CG19]. For theories that are mirror dual to the S^1-reduction of a four-dimensional $\mathcal{N} = 2$ theory, there is a natural boundary condition that recovers the four-dimensional VOA. However, the quivers without the extra fundamental matter are not SCFTs. As such, the boundary VOA construction is obstructed by an anomaly that must be ameliorated by adding in some number of free fermions. Such considerations mean that a precise conjecture for what physics to associate to the $\mathbf{V}_{G,b}$ eludes us.

As an example, consider the $\mathfrak{d}_n$ theory on $\mathcal{C}_{0,2}$. The corresponding vertex algebra has four actions of $\mathfrak{g}_t = \mathfrak{usp}(2n-2)$, and the three dimensional mirror of the theory is given on the left-hand side of Figure 3.5. We propose to identify Arakawa's VOA $\mathbf{V}_{\mathfrak{c}_{n-1},4}$ as some kind of boundary VOA for the three dimensional quiver on the right-hand side of Figure 3.5.

It would be interesting to identify some indication that the $\mathbf{V}_{G_t,s}$ VOAs are not related to four-dimensional physics. At face value they have no serious pathologies—they are conical with negative central charge and satisfy all known unitarity bounds. These vertex algebras, therefore, may warrant some attention with an eye towards characterising precisely what vertex algebras have parent four-dimensional SCFTs.

3.3.3 Moore–Tachikawa varieties for twisted class $\mathcal{S}$

We note that our construction also descends to a construction of the Moore–Tachikawa varieties of $\mathcal{C}_{m,n}$—via the associated variety functor. Indeed, the associated variety functor commutes with DS reduction [Ara15], in the sense that $X_{H^0_{DS}(V)} \cong H^0_{DS}(X_V)$ for any $V \in \mathrm{KL}$, where finite Drinfel'd–Sokolov reduction is used on the right.

For the special case of $\mathcal{C}_{m,1}$ we have a conjectural description, in terms of the geometric Satake correspondence. Let G_u be the simply connected group with Lie algebra $\mathfrak{g}_u$, then the action of σ on $\mathfrak{g}_u$ lifts to an action on G_u. The fixed points under this action of σ form a Lie group and we set

$$G_t \coloneqq {}^L(G_u^\sigma) \ , \tag{3.3.6}$$

which comes equipped with a proper immersion ${}^LG_t \hookrightarrow G_u$. This induces a proper morphism

$$\iota_\sigma : \mathrm{Gr}_{{}^LG_t} \to \mathrm{Gr}_{G_u} \ , \tag{3.3.7}$$

on the affine Grassmannians. We can pushforward along $\iota_{\sigma *}$ and apply induction to obtain a $G_u(\mathcal{O})$ equivariant sheaf on $\mathrm{Gr_{G_u}}$—denote this composition by $\bar{\iota}_{\sigma *}$.

Now, let $\mathcal{A}_t$ be the regular sheaf on $\mathrm{Gr}_{{}^LG_t}$ and let $\mathcal{A}_u$ be the regular sheaf on Gr_{G_u}. Let $\Delta_t : \mathrm{Gr}_{{}^LG_t} \to \mathrm{Gr}_{{}^LG_t} \times \mathrm{Gr}_{{}^LG_t}$ be the diagonal map and similarly let $\Delta_b : \mathrm{Gr}_{G_u} \to (\mathrm{Gr}_{G_u})^b$ be the diagonal map for the untwisted case.

The sheaf $\bar{\iota}_{\sigma *}\Delta^!(\mathcal{A}_t \boxtimes \mathcal{A}_t)$ is a $G_u(\mathcal{O})$ equivariant sheaf on Gr_{G_u}. We define,

$$X_{b,1} \coloneqq \mathrm{mSpec}\ H^\bullet_{G_u(\mathcal{O})}\left(\mathrm{Gr_{G_u}}, \Delta^!_{\mathrm{b+1}}\big((\boxtimes^{\mathrm{m}}_{\mathrm{k=1}}\mathcal{A}_\mathrm{u}) \boxtimes \bar{\iota}_{\sigma^*}\Delta^!_\mathrm{t}(\mathcal{A}_\mathrm{t} \boxtimes \mathcal{A}_\mathrm{t})\big)\right) \ . \tag{3.3.8}$$

In ongoing work with D. Butson [BN], we hope to prove the following conjectural description of the associated varieties of the $\mathbf{V}_{m,1}$.

Conjecture 3.3.1. *The associated variety of $\mathbf{V}_{m,1}$ is isomorphic to $X_{b,1}$ as Poisson varieties, i.e.*

$$R_{\mathbf{V}_{m,1}} \cong H^\bullet_{G_u(\mathcal{O})}\left(\mathrm{Gr_{G_u}}, \Delta^!_{\mathrm{b+1}}\big((\boxtimes^{\mathrm{m}}_{\mathrm{k=1}}\mathcal{A}_\mathrm{u}) \boxtimes \bar{\iota}_{\sigma^*}\Delta^!_\mathrm{t}(\mathcal{A}_\mathrm{t} \boxtimes \mathcal{A}_\mathrm{t})\big)\right) \ ,$$

as Poisson algebras, where $R_{\mathbf{V}_{m,1}}$ is the Zhu's C_2 algebra of $\mathbf{V}_{m,1}$.

For the most general $\mathcal{C}_{m,n}$, this small extension cannot capture the extra information present in the Hall–Littlewood operators and so one would require a different approach.

Appendices

Besides, proofs are no help to believing,
especially material proofs

Fyodor Dostoevsky
The Brothers Karamazov

A.1 Nilpotent orbits in Lie algebras

This appendix will be a review of some concepts in Lie theory, with particular focus on the structure of nilpotent orbits.

Throughout this section, $\mathfrak{g}$ shall refer to a finite dimensional, simple Lie algebra, over $\mathbb{C}$. We let G denote a simple algebraic group whose Lie algebra is $\mathfrak{g}$—note that we relax our convention that G is simply connected.

Let $\mathcal{R}$ be the root system of $\mathfrak{g}$, with $\mathcal{R}_\pm$ the positive and negative roots and Δ the set of simple roots. We fix, once and for all, some Cartan decomposition $\mathfrak{g} = \mathfrak{n}_- \oplus \mathfrak{h} \oplus \mathfrak{n}$, with $\mathfrak{b} = \mathfrak{h} \oplus \mathfrak{n}$ the upper Borel subalgebra and $\mathfrak{b}_- = \mathfrak{h} \oplus \mathfrak{n}$ the lower Borel subalgebra. The Killing form on $\mathfrak{g}$ is denoted by $(\cdot,\cdot)$.

Let $\mathfrak{g}^* := \mathrm{Hom}(\mathfrak{g},\mathbb{C})$ be the linear dual of $\mathfrak{g}$. We fix an isomorphism $\mathfrak{g} \xrightarrow{\sim} \mathfrak{g}^*$ by $x \mapsto (x,\cdot)$. The adjoint action of $x \in \mathfrak{g}$ will be denoted by ad_x while the coadjoint action on $\mathfrak{g}^*$ is denoted by ad^*_x. Similarly, we denote the adjoint action of $g \in G$ on $\mathfrak{g}$ by $\mathrm{A}d_g$ and on $\mathfrak{g}^*$ by Ad^*_g.

A.1.1 Poisson structure of $\mathfrak{g}^*$ and coadjoint orbits

The dual space $\mathfrak{g}^*$ is naturally an affine variety with $\mathfrak{g}^* = \operatorname{Spec}\operatorname{Sym}(\mathfrak{g})$. The co-ordinate ring $\operatorname{Sym}(\mathfrak{g})$ is naturally a Poisson algebra, with Poisson bracket defined by

$$\{x, y\} = [x, y], \quad \text{for } x, y \in \mathfrak{g} \subset \operatorname{Sym}(\mathfrak{g}) \tag{A.1.1}$$

and extended by the Leibniz rule to arbitrary polynomials. This Poisson bracket is known as the Kirillov–Kostant–Souriau (KKS) Poisson bracket. The variety $\mathfrak{g}^*$ is therefore a Poisson variety, however it is not symplectic. This is somewhat obvious since $\dim \mathfrak{g}^*$ can be odd.

Moreover, note that $\operatorname{Sym}(\mathfrak{g})$ contains a large Poisson centre; the subalgebra

$$\operatorname{Sym}(\mathfrak{g})^G \subset \operatorname{Sym}(\mathfrak{g}) \ , \tag{A.1.2}$$

Poisson commutes with all of $\operatorname{Sym}(\mathfrak{g})$, since it is ad-invariant. The putative symplectic form induced by the KKS bracket is, therefore, degenerate.

Coadjoint orbits in $\mathfrak{g}^*$, however, are symplectic. Let $\xi \in \mathfrak{g}^*$ with $\xi = (x, \cdot)$ for $x \in \mathfrak{g}$ and let $\mathbb{O}_\xi$ be the (open) G-orbit of ξ in $\mathfrak{g}^*$. We can use the Killing isomorphism to identify the fibre of $T\mathbb{O}_\xi$ above ξ with a quotient of $\mathfrak{g}$

$$0 \to \mathfrak{g}_x \to \mathfrak{g} \xrightarrow{\operatorname{ad}_x} T_\xi \mathbb{O}_\xi \to 0 \ , \tag{A.1.3}$$

where $\mathfrak{g}_x$, the kernel of ad_x is equal to $\mathfrak{g}_\xi$, the isotropy subalgebra of ξ. The KKS bracket induces a two-form $\omega_{KKS,\xi} : T_\xi\mathbb{O}_\xi \wedge T_\xi\mathbb{O}_\xi \to \mathbb{C}$. on $T_\xi\mathbb{O}_\xi$. Let $x, y \in T_\xi\mathbb{O}_\xi$ with representatives $\widetilde{x}, \widetilde{y} \in \mathfrak{g}$, then

$$\omega_{KKS,\xi}(x, y) = (\xi, [\widetilde{x}, \widetilde{y}]) \ , \tag{A.1.4}$$

is a non-degenerate two form on the tangent space at ξ. This extends to a closed two-form ω_{KKS} on $\mathbb{O}_\xi$ and is non-degenerate on each fibre—hence it is a symplectic form.

Since $\mathfrak{g}^*$ has infinitely many G-orbits, there are infinitely many symplectic leaves with dimension $\dim\mathfrak{g} - \operatorname{rk}\mathfrak{g}$. Furthermore, there is a singular locus inside $\mathfrak{g}^*$, where coadjoint orbits have codimension greater than $\operatorname{rk}\mathfrak{g}$.

Almost by construction, the coadjoint action of G on any orbit $\mathbb{O}_\xi$, $\xi \in \mathfrak{g}^*$ is Hamiltonian with moment map $\mu_\xi : \mathbb{O}_\xi \to \mathfrak{g}^*$ given by the natural immersion of the orbit into $\mathfrak{g}^*$.

A.1.2 The Harish Chandra centre and the BGG category

Recall that the Poisson algebra of functions on $\mathfrak{g}^*$ had a Poisson central subalgebra $\mathrm{Sym}(\mathfrak{g})^G$. As an $\mathbb{C}$-algebra $\mathrm{Sym}(\mathfrak{g})^G$ is generated by monomials P_{d_i} of degrees d_i+1, where d_i are the exponents of $\mathfrak{g}$ for $i = 1, \dots, \operatorname{rk}\mathfrak{g}$. These generators are called the *fundamental invariants* of $\mathfrak{g}$ and the set of d_{i+1} are the *degrees of the fundamental invariants.* For any $\mathfrak{g}$, P_1 is the lowest degree generator with

$$P_1 = \sum_{a,b} \kappa^{ab} J_a J_b \ , \tag{A.1.5}$$

where J_a is a basis of $\mathfrak{g}$ and κ^{ab} is the inverse of the Killing form $\kappa_{ab} = (J_a, J_b)$.

Moreover, by looking at the G-orbits of the Cartan subalgebra $\mathfrak{h}^*$, we have that

$$\mathbb{C}[\mathfrak{g}^* /\!\!/ G] \cong \mathbb{C}[\mathfrak{h}^* /\!\!/ W] \ . \tag{A.1.6}$$

Definition A.1.1. The universal enveloping algebra, $U(\mathfrak{g})$, has a centre $Z(\mathfrak{g})$ called the *Harish-Chandra* centre. Moreover,

$$Z(\mathfrak{g}) \cong \mathrm{Sym}(\mathfrak{g})^G \cong \mathbb{C}[\mathfrak{h}^* /\!\!/ W] \cong \mathbb{C}[P_{d_i} \,|\, i = 1, \dots, \operatorname{rk}\mathfrak{g}] \ . \tag{A.1.7}$$

This centre plays an important role in the representation theory of $\mathfrak{g}$. By Schur's lemma, $Z(\mathfrak{g})$ must act by scalar multiplication on any finite dimensional, highest-weight module V_λ, for $\lambda \in P^+$. So we define the central character, $\chi_\lambda : Z(\mathfrak{g}) \to \mathbb{C}$ by $z \cdot v = \chi_\lambda(z) v$ for any $v \in V_\lambda$.

Definition A.1.2. Let $\mathcal{O}_\mathfrak{g}$ be the Bernstein–Gelfand–Gelfand subcategory of $\mathfrak{g}$-mod whose objects are modules M such that

- M is finitely generated over $\mathfrak{g}$
- $\mathfrak{h}$ is diagonal on M
- M is locally $\mathfrak{n}$-finite, *i.e.*, for any $m \in M$, $U(\mathfrak{n}) \cdot m$ is finite dimensional.

This category has a block decomposition

$$\mathcal{O}_{\mathfrak{g}} = \bigoplus_{\lambda \in P^+} \mathcal{O}_{\mathfrak{g},[\lambda]} \ , \tag{A.1.8}$$

where $\mathcal{O}_{\mathfrak{g},[\lambda]}$ is the subcategory where $Z(\mathfrak{g})$ acts as the *generalised* eigenvalue χ_λ. Equivalently, these can be thought of as the subcategory of modules in $\mathcal{O}_{\mathfrak{g}}$ that are supported *set-theoretically* at, *i.e.*, in a formal neighbourhood of, the point $(z - \chi_\lambda(z))$ in $\operatorname{Spec} Z(\mathfrak{g})$.

When we introduce the affine analogue of $\mathcal{O}_{\mathfrak{g}}$, the Kazhdan–Lusztig category, KL, we shall observe a similar decomposition.

A.1.3 Nilpotent orbits in $\mathfrak{g}^*$

An element $x \in \mathfrak{g}$ is called nilpotent if $\mathrm{ad}_x : \mathfrak{g} \to \mathfrak{g}$ is nilpotent. For the case of $x \in \mathfrak{sl}_n \subset \mathfrak{gl}_n$ this agrees with the usual notion of a nilpotent matrix in $\mathfrak{gl}_n$.

We prefer to think of nilpotent elements as living in $\mathfrak{g}^*$. To that end we give three equivalent definitions of nilpotent elements in $\mathfrak{g}^*$, following [CG97].

Definition A.1.3. An element $\xi \in \mathfrak{g}^*$ is nilpotent if any of the following equivalent conditions are met

- $\xi = (x, \cdot)$ for some nilpotent element $x \in \mathfrak{g}$
- $P(\xi) = 0$ for any polynomial $P \in \mathrm{Sym}(\mathfrak{g})^G$ with no constant term
- $\xi(\mathfrak{g}_\xi) = 0$ where $\mathfrak{g}_\xi = \{x \in \mathfrak{g} \,|\, \mathrm{ad}^*_x(\xi) = 0\}$ is the isotropy subalgebra

The set of nilpotent elements $\mathcal{N} \subset \mathfrak{g}^*$ is called the nilpotent cone, or nilcone for short. Condition two in the above definition tells us that $\mathcal{N}$ is an algebraic subvariety of $\mathfrak{g}^*$, with ideal of definition generated by the fundamental invariants P_{d_i} for $i = 1, \ldots, \mathrm{rk}\,\mathfrak{g}$. Moreover, it is stable under dilatation by $\mathbb{C}^*$, *i.e.*, the nilpotent cone is actually a cone. The nilcone has codimension $\operatorname{rk}\mathfrak{g}$ inside $\mathfrak{g}^*$.

