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Special Issue
Symmetry in Gravity Theories and Cosmology

Edited by
Dr. Sunil Kumar Tripathy, Dr. Dipanjali Behera and Dr. Hooman Moradpour

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https://doi.org/10.3390/sym16121672

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$SU(\infty)$ Quantum Gravity and Cosmology

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Abstract: In this letter, we highlight the structure and main properties of an abstract approach to quantum cosmology and gravity, dubbed $SU(\infty)$-QGR. Beginning from the concept of the Universe as an isolated quantum system, the main axiom of the model is the existence of an infinite number of mutually commuting observables. Consequently, the Hilbert space of the Universe represents $SU(\infty)$ symmetry. This Universe as a whole is static and topological. Nonetheless, quantum fluctuations induce local clustering in its quantum state and divide it into approximately isolated subsystems representing $G \times SU(\infty)$, where G is a generic finite-rank internalsymmetry. Due to the global $SU(\infty)$ each subsystem is entangled to the rest of the Universe. In addition to parameters characterizing the representation of G, quantum states of subsystems depend on four continuous parameters: two of them characterize the representation of $SU(\infty)$, a dimensionful parameter arises from the possibility of comparing representations of $SU(\infty)$ by different subsystems, and the fourth parameter is a measurable used as time registered by an arbitrary subsystem chosen as a quantum clock. It introduces a relative dynamics for subsystems, formulated by a symmetry-invariant effective Lagrangian defined on the (3+1)D space of the continuous parameters. At lowest quantum order, the Lagrangian is a Yang–Mills field theory for both $SU(\infty)$ and internal symmetries. We identify the common $SU(\infty)$ symmetry and its interaction with gravity. Consequently, $SU(\infty)$-QGR predicts a spin-1 mediator for quantum gravity (QGR). Apparently, this is in contradiction with classical gravity. Nonetheless, we show that an observer who is unable to detect the quantumness of gravity perceives its effect as curvature of the space of average values of the continuous parameters. We demonstrate Lorentzian geometry of this emergent classical *spacetime*.

Keywords: quantum gravity; quantum cosmology; symmetry

Citation: Ziaeepour, H. $SU(\infty)$ Quantum Gravity and Cosmology. *Symmetry* **2024**, *16*, 1672. https://doi.org/10.3390/sym16121672

Academic Editor: Hooman Moradpour

Received: 25 November 2024
Revised: 11 December 2024
Accepted: 13 December 2024
Published: 17 December 2024

1. Introduction

In the absence of a satisfactory quantum model for spacetime, gravity, and their relationship with other fundamental interactions, the model called $SU(\infty)$-QGR takes an abstract approach to QGR.

Since the first attempts to quantize gravity, model makers have tried to quantize either one of the various formulations of the Einstein gravity, such as the Arnowitt, Deser, and Misner (ADM) canonical (3+1)D formulation [1] and Loop Quantum Gravity (LQG) [2], or another classical model in which a spin-2 field—presumably a graviton field—emerges. Such a field is usually associated to the Riemannian geometry of a spacetime, which sometimes is extended to include matter fields. The most popular version of such models is (super)string theory and its many variants. Despite many differences between these approaches, they have at least one common feature: the space (time) and other ingredients of these models should a priori persist if the Planck constant $\hbar \to 0$. However, in some cases, such as in LQG, the existence of a classical limit is not certain. Indeed, it is very difficult to prove the existence of a classical limit for the non-perturbative canonical formulation of LQG. Even in its perturbative form, that is, the spin foam formulation, finding a semi-classical limit is not straightforward or unique—see, e.g., [3–5]. In any case, decades of investigation into these models show that they suffer from various other issues, such as non-renormalizability; the lack of a genuine and fundamental relation between spacetime and matter (the source of gravitation); the prediction of Lorentz symmetry violation and modified dispersion relation, which are both stirringly constrained [6,7]; the presence of extra dimensions that need compactification and leads to an exponentially large number of possible vacua and models; and finally, the lack of a convincing explanation for the observed (3+1)D classical spacetime. These issues have encouraged the suggestion of an emergent spacetime—see, e.g., [8]—and even the suggestion of a fundamentally classical nature for gravity [9].

On the other hand, $\hbar \rightarrow 0$ does not correspond to our Universe. Recent quantum technological achievements have shown that quantum mechanics should be applied to all scales and that classicality is an approximation. The evidence of macroscopic quantum effects is everywhere: the realization of quantum entanglement between billions of atoms [10]; the remote entanglement of micro-mechanical oscillators [11]; the crucial role of quantum mechanics in explaining exotic phenomena in condensed matter, such as superconductivity, superfluidity, and topological effects (not only in a laboratory setting, but also in astronomical objects such as giant planets, white dwarfs, and neutron stars); and the necessity of a quantum-generated inflation to explain cosmological observations.

Considering the difficulties in standard quantization approaches to QGR and the prominence of quantum mechanics in the fabric of the Universe, it seems reasonable to seek a fundamentally quantum approach to cosmology and QGR. The $SU(\infty)$-QGR is not the first model in this category—called Quantum First Models (QFM) by some authors [12]. Nonetheless, it is significantly different from QFM that try to introduce an intrinsic concept of locality in Quantum Field Theory (QFT)—assumed to be important for the physics of black holes [13–15]. It also significantly diverges from models that use information, graph theory and/or quantum tensor networks in place of a spacetime, and interpret gravity as their entanglement [16–18] (and references therein). A main topic that is crucial for the introduction of structure, locality, matter, and internal symmetries in the QFM is the concept of subsystem. This issue is not addressed in a systematic way in the previous models. One of the goals of $SU(\infty)$-QGR is to introduce and study this fundamental concept.

The $SU(\infty)$-QGR was first reported in [19] and in more detail in [20]. Some of the technical details and demonstrations are reported in [21], and the model is compared with some of other approaches to quantum gravity, including QFM mentioned above in [22]. The aim of this letter is to provide a concise description of the essential features of the model as studied so far. In-depth explanations of the reviewed topics and their mathematical derivation can be found in [19–22].

In Section 2, we briefly describe the axioms of $SU(\infty)$-QGR and how they lead to a Hilbert space for the Universe, representing $SU(\infty)$ symmetry. Section 3 reviews the properties of this Hilbert space and its vectors as a quantum state of the Universe. In Section 4, we define a symmetry-invariant functional on the 2D space of parameters that characterize the representation of $SU(\infty)$ by the Hilbert space. We show that, as a Lagrangian for the whole Universe, this functional is static and the only difference between possible universes is the topology of their parameter space. In Section 5, we argue that quantum fluctuations can locally—in the Hilbert space—break the symmetry of states and induce structures representing generic finite-rank symmetries G. Consequently, according to well-defined criteria for the division of a quantum system to subsystems [23], these structures can be interpreted as approximately isolated subsystems. Indeed, in Section 6, we show that, due to the global $SU(\infty)$ symmetry, these subsystems are not separable and each of them is quantum-entangled to the rest of the Universe (Proposition 1). In Section 7, we demonstrate that, due to this global entanglement, the Hilbert spaces of subsystems represent $G \times SU(\infty)$. As $SU(\infty)$ symmetry is shared by all subsystems, the corresponding interaction can be identified with gravity. The space of parameters characterizing the states of subsystems and their relative dynamics is discussed in Section 8. We demonstrate that they include four continuous parameters. But the geometry of their space Ξ is arbitrary and its reparameterization is equivalent to a $SU(\infty)$ gauge transformation, under which subsystems should be invariant (Proposition 2). Hence, despite apparent similarity to the classical spacetime, Ξ cannot be identified with it. Nonetheless, in Section 9, we use Quantum Speed Limits (QSLs), originating from quantum uncertainty relations, to define the average/expectation values of parameters and an average path in the parameter space for the subsystems. These average/expectation quantities define a classical space Σ, and we demonstrate that its geometry is Lorentzian and related to the quantum state of subsystems. We identify this emerged (3+1)-dimensional space with the perceived classical spacetime. In Section 10, we construct a symmetry-invariant effective Lagrangian for subsystems on their (3+1)D space of continuous parameters Ξ and demonstrate that, at lowest quantum order, it has the form of a Yang–Mills theory for both G and $SU(\infty)$ symmetries. Finally, in Section 11, after clarifying the meaning of a classical limit in $SU(\infty)$-QGR, we show that, in this limit, its Lagrangian is similar to a QFT in curved spacetime with Einstein–Hilbert action for gravity. Section 12 includes concluding remarks.

2. Axioms of $SU(\infty)$-QGR

The construction of $SU(\infty)$-QGR begins with considering the Universe as an isolated quantum system satisfying the following rules and properties:

I. Quantum mechanics is valid at all scales and applies to every entity, including the Universe as a whole;
II. Every quantum system is exclusively described by its symmetries, and its Hilbert space represents them;
III. The Universe has an infinite number of independent degrees of freedom associated to as many mutually commuting quantum observables.

