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On Carrollian and Galilean contractions of BMS algebra in 3 and 4 dimensions

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Abstract

In this paper, we find a class of Carrollian and Galilean contractions of (extended) Bondi–van der Burg–Metzner–Sachs (BMS) algebra in 3+1 and 2+1 dimensions. To this end, we investigate possible embeddings of 3D/4D Poincaré into the BMS_3 and BMS_4 algebras, respectively. The contraction limits in the 2+1-dimensional case are then enforced by appropriate contractions of its Poincaré subalgebras. In 3+1 dimensions, we have to apply instead the analogy between the structures of Poincaré and BMS algebra. In the case of non-vanishing cosmological constant in 2+1 dimensions, we consider the contractions of Λ-BMS_3 algebra in an analogous manner. As a by-product, we have also analyzed reality conditions on the Witt algebra and obtained new results.

Keywords: asymptotic symmetries, extended BMS, Carroll, Galilei, cosmological constant

1. Introduction

The symmetries of space and time are concepts at the heart of modern physics. Poincaré symmetry is the cornerstone of classical relativistic physics and quantum field theory. Diffeomorphism invariance, expressed by the equivalence principle is the fundamental concept

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underlying general relativity. As shown by Arnowitt *et al* [1], and Regge and Teitelboim [2] in the case of asymptotically flat spacetime, at spatial infinity, the diffeomorphisms quite expectantly reduce to Poincaré symmetry. It came as a great surprise when shortly afterwards it was discovered that the relevant symmetry algebra at null infinity is not Poincaré but the infinite-dimensional Bondi–van der Burg–Metzner–Sachs (BMS) algebra [3, 4]. After decades when asymptotic symmetries have been largely forgotten, a few years ago they reemerged as one of the most actively studied topics in high energy physics and gravity. The breakthrough here was the research carried out by Strominger and his collaborators that resulted in revealing a close relation between BMS symmetry, Weinberg's soft theorems, and memory effect (see [5] for a review).

The original BMS algebra is a semidirect product of the standard Lorentz algebra and the infinite-dimensional commuting algebra of supertranslations. This algebra contains the Poincaré subalgebra, spanned by the Lorentz algebra and four supertranslation generators. It should be noted however that the form of asymptotic symmetry algebra depends crucially on the adopted asymptotic conditions for the spacetime metric. Indeed, it was shown in [6, 7] that by relaxing these boundary conditions, one can extend BMS algebra so as to make it composed of infinite supertranslational and superrotation sectors. From a different point of view, the original BMS algebra becomes extended by superrotations when we go down the tower of subleading asymptotic structures [5, 8]. Meanwhile, it turned out that by relaxing boundary conditions at spatial infinity, the asymptotic symmetry algebra there can also be extended from Poincaré to BMS [9, 10]. Similarly, the asymptotic symmetries can be made to emerge at non-extremal horizons [11].

The 2+1 dimensional counterpart of BMS algebra in 3+1 dimensions has also attracted a lot of attention. It was originally analyzed in [12] and these investigations are recognized as a predecessor of the AdS/CFT research program. Comparing the derivations for the 2+1- and 3+1-dimensional cases is very instructive, see [13]. Recently, the BMS_3 was investigated in detail [14, 15]. From our perspective, the case of BMS_3 is particularly interesting because this algebra can be extended to the case of non-vanishing cosmological constant [16], which is impossible in the case of BMS_4 [17], where the corresponding structure becomes a Lie algebroid [18].

In all the cases discussed above, Poincaré algebra played a crucial role, being the largest finite-dimensional subalgebra of BMS algebra. It is well known however that Poincaré algebra is not a unique algebra to describe the kinematics of particles and fields. In fact, the classification of such 'kinematical algebras' (see [19, 20] for reviews) was provided in the seminal work by Bacry and Lévy-Leblond [21], later extended in [22] and generalized to 2+1 dimensions only recently in [23]. The idea was to find a set of 10-parameter algebras that contains space and time translations, rotations and boosts, satisfying some physically appealing conditions: the isotropy of space (the standard action of rotations on energy, momenta, and boosts), the invariance under discrete symmetries of time reversal and parity (this condition can be lifted, as done in [22]), and the noncompactness of boosts—the transformations between inertial observers. Among the kinematical algebras, apart from the Lorentzian algebras (i.e. Poincaré or (Anti)-de Sitter), two families are particularly physically relevant: the Galilean and Carrollian one [24, 25]. Both can be obtained as contraction limits of Lorentzian algebras, with speed of light being the contraction parameter; the limits are opposite, however: $c \to \infty$ in the Galilean case and $c \to 0$ in the Carrollian one.

The investigation of these regimes in the theory of gravity, where they are closely related to expansions around the weak and strong gravitational coupling, respectively, is currently a subject of great interest, see e.g. [26–28], or a review [29] and references therein (in particular, it was finally found [27, 30, 31] how to correctly recover Newtonian gravity). Similar studies have been developed in 2+1-dimensional gravity [32–35]. On the other hand, a geometry with the Galilean symmetries can also be obtained via the null reduction of a Lorentzian geometry [36, 37], while null hypersurfaces in a Lorentzian geometry are geometries with the Carrollian symmetries [24, 38]. Moreover, such structures have applications going beyond general relativity, including the non-AdS holography [39, 40].

Since Poincaré algebra is contained in BMS algebra, it should also be possible to extend its Carrollian and Galilean contractions to the latter. This has recently been achieved [41–43] for the original BMS, while the aim of the present paper is to define such contractions for the extended BMS algebra, both in 3+1 and 2+1 spacetime dimensions.

The Galilean contraction of BMS algebra corresponds physically to the non-relativistic limit of the theory. This contraction does not seem to be of any physical relevance in the case of the symmetries of asymptotically flat gravitational systems in the neighborhood of null infinity in four spacetime dimensions. It is however of physical interest in the case of 3+1-dimensional asymptotically flat solutions of non-relativistic gravity [27] at spatial infinity[3], as well as, consequently, it may play a role in the non-relativistic holography (in the sense of holography with the non-relativistic gravity in the bulk) and various models within condensed matter physics. The Carrollian contraction, on the other hand, is of physical interest not only in the neighbourhood of null infinity but also in the context of Carrollian gravity [28] and, more generally, gravity on manifolds with null boundaries (see e.g. [42–46] and references therein). This should not be confused with an interesting fact that the original BMS algebra itself turns out to be a conformal extension of the lower-dimensional Carroll algebra [47, 48], which forms the foundation for the so-called Carrollian holography as a potential realization of the flat-space holography [49, 50].

The plan of this paper is as follows. In the following section 2, we shortly recall the Carrollian and Galilean contractions of (finite-dimensional) kinematical algebras for any value of the cosmological constant, taking 2+1 spacetime dimensions as an example. Section 3 is devoted to a discussion of BMS_3 algebra and the embeddings of the (2+1)d Poincaré algebra into it. Then, in section 4, we discuss contractions of 2+1-dimensional BMS algebras without and (section 4.1) with the cosmological constant, while in section 5, we consider the 3+1 dimensional (extended) BMS algebra and its contraction limits. Finally, section 6 summarizes the obtained results.

2. Carrollian and Galilean kinematical Lie algebras in (2+1)d

The Lie brackets of 2+1-dimensional Poincaré and (Anti-)de Sitter algebras, $\mathfrak{iso}(2,1) = \mathfrak{so}(2,1) \ltimes \mathcal{T}^{2,1}$, $\mathfrak{so}(3,1)$ and $\mathfrak{so}(2,2)$ (for the cosmological constant $\Lambda = 0$, $\Lambda > 0$ or $\Lambda < 0$, respectively), can be expressed in a unified fashion:

$$[\mathcal{J}_\mu, \mathcal{J}_\nu] = \epsilon_{\mu\nu}{}^{\sigma} \mathcal{J}_\sigma\,, \qquad [\mathcal{J}_\mu, \mathcal{P}_\nu] = \epsilon_{\mu\nu}{}^{\sigma} \mathcal{P}_\sigma\,, \qquad [\mathcal{P}_\mu, \mathcal{P}_\nu] = -\Lambda\, \epsilon_{\mu\nu}{}^{\sigma} \mathcal{J}_\sigma\,, \tag{1}$$

where $\mu = 0,1,2$, $\epsilon_{012} = 1$ and indices are raised with Minkowski metric $(1,-1,-1)$). Let us perform a change of basis

$$J := -\mathcal{J}_0\,, \qquad K_a := -\mathcal{J}_a\,, \qquad P_0 := \mathcal{P}_0\,, \qquad P_1 := -\mathcal{P}_2\,, \qquad P_2 := \mathcal{P}_1\,, \tag{2}$$

[3] Notice that the BMS algebra arises not only at null infinity but, if the appropriate asymptotic conditions are adapted, also at the spatial one [9, 10].

so that the brackets (1) become

$$\begin{aligned} &[J,K_a]=\epsilon_a{}^bK_b\,, && [K_1,K_2]=-J\,, && [J,P_a]=\epsilon_a{}^bP_b\,, && [J,P_0]=0\,,\\ &[K_a,P_b]=\delta_{ab}P_0\,, && [K_a,P_0]=P_a\,, && [P_1,P_2]=\Lambda J\,, && [P_0,P_a]=-\Lambda K_a\,, \end{aligned} \tag{3}$$

where $a=1,2$ and ${\epsilon_1}^2=1$. In order to perform the Carrollian contraction of either of the considered Lorentzian algebras, one needs to denote $R:=J$, $\mathcal{T}_a:=P_a$, define the rescaled generators $Q_a:=cK_a$, $\mathcal{T}_0:=cP_0$ and take the limit $c\to 0$. As a result, we obtain the brackets of 2+1-dimensional Carroll (for $\Lambda=0$) or (Anti-)de Sitter–Carroll algebra (also called the para-Poincaré if $\Lambda<0$ and para-Euclidean if $\Lambda>0$, due to their respective isomorphisms with 2+1-dimensional Poincaré and Euclidean algebras, see e.g. [51]):

$$\begin{aligned} &[R,Q_a]=\epsilon_a{}^bQ_b\,, && [Q_1,Q_2]=0\,, && [R,\mathcal{T}_a]=\epsilon_a{}^b\mathcal{T}_b\,, && [R,\mathcal{T}_0]=0\,,\\ &[Q_a,\mathcal{T}_b]=\delta_{ab}\mathcal{T}_0\,, && [Q_a,\mathcal{T}_0]=0\,, && [\mathcal{T}_1,\mathcal{T}_2]=\Lambda R\,, && [\mathcal{T}_0,\mathcal{T}_a]=-\Lambda Q_a\,. \end{aligned} \tag{4}$$

On the other hand, the Galilean contraction of a Lorentzian algebra consists in denoting $R:=J$, $\mathcal{T}_0:=P_0$, introducing the rescaled generators $Q_a:=c^{-1}K_a$, $\mathcal{T}_a:=c^{-1}P_a$ and taking the limit $c\to\infty$. It allows to obtain the brackets of 2+1-dimensional Galilei (for $\Lambda=0$) or (Anti-)de Sitter–Galilei algebra (also called the oscillating Newton–Hooke if $\Lambda<0$ and the expanding Newton–Hooke if $\Lambda>0$):

$$\begin{aligned} &[R,Q_a]=\epsilon_a{}^bQ_b\,, && [Q_1,Q_2]=0\,, && [R,\mathcal{T}_a]=\epsilon_a{}^b\mathcal{T}_b\,, && [R,\mathcal{T}_0]=0\,,\\ &[Q_a,\mathcal{T}_b]=0\,, && [Q_a,\mathcal{T}_0]=\mathcal{T}_a\,, && [\mathcal{T}_1,\mathcal{T}_2]=0\,, && [\mathcal{T}_0,\mathcal{T}_a]=-\Lambda Q_a\,. \end{aligned} \tag{5}$$

Finally, let us recall that Carroll and Galilei algebras are also directly related with Poincaré algebra one dimension higher [25]. In particular, in the 2+1-dimensional case, Carroll algebra can be embedded as a subalgebra of (3+1)d Poincaré (see [51] for the explicit formulae), while Galilei algebra can be obtained as a quotient subalgebra of (3+1)d Poincaré.