Condition one tells us that nilpotent elements in $\mathfrak{g}^*$ are equivalent to nilpotent elements in $\mathfrak{g}$ and so we shall frequently use the Killing isomorphism to think of the nilcone as embedded in $\mathfrak{g}^*$ or $\mathfrak{g}$ as is convenient.

It follows from the definition that $\mathcal{N}$ is stable under the coadjoint action of G, and so we define a nilpotent orbit to be the coadjoint orbit of some $\xi \in \mathcal{N}$.

Theorem A.1.4 (Jacobson–Morozov)**.** *Every nilpotent element $f \in \mathfrak{g}$ may be completed to an $\mathfrak{sl}_2$ triple (e, h, f), i.e., there exist elements $h, e, \in \mathfrak{g}$, such that*

$$[h, e] = 2e, \quad [h, f] = -2f, \quad [e, f] = h \ , \tag{A.1.9}$$

with h semisimple and e nilpotent. Thus, nilpotent orbits in $\mathfrak{g}$ are in one-to-one correspondence with Lie algebra morphism $\mathfrak{sl}_2 \to \mathfrak{g}$, up to G-conjugation.

The theorem of Jacobson–Morozov allows us to define some auxiliary data associated to a nilpotent orbit. Suppose $\xi \in \mathcal{N}$ is of the form $(f, \cdot)$ for a nilpotent $f \in \mathfrak{g}$. We may complete this to an $\mathfrak{sl}_2$ triple e, h, f where h is diagonal on $\mathfrak{g}$. The operator ad^*_h defines an integral grading on $\mathfrak{g}$ with $\mathfrak{g} = \bigoplus_i \mathfrak{g}_i$. We define a parabolic subalgebra $\mathfrak{b}_\xi \subset \mathfrak{g}$ and its nilpotent radical $\mathfrak{n}_\xi \subset \mathfrak{g}$ by

$$\mathfrak{b}_\xi := \bigoplus_{i \geq 0} \mathfrak{g}_i \ , \quad \mathfrak{n}_\xi := \bigoplus_{i > 0} \mathfrak{g}_i \ . \tag{A.1.10}$$

The Levi subalgebra exponentiates to a unipotent Lie group $N_\xi \subset G$ which stabilises ξ and acts on $\mathfrak{b}_\xi$—in fact we shall show that this action is free if $\xi \neq 0$. Note that these subalgebras are not unique and they depend on the choice of completion of ξ to an $\mathfrak{sl}_2$-triple.

Proposition A.1.5. *The nilcone $\mathcal{N}$ is a stratified algebraic variety*

$$\mathcal{N} = \bigsqcup_\rho \mathbb{O}_\rho \ , \tag{A.1.11}$$

with finitely many strata, corresponding to disjoint coadjoint orbits [Dix96].

The largest stratum is the principal orbit and has dimension $\dim \mathfrak{g} - \mathrm{rk}\, \mathfrak{g}$. *This corresponds to elements in* $\mathcal{N}$ *whose* G*-stabiliser is of dimension* $\mathrm{rk}\, \mathfrak{g}$. *The principal orbit is Zariski-open inside* $\mathcal{N}$.

Furthermore, there is always a stratum of codimension 2 inside $\mathcal{N}$ *called the subregular orbit. The smallest, non-trivial orbit is called the minimal orbit.*

The strata of $\mathcal{N}$ *have a natural partial ordering where* $\rho' \leq \rho$ *if the closure of* $\mathbb{O}_\rho$ *in* $\mathcal{N}$ *contains* $\mathbb{O}_{\rho'}$.

Remark A.1.6. The principal orbit always contains the element $p^*_{-1} = \sum_{\alpha \in \Delta} e^*_{-\alpha}$, where $e_{-\alpha}$ are the Chevalley generators conjugate to the negative simple roots and * is the image under the Killing isomorphism. The element $p_{-1} \in \mathfrak{g}$ can be completed to a number of $\mathfrak{sl}_2$ triples, let us choose a special triple. Let $\check{\rho} = \sum_i^{\mathrm{rk}\, \mathfrak{g}} \omega_i$ be the sum of the fundamental coweights of $\mathfrak{g}$. This defines an element of $\mathfrak{h}$ by

$$p_0 = 2 \sum_{i-1}^{\mathrm{rk}\, \mathfrak{g}} \omega_i(\alpha_i) h_i \ , \tag{A.1.12}$$

where h_i is a basis of $\mathfrak{h}$ and α_i is the root $(h_i, \cdot)$. Note that $[p_0, p_{-1}] = -2p_{-1}$. There is a unique $p_1 \in \mathfrak{g}$ such that (p_1, p_0, p_{-1}) is an $\mathfrak{sl}_2$ triple. We call this distinguished $\mathfrak{sl}_2$ triple the *canonical principal* $\mathfrak{sl}_2$ triple.

The canonical principal triple is a very nice choice of triple in that $\mathfrak{b}_{p_{-1}} = \mathfrak{b}$ and $\mathfrak{n}_{p_{-1}} = \mathfrak{n}$, and of course $N_{p_{-1}} = N$.

For $\mathfrak{g} = \mathfrak{sl}_n$, every nilpotent element is conjugate to a matrix of nilpotent Jordan blocks. A nilpotent Jordan block of size k is a $k \times k$ whose $k-1$ many super-diagonal terms are all equal to one. For example a nilpotent Jordan block of size 3 is a submatrix of form

$$\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} . \tag{A.1.13}$$

Choosing a matrix of nilpotent Jordan blocks in $\mathfrak{sl}_n$ amounts to choosing positive integers $\lambda_1, \ldots, \lambda_k$, specifying the sizes of each block and with $\sum_i \lambda_i = n$.

Remark A.1.7. Nilpotent orbits in $\mathfrak{sl}_n$ are in one-to-one correspondence with partitions of n. Furthermore, the partial order on the strata of $\mathcal{N}$ is precisely the dominance ordering on partitions.

Example A.1.8. Suppose $\mathfrak{g} = \mathfrak{sl}_2$, which we realise as the space of matrices of form

$$\begin{pmatrix} h & e \\ f & -h \end{pmatrix} . \tag{A.1.14}$$

Nilpotency for an $\mathfrak{sl}_n$ element is equivalent to being a nilpotent matrix. The nilcone of $\mathfrak{sl}_2^*$ can therefore be identified with the space of singular matrices satisfying

$$\det \begin{pmatrix} h & e \\ f & -h \end{pmatrix} = -h^2 - ef \overset{!}{=} 0 , \tag{A.1.15}$$

where h^2+ef is the degree two fundamental invariant of $\mathfrak{sl}_2$. We note that $\mathbb{C}[h,e,f]/(h^2+ef)$ is the co-ordinate ring of the A_1-singularity $\mathbb{C}^2/(\mathbb{Z}/2)$, and so $\mathcal{N}_{\mathfrak{sl}_2} \cong \mathbb{C}^2/\mathbb{Z}/2$ as algebraic varieties—and, indeed, as symplectic singularities.

There are two nilpotent orbits inside $\mathcal{N}_{\mathfrak{sl}_2}$: the trivial orbit of 0, and the principal orbit containing $\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$. These correspond to the partitions $[1,1]$ and $[2]$, respectively. In terms of $\mathfrak{sl}_2$ triples, these correspond to the trivial triple $(0,0,0)$ and the principal triple (h,e,f).

Example A.1.9. Suppose $\mathfrak{g} = \mathfrak{sl}_3$, we expect $\mathcal{N}_{\mathfrak{sl}_3}$ to have three strata—corresponding to the three partitions of 3, *i.e.*,

$$\mathcal{N}_{\mathfrak{sl}_3} = \mathbb{O}_{[1,1,1]} \sqcup \mathbb{O}_{[2,1]} \sqcup \mathbb{O}_{[3]} . \tag{A.1.16}$$

We parameterise $\mathfrak{sl}_3$ as the space of matrices

$$\begin{pmatrix} h_1 & e_1 & e_3 \\ f_1 & h_2 - h_1 & e_2 \\ f_3 & f_2 & -h_2 \end{pmatrix} . \tag{A.1.17}$$

The full nilcone is the vanishing locus of the fundamental invariants,

$$\begin{aligned} P_1 &= e_1 f_1 + e_2 f_2 + e_3 f_3 + h_1^2 - h_2 h_1 + h_2^2 \ , \\ P_2 &= e_3(f_3(h_1 - h_2) + f_1 f_2) + e_1(e_2 f_3 + f_1 h_2) + h_1((h_1 - h_2)h_2 - e_2 f_2) \ . \end{aligned} \tag{A.1.18}$$

The smallest stratum, $\mathbb{O}_{[1,1,1]}$ is the trivial orbit of $0 \in \mathfrak{sl}_3$. The *minimal* orbit, $\mathbb{O}_{[2,1]}$, is the SL_3-conjugacy class of

$$\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \ , \tag{A.1.19}$$

and its closure can be presented as the vanishing locus of

$$h_2 e_1 - e_3 f_2 \ , \quad f_1 e_1^2 + e_3 f_2 h_1 + e_1 h_1^2 \ , \quad f_3 e_1 - f_2 h_1, \quad e_2 e_1^2 + e_3^2 f_2 + e_1 e_3 h_1 \ . \tag{A.1.20}$$

Note that for $\mathfrak{sl}_3$, the minimal orbit is the same as the subregular orbit. The *principal* orbit, $\mathbb{O}_{[3]}$, is the SL_3-conjugacy class of

$$\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} \ , \tag{A.1.21}$$

and its closure is the vanishing locus of P_1 and P_2, *i.e.*, the full nilcone.

A.1.4 Slices to nilpotent orbits

The ideal of definition of $\mathcal{N}$ is a Poisson ideal, since the generators of this ideal are Poisson-central. Therefore, the Kostant–Kirillov–Souriau bracket restricts to a Poisson bracket on $\mathcal{N}$ making it a Poisson subvariety of $\mathfrak{g}^*$. The KKS form, restricted to coadjoint orbits is non-degenerate and so the strata of $\mathcal{N}$ are symplectic. Therefore, $\mathcal{N}$ is a stratified symplectic singularity. In many ways, nilcones of a simple Lie algebra are the prototypical examples of symplectic singularities.

Definition A.1.10. Given a coadjoint orbit $\mathbb{O}_\xi$, containing $\xi \in \mathfrak{g}^*$, a *transverse slice* to $\mathbb{O}_\xi$ at ξ is a closed subvariety $S_\xi \subset \mathfrak{g}^*$ containing ξ such that $\mathbb{O}_\xi$ and S intersect only once at ξ and they intersect transversally, *i.e.*, $T_\xi \mathfrak{g}^* = T_\xi \mathbb{O}_\xi \oplus T_\xi S$.

Given two nilpotent orbits $\mathbb{O}_\xi \leq \mathbb{O}_{\xi'}$, we denote the intersection $S_\xi \cap \mathbb{O}_{\xi'}$ by $S_\xi^{\xi'}$.

There are a number of ways to construct a transverse slice to a nilpotent orbit $\mathbb{O}_\xi$, and (except for the trivial orbit where the slice is $\mathfrak{g}^*$) such a slice is not unique. However, there is a particularly nice slice that one can define called the *Slodowy slice*.

Definition A.1.11. Let $\xi \in \mathcal{N}$ with $\xi = (f, \cdot)$ for some nilpotent $x \in \mathfrak{g}$ and let $\mathbb{O}_\xi$ be it's coadjoint orbit. By Jacobson–Morozov, we can complete f to an $\mathfrak{sl}_2$ triple (e, h, f) where h is diagonal on $\mathfrak{g}^*$ and e is nilpotent. The Slodowy slice is

$$S_\xi = \xi + (\ker \mathrm{ad}_e)^* \ , \tag{A.1.22}$$

where $(\ker \mathrm{ad}_e)^*$ is the subspace $\{(a, \cdot) \in \mathfrak{g}^* \mid a \in \ker \mathrm{ad}_e\}$.

Unless explicitly stated otherwise, the notation S_ξ shall, henceforth, always refer to the Slodowy slice

Proposition A.1.12 (Gan–Ginzburg)**.** *Let $\xi \in \mathfrak{g}^* \backslash \{0\}$ be nilpotent, and let $\mathbb{O}_\xi$ be its coadjoint orbit. Recall the parabolic subalgebra $\mathfrak{b}_\xi$ and $\mathfrak{n}_\xi$ and denote their images in $\mathfrak{g}^*$ under the Killing isomorphism by $\mathfrak{b}_\xi^*$ and $\mathfrak{n}_\xi^*$ respectively. Let S_ξ be the Slodowy slice, then we have an isomorphism*

$$\begin{gathered} N_\xi \times S_\xi \xrightarrow{\sim} \xi + \mathfrak{b}_\xi^* \\ (n, s) \mapsto \mathrm{Ad}_n^* s \end{gathered} \tag{A.1.23}$$

As a consequence, $\xi + \mathfrak{b}_\xi^*$ is an N_ξ-torsor over S_ξ.

To finish, we focus in on the Slodowy slice of the principal orbit. Recall that there is a canonical $\mathfrak{sl}_2$ triple, (p_1, p_0, p_{-1}) corresponding to the principal orbit. Denote the kernel of p_1 by $\mathfrak{s}$. It $\mathrm{rk}\,\mathfrak{g}$ dimensional and has a grading under ad_{p_0} with isotypic components

$$\mathfrak{s} = \bigoplus_{i=1}^{\mathrm{rk}\mathfrak{g}} \mathfrak{s}_{d_i} \tag{A.1.24}$$

where $\{d_i\}_{i=1}^{\mathrm{rk}\,\mathfrak{g}}$ are the exponents of $\mathfrak{g}$. The dimensions of each graded component is equal to the multiplicity of d_i in $\mathfrak{g}$. The subspace $\mathfrak{s}_1$, always exists for any $\mathfrak{g}$, and is spanned by p_1.

Remark A.1.13. For $\mathfrak{g}$ not of D-type, the multiplicity of d_i is always one so we pick a basis $(p_{d_i})_{i=1}^{\mathrm{rk}\,\mathfrak{g}}$ for V^{can} with p_{d_i} spanning $V^{\mathrm{can}}_{d_i}$. In the case where $\mathfrak{g} = \mathfrak{d}_n$, then the exponent $d_j = 2n$ has multiplicity two and in this case we have to choose two linearly independent vectors p_{2n} and $\widetilde{p}_{2n}$.

Definition A.1.14. The *canonical* principal Slodowy slice is a transverse slice to the principal orbit at p^*_{-1} defined as

$$S_{\mathrm{prin}} \coloneqq p^*_{-1} + (\ker \mathrm{ad}_{p_1})^* \ . \tag{A.1.25}$$

The Gan–Ginzburg isomorphism tells us that

$$N \times S_{\mathrm{prin}} \cong p^*_{-1} + \mathfrak{b}^* \ . \tag{A.1.26}$$

Furthermore, any slice to the principal orbit is special in that any regular coadjoint orbit intersects the slice precisely once. Here by regular, we mean that the orbit has dimension $\dim\mathfrak{g} - \mathrm{rk}\,\mathfrak{g}$. In other words, S_{prin} is a global slice to the coadjoint action of G on $\mathfrak{g}^*$, giving an isomorphism

$$\mathfrak{g}^*//G \cong S_{\mathrm{prin}} \cong \mathfrak{h}//W \ . \tag{A.1.27}$$

A.2 Vertex algebras

We shall discuss vertex algebras in some generality and introduce some of the techniques and terminology used in [Ara18].