Examples of macroscopic quantum systems and effects discussed in the Introduction justify the axiom I. As for the axiom II, the postulates of quantum mechanics, à la Dirac and von Neumann, do not specify how the Hilbert space should be chosen. Nonetheless, in practice, it always represents the symmetries of the quantum system. Indeed, it is possible to reformulate quantum mechanics postulates with symmetry as a foundational concept [24]. Finally, the huge number of observed approximately separable elementary particles, each having a number of independent quantum observables, motivates the axiom III.

Considering the condition of the hermitianity of operators associated to quantum observables and the unitarity of basis transformation, the Hilbert space $\mathcal{H}_U$ and the space of (bounded) linear operators $\mathcal{B}[\mathcal{H}_U]$ are infinite-dimensional and represent $SU(\infty)$ symmetry. It is shown [25] that all simple compact Lie groups converge to $SU(\infty)$ when their rank $N \to \infty$. Therefore, $SU(\infty)$, as the symmetry of such a Universe, is unique.

3. Hilbert Space of the Universe and Quantization

Representations of $SU(\infty)$ are homomorphic to area-preserving diffeomorphisms of 2D compact Riemann surfaces D_2; their associated algebra is homomorphic to that of the Poisson brackets [26–31] (see the appendices in [19–21] for a review of these representations, other properties of $SU(\infty)$ and its algebra, and more references). Thus, $\mathcal{B}[\mathcal{H}_U] \cong SU(\infty) \cong ADiff(D_2)$, where, throughout this work, the symbol $\cong$ means "homomorphic to", and $\mathcal{B}[\mathcal{H}_U]$ is the space of (bounded) linear operators acting on $\mathcal{H}_U$. We call the 2D compact surface associated to a representation of $SU(\infty)$ its "diffeo-surface".

Generators of $SU(\infty)$ have the following general expression [30]:

$$\hat{L}_f = \frac{\partial f}{\partial \eta}\frac{\partial}{\partial \zeta} - \frac{\partial f}{\partial \zeta}\frac{\partial}{\partial \eta} \quad , \quad \hat{L}_f\, g = \{f, g\}, \quad [\hat{L}_f, \hat{L}_g] \equiv \hat{L}_{\{f,g\}} \tag{1}$$

where f and g are any C^∞ scalar function on the diffeo-surface D_2 and (η, ζ) are local coordinates on the surface. It is more convenient to use the decomposition of functions f and g to orthonormal functions. Two popular decompositions, called "spherical" and "torus" bases, use spherical harmonic [26] and 2D Fourier transform [27,29,30], respectively. We indicate the decomposed generators of $SU(\infty)$ as $\hat{L}_a(\eta,\zeta)$. In sphere basis, $a \equiv (l,m)$, $l \in \mathbb{Z}^+$, $-l \leqslant m \leqslant l$ and, in torus basis, $a \equiv (m,\ n)$, $m,n \in \mathbb{Z}$.

In the absence of a background spacetime in $SU(\infty)$-QGR, the non-Abelian algebra in (1) replaces the usual quantization relations [32,33]. Nonetheless, as this algebra has an infinite rank and is characterized by two continuous variables, it is also possible to find conjugate operators $\hat{J}_a$ for $\hat{L}_a$ such that $[\hat{J}_a, \hat{L}_b] = -i\hbar\delta_{ab}\mathbb{1}$, where $\hat{J}_a \in \mathcal{B}[\mathcal{H}^*_U]$ and$\mathcal{H}^*_U$ is the dual Hilbert space. See the appendices in [20] for the decomposition of $\hat{L}_a$ and $\hat{J}_a$, and their analogy with position and momentum operators in models with a background spacetime according to the Weyl quantization scheme. Notice that, in the above commutation relation, we have explicitly shown the Planck constant $\hbar$. Indeed, as the generators of the Hilbert space of a quantum system, operators $\hat{L}_f$ and $\hat{L}_g$ in (1), and their homologous $\hat{L}_a$ in a specific basis for $SU(\infty)$ algebra, should be normalized such that the r.h.s. of the commutation relation becomes proportional to $\hbar$. Moreover, later in this work, we show that a dimensionful constant, which can be chosen to be the Planck mass M_P or the Planck length $L_P \propto 1/M_P$, arises in the model when the Universe is divided into subsystems. This constant can be included in the generators such that the r.h.s. of (1) becomes proportional to $\hbar/M_P$. This normalization corresponds to using $SU(\infty)$ gauge fields, defined in (3), as generators of the $\mathcal{SU}(\infty)$ algebra and shows that, for $\hbar \to 0$ or $M_P \to \infty$—corresponding to classical limit or no gravity, respectively—the algebra in (1) becomes Abelian and its associated symmetry group $\otimes^{N\to\infty}U(1) \cong \otimes^{N\to\infty}\mathbb{R}$. This is the symmetry of a classical system with $N \to \infty$ independent observables. However, in this case, the diffeo-surface as a parameter space for the states of the system would not be present. Thus, to have a meaningful quantum model, $\hbar/M_P$ must remain finite. In other words, according to $SU(\infty)$-QGR, quantumness and gravity are inseparable.

4. A Globally Static and Topological Universe

This isolated Universe is, by construction, static because there is no external system that can be used as a clock. Indeed, it is a trivial quantum system because every state vector $\psi \in \mathcal{H}_U$ can be transformed to another state by a unitary transformation $U \in \mathcal{B}[\mathcal{H}_U] \cong SU(\infty)$. Such transformations can be considered as changes in the Hilbert space basis. But, as there is no preferred (pointer) basis, all bases and states are physically equivalent. The triviality of the model can be also verified by considering an effective Lagrangian functional $\mathcal{L}_U$ that is invariant under $SU(\infty)$ [19]. Such a functional can be constructed from the traces of the products of symmetry generators. In analogy with QFT, we only consider the lowest-order traces because higher order functionals can be constructed through path integral formalism. The application of the variational principle shows that the solution of dynamical equations of the "fields"—the coefficients of trace terms—is locally trivial, but unstable under fluctuations [19]. This means that states can locally—in the Hilbert space—become clustered. We discuss this process in more detail in the following sections.

Using the logical requirement that the Lagrangian should be invariant under the reparameterization of the diffeo-surface of $SU(\infty)$, we find [20] that, at lowest order in traces, $\mathcal{L}_U$ has an expression similar to a 2D Yang–Mills theory on the diffeo-surface D_2:

$$\mathcal{L}_U = \kappa \int d^2\Omega \left[\tfrac{1}{2}\,\mathrm{tr}(F^{\mu\nu}F_{\mu\nu}) + \tfrac{1}{2}\mathrm{tr}(\not{D}\hat{\rho}_U) \right], \qquad \mu,\nu \in \theta,\phi \tag{2}$$

$$F_{\mu\nu} \equiv F^a_{\mu\nu}\hat{L}^a \equiv [D_\mu, D_\nu], \quad D_\mu = (\partial_\mu - \Gamma_\mu)\mathbb{1} + \textstyle\sum_a i\lambda A^a_\mu \hat{L}^a, \quad a \equiv (l,m) \tag{3}$$

$$F^a_{\mu\nu}F^{\mu\nu}_a = L^*_a L^a, \quad \forall a. \tag{4}$$

where as an example index a is expanded for sphere basis. The first term in (2) does not depend on the geometric connection of D_2. Indeed, it is well-known that, in any dimension, only the covariant derivatives of Yang–Mills gauge fields depend on the geometry of background space. In QFT on an independent background, the second term is not topological. However, in $\mathcal{L}_U$, the space D_2 is the diffeo-surface of its $SU(\infty)$ symmetry.

Every deformation of D_2 can be decomposed to an area-preserving deformation and a global scaling that changes the irrelevant constant κ. Such operation rescales the area of D_2, but the area of the diffeo-surface is irrelevant for its relationship with $SU(\infty)$ and its representation—see Diagrams (28) and (29) below. Therefore, a change in D_2 geometry can be neutralized by an $SU(\infty)$ transformation; see [21] for a detailed demonstration. Consequently, $\mathcal{L}_U$ does not depend on the local geometry of the diffeo-surface D_2 and can be considered as a topological action.

The 2D Riemannian surfaces D_2 are topologically classified by the Euler characteristic $\chi(D_2)$:

$$\int d^2\Omega\, \mathcal{R}^{(2)} = 4\pi\, \chi(D_2), \qquad \chi(D_2) = 2 - \mathcal{G}(D_2), \qquad \chi(D_2) = 2 - \mathcal{G}(D_2) \tag{5}$$

where $\mathcal{R}^{(2)}$ is the scalar curvature of D_2 and $\mathcal{G}$ is its genus. The topological nature of the Lagrangian in (2)—in particular, the pure gauge term—implies that, without any loss of generality, its integrand must be proportional to $\mathcal{R}^{(2)}$:

$$\mathrm{tr}(F^{\mu\nu}F_{\mu\nu}) \;\propto\; \mathcal{R}^{(2)} \tag{6}$$

This relation becomes an equality by changing the arbitrary normalization of $SU(\infty)$ gauge fields. In addition, Equation (6) is the first hint about the relationship between the action $\mathcal{L}_U$ with gravity because, in contrast to 2D Yang–Mills on a classical background, the space D_2, on which this action is defined, is related to the Yang–Mills gauge symmetry itself.

