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Feynman formulas for *qp*- and *pq*-quantization of some Vladimirov type time-dependent Hamiltonians on finite adeles

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Abstract

Let Q be the d-dimensional space of finite adeles over the algebraic number field K and let $P = Q^*$ be its dual space. For a certain type of Vladimirov type time-dependent Hamiltonian $H_V(t) : Q \times P \to \mathbb{C}$ we construct the Feynman formulas for the solution of the Cauchy problem with the Schrödinger operator $\widehat{-H_V(t)}$, where the caret operator stands for the qp- or pq-quantization.

Keywords Pseudodifferential operators · Vladimirov operator · Time-dependent Schrödinger equation · Quantization · Feynman formula · Chernoff product formula · Finite adeles

Mathematics Subject Classification 81S40 · 81S99 · 35S05 · 47D08 · 11R56

1 Introduction

The idea of applying p-adic analysis in physics is due to Volovich. In his 1987 CERN preprint[1] (published later as a paper [54]) he proposed that non-Archimedean space-time geometry should be considered, and also advocated the development of p-adic quantum mechanics. The first papers on p-adic quantum systems by Vladimirov and Volovich appeared in 1989 [50, 51]. Since then, work on p-adic physics has been developing at a great pace and a substantial number of papers have appeared in this area.

To get acquainted with the development of p-adic physics we recommend the reader [16, 23, 38] and the monograph [52] by Vladimirov, Volovich and Zelenov. Relatively recent review of literature concerning non-Archimedean mathematical physics can be

[1] I. V. Volovich. CERN-TH. 4781/87, Geneva, 11 pp., 1987.

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found in [17]. For a variety of applications of p-adic analysis in physics the reader may consult, for example, some of the recent papers on the porous medium equation [3, 24], energy landscapes [29, 61], image processing [59], and on p-adic Laplacian on graphs [10].

Since no prime number p is in any particular way special it seems reasonable to study physical systems in all p-adic fields. This observation is expressed in Manin's article [32], where he writes: "On the fundamental level our world is neither real nor p-adic, it is adelic."

Consequently, many works have appeared in which the phase space of the physical system is a ring of rational adeles $\mathbb{A}_{\mathbb{Q}}$. We mention here only a few papers: [15] on adelic quantum oscillator, [39] on quantum fields and strings on adeles, and an application of adelic quantum mechanics to adelic quantum cosmology [18]. There is a strong connection between quantum mechanics and probability theory. Stochastic processes with values in $\mathbb{A}_{\mathbb{Q}}$ (or more generally in $\mathbb{A}_K$, where K is an algebraic number field), in particular adelic Brownian motion, has recently attracted renewed interest of researchers [46, 47, 57, 58]. Early works on non-Archimedean diffusion processes include [2, 26–28, 45, 49]. The recent works [25, 60, 62] give background on the applications of ultrametric diffusions.

Our research presented here is motivated by the Smolyanov and Shamarov paper [42]. They consider the Vladimirov operator with variable coefficient and the corresponding Schrödinger operator acting on functions defined on the p-adic configuration space $\mathbb{Q}_p^d$ and give the representation of the solution of the Schrödinger equation in terms of Feynman-type path integrals.

In [48] the space of adeles $\mathbb{A}_K^d$ over the algebraic number field as the configuration space is considered and the results from [42] are generalized. Specifically, in [48], we consider a class of Hamiltonians $H_V = H_0 + V$ on $\mathbb{A}_K^d \times \mathbb{A}_K^d$ such that their qp-quantization $\widehat{H}^{qp} = M_g D^\alpha + V$, where D^α is the Vladimirov operator and M_g is the operator of multiplication by a real-valued function g. We obtain the Feynman-Kac formula for the propagator of a quantum mechanical system with the space $\mathbb{A}_K^d$ generated by the Schrödinger operator $-\widehat{H_V}^{qp}$.

The main aim of this paper is to generalize the results obtained in [48] to the case of time-dependent Hamiltonians as well as to consider the pq-quantization.

1.1 Setting and main results

Let K be an algebraic number field (i.e. a finite extension of $\mathbb{Q}$). Let $\mathscr{P}(K)$ ($\mathscr{P}_{\mathrm{f}}(K)$, resp.) denote the set of places (finite places, resp.) of K. By K_v we denote the completion of K with respect to the place v, and we let $|\cdot|_v$ be the normalized valuation (see (2.2)). The space K_v^d is endowed with the supremum norm $\|x\|_v = \max_{1\le i\le d} |x_i|_v$. The *finite* (*d-dimensional) adele ring* of K is defined as

$$\mathbb{A}_K^d = \left\{ x = (x_v) \in \prod_{v\in\mathscr{P}_{\mathrm{f}}(K)} K_v^d \,\middle|\, \|x_v\|_v \le 1 \text{ for almost all } v \in \mathscr{P}_{\mathrm{f}} \right\}.$$

Thus, the ring $\mathbb{A}_K^d$ is a restricted direct product, i.e. the product

$$\prod_{v \in \mathscr{P}_{\mathrm{f}}(K)} K_v^d$$

relative to

$$R_v^d = \{x \in K_v^d \mid \|x_v\|_v \leq 1\}, \quad v \in \mathscr{P}_{\mathrm{f}}(K).$$

We define a *restricted direct product topology* on $\mathbb{A}_K^d$ by specifying a base of neighborhoods of the identity, consisting of sets of the form $\prod \mathscr{N}_v$, where $\mathscr{N}_v$ is a neighborhood of the identity in K_v^d and $\mathscr{N}_v = R_v^d$ for almost all $v \in \mathscr{P}_{\mathrm{f}}$. The space $\mathbb{A}_K^d$ is a second countable, locally compact Haussdorff topological space. The elements of $\mathbb{A}_K^d$ are called the (d-dimensional) *adeles*.

The ring of adeles of K is a locally compact Abelian group under its addition, while

$$R_{\mathbb{A}_K^d} = \{a \in \mathbb{A}_K^d : a_v \in R_v^d \text{ for all } v \in \mathscr{P}_{\mathrm{f}}\}$$

is an open and compact subring of $\mathbb{A}_K^d$. By $\mu_{\mathbb{A}_K^d}$ we denote the Haar measure on $\mathbb{A}_K^d$ normalized so that $\mu_{\mathbb{A}_K^d}(R_{\mathbb{A}_K^d}) = 1$. Let $Q = \mathbb{A}_K^d$. When Q is considered a copy of its dual we denote this space by P.

Definition 1.1 (qp-quantization) Let $H : Q \times P \to \mathbb{C}$ be a continuous function. Let $\widehat{H}^{qp}$ be a pseudo-differential operator (PDO, for short) with symbol H defined as follows. The value of $\widehat{H}^{qp}$ on a function φ from the Bruhat-Schwartz space $\mathscr{D}(Q)$ is a continuous function $\widehat{H}^{qp}\varphi : Q \to \mathbb{C}$, given by

$$\widehat{H}^{qp}\varphi(q) = \mathscr{F}^{-1}_{\mathbb{A}_K^d}\left(H(q,\cdot)\mathscr{F}_{\mathbb{A}_K^d}\varphi(\cdot)\right)(q),$$

where $\mathscr{F}_{\mathbb{A}_K^d}\varphi$ is the Fourier transform of φ.

Definition 1.2 (pq-quantization) The value of the PDO $\widehat{H}^{pq}$ with pq-symbol is a (generalized) function $\widehat{H}^{pq}\varphi \in \mathscr{D}(Q)'$ whose Fourier transform $\mathscr{F}_{\mathbb{A}_K^d}(\widehat{H}^{pq}\varphi)$ is a regular generalized function with continuous density

$$p \mapsto \mathscr{F}_{\mathbb{A}_K^d}(H(\cdot, p)\varphi(\cdot))(p).$$

If a vector subspace $S \subset \mathscr{D}'$ is a Banach space with respect to some norm and contains $\mathscr{D}(Q)$, and if the set $D^S_{\widehat{H}^{qp}} = \{\varphi \in \mathscr{D}(Q) \mid \widehat{H}^{qp}\varphi \in S\}$ is dense in S and the restriction $\widehat{H}^{qp}|_{D^S_{\widehat{H}^{qp}}}$ is closable as an operator on S, then such a closure is denoted by $\widehat{H}^{qp}_S$ and is called a PDO with qp-symbol H on the space S. The PDO with pq-symbol H on the space S is defined in a similar way.

For $\alpha > 1$ define

$$f^{\alpha}(q, p) = \|p\|^{\alpha}_{\mathbb{A}_K^d},$$

where $\| \cdot \|_{\mathbb{A}_K^d}$ is the "norm" on $\mathbb{A}_K^d$ such that the topology generated by it (i.e. by the metric $\|x - y\|_{A_K^d}$) agrees with the restricted product topology (see Proposition 2.1).

Definition 1.3 (Vladimirov operator) Since

$$\widehat{f^{\alpha}}^{qp} = \widehat{f^{\alpha}}^{pq} \tag{1.1}$$

we denote the PDO in (1.1) by D^{α}, i.e.

$$D^{\alpha}\varphi(q) = \mathscr{F}^{-1}_{\mathbb{A}_K^d}\left(\| \cdot \|^{\alpha}_{\mathbb{A}_K^d} \mathscr{F}_{\mathbb{A}_K^d}\varphi(\cdot)\right)(q).$$

We refer to D^{α} (as well as to the corresponding operators on some S) as the *Vladimirov operator* (*of fractional differentiation of order* α).

The operator $D^{\alpha}_{L^2(Q)} = (\widehat{f^{\alpha}})_{L^2(Q)}$ is self-adjoint and positive definite [52]. The density of the Fourier transform of $D^{\alpha}\varphi$ is given by

$$\|p\|^{\alpha}_{\mathbb{A}_K^d} \mathscr{F}_{\mathbb{A}_K^d}\varphi(p).$$

Remark 1.4 The Vladimirov operator and its corresponding Green function appear in many places in mathematical physics, for example in the p-adic string theory [20, 53] and p-adic AdS/CFT correspondence [9, 21]. A role of Tate's thesis [44] in adelic physics is pointed out in [22] where it is shown that the Green function for the Vladimirov operator is given by the local functional equation for zeta integrals.

Let $C_0(Q, \mathbb{C})$ be the space of all continuous functions from Q to $\mathbb{C}$ vanishing at infinity considered with the uniform norm $\| \cdot \|_{L^{\infty}(\mathbb{A}_K^d)}$ and let $C_b(Q, \mathbb{R})$ be the space of all real-valued continuous bounded function on Q. Let, for $t \in \mathbb{R}_+$,

$$g(t, \cdot) = g(t)(\cdot) \in C_b(Q, \mathbb{R}).$$

We set, for $q \in Q,\ p \in P$,

$$H_0(t, q, p) = g(t, q)\|p\|^{\alpha}_{\mathbb{A}_K^d}.$$

Let $V : \mathbb{R}_+ \times Q \to \mathbb{C}$ be a continuous function. Define the Vladimirov type time-dependent Hamilton function on $Q \times P$ by

$$H_V(t, q, p) = H_0(t, q, p) + V(t, q). \tag{1.2}$$

From now on we work under the following two assumptions about g and V, respectively.

Remark 1.5 The context in which we work has strong ties to number theory. The Hilbert-Pólya conjecture states that the non-trivial zeros of the Riemann zeta function correspond to eigenvalues of a self-adjoint operator. There has been recent work suggesting a connection between the non-trivial zeros of the Riemannian function and a spectrum of operators that are quantizations of certain classical operators used in quantum mechanics [5–8]. Although these results are obtained in the case of Archimedean fields, they still seem likely to give direction to new research in non-Archimedean number theory.

Assumption 1.6 We assume that for every $t \geq 0$, $g(t, \cdot) \in C_b(Q, \mathbb{R}_+)$. Moreover, for every $t \geq 0$, there exists $c(t) > 0$ such that, for every $x \in Q = \mathbb{A}_K^d$,

$$g(t, x) \geq c(t) \tag{1.3}$$

and there exists $C > 0$ such that for all $t \geq 0$,

$$c(t) \geq c > 0,$$

Assumption 1.7 The function $V : \mathbb{R}_+ \times Q \to \mathbb{C}$ is a continuous and bounded function with positive real part $\operatorname{Re} V$, which is separated from 0, i.e. there exist constants $C > 0$ and $c > 0$ such that

$$\|V(\cdot, \cdot)\|_{L^\infty(\mathbb{R}_+ \times \mathbb{A}_K^d)} \leq C, \text{ and } \operatorname{Re} V(t, q) > c \text{ for all } q \in Q \text{ and } t \in \mathbb{R}_+. \tag{1.4}$$

Definition 1.8 Let $A(t)$ be a generator of an evolution $U(t, s)$ in a Banach space $S \subset \mathscr{D}'$ and let $\psi_s \in S$. A solution of the *Cauchy problem (or* the initial value problem*)* $(A(t), \psi_s)$:

$$\partial_t \psi(t, x) = A(t)\psi(t, x) \text{ and } \psi(s, x) = \psi_s(x) \tag{1.5}$$

is the mapping $\Psi : [s, +\infty) \to S$ defined by $\Psi(t) = U(t, s)\psi_s$ as well as the corresponding function $\psi(t, x)$ for which $\psi(t, \cdot) = \Psi(t)$.

Definition 1.9 A *Feynman formula* is a representation of a solution of the Cauchy problem (1.5) (or, equivalently, a representation of the evolution $U(t, s)$ generated by $A(t)$) by a limit of n-fold iterated integrals, i.e. if $\max |t_{i+1} - t_i| \to 0$ then

$$U(t, s)\psi_s = \lim_{n\to\infty} R(t_n, t_{n-1}) \ldots R(t_1, t_0)\psi_s, \tag{1.6}$$

where $t_0 = s$, $t_n = t$ and $R(t_k, t_{k-1})$ are integral operators.

These approximations in many cases contain only elementary functions as integrands and, therefore, can be used for direct calculations and simulations.

Definition 1.10 We call identity (1.6) a *Lagrangian Feynman formula*, if the $R(t_k, t_{k-1})$ are integral operators with elementary kernels; if the $R(t_k, t_{k-1})$ are pseudo-differential operators, we speak of *Hamiltonian Feynman formulas*.

The main objective of this work is to find a representation of the solution of the Cauchy problem (1.5) with $A(t) = \widehat{-H_V(t)}^{qp}_{C_0}$ and $A(t) = \widehat{-H_V(t)}^{pq}_{L^1}$ under assumptions (1.3) and (1.4) or equivalently, to find a representation of the evolution operators $U(t,s)$ generated by $A(t)$, $t \geq 0$, in the form of the Feynman formula.

The following theorems are two of our main results and are the starting points for getting the other Feynman formulas.