3. BMS$_3$ algebra $\mathfrak{B}_3$, its real forms and embeddings of $\mathfrak{iso}(2,1)$

The brackets of BMS algebra in 2+1 dimensions (which we will denote by $\mathfrak{B}_3$) in terms of the generators of superrotations l_n and supertranslations T_n have the form

$$[l_n,l_m]=(n-m)\,l_{n+m}\,,\qquad [l_n,T_m]=(n-m)\,T_{n+m}\,, \tag{6}$$

where $n,m\in\mathbb{Z}$, while the supertranslation subalgebra is commutative.

From the mathematical perspective, $\mathfrak{B}_3$ is also known as the inhomogeneous centreless two-sided Witt algebra, $\mathfrak{W}(2,2)$. Moreover, it is actually a complex algebra and in order to impose the reality conditions on its generators, one needs to introduce a real structure, i.e. a $*$-conjugation (an involutive, antilinear antiautomorphism). We find (see appendix) that the $\mathfrak{B}_3$ algebra has four such possible real forms, inherited from real forms of the $\mathfrak{sl}(2,\mathbb{C})$ algebra:

(i) Type $\mathfrak{sl}(2,\mathbb{R})$, with the reality conditions $l_n^*=-l_n\,,\quad T_n^*=-T_n\,,\quad n\in\mathbb{Z}$,
(ii) Type $\mathfrak{su}(1,1)$, with the reality conditions $l_n^*=l_{-n}\,,\quad T_n^*=T_{-n}\,,\quad n\in\mathbb{Z}$,
(iii) Mixed type, with the reality conditions $l_n^*=(-1)^{n+1}l_n\,,\quad T_n^*=(-1)^{n+1}T_n\,,\quad n\in\mathbb{Z}$,
(iv) Compact-mixed type, with the reality conditions $l_n^*=(-1)^nl_{-n}\,,\quad T_n^*=(-1)^nT_{-n}\,,\quad n\in\mathbb{Z}$,

In contrast to the real forms of $\mathfrak{sl}(2,\mathbb{C})$, it can be shown (as we do in appendix) that the cases (i)–(iii) are non-isomorphic to each other and, as we will see, this makes a difference when one considers contractions of the algebra). The reality conditions of the $\mathfrak{su}(2)$-type ($\mathfrak{su}(2)$ is the remaining, compact real form of $\mathfrak{sl}(2,\mathbb{C})$): $l_0^* = l_0\,,\ l_n^* = -l_{-n}\,,\ n \neq 0$, are incompatible with the first bracket (6) for $n \pm m \neq 0$ and hence they are also not valid for the full $\mathfrak{B}_3$ algebra. However, case (iv) combines the real structure of type $\mathfrak{su}(1,1)$ for even n and type $\mathfrak{su}(2)$ for odd n.

Let us now recall that the non-vanishing brackets of Poincaré algebra in arbitrary dimension $d+1$ (with the flat metric η of any signature and generators satisfying the anti-Hermitian reality conditions $X^* = -X$) can be written as

$$\begin{aligned}[][M_{\mu\nu}, M_{\rho\sigma}] &= \eta_{\mu\sigma} M_{\nu\rho} + \eta_{\nu\rho} M_{\mu\sigma} - \eta_{\mu\rho} M_{\nu\sigma} - \eta_{\nu\sigma} M_{\mu\rho}\,,\\ [M_{\mu\nu}, P_\rho] &= \eta_{\nu\rho} P_\mu - \eta_{\mu\rho} P_\nu\,,\end{aligned} \tag{7}$$

where indices run from 0 to d. In particular, we may change the basis in 2+1 dimensions to $J = M_{12}$, $K_i = M_{i0}$, P_0, P_i, with the diagonal metric $(\eta_{00}, \eta_{11}, \eta_{22})$, so that (7) becomes

$$\begin{aligned}[][J, K_i] &= -\eta_{ii}\epsilon_{ij} K_j\,, & [K_i, K_j] &= -\eta_{00}\epsilon_{ij} J\,, & &\\ [J, P_i] &= -\eta_{ii}\epsilon_{ij} P_j\,, & [K_i, P_j] &= -\eta_{ij} P_0\,, & [K_i, P_0] &= \eta_{00} P_i\,,\end{aligned} \tag{8}$$

which is a generalization of (3) for $\Lambda = 0$.

There exists a one-parameter family $\mathcal{P}_n(1,2) = \mathrm{span}\{l_0, l_{\pm n}, T_0, T_{\pm n}\}$, $n \in \mathbb{N}$ (in this paper we adopt the convention $0 \neq \mathbb{N}$) of maximal finite-dimensional subalgebras of $\mathfrak{B}_3$, each of them isomorphic to the Poincaré algebra $\mathfrak{iso}(2,1)$. In particular, if we equip $\mathfrak{B}_3$ with real structure of the $\mathfrak{su}(1,1)$-type, such a Poincaré subalgebra can be described as the image of an embedding $\mathrm{P}^{(n)}(1,2)$ of $\mathfrak{iso}(2,1)$ into $\mathfrak{B}_3$, spanned by the generators:

$$\begin{aligned} J &= i\,l_0\,, & K_1^{(n)} &= \frac{1}{2}(l_n - l_{-n})\,, & K_2^{(n)} &= -\frac{i}{2}(l_n + l_{-n})\,,\\ P_0 &= i\,T_0\,, & P_1^{(n)} &= \frac{i}{2}(T_n + T_{-n})\,, & P_2^{(n)} &= \frac{1}{2}(T_n - T_{-n})\,,\end{aligned} \tag{9}$$

which satisfy the brackets (8) with the metric $\eta = n\,\mathrm{diag}(1,-1,-1)$ (if needed, the factor n can eliminated with the help of rescaling of J, $K_{i(n)}$ by $1/n$ and P_0, $P_{i(n)}$ by n) and the anti-Hermitian reality conditions, i.e. $X^* = -X$. The family of all embeddings $\mathrm{P}^{(n)}(1,2)$ determines a special basis of the $\mathfrak{B}_3$ algebra $\bigcup_{n\in\mathbb{N}}\{K_1^{(n)}, K_2^{(n)}, P_1^{(n)}, P_2^{(n)}\} \cup \{J, P_0\}$, which will play the central role in defining the Carrollian/Galilean contractions of the latter.

If we consider real form of the $\mathfrak{sl}(2,\mathbb{R})$-type instead, Poincaré subalgebras can be identified with another family of embeddings $\mathrm{P}'^{(n)}(1,2)$, spanned by the anti-Hermitian generators:

$$\begin{aligned} J^{(n)} &= \frac{1}{2}(l_n + l_{-n})\,, & K_1^{(n)} &= \frac{1}{2}(l_n - l_{-n})\,, & K_2 &= -l_0\,,\\ P_0^{(n)} &= \frac{1}{2}(T_n + T_{-n})\,, & P_2^{(n)} &= \frac{1}{2}(T_n - T_{-n})\,, & P_1 &= T_0\,,\end{aligned} \tag{10}$$

which satisfy the brackets (8) with the metric $\eta = n\,\mathrm{diag}(1,-1,-1)$. The family corresponds to a differently arranged basis of the $\mathfrak{B}_3$ algebra $\bigcup_{n\in\mathbb{N}}\{J^{(n)}, K_1^{(n)}, P_0^{(n)}, P_2^{(n)}\} \cup \{K_2, P_1\}$. Consequently, as we will see, the contractions performed using this basis work in a different way than for the $\mathfrak{su}(1,1)$-type real form.

Both families of embeddings allow us to decompose (a real form of) $\mathfrak{B}_3$ into the union of its maximal finite-dimensional subalgebras isomorphic to $\mathfrak{iso}(2,1)$, which have the non-empty intersection spanned by l_0 and T_0. Real forms of the mixed and compact-mixed type (i.e. cases iii) and iv)) are less appealing in the sense that they are inconsistent with such a decomposition into Poincaré subalgebras covering the whole $\mathfrak{B}_3$. A family of embeddings associated to the former real form is given by $\mathrm{P}'^{(n)}(1,2)$ but with $n \in 2\mathbb{N}$, while embeddings associated to the latter are $\mathrm{P}^{(n)}(1,2)$ with $n \in 2\mathbb{N}$.