5. Emergence of Structures and Subsystems

The assumed global $SU(\infty)$ symmetry prevents an exact division of the Universe into separable subsystems according to the criteria defined in [34]. Nonetheless, here, we show that its infinite number of observables—degrees of freedom—are sufficient for an approximate blockization of states and a description of $\mathcal{H}_U$ as a tensor product.

A divisible quantum system must fulfill specific conditions [23]. In particular, linear operators applied to its state should consist of mutually commuting subsets $\{\hat{A}_i\}$s, where each subset represents an internal symmetry G_i. Another way to distinguish subsystems in a quantum system is through the factorization of its state. In [21], we show that these two definitions are equivalent.

At this stage, there is no concept of time in $SU(\infty)$-QGR and, therefore, no "order of events". For this reason, to show how approximately isolated subsystems arise in the $SU(\infty)$-QGR Universe, we use an operational approach. Consider the Universe in a completely coherent state, defined as $\hat{\rho}^{CC} \equiv \mathcal{N} \sum_{a,b} |a\rangle\langle b|$ in an arbitrary basis ,where $\mathcal{N}$ is a normalization constant. By definition, this state has maximum coherence according to any of the coherence measures suggested in [35]. Thus, the application of a unitary operator to it transforms $\hat{\rho}^{CC}$ to a less coherent state. For instance, using the sphere basis for $SU(\infty)$ representation, a general state can be written as

$$|\psi_U\rangle = \int d\Omega \sum_{\substack{l\geqslant 0,\\ -l\leqslant m\leqslant l}} \psi_U^{lm} |\mathcal{Y}_{lm}(\theta,\phi)\rangle, \qquad |\mathcal{Y}_{lm}(\theta,\phi)\rangle \equiv Y_{lm}(\theta,\phi)|\theta,\phi\rangle \tag{7}$$

In $|\mathcal{Y}_{lm}(\theta,\phi)\rangle$ basis, the completely coherent state corresponds to $\psi_U^{lm} = const$. Assume that quantum fluctuations lead to the application of $\hat{L}_{l_1 m_1}$ on $|\psi^{cc}\rangle$ and changes it to

$$
\begin{aligned}
\int d\Omega\, \hat{L}_{l_1 m_1}(\theta,\phi)|\psi^{cc}\rangle &= \mathcal{N}\int d\Omega \sum_{\substack{l\geqslant 0,\\ -l\leqslant m\leqslant l}} \hat{L}_{l_1 m_1}(\theta,\phi)|\mathcal{Y}_{lm}(\theta,\phi)\rangle \\
&= -i\hbar\mathcal{N}\int d\Omega \sum_{\substack{l\geqslant 0,-l\leqslant m\leqslant l\\ l'\geqslant 0,-l'\leqslant m'\leqslant l'}} f^{l'm'}_{l_1 m_1,lm} Y_{l'm'}(\theta,\phi)|\theta,\phi\rangle \\
&= -i\hbar\mathcal{N}\int d\Omega \sum_{\substack{l\geqslant 0,-l\leqslant m\leqslant l\\ l'\geqslant 0,-l'\leqslant m'\leqslant l'}} f^{l'm'}_{l_1 m_1,lm} |\mathcal{Y}_{l'm'}(\theta,\phi)\rangle \equiv |\mathbf{g}_{l_1 m_1}\rangle \qquad (8)
\end{aligned}
$$

where the $SU(\infty)$ structure constants $f^{l'm'}_{l_1 m_1,lm}$ are proportional to 3j symbols and depend on the indices $(l,m),(l',m'),(l'_1,m'_1)$ [26]. In general, the new state $|\mathbf{g}_{l_1 m_1}(\theta,\phi)\rangle$ and its corresponding density matrix are not completely coherent anymore, but more structured. Specifically, using the norm of the off-diagonal components of the density matrices of $|\psi_U\rangle$ and $|\mathbf{g}_{l_1 m_1}\rangle$ as a measure of coherence [35] and the boundedness of the integrals of spherical harmonic functions [36], on which the coefficients $f^{l'm'}_{l_1 m_1,lm}$ depend, it is straightforward to show that $|\psi_U\rangle$ is maximally coherent, but $|\mathbf{g}_{l_1 m_1}\rangle$ is much less so—in other words, it is more clustered/blockized (details of this calculation will be reported elsewhere). Moreover, the blockization of the density matrix is more probable to grow with the successive application of $\hat{L}_a$s because there are an infinite number of $\hat{L}_a$ operators with $a \equiv (l,m) \neq (l_1,m_1)$, and the probability of a random occurrence of $\hat{L}^{-1}_{l_1 m_1}$ after operation (8) is extremely small. Of course, as we explained earlier, all these states are globally equivalent and can be transformed to each others without changing physical observables. But, locally and approximately, these blocks satisfy the subsystem criteria defined in [23]. Therefore, $\mathcal{H}_U$ and $\mathcal{B}[\mathcal{H}_U]$ can be approximately decomposed as:

$$
\mathcal{H}_U \rightsquigarrow \bigoplus_i \mathcal{H}_i \rightsquigarrow \bigotimes_i \mathcal{H}_i \qquad (9)
$$

$$
\mathcal{B}[\mathcal{H}_U] \rightsquigarrow \bigoplus_i \mathcal{B}[\mathcal{H}_i] \rightsquigarrow \bigotimes_i \mathcal{B}[\mathcal{H}_i] \qquad (10)
$$

where the symbol $\rightsquigarrow$ means "approximately leads to". To demonstrate the emergence of this approximate tensor product more explicitly and in a representation-independent manner, we use the properties of the Cartan decomposition of $SU(\infty)$. Specifically, $SU(\infty)$ can be decomposed to an infinite tensor product of any finite-rank Lie group [25]:

$$
SU(\infty) \cong \bigotimes_i^{\infty} G_i \qquad (11)
$$

$$
SU(\infty)^n \cong SU(\infty)\ \forall n \qquad (12)
$$

where G_is are finite-rank Lie groups and can be different from each others. Moreover, G_i's themselves can be the products of groups with smaller ranks. However, here, for the sake of the simplicity of notation, we generically call them G.

Considering (11) and (12), the Hilbert space $\mathcal{H}_U$ can be decomposed to

$$
G \times SU(\infty) \cong SU(\infty) \qquad (13)
$$

Notice that, if G had an infinite rank, the subsystem would be indistinguishable from the whole Universe. Moreover, Equations (11) and (12) can be combined to

$$
SU(\infty) \cong \bigotimes^{\infty}(G \times SU(\infty)) \qquad (14)
$$

Following these decompositions, the state of the Universe can be written as a tensor product:

$$|\Psi_U\rangle = \sum\!\!\!\!\!\!\int_{(\eta,\zeta,)} A_U(\eta,\zeta)|\psi_U(\eta,\zeta)\rangle = \sum\!\!\!\!\!\!\int_{\{k_G\},\{y\}} A(k_G;y)\,|\psi_G(k_G)\rangle \times |\psi_\infty(y)\rangle,$$
$$y \equiv (\eta,\zeta;\cdots) \quad (15)$$

and its corresponding density matrix as

$$\begin{aligned}\hat{\varrho}_U &= \sum\!\!\!\!\!\!\int_{(\eta,\zeta,\eta',\zeta')} A_U(\eta,\zeta)A^*_U(\eta',\zeta')\,\hat{\rho}_U(\eta,\zeta,\eta',\zeta') \\ &= \sum\!\!\!\!\!\!\int_{\substack{\{k_G,k'_G\}\\ \{y,y'\}}} A(k_G;y)A^*(k'_G,y')\,\hat{\rho}_G(k_G,k'_G) \times \hat{\rho}_\infty(y,y') \end{aligned} \quad (16)$$

$$\hat{\rho}_U(\eta,\zeta,\eta',\zeta') \equiv |\psi_U(\eta,\zeta)\rangle\langle\psi_U(\eta',\zeta')|, \quad (17)$$

$$\hat{\rho}_G(k_G,k'_G) \equiv |\psi_G(k_G)\rangle\langle\psi_G(k'_G)|, \quad \hat{\rho}_\infty(y,y') \equiv |\psi_\infty(y)\rangle\langle\psi_\infty(y')| \quad (18)$$

$$\sum\!\!\!\!\!\!\int_{(\eta,\zeta)} |A_U(\eta,\zeta)|^2 = 1, \qquad \sum\!\!\!\!\!\!\int_{\substack{\{k_G\}\\ \{y\}}} |A(k_G;y)|^2 = 1 \quad (19)$$