For every $T \geq t \geq s \geq 0$, define the operator

$$F^{qp}(t,s) = \widehat{(e^{-\int_s^t H_V(u)du})}^{qp}_{C_0}. \tag{1.7}$$

Theorem 1.11 (The qp-Feynman formula) *The family* $\{\widehat{-H_V(t)}^{qp}_{C_0}\}_{t\in[0,T]}$ *generates a family of evolution operators* $U^{qp}(t,s)$ *on the space* $C_0 = C_0(\mathbb{A}_K^d, \mathbb{C})$ *which gives the solution of the Cauchy problem*

$$\partial_t \psi(t,x) = \widehat{-H_V}^{qp}_{C_0}\psi(t,x), \qquad \psi(s,x) = \psi_s(x).$$

Moreover,

$$F^{qp}(t_n, t_{n-1}) \dots F^{qp}(t_1, t_0)\varphi \to U^{qp}(t_n, t_0)\varphi \text{ in } C_0(\mathbb{A}_K^d, \mathbb{C}) \tag{1.8}$$

as $\max|t_{i+1} - t_i| \to 0$ *uniformly with respect to* $t_n, t_0 \in [0,T]$ *for every function* $\varphi \in C_0(\mathbb{A}_K^d, \mathbb{C})$.

Let, for $T \geq t \geq s \geq 0$,

$$F^{pq}(t,s) = \widehat{\left(e^{-\int_s^t H_V(u)du}\right)}^{pq}_{L^1}. \tag{1.9}$$

Theorem 1.12 (The pq-Feynman formula) *The family* $\{\widehat{-H_V(t)}^{pq}_{L^1}\}_{t\in[0,T]}$ *generates the family of evolution operators* $U^{pq}(t,s)$ *on the space* $L^1(\mathbb{A}_K^d)$ *which gives the solution of the Cauchy problem*

$$\partial_t \psi(t,x) = \widehat{-H_V}^{pq}_{L^1}\psi(t,x), \qquad \psi(s,x) = \psi_s(x).$$

Moreover,

$$F^{pq}(t_n, t_{n-1}) \dots F^{pq}(t_1, t_0)\varphi \to U^{pq}(t_n, t_0)\varphi \text{ in } L^1(\mathbb{A}_K^d) \tag{1.10}$$

as $\max|t_{i+1} - t_i| \to 0$ *uniformly with respect to* $t_n, t_0 \in [0,T]$ *for every function* $\varphi \in L^1(\mathbb{A}_K^d)$.

Remark 1.13 Note that formulas (1.8) and (1.10) allow us to numerically find the solution of the corresponding Schrödinger equation. Let us note that in some cases the Feynman formula can lead to a solution of the Schrödinger equation in the form of the Feynman-Kac formula. This is the case, for example, when the Hamiltonian does not depend on time or when only the potential V depends on time. In the former case, we can express the solution as an integral over the trajectories of the process generated by the operator $\widehat{H}^{qp}$, see [48, Theorem VII.6],

$$\psi_t(x) = \mathbf{E}_x e^{-\int_0^t V(\gamma(u))du} \psi_0(\gamma(t)), \qquad \gamma(0) = x.$$

It is widely known that in the Archimedean case (i.e. when the analysis is done over $\mathbb{R}$ or $\mathbb{C}$) the Feynman-Kac formula establishes a link between parabolic partial differential equations and stochastic processes. The probabilistic aspect of solving differential equations has many applications - both in pure mathematics and in applications. For applications in quantum physics, see the monograph [30, 31].

For the reader's convenience, we give below examples of fairly simple adelic Hamiltonians, for which the analysis of their quantization is relatively uncomplicated. We must limit ourselves to operators that do not depend on time.

Example 1.14 For details see [57]. Let $K = \mathbb{Q}$, $d = 1$, $Q \times P = \mathbb{A}_{\mathbb{Q}} \times \mathbb{A}_{\mathbb{Q}}$. Let, for $j = 1, 2, \ldots$, $|\cdot|_j$ be the p-adic absolute value corresponding to the j-th prime p. Thus $|\cdot|_1$ is the 2-adic absolute value, $|\cdot|_2$ is the 3-adic absolute value, and so on. Consider the following free Hamilltonian which is time-independent,

$$H_0(q, p) = \sum_{j=1}^{\infty} \sigma_j |p_j|_j^{\alpha}, \qquad \sigma_j \geq 0, \qquad \sum_{j=1}^{\infty} \sigma_j < +\infty.$$

Let

$$H(q, p) = H_0(q, p) + V(q).$$

Notice that if φ is from the adelic Bruhat-Schwartz space $\mathscr{D}(\mathbb{A}_{\mathbb{Q}})$ (see Sect. 2.3) and depends only on one "coordinete" j then $\widehat{H}^{qp}$ acts on φ as the standard Vladimirov operator on $\mathbb{Q}_{p_j}$ multiplied by a constant term σ_j. The analysis of the Vladimirov operator on $\mathbb{Q}_{p_j}$ is very well known [52].

The operator $\widehat{H}^{qp}$ generates a semigroup T_t of operators on $L^2(\mathbb{A}_{\mathbb{Q}})$ and the corresponding stochastic process whose trajectories $\gamma(t)$ are in the Skorohod space of càdlàg (continue à droite, limite à gauche) functions from $\mathbb{R}_+$ to $\mathbb{A}_{\mathbb{Q}}$. Moreover, for $\varphi \in \mathscr{D}(\mathbb{A}_Q)$, the following Feynman-Kac formula holds,

$$T_t \varphi(x) = \mathbf{E}_x e^{-\int_0^t V(\gamma(u))du} \varphi(\gamma(t)). \tag{1.11}$$

Example 1.15 This is simplified version of the setting from [46]. Let $K = \mathbb{Q}$, $d = 1$, $Q \times P = \mathbb{A}_{\mathbb{Q}} \times \mathbb{A}_{\mathbb{Q}}$. Let

$$H_0(q, p) = \|p\|^{\alpha},$$

where $\|\cdot\|$ is a certain "norm" on $\mathbb{A}_{\mathbb{Q}}$ (see (2.3)). Then $\widehat{H}^{qp}$ is a natural generalization of the Vladimirov operator from $\mathbb{Q}_p$ to $\mathbb{A}_{\mathbb{Q}}$. Let

$$H(q, p) = H_0(q, p) + V(q)$$

Then for the semigroup of operators T_t generated by $\widehat{H}^{qp}$ the Feynman-Kac formula (1.11) holds.

The main tool in the proofs of Theorem 1.11 and Theorem 1.12 is the Chernoff product formula for evolutions (see Theorem 3.6) proved by Vuillermot [55] and Plyashechnik [36, Theorem 4].

Other types of Feynman formulas for the solutions of the Cauchy problem (1.5) (of the Schrödinger equation) with $A(t) = \widehat{-H_V(t)}^{qp}_{C_0}$ and $A(t) = \widehat{-H_V(t)}^{pq}_{L^1}$ under assumptions (1.3) and (1.4) (i.e. Theorems: 6.1, 6.2, 6.4, 6.5, 6.6) are presented in Sect. 6. They are obtained using the evolution perturbation theorem (Theorem 3.7).

1.2 Structure of the paper

In Sect. 2 we recall basic facts about algebraic number fields, define the Fourier transform for functions defined on $\mathbb{A}_K^d$, as well as the corresponding function spaces, which will be used later on in the paper.

The generalized Chernoff product formula and perturbation theorem for evolutions are presented in Sect. 3.

In Sect. 4, we study family $F^{qp}(t, s)$ and family $F^{pq}(t, s)$ and prove their properties, which we will then use in Sect. 5 to prove Theorem 1.11 and Theorem 1.12.

Finally, in Sect. 6, Hamiltonian and Lagrangian Feynman formulas for the Schrödinger equation corresponding to the qp- and pq-quantizations of the Hamiltonian H_V (defined in (1.2)) are obtained.

2 Preliminaries

2.1 Basic facts on $\mathfrak{p}$-adic fields

For more details, we recommend the reader the following monographs [33, 34, 41, 56]. Certain passages in this paragraph closely follow [41, p. 61]. Let K be an algebraic number field (i.e. a finite extension of $\mathbb{Q}$). A valuation v of K is a homomorphism $v : K \to \mathbb{R}^+ \cup \{0\}$ such that $v(x) = 0$ if and only if $x = 0$, and there is a real number $c \geq 1$ such that for all $x, y \in K$, $v(xy) = v(x)v(y)$ and $v(x+y) \leq c \max\{v(x), v(y)\}$. The valuation v is non-trivial if $v(K) \supsetneq \{0, 1\}$. The valuation v is non-Archimedean if v is non-trivial and we can set $c = 1$, and is said to be Archimedean otherwise.

We say that two valuations v_1 and v_2 of K are equivalent if there is an $s > 0$ such that $v_1(x) = v_2(x)^s$ for every $x \in K$. An equivalence class v of a non-trivial absolute value of K is called a place of K. A place v is finite if v contains a non-Archimedean absolute value, and infinite otherwise. The set of places, finite places and infinite places of K are denoted by $\mathscr{P} = \mathscr{P}(K)$, $\mathscr{P}_{\mathrm{f}} = \mathscr{P}_{\mathrm{f}}(K)$ and $\mathscr{P}_\infty = \mathscr{P}_\infty(K)$, respectively.

By Ostrovski's theorem every non-trivial valuation of $\mathbb{Q}$ is either equivalent to the usual absolute value $|\cdot|_\infty$, or to the p-adic absolute value $|\cdot|_p$ for some rational prime $p > 1$, defined by $|0|_p = 0$ and $|p^k \frac{n}{m}|_p = p^{-k}$ for $k, n, m \in \mathbb{Z}$ and $p \nmid nm$.

For every valuation ϕ of K, the restriction of ϕ to $\mathbb{Q} \subset K$ is a valuation of $\mathbb{Q}$ and is equivalent either to $|\cdot|_\infty$ or to $|\cdot|_p$ for some rational prime p. In the first case the place $v \ni \phi$ is called *infinite* (or is said to *lie above* ∞) - in this case v is either *real* (if $K_v = \mathbb{R}$) or *complex* (if $K_v = \mathbb{C}$) - and in the second case v *lies above* p (or p *lies below* v). We denote by w the place of $\mathbb{Q}$ below v and observe that K_v is a finite-dimensional vector space over the locally compact, metrizable field $\mathbb{Q}_w$ and hence locally compact and metrizable in its own right. Choose a Haar measure λ_v on K_v (with respect to addition), fix a compact set $C \subset K_v$ with non-empty interior, and write $\mathrm{mod}_{K_v}(a) = \lambda_v(aC)/\lambda_v(C)$ for the module of an element $a \in K_v$. The map $\mathrm{mod}_{K_v} : K \to \mathbb{R}_+$ is continuous, independent of the choice of λ_v, and its restriction to K is a valuation in v which is denoted by $|\cdot|_v$.

Above every place v of $\mathbb{Q}$ there are at least one and at most finitely many places of K.

Let R_K be the ring of integers of an algebraic number field K. Let $\mathfrak{p}$ be a prime ideal of R_K, v the (discrete) valuation associated with $\mathfrak{p}$ ([33, Theorem 3.3]). By $K_\mathfrak{p}$ or K_v we denote the completion of K under v, and we call $K_\mathfrak{p}$ the $\mathfrak{p}$-adic field. By k we denote the quotient field $R_K/\mathfrak{p}$, the residue class field. The cardinality of this residue field we denote by $\mathfrak{q} = \mathfrak{q}_\mathfrak{p} = \mathfrak{q}_v$. The extension of v to $K_\mathfrak{p}$ will be also denoted by v. The ring of integers of $K_\mathfrak{p}$, $R_\mathfrak{p} = \{x \in K_\mathfrak{p} : v(x) \leq 1\}$ is the closure of the ring $R = \{x \in K : v(x) \leq 1\}$, and $\mathfrak{P} = \{x \in K_\mathfrak{p} : v(x) < 1\} = \mathfrak{p}R_\mathfrak{p}$ is a prime ideal of $R_\mathfrak{p}$, which is the closure of the prime ideal $\{x \in K : v(x) < 1\}$ of R. The invertible elements of $R_\mathfrak{p}$ form a group $U(R_\mathfrak{p})$ of units of $K_\mathfrak{p}$. The quotient fields $R_K/\mathfrak{p}$ and $R_\mathfrak{p}/\mathfrak{P}$ are isomorphic ([33, Proposition 5.1]).

We define a uniformizer for v, or a local parameter, to be an element π, also denoted by π_v or $\pi_\mathfrak{p}$ of $K_\mathfrak{p}$ of maximal $v(\pi)$ less than 1. If we fix a uniformizer π, every element of $K_\mathfrak{p}^*$ can be written uniquely as $x = u\pi^m$ for some u with $v(u) = 1$ and $m \in \mathbb{Z}$. Moreover, each element $x \in K_\mathfrak{p}^*$ can be expressed in one and only one way as a convergent series

$$x = \sum_{i=m}^{\infty} r_i \pi^i, \tag{2.1}$$

where the coefficients r_i are taken from a set $\mathscr{R} \subset R_\mathfrak{p}$ (of cardinality $\mathfrak{q}$) of representatives of the residue classes in the field $k_\mathfrak{p} := R_\mathfrak{p}/\mathfrak{P}$ (i.e. the canonical map $R_\mathfrak{p} \to k_\mathfrak{p}$ induces a bijection of $\mathscr{R}$ onto $k_\mathfrak{p}$).

In what follows we consider the normalized valuation

$$|x|_v = v(x) := \mathfrak{q}^{-k}, \tag{2.2}$$

where k is the unique integer such that $x = u\pi^k$ for some unit u.

Let K be a field with a valuation v. Then K is a $\mathfrak{p}$-adic field with the $\mathfrak{p}$-adic valuation if and only if K is a finite extension of $\mathbb{Q}_p$ for a suitable p. (See [33, Theorem 5.10].) In this case an absolute value of K extending $|\cdot|_p$ on $\mathbb{Q}_p$ can be defined by

$$|x|_K = |N_{K/\mathbb{Q}_p}(x)|_p^{1/m}, \text{ where } m = (K : \mathbb{Q}_p)$$

and the *determinant* $N_{K/\mathbb{Q}_p(x)}$ is the determinant of the multiplication by x in K, i.e. the determinant of the linear map from (the vector space over $\mathbb{Q}_p$) K to K given by $\xi \mapsto x\xi$.

One can also define the absolute value setting

$$\|x\|_K = |N_{K/\mathbb{Q}_p}(x)|_p.$$

Clearly, $|\cdot|_K$ and $\|\cdot\|_K$ are in the same equivalence class.