Let us note in passing that one can construct vacuum spacetimes associated with different choices of an embedding of $\mathfrak{iso}(2,1)$ into $\mathfrak{B}_3$. To this end, we consider the general asymptotic solution of the vacuum Einstein equations in the Bondi gauge [7]

$$\mathrm{d}s^2 = \Theta(\phi)\,\mathrm{d}u^2 - 2\mathrm{d}u\mathrm{d}r + \left(u\,\Theta'(\phi) + \Xi(\phi)\right)\mathrm{d}u\mathrm{d}\phi + r^2\mathrm{d}\phi^2\,, \tag{11}$$

where Θ, Ξ are arbitrary periodic functions. It is then convenient to introduce a new variable $z = e^{i\phi}$, bringing the metric to the form

$$\mathrm{d}s^2 = \Theta(z)\,\mathrm{d}u^2 - 2\mathrm{d}u\mathrm{d}r + \left(u\,\Theta'(z) - i\,\Xi(z)\,z^{-1}\right)\mathrm{d}u\mathrm{d}z - \frac{r^2}{z^2}\mathrm{d}z^2\,. \tag{12}$$

It can be shown that in the case of the metric (12) the vector fields corresponding to the Lie algebra generators l_n, T_n have asymptotically (for large r) the following form

$$\begin{aligned} T_n &= z^n\partial_u - \left(n^2z^n - n\frac{z^{n+1}}{r}g_{zu}\right)\partial_r + n\frac{z^{n+1}}{r}\partial_z\,, \\ l_n &= -nz^nu\partial_u + \left(nz^nr + n^3uz^n + n^2\frac{uz^{n+1}}{r}g_{uz}\right)\partial_r - \left(1 + n^2\frac{u}{r}\right)z^{n+1}\partial_z \end{aligned} \tag{13}$$

and satisfy (in the limit of large r) the algebra (6). One sees immediately that the vector $T_0 = \partial_u$ is a Killing vector of the metric (12) if $\Theta'(z) = 0$. Next, the vector $l_0 = z\partial_z$ is a Killing vector of this metric if $\Xi(z) = 0$. One can further show [70] that the condition under which the vectors $l_{\pm n}$ and $T_{\pm n}$ are Killing vectors reads

$$\Xi = 0\,, \quad \Theta = -n^2\,. \tag{14}$$

We see therefore that for each $n \geqslant 1$ there exists a class of metrics possessing the maximal number of six Killing vectors. These metrics in polar coordinates (t, r, ϕ) have the form

$$\mathrm{d}s^2 = -\mathrm{d}t^2 + \mathrm{d}r^2 + r^2n^2\mathrm{d}\phi^2\,. \tag{15}$$

For $|n| = 1$, we have to do here with the standard Minkowski metric. For $|n| > 1$, the metric (15) is still locally Minkowski but has a conical singularity at the origin [70] (see also [15]). This metric, parametrized by an integer n, $|n| > 1$ can be interpreted as describing (2+1)-dimensional spacetime containing a massive particle (see [52]) with the quantized, negative mass $M_n = -(|n| - 1)/(4G)$, where G is the three-dimensional Newton's constant. The deficit angle of such a conical spacetime geometry, $2\pi - 2M_n$ is larger than 2π. As a result, curvature at the singularity is negative, which corresponds to the negative energy density at the origin of coordinates, bending away the worldlines of test particles. Therefore, although one may be tempted to call the spacetimes (15) the asymptotic vacua, their physical meaning is not clear and deserves further studies.

Finally, let us note that since the Poincaré embeddings (9) or (10) differ only by arrangement of the l_n and T_n generators for given n, they both lead to the same family of 'vacuum' spacetimes.

4. Contractions of the BMS_3 and Λ-BMS_3 algebras

Based on the family of embeddings (9) or (10), associated with the respective real form, we want to extend the Carrollian and Galilean contractions of Poincaré algebra $\mathfrak{iso}(2,1)$ discussed in section 2 to the BMS_3 algebra $\mathfrak{B}_3$ (6). Our guiding principle is that each subalgebra of the contracted $\mathfrak{B}_3$ that was isomorphic to $\mathfrak{iso}(2,1)$ before a contraction (within the chosen family of embeddings) should be isomorphic to Carroll (4) or Galilei (5) algebra after the respective contraction. An equivalent point of view is to consider this a consistent extension of Carroll/Galilei algebra by superrotations and supertranslations, which aligns with the approach of [43] to the Carrollian and Galilean contractions of the original (non-extended) BMS_4 algebra. In practice, it means that, in the contraction procedure, we need to appropriately rescale all generators of $\mathfrak{B}_3$ that play the role of generators of boosts and time translation (in the Carrollian case) or generators of boosts and spatial translations (in the Galilean case) in any subalgebra spanned by (9)/(10).

Let us first consider the family (9), which corresponds to the choice of the $\mathfrak{su}(1,1)$-type real form. It can be shown that the rescaling of generators of Poincaré subalgebras can equivalently be applied to the standard basis of $\mathfrak{B}_3$, i.e. $\{l_n, T_n;\, n\in\mathbb{Z}\}$ with the brackets (6), and it follows that:

- In the case of Carrollian contraction, we should perform the rescaling $l_n \mapsto c\, l_n = \tilde{l}_n$, $T_0 \mapsto c\, T_0 = \widetilde{T}_0$ for all $n \neq 0$ and take the limit $c \to 0$ to obtain

$$\begin{aligned} &\left[\tilde{l}_n, \tilde{l}_m\right] = 0\,, && \left[l_0, \tilde{l}_n\right] = -n\,\tilde{l}_n\,, && [l_0, T_n] = -n\,T_n\,, \\ &\left[l_0, \widetilde{T}_0\right] = 0\,, && \left[\tilde{l}_n, \widetilde{T}_0\right] = 0\,, && \left[\tilde{l}_n, T_m\right] = 2\delta_{m,-n}\, n\,\widetilde{T}_0\,, \end{aligned} \tag{16}$$

  which we will call the Carroll-BMS_3 algebra, $CBMS_3$;
- in the case of Galilean contraction, we perform the rescaling $l_n \mapsto c^{-1} l_n = \hat{l}_n$, $T_n \mapsto c^{-1} T_n = \widehat{T}_n$ for all $n \neq 0$ and take the limit $c \to \infty$ to obtain

$$\begin{aligned} &\left[\hat{l}_n, \hat{l}_m\right] = 0\,, && \left[l_0, \hat{l}_n\right] = -n\,\hat{l}_n\,, && \left[l_0, \widehat{T}_n\right] = -n\,\widehat{T}_n\,, \\ &[l_0, T_0] = 0\,, && \left[\hat{l}_n, T_0\right] = n\,\widehat{T}_n\,, && \left[\hat{l}_n, \widehat{T}_m\right] = 0\,, \end{aligned} \tag{17}$$

  which we will call the Galilei-BMS_3 algebra, $GBMS_3$.

Comparing the brackets (16), (17) with those of (2+1)d Carroll and Galilei algebras (given by (4), (5) with $\Lambda = 0$), we observe that indeed, the structures of the considered infinite- and finite-dimensional algebras are completely analogous. In particular, the generators l_0, l_n, T_0 and T_n appear in the same places as the Carroll/Galilei generators of rotation, boosts, time translation and spatial translations, respectively. The underlying reason is that if we change the basis of $CBMS_3$/$GBMS_3$ to the one analogous to (9) or, equivalently, perform either of the contractions in the basis (9), it turns out that the embeddings of Carroll and Galilei algebras into $CBMS_3$ and $GBMS_3$, respectively, can be defined analogously to the embeddings of Poincaré into the $\mathfrak{B}_3$ algebra. Moreover, in contrast to what happens for $\mathfrak{B}_3$ (the relevant formulae can be obtained by changing the basis in (18) below), all commutation relations between generators of different Carroll/Galilei subalgebras vanish, except the ones involving the two generators that are shared by all of them, l_0 and T_0 (or $\widetilde{T}_0$). If the latter was not the case, $CBMS_3$ and

GBMS_3 would have structure of the direct product of infinite number of Carroll or Galilei subalgebras, respectively.

If we choose the $\mathfrak{sl}(2,\mathbb{R})$-type real form, the contractions based on the family of embeddings (10) cannot be performed in the basis $\{l_n, T_n; n \in \mathbb{Z}\}$ (since e.g. both rotation and boost generators now depend on all l_n, $n \in \mathbb{Z}\setminus\{0\}$). Instead, we stay in the basis $\{K_2, P_1, K_1^{(n)}, J^{(n)}, P_2^{(n)}, P_0^{(n)}; n \in \mathbb{N}\}$ and calculate the contraction limits of commutation relations between different Poincaré subalgebras

$$
\begin{aligned}
\left[J^{(n)}, J^{(m)}\right] &= \frac{1}{2}\left((n-m)\,K_1^{(n+m)} + (n+m)\,\sigma_2 K_1^{(|n-m|)}\right),\\
\left[J^{(n)}, K_1^{(m)}\right] &= \frac{1}{2}\left((n-m)\,J^{(n+m)} - (n+m)\,J^{(|n-m|)}\right),\\
\left[K_1^{(n)}, K_1^{(m)}\right] &= \frac{1}{2}\left((n-m)\,K_1^{(n+m)} - (n+m)\,\sigma_2 K_1^{(|n-m|)}\right),\\
\left[J^{(n)}, P_{0/2}^{(m)}\right] &= \frac{1}{2}\left((n-m)\,P_{2/0}^{(n+m)} \pm (n+m)\,\sigma_{2/0} P_{2/0}^{(|n-m|)}\right),\\
\left[K_1^{(n)}, P_{0/2}^{(m)}\right] &= \frac{1}{2}\left((n-m)\,P_{0/2}^{(n+m)} \pm (n+m)\,\sigma_{0/2} P_{0/2}^{(|n-m|)}\right),
\end{aligned}
\tag{18}
$$

where $n \neq m$ and, for brevity, $\sigma_0 := 1$, $\sigma_2 := \mathrm{sgn}(n-m)$ (the remaining brackets for each n are the ones of Poincaré algebra (8) with $\eta = n\,\mathrm{diag}(1,-1,-1)$). It turns out that:

- In the case of Carrollian contraction, the rescaling $K_2 \mapsto c\,K_2$, $K_1^{(n)} \mapsto c\,K_1^{(n)}$, $P_0^{(n)} \mapsto c\,P_0^{(n)}$ (for all $n \in \mathbb{N}$) and taking the limit $c \to 0$ lead to the divergence of the first and fourth bracket in (18);
- In the case of Galilean contraction, the rescaling $P_1 \mapsto c^{-1}P_1$, $P_2^{(n)} \mapsto c^{-1}P_2^{(n)}$, $K_2 \mapsto c^{-1}K_2$, $K_1^{(n)} \mapsto c^{-1}K_1^{(n)}$ (for all $n \in \mathbb{N}$) and taking the limit $c \to \infty$ lead to the divergence of the first and fourth bracket in (18).