The bases $\{|\psi_U(\eta,\zeta)\rangle\}$, $\{|\psi_G(k_G)\rangle\}$, and $\{|\psi_\infty(y)\rangle\}$ generate the Hilbert spaces of the Universe $\mathcal{H}_U$, and subspaces $\mathcal{H}_G \subset \mathcal{H}_U$ and $\mathcal{H}_\infty \subseteq \mathcal{H}_U$ that represent G and $SU(\infty)$, respectively. Accordingly, operators $\hat{\rho}_G(k_G,k'_G) \in \mathcal{B}[\mathcal{H}_G]$ and $\hat{\rho}_\infty(y) \in \mathcal{B}[\mathcal{H}_\infty]$, such that $\mathcal{B}[\mathcal{H}_G] \times \mathbb{1}_\infty \subset \mathcal{B}[\mathcal{H}_U]$ and $\mathbb{1}_G \times \mathcal{B}[\mathcal{H}_\infty] \subset \mathcal{B}[\mathcal{H}_U]$ are the bases of $\mathcal{B}[\mathcal{H}_G]$ and $\mathcal{B}[\mathcal{H}_\infty]$, respectively. The operator set $\{\hat{\rho}_U(\eta,\zeta,\eta',\zeta')\}$ is a basis for $\mathcal{B}[\mathcal{H}_U]$. The set $\{k_G\}$ parameterizes the representation of G. For finite-rank Lie groups, the number of independent k_Gs, that is the dimension d_G of the parameter space $\{k_G\}$, is finite and k_Gs usually take discrete values. For example, for $G = SU(2)$, $k_G = (l,m)$, $l \in \mathbb{Z}^+$, $-l \leqslant m \leqslant l$. For a fixed l—corresponding to a super-selected representation of $SU(2)$—the dimension $d_G = 2l+1$. The continuous parameters (η,ζ) are the coordinates of the diffeo-surface and characterize the generators of $SU(\infty)$. The extension dots in y and y' indicate emergent parameters when the Universe is perceived through the ensemble of its subsystems. They are described in the following sections.

It is easy to verify that $\mathcal{H}_G$ and $\mathcal{H}_\infty$ fulfill the requirements for subsystems defined in [23]. In a given basis for $\mathcal{H}_U$, they are, by construction, orthogonal to each other. Considering (13), endomorphism condition $\mathcal{H}_U \cong \mathcal{H}_G \times \mathcal{H}_\infty$ is fulfilled. In the absence of a background spacetime, the locality condition in the usual sense is irrelevant. Nonetheless, due to the contractivity of distance functions [37], the distance between states belonging to $\mathcal{H}_G$ is always smaller than their distance from similar states with non-zero projection in the complementary subspace $\mathcal{H}_\infty$, and vice versa. Hence, the decomposition in (16) induces a "locality" concept and structure in the Hilbert space $\mathcal{H}_U$. This is in addition to the geometrical locality, which can be defined for any Hilbert space by associating a Fubini–Study metric and distance to its states.

It is important to notice that the concept of subsystem is more general than elementary particles or fields, which can be defined as a subsystem that cannot be decomposed to smaller parts. By contrast, a general subsystem can include many or even an infinite number of elementary subsystems. Similar to classical gravity, the $SU(\infty)$-QGR formulation is independent of internal symmetries and the properties of subsystems. Therefore, it fully respects the equivalence principle.

6. Global Entanglement

The difference between the Hilbert space of the Universe $\mathcal{H}_U$ that represents $SU(\infty)$ and its states $|\psi_U\rangle$, and those of $\mathcal{H}_\infty$, is better understood if we use the properties of $SU(N \to \infty)$ Cartan decomposition and write (13) as

$$SU(N \to \infty) \supseteq SU(K) \times SU(N - K \to \infty) \supseteq G \times SU(N - K \to \infty) \quad (20)$$

in which $\infty > K \in \mathbb{Z}^+$ is chosen such that $SU(K) \supseteq G$. From this relation, it is clear that symmetry in the r.h.s. of (20) is smaller and presents a broken version of the l.h.s. Only when $N \to \infty$ are the two sides are homomorphic. Thus, only in this limit, $\mathcal{H}_U \cong \mathcal{H}_G \times \mathcal{H}_\infty$. Otherwise, the factorization of G from $SU(N \gg 1)$ due to the clustering of states presents an (approximate) breaking of the symmetry of the Universe. In the limit of $N \to \infty$, the Hilbert space $\mathcal{H}_U$ can be decomposed to an infinite number of subsystems representing the generic finite-rank symmetry G. As discussed in Section 5, the finite-rank internal symmetry of subsystems do not need to be the same because $SU(N - K \to \infty)$ in (20) can be in turn decomposed, as long as the total rank of factorized groups $K' < \infty$ and the remaining $SU(N - K' \to \infty)$. Thus, Equations (11)–(14) and (20) show that the $SU(\infty)$-QGR Universe can be constructed either top-down, that is, by dividing it into an infinite number of finite-rank subsystems (contents), or bottom-up, by considering it as the ensemble of an infinite number of quantum systems representing a symmetry, which can have any rank, including infinity. However, in the latter case, one should also impose the global $SU(\infty)$ symmetry to connect everything together, as we show in the next sections. Thus, the top-down approach is more economical in the number of axioms.

As we discussed earlier, despite the tensor product structure of the bases in (15) and (16), and their corresponding Hilbert spaces, due to the global $SU(\infty)$, amplitudes $A(k_G; \eta, \zeta, \cdots)$ are not factorizable. Consequently, G-representing subsystems are not separable—they are entangled. This observation is formulated in the following proposition.

Proposition 1. *In $SU(\infty)$-QGR, every subsystem is entangled to the rest of the Universe.*

In [20], mutual information is used to prove this proposition. We call this attribute of the model "the global entanglement". A more explicit demonstration of Proposition 1 consists of tracing out the $SU(\infty)$-representing component of $\hat{\varrho}_U$ [21]:

$$\hat{\varrho}_G \equiv \mathrm{tr}_\infty \hat{\varrho}_U = \int dy^D \sum_{\{k_G, k'_G\}} A_G(k_G; y) A^*_G(k'_G, y)\, \hat{\rho}_G(k_G, k'_G), \qquad y \equiv (\eta, \zeta, \cdots) \quad (21)$$

It is shown that $\hat{\varrho}_G$ is a mixed state and has a non-zero von Neumann entropy [21]. This result is not a surprise because, due to the global $SU(\infty)$ symmetry, amplitudes $A_G(k_G; y)$ are not factorizable to k_G and y dependent functions. Therefore, $\mathcal{H}_G$ and $\mathcal{H}_\infty$ cannot be considered Hilbert spaces of separable subsystems. Nonetheless, the subspace $\mathcal{H}_G$ is approximately isolated by its "local" symmetry G. Moreover, considering the finite rank of G and the entanglement of $\hat{\varrho}_G$ with the rest of the Universe, it can be interpreted as the mixed state of a subsystem approximately isolated from its infinite-dimensional "environment" due to their approximate inaccessibility,. Therefore, decompositions of the type presented in (20) induce a concept of "locality" for the subsystems. In addition, we notice that the amplitudes $A_G(k_G; y)$ have a structure similar to gauge fields, that is, they depend on the parameters of a finite-rank Lie group and a continuous "background". Observables of the state $\hat{\varrho}_G$—that is, Hermitian operators in $\mathcal{B}[\mathcal{H}_G]$—are, by construction, invariant under application of G and reparameterization of the "external parameters" y. We discuss the meaning and importance of these properties when relative dynamics for subsystems are introduced in Section 10.

Similarly, tracing out the $\hat{\rho}_G$ component of $\hat{\varrho}_U$ leads to a mixed state $\hat{\varrho}_\infty$:

$$\hat{\varrho}_\infty \equiv \text{tr}_G \, \hat{\varrho}_U = \sum\!\!\!\!\!\!\int_{\{(\eta,\zeta,\cdots)\},\{(\eta',\zeta',\cdots)\}} \mathcal{A}_\infty(\eta,\zeta;\eta',\zeta',\cdots) \, \hat{\rho}_\infty(\eta,\zeta;\eta',\zeta',\cdots) \quad (22)$$

$$\mathcal{A}_\infty(\eta,\zeta;\eta',\zeta',\cdots) \equiv \sum_{\{k_G\}} A_\infty(k_G;\eta,\zeta,\cdots) A^*_\infty(k_G;\eta',\zeta',\cdots) \quad (23)$$

$$\sum\!\!\!\!\!\!\int_{\{(\eta,\zeta,\cdots)\},\{(\eta',\zeta',\cdots)\}} \mathcal{A}_\infty(\eta,\zeta;\eta,\zeta,\cdots) = 1 \quad (24)$$

for the $SU(\infty)$-representing environment. We notice that $\hat{\varrho}_\infty$ also depends on a set of external parameters $\{k_G\}$, which are not related to the $SU(\infty)$ symmetry of this state. The physical meaning of this dependence is clarified once we establish, in Section 9, an effective path for the subsystems in their parameter space.

The entanglement of mixed states $\hat{\varrho}_G$ and $\hat{\varrho}_\infty$ can be quantified using usual entanglement measures and are calculated in [21] for future applications of $SU(\infty)$-QGR.