2.2 Metrizability of $\mathbb{A}_K^d$

Define the following two functions on $\mathbb{A}_K^d$,

$$\|x\|_{\mathbb{A}_K^d}^{(1)} = \max_{v\in\mathscr{P}_\mathrm{f}} \|x_v\|_v, \qquad \|x\|_{\mathbb{A}_K^d}^{(0)} = \max_{v\in\mathscr{P}_\mathrm{f}} \frac{\|x_v\|_v}{\mathfrak{q}_v},$$

where $\mathfrak{q}_v$ is the cardinality of the residue field and

$$\|x_v\|_v = \|(x_1,\ldots,x_d)\|_v = \max_{1\le j\le d} |x_j|_v.$$

Let

$$\|x\|_{\mathbb{A}_K^d} = \begin{cases} \|x\|_{\mathbb{A}_K^d}^{(0)} & \text{for } x \in \prod_{v\in\mathscr{P}_\mathrm{f}} R_v^d, \\ \|x\|_{\mathbb{A}_K^d}^{(1)} & \text{for } x \notin \prod_{v\in\mathscr{P}_\mathrm{f}} R_v^d. \end{cases}$$

Proposition 2.1 *The restricted product topology on $\mathbb{A}_K^d$ is metrizable. The metric is given by*

$$d_{\mathbb{A}_K^d}(x,y) = \|x-y\|_{\mathbb{A}_K^d}. \tag{2.3}$$

Furthermore, $\mathbb{A}_K^d$ with $d_{\mathbb{A}_K^d}$ is a complete non-Archimedean metric space.

Proof See [48, 60]. □

2.3 Function spaces on $\mathbb{A}_K^d$

Let, for $v\in\mathscr{P}_\mathrm{f}$, $\mathscr{H}_v$ be the Hilbert space $L^2(K_v^d) = L^2(K_v^d,\mu_v)$, where μ_v is the Haar measure on K_v^d normalized so that $\mu_v(R_v^d) = 1$. The linear space K_v^d is endowed with the norm $\|x\|_v = \max_{1\le i\le d}|x_i|_v$. A function $f : K_v^d \to \mathbb{C}$ is said to be *locally constant* if there exists such an integer $\ell \ge 0$ that for any $x\in K_v^d$

$$f(x+y) = f(x) \qquad \text{if} \qquad \|y\|_v \le \mathfrak{q}_v^{-\ell}.$$

Let, for $v\in\mathscr{P}_\mathrm{f}$, Ω_v be the *vacuum vector* of R_v^d, i.e. the characteristic function of R_v^d. Define the algebraic tensor product

$$\mathscr{H}_{\mathrm{alg}} = \bigotimes_{v\in\mathscr{P}_\mathrm{f}} \mathscr{H}_v.$$

We say that an $f \in \mathscr{H}_{\mathrm{alg}}$ is *simple* if f is of the form $f = \otimes_{v\in\mathscr{P}_{\mathrm{f}}} f_v$, where $f_v = \Omega_v$ for almost all $v \in \mathscr{P}_{\mathrm{f}}$. The space $\mathscr{H}_{\mathrm{alg}}$ consists of finite linear combination of simple elements.

For simple elements f, g we define the inner product and the corresponding norm:

$$\langle f, g\rangle_{L^2(\mathbb{A}_K^d)} = \prod_{v\in\mathscr{P}} \langle f_v, g_v\rangle_v, \qquad \|f\|_{L^2(\mathbb{A}_K^d)} = \prod_{v\in\mathscr{P}} \|f_v\|_{L^2(K_v^d)}$$

and then we extend the above formulas for arbitrary element in $\mathscr{H}_{\mathrm{alg}}$ by linearity getting the inner product and the corresponding norm on $\mathscr{H}_{\mathrm{alg}}$. *Restricted tensor product*

$$\mathscr{H} = \bigoplus_{v\in\mathscr{P}_{\mathrm{f}}} \mathscr{H}_v$$

is the completion of the space $\mathscr{H}_{\mathrm{alg}}$ in the norm $\|\cdot\|_{L^2(\mathbb{A}_K^d)}$. We identify the space $L^2(\mathbb{A}_K^d, \mu_{\mathbb{A}_K^d})$ with $\mathscr{H}$.

We say that an element $f \in \mathscr{H}$ is *locally constant simple adelic function* if it is a simple element of $\mathscr{H}_{\mathrm{alg}}$ and for every $v \in \mathscr{P}_{\mathrm{f}}$, $f_v \in \mathscr{D}(K_v^d)$, i.e. f_v is a locally constant function on K_v^d. The *adelic Bruhat-Schwartz space* $\mathscr{D}(\mathbb{A}_K^d)$ is the set of finite sums of locally constant simple adelic functions with compact support. The set $\mathscr{D}(\mathbb{A}_K^d)$ is dense in $\mathscr{H} = L^2(K_v^d)$.

The Fourier transform $\mathscr{F}_v$ leaves $\mathscr{D}(K_v^d)$ invariant.

We say that a function $f : \mathbb{A}_K^d \to \mathbb{C}$ is *locally constant on* $\mathbb{A}_K^d$ if for any $x \in \mathbb{A}_K^d$ there exists a constant $\ell(x) > 0$ such that $f(x+y) = f(x)$ for any $y \in B_{\ell(x)}(0)$.

Proposition 2.2 *The function f belongs to $\mathscr{D}(\mathbb{A}_K^d)$ if and only if it is locally constant with compact support.*

Let f be a non-zero compactly supported function. We define the *parameter of constancy* ℓ of f as the largest non-zero integer power of a number q_v, $v \in \mathscr{P}_{\mathrm{f}}$ such that

$$f(x+y) = f(x) \text{ for every } x \in \mathbb{A}_K^d,\ y \in B_\ell(0).$$

By definition we set the parameter of constancy of function 0 to be equal $+\infty$.

We denote by $\mathscr{D}_R^\ell(\mathbb{A}_K^d)$ the subspace of functions from $\mathscr{D}(\mathbb{A}_K^d)$ with supports contained in the ball B_R and parameters of constancy greater than or equal to ℓ.

We have the following embedding

$$\mathscr{D}_R^\ell(\mathbb{A}_K^d) \subset \mathscr{D}_{R'}^{\ell'}(\mathbb{A}_K^d) \text{ whenever } R \le R',\ \ell \ge \ell'.$$

We define the convergence in $\mathscr{D}(\mathbb{A}_K^d)$. We say that f_k tends to 0 as $k \to +\infty$ in $\mathbb{A}_K^d$ if and only if

(i) $f_k \in \mathscr{D}_R^\ell(\mathbb{A}_K^d)$, where R and ℓ do not depend on k;
(ii) $f_k \to 0$ uniformly as $k \to +\infty$.

With this notion of convergence $\mathscr{D}(\mathbb{A}_K^d)$ becomes a complete topological vector space.

2.4 Fourier transform on $\mathbb{A}_K^d$

Let K be an algebraic number field, $v \in \mathscr{P}_{\mathrm{f}}$, and let $\mathbb{C}^\times$ stands for the multiplicative group of $\mathbb{C}$. Consider the additive group $(K_v, +)$. Since K_v is locally compact it is self-dual, that is if $\chi : K_v \to \mathbb{C}^\times$ is a non-trivial additive character on $(K, +)$, then any other character φ is of the form $\varphi(x) = \chi(ax)$ for some $a \in K_v$ [37, 40].

Recall that the *rank* of a character χ is the largest integer r such that $\chi|_{B_r} \equiv 1$. Let μ_v be the normalized Haar measure on $(K_v, +)$. For a fixed non-trivial character χ of rank zero (see [4, Subsection 2.1] for a construction of rank zero characters) we define the Fourier transform of $f \in L^1(K_v)$ as (cf. [37])

$$\mathscr{F}_v f(x) = \int_{K_v} \chi(-x\xi) f(\xi) d\mu_v(\xi),$$

where μ_v is the Haar measure normalized so that $\mu_v(R_v) = 1$. Then the inverse Fourier transform is given by

$$\mathscr{F}_v^{-1} f(x) = \mathscr{F}_v^* f(x) = \int_{K_v} \chi(x\xi) f(\xi) d\mu_v(\xi).$$

The above definition of the Fourier transform $\mathscr{F}_v$ carry over to K_v^d (see below).

The function $e_p(x) = e^{2\pi i \{x\}_p}$ is an additive character of $\mathbb{Q}_p$ (the *canonical additive character*). It is clear that $e_p(x) = 1$ if $|x|_p < 1$. If K is a finite extension of $\mathbb{Q}_p$, we can obtain a non-trivial additive character of K taking the composition $e_p \circ \mathrm{Tr}_{K/Q_p}$.

Let $\mathbb{A}_K^d$ be the adele ring of K. An adelic additive character

$$\chi = \chi^\xi : \mathbb{A}_K^d \to \mathbb{C}^\times, \qquad \xi \in \mathbb{A}_K^d$$

corresponds to some $\xi \in \mathbb{A}_K^d$ (since $\mathbb{A}_K^d$ is self-dual) and is defined as a product over local characters,

$$\otimes_v \chi_v, \qquad \chi_v : K_v^d \to \mathbb{C}^\times,$$

and each local character χ_v (finite or infinite) which lies over p, i.e. $v \mid p$, corresponds to some $\xi_v \in K_v$ (K_v is self-dual) and is given by

$$\chi_v(\cdot) = \chi_v^{\xi_v}(\cdot) = e^{2\pi i \{\mathrm{Tr}_{K_v/\mathbb{Q}_p} \langle \cdot, \xi_v \rangle\}_p}, \qquad v \mid p, \tag{2.4}$$

where $\langle \cdot, \cdot \rangle$ is a symmetric non-degenerate K_v-bilinear form on $K_v^d \times K_v^d$.

By $\mathscr{F}_v$ we denote the Fourier transform on $L^2(K_v^d)$. The Fourier transform $\mathscr{F}_v$ is an isometry on $L^2(K_v^d) \cap L^1(K_v^d)$ and can be extended to an unitary operator on $L^2(K_v^d)$.

The Fourier transform on $\mathbb{A}_K^d$ is defined, for $f \in L^1(\mathbb{A}_K^d)$, by

$$\mathscr{F}_{\mathbb{A}_K^d} f(\xi) = \int_{\mathbb{A}_K^d} \chi^\xi(x) f(x) d\mu_{\mathbb{A}_K^d}(x).$$

Further in the text, we often use a more intuitive and convenient notation and write

$$\chi(x \cdot \xi) := \chi^{\xi}(x).$$

Using this notation, the Fourier transform is written as

$$\mathscr{F}_{\mathbb{A}_K^d} f(\xi) = \int_{\mathbb{A}_K^d} \chi(x \cdot \xi) f(x) d\mu_{\mathbb{A}_K^d}(x).$$

For a locally constant simple adelic function f in $\mathscr{D}(\mathbb{A}_K^d)$ its Fourier transform is given by

$$(\mathscr{F}_{\mathbb{A}_K^d} f)(a) = \prod_v (\mathscr{F}_v f_v)(a_v), \qquad a \in \mathbb{A}_K^d. \tag{2.5}$$

We extend (2.5) to $\mathscr{D}(\mathbb{A}_K^d)$ by linearity. The operator $\mathscr{F}_{\mathbb{A}_K^d}$ is a unitary operator on a dense subspace $\mathscr{D}(\mathbb{A}_K^d)$ of $\mathscr{H}$. Thus, $\mathscr{F}_{\mathbb{A}_K^d}$ extends to a unitary operator on $\mathscr{H}$.

3 The Chernoff product formula and perturbation theorem

3.1 Evolution system

Let $(X, \|\cdot\|_X)$ be a Banach space (over $\mathbb{R}$ or $\mathbb{C}$), $\mathscr{L}(X)$ be the space of all continuous linear operators on X equipped with the strong operator topology. Let $\|\cdot\|$ denote the operator norm on $\mathscr{L}(X)$, and I be the identity operator on X. We construct an evolution system $U(t, s)$ for the initial value problem

$$\partial_t \psi(t, x) = A(t)\psi(t, x)$$

and

$$\psi(0, x) = \psi_0(x).$$

Definition 3.1 Let X be a Banach space. An *evolution system* (or simply *evolution*) is a family of operators $U(t, s)$ in $\mathscr{L}(X)$, defined for $0 \leq r \leq s \leq t \leq T$, such that

(i) $U(s, s) = I$,
(ii) $U(t, s)U(s, r) = U(t, r)$,
(iii) the map $(t, s) \mapsto U(t, s)$ is strongly continuous.

For a linear operator A, let $\rho(A)$ stands for its resolvent set.

Definition 3.2 Let, for every $t \in [0, T]$, a linear operator $A(t)$ on a Banach space X be the infinitesimal generator of a strongly continuous semigroup $S_t(s)$. The family $\{A(t)\}_{t\in[0,T]}$ is said to be *stable* if there exist constants M and ω such that

(i) for every $t \in [0, T]$, $(\omega, \infty) \subset \rho(A(t))$,
(i) for any finite sequences $0 \leq t_1 \leq \ldots \leq t_k \leq T$, and $s_j \geq 0$,

$$\left\| \prod_{j=1}^{k} S_{t_j}(s_j) \right\| \leq M \exp\left(\omega \sum_{j=1}^{k} s_j\right)$$

Remark 3.3 If for $t \in [0, t]$, $A(t)$ is the infinitesimal generator of a C_0 semigroup $S_t(s)$, $s \geq 0$, satisfying $\|S_t(s)\| \leq e^{\omega s}$ then the family $\{A(t)\}_{t\in[0,T]}$ is stable with constants $M = 1$ and ω. In particular any family $\{A(t)\}_{t\in[0,T]}$ of infinitesimal generators of C_0 semigroups of contractions is stable.

Theorem 3.4 *Let $T > 0$ be fixed. Let X be a Banach space and let $\{A(t)\}_{t\in[0,T]}$ be a stable family of infinitesimal generators of strongly continuous semigroups with stability constants M and ω. Suppose that the domain of $A(t)$, $\mathrm{Dom}(A(t)) = D$ does not depend on t. Suppose that the function $A(t)\varphi$ is strongly continuously differentiable for every $\varphi \in D$. Then there is an evolution system $U(t, s)$, $0 \leq s \leq t \leq T$, satisfying the following conditions:*

(1) $U(t, s) \leq Me^{\omega(t-s)}$,
(2) $U(t, s)D \subset D$,
(3) *for any $\varphi \in D$, the function $U(t, s)\varphi$ is continuous as a function with values in D with respect to the graph norm of $A(0)$,*
(4) *for any $\varphi \in D$,*

$$(\partial_t^+)U(t, s)\varphi|_{t=s} = A(s)\varphi$$

and

$$\partial_s U(t, s)\varphi = -U(t, s)A(s)\varphi.$$

Proof See e.g. [35, Ch. 5, Sec. 4, Theorem 4.8]. □

Proposition 3.5 *Let the assumptions of Theorem 3.4 hold. Then, for every $\varphi \in D$,*

$$\lim_{h\searrow 0} \frac{U(t+h, t)\varphi - \varphi}{h} = A(t)\varphi$$

uniformly with respect to t.

Proof See [36, Proposition 15] □

The evolution system $U(t, s)$ from Theorem 3.4 is called a *a solution of the Cauchy problem*.