A.2.1 Preliminaries

There are many, equivalent, definitions of a vertex algebra. We shall follow the definition by Frenkel–Ben-Zvi—as formulated in [FBZ04].

Definition A.2.1. A vertex algebra over $\mathbb{C}$ is the collection $(V, Y, \partial, |0\rangle)$ where:

- V is a vector space over $\mathbb{C}$, i.e. the *space of states*
- $|0\rangle \in V$ is a distinguished vector called the *vacuum vector*
- $\partial : V \to V$, is the *translation operator*
- $Y(\cdot, z) : V \to \mathrm{End}(V)[\![z, z^{-1}]\!]$, is the *vertex operator*

Satisfying

- $Y(|0\rangle, z) = id_V$ and $Y(a, z)|0\rangle \in V[[z]]$, the *vacuum* axiom
- $\forall a \in V$, $[\partial, Y(a, z)] = \partial_z Y(a, z)$, the *translation* axiom
- $T|0\rangle = 0$
- $\forall a, b \in V$, $\exists N \in N(z - w)^N [Y(a, z), Y(b, w)] = 0$, the *locality* axiom

Remark A.2.2. Let $a \in V$, then

$$a(z) \coloneqq Y(a, z) = \sum_{n \in \mathbb{Z}} a_{(n)} z^{-n-1}, \tag{A.2.1}$$

with each $a_{(n)} \in \mathrm{End}(V)$. We can therefore think of the data of Y as a family of noncommutative, nonassociative products $\mu_n : V \otimes V \to V$ indexed by $n \in \mathbb{Z}$ with $\mu_n(a, b) = a_{(n)} b$ for any $a, b \in V$.

Note that in physics literature, one conventionally labels the Fourier modes as

$$a(z) = \sum_{n \in \mathbb{Z}} a_n z^{-n-\Delta-1} \ , \tag{A.2.2}$$

where Δ is the conformal weight of a (to be defined later). We distinguish between these two conventions by including brackets for the grading of (A.2.1).

A distribution $a(z \in \mathrm{End}(V)[\![z, z^{-1}]\!])$ is called a field if for any $v \in V$,

$$a(z)v \in V(\!(z)\!) \ . \tag{A.2.3}$$

The subspace $\mathrm{End}(V)(\!(z)\!) \subset \mathrm{End}(V)[\![z, z^{-1}]\!]$ is often called the space of *fields*. For any state $a \in V$, the axioms force $Y(a,z)v$ to be bounded below in powers of z. The vertex operator $Y(\cdot, z)$ is an algebraic manifestation of the *state-field correspondence* of conformal field theory.

Vertex algebras have a normally ordered product on states and fields, given by

$$ab \coloneqq a_{(-1)}b \quad \text{or on fields as} (ab)(z) :\coloneqq (a_{(-1)}b)(z) \ . \tag{A.2.4}$$

This product is neither associative not commutative so we adopt conventions of nesting from the right, *i.e.*,

$$abcd \equiv a(b(cd)) = a_{(-1)}b_{(-1)}c_{(-1)}d \ . \tag{A.2.5}$$

A few words on locality

The locality axiom is subtler than it looks. It does not, for example imply that all vertex operators commute.

Given two fields $a(z)$ and $b(w)$, their products $a(z)b(w)$ and $b(w)a(z)$ are power series in the space $\mathrm{End}(V)[\![z, z^{-1}, w, w^{-1}]\!]$. Picking a test vector $v \in V$ and a linear functional $\phi : V \to \mathbb{C}$, we can construct two power series, $\phi(a(z)b(w)v))$ and $\phi(b(w)a(z)v)$ in the spaces $\mathbb{C}(\!(z)\!)(\!(w)\!)$ and $\mathbb{C}(\!(w)\!)(\!(z)\!)$.

The two spaces are not the same; the first has bounded below powers of w but powers of z are not uniformly bounded below and the second has bounded below powers of z but powers of w are not uniformly bounded below. Their intersection, is the space $\mathbb{C}[\![z, w]\!][z^{-1}, w^{-1}$, in which powers of z^{-1} and w^{-1} are uniformly bounded. Given a rational function in the fraction field, $\mathbb{C}(\!(z, w)\!)$, of $\mathbb{C}[\![z, w]\!][z^{-1}, w^{-1}$, we can expand in the region $|z| > |w|$ by expanding in positive powers of z/w. This results in a power series expansion in $\mathbb{C}(\!(z)\!)(\!(w)\!)$. Similarly, by expanding around $|w| > |z|$, *i.e.*, in positive powers of w/z, we have a power series in $\mathbb{C}(\!(w)\!)(\!(z)\!)$.

The locality axiom implies that the two power series, $\phi(a(z)b(w)v)$ and $\phi(b(w)a(z)v)$, are expansions of the same function in $\mathbb{C}(\!(z,w)\!)$ but in the domains $|z| > |w|$ and $|w| > |z|$ respectively.

A consequence of the locality axiom, is that the vertex operator $Y(-,z)$ is associative, in the sense that if we have two fields $Y(a,z)$ and $Y(b,w)$, the product $Y(a,z)Y(b,w)$ is equal to the expansion $Y(Y(a,z-w)b,w)$.

Definition A.2.3. Given two fields $a(z)$ and $b(w)$ we can define the OPE (Operator-Product-Expansion)

$$a(z)b(w) = \sum_{n\in\mathbb{Z}} \frac{1}{(z-w)^{n+1}}(a_{(n)}b)(w), \tag{A.2.6}$$

where the equality should be understood as saying that the two sides represent the expansions of the same ``function'' in two different domains—we leave the subtleties to Section 3.3 of [FBZ04].

Given two fields $a(z), b(w)$, we shall often write the OPE as

$$a(z)b(w) \sim \sum_{n\geq 0} \frac{1}{(z-w)^{n+1}}(a_{(n)}b)(w)\ , \tag{A.2.7}$$

and suppress all terms that are regular in the limit $z \to w$.

A simple way of satisfying the axiom of locality is if the image of $Y(\cdot,z)$ lives in End$V[\![z]\!]$, or in other words, the modes $a_(n)$ vanish for all $n \geq 0$ for any state a. In such a vertex algebra the products $a(z)b(w)$ and $b(w)a(z)$ are both in End$[\![z,w]\!]$ and $(z-w)^N$ has no zero divisors in $\mathbb{C}[\![z,w]\!]$ for any $N \in \mathbb{N}$. Thus locality enforces that $[a(z), b(w)] = 0$.

Definition A.2.4. A vertex algebra is called commutative, if all fields commute or, equivalently by the preceding discussion, if the image of $Y(\cdot,z)$ is contained in End$V[\![z]\!]$.

Remark A.2.5. The normal ordered product is commutative and associative in a commutative vertex algebra. Therefore the data of a commutative vertex algebra is equivalent to that of a unital associative commutative $\mathbb{C}$-algebra with a derivation.

To finish, we generalise our definitions to a vertex superalgebra.

Definition A.2.6. A vertex superalgebra is a collection $(V, |0\rangle, \partial, Y)$, where

- $V = V_0 \oplus V_1$, is a superspace, *i.e.*, a $\mathbb{Z}/2\mathbb{Z}$-graded vector space
- $|0\rangle \in V_0$, is the vacuum
- $\partial : V \to V$ is a linear map with even parity
- $Y(\cdot, z) : V \to \mathrm{End}(V)[\![z, z^{-1}]\!]$ such that if $a \in V$ has parity $|a|$ then $Y(a, z)$ has parity $|a|$, *i.e.*, for all $n \in \mathbb{Z}$. the modes $a_{(n)}$ have parity $|a|$

satisfying all the axioms of a vertex algebra except for locality. Instead we require that for all $a, b \in V$ there exists some $N \in \mathbb{N}$ such that

$$(z-w)^N a(z)b(w) = (-1)^{|a||b|}(z-w)^N b(w)a(z) \ , \tag{A.2.8}$$

where $|a|$ and $|b$ are the parities of a and b, respectively.

A.2.2 Morphisms, ideals and modules

We collect definitions for some basic algebraic notions below.

Definition A.2.7. A morphism between two vertex (super)algebras $(V, |0\rangle_V, \partial_V, Y_V)$ and $(W, |0\rangle_W, \partial_W, Y_W)$ is a linear map $\phi : V \to W$ of even parity satisfying the following:

- ϕ intertwines the actions of ∂_V and ∂_W
- $\phi(|0\rangle_V) = |0\rangle_W$
- for any $a, b \in V$, $\rho(a_{(n)}b) = \rho(a)_{(n)}\rho(b)$ for all modes $n \in \mathbb{Z}$.

A vertex subalgebra of V is a ∂-invariant subspace $W \subset V$ containing $|0\rangle$, such that $Y(W, z) \subset \mathrm{End}(W)[\![z]\!]$.

A (left) vertex ideal of V is a ∂-invariant subspace $I \subset V$ such that $a_{(n)}V \subset I$ for any $a \in I$ and any $n \in \mathbb{Z}$. All ideals are two-sided in the sense that, for any $v \in V$, $v_{(n)}I \subset I$ must also hold.

Lemma A.2.8. *If V is a vertex (super)algebra and I a vertex ideal, then the quotient space*

V/I inherits a natural vertex (super)algebra structure.

In the context of physics, vertex ideals appear as so called *null* states, *i.e.*, states where some chosen inner product becomes degenerate.

Definition A.2.9. A module over a vertex algebra, $(V, |0\rangle, \partial, Y)$, is the collection (M, Y_M) where M is a vector space and $Y_M(-, z) : V \to End(M)[\![z, z^{-1}]\!]$ satisfying

- $Y_M(|0\rangle, z) = \mathbb{1}_M$
- For any $a, b \in V$ and any $m \in M$, the expressions

$$\begin{aligned} Y_M(a,z)Y_M(b,w)m &\in M(\!(z)\!)(\!(w)\!) \ , \\ Y_M(b,w)Y_M(a,z)m &\in M(\!(w)\!)(\!(z)\!) \ , \\ Y_M(Y(a,z-w)b,w)m &\in M(\!(w)\!)(\!(z-w)\!) \ , \end{aligned} \tag{A.2.9}$$

  are expansions of the same series—as in the discussion on locality—in $M[\![z, w]\!][z^{-1}, w^{-1}, (z-w)^{-1}]$ in the domains $|z| > |w| > 0$, $|w| > |z| > 0$, and $|w| > |z - w| > 0$ respectively.

Modules over vertex superalgebras are analogously defined, but once again locality is modified to a suitable ``super'' version. We will primarily be interested in modules over ordinary (non-super) vertex algebras so we shall forgo the details.

We shall be interested in (co)chain complexes of vertex algebras and so we wish to define a notion of a differential on a vertex algebra.

Definition A.2.10. A derivation on a vertex superalgebra $(V, |0\rangle, \partial, Y)$ is a linear map $Q : V \to V$ of parity $|Q|$, which intertwines ∂ and for any $a, b \in V$

$$Q(a_{(p)}b) = (-1)^{|Q|}a_{(p)}Q(b) + (Q(a))_{(p)}b \ . \tag{A.2.10}$$

A differential on a vertex algebra is a derivation d such that $d^2 = 0$.

Lemma A.2.11. *Let d be a differential on a vertex algebra, then* $\ker d$ *is a vertex subalgebra* *of V and $\operatorname{im} d \subset \ker d$ is a vertex ideal of $\ker d$. Therefore, the cohomology*

$$H(V,d) = \frac{\ker d}{\operatorname{im} d} \ , \tag{A.2.11}$$

naturally inherits the structure of a vertex algebra.

Remark A.2.12. A particularly nice source of such differentials are from the BRST construction. Let V be a vertex superalgebra with an auxiliary $\mathbb{Z}$-grading, $V = \bigoplus_{i\in\mathbb{Z}} V_i$. Choose some homogeneous element $J_{BRST} \in V_1$, such that $Q_{BRST} = J_{BRST,(0)}$ squares to zero. Then Q is a differential of degree one on the chain complex $V = \bigoplus_{i\in\mathbb{Z}} V_i$ and the cohomology $H^{\bullet}(V, Q_{BRST})$ is a $\mathbb{Z}$-graded vertex superalgebra.

This construction will be used, repeatedly, to construct new vertex algebras out of simpler ones.

To finish off, let us discuss the matter of how to give a presentation of a vertex algebra.

Definition A.2.13. Let V be a vertex (super)algebra and let $B \subset V$ be a linearly independent subset of V that does not contain the vacuum. We say that V is *weakly generated* by B if every $v \in V$ can be written as a linear combination of monomials of the form

$$b^1_{(n_1)} b^2_{(n_2)} \dots b^k_{(n_k)} |0\rangle \tag{A.2.12}$$

for $b^i \in B$ and $n_i \in \mathbb{Z}$ and some $k \in \mathbb{N}$. Similarly, we say that V is *strongly generated* if we can restrict to $n_i \in \mathbb{Z}_{<0}$.

A.2.3 Conformal vertex algebras

Owing to their origins in two-dimensional conformal field theory, vertex algebras are often equipped with the structure of a module over the Virasoro algebra. The modes of the Virasoro algebra appear as the Fourier modes of a particular state.

Definition A.2.14. A state $T \in V$ is called the *conformal* vector if

$$T(z)T(w) \sim \frac{c/2}{(z-w)^4} + \frac{2T(z)}{(z-w)^2} + \frac{\partial T(w)}{z-w} \ , \tag{A.2.13}$$

and if $T_{(0)} = \partial$. This is equivalent to demanding that the modes of T satisfy the Virasoro algebra. The parameter c is the central charge. A vertex algebra is conformal if it has a (not necessarily unique) conformal vector. We define the conformal weight, Δ, of a state a, by $T_{(1)}a = \Delta a$

Note that the vacuum vector always has weight $\Delta = 0$. We say that a conformal vertex algebra V is *conical* if the grading by conformal weight is non-negative and the weight zero subspace is spanned by the vacuum vector. A conformal vertex algebra, V, is of CFT type if the following conditions are all satisfied

- V is $\frac{1}{2}\mathbb{N}$-graded by conformal weight
- V is conical
- The contragredient module $V^* = \mathrm{Hom}(V, \mathbb{C})$ is isomorphic to V as V-modules.

A.2.4 Li's filtration and associated varieties

It was shown by Li [Li05] that every vertex algebra has a canonical, decreasing filtration.

Definition A.2.15. The Li filtration on a vertex algebra V is a descending filtration $V = F^0V \supset F^1V \supset \ldots$ whose subspaces are defined by

$$F^pV = \mathrm{Span}_{\mathbb{C}}\{a_{(-i-1)}b \,|\, a \in V, b \in F^{p-i}V, i \geq 1\} \;, \tag{A.2.14}$$

It is compatible with the vertex algebra structure on V, in the sense that

$$\begin{aligned} a_{(n)}F^pV &\subset F^{p+q-n-1},\; a \in F^qV,\; n \in \mathbb{Z}\,, \\ a_{(n)}F^pV &\subset F^{p+q-n},\; a \in F^qV, n \in \mathbb{N}\,, \\ \partial F^pV &\subset F^{p+1}V\,. \end{aligned} \tag{A.2.15}$$

We may then define the associated graded space (with respect to the filtration $F^\bullet V$) of V

to be

$$\mathrm{gr}_F V \coloneqq \bigoplus_{p\geq 0} F^p V / F^{p+1} V\,. \tag{A.2.16}$$

It can be shown, see for instance [Li05], that $\mathrm{gr}_F V$ has the structure of a Poisson vertex algebra.

Definition A.2.16. The *Zhu's C_2 algebra* of a vertex algebra V is the subspace $R_V = V/F^1 V$. It inherits the structure of a Poisson algebra from V.