7. The Full Symmetry of Subsystems

In the mixed state $\hat{\varrho}_G$, the parameter vector y is in part the footprint of $SU(\infty)$ symmetry and plays the role of a classical background for an observer who does not have access to the full extent of the quantum state of the Universe $\hat{\varrho}_U$. On the other hand, considering axioms I and II of $SU(\infty)$-QGR about the direct or indirect quantum origin of all processes and observables, and their association to symmetries represented by the Hilbert space, the observer can associate two of the four components of y—we show this later—to a representation of $SU(\infty)$ symmetry and use them to purify $\hat{\varrho}_G$ by extending the Hilbert space with an auxiliary space representing this symmetry. In [21], we showed that $\hat{\varrho}_G$ satisfies the conditions for faithful purification [38]. The purified state will have the following form:

$$|\psi_{G_\infty}\rangle \equiv \sum\!\!\!\!\!\!\int_{\{k_G\};\{y\}} A_{G_\infty}(k_G;y) \, |\psi_G(k_G)\rangle \times |\psi_\infty(y)\rangle \quad (25)$$

where $|\psi_\infty(y)$ has the same definition as in (15), but is not necessarily the same basis. Although $|\psi_{G_\infty}\rangle$ looks like the state of the Universe $|\psi_U\rangle$ in (15), according to the Schrödinger–HJW theorem [39–41] about the degeneracy of purification, in general, $|\psi_{G_\infty}\rangle \neq |\psi_U\rangle$. The state $|\psi_{G_\infty}\rangle$ can be also considered as a purification of $\hat{\varrho}_\infty$. In both cases, the state is a vector in a Hilbert space that represents $G \times SU(\infty)$, which is the full symmetry of the subsystems. This shows the reciprocity of the state of a subsystem and its environment—any of them can be considered as a subsystem or environment. In particular, their entanglement means that they have to share at least one common symmetry through which they can be entangled. Considering the fact that the finite-rank symmetry G can be different for different subsystems, the common symmetry that ensures the global entanglement is necessarily $SU(\infty)$. Indeed, considering (13), it is possible in $\hat{\varrho}_\infty$ to include $\{k_G\}$ parameters into the infinite set of $SU(\infty)$ parameters. Nonetheless, the explicit dependence of $\hat{\varrho}_\infty$ on $\{k_G\}$ shows the perspective dependence of the environment [42].

8. Parameter Space of Subsystems

As the perception of "environment" by different subsystems is not the same, there are an infinite number of representations of $SU(\infty)$, one for each subsystem. Moreover, the algebra of $ADiff(D_2)$ in (1) is invariant under scaling of parameters. Thus, the area of the diffeo-surface D_2 is irrelevant for $\mathcal{SU}(\infty) \cong ADiff(D_2)$. This means that (1) is indeed homomorphic to the algebra $\mathcal{SU}(\infty) + \mathcal{U}(1)$ [31].

In the presence of multiple representations of $SU(\infty)$, the homomorphism in (12) implies the following relation between the $ADiff$ of their diffeo-surfaces:

$$\bigcup_{i=1}^{n} ADiff(D_2^{(i)}) \cong ADiff(D_2), \qquad D_2 \equiv \bigcup_{i=1}^{n} D_2^{(i)} \quad (26)$$

where D_2 is, by definition, the diffeo-surface of $SU(\infty)$ in the r.h.s. of (12). Although the area of D_2 is arbitrary, once diffeo-surfaces are stuck together, only the area of their ensemble D_2 can be arbitrarily scaled and those of the components $D_2^{(i)}$ should be adjusted such that the area of D_2 is preserved. Consequently, the quantum states of subsystems will depend on an additional continuous parameter $r > 0$ that represents the relative area of the compact diffeo-surfaces of their $SU(\infty)$ subsystems, or any function of the area, such as its square-roots, as a size indicator. In this interpretation, $r = 0$ means a trivial representation of $SU(\infty)$ and, according to the axioms and construction of the model, it should be excluded.

Algebraically, (26) is equivalent to breaking of the scaling $\mathbb{R} \cong \mathcal{U}(1)$ symmetry of the component subsystems:

$$\bigotimes^{n}\Big(\mathcal{SU}(\infty)+\mathcal{U}(1)\Big)\xrightarrow[\text{area is preserved}]{\text{Only total}}\bigotimes^{n}\mathcal{SU}(\infty)+\mathcal{U}(1)\cong\mathcal{SU}(\infty)+\mathcal{U}(1) \tag{27}$$

Diagrams (28) and (29) summarize the relationship of $SU(\infty)$ and $ADiff$ for single and multiple representations. A depiction of the diffeo-surfaces of subsystems in Figure 1 shows how the rescaling of one affects others, such that the total area is preserved

$$\begin{array}{c}\text{Single subsystem}\\ \begin{array}{ccccc} SU(\infty) & \longrightarrow & \mathcal{SU}(\infty) & \xrightarrow{\text{area irrel.}} & \mathcal{SU}(\infty)+\mathcal{U}(1)\\ & & {\scriptstyle\cong}\downarrow & & \downarrow{\scriptstyle\cong}\\ & & ADiff(D_2) & \xrightarrow[\text{area irrel.}]{} & ADiff(D_2)\times U(1)\end{array}\end{array} \tag{28}$$

$$\begin{array}{c}\text{Multiple subsystems}\\ \begin{array}{ccc} SU(\infty)\times\cdots\times SU(\infty)\cong SU(\infty)\longrightarrow\mathcal{SU}(\infty) & \xrightarrow{\text{area irrel.}} & \mathcal{SU}(\infty)+\mathcal{U}(1)\\ {\scriptstyle\text{area irrel.}}\downarrow & & \downarrow{\scriptstyle\text{area irrel.}}\\ (ADiff(D_2^{(1)})\times U(1))\times\cdots\times(ADiff(D_2^{(n)})\times U(1)) & \xrightarrow[\text{symm. break}]{\text{area preserv.}} & ADiff(D_2)\times U(1)\end{array}\end{array} \tag{29}$$

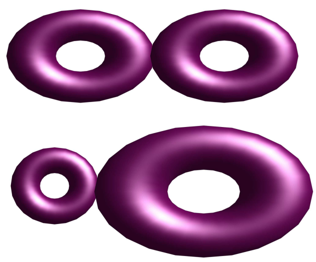

Figure 1. Rescaling of the diffeo-surface of left (**right**) subsystem induces rescaling to the right (**left**) subsystem such that their total area is preserved.

The breaking of the scaling symmetry does not mean that the value of r, or, in other words, the areas of diffeo-surfaces of subsystems, are fixed. For each global area and/or its rescaling in the r.h.s. of (29), there are infinite number of rescaling combinations of the l.h.s. that satisfy the constraint on their total area. Therefore, in general, the quantum states of subsystems are in a coherent superposition of r eigen states.

8.1. Time Parameter and Relative Dynamics

In addition to the emergence of an are or size parameter, the division of the Universe into subsystems makes it possible to choose one of them as a quantum clock. Then, one of its observables can be chosen as a time parameter t, and a relative dynamics—à la Page and Wootters [43] or equivalent methods [44]—arises in an operational manner as the following: a random application of an operator $\hat{O}$ to the state $\hat{\varrho}_s \in \mathcal{B}[\mathcal{H}_s]$ of a subsystem with Hilbert space $\mathcal{H}_s$—in other words, a quantum fluctuation—changes it to $\hat{O}\hat{\varrho}_s\hat{O}^\dagger$. The global entanglement conveys this change to other subsystems, in particular to the clock, and its time parameter t changes—"the clock ticks". As the clock state is, by definition, a reference, the states of subsystems are tagged by t, and variation of their states along with the time variation generates a dynamics.

Of course, the change of the clock's state and, therefore, the time does not need to be projective. In addition, other subsystems will have their own change of state, both coherently and through reciprocal interactions. Consequently, an arrow of time arises and persists eternally because, although inverse processes are, in principle, possible, giving the infinite number of subsystems, operators in $\mathcal{B}[\mathcal{H}_s]$, and the global entanglement, bringing back the states of all subsystems to their initial one—in other words, inverting the arrow of time—is extremely improbable.

8.2. Properties of Continuous Parameters

We can now complete the list of continuous parameters $(\eta, \zeta, \cdots)$ in (15)–(19). With a new ordering, we write them as $x \equiv (t, r, \eta, \zeta)$. The last two parameters characterize the representation of $SU(\infty)$ by a subsystem and generate its diffeo-surface as a compact 2D subspace D_2 of the 4D parameter space $\Xi \equiv \{x\}$. Parameter r is dimensionful and presents the area (or a characteristic length, for example, square-root of area) of the diffeo-surfaces of $SU(\infty)$ representations. This is a relative value and is defined with respect to that of a reference subsystem. Finally, as we described above, t is a time parameter—an observable of a clock subsystem.