3.2 The Chernoff type product formula

The following generalization of the Chernoff product formula [13] is due to Vuillermot [55] and Plyashechnik [36, Theorem 4].

Theorem 3.6 (Generalized Chernoff theorem) *Suppose that the conditions of Theorem 3.4 hold. Let a family of bounded operators $R(t, s)$, $0 \leq s \leq t \leq T$, be such that*

$$\|R(t_k, t_{k-1}) \ldots R(t_2, t_1)\| \leq C \tag{3.1}$$

for any family of points $0 \leq t_1 < \ldots < t_k \leq T$, and let

$$\frac{R(t + \Delta t, t) - I}{\Delta t} g \to A(t)g \tag{3.2}$$

for any $g \in D$ *as* $\Delta t \searrow 0$ *uniformly with respect to* t. *Then,*

$$R(t_n, t_{n-1}) \dots R(t_1, t_0) f \to U(t_n, t_0) f \tag{3.3}$$

for any $f \in X$ *as* $\max |t_{i+1} - t_i| \to 0$ *uniformly with respect to* $t_n, t_0 \in [0, T]$.

Proof See Vuillermot [55] or Plyashechnik [36, Theorem 4]. □

3.3 Additive perturbation

Let $T > 0$ be fixed. Let X be a Banach space and let $\{A(t)\}_{t\in[0,T]}$ and $\{B(t)\}_{t\in[0,T]}$ be stable families of infinitesimal generators of strongly continuous semigroups with stability constants M_A, ω_A and M_B, ω_B, respectively. Suppose that the domain $\mathrm{Dom}(A(t)) = \mathrm{Dom}(B(t)) = D$ does not depend on t. Suppose that the functions $A(t)\varphi$ and $B(t)\varphi$ are strongly continuously differentiable for every $\varphi \in D$. Then, by Theorem 3.4 there are evolution systems $U_A(t, s)$, and $U_B(t, s)$, $0 \le s \le t \le T$.

Let

$$L(t) = A(t) + B(t).$$

Suppose that $\{L(t)\}_{t\in[0,T]}$ satisfies assumptions of Theorem 3.4. Denote by $U(t, s)$ its evolution system.

The following theorem is a generalization of the corresponding result for semigroups [11].

Theorem 3.7 (Additive perturbation) *Let the families of bounded operators* $R_A(t, s)$, *and* $R_B(t, s)$, $0 \le s < t \le T$, *be such that*

$$R_A(t, s) R_B(t, s) = R_B(t, s) R_A(t, s) \tag{3.4}$$

and

$$\|R_A(t_k, t_{k-1}) \dots R_A(t_2, t_1)\| \le C_A \text{ and } \|R_B(t_k, t_{k-1}) \dots R_B(t_2, t_1)\| \le C_B \tag{3.5}$$

for any finite sequence of points $0 \le t_1, \dots, t_k \le T$, *and let*

$$\frac{R_A(t + \Delta t, t) - I}{\Delta t} g \to A(t) g \text{ and } \frac{R_B(t + \Delta t, t) - I}{\Delta t} g \to B(t) g \tag{3.6}$$

for any $g \in D$ *as* $\Delta t \searrow 0$ *uniformly with respect to* t.

Let

$$R(t, s) = R_A(t, s) R_B(t, s).$$

Then,

$$R(t_n, t_{n-1}) \dots R(t_1, t_0) f \to U(t_n, t_0) f \tag{3.7}$$

for any $f \in X$ *as* $\max |t_{i+1} - t_i| \to 0$ *uniformly with respect to* $t_n, t_0 \in [0, T]$.

Proof It follows from (3.4) and (3.5) that the family $R(t,s)$, $0 \le s \le t \le T$, satisfies (3.1) with $C = C_A + C_B$.

For each $g \in D$, we have

$$\begin{aligned}\frac{R(t+\Delta t,t)-I}{\Delta t}g - L(t)g &= \frac{R_A(t+\Delta t,t)R_B(t+\Delta t,t)g-g}{\Delta t} - A(t)g - B(t)g\\ &= R_A(t+\Delta t,t)\left(\frac{R_B(t+\Delta t,t)g-g}{\Delta t} - B(t)\right)\\ &\quad +(R_A(t+\Delta t,t)-I)B(t)g + \frac{R_A(t+\Delta t,t)g-g}{\Delta t} - A(t)g.\end{aligned}$$

Therefore, by (3.6),

$$\lim_{\Delta t\searrow 0}\left\|\frac{R(t+\Delta t,t)-I}{\Delta t}g - L(t)g\right\|_X = 0.$$

Thus all requirements of Theorem 3.6 are fulfilled and hence (3.7) holds. □

4 Two approximating families of operators: $F^{qp}(t,s)$ and $F^{pq}(t,s)$

In this section we study two families of operators: $F^{qp}(t,s)$ and $F^{pq}(t,s)$, defined in (1.7) and (1.9), respectively, and prove their properties, which will be then used in Sect. 5 to prove Theorem 1.11 and Theorem 1.12.

To start with, we need some facts about the heat kernel for the Vladimirov operator D^α, which we have collected below.

4.1 Heat kernel for the Vladimirov operator D^α

Theorem 4.1 (Bochner) *A function $\varphi : \mathbb{A}_K^d \to \mathbb{C}$ is continuous and positive definite if and only if it is the Fourier transform of a bounded Radon measure μ on $\mathbb{A}_K^d$.*

Lemma 4.2 *For every $t > 0$ and $\alpha > 1$ the function*

$$x \mapsto e^{-t\|x\|^\alpha_{\mathbb{A}_K^d}}$$

is positive definite.

Proof By [48, Theorem 5.7], $Z(\alpha,t,x) = \mathscr{F}^{-1}_{\mathbb{A}_K^d}\left(e^{-t\|\cdot\|^\alpha_{\mathbb{A}_K^d}}\right)(x)$ is a transition function of a Markov process with space state $\mathbb{A}_K^d$. Thus the result follows from Theorem 4.1. □

We will need some properties of the kernel $Z(\alpha,t,x)$. Let, for $x > 0$,

$$\Phi(x) = \prod_{v\in\mathscr{P}_\mathrm{f}} \mathfrak{q}_v^{[[\log_{\mathfrak{q}_v} x]]}, \tag{4.1}$$

where, for $t \in \mathbb{R}$,

$$[[t]] = \begin{cases} [t] & \text{if } t \geq 0, \\ [t]+1 & \text{if } t < 0, \end{cases}$$

where $[t]$ is the integer part of t.

For $n \in \mathbb{R}$, $n > 0$ we define

$$n_+ = \min\{\mathfrak{q}_v^\beta \mid n < \mathfrak{q}_v^\beta, \mathfrak{q}_v \in \mathscr{P}_{\mathrm{f}}, \beta \in \mathbb{Z} \setminus \{0\}\}$$

and

$$n_- = \max\{\mathfrak{q}_v^\beta \mid \mathfrak{q}_v^\beta < n, \mathfrak{q}_v \in \mathscr{P}_{\mathrm{f}}, \beta \in \mathbb{Z} \setminus \{0\}\}.$$

Proposition 4.3 *Let $\alpha > 1$. For every $t > 0$ and every $x \in \mathbb{A}_K^d$ we have:*

$$Z(\alpha, t, x) = \sum_{\{\mathfrak{q}_v^j \mid \mathfrak{q}_v^j < \|x\|_{\mathbb{A}_K^d}^{-1}, v \in \mathscr{P}_{\mathrm{f}}, j \in \mathbb{Z}\setminus\{0\}\}} \Phi(\mathfrak{q}_v^j) \left(e^{-t\mathfrak{q}_v^{j\alpha}} - e^{-t(\mathfrak{q}_{v+}^j)^\alpha} \right). \tag{4.2}$$

Lemma 4.4 *The adelic heat kernel $Z(\alpha, t, x)$ on $\mathbb{A}_K^d$ satisfies the following:*

(i) $Z(\alpha, t, x) \geq 0$,
(ii) $\int_{\mathbb{A}_K^d} Z(\alpha, t, x) d\mu_{\mathbb{A}_K^d}(x) = 1$,
(iii) $Z(\alpha, t, \cdot) \in L^1(\mathbb{A}_K^d)$,
(iv) $Z(\alpha, t, \cdot) * Z(\alpha, t', x) = Z(\alpha, t + t', x)$,
(v) $\lim_{t \to 0+} Z(\alpha, t, x) = \delta_x$ *in* $\mathscr{D}'(\mathbb{A}_K^d)$,
(vi) $Z(\alpha, t, \cdot)$ *is a uniformly continuous function for any fixed* $t > 0$,
(vii) $Z(\alpha, t, x)$ *is uniformly continuous in* t, *i.e.* $Z(\alpha, t, x) \in C\left((0, +\infty), C(\mathbb{A}_K^d, \mathbb{R})\right)$ *or* $\lim_{t' \to t} \max_{x \in \mathbb{A}_K^d} |Z(\alpha, t, x) - Z(\alpha, t', x)| = 0$ *for any* $t > 0$.

Proof See [60, Theorem 105] and [48, Theorem V.7]. □

Lemma 4.5 *The following estimate holds for the heat kernel:*

$$Z(\alpha, t, x) \leq 2t \|x\|_{\mathbb{A}_K^d}^{-\alpha} \Phi\left((\|x\|_{\mathbb{A}_K^d}^{-1})_- \right), \qquad x \in \mathbb{A}_K^d \setminus \{0\}, \quad t > 0.$$

Proof See [60, Lemma 103] and [48, Lemma V.6]. □

By Lemma 4.5 and definition (4.1) of $\Phi(x)$ we get immediately the following corollary.

Corollary 4.6 *There exists $C > 0$ such that for all x with $\|x\|_{\mathbb{A}_K^d} \geq 1$,*

$$Z(\alpha, t, x) \leq Ct \|x\|_{\mathbb{A}_K^d}^{-\alpha}.$$

4.2 Properties of $F^{qp}(t,s)$

Recall that, for every $0 \le s \le t \le T$,

$$F^{qp}(t,s) = \widehat{(e^{-\int_s^t H_V(u)du})}^{qp}_{C_0}.$$

Let, for $0 \le s \le t \le T$, $q \in Q$, $p \in P$,

$$f_{s,t,q}(p) = e^{-\int_s^t H_V(u)(q,p)du}. \tag{4.3}$$

Remark 4.7 We have the following upper bound

$$\begin{aligned} \sup_q \left| e^{-\int_s^t H_V(u)(q,p)du} \right| &= \sup_q \left| e^{-\int_s^t g(u,q)\|p\|^\alpha - V(u,q)du} \right| \\ &\le \sup_q \left| e^{-\int_s^t g(u,q)du\|p\|^\alpha} \right| \sup_q \left| e^{-\int_s^t V(u,q)du} \right| \\ &\le e^{-(t-s)\inf_{q,u} \operatorname{Re} g(q,u)\|p\|^\alpha} e^{-(t-s)\inf_{q,u} \operatorname{Re} V(u,q)}. \end{aligned} \tag{4.4}$$

Lemma 4.8 *The function* $f_{s,t,q}(\cdot) \in L^1(\mathbb{A}_K^d)$ *and*

$$\mathscr{F}^{-1}_{\mathbb{A}_K^d} f_{s,t,q}(\cdot) = e^{-\int_s^t V(u,q)du} P^q_{s,t}(\cdot), \tag{4.5}$$

where $P^q_{s,t}$ *is a density of a probability measure on* $\mathbb{A}_K^d$. *Explicitly,*

$$P^q_{s,t}(x) = Z\left(\alpha, \int_s^t g(u,q)du, x\right), \tag{4.6}$$

where $Z(\alpha,t,x)$ *is the heat kernel for the Vladimirov operator* D^α.

Proof By (4.4) and the assumptions on g and V we get that for every $0 \le s \le t \le T$ and for every $q \in \mathbb{A}_K^d$,

$$\begin{aligned} f_{s,t,q}(p) &= e^{-\int_s^t H_V(u)(q,p)du} \\ &= e^{-\|p\|^\alpha \int_s^t g(u,q)du} e^{-\int_s^t V(u,q)du} \in L^1(\mathbb{A}_K^d). \end{aligned}$$

By Lemma 4.2, $f_{s,t,q}$ is a continuous, positive definite function of p for every q and $s \le t$. Therefore, from the Bochner Theorem 4.1 and the fact that Fourier transform maps $L^1(\mathbb{A}_K^d)$ into $\mathscr{D}(\mathbb{A}_K^d)$, we get that $\mathscr{F}^{-1}_{\mathbb{A}_K^d} f_{s,t,q}$ is of the form (4.5). By [48, Theorem 5.7] $Z(\alpha,t,x) = \mathscr{F}^{-1}_{\mathbb{A}_K^d}\left(e^{-t\|\cdot\|^\alpha_{\mathbb{A}_K^d}}\right)(x)$ and (4.6) follows. □

Definition 4.9 Let

$$\mathscr{L}_0 = \{f \in \mathscr{D}(\mathbb{A}_K^d) \mid \mathscr{F}_{\mathbb{A}_K^d} f(0) = 0\}.$$

Following [60], we call $\mathscr{L}_0$ a *Lizorkin space*.

Remark The space $\mathscr{L}_0$ is an adelic analogue of the Lizorkin space of the second kind (see [1]).

Lemma 4.10 *The following facts are true:*

(i) *the set* $\mathscr{L}_0 \cap C_0$ *is a dense subset of* C_0 *with respect to the supremum norm;*
(ii) *the set* $\mathscr{L}_0 \cap L^1$ *is a dense subset of* L^1.