Therefore, both contractions are ill-defined. On the other hand, it is still possible to arrive at the algebra (16) or (17) starting from (10) if we perform a different kind of contractions, which are equivalent to the ones performed for the $\mathfrak{su}(1,1)$-type real form when expressed in the basis $\{l_n, T_n; n \in \mathbb{Z}\}$ but their interpretation for the $\mathfrak{sl}(2,\mathbb{R})$-type real form is different. To this end, let us first change the basis of $\mathfrak{B}_3$ from (10) (keeping P_1) to

$$
M_{\pm 1}^{(n)} = \mp\frac{1}{\sqrt{2}}\left(J^{(n)} \pm K_1^{(n)}\right), \qquad M_{+-} = K_2, \qquad P_\pm^{(n)} = \frac{1}{\sqrt{2}}\left(P_0^{(n)} \pm P_2^{(n)}\right),
\tag{19}
$$

where $n \in \mathbb{N}$. In a given embedding $\mathrm{P}'^{(n)}(1,2)$ of Poincaré algebra $\mathfrak{iso}(2,1)$, $M_{\pm 1}^{(n)}$ are actually two generators of null rotations (parabolic Lorentz transformations) and $P_\pm^{(n)}$ are two generators of translations along the null directions. The non-vanishing brackets of (the image of) $\mathrm{P}'^{(n)}(1,2)$ become

$$
\begin{aligned}
&\left[M_{+-}, M_{\pm 1}^{(n)}\right] = \pm\eta_{+-} M_{\pm 1}^{(n)}, \qquad \left[M_{+1}^{(n)}, M_{-1}^{(n)}\right] = -\eta_{11} M_{+-},\\
&\left[M_{+-}, P_\pm^{(n)}\right] = \pm\eta_{+-} P_\pm^{(n)}, \qquad \left[M_{\pm 1}^{(n)}, P_\mp^{(n)}\right] = -\eta_{+-} P_1, \qquad \left[M_{\pm 1}^{(n)}, P_1\right] = \eta_{11} P_\pm^{(n)},
\end{aligned}
\tag{20}
$$

where η is a Lorentzian metric with the non-zero components $\eta_{+-} = \eta_{-+} = -\eta_{11} = n$. For brevity, the commutation relations of generators with different n (analogous to (18)) are not

$$\begin{tikzcd}
 & \text{Carroll-BMS}_3 & \\
\text{“type } \mathfrak{su}(1,1)\text{”} \arrow[ur, "c\to 0"] \arrow[dr, "c\to\infty"'] & \text{BMS}_3\ (\mathfrak{B}_3) \arrow[l, dashed] \arrow[r, dashed] & \text{“type } \mathfrak{sl}(2,\mathbb{R})\text{”} \arrow[ul, "(\text{quasi})\ c\to 0"'] \arrow[dl, "(\text{quasi})\ c\to\infty"] \\
 & \text{Galilei-BMS}_3 &
\end{tikzcd}$$

Figure 1. Schematic representation of the contractions of $\mathfrak{B}_3$, performed by choosing a family of (2+1)d Poincaré subalgebras associated with one of its real forms and assuming that in the limit $c \to 0/c \to \infty$ each of these subalgebras is contracted to (2+1)d (quasi-) Carroll/Galilei algebra.

shown. If we now rescale $M^{(n)}_{\pm 1} \mapsto c M^{(n)}_{\pm 1} = \widetilde{M}^{(n)}_{\pm 1}$, $P_1 \mapsto c P_1 = \widetilde{P}_1$ (for all $n \in \mathbb{N}$) and take the limit $c \to 0$, it allows to recover the CBMS$_3$ algebra (16). Similarly, rescaling $M^{(n)}_{\pm 1} \mapsto c^{-1} M^{(n)}_{\pm 1} = \widehat{M}^{(n)}_{\pm 1}$, $P^{(n)}_{\pm} \mapsto c^{-1} P^{(n)}_{\pm} = \widehat{P}^{(n)}_{\pm}$ (for all $n \in \mathbb{N}$) and taking the limit $c \to \infty$ allows to recover the GBMS$_3$ algebra (17). Let us stress that these contractions (in the sense of their meaning in terms of Poincaré embeddings) cannot be seen as a BMS-generalization of the Carrollian or Galilean contraction—the rescaled Poincaré generators do not describe boosts and time translation, or boosts and spatial translations, respectively. We may call them ‘quasi-Carrollian’ and ‘quasi-Galilean’. The situation is summarized by the diagram in figure 1.

We can shed some light on the physical meaning of the latter contractions by making the following observation. If the quasi-Carrollian contraction is applied to the stand-alone Poincaré algebra, leading to the contraction limit:

$$\begin{aligned}
&\left[M_{+-}, \widetilde{M}_{\pm 1}\right] = \pm \widetilde{M}_{\pm 1}\,, && \left[\widetilde{M}_{+1}, \widetilde{M}_{-1}\right] = 0\,, && \\
&[M_{+-}, P_{\pm}] = \pm P_{\pm}\,, && \left[\widetilde{M}_{\pm 1}, P_{\mp}\right] = -\widetilde{P}_1\,, && \left[\widetilde{M}_{\pm 1}, \widetilde{P}_1\right] = 0\,, \qquad (21)
\end{aligned}$$

it can be shown to have an unexpected connection to certain our results [53, 54] (see also [55]) obtained for (2+1)d gravity in the Chern–Simons formulation. Namely, using an isomorphism given by

$$\widetilde{M} := M_{+-}\,, \qquad \widetilde{K}_a := \frac{1}{\sqrt{2}}\left(\widetilde{M}_{+1} \pm P_{+}\right), \qquad \widetilde{T}_0 := -\widetilde{P}_1\,, \qquad \widetilde{T}_a := \frac{1}{\sqrt{2}}\left(P_{-} \mp \widetilde{M}_{-1}\right), \tag{22}$$

we recover the algebra (in the form given in [54]) that we derived by contracting the Lorentz subgroup in the local Iwasawa decomposition of (2+1)d de Sitter group. Particles coupled to Chern–Simons theory with such a contracted gauge group turn out to satisfy the equations of motion of a Carroll particle, while symmetries of the action are a deformation of Carroll symmetries [53], which further justifies the name ‘quasi-Carrollian’ introduced above. We should also note that the algebra (21) is not a kinematical algebra and hence it is not included in the classification of [22].

Finally, turning to the remaining real forms of $\mathfrak{B}_3$, of mixed and compact-mixed type, let us try to impose the same principle of defining the Carrollian/Galilean contractions that lead to the Carroll/Galilei algebra when restricted to any Poincaré subalgebra. As we mentioned in section 3, the subalgebras are now parametrized by $n \in 2\mathbb{N}$. However, since taking the brackets of two generators with odd values of n leads to the generators with even values of n (cf (18)), in order to avoid divergences of the considered contractions, we would have to rescale the appropriate generators with odd values of n instead of even, contradicting our assumption from the beginning of this section that the rescalings are performed for the generators spanning all the Poincaré subalgebras.

4.1. Λ-*BMS*$_3$

A special feature of BMS algebra in 2+1 dimensions is that it can be generalized to the symmetry algebra of an asymptotically (Anti-)de Sitter spacetime, with the brackets ($n,m \in \mathbb{Z}$)

$$\begin{aligned} &[l_n, l_m] = (n-m)\, l_{n+m}\,, \qquad [l_n, T_m] = (n-m)\, T_{n+m}\,, \\ &[T_n, T_m] = -\Lambda\,(n-m)\, l_{n+m}\,. \end{aligned} \tag{23}$$

In general, it should be treated as a complex algebra known as Λ-BMS$_3$. Depending on the sign of Λ, Λ-BMS$_3$ can be equipped with one of several real structures, which allows to embed into it the (2+1)d de Sitter or Anti-de Sitter algebra (see below). The $\Lambda \to 0$ contraction limit of (23) recovers the usual BMS$_3$ algebra $\mathfrak{B}_3$, i.e. (6).

Actually, transforming the brackets (23) by an isomorphism (which is complex for $\Lambda > 0$)

$$L_n := \frac{1}{2}\left(l_n + \frac{1}{\sqrt{-\Lambda}}\, T_n\right), \qquad \bar{L}_n := \frac{1}{2}\left(l_n - \frac{1}{\sqrt{-\Lambda}}\, T_n\right) \tag{24}$$

shows that the Λ-BMS$_3$ algebra is equivalent to the direct sum of two copies of Witt algebra, $\mathfrak{W} \oplus \mathfrak{W}$:

$$[L_n, L_m] = (n-m)\, L_{n+m}\,, \qquad [\bar{L}_n, \bar{L}_m] = (n-m)\, \bar{L}_{n+m}\,, \qquad [L_n, \bar{L}_m] = 0\,. \tag{25}$$

It is then easy to see that there exist infinitely many embeddings of the $\mathfrak{o}(4,\mathbb{C})$ algebra into Λ-BMS$_3$, given by any subalgebra of the form

$$\mathfrak{o}\,(4,\mathbb{C}) \cong \mathrm{span}\,\{L_0, L_{\pm n}, \bar{L}_0, \bar{L}_{\pm n}\} \subset \mathfrak{W} \oplus \mathfrak{W}\,, \quad n \in \mathbb{N}\,, \tag{26}$$

up to a rescaling of $L_n, \bar{L}_n$ by $1/n$. In terms of the generators l_n, T_n, these embeddings are given by

$$\mathfrak{o}\,(4,\mathbb{C}) \cong \mathrm{span}\,\{l_0, l_{\pm n}, T_0, T_{\pm n}\} \subset \mathfrak{W} \oplus \mathfrak{W}\,, \quad n \in \mathbb{N}\,, \tag{27}$$

where (as one could expect due to the existence of $\Lambda \to 0$ contraction limit) the generators are the exact counterparts of those that span the embeddings of $\mathfrak{iso}(2,1)$ into $\mathfrak{B}_3$.

As a consequence, possible real structures on the Λ-BMS$_3$ algebra can be obtained by extending (non-compact) real forms of $\mathfrak{o}(4,\mathbb{C}) \cong \mathfrak{sl}(2,\mathbb{C}) \oplus \mathfrak{sl}(2,\mathbb{C})$ (see [56, 57]). Namely, we find the following real forms of Λ-BMS$_3$:

(i) Type $\mathfrak{sl}(2,\mathbb{R}) \oplus \mathfrak{sl}(2,\mathbb{R})$, for which $L_n^* = -L_n\,, \quad \bar{L}_n^* = -\bar{L}_n\,, \quad n \in \mathbb{Z}$,
(ii) Type $\mathfrak{su}(1,1) \oplus \mathfrak{su}(1,1)$, for which $L_n^* = L_{-n}\,, \quad \bar{L}_n^* = \bar{L}_{-n}\,, \quad n \in \mathbb{Z}$,
(iii) Type $\mathfrak{sl}(2,\mathbb{R}) \oplus \mathfrak{su}(1,1)$, for which $L_n^* = -L_n\,, \quad \bar{L}_n^* = \bar{L}_{-n}\,, \quad n \in \mathbb{Z}$,
(iv) Type $\mathfrak{so}(3,1)_{\mathrm{a}}$, for which $L_n^* = -\bar{L}_n\,, \quad n \in \mathbb{Z}$,
(v) Type $\mathfrak{so}(3,1)_{\mathrm{b}}$, for which $L_n^* = \bar{L}_{-n}\,, \quad n \in \mathbb{Z}$.