Although (t, r, η, ζ) parameters have different origins, due to quantum coherence, indistinguishability of subsystems having the same symmetries and their representation, and arbitrariness of the choice of clock and reference subsystems, they are related to each others and the space Ξ cannot be factorized. Thus, diffeo-surfaces of the $SU(\infty)$ symmetry of subsystems can be arbitrarily embedded in Ξ and their induced coordinates would be functions of all these parameters. Moreover, as described in Section 3, in analogy with quantum mechanics on a background spacetime, it is possible to associate an operator to each component of x and to their duals. It is shown [19,20] that these operators satisfy the Heisenberg commutation relations. In addition, they can be expanded with respect to the generators $\hat{L}_a$ of $SU(\infty)$ [20]. However, their expansion is not unique. Consequently, the amount of information carried by $\hat{L}_a$ is much larger than what can be expressed by the four observables associated to $x \in \Xi$.

The observables of subsystems should be invariant or transformable under a reparameterization of Ξ. On the other hand, a basis transformation of the $SU(\infty)$ sector of the Hilbert spaces $\mathcal{H}_s$ is equivalent to a diffeomorphism of the parameter space Ξ. This is because Ξ can be considered as the collection of 2D diffeo-surfaces [20,21]. Inversely, a deformation or reparameterization of Ξ can be compensated by a $SU(\infty)$ transformation. As $SU(\infty) \cong ADiff(D_2)$, such transformations are equivalent to the deformation of 2D subspaces Ξ—the diffeo-surfaces. Therefore, the geometry of Ξ is irrelevant for physical observables. This feature of $SU(\infty)$-QGR can be formulated as the following proposition.

Proposition 2. *The curvature of the parameter space Ξ of subsystems can be made trivial by an $SU(\infty)$ gauge transformation, under which the Universe and its subsystems are invariant.*

In [20], we used the relationship between Riemann and Ricci curvature tensors, Ricci scalar curvature, and the sectional curvature of embedded 2D diffeo-surfaces in Ξ to prove this proposition. In [21], we confirmed this demonstration by applying $SU(\infty)$ transformations on the lowest-order effective Lagrangian of subsystems reviewed in Section 10.

Proposition 2 shows that, despite the similarity of Ξ with what we perceive as classical spacetime—especially its dimension—it cannot be identified with the latter, because various astronomical observations have shown the influence of the curvature and geometry of the classical spacetime on physical phenomena and observables. Nonetheless, in the next section, we find observables that can be identified with the classical spacetime.

9. Classical Geometry as an Effective Path in the Hilbert Space

Following the designation of a clock subsystem and a time parameter, the unitary evolution of states of subsystems is determined by a Hamiltonian H_s and a Liouvillian operation: $d\hat{\varrho}_s/dT = -i/\hbar[\varrho, \hat{H}_s]$. On the other hand, considering Proposition 1 and global entanglement, the evolution of $\hat{\varrho}_s$ should be more realistically formulated by a superoperator $\hat{\mathcal{L}} \in \mathcal{B}[\mathcal{B}[\mathcal{H}_s]]$, such that $d\hat{\varrho}_s/dT = -i/\hbar\hat{\mathcal{L}}(\hat{\varrho}_s)$. The variable T is either the outcome of the measurement of the time parameter t of the clock or the expectation value of such measurements. In the next section, we explain how the evolution of subsystems are formulated, that is, how $\hat{H}_s$ or $\mathcal{L}(\hat{\varrho}_s)$ are determined. Meanwhile, we use the Mandelstam–Tamm Quantum Speed Limit (QSL) for the unitary evolution of pure states [45,46] and analogous relations for the unitary, Markovian, or non-Markovian evolution of mixed states [45,47–50] to find an effective classical spacetime in the framework of $SU(\infty)$-QGR. We remind readers that QSL relations are extensively studied in the literature and the references cited here are only a small sample of relevant works.

The QSL inequalities attribute a minimum time to the evolution of a quantum state to another completely or partially distinguishable state. The Mandelstam–Tamm QSL (MTQSL) [45,46] is a consequence of the uncertainty relations between non-commuting observables and their unitary evolution according to the Schrödinger equation. For mixed states and the non-unitary evolution of open systems, geometrical properties of the space of density matrices and their relationship with probability distributions [51,52] provide easier ways to find QSL relations. In this approach [47,53–55], after assigning a distance function $\mathcal{D}$ to two states $\hat{\varrho}(T_0)$ and $\hat{\varrho}(T)$ and the corresponding metric $g_{tt}(\hat{\mathcal{L}}, \hat{\varrho})$ for the geometry of the space of density operators, the QSL can be written as

$$\Delta T \geqslant \frac{\mathcal{D}(\hat{\varrho}(T_0), \hat{\varrho}(T))}{\langle\!\langle \sqrt{g_{tt}} \rangle\!\rangle} \tag{30}$$

where the double bracket means averaging over the measured time interval $\Delta T = T - T_0$ along the evolution path in the Hilbert space. However, these QSLs are not all tight, that is, the minimum time is not always attainable [55,56]. Here, we only consider tight QSLs. In any case, the relationship between the geometry of the space of density operators and their statistical properties is the evidence that uncertainty relations are behind the existence of a speed limit for the evolution of quantum states. This is in contrast to the classical physics, in which the speed limit is empirical and an axiom of special and general relativity. For example, in the case of pure states, the distance $\mathcal{D}$ and the corresponding g_{tt} are unique and correspond to the Fubini–Study distance and metric, respectively [57]. For mixed states, the distance function is not unique and the metric for a given definition of distance is not always known [51,52,58]. An exception is the Bures distance [47,59], whose corresponding metric is the Wigner–Yanase skew information [60]. In non-geometric QSL relations, the denominator in (30) is usually a non-geometric quantity. For example, for mixed states, the relative purity of the state can be used to find an attainable QSL [49].

Here, a remark about the physical meaning of time in QSLs is necessary. In the literature, the QSL relations, such as (30), are studied in the framework of quantum mechanics with a background classical spacetime. By contrast, for $SU(\infty)$-QGR, we have to employ them in the context of relative time and dynamics, where T should be interpreted as an expectation value or conditional outcome of the measurement of the time parameter t [43]. For this reason, we indicate time as T and not the component t of $x \in \Xi$. Accordingly, traces in $\mathcal{D}$ and g_{tt} (see [19,20] for examples), which are calculated at a given time, should take into account the meaning of T. For instance, it is clear that the tracing operation leading to (25) includes all components of $x \in \Xi$, including t. For using such states in (30), one has to project amplitudes on $t = T$ for a projective measurement of time or, more generally, add the condition $\mathrm{tr}(\hat{\varrho}\hat{T}) = T$, where $\hat{T}$ is the operator associated to the time observable of the clock.

In the framework of $SU(\infty)$-QGR, consider an infinitesimal variation of the state of subsystems after tracing out the contribution of internal symmetries, that is, $\hat{\varrho}_\infty \to \hat{\varrho}_\infty + d\hat{\varrho}_\infty$. Assume that the clock, its time parameter t, measured time T, and the corresponding evolution superoperator $\hat{\mathcal{L}}_\infty$ are such that, in the QSL in (30), equality is achieved:

$$\langle\!\langle \sqrt{g_{tt}} \rangle\!\rangle^2 dT^2 = \mathcal{D}^2[\hat{\varrho}_\infty, \hat{\varrho}_\infty + d\hat{\varrho}_\infty] \equiv ds^2 \tag{31}$$

Although (22) shows that $\hat{\varrho}_\infty$, $d\hat{\varrho}_\infty$, and, thereby, the r.h.s. of (31) are characterized by continuous parameters $x \in \Xi$, it also demonstrates that they are independent of Ξ's parameterization. Therefore, the introduced parameter s and its variation ds depend only on the state and its variation—see the appendices in [20] for an explicit description of ds for the QSL examples mentioned above. For this reason (and because of the geometric interpretation of QSLs), s is analogous to the affine separation in the classical spacetime. In fact, it is indeed the affine separation for the geometry of the space of density matrices [47,54].

If we choose another clock with time parameter t', and measured time T', the evolution superoperator changes to $\hat{\mathcal{L}}'_s$, and, in general, in (30), the equality is not attained, because both $\Delta T'$ and the denominator in (30), which depends on $\hat{\mathcal{L}}'_s$ (see examples in [20]), change. Nonetheless, according to (31), the affine parameter ds only depends on the state and its variation. Thus, it remains unchanged, and, in general,

$$\langle\!\langle \sqrt{g'_{tt}} \rangle\!\rangle^2 dT''^2 \geqslant ds^2 \tag{32}$$

This inequality can be changed back to equality by adding a term $-d\mathcal{F}^2$ to its l.h.s.:

$$\langle\!\langle \sqrt{g'_{tt}} \rangle\!\rangle^2 dT''^2 - d\mathcal{F}^2 = ds^2 \tag{33}$$

To understand the nature of $d\mathcal{F}$, we should remind readers that, for any state $\hat{\varrho}$, the affine variation ds^2 is a scalar functional of $\hat{\varrho}$ and $d\hat{\varrho}$. Thus, it has the general expression $ds^2 = \mathcal{S}[\mathrm{tr}(f_1(\hat{\varrho})f_2(d\hat{\varrho})]$, where $\mathcal{S}$, f_1, and f_2 are some functions. A tracing operation on the functionals of the density operator and its variation is necessary for changing them to C-numbers. Therefore, considering the relationship of density matrices with the probability distribution of the outcomes of quantum measurements, ds^2 presents some sort of statistical averaging. For example, in the QSL based on the relative purity [49] and when Fubini–Study distance is used in (30), at the lowest order, ds^2 has the following form [20,21]:

$$ds^2 = (\mathrm{tr}(\hat{\varrho} d\hat{\varrho}))^2 \tag{34}$$

In these cases, $|ds|$ has a clear interpretation as the average variation of state. In the context of $SU(\infty)$-QGR, the state $\hat{\varrho}_\infty$ is characterized by $x \in \Xi$ and the pushback of averaging in (34) leads to an average or effective value X for x. We call Σ the space of these average/expectation values. Notice that both $\hat{\varrho}_\infty$ and its purification $|\psi_{G_\infty}\rangle$ in (25) are in a superposition of $x \in \Xi$ parameters. Therefore, in the framework of quantum mechanics, the average value of x is both mathematically and physically meaningful.