There is a canonical way of associating a variety, or more generally a scheme, to a vertex algebra that has been detailed in [Ara10].

Definition A.2.17. Given a vertex algebra V, the associated scheme, $\tilde{X}_V$, and associated variety, X_V, are defined as

$$\tilde{X}_V = \mathrm{Spec}\, R_V\,, \qquad X_V = \mathrm{mSpec}\, R_V = (\tilde{X}_V)_{\mathrm{red}} \tag{A.2.17}$$

where R_V is the Zhu's C_2 algebra.

In [BR18], it was conjectured that the Higgs branch of the SCFT is precisely X_V, for the associated vertex algebra V. The Zhu's C_2 algebra is then the coordinate ring of the Higgs branch.

A.3 Affine Kac–Moody algebras and universal affine vertex algebras

A.3.1 Affine Kac–Moody algebras and their modules

Let $\mathfrak{g}$ be a finite dimensional simple Lie algebra with a basis (J^a). We can endow the loop space $\mathfrak{g}[\![t,t^{-1}]\!]$ with a Lie algebra structure by defining the bracket

$$[J^a_n, J^b_m] = [J^a, J^b] \otimes t^{m+n}\ , \tag{A.3.1}$$

where $J^a_m = J^a \otimes t^m$ forms a basis of $\mathfrak{g}[\![t,t^{-1}]\!]$ and the bracket on the right hand side should be understood as the Lie bracket on $\mathfrak{g}$.

The affine Kac–Moody algebra $\hat{\mathfrak{g}}$, is defined as the central extension

$$0 \to \mathbb{C}K \to \hat{\mathfrak{g}} \to \mathfrak{g}[\![t,t^{-1}]\!] \to 0\,, \tag{A.3.2}$$

where $\mathbb{C}K$ is a one dimensional abelian Lie algebra spanned by K. Such extensions are classified by $H^2(\mathfrak{g}[\![t,t^{-1}]\!],\mathbb{C}) \cong \mathbb{C}$, or equivalently a choice of ad-invariant symmetric bilinear form on $\mathfrak{g}$. The space of such forms is one dimensional and we choose as our basis element $\langle\cdot,\cdot\rangle = \frac{1}{2h^\vee}(\cdot,\cdot)$, where $(\cdot,\cdot)$ is the Killing form on $\mathfrak{g}$ and $h^\vee$ is the dual Coxeter number.

As vector spaces, this sequence splits and we have the isomorphism $\hat{\mathfrak{g}} = \mathfrak{g}[\![t,t^{-1}]\!] \oplus \mathbb{C}K$, with the bracket

$$\begin{aligned} [J^a_m, J^b_n] &= [J^a, J^b] \otimes t^{m+n} + m\delta_{m+n,0}\langle J^a, J^b\rangle K \\ [K,a] &= 0 \qquad \forall a \in \hat{\mathfrak{g}}\ . \end{aligned} \tag{A.3.3}$$

Let $\widehat{\mathfrak{g}}-$mod denote the category of left $\widehat{\mathfrak{g}}$-modules. We denote by $\widehat{\mathfrak{g}}_\kappa-$mod, the subcategory of left $\widehat{\mathfrak{g}}$ modules on which K acts as multiplication by some scalar $\kappa \in \mathbb{C}$. We often abuse notation and say that these are modules over $\widehat{\mathfrak{g}}_\kappa$, the subscript denoting the fact that K acts as κ on these modules. A module M is *smooth* if $t\mathfrak{g}[\![t]\!]$ acts locally nilpotently, *i.e.*, for any $m \in M$ and any $x \in \mathfrak{g}$

$$x \otimes t^n m = 0 \tag{A.3.4}$$

for some $n \gg 0$.

Definition A.3.1. The Kazhdan–Lusztig category, $\mathrm{KL}_\kappa \subset \widehat{\mathfrak{g}}_\kappa$-mod, is the subcategory of $G(\mathcal{O})$-integrable, $\mathbb{Z}$-graded modules of $\widehat{\mathfrak{g}}_\kappa$. Equivalently, KL_κ is the subcategory smooth $\mathbb{Z}$-graded $\widehat{\mathfrak{g}}_\kappa$-modules, M, on which the subalgebra $\mathfrak{g} \subset \widehat{\mathfrak{g}}_\kappa$ acts locally finitely, *i.e.*,

$$\forall m \in M, \quad \dim(U(\mathfrak{g})m) < \infty\ , \tag{A.3.5}$$

where $U(\mathfrak{g})$ is the universal enveloping algebra of $\mathfrak{g}$.

Since we allow a full $\mathbb{Z}$-grading, then KL_κ includes objects that are unbounded below by weight, which is somewhat pathological. Consider, therefore, KL_κ^{ord}, the full subcategory of KL_κ admitting a $\mathbb{Z}_{\geq 0}$-grading such that each homogeneous subspace is finite dimensional. The subcategory KL_κ^{ord} can be thought of as the category of positive energy representations.

A.3.2 The universal affine vertex algebra

Let $\mathbb{C}_\kappa$ be a module of the subalgebra $\mathfrak{g}[\![t]\!] \oplus \mathbb{C}K$, where $\mathfrak{g}[\![t]\!]$ acts trivially and K acts as the scalar $\kappa \in \mathbb{C}$.

Definition A.3.2. The induced module,

$$V^\kappa(\mathfrak{g}) \coloneqq \mathrm{Ind}_{\mathfrak{g}[[t]]\oplus\mathbb{C}K}^{\hat{\mathfrak{g}}}\mathbb{C} = U(\hat{\mathfrak{g}}) \otimes_{U(\mathfrak{g}[[t]]\oplus\mathbb{C}K)} \mathbb{C}_\kappa \ . \tag{A.3.6}$$

is called the vacuum representation and has a unique VOA structure $(V^\kappa(\mathfrak{g}), \partial, Y, |0\rangle)$ with

- $|0\rangle = 1 \otimes v$, for some choice of $v \in \mathbb{C}_\kappa$
- $\partial = -\partial_t$
- $Y(J^a_{-1}, z) = \sum_{n\in\mathbb{Z}} J^a_n t^n z^{-n-1}$, with other fields defined by acting with ∂.

Equipped with this vertex algebra structure, $V^\kappa(\mathfrak{g})$ is called the universal affine vertex algebra of $\mathfrak{g}$.

The commutation relations of $\widehat{\mathfrak{g}}$ is now captured in the OPEs

$$J^a(z)J^b(w) \sim \frac{\langle J^a, J^b\rangle \mathbb{1}}{(z-w)^2} + \frac{[J^a, J^b](w)}{(z-w)} \ , \tag{A.3.7}$$

where the J^a, J^b appearing inside the commutator and inner product should be understood as elements of $\mathfrak{g}$.

Similarly, given a highest weight representation, V_λ, we can construct the *Weyl module*, $\mathbb{V}_\lambda$, as

$$\mathbb{V}_\lambda^\kappa \coloneqq \mathrm{Ind}_{\mathfrak{g}[[t]]\oplus\mathbb{C}\mathbb{1}}^{\mathfrak{g}_\kappa} V_\lambda \,. \tag{A.3.8}$$

The Weyl modules $\mathbb{V}_\lambda^\kappa$ admit a Poincare–Birkhoff–Witt basis of lexicographically ordered monomials of the form

$$J^{a_1}_{-n_1} J^{a_2}_{-n_2} \dots J^{a_m}_{-n_m} v_i \ , \tag{A.3.9}$$

where v_i are a basis of V_λ, $n_1 \geq n_2 \geq \dots \geq n_m > 0$ and if $n_i = n_{i+1}$ then $a_i \leq a_{i+1}$.

One reason to study $V^\kappa(\mathfrak{g})$ is that

$$V^\kappa(\mathfrak{g}) - \mathrm{mod} \simeq \widehat{\mathfrak{g}}_\kappa - \mathrm{mod}_{sm} \tag{A.3.10}$$

where $\widehat{\mathfrak{g}}_\kappa - \mathrm{mod}_{sm}$ is the subcategory of smooth modules of $\widehat{\mathfrak{g}}_\kappa$. For the proof of this statement, see Theorem 5.16 and Section 5.18 of [FBZ04]. In particular, this means that the Kazhdan–Lusztig category KL_κ can be thought of as a subcategory of $V^\kappa(\mathfrak{g})-$mod. The Weyl modules introduced above are modules over $V^\kappa(\mathfrak{g})$ and, moreover, are objects of KL_κ.

Remark A.3.3. Following the conventions of Arakawa [Ara18], a vertex algebra object in KL_κ is a vertex algebra V equipped with a vertex algebra homomorphism $\mu_V : V^\kappa(\mathfrak{g}) \to V$ such that V is a limit of objects in KL_κ^{ord}. The decomposition into positive energy representations is highly useful, as the vertex algebras introduced in [Ara18] are limits of Weyl modules.

A.3.3 The Feigin–Frenkel centre

A natural question to ask is whether the algebras, $V^\kappa(\mathfrak{g})$, are conformal? The answer is yes and we can show this by explicitly constructing a conformal vector. Let κ_{ab} be the Killing form with respect to the basis $\{J^a\}$ and let κ^{ab} be its inverse. Then, we may construct the quadratic Casimir

$$P_1 = \frac{1}{2}\sum_{a,b} \kappa^{ab} J^a_{(-1)} J^b_{(-1)} |0\rangle \,. \tag{A.3.11}$$

Suppose $\kappa \neq h^\vee$, then the vector

$$T = \frac{1}{\kappa + h^\vee} P_1 \ , \tag{A.3.12}$$

is conformal, with central charge

$$c_k = \frac{\kappa \dim \mathfrak{g}}{\kappa + h^\vee} \ . \tag{A.3.13}$$

This particular choice of conformal vector is known as the Segal–Sugawara construction. The Segal–Sugawara construction endows the universal affine vertex algebra with a conformal structure in all cases except when

$$\kappa = \kappa_c \equiv -h^\vee \tag{A.3.14}$$

In this case T becomes singular and is no longer a valid conformal vector.

The universal affine vertex algebra at the critical level is, in fact, not conformal but does enjoy a number of properties one would expect from a conformal vertex algebra. For instance, the vertex algebra has a natural $\mathbb{Z}$-grading arising from the degree of the modes of the generators J^a. One may define a degree operator D with weights $D(J^a_n) = -n$ and $D(|0\rangle) = 0$.

The un-normalised Segal–Sugawara vector P_1 is not singular but instead it is central—since its OPEs with any other field will contain vanishing factors of $\kappa - \kappa_c$.

Definition A.3.4. At the critical level $V^{\kappa_c}(\mathfrak{g})$ has a large centre $\mathfrak{z}(\mathfrak{g})$, the *Feigin–Frenkel* (FF) centre. By this we mean that $\mathfrak{z}(\mathfrak{g}) \subset V^{\kappa_c}(\mathfrak{g})$ is a commutative vertex subalgebra with non-singular OPEs with all fields of $V^{\kappa_c}(\mathfrak{g})$.

At non-critical level $\kappa \neq \kappa_c$, the centre of $V^\kappa(\mathfrak{g})$ is spanned by $|0\rangle$, see [Fre07, Proposition 3.3.3] for a proof.

Analogously to the higher order Casimir operators, we can construct higher order Segal–Sugawara vectors $P_{d_i} \in V^{\kappa_c}(\mathfrak{g})$—where d_i are the exponents of $\mathfrak{g}$. The vertex algebra $\mathfrak{z}(\mathfrak{g})$ is strongly generated by these fields [FF92]

By Remark A.2.5, a commutative vertex algebra is equivalent to a commutative algebra with a derivation. An equivalent presentation of $\mathfrak{z}(\mathfrak{g})$ is as the ring

$$\mathfrak{z}(\mathfrak{g}) \cong \mathbb{C}[P_{d_i,(n)},\,|\, d_i = 1, \ldots, \operatorname{rk}\mathfrak{g}\ n \in \mathbb{Z}_{<0}] \tag{A.3.15}$$

with derivation ∂ satisfying $\partial P_{d_i,(n)} = P_{d_i,(n-1)}$.

The topological completion, $\widetilde{U}(\widehat{\mathfrak{g}})/(\kappa - \kappa_c)$, also has a centre, which we denote by $\mathcal{Z}$. This centre has a presentation as

$$\mathcal{Z} \cong \mathbb{C}[P_{d_i,(n)},\,|\, d_i = 1, \ldots, \operatorname{rk}\mathfrak{g}\ n \in \mathbb{Z}]\ . \tag{A.3.16}$$

by identifying this centre with the universal enveloping algebra of the Fourier modes of the fields of $\mathfrak{z}(\mathfrak{g})$.

A module $M \in \mathrm{KL}_{\kappa_c}$ is endowed with a natural action of $\mathfrak{z}(\mathfrak{g})$ and so every object in KL_{κ_c} is also a $\mathfrak{z}(\mathfrak{g})$ module. By passing to the universal enveloping algebra, a module M is also a module over $\mathcal{Z}$.

Following [Ara18], we define $\mathcal{Z}$-Mod to be the category of positive-energy representations of the Feigin-Frenkel centre. Equivalently, the objects of $\mathcal{Z}$-Mod are the $\mathcal{Z}$ modules M such that $M = \bigoplus_{d\in p+\mathbb{N}} M_d$ for some $p \in \mathbb{C}$.

Let us move to describing some distinguished subalgebras and quotients of $\mathcal{Z}$. We define,

$$\mathcal{Z}_{(<0)} \coloneqq \mathbb{C}[P_{d_i,(n)},\,|\, d_i = 1, \ldots, \operatorname{rk}\mathfrak{g}\ n \in \mathbb{Z}_{<0}]\ , \tag{A.3.17}$$

which is isomorphic (as $\mathbb{C}$-algebras) to $\mathfrak{z}(\mathfrak{g})$. Furthermore, we define

$$\mathcal{Z}_{<0} \coloneqq \mathbb{C}[P_{d_i,n},\,|\, d_i = 1, \ldots, \operatorname{rk}\mathfrak{g}\ n \in \mathbb{Z}_{<0}]\ , \tag{A.3.18}$$

where the reader should note that we have used the physicist's gradings on the mode number with $P_{d_i,n} = P_{d_i,(n)-d_i-1}$.

Definition A.3.5. Let $\mathcal{I}_\lambda$ be the annihilator ideal of $\mathbb{V}_\lambda$ inside $\mathcal{Z}_{<0}$, then

$$\mathfrak{z}_\lambda \coloneqq \mathcal{Z}_{<0}/\mathcal{I}_\lambda \ . \tag{A.3.19}$$

These quotients will be integral to our construction and will appear again in the following section, where we shall find a geometric interpretation for them. For now we quote a result.

Proposition A.3.6 ([FG10, Theorem 2]). *Let $H^\bullet_{\mathrm{DS}}$ be the (derived) functor of principal Drinfel'd Sokolov reduction and let $\lambda \in P^+$. We have the following isomorphism of $\mathcal{Z}$-modules,*

$$H^i_{\mathrm{DS}}(\mathbb{V}_\lambda) \cong \delta_{i,0}\mathfrak{z}_\lambda \ . \tag{A.3.20}$$

Recall the characters, χ_λ, of $Z(\mathfrak{g})$ defined by the action of $Z(\mathfrak{g})$ on finite-dimensional highest-weight modules V_λ. We may lift this to a character of $\mathcal{Z}$ via

$$\chi_\lambda(P_{d_i,(n)}) = \delta_{n,d_i}\chi_\lambda(P_{d_i}) \ . \tag{A.3.21}$$

The Kazhdan–Lusztig category (at the critical level) KL has a block decomposition, much like the finite-dimensional BGG category $\mathcal{O}_\mathfrak{g}$, given by

$$\mathrm{KL} = \bigoplus_{\lambda \in P^+} \mathrm{KL}_{[\lambda]} \ , \tag{A.3.22}$$

where $\mathrm{KL}_{[\lambda]}$ is the block where the $P_{d_i,(d_i)}$ act via the generalised eigenvalue χ_λ.