Proof It follows by [43, Proposition 1.3]. □

Proposition 4.11 *We have the following upper bound for the* $C_0 \to C_0$*-norm of* $F(t,s)$,

$$\|F^{qp}(t,s)\|_{C_0\to C_0} \le \sup_q e^{-\operatorname{Re}\int_s^t \operatorname{Re} V(u,q)du}.$$

Proof We proceed similarly to [12, p. 9]). Let $\varphi \in \mathscr{L}_0 \cap C_0$. We have

$$\begin{aligned}
F^{qp}(t,s)\varphi(q) &= \int_{\mathbb{A}_K^d} \chi(q\xi)e^{-\int_s^t H_V(u)(q,\xi)du}\tilde{\varphi}(\xi)d\mu_{\mathbb{A}_K^d}(\xi)\\
&= \int_{\mathbb{A}_K^d} \chi(q\xi)e^{-\int_s^t H_V(u)(q,\xi)du}\int_{\mathbb{A}_K^d} \chi(-\xi p)\varphi(p)d\mu_{\mathbb{A}_K^d}(p)d\mu_{\mathbb{A}_K^d}(\xi)\\
&= \int_{\mathbb{A}_K^d}\int_{\mathbb{A}_K^d} \chi(\xi(q-p))e^{-\int_s^t H_V(u)(q,\xi)du}d\mu_{\mathbb{A}_K^d}(\xi)\varphi(p)d\mu_{\mathbb{A}_K^d}(p)\\
&= \int_{\mathbb{A}_K^d} \mathscr{F}^{-1}_{\mathbb{A}_K^d}\left(e^{-\int_s^t H_V(u)(q,\cdot)du}\right)(q-p)\varphi(p)d\mu_{\mathbb{A}_K^d}(p)\\
&= \int_{\mathbb{A}_K^d} (\mathscr{F}^{-1}_{\mathbb{A}_K^d} f_{s,t,q})(q-p)\varphi(p)d\mu_{\mathbb{A}_K^d}(p).
\end{aligned}$$

By Lemma 4.8,

$$F^{qp}(s,t)\varphi(q) = \int_{\mathbb{A}_K^d} e^{-\int_s^t V(u,q)du}P^q_{s,t}(q-p)\varphi(p)d\mu_{\mathbb{A}_K^d}(p)$$

and consequently

$$\begin{aligned}
&\|F^{qp}(s,t)\varphi\|_{L^\infty(\mathbb{A}_K^d)}\\
&= \sup_q\left|\int_{\mathbb{A}_K^d} e^{-\int_s^t V(u,q)du}P^q_{s,t}(q-p)\varphi(p)d\mu_{\mathbb{A}_K^d}(p)\right|\\
&\le \sup_q\left|e^{-\int_s^t V(u,q)du}\right|\sup_q\int_{\mathbb{A}_K^d}\left|P^q_{s,t}(q-p)\varphi(p)\right|d\mu_{\mathbb{A}_K^d}(p)\\
&\le \sup_q e^{-\operatorname{Re}\int_s^t \operatorname{Re} V(u,q)du}\sup_q\int_{\mathbb{A}_K^d}\left|P^q_t(q-p)\right|d\mu_{\mathbb{A}_K^d}(p)\|\varphi\|_{L^\infty(\mathbb{A}_K^d)}\\
&= \sup_q e^{-\operatorname{Re}\int_s^t \operatorname{Re} V(u,q)du}\|\varphi\|_{L^\infty(\mathbb{A}_K^d)}.
\end{aligned}$$

□

Proposition 4.12 *For every* $\varphi \in \mathscr{L}_0 \cap C_0$,

$$\lim_{t \searrow s} \left\| \frac{F^{qp}(t,s)\varphi - \varphi}{t-s} + \widehat{H_V(s)}^{qp} \varphi \right\|_{L^\infty(\mathbb{A}_K^d)} = 0.$$

Proof Let $\varphi \in \mathscr{L}_0 \cap C_0$. By Lemma 4.10 (i), the set $\mathscr{L}_0 \cap C_0$ is a dense subset of C_0. We start with

$$\begin{aligned} &\frac{F^{qp}(t,s)\varphi - \varphi}{t-s} + \widehat{H_V(s)}^{qp} \varphi \\ &= \int_{\mathbb{A}_K^d} \chi(\cdot\xi) \left(\frac{e^{-\int_s^t H_V(u)(\cdot,\xi)du} - 1}{t-s} \right) \tilde{\varphi}(\xi) d\mu_{\mathbb{A}_K^d}(\xi) \\ &\quad + \int_{\mathbb{A}_K^d} \chi(\cdot\xi) H_V(s)(\cdot,\xi) \tilde{\varphi}(\xi) d\mu_{\mathbb{A}_K^d}(\xi). \end{aligned} \tag{4.7}$$

By the mean value theorem the first integral on the right above is equal to

$$-\int_{\mathbb{A}_K^d} \chi(\cdot\xi) e^{-\int_s^{t'} H_V(u)(\cdot,\xi)du} H_V(t')(\cdot,\xi) \tilde{\varphi}(\xi) d\mu_{\mathbb{A}_K^d}(\xi), \tag{4.8}$$

where $s < t' < t$. We add and subtract the term

$$\int_{\mathbb{A}_K^d} \chi(\cdot\xi) e^{-\int_s^{t'} H_V(u)(\cdot,\xi)du} H_V(s)(\cdot,\xi) \tilde{\varphi}(\xi) d\mu_{\mathbb{A}_K^d}(\xi)$$

to the right hand side of (4.7) and get that the L^∞-norm of (4.7) is bounded by

$$\begin{aligned} &\left\| \int_{\mathbb{A}_K^d} \chi(\cdot\xi) \left(1 - e^{-\int_s^{t'} H_V(u)(\cdot,\xi)du} \right) H_V(s)(\cdot,\xi) \tilde{\varphi}(\xi) d\mu_{\mathbb{A}_K^d}(\xi) \right\|_{L^\infty(\mathbb{A}_K^d)} \\ &\quad + \left\| \int_{\mathbb{A}_K^d} \chi(\cdot\xi) e^{-\int_s^{t'} H_V(u)(\cdot,\xi)du} \left(H_V(s)(\cdot,\xi) \right. \right. \\ &\quad \left. \left. - H_V(t')(\cdot,\xi) \right) \tilde{\varphi}(\xi) d\mu_{\mathbb{A}_K^d}(\xi) \right\|_{L^\infty(\mathbb{A}_K^d)}. \end{aligned} \tag{4.9}$$

Again, by the mean value theorem, the first norm in (4.9) is equal to

$$(t'-s) \left\| \int_{\mathbb{A}_K^d} \chi(\cdot\xi) e^{-\int_s^{t''} H_V(u)(\cdot,\xi)du} H_V(s)(\cdot,\xi) H_V(t'')(\cdot,\xi) \tilde{\varphi}(\xi) d\mu_{\mathbb{A}_K^d}(\xi) \right\|_{L^\infty(\mathbb{A}_K^d)}, \tag{4.10}$$

where $s < t'' < t' < t$. By definition (1.2) of H_V we have

$$
\begin{aligned}
&H_V(s)(q,p)H_V(t'')(q,p)\\
&\quad = g(q,s)g(q,t'')\|p\|^{2\alpha}_{\mathbb{A}^d_K} + g(q,s)V(q,t'')\|p\|^{\alpha}_{\mathbb{A}^d_K}\\
&\quad\quad + g(q,t'')V(q,s)\|p\|^{\alpha}_{\mathbb{A}^d_K} + V(q,s)V(q,t''). \qquad (4.11)
\end{aligned}
$$

Thus the $L^\infty(\mathbb{A}^d_K)$-norm of the first term in (4.9) is dominated by

$$
\begin{aligned}
&\left\| g(\cdot,s)g(\cdot,t'')\int_{\mathbb{A}^d_K}\chi(\cdot\xi)e^{-\int_s^{t''}H_V(u)(\cdot,\xi)du}\|\xi\|^{2\alpha}_{\mathbb{A}^d_K}\tilde{\varphi}(\xi)d\mu_{\mathbb{A}^d_K}(\xi)\right\|_{L^\infty(\mathbb{A}^d_K)}\\
&\quad + \left\| g(\cdot,s)V(\cdot,t'')\int_{\mathbb{A}^d_K}\chi(\cdot\xi)e^{-\int_s^{t''}H_V(u)(\cdot,\xi)du}\|\xi\|^{\alpha}_{\mathbb{A}^d_K}\tilde{\varphi}(\xi)d\mu_{\mathbb{A}^d_K}(\xi)\right\|_{L^\infty(\mathbb{A}^d_K)}\\
&\quad + \left\| g(\cdot,t'')V(\cdot,s)\int_{\mathbb{A}^d_K}\chi(\cdot\xi)e^{-\int_s^{t''}H_V(u)(\cdot,\xi)du}\|\xi\|^{\alpha}_{\mathbb{A}^d_K}\tilde{\varphi}(\xi)d\mu_{\mathbb{A}^d_K}(\xi)\right\|_{L^\infty(\mathbb{A}^d_K)}\\
&\quad + \left\| V(\cdot,s)V(\cdot,t'')\int_{\mathbb{A}^d_K}\chi(\cdot\xi)e^{-\int_s^{t''}H_V(u)(\cdot,\xi)du}\tilde{\varphi}(\xi)d\mu_{\mathbb{A}^d_K}(\xi)\right\|_{L^\infty(\mathbb{A}^d_K)}.
\end{aligned}
$$

If $t \to s$ then $t'' \to s$ and the above sum of norms tends to

$$
\left\|g^2(s)D^{2\alpha}f\right\|_{L^\infty(\mathbb{A}^d_K)} + 2\left\|g(s)V(s)D^\alpha f\right\|_{L^\infty(\mathbb{A}^d_K)} + \left\|V^2(s)f\right\|_{L^\infty(\mathbb{A}^d_K)},
$$

so by (4.10), the first norm in (4.9) tends to 0. The second norm in (4.9) also tends to 0 by the assumptions on H_V. □

Proposition 4.13 *The family $F^{qp}(t,s)$ is strongly continuous at $t = s$, i.e.*

$$
\lim_{t\searrow s}\|F^{qp}(t,s)\varphi - \varphi\|_{L^\infty(\mathbb{A}^d_K)} = 0.
$$

Proof By the mean value theorem,

$$
\begin{aligned}
&\|F^{qp}(t,s)\varphi - \varphi\|_{L^\infty(\mathbb{A}^d_K)}\\
&\quad = \left\|\int_{\mathbb{A}^d_K}\chi(\cdot\xi)\left(e^{-\int_s^t H_V(u)(\cdot,\xi)du} - 1\right)\tilde{\varphi}(\xi)d\mu_{\mathbb{A}^d_K}(\xi)\right\|_{L^\infty(\mathbb{A}^d_K)}\\
&\quad = (t-s)\left\|\int_{\mathbb{A}^d_K}\chi(\cdot\xi)e^{-\int_s^{t'}H_V(u)(\cdot,\xi)du}H_V(t')(\cdot,\xi)\tilde{\varphi}(\xi)d\mu_{\mathbb{A}^d_K}(\xi)\right\|_{L^\infty(\mathbb{A}^d_K)}
\end{aligned}
$$

where $t' \in (s, t)$. Thus, by definition of H_V,

$$\begin{aligned} &\|F^{qp}(t,s)\varphi - \varphi\|_{L^\infty(\mathbb{A}_K^d)} \\ &\quad \le (t-s)\left\| g(t',\cdot)\int_{\mathbb{A}_K^d} \chi(\cdot\xi)e^{-\int_s^{t'} H_V(u)(\cdot,\xi)du}\|\xi\|^\alpha_{\mathbb{A}_K^d}\tilde{\varphi}(\xi)d\mu_{\mathbb{A}_K^d}(\xi)\right\|_{L^\infty(\mathbb{A}_K^d)} \\ &\qquad + (t-s)\left\| V(t',\cdot)\int_{\mathbb{A}_K^d} \chi(\cdot\xi)e^{-\int_s^{t'} H_V(u)(\cdot,\xi)du}\|\xi\|^\alpha_{\mathbb{A}_K^d}\tilde{\varphi}(\xi)d\mu_{\mathbb{A}_K^d}(\xi)\right\|_{L^\infty(\mathbb{A}_K^d)} \end{aligned}$$

and the above expression tends to 0 as $t \to s$ by the assumptions on g and V. □

4.3 Definition and properties of $F^{pq}(t, s)$

Recall that, for $t \ge s \ge 0$, the operator

$$F^{pq}(t,s) \in \mathscr{L}\left(L^1(\mathbb{A}_K^d)\right)$$

is defined by the formula

$$F^{pq}(t,s) = \overline{\left(e^{-\int_s^t H_V(u)du}\right)}^{pq}_{L^1}.$$

The Fourier transform of the distribution $F^{pq}(t,s)\varphi(q)$ is a regular distribution with a continuous density given by

$$p \mapsto \mathscr{F}_{\mathbb{A}_K^d}(e^{-\int_s^t H_V(u)(\cdot,p)du}\varphi(\cdot))(p).$$

That is

$$F^{pq}(t,s)\varphi(q) = \mathscr{F}^{-1}_{\mathbb{A}_K^d}\left(\mathscr{F}_{\mathbb{A}_K^d}(e^{-\int_s^t H_V(u)(\cdot,p)du}\varphi(\cdot))(p)\right)(q).$$

In the integral form

$$\begin{aligned} &F^{pq}(t,s)\varphi(q) \\ &\quad = \int_{\mathbb{A}_K^d} \chi(q\cdot p)\mathscr{F}_{\mathbb{A}_K^d}(e^{-\int_s^t H_V(u)(\cdot,p)du}\varphi(\cdot))(p)d\mu_{\mathbb{A}_K^d}(p) \\ &\quad = \int_{\mathbb{A}_K^d}\int_{\mathbb{A}_K^d} \chi(q\cdot p)\chi(-p\cdot q')e^{-\int_s^t H_V(u)(q',p)du}\varphi(q')d\mu_{\mathbb{A}_K^d}(q')d\mu_{\mathbb{A}_K^d}(p) \\ &\quad = \int_{\mathbb{A}_K^d}\int_{\mathbb{A}_K^d} \chi\left(p\cdot(q-q')\right)e^{-\int_s^t H_V(u)(q',p)du}\varphi(q')d\mu_{\mathbb{A}_K^d}(q')d\mu_{\mathbb{A}_K^d}(p). \end{aligned} \tag{4.12}$$

Proposition 4.14 *There is a non-negative constant c such that for all* $0 \leq s \leq t \leq T$,

$$\|F^{pq}(t,s)\|_{L^1(\mathbb{A}_K^d)\to L^1(\mathbb{A}_K^d)} \leq e^{-c(t-s)}. \tag{4.13}$$

Proof Let $\varphi \in \mathscr{L}_0 \cap L^1$ (by Lemma 4.10 (ii), $\mathscr{L}_0 \cap L^1$ is a dense subset of L^1). By (4.12) we can write

$$\begin{aligned}
&\|F^{pq}(t,s)\varphi\|_{L^1(\mathbb{A}_K^d)} \\
&= \left\| \int_{\mathbb{A}_K^d} \int_{\mathbb{A}_K^d} \chi\left(p\cdot(\cdot - q')\right) e^{-\int_s^t H_V(u)(q',p)du} \varphi(q') d\mu_{\mathbb{A}_K^d}(q') d\mu_{\mathbb{A}_K^d}(p) \right\|_{L^1(\mathbb{A}_K^d)} \\
&= \left\| \int_{\mathbb{A}_K^d} \varphi(q') \int_{\mathbb{A}_K^d} \chi\left(p\cdot(\cdot - q')\right) e^{-\int_s^t H_V(q',p)(u)} d\mu_{\mathbb{A}_K^d}(p) d\mu_{\mathbb{A}_K^d}(q') \right\|_{L^1(\mathbb{A}_K^d)} \\
&= \left\| \int_{\mathbb{A}_K^d} \varphi(q') \mathscr{F}^{-1}_{\mathbb{A}_K^d} f_{s,t,q'}(\cdot - q') d\mu_{\mathbb{A}_K^d}(q') \right\|_{L^1(\mathbb{A}_K^d)}.
\end{aligned} \tag{4.14}$$