Considering (31)–(34) and the above discussion, we can associate a Riemann metric to Σ with s as its affine parameter:

$$ds^2 = g_{\mu\nu}(X) dX^\mu dX^\nu, \mu,\ \nu = 0, \cdots, 3 \tag{35}$$

A reparameterization of Ξ, under which observables are invariant, is transferred to the space of their expectation values Σ. Therefore, 4 of 10 components in the metric $g_{\mu\nu}(X)$ of Σ are arbitrary and we can choose x, and thereby X, such that (35) has the following form:

$$ds^2 = g_{00}(X) dT'^2 - g_{ij}(X) dX^i dX^j, \qquad g_{00}(X) = \langle\!\langle \sqrt{g'_{tt}} \rangle\!\rangle^2 > 0, \quad i, j = 1, 2, 3 \tag{36}$$

In this gauge, X^i, $i = 1, 2, 3$ are related to parameters (r, η, ζ) or, equivalently, their Cartesian form (x^1, x^2, x^3). As they are associated to the geometry of 2D compact diffeo-surfaces, they and their expectation values X^is should be interchangeable. Consequently, $g_{ij}(X), i, j = 1, 2, 3$ must have the same sign. On the other hand, as, in (33), $d\mathcal{F}^2 \geqslant 0$, the metric components in (36) must be positive, that is, $g_{ij}(X) > 0$. Therefore, the signature of the metric $g_{\mu\nu}(X)$ is negative and Σ has a Lorentzian (pseudo-Riemannian) geometry. In conclusion, the metric (35) has the geometrical properties of the classical spacetime. Moreover, Equation (34) shows that Σ and its metric are related to the quantum state of subsystems—the content of the Universe—and its evolution. For these reasons, we identify the (3+1)-dimensional Σ with the perceived classical spacetime. In summary, $SU(\infty)$-QGR explains the origin of both the dimensionality and Lorentzian geometry of the classical spacetime.

If the contribution of internal symmetries in the density matrix is not traced out, one can define a metric and an affine parameter that includes parameters of the internal symmetries. However, finite-rank Lie groups are usually characterized by discrete parameters. Therefore, in contrast to some other QGR proposals, there is no continuous extra-dimension in $SU(\infty)$-QGR.

10. Dynamics of Subsystems

In the same manner as we did for the whole Universe, we can construct a symmetry-invariant Lagrangian on the parameter space Ξ by using symmetry-invariant traces of the products of the generators of $SU(\infty)$ and the internal symmetry G of subsystems [19,20]. The invariance of the coefficients of these traces under the reparameterization of Ξ and under $G \times SU(\infty)$ symmetry constrains their expression, and we find that, at the lowest order in the number of traced generators, the effective Lagrangian of subsystems $\mathcal{L}_{U_s}$ must have the form of a Yang–Mills theory for both $SU(\infty)$ and G symmetries:

$$\mathcal{L}_{U_s} = \int d^4x \sqrt{|\eta|} \left[\frac{1}{16\pi L_P^2} \text{tr}(F^{\mu\nu} F_{\mu\nu}) + \frac{\lambda}{4} \text{tr}(G^{\mu\nu} G_{\mu\nu}) + \frac{1}{2} \sum_s \text{tr}(\not{D} \hat{\rho}_s) \right] \tag{37}$$

$$F_{\mu\nu} \equiv F^{lm}_{\mu\nu} \hat{L}^{lm} \equiv [D_\mu, D_\nu], \quad D_\mu = (\partial_\mu - \Gamma_\mu)\mathbb{1} - \sum_{lm} i\lambda A^{lm}_\mu \hat{L}^{lm} \tag{38}$$

$$G_{\mu\nu} \equiv G^a_{\mu\nu} \hat{T}^a \equiv [D'_\mu, D'_\nu], \quad D'_\mu = (\partial_\mu - \Gamma_\mu)\mathbb{1} - \sum_a i\lambda_G B^a_\mu \hat{T}^a \tag{39}$$

The symmetric tensor $\eta_{\mu\nu}$ is the metric of the parameter space Ξ and should not be confused with the metric $g_{\mu\nu}$ of the emergent classical spacetime Σ. According to Proposition 2, $\eta_{\mu\nu}$ is arbitrary because the geometry of Ξ is not a physical observable. The first and second terms in (37) are the Lagrangian density for the $SU(\infty)$ and internal symmetry G gauge fields A^{lm}_μ and B^a_μ, respectively. The $SU(\infty)$ generators $\hat{L}^{lm}$ and range of (l, m) are defined in Section 3, and Γ_μ is the geometric connection of the parameter space Ξ. Operators T^a are the generators of the internal symmetry G of subsystems. Their number is determined by the range of index a, which must be finite because, as we explain in Section 5, the rank of G is finite. The density matrix $\hat{\rho}_s$ is the quantum state of a subsystem. The covariant operator $\not{D}$ is a differential operator and its exact definition depends on the representation of symmetries of Ξ by the states of subsystems—see [20] for details and examples. In [21], it is shown that geometry connection terms in $\not{D}$ and field equations can be neutralized by an $SU(\infty)$ gauge transformation [21]. Therefore, as mentioned above, the Lagrangian in (37) does not depend on $\eta_{\mu\nu}$.

A crucial difference between the $SU(\infty)$ sector of $\mathcal{L}_{U_s}$ and $SU(\infty)$ Yang–Mills theory on a background spacetime, first studied in [61], is that, in the latter case the fields depend on two additional continuous parameters, constituting so-called "internal space" by [61]. These variables correspond to the coordinates of the compact diffeo-surface of $SU(\infty)$ representation. As we discussed earlier, in $SU(\infty)$-QGR, the parameter space Ξ is indeed the collection of these so-called internal spaces or, as we called them, the diffeo-surfaces of subsystems. For this reason, they can be identified with two of the four parameters of

Ξ or two functions of these parameters as the induced coordinates on the diffeo-surface. However, the exact expression of these functions is irrelevant, because their variation can be considered as a gauge transformation. In addition, Equation (38) shows in a simple way why the geometry of parameter space Ξ is irrelevant, that is the Proposition 2. The geometrical connection Γ_μ of Ξ can be decomposed to spherical harmonic functions. Moreover, as discussed in Section 3, the Poisson algebra of these functions is homomorphic to $\mathcal{SU}(\infty)$ algebra and can be identified with generators $\hat{L}^{lm}$. Therefore, in the covariant derivative D_μ, the geometric connection can be included in the $SU(\infty)$ term.

The coefficients of the traces in (37) can be called "fields" because they depend on continuous parameters $x \in \Xi$. But they do not need to be quantized because, by construction, the effective Lagrangian $\mathcal{L}_{U_s}$ presents the lowest-order interactions of a quantum system. They should rather be considered as probability distributions. It is evidently trivial to change this quantum mechanical interpretation to a QFT one; see the appendices in [21]. A QFT description would be more useful for formulating interactions as a scattering problem, which is useful for testing the model in a high-energy collider setting. It is important to remind readers that, like all Yang–Mills theories, as a QFT, $SU(\infty)$-QGR is renormalizable. This is a crucial criterion for any QGR candidate model and the main point of failure for many of them. Several other topics, such as parity (P), charge conjugate (C), CP symmetries, the possibility of their breaking by adding a topological term to the Lagrangian (37), and the possibility of the existence of an "$SU(\infty)$-axion", are discussed in [20].

11. Classical Limit of Gravity

The universal representation of $SU(\infty)$ by all subsystems and their interaction through this symmetry according to the Lagrangian $\mathcal{L}_{U_s}$ make $SU(\infty)$ Yang–Mills a good candidate for the formulation of quantum gravity, except that, according to observations (in particular, the recent detections of gravitational waves), the mediator of classical gravity is a spin-2 field. This is in clear contradiction with Yang–Mills gauge theories, in which the gauge field is a spin-1 field. In this section, we demonstrate that, if the quantum properties of $SU(\infty)$-Yang–Mills are not detectable by the observer—the case we call the classical limit of $SU(\infty)$-QGR—its effects would be perceived as classical gravity formulated according to the Einstein general relativity. A historical parallel of such misperception due to the lack of resolution is the sigma model for strong nuclear interaction, in which mesons are the mediator particles. Before the discovery of the Quantum Chromo-Dynamics (QCD), the sigma model was the favorite because experiments did not have sufficient energy or resolution to detect the internal structure of mesons and baryons.