Let us recall that $\mathfrak{so}(3,1)$ is the (2+1)d de Sitter algebra, while the real forms of $\mathfrak{o}(4,\mathbb{C})$ from the cases (i)–(iii) are isomorphic to the (2+1)d Anti-de Sitter algebra $\mathfrak{so}(2,2)$; types $\mathfrak{so}(3,1)_{\mathrm{a}}$ and $\mathfrak{so}(3,1)_{\mathrm{b}}$ are two BMS-extensions of $\mathfrak{so}(3,1)$. The remaining real forms of $\mathfrak{o}(4,\mathbb{C})$ ($\mathfrak{o}(4)$, $\mathfrak{o}^{\star}(4)$ and $\mathfrak{o}'\star(4)$) cannot be extended to Λ-BMS$_3$ because it leads to the $\mathfrak{su}(2)$-type reality conditions on its subalgebra spanned by the generators L_n, which is incompatible with the Lie bracket on $\mathfrak{W}$ (as we already noted in our discussion of real forms of $\mathfrak{B}_3$).

Based on this whole discussion, it is justified to apply to (23) the same procedure of the Carrollian/Galilean contraction as we defined for the $\mathfrak{B}_3$ algebra in the basis $\{l_n, T_n; n\in\mathbb{Z}\}$. In the Carrollian case, the contraction leads to the brackets identical to (16) plus the additional non-trivial ones in the supertranslation sector:

$$\left[T_n,\widetilde{T}_0\right]=-\Lambda n\tilde{l}_n\,,\qquad [T_n,T_{-n}]=-2\Lambda n\,l_0\,,\qquad [T_n,T_m]\to\infty\,,\ \ m\neq -n \tag{28}$$

but the divergence means that, surprisingly, the contraction limit does not exist. On the other hand, in the Galilean case, we obtain the brackets identical to (17) plus the following ones in the supertranslation sector:

$$\left[\widehat{T}_n,T_0\right]=-\Lambda n\widehat{l}_n\,,\qquad \left[\widehat{T}_n,\widehat{T}_{-n}\right]=0\,,\qquad \left[\widehat{T}_n,\widehat{T}_m\right]=0\,,\ \ m\neq -n\,. \tag{29}$$

The algebra with such brackets may be called Galilei-Λ-BMS$_3$. However, similarly to what we showed for the $\mathfrak{B}_3$ algebra, the interpretation of the above contractions depends on a chosen real form of Λ-BMS$_3$. Let us now discuss that.

It is quite obvious that the reality conditions for the generators l_n, T_n in the $\mathfrak{su}(1,1)$-type real form of the $\mathfrak{B}_3$ algebra are the same as in the Λ-BMS$_3$ real forms: type $\mathfrak{su}(1,1)\oplus\mathfrak{su}(1,1)$ if $\Lambda<0$, or type $\mathfrak{so}(3,1)_{\mathrm{b}}$ if $\Lambda>0$. Consequently, the formulae (9) can also be used to describe a family of embeddings of either $\mathfrak{so}(2,2)$ or $\mathfrak{so}(3,1)$ into Λ-BMS$_3$. Generalizing our approach applied in the case of $\mathfrak{B}_3$ in (16), (17), if we consider one of those two real forms of Λ-BMS$_3$, the contractions leading to (28), (29) can actually be inferred as the natural extension of the Carrollian/Galilean contractions of (Anti-)de Sitter algebra leading to (4), (5).

Meanwhile, the $\mathfrak{sl}(2,\mathbb{R})$-type real form of $\mathfrak{B}_3$ is in agreement with the Λ-BMS$_3$ real forms: type $\mathfrak{sl}(2,\mathbb{R})\oplus\mathfrak{sl}(2,\mathbb{R})$ if $\Lambda<0$, or type $\mathfrak{so}(3,1)_{\mathrm{a}}$ if $\Lambda>0$. Defining a family of embeddings of either $\mathfrak{so}(2,2)$ or $\mathfrak{so}(3,1)$ into Λ-BMS$_3$, given by the formulae identical to (10), we find that the contractions leading to (28), (29) have to be interpreted as quasi- Carrollian/Galilean, in terms of the Λ-BMS$_3$-generalization of the basis (20). The latter includes additional non-vanishing brackets

$$\left[P^{(n)}_{\pm},P_1\right]=-\Lambda\eta_{11}M^{(n)}_{\pm 1}\,,\qquad \left[P^{(n)}_{+},P^{(n)}_{-}\right]=-\Lambda\eta_{11}M_{+-}\,, \tag{30}$$

as well as the ones for the $P^{(n)}_{\pm}$ generators with different n. Naturally, the calculations made in this basis confirm that the (quasi-)Carrollian contraction limit (28) is divergent.

Finally, the $\mathfrak{sl}(2,\mathbb{R})\oplus\mathfrak{su}(1,1)$-type reality conditions are given by more complicated expressions:

$$l^*_n=\frac{1}{2}\left(l_{-n}-l_n\right)-\frac{1}{2\sqrt{-\Lambda}}\left(T_{-n}+T_n\right),\qquad T^*_n=\frac{1}{2}\left(T_{-n}-T_n\right)-\frac{\sqrt{-\Lambda}}{2}\left(l_{-n}+l_n\right), \tag{31}$$

which are different than for any real form of $\mathfrak{B}_3$. They are also inconsistent with the rescalings applied in the Carrollian contraction, due to the mixing of l_n's with T_n's. Therefore, this case seems to be of lesser interest and we will not delve deeper into it. The well-defined contractions

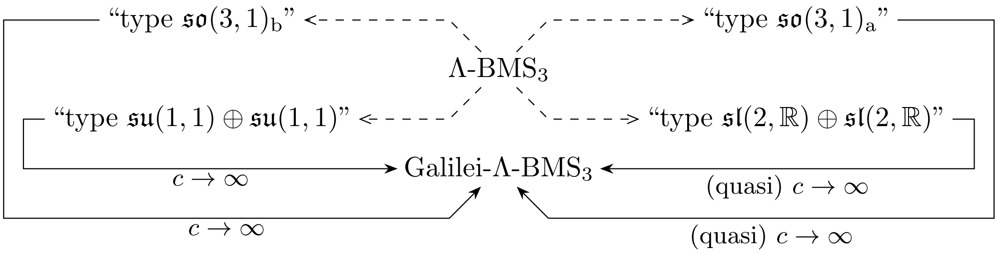

Figure 2. Schematic representation of the contractions of Λ-BMS$_3$, performed by choosing a family of (2+1)d (Anti-)de Sitter subalgebras associated with one of its real forms and assuming that in the limit $c\to\infty$ each of these subalgebras is contracted to (2+1)d (quasi-) (Anti-)de Sitter-Galilei algebra (while contractions in the Carroll limit $c\to 0$ do not exist).

that we found for the previously discussed real forms of Λ-BMS$_3$ are collected in the diagram in figure 2.

One might still imagine that the Carrollian counterpart of Λ-BMS$_3$ exists but cannot be reached by the contraction procedures considered here. In order to find such a hypothetical Carroll-Λ-BMS$_3$ algebra, we could assume that the well-known isomorphism between Poincaré and AdS-Carroll algebras generalizes to the BMS case. Namely, based on the form of Poincaré embeddings (9) into the $\mathfrak{B}_3$ algebra, let us define the map that transforms the standard basis of the latter $\{l_n, T_n;\, n\in\mathbb{Z}\}$ to a new basis $\{l'_n, T'_n;\, n\in\mathbb{Z}\}$:

$$l_k\mapsto \mathrm{sgn}\,(k)\, i\sqrt{-\Lambda}^{-1}T'_k\,,\qquad T_k\mapsto \mathrm{sgn}\,(k)\, i\sqrt{-\Lambda}\,l'_k\,,\qquad l_0\mapsto l'_0\,,\qquad T_0\mapsto T'_0 \tag{32}$$

($k\neq 0$), so that it reduces to

$$K_a\mapsto\sqrt{-\Lambda}^{-1}\mathcal{T}_a\,,\quad P_a\mapsto -\sqrt{-\Lambda}\,Q_a\,,\quad J\mapsto R\,,\quad P_0\mapsto\mathcal{T}_0 \tag{33}$$

for each Poincaré subalgebra (cf (3), (4)). It turns out that the brackets of $\mathfrak{B}_3$ in the new basis are identical to (16), (28) with two exceptions: $[T'_n, T'_m]$ is no longer divergent but $[l'_n, T'_m]$ differs from its counterpart given in (16). It follows that the map (32) does not provide us with what one could call the Carroll-Λ-BMS$_3$ algebra. We comment on the non-existence of such an algebra in Conclusions.

5. Contractions of (extended) BMS$_4$ algebra $\mathfrak{B}_4$

The extended BMS algebra in 3+1 dimensions (introduced by Barnich and Troessaert [6]), which we will denote $\mathfrak{B}_4$, has the brackets

$$\begin{gathered}
[L_n, L_m]=(n-m)\,L_{n+m}\,,\qquad [\bar{L}_n,\bar{L}_m]=(n-m)\,\bar{L}_{n+m}\,,\qquad [L_n,\bar{L}_m]=0\,,\\
[L_k, T_{nm}]=\left(\frac{k+1}{2}-n\right)T_{n+k,m}\,,\quad [\bar{L}_k, T_{nm}]=\left(\frac{k+1}{2}-m\right)T_{n,m+k}\,,\quad [T_{nm}, T_{n'm'}]=0
\end{gathered} \tag{34}$$

in terms of the generators of superrotations L_n, $\bar{L}_n$ and supertranslations T_{nm}, where $n,m\in\mathbb{Z}$. Similarly to $\mathfrak{B}_3$ and Λ-BMS$_3$, $\mathfrak{B}_4$ is in general a complex algebra. Its subalgebra spanned by $\{L_n,\bar{L}_n;\, n\in\mathbb{Z}\}$ is obviously isomorphic to Λ-BMS$_3$, cf (25). There are four real structures that one can introduce here as generalizations of the Λ-BMS$_3$ real forms:

(i) Type $\mathfrak{sl}(2,\mathbb{R}) \oplus \mathfrak{sl}(2,\mathbb{R})$, for which $L_n^* = -L_n\,, \quad \bar{L}_n^* = -\bar{L}_n\,, \quad T_{nm}^* = -T_{nm}\,,$
(ii) Type $\mathfrak{su}(1,1) \oplus \mathfrak{su}(1,1)$, for which $L_n^* = L_{-n}\,, \quad \bar{L}_n^* = \bar{L}_{-n}\,, \quad T_{nm}^* = -T_{1-n,1-m}\,,$
(iii) Type $\mathfrak{so}(3,1)_{\mathrm{a}}$, for which $L_n^* = -\bar{L}_n\,, \quad T_{nm}^* = T_{mn}\,,$
(iv) Type $\mathfrak{so}(3,1)_{\mathrm{b}}$, for which $L_n^* = \bar{L}_{-n}\,, \quad T_{nm}^* = T_{1-m,1-n}\,.$

The real form of the $\mathfrak{sl}(2,\mathbb{R}) \oplus \mathfrak{su}(1,1)$-type, for which $L_n^* = -L_n\,,\ \bar{L}_n^* = \bar{L}_{-n}$, does not extend to the supertranslation sector.