In view of Lemma 4.8 we have

$$\|F^{pq}(t,s)\varphi\|_{L^1(\mathbb{A}_K^d)} = \left\| \int_{\mathbb{A}_K^d} \varphi(q') e^{-\int_s^t V(u)(q')du} P_{s,t}^{q'}(\cdot - q') d\mu_{\mathbb{A}_K^d}(q') \right\|_{L^1(\mathbb{A}_K^d)},$$

where $P_{s,t}^q$ is a density of a probability measure on $\mathbb{A}_K^d$. Consequently,

$$\begin{aligned}
&\|F^{pq}(t,s)\varphi\|_{L^1(\mathbb{A}_K^d)} \\
&= \left\| \int_{\mathbb{A}_K^d} \varphi(q') e^{-\int_s^t V(u)(q')du} P_{s,t}^{q'}(\cdot - q') d\mu_{\mathbb{A}_K^d}(q') \right\|_{L^1(\mathbb{A}_K^d)} \\
&\leq e^{-(t-s)\inf_{q',u} \operatorname{Re} V(u)(q')} \int_{\mathbb{A}_K^d} \int_{\mathbb{A}_K^d} |\varphi(q')| P_t^{q'}(q - q') d\mu_{\mathbb{A}_K^d}(q') d\mu_{\mathbb{A}_K^d}(q) \\
&= e^{-c(t-s)} \int_{\mathbb{A}_K^d} |\varphi(q')| \int_{\mathbb{A}_K^d} P_t^{q'}(q - q') d\mu_{\mathbb{A}_K^d}(q) d\mu_{\mathbb{A}_K^d}(q') \\
&= e^{-c(t-s)} \int_{\mathbb{A}_K^d} |\varphi(q')| d\mu_{\mathbb{A}_K^d}(q') \\
&= e^{-c(t-s)} \|\varphi\|_{L^1(\mathbb{A}_K^d)}
\end{aligned}$$

and (4.13) is proved. □

Proposition 4.15 *For every* $0 \leq s \leq t \leq T$,

$$\text{if } \varphi \in C_0(\mathbb{A}_K^d, \mathbb{R}) \Longrightarrow F^{pq}(t,s)\varphi \in C_0(\mathbb{A}_K^d, \mathbb{R}).$$

Proof First we show that $F^{pq}(t,s)\varphi \in C(\mathbb{A}_K^d, \mathbb{R})$, i.e. is continuous. By (4.14),

$$F^{pq}(t,s)\varphi(q) = \int_{\mathbb{A}_K^d} \varphi(q')\mathscr{F}^{-1}_{\mathbb{A}_K^d} f_{s,t,q'}(q-q')d\mu_{\mathbb{A}_K^d}(q'). \tag{4.15}$$

By Lemma 4.8,

$$\mathscr{F}^{-1}_{\mathbb{A}_K^d} f_{s,t,q'}(q-q') = e^{-\int_s^t V(u,q')du} Z\left(\alpha, \int_s^t g(u,q')du, q-q'\right). \tag{4.16}$$

Denote,

$$r_{s,t}(q) = \int_s^t g(u,q)du.$$

Clearly, there are positive $c_{s,t}$ and $C_{s,t}$ such that for all $q \in \mathbb{A}_K^d$,

$$0 < c_{s,t} \leq r_{s,t}(q) \leq C_{s,t}.$$

Then

$$\begin{aligned}
&\left|F^{pq}(t,s)\varphi(q) - F^{pq}(t,s)\varphi(q_0)\right| \\
&= \left|\int_{\mathbb{A}_K^d} \varphi(q')e^{-\int_s^t V(u,q')du}\left(Z\left(\alpha, r_{s,t}(q'), q-q'\right)\right.\right. \\
&\quad \left.\left. - Z\left(\alpha, r_{s,t}(q'), q_0-q'\right)\right) d\mu_{\mathbb{A}_K^d}(q')\right| \\
&\leq \int_{\mathbb{A}_K^d} \left|\varphi(q')\left(Z\left(\alpha, r_{s,t}(q'), q-q'\right) - Z\left(\alpha, r_{s,t}(q'), q_0-q'\right)\right)\right| d\mu_{\mathbb{A}_K^d}(q').
\end{aligned} \tag{4.17}$$

Let $\mathscr{K}$ be a compact subset of $\mathbb{A}_K^d$. We split the integral on the right hand side of (4.17) into two part: over $\mathscr{K}$ and its complement $\mathscr{K}^c$. Denote by $UC(\mathbb{A}_K^d, \mathbb{R})$ the sapce of all real valued uniformly continuous functions on $\mathbb{A}_K^d$. By Lemma 4.4 the following map

$$[c_{s,t}, C_{s,t}] \ni u \mapsto Z(\alpha, u, \cdot) \in UC(\mathbb{A}_K^d, \mathbb{R})$$

is uniformly continuous. Therefore, by Corollary 4.6,

$$\begin{aligned}
&\int_{\mathscr{K}^c} Z\left(\alpha, r_{s,t}(q'), q-q'\right) d\mu_{\mathbb{A}_K^d}(q') \\
&\leq \int_{\mathscr{K}^c} \max_{u \in [c_{s,t}, C_{s,t}]} Z\left(\alpha, u, q-q'\right) d\mu_{\mathbb{A}_K^d}(q') \\
&\leq 2C_{s,t} \int_{\mathscr{K}^c} \|q-q'\|^{-\alpha}_{\mathbb{A}_K^d} d\mu_{\mathbb{A}_K^d}(q').
\end{aligned} \tag{4.18}$$

Let $\varepsilon > 0$. Taking appropriate $\mathscr{K}$, i.e. "large enough" we get that

$$\int_{\mathscr{K}^c} Z\left(\alpha, r_{s,t}(q'), q-q'\right) d\mu_{\mathbb{A}_K^d}(q') < \varepsilon \text{ and } \int_{\mathscr{K}^c} Z\left(\alpha, r_{s,t}(q'), q_0-q'\right) d\mu_{\mathbb{A}_K^d}(q') < \varepsilon.$$

Therefore, the integral on the right in (4.17) over $\mathscr{K}^c$ is smaller than $2\|\varphi\|_{L^\infty(\mathbb{A}_K^d)}\varepsilon$. Finally, consider the integral in (4.17) taken over $\mathscr{K}$. This integral is bounded by

$$\|\varphi(q')\|_{L^\infty(\mathbb{A}_K^d)} \int_{\mathscr{K}} \left|Z\left(\alpha, r_{s,t}(q'), q-q'\right) - Z\left(\alpha, r_{s,t}(q'), q_0-q'\right)\right| d\mu_{\mathbb{A}_K^d}(q').$$

Clearly, for all $q' \in \mathscr{K}$,

$$\lim_{q\to q_0} \left|Z\left(\alpha, r_{s,t}(q'), q-q'\right) - Z\left(\alpha, r_{s,t}(q'), q_0-q'\right)\right| = 0.$$

Hence, by dominated convergence theorem,

$$\int_{\mathscr{K}} \left|\varphi(q')\left(Z\left(\alpha, r_{s,t}(q'), q-q'\right) - Z\left(\alpha, r_{s,t}(q'), q_0-q'\right)\right)\right| d\mu_{\mathbb{A}_K^d}(q') \to 0$$

as $q \to q_0$. Consequently $F^{pq}(t,s)\varphi \in C(\mathbb{A}_K^d, \mathbb{R})$.

Now we show that $F^{pq}(t,s)\varphi$ vanishes at infinity. To do this we write as in (4.17)

$$|F^{pq}(t,s)\varphi(q)| = \left|\int_{\mathscr{K}\cup\mathscr{K}^c} \varphi(q')e^{-\int_s^t V(u,q')du} Z\left(\alpha, r_{s,t}(q'), q-q'\right) d\mu_{\mathbb{A}_K^d}(q')\right|.$$

Consider the integral over a compact set $\mathscr{K}$,

$$\begin{aligned}
&\left|\int_{\mathscr{K}} \varphi(q')e^{-\int_s^t V(u,q')du} Z\left(\alpha, r_{s,t}(q'), q-q'\right) d\mu_{\mathbb{A}_K^d}(q')\right| \\
&\quad\le \int_{\mathscr{K}} \left|\varphi(q')Z\left(\alpha, r_{s,t}(q'), q-q'\right)\right| d\mu_{\mathbb{A}_K^d}(q') \\
&\quad\le \|\varphi\|_{L^\infty(\mathbb{A}_K^d)} \int_{\mathscr{K}} \max_{u\in[c_{s,t}, C_{s,t}]} Z\left(\alpha, u, q-q'\right) d\mu_{\mathbb{A}_K^d}(q'). \qquad (4.19)
\end{aligned}$$

By Lemma 4.5,

$$\begin{aligned}
&\int_{\mathscr{K}} \max_{u\in[c_{s,t}, C_{s,t}]} Z\left(\alpha, u, q-q'\right) d\mu_{\mathbb{A}_K^d}(q') \\
&\quad\le 2C_{s,t} \int_{\mathscr{K}} \|q-q'\|_{\mathbb{A}_K^d}^{-\alpha} \Phi\left((\|q-q'\|_{\mathbb{A}_K^d}^{-1})_-\right) d\mu_{\mathbb{A}_K^d}(q') \qquad (4.20)
\end{aligned}$$

and consequently from (4.19) and (4.20) we get that

$$\lim_{q\to\infty} \left|\int_{\mathscr{K}} \varphi(q')e^{-\int_s^t V(u,q')du} Z\left(\alpha, r_{s,t}(q'), q-q'\right) d\mu_{\mathbb{A}_K^d}(q')\right| = 0. \qquad (4.21)$$

Proceeding similarly to (4.19) and using (4.18) we get

$$\begin{aligned}&\left|\int_{\mathcal{K}^c} \varphi(q')e^{-\int_s^t V(u,q')du} Z\left(\alpha, r_{s,t}(q'), q-q'\right) d\mu_{\mathbb{A}_K^d}(q')\right| \\ &\le \max_{x\in\mathcal{K}^c} |\varphi(x)| \int_{\mathcal{K}^c} \max_{u\in[c_{s,t},C_{s,t}]} Z\left(\alpha, u, q-q'\right) d\mu_{\mathbb{A}_K^d}(q') \\ &\le \max_{x\in\mathcal{K}^c} |\varphi(x)| 2C_{s,t} \int_{\mathcal{K}^c} \|q-q'\|_{\mathbb{A}_K^d}^{-\alpha} \Phi\left((\|q-q'\|_{\mathbb{A}_K^d}^{-1})_-\right) d\mu_{\mathbb{A}_K^d}(q').\end{aligned}$$

Since $\alpha > 1$ the above integral tends to 0 as $q \to \infty$. This and (4.21) implies that $\lim_{q\to\infty} F^{pq}(t,s)(q) = 0$. Thus $F^{pq}(t,s)\varphi \in C_0(\mathbb{A}_K^d, \mathbb{R})$. □

Proposition 4.16 *For every* $\varphi \in L^1(\mathbb{A}_K^d)$,

$$\lim_{t\searrow s} \left\| \frac{F^{pq}(t,s)\varphi - \varphi}{t} + \widehat{H_V(s)}^{pq}_{L^1}\varphi \right\|_{L^1(\mathbb{A}_K^d)}. \tag{4.22}$$

Proof For every $\varphi \in \mathcal{L}_0 \cap L^1$ (by [43, Proposition 1.3], $\mathcal{L}_0 \cap L^1$ is a dense subset of L^1) we write

$$\begin{aligned}&\frac{F^{pq}(t,s)\varphi(q) - \varphi(q)}{t-s} + (\widehat{H_V(s)})^{pq}\varphi(q) \\ &= \int_{(\mathbb{A}_K^d)^2} \chi\left(p\cdot(q-q')\right)\left(\frac{e^{-\int_s^t H_V(u)(q',p)du} - 1}{t-s}\right)\varphi(q')d\mu_{\mathbb{A}_K^d}(q')d\mu_{\mathbb{A}_K^d}(p) \\ &\quad + \int_{(\mathbb{A}_K^d)^2} \chi\left(p\cdot(q-q')\right) H_V(s)(q',p)\varphi(q')d\mu_{\mathbb{A}_K^d}(q')d\mu_{\mathbb{A}_K^d}(p).\end{aligned} \tag{4.23}$$

By the mean value theorem the first integral on the right above is equal to

$$-\int_{(\mathbb{A}_K^d)^2} \chi\left(p\cdot(q-q')\right) e^{-\int_s^{t'} H_V(u)(q',p)du} H_V(t')(q',p)\varphi(q')d\mu_{\mathbb{A}_K^d}(q')d\mu_{\mathbb{A}_K^d}(p),$$

where $s < t' < t$. We add and subtract the term

$$\int_{(\mathbb{A}_K^d)^2} \chi\left(p\cdot(q-q')\right) e^{-\int_s^{t'} H_V(u)(q',p)du} H_V(s)(q',p)\varphi(q')d\mu_{\mathbb{A}_K^d}(q')d\mu_{\mathbb{A}_K^d}(p)$$

to the right hand side of (4.23) and get that (4.23) is equal to

$$\begin{aligned}&\int_{(\mathbb{A}_K^d)^2} \chi\left(p\cdot(q-q')\right) e^{-\int_s^{t'} H_V(u)(q',p)du} \left(H_V(s) - H_V(t')\right) \\ &\quad (q',p)\varphi(q')d\mu_{\mathbb{A}_K^d}(q')d\mu_{\mathbb{A}_K^d}(p) \\ &\quad + \int_{(\mathbb{A}_K^d)^2} \chi\left(p\cdot(q-q')\right)\left(1 - e^{-\int_s^{t'} H_V(u)(q',p)du}\right) H_V(s) \\ &\quad (q',p)\varphi(q')d\mu_{\mathbb{A}_K^d}(q')d\mu_{\mathbb{A}_K^d}(p).\end{aligned} \tag{4.24}$$