As we discussed in Section 3, if we define the classical limit as $\hbar \to 0$, the $SU(\infty)$-QGR becomes trivial and meaningless. This is an expected outcome because the model is constructed as being intrinsically quantum, needing $\hbar \neq 0$. Moreover, it does not include a classical background spacetime to play the role of a "container" for other entities. For these reasons, we define the classical limit as the situation where the observer is not able to detect non-commutative $SU(\infty)$ symmetry and associated quantum phenomena. In this case, the observer can only perceive (measure) the expectation values $X \in \Sigma$ of the parameters x defined in (35). Consequently, the terms in the action in (37) are perceived as functions of X, and integration should be performed over the effective space Σ using its Lorentzian metric $g_{\mu\nu}$. Moreover, in the absence of information about the $SU(\infty)$ gauge field, the first term in $\mathcal{L}_{U_s}$ is perceived as a scalar function on the manifold Σ. According to Theorem 4.35 in [62], every scalar function defined on a (pseudo-)Riemannian manifold with dimension $D \geq 3$ is the scalar curvature for a (pseudo-)Riemannian metric. Therefore, when the structure of the $SU(\infty)$ gauge term in (37) is not discernible, it can be considered as being proportional to the scalar curvature of a metric, specifically that of $g_{\mu\nu}$. Therefore, in the classical limit, as defined above, the Lagrangian $\mathcal{L}_{U_s}$ is perceived as

$$\mathcal{L}_{U_s} \xrightarrow[\text{limit}]{\text{Classical}} \mathcal{L}_{cl.gr} = \int_\Sigma d^4X \sqrt{|g|} \left[\kappa R^{(4)} + \frac{1}{4} \text{tr}(G^{\mu\nu} G_{\mu\nu}) + \frac{1}{2} \sum_s \text{tr}(\not{D} \hat{\rho}_s) \right] \quad (40)$$

where the dimensionful constant $\kappa \propto M_P^2$ is necessary to make the pure gravity term dimensionless. This Lagrangian has the form of G-symmetry Yang–Mills QFT with matter in a classical curved spacetime. It is also clear that other possible scalars obtained from the curvature tensor of the same metric, such as R^n, $n > 1$, or $R^{\mu\nu} R_{\mu\nu}$ should be related to higher quantum orders—not included in the lowest-order actions in (37) and (40). They are smaller than the Einstein–Hilbert term by a factor of at least $\mathcal{O}(1)\hbar^2/M_P^2$; hence, currently undetectable. We also remind readers that the definition of $g_{\mu\nu}$ in (35) and the relationship of the affine parameter with the quantum state of subsystems in (34) do not allow one to determine $g_{\mu\nu}$. Therefore, there is no conflict or duplicity in the calculation of effective metric $g_{\mu\nu}$.

Because we interpret the $SU(\infty)$ sector of (37) as gravity, it is important to check whether it has the same number of degrees of freedom (d.o.f.) as the classical Einstein gravity. It is easy to see that they are indeed the same. In a space of dimension D, the antisymmetric 2-form $F_{\mu\nu} \equiv F_{\mu\nu}^{lm} \hat{L}^{lm}$ has $D(D-1)/2$ components. This field is reparameterization-invariant, meaning that it is independent of the connection Γ_μ. In classical gravity $g_{\mu\nu}$ (or, equivalently, the Ricci tensor $R_{\mu\nu}$) is a symmetric tensor and has D degrees of redundancy due to reparameterization. This leaves $D(D+1)/2 - D = D(D-1)/2$ independent d.o.f. Therefore, as tensor quantities on a parameter space or on a classical spacetime, they have the same number of degrees of freedom and there is no inconsistency in this respect between $SU(\infty)$-QGR and its classical limit as Einstein gravity. Thus, a fundamental spin-1 mediator for quantum gravity is not in contradiction with the observed classical spin-2 graviton. Indeed, the equality of the d.o.f. is one of the reasons for the gauge-gravity duality conjecture used in the construction of some QGR models [63–65]. Also notice that modified gravity models such as $F(T)$, $F(Q)$, $F(T,Q)$, which are recently proposed as a modified teleparallel formulation of the classical gravity—see, e.g., [66] for a review—depend on additional d.o.f. for torsion and nonmetricity. Consequently, they break the required equality of the number of d.o.f. in the quantum and classical limit of $SU(\infty)$-QGR. Moreover, the Besse theorem used for obtaining the classical limit action (40) is proven for the (pseudo-)Riemannian manifold without torsion. Therefore, in the framework of $SU(\infty)$-QGR, additional terms such as $F(R)$, $F(T)$, $F(Q)$, $F(T,Q)$ in the lowest order classical limit effective action of gravity cannot be fundamental. Nonetheless, as we explain below, they may arise as special configurations of $SU(\infty)$ fields.

The classical gravity Lagrangian $\mathcal{L}_{cl.gr}$ does not include a cosmological constant, and there is no trivial candidate in $SU(\infty)$-QGR that add such a term to $\mathcal{L}_{U_s}$ or $\mathcal{L}_{cl.gr}$. However, considering the inconsistencies between the measured cosmological parameters from the Cosmic Microwave Background (CMB) [67] and those estimated from late Universe probes such as supernovae, micro-lensing, and Baryon Acoustic Oscillation (BAO)—known as "Hubble and S_8 tensions" in the literature [68]—dark energy may be dynamical rather than a cosmological constant. Indeed, results of the BAO measurement by the DESI survey [69] seem to be more consistent with an evolving dark energy density rather than a cosmological constant. Nonetheless, in [20], a few processes are suggested that may generate an effective cosmological constant in $SU(\infty)$-QGR. In particular, in [20], it is suggested that topologically nontrivial $SU(\infty)$ field configurations such as instantons may behave as a dynamical dark energy or an effective modified gravity. In fact, it is shown that the main difference between modified classical gravity $F(R)$, $F(T)$, $F(Q)$, $F(T,Q)$ models, which are also dark energy candidates, is the boundary configuration of the metric $g_{\mu\nu}$ [70]. Non-trivial configurations of the $SU(\infty)$ gauge field A_{lm}^{μ} can be the fundamental quantum origin for what may classically be interpreted as torsion or nonmetricity and, thereby, modified classical gravity. In particular, it may be possible to embed the $\mathbb{R}^4$ Abelian gauge field of teleparallel formulation in the $SU(\infty)$ gauge field. A thorough study of these possibilities is left to future works. Finally, many other dark energy models studied in the literature are also relevant in the $SU(\infty)$-QGR framework.

12. Concluding Remarks

As a candidate model for quantum gravity and cosmology, $SU(\infty)$-QGR differs both in its construction and predictions from other QGR proposals. It is constructed axiomatically as a quantum system and its postulates include neither an interaction similar to gravity nor a spacetime or fields that play similar roles, as is the case in string theory. All these concepts are emergent. The most distinctive prediction of $SU(\infty)$-QGR is a spin-1 mediator boson at a quantum level for the interaction that is classically perceived as gravity with a spin-2 mediator. On the other hand, the origin and properties of this spin-1 mediator diverge from those in gauge-gravity duality models [63,63–65,71–74], or the gauging of translation symmetry in the frame-based classical gravity models, such as teleparallelism and its extensions. The $SU(\infty)$ gauge symmetry is not directly related to Lorentz invariance and the diffeomorphism of the perceived classical spacetime.

Considering the differences between $SU(\infty)$-QGR and what we know about gravity and our classical perception of spacetime, the question arises as to whether it is a viable model for quantum cosmology and gravity. Our answer is that it is a plausible model because it is self-consistent and, despite its abstract axioms, it approaches Einstein gravity and some of its extensions, in the sense discussed in Section 11. Given the fact that, at present, there is no observed evidence of QGR and its properties, any intrinsically consistent model that predicts Einstein gravity in the classical limit is a potentially viable candidate. As we explained in the Introduction, some of the popular QGR candidates, which seem closer and more inspired by what we know about gravity, do not go as far as the newcomer and abstract $SU(\infty)$-QGR in the demonstration of their consistency with the Einstein gravity and testable predictions.

Future works should investigate predictions of $SU(\infty)$-QGR for processes and phenomena in which quantum gravity is considered to be important, such as the quantum structure of black holes and the puzzle of apparent information loss in these objects, inflation, particle physics beyond the standard model, and laboratory tests of quantum gravity. Another important topic for future works is an in-depth study of $SU(\infty)$-QGR-specific dark energy models and their relation with phenomenological models, such as modified gravity proposals.

Funding: This research received no external funding.

Data Availability Statement: Data are contained within the article.

Conflicts of Interest: The author declares no conflict of interest.

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