The real forms of types $\mathfrak{so}(3,1)_{\mathrm{a}}$ and $\mathfrak{so}(3,1)_{\mathrm{b}}$ are physically the most interesting ones, since then the subalgebra spanned by $\{L_n, \bar{L}_n; n \in \mathbb{Z}\}$ can be seen as the BMS generalization of Lorentz algebra $\mathfrak{so}(3,1)$. We will restrict here to the type $\mathfrak{so}(3,1)_{\mathrm{a}}$ and express the $\mathfrak{B}_4$ algebra in a basis consisting of the following anti-Hermitian generators

$$
\begin{aligned}
R_n &:= L_n + \bar{L}_n\,, & \bar{R}_n &:= -i\,(L_n - \bar{L}_n)\,, \\
S_{nm} &:= \frac{i}{2}\,(T_{nm} + T_{mn})\,, & A_{nm} &:= \frac{1}{2}\,(T_{nm} - T_{mn})
\end{aligned}
\tag{35}
$$

(let us note that S_{nm}'s are symmetric under the exchange of indices $n \leftrightarrow m$, while A_{nm}'s are antisymmetric and hence there are no such generators with $n = m$; however, for brevity, the symbol A_{nn} is implicitly used below in the sense of $A_{nn} = 0$), so that the brackets (34) become

$$
\begin{aligned}
&[R_n, R_m] = (n-m)\,R_{n+m}\,, \qquad [\bar{R}_n, \bar{R}_m] = -\,(n-m)\,R_{n+m}\,, \\
&[R_n, \bar{R}_m] = (n-m)\,\bar{R}_{n+m}\,, \\
&[R_k, S_{nm}] = \left(\frac{k+1}{2} - n\right) S_{n+k,m} + \left(\frac{k+1}{2} - m\right) S_{n,m+k}\,, \\
&[\bar{R}_k, S_{nm}] = \left(\frac{k+1}{2} - n\right) A_{n+k,m} - \left(\frac{k+1}{2} - m\right) A_{n,m+k}\,, \\
&[R_k, A_{nm}] = \left(\frac{k+1}{2} - n\right) A_{n+k,m} + \left(\frac{k+1}{2} - m\right) A_{n,m+k}\,, \\
&[\bar{R}_k, A_{nm}] = -\left(\frac{k+1}{2} - n\right) S_{n+k,m} + \left(\frac{k+1}{2} - m\right) S_{n,m+k}\,.
\end{aligned}
\tag{36}
$$

The new basis makes it evident that the algebraic structure of $\mathfrak{B}_4$ (albeit infinite-dimensional) is analogous to that of a kinematical (Lie) algebra. It follows from the definition [20] that a kinematical algebra contains the subalgebra of rotations, that the generators of boosts and spatial translations transform under rotations as vectors, and that the generator of time translations transforms under rotations as a scalar. Accordingly, (36) contains the rotation-like subalgebra spanned by the generators R_n, which act on $\bar{R}_n$, S_{pq} and A_{pq} in a vector-like way; the only missing piece seems to be an analogue of the time translation generator. If we narrow down our analogy to Poincaré algebra, we can identify $\bar{R}_n$'s as corresponding to the boost generators and S_{pq}'s—to the spatial translation generators, while A_{pq}'s fill the spot of the time translation generator, with a caveat that they do not commute with R_n's (in particular, let us note that if $p, q \in \{-n, n\}$, there are three generators S_{pq} and only one A_{pq}; however, this does not allow to construct an embedding of $\mathfrak{iso}(3,1)$ into $\mathfrak{B}_4$). Consequently, it turns out that one can perform two contractions of $\mathfrak{B}_4$ similar to the Carrollian and Galilean contraction of Poincaré algebra. As we will see, however, these contractions are not generalizations of the corresponding ones for Poincaré algebra and hence we call them quasi- Carrollian/Galilean.

The quasi-Carrollian contraction consists in the rescalings $\bar{R}_n \mapsto c\bar{R}_n$ and $A_{nm} \mapsto cA_{nm}$, $\forall n,m \in \mathbb{Z}$, and then taking the limit $c \to 0$, which leads to the algebra:

$$
\begin{aligned}
&[R_n,R_m] = (n-m)\,R_{n+m}\,, \qquad [R_n,\bar{R}_m] = (n-m)\,\bar{R}_{n+m}\,, \qquad [\bar{R}_n,\bar{R}_m] = 0\,,\\
&[R_k,S_{nm}] = \left(\frac{k+1}{2}-n\right)S_{n+k,m} + \left(\frac{k+1}{2}-m\right)S_{n,m+k}\,,\\
&[\bar{R}_k,S_{nm}] = \left(\frac{k+1}{2}-n\right)A_{n+k,m} - \left(\frac{k+1}{2}-m\right)A_{n,m+k}\,,\\
&[R_k,A_{nm}] = \left(\frac{k+1}{2}-n\right)A_{n+k,m} + \left(\frac{k+1}{2}-m\right)A_{n,m+k}\,, \qquad [\bar{R}_k,A_{nm}] = 0\,.
\end{aligned}
\tag{37}
$$

Meanwhile, the quasi-Galilean contraction of (36) is performed by rescaling $\bar{R}_n \mapsto c^{-1}\bar{R}_n$ and $S_{nm} \mapsto c^{-1}S_{nm}$, $\forall n,m \in \mathbb{Z}$, and then taking the limit $c \to \infty$, which gives us the algebra:

$$
\begin{aligned}
&[R_n,R_m] = (n-m)\,R_{n+m}\,, \qquad [R_n,\bar{R}_m] = (n-m)\,\bar{R}_{n+m}\,, \qquad [\bar{R}_n,\bar{R}_m] = 0\,,\\
&[R_k,S_{nm}] = \left(\frac{k+1}{2}-n\right)S_{n+k,m} + \left(\frac{k+1}{2}-m\right)S_{n,m+k}\,, \qquad [\bar{R}_k,S_{nm}] = 0\,,\\
&[R_k,A_{nm}] = \left(\frac{k+1}{2}-n\right)A_{n+k,m} + \left(\frac{k+1}{2}-m\right)A_{n,m+k}\,,\\
&[\bar{R}_k,A_{nm}] = -\left(\frac{k+1}{2}-n\right)S_{n+k,m} + \left(\frac{k+1}{2}-m\right)S_{n,m+k}\,.
\end{aligned}
\tag{38}
$$

The use of terms 'quasi-Carrollian' and 'quasi-Galilean' can be justified by referring to the analogy between the $\mathfrak{B}_4$ and Poincaré algebra discussed earlier. We observe that both algebras (37) and (38) have commuting boost-like generators $\bar{R}_n$, as well as $\bar{R}_n$ commuting with A_{nm}'s in the first case and $\bar{R}_n$ commuting with S_{nm}'s in the second case, while the remaining brackets are not changed with respect to (36). If A_{nm}, S_{nm} are interpreted as time and spatial 'translation-like' generators (as we already postulated for $\mathfrak{B}_4$), respectively, the above-mentioned commutation relations have exactly the same structure as for Carroll and Galilei algebras, respectively. Therefore, we will call the algebra defined by (37) quasi-Carroll-BMS$_4$ and the algebra defined by (38) – quasi-Galilei-BMS$_4$.

Let us now recall that one can identify the Poincaré algebra $\mathfrak{iso}(3,1)$ with a maximal finite-dimensional subalgebra of $\mathfrak{B}_4$. Similarly as in the case of 2+1 dimensions, there exist infinitely many such subalgebras, spanned by 10 generators that satisfy the commutation relations generalizing (8) to 4 dimensions:

$$
\begin{aligned}
&[J_a,J_b] = \eta_{00}\epsilon_{abc}J_c\,, \quad [K_a,K_b] = -\eta_{00}\epsilon_{abc}J_c\,, \quad [J_a,K_b] = \eta_{00}\epsilon_{abc}K_c\,, \quad [J_a,P_0] = 0\,,\\
&[J_a,P_b] = \eta_{00}\epsilon_{abc}P_c\,, \quad [K_a,P_0] = \eta_{00}P_a\,, \quad [K_a,P_b] = -\eta_{ab}P_0\,, \quad [P_\mu,P_\nu] = 0\,.
\end{aligned}
\tag{39}
$$

(This form of the brackets can be recovered from (7) by taking $M_{ab} = \epsilon_{abc}J_c$, $M_{a0} = K_a$.) The above-mentioned subalgebras correspond to a family of embeddings of $\mathfrak{iso}(3,1)$ into $\mathfrak{B}_4$, parametrized by $n \in 2\mathbb{N}-1$ (see [70]):

$$
\begin{aligned}
&J_1^{(n)} = \frac{1}{2}\left(\bar{R}_{-n}-\bar{R}_n\right), \qquad J_2^{(n)} = \frac{1}{2}\left(R_{-n}+R_n\right), \qquad J_3 = -\bar{R}_0\,,\\
&K_1^{(n)} = \frac{1}{2}\left(R_n-R_{-n}\right), \qquad K_2^{(n)} = \frac{1}{2}\left(\bar{R}_n+\bar{R}_{-n}\right), \qquad K_3 = R_0\,,
\end{aligned}
$$

$$P_0^{(n)} = \frac{1}{2}\left(S_{qq} + S_{pp}\right), \qquad P_3^{(n)} = \frac{1}{2}\left(S_{qq} - S_{pp}\right), \qquad P_1^{(n)} = S_{pq}, \qquad P_2^{(n)} = A_{pq}, \tag{40}$$

where the indices $p = (1+n)/2$, $q = (1-n)/2$, so that the new generators satisfy the brackets (39) with the metric $\eta = n\,\mathrm{diag}(1,-1,-1,-1)$. The Poincaré generators are anti-Hermitian ($X^* = -X$) if we impose the reality conditions of the type $\mathfrak{so}(3,1)_{\mathrm{a}}$ on the $\mathfrak{B}_4$ algebra.