Applying the mean value theorem the second integral is equal to

$$(t'-s)\times\int_{(\mathbb{A}_K^d)^2}\chi\left(p\cdot(q-q')\right)e^{-\int_s^{t''}H_V(u)(q',p)du}H_V(s)H_V(t'')$$
$$(q',p)\varphi(q')d\mu_{\mathbb{A}_K^d}(q')d\mu_{\mathbb{A}_K^d}(p) \tag{4.25}$$

with $t' < t'' < t$. Multiplying $H_V(s)$ by $H_V(t'')$ (see (4.11)) we get that the integral in (4.25) is equal to

$$\int_{(\mathbb{A}_K^d)^2}\chi(p\cdot(q-q')e^{-\int_s^{t''}H_V(u)(q',p)du}g(q',s)g(q',t'')\|p\|_{\mathbb{A}_K^d}^{2\alpha}\varphi(q')d\mu_{\mathbb{A}_K^d}(q')d\mu_{\mathbb{A}_K^d}(p)$$
$$+\int_{\mathbb{A}_K^d}\chi(p\cdot(q-q')e^{-t''H_V(q',p)}\|p\|_{\mathbb{A}_K^d}^{\alpha}g(q',s)V(q',t'')\varphi(q')d\mu_{\mathbb{A}_K^d}(q')d\mu_{\mathbb{A}_K^d}(p)$$
$$+\int_{\mathbb{A}_K^d}\chi(p\cdot(q-q')e^{-t''H_V(q',p)}\|p\|_{\mathbb{A}_K^d}^{\alpha}g(q',t'')V(q',s)\varphi(q')d\mu_{\mathbb{A}_K^d}(q')d\mu_{\mathbb{A}_K^d}(p)$$
$$+\int_{\mathbb{A}_K^d}\chi(p\cdot(q-q')e^{-t''H_V(q',p)}V(q',s)V(q',t'')\varphi(q')d\mu_{\mathbb{A}_K^d}(q')d\mu_{\mathbb{A}_K^d}(p).$$

The L^1-norm (with respect to the variable q) of the above sum of integrals is dominated by the sum of its norms which tend to

$$\left\|D^{2\alpha}(g^2(s)\varphi)\right\|_{L^1(\mathbb{A}_K^d)}+2\left\|g(s)V(s)D^{\alpha}\varphi\right\|_{L^1(\mathbb{A}_K^d)}+\left\|V^2(s)\varphi\right\|_{L^1(\mathbb{A}_K^d)}$$

as $t\to s$. Since $g(s)$ and $V(s)$ are bounded we get by [60, Lemma 93] (see also [48, Lemma 5.3]) that $D^{\alpha}\varphi$ and $D^{2\alpha}g^2(s)\varphi\in\mathscr{L}_0$. Thus the above norms are finite. Consequently, (4.25) tends to 0 when $t\to s$. Since the first integral in (4.24) tends to zero, as $t\to s$, the equality (4.22) is proved. □

Proposition 4.17 *The family $F^{pq}(t,s)$ is strongly continuous at $t=s$.*

Proof Let $\varphi\in\mathscr{L}_0\cap L^1$. By the mean value theorem we have

$$\|F^{pq}(t)\varphi-\varphi\|_{L^1(\mathbb{A}_K^d)}$$
$$=\left\|\int_{(\mathbb{A}_K^d)^2}\chi(p\cdot(\cdot-q')\left(e^{-\int_s^t H_V(u)(q',p)du}-1\right)\varphi(q')d\mu_{\mathbb{A}_K^d}(q')d\mu_{\mathbb{A}_K^d}(p)\right\|_{L^1(\mathbb{A}_K^d)}$$
$$=(t-s)\times$$
$$\times\left\|\int_{(\mathbb{A}_K^d)^2}\chi(p\cdot(\cdot-q')e^{-\int_s^{t'}H_V(u)(q',p)du}H_V(t')(q',p)\varphi(q')d\mu_{\mathbb{A}_K^d}(q')d\mu_{\mathbb{A}_K^d}(p)\right\|_{L^1(\mathbb{A}_K^d)}.$$

If $t \to s$ then $t' \to s$ and the above norm tends to

$$\left\| \int_{(\mathbb{A}_K^d)^2} \chi(p \cdot (\cdot - q') H_V(s)(q', p) \varphi(q') d\mu_{\mathbb{A}_K^d}(q') d\mu_{\mathbb{A}_K^d}(p) \right\|_{L^1(\mathbb{A}_K^d)}$$
$$\leq \|D^\alpha(g)\varphi\|_{L^1(\mathbb{A}_K^d)} + \|V\varphi\|_{L^1(\mathbb{A}_K^d)}.$$

□

5 Proof of Theorem 1.11 and Theorem 1.12

Theorem 5.1 (Dorroh [14]) *Let X be a Banach space (under the supremum norm) of bounded complex valued functions on a set S and $a : S \to \mathbb{R}$ be a bounded positive function on S which is bounded away from zero. Suppose that $aX \subset X$, and let A be the infinitesimal generator of a strongly continuous semigroup of contraction operators in X, then aA is also the infinitesimal generator of a contraction C_0-semigroup in X.*

Theorem 5.2 *Let A be a linear operator in a Banach space X with domain $D(A) \subset X$. Suppose that its resolvent set $\rho(A) \neq \emptyset$ and let L be a bounded linear operator on X. If LA with domain $D(A)$ is a generator of the one-parameter semigroup on X, then AL with domain $D(AL) = \{x \in X \mid Lx \in D(A)\}$ is a generator of the one-parameter semigroup on X.*

Proof See [19, Theorem 3.20 (i), Chapter III, p. 202]. □

We will need the following result from the perturbation theory of semigroups.

Theorem 5.3 *Let X be a Banach space and let A be the infinitesimal generator of a C_0-semigroup $T(t)$ on X, satisfying $\|T(t)\| \leq Me^{\omega t}$. If B is a bounded linear operator on X then $A + B$ is the infinitesimal generator of a C_0-semigroup $S(t)$ on X, satisfying $\|S(t)\| \leq Me^{t(\omega M + \|B\|)}$.*

Proof See e.g. [35, Ch. 3, Sect. 1, Theorem 1.1]. □

Lemma 5.4 *The $\widehat{-H_V(t)}^{qp}$ on the space C_0 exists and satisfies*

$$\widehat{-H_V(t)}^{qp} \varphi(q) = -g(t, q) D^\alpha \varphi(q) - V(t, q)\varphi(q)$$

with domain coinciding with the domain D_α of the operator D^α.

Proof We have, by definition of $\widehat{-H_V(t)}^{qp}$,

$$\widehat{-H_V(t)}^{qp} \varphi(q) = \mathscr{F}^{-1}_{\mathbb{A}_K^d} \left((-g(t, q)\| \cdot \|^\alpha_{\mathbb{A}_K^d} - V(t, q)) \mathscr{F}_{\mathbb{A}_K^d} \varphi(\cdot) \right)(q)$$
$$= -g(t, q) \mathscr{F}^{-1}_{\mathbb{A}_K^d} \left(\| \cdot \|^\alpha_{\mathbb{A}_K^d} \mathscr{F}_{\mathbb{A}_K^d} \varphi(\cdot) \right) - V(t, q)\varphi(q)$$

and the statement follows. □

Proposition 5.5 *For every $t \in [0, T]$ the operator $\widehat{-H_V(t)}^{qp}_{C_0}$ is the infinitesimal generator of a C_0-semigroup $S_t^{qp}(s)$, $s \geq 0$, satisfying*

$$\left\| S_t^{qp}(s) \right\|_{C_0 \to C_0} \leq e^{s\|V(t)\|_{C_0 \to C_0}}.$$

Proof We start with the case $V = 0$. By Lemma 5.4

$$\widehat{-H_0(t)}^{qp} \varphi(q) = -g(t, q) D^\alpha \varphi(q).$$

By [48] the operator $-D^\alpha$ is the infinitesimal generator of a contraction C_0-semigroup in $C_0(Q, \mathbb{C})$. Thus the result for $V = 0$ follows from Theorem 5.1.

The operator $\widehat{-H_V(t)}^{qp}$ is an additive perturbation of $\widehat{-H_0(t)}^{qp}$ by a bounded operator of multiplication by $V(t)$. Thus the result follows from Theorem 5.3. □

Lemma 5.6 *The operator $\widehat{-H_V(t)}^{pq}$ on L^1 exists and satisfies*

$$\widehat{-H_V(t)}^{pq}(\cdot) = -D^\alpha \circ (g(t)\cdot) - (V(t)\cdot)$$

with domain $\{f \in L^1(\mathbb{A}_K^d) \mid gf \in \mathrm{Dom}(D^\alpha)\} = g(\cdot)^{-1}\mathrm{Dom}(D^\alpha)$.

Proof By definition of $\widehat{-H_V(t)}^{pq}$,

$$\begin{aligned}
\langle \mathscr{F}_{\mathbb{A}_K^d} \widehat{-H_V(t)}^{pq} \varphi, \psi \rangle &= \int_{\mathbb{A}_K^d} \psi(p) \mathscr{F}_{\mathbb{A}_K^d} \left((-g(t, \cdot) \|p\|^\alpha_{\mathbb{A}_K^d} - V(t, \cdot))\varphi(\cdot) \right)(p) \\
&= -\int_{\mathbb{A}_K^d} \psi(p) \|p\|^\alpha_{\mathbb{A}_K^d} \mathscr{F}_{\mathbb{A}_K^d}(g(t)\varphi)(p) d\mu_{\mathbb{A}_K^d}(p) \\
&\quad - \int_{\mathbb{A}_K^d} \psi(p) \mathscr{F}_{\mathbb{A}_K^d}(V(t)\varphi)(p) d\mu_{\mathbb{A}_K^d}(p) \\
&= -\langle \mathscr{F}_{\mathbb{A}_K^d}(D^\alpha(g(t)\varphi)), \psi \rangle - \langle \mathscr{F}_{\mathbb{A}_K^d}(V(t)\varphi), \psi \rangle.
\end{aligned}$$

□

Proposition 5.7 *For every $t \in [0, T]$ the operator $\widehat{-H_V(t)}^{pq}_{L^1}$ is the infinitesimal generator of a C_0-semigroup of operators $S_t^{pq}(s)$, $s \geq 0$, satisfying*

$$\left\| S_t^{pq}(s) \right\|_{L^1 \to L^1} \leq e^{s\|V(t)\|_{L^1 \to L^1}}.$$

Proof First we assume that $V = 0$. It is known that $-D^\alpha$ is a generator of a C_0-semigroup of contractions in each of the spaces C_0 and L^ρ, $\rho \geq 1$ (see [48], cf. [42]). Therefore the resolvent set of $-D^\alpha$ is nonempty. From the proof of Proposition 5.5 the composition of operators $-(g\cdot) \circ D^\alpha_{C_0}$ considered with the domain of the operator D^α is a generator of a C_0-semigroup of contractions in $C_0 = C_0(Q, \mathbb{C})$. The operator $(g(t)\cdot)$ of pointwise multiplication in $L^1(\mathbb{A}_K^d)$ by the function g is defined and bounded everywhere on $L^1(\mathbb{A}_K^d)$. Therefore, by Theorem 5.2 the operator $-D^\alpha \circ (g\cdot)$ with its domain equal to $g(\cdot)^{-1}\mathrm{Dom}(D^\alpha)$ is a generator of a one-parameter semigroup of operators in $L^1(\mathbb{A}_K^d)$. Since by Lemma 5.6

$$\widehat{-H_0(t)}_{L^1}^{pq} = -D^\alpha \circ (g(t)\cdot)$$

the result for $V = 0$ is proved.

Now consider a non-zero V. By Lemma 5.6, $(\widehat{-H_V})_{L^1}^{pq} = (\widehat{-H_0})_{L^1}^{pq} - (V\cdot)$. Thus the result follows from Theorem 5.3. □

Proposition 5.8 *The families of operators* $\{\widehat{-H_V(t)}_{C_0}^{qp}\}_{t\in[0,T]}$ *and* $\{\widehat{-H_V(t)}_{L^1}^{pq}\}_{t\in[0,T]}$ *are stable.*

Proof By Proposition 5.5 (Proposition 5.7, resp.) for every $t \in [0, t]$ the operator $\widehat{-H_V(t)}_{C_0}^{qp}$ (the operator $\widehat{-H_V(t)}_{L^1}^{pq}$, resp.) is the infinitesimal generator of a C_0-semigroup $\{S_t^{qp}(s)\}_{s\geq 0}$ ($\{S_t^{pq}(s)\}_{s\geq 0}$) satisfying $\|S_t^{qp}(s)\|_{C_0\to C_0} \leq e^{s\|V(t)\|_{C_0\to C_0}}$ ($\|S_t^{pq}(s)\|_{L^1\to L^1} \leq e^{s\|V(t)\|_{L^1\to L^1}}$, resp.). By our assumption $c \leq \|V(t)\|_{C_0\to C_0} \leq C$, for all $t \geq 0$. Thus the statement follows from Remark 3.3. □

Proof of Theorems 1.11 and 1.12 By Proposition 5.8 the families $\{\widehat{-H_V(t)}_{C_0}^{qp}\}_{t\in[0,T]}$ and $\{\widehat{-H_V(t)}_{L^1}^{pq}\}_{t\in[0,T]}$ are stable. Thus the existence of the families of evolution operators $U^{qp}(t, s)$ and $U^{pq}(t, s)$ follows from Theorem 3.4. To prove the convergence in (1.8) for F^{qp} (convergence in (1.10) for F^{pq}, resp.) it suffices to verify the validity of the conditions of Theorem 3.6 with $Q = F^{qp}$ (with $Q = F^{pq}$, resp.). By Proposition 4.11 (Proposition 5.7, resp.)

$$\|F^{qp}(t, s)\|_{C_0\to C_0} \leq e^{-(t-s)c} \text{ and } \|F^{pq}(t, s)\|_{L^1(\mathbb{A}_K^d)\to L^1(\mathbb{A}_K^d)} \leq e^{-(t-s)c}, \text{ resp.}$$

Therefore, for all $T \geq t_k \geq \ldots \geq t_1 \geq 0$,

$$\|F^{qp}(t_k, t_{k-1})\ldots F^{qp}(t_2, t_1)\|_{C_0\to C_0} \leq C \text{ and } \|F^{pq}(t_k, t_{k-1})\ldots F^{pq}(t_2, t_1)\|_{L^1\to L^1} \leq C$$

and (3.1) with $R = F^{qp}$ ($R = F^{pq}$, resp.) in Theorem 3.6 is satisfied. Finally, by Theorem 4.12 (Theorem 4.16, resp.) (3.2) is also satisfied with $R = F^{qp}$ and $A(t) = \widehat{-H_V(t)}_{C_0}^{qp}$ (with $R = F^{pq}$ and $A(t) = \widehat{-H_V(t)}_{L^1}^{pq}$, resp.). □

6 Hamiltonian and Lagrangian Feynman formulas

6.1 The *qp*-quantization

Theorem 6.1 *For any $t > 0$ and any $q_0 \in Q$, the following Hamiltonian Feynman formula for the Cauchy problem*

$$\partial_t\psi(t, x) = \widehat{-H_V}_{C_0}^{qp}\psi(t, x), \qquad \psi(s, x) = \psi_s(x) \in C_0(Q, \mathbb{C}),$$

holds:

$$\psi(t, q_0) = \lim_{n\to\infty} \prod_{p\in\mathbb{P}} \prod_{v|p} \int_{(Q\times P)^n} e^{-2\pi i \sum_{k=1}^{n} \mathrm{Tr}_{K_v/\mathbb{Q}_p}\{p_k\cdot(q_k-q_{k-1})\}_p}$$
$$\times e^{-\sum_{k=1}^{n} \int_{t-k(t-s)/n}^{t-(k-1)(t-s)/n} g(u,q_{k-1})\|p_k\|^{\alpha}_{\mathbb{A}_K^d} + V(u,q_{k-1})du}$$
$$\times \psi_s(q_n) d\mu_{\mathbb{A}_K^d}(q_n) d\mu_{\mathbb{A}_K^d}(p_n) \dots d\mu_{\mathbb{A}_K^d}(q_1) d\mu_{\mathbb{A}_K^d}(p_1). \tag{6.1}$$