A.4 Opers and the Feigin–Frenkel centre

Our main theorems, Theorem 3.2.1 and 3.2.7, rely heavily on the machinery of opers on $\mathbb{D}$ and $\mathbb{D}^\times$. Here we shall review the machinery required for the proofs of the aforementioned theorems. What follows will largely be a paraphrasing of [Fre07], and we recommend that text along with [Fre02] for a more holistic review.

We adopt the following conventions for this section. Let X be a smooth algebraic curve [6] over $\mathbb{C}$, and let G be a simple algebraic Lie group, with ${}^LG$ its Langlands dual. Let $B \subset G$ be a choice of Borel subgroup that splits as $B = H \ltimes N$, where H is the maximal torus and N is the maximal unipotent subgroup. We denote $\mathfrak{g}$ for the Lie algebra of $G^\vee$. The Borel subalgebra is $\mathfrak{b} \subset \mathfrak{g}$ with splitting $\mathfrak{b} = \mathfrak{h} \oplus \mathfrak{n}$, where $\mathfrak{h}$ is the Cartan subalgebra and $\mathfrak{n}$ is the nilpotent radical.

Define $[\mathfrak{n}, \mathfrak{n}]^\perp$ to be the orthogonal subspace to $[\mathfrak{n}, \mathfrak{n}]$ with respect to the Killing form. The quotient $[\mathfrak{n}, \mathfrak{n}]^\perp \cong \bigoplus_{\alpha\in\Delta} \mathfrak{g}_{-\alpha}$ retains an adjoint action of B. This action factors through an action of the maximal torus H and we define $\mathbb{O}$ to be the Zariski open H-orbit inside $[\mathfrak{n}, \mathfrak{n}]^\perp/\mathfrak{b}$. This orbit is isomorphic to the intersection

$$\mathbb{O} = \mathbb{O}_{\text{prin}} \cap \bigoplus_{\alpha\in\Delta} \mathfrak{g}_{-\alpha} \tag{A.4.1}$$

where $\mathbb{O}_{\text{prin}}$ is the principal nilpotent orbit inside $\mathfrak{g}^*$, considered as an adjoint orbit inside $\mathfrak{g}$ via the Killing form.

A.4.1 Opers—a first definition

First, we recall a basic concept.

Definition A.4.1. Suppose $\mathcal{P}$ is a principal G-bundle on X and let $\iota : H \to G$ be a homomorphism of algebraic Lie groups. We say that a principle H-bundle $\mathcal{P}_H \to X$ is a H-*reduction* of $\mathcal{P}$ if there is an isomorphism $\mathcal{P}_H \underset{H}{\times} G \to \mathcal{P}$, where $\underset{H}{\times}$ corresponds to quotienting by the free diagonal action of H.

For $H \subset G$ a closed subgroup, the choices of such a H-reduction are in bijection with $\Gamma(X, \mathcal{P}/H)$. To see this, note that $\mathcal{P} \to \mathcal{P}/H$ is a principal H-bundle over $\mathcal{P}/H$ and so we can take the base change along a section $s : X \to \mathcal{P}/H$ to define a pullback bundle $s^*(\mathcal{P})$ on X with structure group H.

We need a couple more ingredients before we introduce the definition of a G-oper.

[6] Note that dimensionality will mean that all principal connections will be flat

Definition A.4.2. Suppose $\mathcal{P}$ is a principal G-bundle on X with ∇ a connection on $\mathcal{P}$ and let $\mathcal{P}_B$ be a B-reduction of $\mathcal{P}$. Let $\mathcal{L}_{\mathfrak{g}}$ ($\mathcal{L}_{\mathfrak{b}}$) denote the Lie algebroid, over X, of G (resp. B)-invariant vector fields on $\mathcal{P}$ (resp. $\mathcal{P}_B$). Note that $\mathcal{L}_{\mathfrak{b}} \subset \mathcal{L}_{\mathfrak{g}}$ is a sub Lie-algebroid and let $(\mathfrak{g}/\mathfrak{b})_{\mathcal{P}_B} \equiv \mathcal{L}_{\mathfrak{g}}/\mathcal{L}_{\mathfrak{b}}$.

The connection ∇ is, by definition, a map of vector bundles, $\nabla : TX \to \mathcal{L}_{\mathfrak{g}}$ over X. The composition $\nabla : TX \to \mathcal{L}_{\mathfrak{g}} \to \mathcal{L}_{\mathfrak{g}}/\mathcal{L}_{\mathfrak{b}} \equiv (\mathfrak{g}/\mathfrak{b})_{\mathcal{P}_B}$ gives a section $c(\nabla)$ of the bundle $(\mathfrak{g}/\mathfrak{b})_{\mathcal{P}_B} \otimes \Omega_X$ called the *relative position* of ∇ to $\mathcal{P}_B$.

For a connection ∇ to preserve the reduced bundle $\mathcal{P}_B$ under parallel transport, we must have that $c(\nabla) = 0$.

Definition A.4.3. Let $\mathcal{P}$ be a principal G-bundle on X with ∇ a connection on $\mathcal{P}$ and let $\mathcal{P}_B$ be a B-reduction of $\mathcal{P}$. Recall the relative position of ∇ to $\mathcal{P}_B$ is measured by a section $c(\nabla) \in \Gamma((\mathfrak{g}/\mathfrak{b})_{\mathcal{P}_B} \otimes \Omega_X)$. We say that ∇ is *transversal* to $\mathcal{B}$ if $c(\nabla)$ is in the subset $\Gamma(\mathbb{O}_{\mathcal{P}_B} \otimes \Omega_X) \subset \Gamma((\mathfrak{g}/\mathfrak{b})_{\mathcal{P}_B} \otimes \Omega_X)$.

Suppose we look at the formal neighbourhood of a point $x \in X$; this neighbourhood can be identified with the formal disc $\mathbb{D} = \operatorname{Spf} \mathbb{C}[\![t]\!]$, by choosing a co-ordinate t. On this neighbourhood, $\mathcal{P}$ and $\mathcal{P}_B$ can be trivialised. Transversality of ∇ is equivalent to saying that

$$\nabla = \partial_t + \sum_{\alpha \in \Delta} \phi_\alpha(t) e_{-\alpha} + v(t) \ , \tag{A.4.2}$$

where $\phi_\alpha(t)$ are nowhere vanishing, $v(t) \in \mathfrak{b}[\![t]\!]$, and $e_{-\alpha}$ are the negative simple root Chevalley generators of $\mathfrak{n}_-$.

Definition A.4.4. The space of G-opers $\mathrm{Op}_G(X)$ on X is the moduli space of triples $(\mathcal{P}, \nabla, \mathcal{P}_B)$ where

- $\mathcal{P}$ is a principal G-bundle on X
- ∇ is a connection on $\mathcal{P}$
- $\mathcal{P}_B$ is a B-reduction of G that is *transversal* to ∇.

Any such triple $(\mathcal{P}, \nabla, \mathcal{P}_B)$, satisfying the conditions above, is called a G-oper.

A.4.2 Opers on the disc and the Feigin–Frenkel isomorphism

We shall primarily be interested in the case of opers on the formal disc $\mathbb{D}$ and the formal punctured disc $\mathbb{D}^\times$, both equipped with a co-ordinate t. In these cases, we can produce a far more concrete description of moduli space of opers.

First, note that all G bundles over $\mathbb{D}$ are trivial[7] , therefore the space of opers should be identified with the space of connections of form (A.4.2), modulo the group $B[\![t]\!]$ of gauge transformations coming from the choice of trivialisation of $\mathcal{P}_B$. If $g \in B[\![t]\!]$, then the gauge transformation acts on a connection $\partial_t + A(t)$ as

$$g \cdot (\partial_t + A(t)) = \partial_t + gA(t)g^{-1} - g^{-1}\partial_t g \ . \tag{A.4.3}$$

The orbit $\mathbb{O}$ is a H-torsor and so we may use H-valued gauge transformations to partially gauge fix and set all ϕ_α equal to unity. Let $\widetilde{\mathrm{Op}_G}(\mathbb{D})$ be the space of connections of the form

$$\nabla = \partial_t + \sum_{\alpha\in\Delta} e_{-\alpha} + v(t) \ , \quad v(t) \in \mathfrak{b}[\![t]\!] \ . \tag{A.4.4}$$

Then $\mathrm{Op}_G(\mathbb{D})$ is $\widetilde{\mathrm{Op}_G}(\mathbb{D})/N[\![t]\!]$.

We shall now detail how to pick canonical representatives for each $N[\![t]\!]$-gauge class. Note that our special representative for the principal orbit $p_{-1} = \sum_{\alpha\in\Delta} e_{-\alpha}$ is in $\mathbb{O}$. Recall that we can complete this element to the canonical principal $\mathfrak{sl}_2$ triple (p_1, p_0, p_{-1}) such that the grading induced by ad_{p_0} satisfies $\mathfrak{n} = \sum_{i>0} \mathfrak{g}_i$ and $\mathfrak{b} = \sum_{i\geq 0} \mathfrak{g}_i$. As before, we denote the subspace $\ker \mathrm{ad}_{p_1}$ by $\mathfrak{s}$. This subspace inherits the grading on $\mathfrak{g}$, and its components are

$$\mathfrak{s} = \bigoplus_{i=1}^{\mathrm{rk}\,\mathfrak{g}} \mathfrak{s}_{d_i} \ , \tag{A.4.5}$$

where d_i are the exponents of $\mathfrak{g}$ and the dimension of $\mathfrak{s}_i$ is equal to the multiplicity of d_i. Let $(p_i)_{i=1}^{\mathrm{rk}\,\mathfrak{g}}$ be the basis of Remark A.1.13.

Lemma A.4.5 ([DS85, Proposition 6.1])**.** *The action of $N[\![t]\!]$ on $\widetilde{\mathrm{Op}_G}(\mathbb{D})$ is free and admits*

[7] This statement is a bit quick. If one has a co-ordinate, t on $\mathbb{D}$, one can always construct a section to $\mathcal{P} \to \mathrm{Spf}\,\mathbb{C}[\![t]\!]$ by appealing to the formal smoothness of $\mathcal{P}$ over $\mathrm{Spf}\,\mathbb{C}[\![t]\!]$.

a global slice consisting of connections of the form

$$\nabla = \partial_t + p_{-1} + \sum_{j=1}^{\mathrm{rk}\,\mathfrak{g}} v_{d_j}(t) p_{d_j} \ , \ \textit{where } v_j(t) \in \mathbb{C}[\![t]\!] \ . \tag{A.4.6}$$

As an immediate corollary, we have a very concrete presentation for $\mathrm{Op}_G(\mathbb{D})$ as an affine scheme (of infinite type), that is

$$\mathrm{Op}_G(\mathbb{D}) = \mathrm{Spec}\, \mathbb{C}[v_{d_i,n} \,|\, i = 1, \ldots, \mathrm{rk}\,\mathfrak{g},\, n \in \mathbb{N}] \ , \tag{A.4.7}$$

and we denote the co-ordinate ring as $\mathrm{Fun}\,\mathrm{Op}_G(\mathbb{D})$. This gives an identification $\mathrm{Op}_G(\mathbb{D}) \cong \mathrm{Hom}(\mathbb{D}, S_{\mathrm{prin}})$.

The ring of functions $\mathrm{Fun}\,\mathrm{Op}_G(\mathbb{D})$ is a unital associative algebra over $\mathbb{C}$ and one can define a derivation ∂ by the action $\partial v_{d_i,0} = 0$ and $\partial v_{d_i,n} = v_{d_i,n-1}$ and extending by Leibniz. Therefore, by Remark A.2.5, $\mathrm{Fun}\,\mathrm{Op}_G(\mathbb{D})$ is a commutative vertex algebra over $\mathbb{C}$.

In fact the commutative vertex algebra $\mathrm{Fun}\,\mathrm{Op}_G(\mathbb{D})$ is known, via a celebrated theorem of Feigin and Frenkel, to be related to one that we have already met.

Theorem A.4.6 (The Feigin–Frenkel isomorphism)**.** *Let G be a simple algebraic group with Lie algebra $\mathfrak{g}$. Let ${}^L G$ denote its Langlands dual. We have an isomorphism of commutative vertex algebras*

$$\begin{aligned} \mathrm{Fun}\,\mathrm{Op}_{{}^L G}(\mathbb{D}) &\to \mathfrak{z}(\mathfrak{g}) \\ v_{d_i,n} &\mapsto p_{d_i,(n)} \end{aligned} \tag{A.4.8}$$

moreover this isomorphism intertwines the actions of $\mathrm{Aut}\,\mathcal{O}$ *and* $\mathrm{Der}\,\mathcal{O}$*.*

A.4.3 Infinitesimal co-ordinate changes

In the previous subsection, we had (non-canonically) chosen some co-ordinate, t, on $\mathbb{D}$. What happens if we were to change co-ordinates $t = \phi(s)$, where $\phi \in \mathrm{Aut}\,\mathcal{O}$? A connection of the form (A.4.2) will now be in the form

$$\nabla = (\phi'(s))^{-1}\partial_s + p_{-1} + v(\phi(s)) \ , \tag{A.4.9}$$

So the corresponding connection for ∂_s is $\nabla_s = \partial_s + (\phi'(s))p_{-1} + \phi'(s)v(\phi(s))$. However, this is not a canonical representative. To bring it to the form in (A.4.6), we have to twist by the gauge action of $H[\![s]\!]$. In particular we should twist by $\check{\rho}(\phi'(s))$, where $\check{\rho}$ is the sum of fundamental coweights of G. Doing so, gives us a transversal representative.

$$\check{\rho}(\phi'(s)) \cdot (\nabla_s) = \partial_s + p_{-1} + \phi'(s)\check{\rho}(\phi'(s)) \cdot v(\phi(s)) \cdot \check{\rho}(\phi'(s))^{-1} - \left(\frac{\phi''(s)}{\phi'(s)}\right)\check{\rho} \ , \tag{A.4.10}$$

where $\left(\frac{\phi''(s)}{\phi'(s)}\right)\check{\rho} \in \mathfrak{h}[\![s]\!]$ by viewing $\check{\rho}$ as an element of the Cartan subalgebra, $\mathfrak{h}$.

Therefore the group of co-ordinate changes, $\operatorname{Aut}\mathcal{O}$, acts on $\widetilde{Op}_G(\mathbb{D})$ via this gauge action. What does the action of $\operatorname{Aut}\mathcal{O}$ look like for the canonical representatives of the form (A.4.6)? Suppose $v(t) = \sum_{j=1}^{\operatorname{rk}\mathfrak{g}} v_{d_j}(t)p_{d_j}$, then (A.4.10) is almost of the right form—except for the $\left(\frac{\phi''(s)}{\phi'(s)}\right)\check{\rho}$ term. We can fix this by a further gauge transformation by $n \in N[\![s]\!]$

$$g = \operatorname{Exp}\left(\frac{1}{2}\frac{\phi''(s)}{\phi'(s)}\right)p_1 \ , \tag{A.4.11}$$

where $\operatorname{Exp} : \mathfrak{n} \xrightarrow{\sim} N$ is the exponential map. This gives an oper in the form

$$\nabla_s = \partial_s + p_{-1} + \sum_{i=1}^{\operatorname{rk}\mathfrak{g}} \tilde{v}_{d_j}(s)p_{d_j} \ , \tag{A.4.12}$$

where

$$\begin{aligned} \tilde{v}_1(s) &= v_1(\phi(s))(\phi'(s))^2 - \tfrac{1}{2}\mathrm{Schw}\{\phi, s\} \ , \\ \tilde{v}_{d_j}(s) &= v_{d_j}(\phi(s))(\phi'(s))^{d_j+1} \ , \quad j > 1 \ , \end{aligned} \tag{A.4.13}$$

where $\mathrm{Schw}\{\phi, \mathrm{s}\} = \frac{\phi'''(s)}{\phi'(s)} - \frac{3}{2}\left(\frac{\phi''(s)}{\phi'(s)}\right)^2$ is the Schwarzian derivative. The action of $\operatorname{Aut}\mathcal{O}$ on $\operatorname{Fun}\operatorname{Op}_G(\mathbb{D})$ can be read off from these expressions. This action induces an action of the Lie algebra $\operatorname{Der}\mathcal{O}$, which is isomorphic to the algebra generated by the strictly positive modes $(L_m)_{m>0}$ of the Virasoro algebra.