Such an embedding of the Lorentz algebra is well-defined also for even $n \neq 0$, while the same is not true for the translation generators. Consequently, in contrast to what happens in 2+1 dimensions, the union of all embeddings from the family does not correspond to a basis of the full $\mathfrak{B}_4$ algebra. Furthermore, the brackets between $J_a^{(n)}$ and $K_b^{(m)}$ with odd parameters n, m are given by generators with even parameters $n+m$ and $n-m$ and hence the union of these embeddings is not even a subalgebra. Taking all this into account, we conclude that the embeddings in the 3+1-dimensional case do not allow us to straightforwardly generalize the contractions of Poincaré to BMS algebra; conversely, performing the contractions of BMS is independent from embedding Poincaré algebra into it. In particular, the form of translation generators in (40) does not preserve our interpretation of A_{nm}'s and S_{nm}'s as the generalizations of time and spatial translation generators, respectively, while the rotation and boost generators mix R_n's and $\bar{R}_n$'s. However, this apparent inconsistency will find a resolution below.

Let us consider a different picture. Choosing two light-like vectors in flat spacetime, an outgoing $\tau_+ = 1/\sqrt{2}(1,0,0,1)$ and incoming $\tau_- = 1/\sqrt{2}(1,0,0,-1)$, one can perform the 2+2 spacetime decomposition and introduce light-cone (a.k.a. light-front) generators of Poincaré algebra (7),

$$\begin{aligned} &M_{\pm a} = \tau_\pm^\mu M_{\mu a} = \frac{1}{\sqrt{2}}\left(M_{0a} \pm M_{3a}\right), \qquad M_{+-} = \tau_+^\mu \tau_-^\nu M_{\mu\nu} = M_{30}, \\ &P_\pm = \frac{1}{\sqrt{2}}\left(P_0 \pm P_3\right), \end{aligned} \tag{41}$$

so that, for a general metric η, their commutation relations have the form

$$\begin{aligned} &[M_{+a}, M_{-b}] = -\eta_{+-} M_{ab} - \eta_{ab} M_{+-}, && [M_{\pm a}, M_{\pm b}] = 0, \\ &[M_{\pm a}, M_{bc}] = \eta_{ab} M_{\pm c} - \eta_{ac} M_{\pm b}, && [M_{+-}, M_{\pm a}] = \pm\eta_{+-} M_{\pm a}, \\ &[M_{\pm a}, P_\mp] = -\eta_{+-} P_a, && [M_{\pm a}, P_\pm] = [M_{+-}, P_a] = 0, \\ &[M_{\pm a}, P_b] = \eta_{ab} P_\pm, && [M_{+-}, P_\pm] = \pm\eta_{+-} P_\pm \end{aligned} \tag{42}$$

$(a,b,c = 1,2)$. Similarly to the 2+1-dimensional case (20), $M_{\pm 1}$, $M_{\pm 2}$ are actually generators of null rotations (transformations generated by $M_{\pm a}$ leave invariant the null directions $\tau_\pm$, as well as the spatial directions x_2, x_1 if $a = 1,2$, respectively) and $P_\pm$ are generators of translations along the null directions $\tau_\pm$, respectively.

Transforming the image of a given embedding (40) to such a light-cone basis, we obtain:

$$\begin{aligned} &M_{\pm 1}^{(n)} = \frac{1}{\sqrt{2}}\left(-K_1^{(n)} \pm J_2^{(n)}\right) = \pm\frac{1}{\sqrt{2}} R_{\mp n}, \qquad M_{+-} = K_3 = R_0, \\ &M_{\pm 2}^{(n)} = \frac{1}{\sqrt{2}}\left(-K_2^{(n)} \mp J_1^{(n)}\right) = -\frac{1}{\sqrt{2}} \bar{R}_{\mp n}, \qquad M_{12} = J_3 = -\bar{R}_0, \\ &P_+^{(n)} = \frac{1}{\sqrt{2}} S_{qq}, \qquad P_-^{(n)} = \frac{1}{\sqrt{2}} S_{pp}, \qquad P_1^{(n)} = S_{pq}, \qquad P_2^{(n)} = A_{pq} \end{aligned} \tag{43}$$

and then the brackets (42) are satisfied with the metric components $\eta_{+-} = \frac{1}{2}(\eta_{00} - \eta_{33}) = -\eta_{aa} = n$ and $\eta_{++} = \eta_{--} = \eta_{ab} = 0$, $a \neq b$.

From this point of view, the quasi-Carrollian contraction involves rescaling of the generators generalizing translations along the spatial direction x_2, while for the quasi-Galilean contraction one needs to rescale the generators generalizing translations along the null directions $\tau_\pm$ and the spatial direction x_1; and, in both cases, rescale the generators generalizing null rotations that leave invariant the directions $\tau_\pm$ and x_1, as well as the generator of rotations in the x_1x_2 plane. This can now be compared with contractions of the $\mathfrak{B}_3$ algebra performed in the basis (20). Indeed, in that case, the quasi-Carrollian contraction also involved rescaling of a spatial translation generator, while the quasi-Galilean contraction—rescaling of the null translation generators, and for both contractions we also rescaled the null rotation generators (while there is no 2+1-dimensional counterpart of the rotation generator M_{12}). The difference is that the limits of such contractions in 2+1 dimensions are actually given by the Carroll-BMS_3 and Galilei-BMS_3 algebras (which are also derivable via the 'non-quasi' Carrollian and Galilean contractions, performed in (16) and (17)), while we do not obtain algebras that could be described as Carroll-BMS_4 and Galilei-BMS_4.

6. Conclusions

The aim of this paper was to try to define the notions of Carrollian and Galilean contractions of BMS algebra in 3+1 and 2+1 dimensions, as well as Λ-BMS in 2+1 dimensions. (As we mentioned in Introduction, the inclusion of cosmological constant in 3+1 dimensions leads to a more complicated case of a Lie algebroid.)

We showed that both types of contractions can be extended to the BMS_3 algebra, leading to the BMS counterparts of Carroll and Galilei algebras, called Carroll-BMS_3 and Galilei-BMS_3. This is achieved by using a family of embeddings of Poincaré algebra $\mathfrak{iso}(2,1)$ into $\mathfrak{B}_3$, which allows us to decompose the latter into (overlapping) subalgebras isomorphic to $\mathfrak{iso}(2,1)$. We adopt the condition that the contraction limit of each such subalgebra should be isomorphic to Carroll/Galilei algebra and find that it can be satisfied if the considered embeddings are associated with a particular real form (type $\mathfrak{su}(1,1)$) of $\mathfrak{B}_3$. On the other hand, CBMS_3 and GBMS_3 can also be recovered by starting with a family of embeddings associated with the other real form (type $\mathfrak{sl}(2,\mathbb{R})$). The contractions of BMS_3 that we perform in such a case are not equivalent to the Carrollian/Galilean contractions when restricted to the corresponding Poincaré subalgebras, hence we call them quasi- Carrollian/Galilean. Moreover, they turn out to have an interesting connection to a particular contraction considered for the gauge group ((2+1)d de Sitter group) of the Chern–Simons theory describing (2+1)d gravity with $\Lambda > 0$ [53, 54]. As we discussed, the algebra of this group after the contraction becomes isomorphic to the limit of the above-mentioned quasi-Carrollian contraction of Poincaré algebra. On the other hand, particles coupled to such a theory have the same dynamics as free particles on a Carroll manifold, although their symmetries are deformed. This allows us to state that at the level of kinematical algebras, the quasi-Carrollian contraction in 2+1 dimensions is indeed leading to a similar physical regime as the Carrollian one, while these two contractions extended to the whole $\mathfrak{B}_3$ are mathematically equivalent.

Each of the above contraction procedures can be straightforwardly generalized to the Λ-BMS$_3$ algebra: Carrollian/Galilean (for the real form type $\mathfrak{su}(1,1)\oplus\mathfrak{su}(1,1)$ or $\mathfrak{so}(3,1)_{\mathrm{b}}$) or quasi- Carrollian/Galilean (for the real form type $\mathfrak{sl}(2,\mathbb{R})\oplus\mathfrak{sl}(2,\mathbb{R})$ or $\mathfrak{so}(3,1)_{\mathrm{a}}$). We find that only the Galilean contraction limit is well-defined in these cases. However, this does not mean that a Carrollian spacetime with $\Lambda<0$ has no consistent algebra of asymptotic symmetries. It has recently been found that the algebra of such symmetries for (2+1)d AdS-Carroll spacetime is actually $\mathfrak{B}_3$ [58]. Therefore, while Poincaré algebra (being isomorphic to AdS-Carroll algebra) describes kinematical symmetries of AdS-Carroll spacetime, as well as it can be obtained via the Carrollian contraction of AdS algebra, its infinite-dimensional generalization $\mathfrak{B}_3$ is the symmetry algebra at infinity of AdS-Carroll spacetime but (in light of our results) cannot be obtained by analogously contracting Λ-BMS$_3$. Moreover, a study of the (3+1)d case has similarly shown [59] that in the Carroll limit of an asymptotically Anti-de Sitter spacetime the corresponding symmetry algebra becomes $\mathfrak{B}_4$ (which reduces to the standard BMS$_4$, without superrotations, if one sets more restrictive boundary conditions); precisely speaking, this happens for only one of the two possible Carroll limits of General Relativity, the so-called magnetic one, while the electric limit leads to an inconsistency.

BMS algebra in 3+1 dimensions ($\mathfrak{B}_4$) is more problematic because the embeddings of Poincaré algebra $\mathfrak{iso}(3,1)$ do not cover the whole algebra and hence do not provide a framework for extending the contractions of $\mathfrak{iso}(3,1)$ to $\mathfrak{B}_4$. As an alternative method, we observe the analogy between the structures of these two algebras and find that (at least, considering the $\mathfrak{so}(3,1)_{\mathrm{a}}$-type real form) it is only possible to perform the so-called quasi-Carrollian and quasi-Galilean contractions of $\mathfrak{B}_4$, which lead to the quasi-Carroll-BMS$_4$ and quasi-Galilei-BMS$_4$ algebras. The structure of these algebras is superficially reminiscent of Carroll and Galilei algebras but does not agree with the embeddings of $\mathfrak{iso}(3,1)$ into $\mathfrak{B}_4$. However, when expressed in an appropriate basis, the contractions turn out to be a straightforward generalization of the quasi-Carrollian and quasi-Galilean contractions of $\mathfrak{B}_3$.