Proof of Theorem 6.1 It is enough to apply Theorem 1.11. Consider the following sequence of times

$$t > \frac{(n-1)(t-s)}{n} + s > \dots > \frac{t-s}{n} + s > s.$$

We have to compute the following product

$$F^{qp}\left(t, t - \frac{t-s}{n}\right) F^{qp}\left(t - \frac{t-s}{n}, t - \frac{2(t-s)}{n}\right) \dots$$
$$\dots F^{qp}\left(\frac{2(t-s)}{n}, \frac{t-s}{n} + s\right) F^{qp}\left(\frac{t-s}{n} + s, s\right), \tag{6.2}$$

where

$$F^{qp}(t,s)\varphi(q) = \widehat{e^{-\int_s^t H_V(u)du}}^{qp}_{C_0} \varphi(q)$$
$$= \int \chi(q\xi) e^{-\int_s^t H_V(u)(q,\xi)du} \tilde{\varphi}(\xi) d\mu(\xi),$$

with

$$H_V(u)(q,\xi) = g(u,q)\|\xi\|^{\alpha}_{\mathbb{A}_K^d} + V(u,q)$$

and check if this product is equal to the right hand side of (6.1). We will leave this as an exercise for the reader. As an example, we only show how two operators in (6.2) are composed. We have

$$F^{qp}\left(\frac{2t}{n} + s, \frac{t}{n} + s\right) F^{qp}\left(\frac{t}{n} + s, s\right)$$
$$= \int_P \int_Q \chi\left(p_1 \cdot (q_0 - q_1)\right) e^{-\int_{s+(t-s)/n}^{s+2(t-s)/n} g(u,q_0)\|p_1\|^{\alpha}_{\mathbb{A}_K^d} - V(u,q_0)du}$$
$$\int_P \int_Q \chi\left(p_2 \cdot (q_1 - q_2)\right)$$
$$\times e^{-\int_s^{s+(t-s)/n} g(u,q_1)\|p_2\|^{\alpha}_{\mathbb{A}_K^d} - V(u,q_1)du} \psi_0$$
$$(q_2) d\mu_{\mathbb{A}_K^d}(q_2) d\mu_{\mathbb{A}_K^d}(p_2) d\mu_{\mathbb{A}_K^d}(q_1) d\mu_{\mathbb{A}_K^d}(p_1).$$

Now it is enough to write explicitly χ (see (2.4)). $\square$

Let $I \subset \mathbb{R}_+ = [0, +\infty)$ be the interval (finite or not). Let $D(I, Q)$ be the Skhorohod space of paths $\gamma : I \to Q = \mathbb{A}_K^d$ which are right-continuous and have left-limits at each point of the half-axis $\mathbb{R}_+ = [0, +\infty)$. By $\mathscr{B}$ we denote the σ-algebra generated by all mappings of the form $\pi_s : \gamma \mapsto \gamma(s)$, $s \in \mathbb{R}_+$.

For every $q_0 \in Q$, the family $\widehat{-H_0(t)}_{C_0}^{qp}$ with $H_0(t, q, p) = g(t, q)\|p\|_{\mathbb{A}_K^d}$, where g satisfies Assumption 1.3, determines the probability measure $d\mathbf{W}_{s,q_0}(\gamma)$ on $\mathscr{B}$ which is the distribution of the corresponding Markov process $\gamma(t)$, $t \in I$, - the evolution generated by $U^{qp}(t, s)$ - starting at time s from q_0, i.e. $\gamma(s) = q_0$ and

$$U^{qp}(t, s)\varphi(q_0) = \int_{\mathbb{A}_K^d} p_{s,t}(q_0, q)d\mu_{\mathbb{A}_K^d}(q) = \mathbf{E}_{s,q_0}\varphi(\gamma(t))$$
$$= \int_{D([s,t],Q)} \varphi(\gamma(t))d\mathbf{W}_{s,q_0}(\gamma),$$

where $p_{s,t}(\cdot, \cdot)$ are the transition densities (propagators) of $\gamma(t)$.

Theorem 6.2 *Let $U^{qp}(t, s)$ be the evolution generated by $\widehat{-H_0(t)}_{C_0}^{qp}$ with $H_0(t, q, p) = g(t, q)\|p\|_{\mathbb{A}_K^d}$, where g satisfies Assumption 1.3. Then*

$$U^{qp}(t, s)\varphi(x) = \lim_{n\to+\infty} \int_{(\mathbb{A}_K^d)^n} \varphi(x_n) \prod_{k=1}^{n} K^g_{s+\frac{(k-1)(t-s)}{n}, s+\frac{k(t-s)}{n}}(x_{k-1}, x_k)$$
$$d\mu_{\mathbb{A}_K^d}(x_1)d\mu_{\mathbb{A}_K^d}(x_2)\dots d\mu_{\mathbb{A}_K^d}(x_n), \tag{6.3}$$

where $x_0 = x$ and

$$K^g_{s,t}(x, y) = Z\left(\alpha, \int_s^t g(u, x)du, x - y\right), \tag{6.4}$$

where $Z(\alpha, t, x)$ is the heat kernel corresponding to D^α.

Proof Since $V = 0$ we have by (4.15) and (4.16) the that

$$F^{qp}(t, s)\varphi(x) = \int_{\mathbb{A}_K^d} \varphi(y)K^g_{t,s}(x, y)d\mu_{\mathbb{A}_K^d}(y).$$

Then, by Theorem 1.11,

$$U^{qp}(t, s)\varphi(x) = \lim_{n\to+\infty} \int_{(\mathbb{A}_K^d)^n} \varphi(x_n) \prod_{k=1}^{n} K^g_{s+\frac{(k-1)(t-s)}{n}, s+\frac{k(t-s)}{n}}(x_{k-1}, x_k)$$
$$d\mu_{\mathbb{A}_K^d}(x_1)d\mu_{\mathbb{A}_K^d}(x_2)\dots d\mu_{\mathbb{A}_K^d}(x_n),$$

where $x_0 = x$. $\square$

Let $y \in \mathbb{A}_K^d$. Taking $\varphi = \delta_y$ in (6.3) (i.e. approximating δ_y by a sequence of elements from $\mathscr{D}(\mathbb{A}_K^d)$) we get the following corollary.

Corollary 6.3 *Let $U^{qp}(t,s)$ be the evolution generated by $\widehat{-H_0(t)}_{C_0}^{qp}$ with $H_0(t,q,p) = g(t,q)\|p\|_{\mathbb{A}_K^d}$, where g satisfies Assumption 1.3. Then transition densities (propagators) of the corresponding Markov process γ are given by*

$$p_{s,t}(x,y) = \lim_{n\to+\infty}\int_{(\mathbb{A}_K^d)^n}\prod_{k=1}^n K^g_{s+\frac{(k-1)(t-s)}{n},s+\frac{k(t-s)}{n}}(x_{k-1},x_k)d\mu_{\mathbb{A}_K^d}(x_1)\dots d\mu_{\mathbb{A}_K^d}(x_{n-1}),$$

where $x_0 = x$, $x_n = y$, and $K^g_{s,t}(x,y)$ is defined in (6.4).

Theorem 6.4 *For any $t > 0$ and any $x_0 \in Q$, the solution ψ for the Cauchy problem $\left(\widehat{-H_V(t)}_{C_0}^{qp}, \psi_s\right)$ is given by*

$$\begin{aligned}\psi(t,x_0) = \lim_{n\to+\infty}\int_{(\mathbb{A}_K^d)^n}&\psi_s(x_n)\\ &\prod_{k=1}^n\left(K^g_{s+\frac{(k-1)(t-s)}{n},s+\frac{k(t-s)}{n}}(x_{k-1},x_k)e^{-\int_{s+(k-1)(t-s)/n}^{s+k(t-s)/n}V(u,x_k)du}\right)\\ &d\mu_{\mathbb{A}_K^d}(x_1)\dots d\mu_{\mathbb{A}_K^d}(x_n).\end{aligned}$$

Proof The family $-V(t,x)$, $t \in \mathbb{R}_+$ generates the evolution

$$G(t,s)\varphi(x) = e^{-\int_s^t V(u,x)du}\varphi(x). \tag{6.5}$$

By Theorem 3.7,

$$\begin{aligned}\psi(t,x_0) = \lim_{n\to\infty}F^{qp}&\left(t,t-\frac{t-s}{n}\right)G\left(t,t-\frac{t-s}{n}\right)\dots\\ \dots F^{qp}&\left(t-\frac{t-s}{n},t-\frac{2(t-s)}{n}\right)G\left(t-\frac{t-s}{n},t-\frac{2(t-s)}{n}\right)\dots\\ \dots F^{qp}&\left(\frac{2(t-s)}{n}+s,\frac{t-s}{n}+s\right)G\left(\frac{2(t-s)}{n}+s,\frac{t-s}{n}+s\right)\dots\\ \dots F^{qp}&\left(\frac{t-s}{n}+s,s\right)G\left(\frac{t-s}{n}+s,s\right)\psi_s(x_0)\end{aligned} \tag{6.6}$$

and the result follows. □

6.2 The *pq*-quantization

Consider the following Cauchy problem

$$\partial_t\psi(t,x) = \widehat{-H_V}_{L^1}^{pq}\psi(t,x), \qquad \psi(0,x) = \psi_s(x) \in \mathscr{D}(\mathbb{A}_K^d). \tag{6.7}$$

Theorem 6.5 *Let*

$$\psi_n(t,q_0) = \prod_{p\in\mathbb{P}}\prod_{v|p}\int_{(Q\times P)^n} e^{-2\pi i\sum_{k=1}^n \mathrm{Tr}_{K_v/\mathbb{Q}_p}\{p_k\cdot(q_k-q_{k-1})\}_p}$$
$$\times e^{-\sum_{k=1}^n \int_{t-k(t-s)/n}^{t-(k-1)(t-s)/n} g(u,q_k)\|p_k\|^{\alpha}_{\mathbb{A}_K^d} + V(u,q_k)du}$$
$$\times\psi_s(q_n)d\mu_{\mathbb{A}_K^d}(q_n)d\mu_{\mathbb{A}_K^d}(p_n)\dots d\mu_{\mathbb{A}_K^d}(q_1)d\mu_{\mathbb{A}_K^d}(p_1). \tag{6.8}$$

Then

$$\psi_n(t,\cdot)\in C_0(\mathbb{A}_K^d,\mathbb{R})$$

and the following Hamiltonian Feynman formula for the solution ψ *of Cauchy problem* (6.7) *holds:*

$$\psi(t,\cdot) = \lim_{n\to\infty}\psi_n(t,\cdot) \text{ in } L^1(\mathbb{A}_K^d).$$

Proof If $\psi_s(x)\in C_0(\mathbb{A}_K^d,\mathbb{R})\cap L^1(\mathbb{A}_K^d)$ then by Proposition 4.15,

$$F^{pq}\left(t,t-\frac{t-s}{n}\right)F^{pq}\left(t-\frac{t-s}{n},t-\frac{2(t-s)}{n}\right)\dots$$
$$\dots F^{pq}\left(\frac{2(t-s)}{n}+s,\frac{t-s}{n}+s\right)F^{pq}\left(\frac{t-s}{n}+s,s\right)\psi_s\in C_0(\mathbb{A}_K^d,\mathbb{R}). \tag{6.9}$$

It is easy to show by induction that (6.9) evaluated at $q_0\in\mathbb{A}_K^d$ is equal to the right hand side of (6.1). Now it is enough to apply Theorem 1.12. □

Remark By Theorem 6.5 there is an increasing sequence of natural numbers $n_k\to+\infty$, $k\in\mathbb{N}$, so that the sequence

$$\psi_{n_k}(t,\cdot)\to\psi(\cdot),\quad \mu_{\mathbb{A}_K^d}\text{-a.e.}$$

Theorem 6.6 *Let* $\psi_s\in\mathscr{D}(\mathbb{A}_K^d)$ *and define*

$$\psi_n(t,x_0)$$
$$=\int_{\mathbb{A}_K^d}\psi_s(x_n)\prod_{k=1}^n\left(K^g_{s+\frac{(k-1)(t-s)}{n},s+\frac{k(t-s)}{n}}(x_{k-1},x_k)e^{-\int_{s+(k-1)(t-s)/n}^{s+k(t-s)/n}V(u,x_k)du}\right)$$
$$d\mu_{\mathbb{A}_K^d}(x_1)\dots d\mu_{\mathbb{A}_K^d}(x_n), \tag{6.10}$$

Then $\psi_n(t,\cdot)\in C_0(\mathbb{A}_K^d,\mathbb{R})$ *and the following Hamiltonian Feynman formula for the solution* ψ *of Cauchy problem* (6.7) *holds:*

$$\psi = \lim_{n\to\infty}\psi_n \text{ in } L^1(\mathbb{A}_K^d).$$

Proof Let $V = 0$. Then, by (4.15) and (4.16),

$$F^{pq}(t,s)\varphi(x) = \int_{\mathbb{A}_K^d} \varphi(y)K_t^y(x,y)d\mu_{\mathbb{A}_K^d}(y),$$

where

$$K_t^y(x,y) = Z\left(\alpha, \int_s^t g(u,y)du, x-y\right).$$

Let $G(t,s)$ be as in (6.5). Then, by Theorem 3.7,

$$\begin{aligned}
&F^{pq}\left(t, t-\frac{t-s}{n}\right)G\left(t, t-\frac{t-s}{n}\right)\dots \\
&\quad \dots F^{pq}\left(t-\frac{t-s}{n}, t-\frac{2(t-s)}{n}\right)G\left(t-\frac{t-s}{n}, t-\frac{2(t-s)}{n}\right)\dots \\
&\quad \dots F^{pq}\left(\frac{2(t-s)}{n}+s, \frac{t-s}{n}+s\right)G\left(\frac{2(t-s)}{n}+s, \frac{t-s}{n}+s\right)\dots \\
&\quad \dots F^{pq}\left(\frac{t-s}{n}+s, s\right)G\left(\frac{t-s}{n}+s, s\right)\psi_s(\cdot) \longrightarrow \psi(t,\cdot) \text{ in } L^1(\mathbb{A}_K^d).
\end{aligned}$$

Thus, (6.10) follows. □

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Declarations

Conflict of interest The authors declare no Conflict of interest.

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