A.4.4 Opers on the punctured disc and monodromies

We move to considering the space of G-opers on the punctured disc $\mathbb{D}^\times$. Following the same analysis as in Section A.4.2, we can write a G-oper on $\mathbb{D}^\times$ as a gauge representative

$$\nabla = \partial_t + p_{-1} + \sum_{j=1}^{\operatorname{rk}\mathfrak{g}} v_{d_j}(t) p_{d_j} \ , \text{ where } v_{d_j}(t) \in \mathbb{C}(\!(t)\!) \ . \tag{A.4.14}$$

This gives an identification,

$$\operatorname{Fun}\operatorname{Op}_G(\mathbb{D}^\times) \cong \mathbb{C}[v_{d_j,(n)}, \mid j = 1, \dots, \operatorname{rk}\mathfrak{g}\, n \in \mathbb{Z}] \ . \tag{A.4.15}$$

Note the similarity to $\mathcal{Z}$. Indeed, combing Proposition 4.3.4 and Lemma 4.3.5 of [Fre07], we have the following result.

Theorem A.4.7 ([Fre07, Theorem 4.3.6]). *We have an isomorphism,*

$$\begin{aligned} \mathcal{Z} &\xrightarrow{\sim} \operatorname{Fun}\operatorname{Op}_{^LG}(\mathbb{D}^\times) \\ v_{d_j,(n)} &\mapsto P_{d_j,(n)} \end{aligned} \tag{A.4.16}$$

that intertwines the $(\operatorname{Der}\mathcal{O}, \operatorname{Aut}\mathcal{O})$ *actions on each side.*

We now have geometric models for both $\mathfrak{z}(\mathfrak{g})$ and $\mathcal{Z}$.

Recall the quotients $\mathfrak{z}_\lambda$, defined as $\mathcal{Z}/\mathcal{I}_\lambda$ for $\mathcal{I}_\lambda = \operatorname{Ann}_{\mathcal{Z}}(\mathbb{V}_\lambda)$. These should correspond to closed subschemes of $\operatorname{Spec}\mathcal{Z}$ carved out by the sheaf of ideals, $\mathcal{I}_\lambda$. What do they look like in terms of opers? To answer this question, we will have to examine the pole and residue structures of a connection at $t = 0$.

Definition A.4.8. Let $\lambda \in P^+$ be a dominant integral weight of G, *i.e.*, a coweight of ^LG. We denote by $\operatorname{Op}^{nilp,\lambda}_{^LG}$ the space of $B[\![t]\!]$-conjugacy classes of connections of the form

$$\nabla = \partial_t + \sum_{\alpha\in\Delta} t^{(\check{\alpha},\lambda)} \psi_i(t) e_{-\check{\alpha}} + v(t) + \frac{n}{t} \ , \tag{A.4.17}$$

where $\psi(t) \in \mathbb{C}[\![t]\!]^\times$, $v(t) \in {}^L\mathfrak{b}[\![t]\!]$ and $n \in {}^L\mathfrak{n}$. This is called the space of *nilpotent* opers with coweight λ.

Furthermore, let $\mathrm{Op}^{\lambda}_{{}^LG} \subset \mathrm{Op}^{nilp,\lambda}_{{}^LG}$ denote the closed subscheme of conjugacy classes of connections of the form

$$\nabla = \partial_t + \sum_{\alpha\in\Delta} t^{\langle\check{\alpha},\lambda\rangle}\psi_i(t)e_{-\check{\alpha}} + v(t) \ , \tag{A.4.18}$$

i.e., where $n = 0$. These are the opers with trivial monodromy.

Both $\mathrm{Op}^{nilp,\lambda}_{{}^LG}$ and $\mathrm{Op}^{\lambda}_{{}^LG}$ can be thought of as subschemes of $\mathrm{Op}_{{}^LG}(\mathbb{D}^\times)$ by taking the $B(\!(t)\!)$ conjugacy classes of connections in the given forms. This gives rise to closed embeddings,

$$\mathrm{Op}^{\lambda}_{{}^LG} \hookrightarrow \mathrm{Op}^{nilp,\lambda}_{{}^LG} \hookrightarrow \mathrm{Op}_{{}^LG}(\mathbb{D}^\times) \ . \tag{A.4.19}$$

Theorem A.4.9 ([FG10, Theorem 1])**.** *We have a commutative diagram*

$$\begin{array}{ccccc} & & \mathcal{Z} & \xrightarrow{\sim} & \mathrm{Fun}\,\mathrm{Op}_{{}^LG}(\mathbb{D}^\times) \\ & & \downarrow & & \downarrow \\ \mathfrak{z}_\lambda & \xrightarrow{\sim} & \mathrm{End}_{\widehat{\mathfrak{g}_{\kappa_c}}}(\mathbb{V}_\lambda) & \xrightarrow{\sim} & \mathrm{Fun}\,\mathrm{Op}^{\lambda}_{{}^LG} \end{array} \tag{A.4.20}$$

In other words, the closed embedding $\mathrm{Op}^{\lambda}_{{}^LG} \hookrightarrow \mathrm{Op}_{{}^LG}(\mathbb{D}^\times) \cong \mathrm{Spec}\,\mathcal{Z}$ has image $\mathrm{Spec}\,\mathfrak{z}_\lambda$. Now that we have a geometric understanding of $\mathfrak{z}_\lambda$, we want to try and give a concrete presentation of it. To do so, we shall introduce Miura opers.

A.4.5 Miura opers and Cartan connections

Definition A.4.10. A *Miura* G-oper on X is a quadruple $(\mathcal{P}, \nabla, \mathcal{P}_B, \mathcal{P}'_B)$, where $(\mathcal{P}, \nabla, \mathcal{P}_B)$ is a G-oper on X and $\mathcal{P}'_B$ is a B-reduction of $\mathcal{P}$ that is preserved by ∇. We denote the space of Miura G-opers on X by $\mathrm{MOp}_G(X)$ and it comes with a natural projection

$$\mathrm{oblv} : \mathrm{MOp}_G(X) \to \mathrm{Op}_G(X) \ . \tag{A.4.21}$$

Once again, we restrict to the case when $X = \mathbb{D}$. A Miura G-oper is *generic* if $\mathcal{P}_B$ and $\mathcal{P}'_B$ are in generic relative positions and we denote the subspace of generic Miura opers on $\mathbb{D}$ by $\mathrm{MOp}_G(\mathbb{D})_{\mathrm{gen}}$.

A B-reduction of $\mathcal{P}$ that is preserved by ∇ is uniquely determined by a choice of B-reduction of the fibre of $\mathcal{P}$ at 0, $\mathcal{P}_0$. The space of such B-reductions is given by the space of sections $\Gamma(\{0\}, \mathcal{P}_0/B)$ which are just the $\mathbb{C}$-points of $\mathcal{P}_0/B$ and may be identified with $\mathcal{P}_0 \underset{G}{\times} G/B$. Therefore, the natural projection $\mathrm{MOp}_G(\mathbb{D}) \to \mathrm{Op}_G(\mathbb{D})$ should be a principal B-bundle over $\mathrm{Op}_G(\mathbb{D})$,

Define $\mathcal{P}_{\mathrm{univ}}$ to be the universal G-bundle on $\mathrm{Op}_G(\mathbb{D})$ with abstract fibre $\mathcal{P}_0$ at the point $(\mathcal{P}, \nabla, \mathcal{P}_B)$. Then, by the preceding argument

$$\mathrm{MOp}_G(\mathbb{D}) \cong \mathcal{P}_{\mathrm{univ}} \underset{G}{\times} G/B \ . \tag{A.4.22}$$

For a fixed $\mathcal{P}_B$, $\mathcal{P}'_B$ is in generic relative position if it lies in the pullback of the big cell $\mathcal{U} \subset G/B$ to $\mathcal{P}_0 \underset{G}{\times} G/B$. By a similar argument as before,

$$\mathrm{MOp}_G(\mathbb{D})_{gen} \cong \mathcal{P}_{B,\mathrm{univ}} \underset{B}{\times} \mathcal{U} \ . \tag{A.4.23}$$

As a corollary, $\mathrm{MOp}_G(\mathbb{D})_{\mathrm{gen}} \to \mathrm{Op}_G(\mathbb{D})$ is a principal N-torsor.

Given a generic Miura oper $(\mathcal{P}, \nabla, \mathcal{P}_B, \mathcal{P}'_B)$ on $\mathbb{D}$, we can define H-bundles $\mathcal{P}_H = \mathcal{P}_B/N$ and $\mathcal{P}'_H = \mathcal{P}'_B/N$.

Lemma A.4.11 ([Fre07, Lemma 8.2.1])**.** *For a generic Miura oper $(\mathcal{P}, \nabla, \mathcal{P}_B, \mathcal{P}'_B)$ on $\mathbb{D}$, $\mathcal{P}_H$ and $\mathcal{P}'_H$ are related by*

$$\mathcal{P}'_T \cong w_0^*(\mathcal{P}_T) \ , \tag{A.4.24}$$

where w_0^* is a bundle morphism that twists the fibres by the longest element of the Weyl group w_0.

The connection ∇ on $\mathcal{P}$ preserves $\mathcal{P}'_B$ and descends to a connection on $\mathcal{P}'_H$. Since, $\mathcal{P}_H$ and $\mathcal{P}'_H$ are related by an automorphism, this defines a connection, $\overline{\nabla}$, on $\mathcal{P}_H$.

Lemma A.4.12 ([Fre07, Lemma 4.2.1]). *The H-bundle $\mathcal{P}_H$ is isomorphic to the H-bundle $\Omega^{\check{\rho}}$, which is characterised by requiring that $\phi \in \operatorname{Aut}\mathcal{O}$ acts on the trivialisation, of $\Omega^{\check{\rho}}$, by the transition function $\check{\rho}(\phi')$.*

Thus, the connection $\overline{\nabla}$ is a connection on $\Omega^{\check{\rho}}$ and we have a map to the space of connections of $\Omega^{\check{\rho}}$ on $\mathbb{D}$.

Proposition A.4.13 ([Fre07, Proposition 8.2.2]). *The map*

$$\begin{aligned} \beta : \mathrm{MOp}(\mathbb{D})_{\mathrm{gen}} &\to \mathrm{Conn}(\Omega^{\check{\rho}})_{\mathbb{D}} \\ (\mathcal{P}, \nabla, \mathcal{P}_B, \mathcal{P}'_B) &\mapsto \overline{\nabla} \end{aligned} \tag{A.4.25}$$

is an isomorphism.

This is nice, since any connection in $\mathrm{Conn}(\Omega^{\check{\rho}})_{\mathbb{D}}$ is of the form $\overline{\nabla} = \partial_t + h(t)$ for some element $h(t) \in \mathfrak{h}[\![t]\!]$. Thus, we can identify $\mathrm{MOp}_G(\mathbb{D})_{\mathrm{gen}} \cong \mathfrak{h}[\![t]\!]$. Composing with the forgetful morphism, $\mathrm{oblv} : \mathrm{MOp}_G(\mathbb{D})_{gen} \to \mathrm{Op}_G(\mathbb{D})$, gives us a morphism

$$\mu_{\mathrm{Miura}} : \mathrm{Conn}(\Omega^{\check{\rho}}) \to \mathrm{Op}_G(\mathbb{D}) \ . \tag{A.4.26}$$

called the Miura transform. This will be highly useful since the simple presentation of $\mathrm{Conn}(\Omega^{\check{\rho}})$ will be invaluable in establishing various properties about $\mathrm{Op}_G(\mathbb{D})$.

Now that we have covered the case where $X = \mathbb{D}$, let us move to discussing the case of the punctured disc. By [FG06, Lemma 3.2.1], we have that every Miura oper on $\mathbb{D}^\times$ is generic and so we have an isomorphism

$$\mathrm{MOp}_G(\mathbb{D}^\times) \cong \mathrm{Conn}(\Omega^{\check{\rho}})_{\mathbb{D}^\times} \ . \tag{A.4.27}$$

Just like with the disc, composing this map with the forgetful morphism gives the Miura map

$$\mu_{\mathrm{Miura}} : \mathrm{Conn}(\Omega^{\check{\rho}})_{\mathbb{D}^\times} \to \mathrm{Op}_{{}^LG}(\mathbb{D}^\times) \ . \tag{A.4.28}$$

We shall abuse notation slightly and use μ_{Miura} for both Miura maps.

Let $\lambda \in P^+$ be a dominant integral weight of G, *i.e.*, a coweight of ${}^LG$. We form the pullback

$$\begin{array}{ccc} \mathrm{MOp}^{\lambda}_{{}^LG}(\mathbb{D}^\times) & \longrightarrow & \mathrm{MOp}_{{}^LG}(\mathbb{D}^\times) \\ \downarrow & & \downarrow \\ \mathrm{Op}^{nilp,\lambda}_{{}^LG} & \hookrightarrow & \mathrm{Op}_{{}^LG}(\mathbb{D}^\times) \end{array} \tag{A.4.29}$$

Similarly, for ρ the sum of fundamental dominant weights in G, we can define the space of Cartan connections, $\mathrm{Conn}(\Omega^\rho)^{-\lambda}_D$ as the space of gauge conjugacy classes of connections of the form

$$\nabla = \partial_t - \frac{\lambda}{t} + u(t) \ , \tag{A.4.30}$$

where $u(t) \in \mathfrak{h}[\![t]\!]$. To reiterate, these are connections for an ${}^LH$ principal bundle, *i.e.*, the dual torus of G, with residue λ at 0. Functions on this space have a very concrete realisation,

$$\mathrm{Fun}(\mathrm{Conn}(\Omega^\rho)^{-\lambda}_D) \cong \mathbb{C}[u_{i,n} \,|\, i = 1, \ldots, \mathrm{rk}\,\mathfrak{g}\; n \in \mathbb{Z}_{<0}] \ . \tag{A.4.31}$$

There is a natural embedding

$$\mathrm{Conn}_{{}^LH}(\Omega^\rho)^{-\lambda}_{\mathbb{D}} \hookrightarrow \mathrm{Conn}_{{}^LH}(\Omega^\rho)_{\mathbb{D}^\times} \ , \tag{A.4.32}$$

by taking ${}^LH(\!(t)\!)$ conjugacy classes.

Now consider the restriction of the Miura map to this subspace,

$$\mu^{\lambda}_{\mathrm{Miura}} : \mathrm{Conn}_{{}^LH}(\Omega^\rho)^{-\lambda}_{\mathbb{D}} \to \mathrm{Op}_{{}^LG}(\mathbb{D}^\times) \ , \tag{A.4.33}$$

where we think of $\mathrm{Conn}_{{}^LH}(\Omega^\rho)^{-\lambda}_{\mathbb{D}}$ as a subspace of $\mathrm{Conn}_{{}^LH}(\Omega^\rho)_{\mathbb{D}^\times}$ via (A.4.32).

Proposition A.4.14 ([FG06, Proposition 3.5.4] [FG10, Lemma 2]). *Let $\lambda \in P^+$ be an integral dominant weight of G, then we have the following pullback square*

$$\begin{array}{ccc} \mathrm{Op}^{\lambda}_{{}^LG} & \xrightarrow{\sim} & \mathrm{Op}^{\lambda}_{{}^LG} \\ \downarrow & & \downarrow \\ \mathrm{Conn}_{{}^LH}(\Omega^\rho)^{-\lambda}_{\mathbb{D}} & \xrightarrow{\mu^{\lambda}_{\mathrm{Miura}}} & \mathrm{Op}_{{}^LG}(\mathbb{D}^\times) \end{array} \ , \tag{A.4.34}$$

i.e., the image of $\mu^{\lambda}_{\mathrm{Miura}}$ *in* $\mathrm{Op}_{{}^LG}(\mathbb{D}^{\times})$ *coincides with the image of the closed embedding* $\mathrm{Op}^{\lambda}_{{}^LG} \hookrightarrow \mathrm{Op}_{{}^LG}(\mathbb{D}^{\times})$ *. Moreover, the map* $\mu^{\lambda}_{\mathrm{Miura}} : \mathrm{Conn}_{{}^LH}(\Omega^{\rho})^{-\lambda}_{\mathbb{D}} \to \mathrm{Op}^{\lambda}_{{}^LG}$*, is a principal* N*-bundle.*

Therefore, we have a map on functions,

$$\mathfrak{z}_{\lambda} \cong \mathrm{Fun}(\mathrm{Op}^{\lambda}_{{}^LG}) \xhookrightarrow{(\mu^{\lambda}_{\mathrm{Miura}})^{\#}} \mathrm{Fun}(\mathrm{Conn}_{{}^LH}(\Omega^{\rho})^{-\lambda}_{\mathbb{D}}) \cong \mathbb{C}[u_{i,n} \,|\, i = 1, \ldots, \mathrm{rk}\,\mathfrak{g}\, n \in \mathbb{Z}_{<0}] \;, \tag{A.4.35}$$

giving a realisation of $\mathfrak{z}_{\lambda}$ inside a free polynomial algebra, in other words a free-field realisation. In the next subsection, we shall describe the image of this embedding as the kernel of certain screening operators. .

A.4.6 Screening charges

We want to recast the image of the Miura transform as the kernel of certain screening operators. To do so, recall that Proposition A.4.14 states that $\mu_{\mathrm{Miura}} : \mathrm{Conn}_{{}^LH}(\Omega^{\rho})_{\mathbb{D}} \to \mathrm{Op}_{{}^LG}(\mathbb{D})$ is a principal ${}^LN$-bundle. In other words, we can identify $\mathrm{Op}_{{}^LG}(\mathbb{D})$ with the orbit space $(\mathrm{Conn}_{{}^LH}(\Omega^{\rho})_{\mathbb{D}}) /\!\!/ {}^LN$.

The infinitesimal action of ${}^LN$ on $(\mathrm{Conn}_{{}^LH}(\Omega^{\rho})) \cong \mathbb{C}[u_{i,n} \,|\, i = 1, \ldots, \mathrm{rk}\,\mathfrak{g}\,,\, n \in \mathbb{Z}_{<0}]$ is generated by the vector fields [Fre07]

$$V_i[1] = \sum_{j=1}^{\mathrm{rk}\,\mathfrak{g}} a_{ji} \sum_{n \geq 0} x_{i,n} \frac{\partial}{\partial u_{j,-n-1}} \;, \tag{A.4.36}$$

for $i = 1, \ldots, \mathrm{rk}\,\mathfrak{g}$ and a_{ji} the Cartan matrix of $\mathfrak{g}$. The $x_{i,n}$ are determined by

$$\sum_{n \leq 0} x_{i,n} t^{-n} = \mathrm{Exp}\left(- \sum_{m>0} \frac{u_{i,-m}}{m} t^{m} \right) \;. \tag{A.4.37}$$

These vector fields generate the Lie algebra ${}^L\mathfrak{n}$. Therefore, we have the isomorphism [Fre07, Proposition 8.2.3].

$$\mathrm{Fun}(\mathrm{Op}_{^LG}(\mathbb{D})) \cong \left(\mathrm{Fun}(\mathrm{Conn}_{^LH}(\Omega^\rho)_{\mathbb{D}}\right)^{^L\mathfrak{n}} \cong \bigcap_{i=1}^{\mathrm{rk}\,\mathfrak{g}} \ker V_i[1] \subset \mathbb{C}[u_{i,n} \,|\, i = 1, \ldots, \mathrm{rk}\,\mathfrak{g}\,,\, n \in \mathbb{Z}_{<0}]\,. \tag{A.4.38}$$

This analysis carries over for the image of the restricted Miura map $\mu^\lambda_{\mathrm{Miura}}$, see Proposition A.4.14. The actions of the vector fields, however are a little different. The action of $^L\mathfrak{n}$ on $\mathrm{Fun}(\mathrm{Conn}_{^LH}(\Omega^\rho)^{-\lambda}_{\mathbb{D}}) \cong \mathbb{C}[u_{i,n} \,|\, i = 1, \ldots, \mathrm{rk}\,\mathfrak{g}\,,\, n \in \mathbb{Z}_{<0}]$ is generated by the vector fields

$$V_i[\lambda_i + 1] = -\sum_{j=1}^{\mathrm{rk}\,\mathfrak{g}} a_{ji} \sum_{n \geq \lambda_i} x_{i,n-\lambda_i} \frac{\partial}{\partial u_{j,-n-1}}\;, \tag{A.4.39}$$

Proposition A.4.15 ([Fre07, Proposition 9.6.3]). *Let $\lambda \in P^+$ be an integral dominant weight of G and so a coweight of ^LG. The Miura embedding can be realised as*

$$\mathfrak{z}_\lambda \cong \mathrm{Fun}\,\mathrm{Op}^\lambda_{^LG} \cong \bigcap_{i=1}^{\mathrm{rk}\,\mathfrak{g}} \ker V_i[\lambda_i + 1] \subset \mathrm{Fun}(\mathrm{Conn}_{^LH}(\Omega^\rho)^{-\lambda}_{\mathbb{D}})\;. \tag{A.4.40}$$

A.5 Semi-infinite homological algebra

Much of the techniques of [Ara18] rely on various forms of BRST reduction. In the context of vertex algebras, the correct formalism for dealing with BRST cohomology is that of semi-infinite homological algebra. Semi-infinite cohomology was introduced by Feigin in [Fei84], and adapted for use in bosonic string theory by Frenkel, Garland and Zuckerman in [FGZ86]. We shall largely follow the formalism of Voronov [Vor93, Vor97], who recast semi-infinite cohomology in the language of homological algebra.

A.5.1 Semi-infinite structure

For the rest of this section, $\mathfrak{g}$ shall be a $\mathbb{Z}$-graded Lie algebra over $\mathbb{C}$. We define the two Lie subalgebras

$$\begin{aligned} \mathfrak{g}_+ &\coloneqq \bigoplus_{i>0} \mathfrak{g}_i \,, \\ \mathfrak{g}_- &\coloneqq \bigoplus_{i\leq 0} \mathfrak{g}_i \,, \end{aligned} \tag{A.5.1}$$

We define $\mathfrak{gl} \subset \mathrm{End}(\mathfrak{g})$ to be the (Lie algebra of the) restricted general linear group on $\mathfrak{g}$, consisting of all matrices, ϕ, whose $\phi_{-+} : \mathfrak{g}_+ \to \mathfrak{g}_-$ block is of finite rank. We then define the space $\widetilde{\mathfrak{gl}}$ to be a one dimensional central extension of $\mathfrak{gl}$, see [Vor93] for details. The adjoint representation $\mathrm{ad} : \mathfrak{g} \to \mathfrak{gl}$, lifts to a homomorphism of Lie algebras $\mathrm{ad} : \mathfrak{g} \to \widetilde{\mathfrak{gl}}$. We define $\beta \in \mathfrak{g}^*$ to be the composition of the lifted ad followed by a left splitting of the short exact sequence

$$0 \to \mathbb{C} \to \widetilde{\mathfrak{gl}} \to \mathfrak{gl} \to 0 \,, \tag{A.5.2}$$

as vector spaces. For any graded Lie algebra $\mathfrak{g}$, we may choose a splitting such that $\beta(\mathfrak{g}_i) = 0$ for $i \neq 0$, see Prop 2.4 of [Vor93].

Definition A.5.1. Let $\mathfrak{g}$ be a $\mathbb{Z}$-graded Lie algebra over $\mathbb{C}$. A semi-infinite structure on $\mathfrak{g}$ is a 1-cocycle $\beta \in \mathfrak{g}^*$ defined as above such that $\beta(\mathfrak{g}_{i\neq 0}) = 0$.

Since β is a one cocycle, we have a natural one dimensional module associated to it: $\mathcal{L}_\beta$. As a vector space, $\mathcal{L}_\beta \cong \mathbb{C}$, with the $\mathfrak{g}$ action given by $x \cdot m = \beta(x)m$ for all Lie algebra elements $x \in \mathfrak{g}$ and $m \in \mathcal{L}_\beta$.

There are two examples of Lie algebras that admit a semi-infinite structure that shall be of great use to us:

- Any Abelian Lie Algebra, with $\beta = 0$.
- Any Kac-Moody algebra with its natural grading admits a semi-infinite structure.

More generally, we can equip any semi-simple Lie algebra $\mathfrak{g}$, with a $\mathbb{Z}$-grading, with a semi-infinite structure by setting $\beta = 0$—as a consequence of the Whitehead Lemmas.

Remark A.5.2. A semi-infinite structure also gives a generalisation of the BGG category $\mathcal{O}_{\mathfrak{g}}$ to non-semisimple Lie algebras. Given a Lie algebra $\mathfrak{g}$ with semi-infinite structure, we define $\mathcal{O}_{\mathfrak{g}}$ to be the category of $\mathfrak{g}$-modules where $\mathfrak{g}_+$ acts locally finitely, *i.e.*, for every element m, the subspace $U(\mathfrak{g}_+)m$ has finite dimension.

A.5.2 The space of semi-infinite forms and the Feigin standard complex

Let Cliff($\mathfrak{g}$) be the Clifford algebra of $\mathfrak{g}$ i.e. the algebra generated by $\mathfrak{g} \oplus \mathfrak{g}^*$ with the symmetric bracket

$$\{x, y\} = \{\alpha, \gamma\} = 0\,, \quad \{\alpha, y\} = \{y, \alpha\} = \alpha(y)\,, \quad \text{for } x, y \in \mathfrak{g}\,, \alpha, \gamma \in \mathfrak{g}^*\,. \tag{A.5.3}$$

The space of semi-infinite forms, $\bigwedge^{\frac{\infty}{2}+\bullet}\mathfrak{g}$ is the representation of Cliff($\mathfrak{g}$) generated by a choice of vacuum vector ω_0 subject to

$$x\omega_0 = \alpha\omega_0 = 0\,, \quad \text{for } x \in \mathfrak{g}_-\,, \alpha \in \mathfrak{g}_+^*\,. \tag{A.5.4}$$

The grading on $\bigwedge^{\frac{\infty}{2}+\bullet}\mathfrak{g}$ is defined by setting $\deg\omega_0 = 0$, $\deg\mathfrak{g} = -1$ and $\deg\mathfrak{g}^* = 1$. Let us choose a basis $\{e_i\}_{i\in\mathbb{Z}}$ of $\mathfrak{g}$ compatible with the $\mathbb{Z}$-grading, such that $\mathfrak{g}_- = \mathrm{Span}\{e_0, e_{-1}, \dots\}$. Let $\{e_i^*\}$ denote the dual basis. In this basis, ω_0 is can be written as a determinant with an infinite tail

$$\omega_0 = e_0^* \wedge e_{-1}^* \wedge e_{-2}^* \wedge \dots\,, \tag{A.5.5}$$

hence the name semi-infinite. Thus, a generic element is given by

$$\omega = \alpha_1 \wedge \alpha_2 \wedge \dots \wedge e_{-n}^* \wedge \dots\,. \tag{A.5.6}$$

Note that ω and ω_0 ``agree'' after a finite number of terms. The action of the Clifford algebra is given by

$$\begin{aligned} \epsilon(\alpha)\omega &= \alpha \wedge \omega\,, \quad \alpha \in \mathfrak{g}^*\,, \\ \iota(x)\alpha_1 \wedge \alpha_2 \wedge \dots &= \sum_{k\geq 1}(-1)^{k+1}\{x, \alpha_k\}\alpha_1 \wedge \alpha_2 \wedge \dots \wedge \widehat{\alpha_k} \wedge \dots\,, \quad x \in \mathfrak{g}\,, \end{aligned} \tag{A.5.7}$$

where the $\widehat{\ }$ denotes an omitted entry.

Following [FGZ86],we define the normal ordering operator on semi infinite forms as

$$:\iota(e_i)\epsilon(e_i^*):=\begin{cases}\iota(e_i)\epsilon(e_i^*) & \text{for } i\leq 0\\ \epsilon(e_i^*)\iota(e_i) & \text{for } i>0\end{cases}\ . \tag{A.5.8}$$

We can now define a $\mathfrak{g}$-module structure on $\bigwedge^{\frac{\infty}{2}+\bullet}\mathfrak{g}$ by the action

$$\rho(x)=\sum_{i\in\mathbb{Z}}:\iota([x,e_i])\epsilon(e_i^*):+\beta(\mathfrak{g})\,,\quad x\in\mathfrak{g}\,. \tag{A.5.9}$$

As complicated as this may look, the action of $\rho(x)$ is actually the natural action [FGZ86]

$$\rho(x)\alpha_1\wedge\alpha_2\wedge\cdots=\sum_{k>1}\alpha_1\wedge\alpha_2\wedge\cdots\wedge(\mathrm{ad}^*(x)\alpha_k)\wedge\ldots\,, \tag{A.5.10}$$

where $\mathrm{ad}^*(x)$ is the coadjoint action of $\mathfrak{g}$ on $\mathfrak{g}^*$. Thus $\bigwedge^{\frac{\infty}{2}+\bullet}\mathfrak{g}$ has the structure of a $\mathfrak{g}$-module and in particular is an object in category $\mathcal{O}_\mathfrak{g}$.

Let $M\in\mathcal{O}_\mathfrak{g}$, then $M\otimes\bigwedge^{\frac{\infty}{2}+\bullet}\mathfrak{g}$ can be given a $\mathfrak{g}$-module structure. The grading on $\bigwedge^{\frac{\infty}{2}+\bullet}\mathfrak{g}^*\otimes M$ is inherited from the grading on $\bigwedge^{\frac{\infty}{2}+\bullet}\mathfrak{g}$ and setting $\deg M=0$. We define a differential d on $\bigwedge^{\frac{\infty}{2}+\bullet}\mathfrak{g}^{\otimes}M$ by

$$d=\sum_i e_i\otimes\epsilon(e_i^*)+\sum_{i<j}:\iota([e_i,e_j])\epsilon(e_i^*)\epsilon(e_j^*):+\epsilon(\beta)\,. \tag{A.5.11}$$

It can be shown that d satisfies $d^2=0$ (see [Vor93, FGZ86]) and has $\deg d=1$ with respect to the grading on $\bigwedge^{\frac{\infty}{2}+\bullet}\mathfrak{g}^*\otimes M$. This is the semi-infinite analogue of the usual Chevalley-Eilenberg differential in finite dimensional Lie algebra cohomology. Thus, as a generalisation of the Chevalley-Eilenberg complex, we define the Feigin standard complex for a $\mathfrak{g}$-module M as the cochain complex $C^{\frac{\infty}{2}+\bullet}(\mathfrak{g},M)=(\bigwedge^{\frac{\infty}{2}+\bullet}\mathfrak{g}^*\otimes M,d)$.

While it may seem arcane, the Feigin standard complex is a very familiar construction in vertex algebraic language. Suppose, for now that $\mathfrak{g}$ is an affine Kac–Moody Lie algebra $\widehat{\mathfrak{g}}$.

Let us introduce a friendlier notation. We write the actions of the Clifford algebra as $\epsilon(e_i^*) = c^i$ and $\iota(e_i) = b_i$, i.e. the Clifford algebra is the (b, c) ghost system of BRST. The space of semi-infinite forms is nothing more than the vacuum module of the ghost system and so has the structure of a vertex algebra.

Let $J^i(z)$ be the generating currents of $V^\kappa(\mathfrak{g})$. In this notation, the differential (A.5.11) is the zero mode of the BRST current

$$J_{BRST}(z) = \sum_i (J^i c^i)(z) + \sum_{i,j,k} f_{ij}{}^k : c^i c^j b_k : (z) \,, \tag{A.5.12}$$

where $f_{ij}{}^k$ are structure constants of $\mathfrak{g}$. Therefore, the Feigin standard complex is equivalent to the usual vertex algebra BRST complex.

A.5.3 Semi-infinite cohomology

Having defined the Feigin standard complex, we define the semi-infinite cohomology of a Lie algebra $\mathfrak{g}$ (with semi-infinite structure) with coefficients in a $\mathfrak{g}$-module M as

$$\mathrm{H}^{\frac{\infty}{2}+\bullet}(\mathfrak{g}, M) = H^\bullet\left(\bigwedge{}^{\frac{\infty}{2}+\bullet}\mathfrak{g}^* \otimes M, d\right). \tag{A.5.13}$$

In ordinary Lie algebra cohomology, one can compute the cohomology of a Lie algebra *relative* to some subalgebra. We can extend this naturally to the semi-infinite case. Let $\mathfrak{h} \subset \mathfrak{g}$ be a subalgebra of a $\mathbb{Z}$-graded Lie algebra, $\mathfrak{g}$, that admits a semi-infinite structure. We define the *relative* Feigin standard complex [FGZ86] as

$$C^{\frac{\infty}{2}+\bullet}(\mathfrak{g}, \mathfrak{h}, M) = \{c \in C^{\frac{\infty}{2}+\bullet}(\mathfrak{g}, M) \,|\, \iota(x)c = (x \otimes 1 + 1 \otimes \rho(x))c = 0) \,, \forall x \in \mathfrak{h}\}\,. \tag{A.5.14}$$

Thus we can define the semi-infinite cohomology of $\mathfrak{g}$ relative to $\mathfrak{h}$ with coefficients in M by

$$\mathrm{H}^{\frac{\infty}{2}+\bullet}(\mathfrak{g}, \mathfrak{h}, M) = H^\bullet(C^{\frac{\infty}{2}+\bullet}(\mathfrak{g}, \mathfrak{h}, M))\,. \tag{A.5.15}$$

This is a useful construction, which allows us to limit ourselves to certain sectors of cohomology.

Computing, semi-infinite cohomology with coefficients in an arbitrary module can be fairly involved. There are two main tools that make this computation tractable. First, we have a spectral sequence that converges on the cohomology $\mathrm{H}^{\frac{\infty}{2}+\bullet}(\mathfrak{g}, M)$.

Theorem A.5.3. *(Theorem 2.3 of [Vor93]) For a Lie algebra* $\mathfrak{g} = \mathfrak{g}_- \oplus \mathfrak{g}_+$ *and* $M \in \mathcal{O}$*, we have a spectral sequence* $(E_r^{p,q}, d_r^{p,q})$ *with* $p \geq 1$*,* $q \leq 0$ *such that*

- $E_1^{p,q} = H^q(\mathfrak{g}_-, \bigwedge^{\frac{\infty}{2}+p}(\mathfrak{g}/\mathfrak{g}_-) \otimes M) = H_{-q}(\mathfrak{g}_-, \bigwedge^{\frac{\infty}{2}+\bullet}(\mathfrak{g}_-) \otimes \bigwedge^{\frac{\infty}{2}+p}(\mathfrak{g}/\mathfrak{g}_-) \otimes M)$*, where we have used Poincaré in the second equality and* H_{-q} *is the Koszul homology*
- $\varprojlim E_r^{p,q} = E_\infty^{p,q}$*, where the limit is taken with respect to the epimorphisms* d_r
- $E_\infty^{p,q} = \mathrm{gr}^p\, \mathrm{H}^{\frac{\infty}{2}+p+q}(\mathfrak{g}, M)$

Thus, the spectral sequence converges to the cohomology $\mathrm{H}^{\frac{\infty}{2}+\bullet}(\mathfrak{g}, M)$*.*

This is just the familiar Hochschild–Leray–Serre spectral sequence in the context of semi-infinite cohomology. There is a similar spectral sequence with respect to $\mathfrak{g}_+$—see Theorem 2.2 of [Vor93].

By using these two spectral sequences, one can establish the second valuable tool, a vanishing theorem[8]

Theorem A.5.4. *(Theorem 2.1 of [Vor93]) Let* $M \in \mathcal{O}$ *such that it is injective as a* $\mathfrak{g}_+$ *module and projective as a* $\mathfrak{g}_-$ *module. Then*

$$\mathrm{H}^{\frac{\infty}{2}+p}(\mathfrak{g}, M) = \begin{cases} M_{\mathfrak{g}_-}^{\mathfrak{g}_+} & \text{for } p = 0\,, \\ 0 & \text{else,} \end{cases} \tag{A.5.16}$$

where $M_{\mathfrak{g}_-}^{\mathfrak{g}_+} = \mathrm{im}((M \otimes \mathcal{L}_\beta)^{\mathfrak{g}_+} \to (M \otimes \mathcal{L}_\beta)_{\mathfrak{g}_-})$ *which is the natural projection of the* $\mathfrak{g}_+$*-invariants onto the* $\mathfrak{g}_-$ *co-invariants. More concretely,*

$$M_{\mathfrak{g}_-}^{\mathfrak{g}_+} = \{m \in M \,|\, \mathfrak{g}_+ m = 0\} / \{m \in M \,|\, \mathfrak{g}_+ m = 0 \quad \text{and } m = (x + \beta x)m' \quad \text{for some } x \in \mathfrak{g}_-\}\,.$$

[8]We believe that there is a misprint in the original text which states that the cohomology vanishes for $p = 0$ and is $M_{\mathfrak{g}_-}^{\mathfrak{g}_+}$ for others. The proof of this theorem in [Vor93] agrees with our statement.

We have met a number of cohomological constructions in relation to the vertex algebra correspondence of [BLL+15, BPRvR15]. Let us recast them in the light of semi-infinite cohomology, in line with the construction of [Ara18].

A.5.4 Gauging

In Section 1.2.5, we have described the gauging prescription for vertex algebras. Given a vertex algebra $V \in \mathrm{KL}_{\kappa_g}$ we can gauge the action of the affine Lie algebra by introducing a (b, c) ghost system and performing BRST reduction. The ghost system is nothing more than the space of semi-infinite forms $\bigwedge^{\frac{\infty}{2}+\bullet}\mathfrak{g}$ that we have introduced and so we can write this in the semi-infinite language as the cohomology

$$\mathrm{H}^{\frac{\infty}{2}+\bullet}(\hat{\mathfrak{g}}_{\kappa_g}, V)\,. \tag{A.5.17}$$

We have the canonical embedding $\mathfrak{g} \hookrightarrow \hat{\mathfrak{g}}_{-\kappa_g}$ via $x \mapsto xt^{-1}$ and thus we have a short exact sequence of Lie algebras

$$0 \to \mathfrak{g} \to \hat{\mathfrak{g}}_{-\kappa_g} \to \hat{\mathfrak{g}}_{-\kappa_g}/\mathfrak{g} \to 0\,. \tag{A.5.18}$$

For any such sequence, we have an associated Hochschild–Serre spectral sequence, which computes the cohomology $\mathrm{H}^{\frac{\infty}{2}+\bullet}(\mathfrak{g}, V)$. The second page is

$$E_2^{p,q} \cong \mathrm{H}^{\frac{\infty}{2}+p}(\hat{\mathfrak{g}}_{-\kappa_g}, \mathfrak{g}, V) \otimes H^q(\mathfrak{g}, \mathbb{C}) \tag{A.5.19}$$

where $H^q(\mathfrak{g}, \mathbb{C})$ is just ordinary Lie algebra cohomology with coefficients in $\mathbb{C}$. In fact, the spectral sequence collapses on the second page [Ara18] and so

$$\mathrm{H}^{\frac{\infty}{2}+\bullet}(\hat{\mathfrak{g}}_{-\kappa_g}, V) \cong \mathrm{H}^{\frac{\infty}{2}+\bullet}(\hat{\mathfrak{g}}_{-\kappa_g}, \mathfrak{g}, V) \otimes H^\bullet(\mathfrak{g}, \mathbb{C})\,. \tag{A.5.20}$$

This does not seem to be particularly helpful, until we introduce:

Theorem A.5.5 ([Ara18, Proposition 3.4]). *Suppose $V \in \mathrm{KL}_{-\kappa_g}$, such that V is projective* *as a $U(t^{-1}\mathfrak{g}[t^{-1}])$ module and injective as a $U(t\mathfrak{g}[t])$ module. Then*

$$\mathrm{H}^{\frac{\infty}{2}+\bullet}(\hat{\mathfrak{g}}_{-\kappa_g}, \mathfrak{g}, V) = 0 \quad \textit{for } i \neq 0\,. \tag{A.5.21}$$

Proof. Follows from Theorem A.5.4. □

A.5.5 Drinfeld-Sokolov reduction

Let f be a nilpotent element of $\mathfrak{g}$, by the Jacobson–Morozov theorem (Theorem A.1.4, this can be completed to an $\mathfrak{sl}_2$ triple, (e, h, f) in $\mathfrak{g}$. The Cartan element h induces an integral grading on $\mathfrak{g}$, see Section A.1.3. We set $\mathfrak{g}_{>0} = \bigoplus_{i>0} \mathfrak{g}_i$ and $\mathfrak{g}_{\geq 2} = \bigoplus_{i \geq 2} \mathfrak{g}_i$. The Lie algebra that will appear in our semi-infinite cohomology is

$$\tilde{\mathfrak{g}} = \mathfrak{g}_{>0}[t, t^{-1}]\,. \tag{A.5.22}$$

with the natural grading by loop-rotation. Let χ be a character of $\mathfrak{g}_{\geq 1}[t^{-1}]$ such that

$$\begin{aligned} &\chi : \mathfrak{g}_{\geq 2}[t, t^{-1}] \to \mathbb{C}\,, \\ &\chi : xt^n \mapsto \delta_{n,-1}(x, f)\,. \end{aligned} \tag{A.5.23}$$

Note that χ is completely determined by f. We define a one dimensional representation $\mathbb{C}_\chi$ of the subalgebra $\mathfrak{g}_{>0}[t] \oplus \mathfrak{g}_{\geq 2}[t^{-1}] \subset \mathfrak{g}_{>0}[t, t^{-1}]$ by letting $\mathfrak{g}_{>0}[t]$ act trivially and $\mathfrak{g}_{\geq 2}[t^{-1}]$ act via χ. This induces a vacuum representation F_χ of $\mathfrak{g}_{>0}[t, t^{-1}]$ via the usual,

$$F_\chi = \mathrm{Ind}_{\mathfrak{g}_{>0}[t] \oplus \mathfrak{g}_{\geq 2}[t^{-1}]}^{\mathfrak{g}_{>0}[t,t^{-1}]} \mathbb{C}_\chi = U(\mathfrak{g}_{>0}[t, t^{-1}]) \otimes_{U(\mathfrak{g}_{>0}[t] \oplus \mathfrak{g}_{\geq 2}[t^{-1}])} \mathbb{C}_\chi\,. \tag{A.5.24}$$

This looks overly abstract but F_χ is nothing more than the $\beta\gamma$-system associated to the symplectic vector space $\mathfrak{g}_1$ and so is a fairly straightforward vertex algebra.

Let $V \in \mathrm{KL}_\kappa$, then, in particular, V is also an object in the category $\mathcal{O}$ of $\mathfrak{g}_{>0}[t, t^{-1}]$ modules. The module $V \otimes F_\chi$ is again a $\mathfrak{g}_{>0}[t, t^{-1}]$ module with the diagonal action.

Definition A.5.6. The *Drinfel'd–Sokolov* reduction of $V \in \mathrm{KL}_\kappa$, with respect to the nilpotent f, is defined as the semi-infinite cohomology,

$$H^{\bullet}_{\mathrm{DS},f}(V) = \mathrm{H}^{\frac{\infty}{2}+\bullet}(\mathfrak{g}_{>0}[t,t^{-1}], V \otimes F_{\chi})\,. \tag{A.5.25}$$

Remark A.5.7. The Drinfel'd–Sokolov only depends on the G-conjugacy class of f. To see this, note that $V \in \mathrm{KL}_\kappa$ is G-integrable and so one can twist the action of $\widehat{\mathfrak{g}}$ by the action of G, taking f to any other element in its orbit. These twists act by automorphisms and so the resulting reductions are isomorphic.

In general, if $V \in \mathrm{KL}_\kappa$ is a vertex algebra object, then $H^{\bullet}_{\mathrm{DS},\Lambda}(V)$ is also a vertex algebra. For the special case where $V = V^\kappa(\mathfrak{g})$, the resulting vertex algebra $H^0_{DS,f}(V^\kappa)$ is the W-algebra associated with $(\mathfrak{g}, f)$ at level κ, $\mathcal{W}^\kappa(\mathfrak{g}, f)$. Therefore we have a functor

$$\mathrm{KL}_\kappa \xrightarrow{H^0_{\mathrm{DS},f}} \mathcal{W}^\kappa(\mathfrak{g}, f)\ . \tag{A.5.26}$$

Once again, we have a vanishing theorem.

Theorem A.5.8 ([Ara10, Theorem 4.3.2])**.** *For any $M \in \mathrm{KL}_\kappa$ and any nilpotent element $f \in \mathcal{N}$, the cohomology $H^i_{\mathrm{DS},f}(M) = 0$ for $i \neq 0$ and so $H^0_{\mathrm{DS},f} : \mathrm{KL}_\kappa \to \mathcal{W}^\kappa(\mathfrak{g}, f)$ is an exact functor.*

Remark A.5.9. Suppose, $V \in \mathrm{KL}_\kappa$ is a vertex algebra object, and also suppose that V is conformal. Then the reduction $H^0_{\mathrm{DS,f}}(V)$ is a conformal vertex algebra.

An explicit expression for the conformal vector can be found in Section 2.2 of [KRW03], but the central charge of $H^0_{\mathrm{DS,f}}(V)$ is related to the central charge c_V of V by

$$c_{H^0_{\mathrm{DS}}(V)} = c_V - \dim \mathbb{O}_f - \tfrac{3}{2}\dim \mathfrak{g}_1 + 12(\rho, h) - 3(\kappa + h^\vee)(h,h) \tag{A.5.27}$$

where h is the Cartan element of the $\mathfrak{sl}_2$ triple, $\mathbb{O}_f$ is the orbit of f and ρ is the Weyl vector.

Suppose $\kappa = \kappa_c$, then we have the isomorphism [FF92] of vertex algebras

$$\mathfrak{z}(\mathfrak{g}) \to \mathcal{W}^{\kappa_c}(\mathfrak{g}, p_{-1})\,, \tag{A.5.28}$$

where p_{-1} is our standard representative for the principal nilpotent orbit. Applying Theorem A.5.8, we have an exact functor

$$H^0_{\mathrm{DS},p_{-1}} : \mathrm{KL}_{\kappa_c} \to \mathcal{Z}\text{-Mod}\,. \tag{A.5.29}$$

We shall primarily be interested in principal DS reduction and so as shorthand we use H^0_{DS} to denote $H^0_{\mathrm{DS},p_{-1}}$.

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