Having obtained the (quasi-)Carrollian and (quasi-)Galilean BMS algebras discussed above, one may further try to construct their quantum deformations. What we mean by such deformations are non-trivial Hopf algebras (also known as quantum groups) that reduce to a given 'classical' algebra in the appropriate limit of their deformation parameters [60, 61], are associated with non-commutative geometry [62] and conjectured to play a role in certain regimes of quantum gravity—see an extensive review [63]. They have been studied for many years in the case of kinematical algebras [64–66], starting from the seminal papers [67, 68] and the framework has quite recently been extended to the (non-contracted) BMS algebras [69–71]. The construction introduced in the latter series of papers could now be applied to the (quasi-)Carrollian and (quasi-)Galilean BMS algebras, analogously to the generalization of deformations of Poincaré and (Anti-)de Sitter algebras to their Carrollian and Galilean counterparts [72]; in particular, as in the latter case, it may happen that certain deformations of a given contracted algebra cannot be obtained by contracting deformations of the original algebra [51]. This will be the subject of our intended follow-up paper.

Finally, let us stress that an extra result of this paper is finding all possible real structures on the BMS$_3$ algebra (known to mathematicians as the Witt algebra); three non-compact ones become isomorphic to each other when restricted to a $\mathfrak{sl}(2,\mathbb{C})$ subalgebra.

Data availability statement

No new data were created or analysed in this study.

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Appendix. Classification of the reality conditions on the Witt algebra

According to the seminal works of E Cartan [73] (see also [74]), all real forms of a finite-dimensional (semi)simple Lie algebra can be obtained from its compact real form by acting with involutive automorphisms. One easily sees that the (complex) Witt algebra defined (in the Cartan-Weyl-like basis) as

$$[l_n, l_m] = (n-m)\, l_{n+m}\,, \tag{44}$$

where $n, m \in \mathbb{Z}$, is a simple Lie algebra with the structure analogous to the finite-dimensional ones. The generator l_0 plays a role of the Cartan element and l_n (resp. l_{-n}), $n \geqslant 1$, are the positive (resp. negative) roots[4]. The most general automorphism of (44) has the form (see [75])

$$\phi_{(a,\omega)}\,(l_n) = \omega a^n l_{\omega n}\,, \tag{45}$$

where $\omega = \pm 1$, $0 \neq a \in \mathbb{C}$. Notice that $\phi_{(a,\omega)}(l_0) = \omega l_0$ is independent of a and that $\phi_{(a,1)}(l_n) = a^n l_n$, $n \in \mathbb{Z}$ is a rescaling (internal) automorphism. For an involutive authomorphism, one also needs $a = \pm 1$. This means that there are only three non-trivial involutive authomorphisms: $\phi_{\mathrm{I}} = \phi_{(-1,-1)}$, $\phi_{\mathrm{II}} = \phi_{(-1,1)}$, $\phi_{\mathrm{III}} = \phi_{(1,-1)}$, inherited from the $\mathfrak{sl}(2,\mathbb{C})$ algebra[5]. More explicitly,

$$\phi_{\mathrm{I}}\,(l_n) = (-1)^{n+1}\, l_{-n}\,, \qquad \phi_{\mathrm{II}}\,(l_n) = (-1)^n\, l_n\,, \qquad \phi_{\mathrm{III}}\,(l_n) = -l_{-n}\,. \tag{46}$$

Extended by the identity map, they form a multiplicative group $\mathbb{Z}_2 \times \mathbb{Z}_2$. Applying to these automorphisms the Cartan's rule $X^\phi = \phi(X^*)$, where X^* denotes the compact conjugation, one obtains all non-compact real forms. In fact, the non-compact ones may in principle be isomorphic to each other, which is the case for $\mathfrak{sl}(2,\mathbb{C})$ algebra (see, e.g. [76]).

We will show here that, unlike in the finite-dimensional $\mathfrak{sl}(2,\mathbb{C})$ case, real forms I)–III) corresponding to the automorphisms $\phi_{\mathrm{I}} - \phi_{\mathrm{III}}$ are not isomorphic, where ($n \in \mathbb{Z}$):

[4] Moreover, Witt algebra contains an infinite family of maximal finite-dimensional subalgebras isomorphic to $\mathfrak{sl}(2,\mathbb{C})$. This can be seen using the following identifications: e.g. $H = l_0$, $E_+ \sim -l_{-n}$, $E_- \sim l_n$, or e.g. $H = -l_0$, $E_+ \sim i\,l_n$, $E_- \sim i\,l_{-n}$; cf the analogous observation for the real forms of $\mathfrak{sl}(2,\mathbb{C})$ in section 3.

[5] An automorphism restricted to a subalgebra has to be an automorphism of the latter.

(I) Type $\mathfrak{sl}(2,\mathbb{R})$ has the reality conditions $l_n^{\#} = \phi_{\mathrm{I}}(l_n^*) = -l_n\,,$
(II) Type $\mathfrak{su}(1,1)$ has the reality conditions $l_n^{\dagger} = \phi_{\mathrm{II}}(l_n^*) = l_{-n}\,,$
(III) Mixed type has the reality conditions $l_n^{\ddagger} = \phi_{\mathrm{III}}(l_n^*) = (-1)^{n+1} l_n\,,$

and automorphisms are acting on the real form (generalizing what we said at the very beginning of this appendix)

(IV) Compact-mixed type, which is defined by the reality conditions $l_n^* = (-1)^n l_{-n}\,.$

One should notice that, as a matter of fact, for even values of n, IV) coincides with II) and III) coincides with I). Let us also clarify that in section 3 we use the same symbol for all conjugations for the simplicity of notation.

In order to introduce the above 'compact' form IV), we determine a new system of the $\mathfrak{su}(2)$-type generators (a basis) that corresponds to compact generators in the finite-dimensional case (see [76])

$$J_1^{(n)} = -\frac{1}{2}\left(l_n + l_{-n}\right), \qquad J_3^{(n)} = \frac{i}{2}\left(l_{-n} - l_n\right), \qquad J_2 = i l_0\,, \tag{47}$$

with $n \geqslant 1$, satisfying the following commutation relations

$$\begin{gathered}\left[J_1^{(n)}, J_2\right] = n J_3^{(n)}\,, \qquad \left[J_2, J_3^{(n)}\right] = n J_1^{(n)}\,,\\ \left[J_3^{(n)}, J_1^{(m)}\right] = -\frac{i}{2}\left((n-m)\, J_1^{(n+m)} + (n+m)\, J_1^{(|n-m|)}\right),\end{gathered} \tag{48}$$

as well as

$$\begin{gathered}\left[J_1^{(n)}, J_1^{(m)}\right] = \frac{i}{2}\left((n-m)\, J_3^{(n+m)} + (n+m)\, J_3^{(n-m)}\right),\\ \left[J_3^{(n)}, J_3^{(m)}\right] = \frac{i}{2}\left((m-n)\, J_3^{(n+m)} + (n+m)\, J_3^{(n-m)}\right),\end{gathered} \tag{49}$$

(here $J_3^{(-n)} = -J_3^{(n)}$, $J_3^{(0)} = 0$, $J_1^{(-n)} = J_1^{(n)}$, $J_1^{(0)} = iJ_2$). In the case of (48), for $n = m \geqslant 1$, one recovers the $\mathfrak{su}(2)$ bracket (with $a,b,c = 1,2,3$)

$$\left[J_a^{(n)}, J_b^{(n)}\right] = n\,\epsilon_{abc}\, J_c^{(n)}\,. \tag{50}$$

However, imposing the compact reality condition $X^* = -X$ is only possible for odd n, due to the conflict with the relations (44) for $n+m \neq 0$ or $n-m \neq 0$[6]. The corrected compact(-mixed) conjugation takes the form

$$J_1^{(n)*} = (-1)^n J_1^{(n)}, \qquad J_3^{(n)*} = (-1)^n J_3^{(n)}, \qquad J_2^* = -J_2\,. \tag{51}$$

In terms of these generators, all three involutive authomorphisms (45) diagonalize into

$$\phi_{\mathrm{I}}\left(J_1^{(n)}\right) = (-1)^{n+1} J_1^{(n)}\,, \qquad \phi_{\mathrm{I}}\left(J_3^{(n)}\right) = (-1)^n J_3^{(n)}\,, \qquad \phi_{\mathrm{I}}\left(J_2\right) = -J_2\,, \tag{52}$$

[6] The conflict can also be seen in the formulae (49), since the commutator of two anti-hermitian generators cannot be hermitian.

$$\phi_{\mathrm{II}}\left(J_1^{(n)}\right)=(-1)^n J_1^{(n)}\,,\qquad \phi_{\mathrm{II}}\left(J_3^{(n)}\right)=(-1)^n J_3^{(n)}\,,\qquad \phi_{\mathrm{II}}\left(J_2\right)=J_2\,,\tag{53}$$

$$\phi_{\mathrm{III}}\left(J_1^{(n)}\right)=-J_1^{(n)}\,,\qquad \phi_{\mathrm{III}}\left(J_3^{(n)}\right)=J_3^{(n)}\,,\qquad \phi_{\mathrm{III}}\left(J_2\right)=-J_2\,.\tag{54}$$

Applying the Cartan's rule $X^{\phi}=\phi(X^*)$, we obtain the following non-isomorphic non-compact real structures

$$J_1^{(n)\#}=-J_1^{(n)}\,,\qquad J_3^{(n)\#}=J_3^{(n)}\,,\qquad J_2^{\#}=J_2\,,\tag{55}$$

$$J_1^{(n)\dagger}=J_1^{(n)}\,,\qquad J_3^{(n)\dagger}=J_3^{(n)}\,,\qquad J_2^{\dagger}=-J_2\,,\tag{56}$$

$$J_1^{(n)\ddagger}=(-1)^{n+1}J_1^{(n)}\,,\qquad J_3^{(n)\ddagger}=(-1)^{n}J_3^{(n)}\,,\qquad J_2^{\ddagger}=J_2\,.\tag{57}$$

We see that for each conjugation only one generator is anti-hermitian and the other two hermitian. In the last case above, the position of anti-hermitian generator depends on the parity of n, indicating the change of metric signature. Therefore (see [76]), all non-compact conjugations are of the $\mathfrak{so}(1,2)$-type.

We are now in position to argue that the Witt algebra is equipped with three non-isomorphic non-compact real structures, in spite of the fact that they are isomorphic on the subalgebras $\mathrm{span}\{l_{-1},l_0,l_1\}\cong\mathfrak{sl}(2,\mathbb{C})$. To this aim, one may consider the (non-involutive) automorphism [76]

$$\psi\left(l_{\pm1}\right)=il_0\pm\frac{1}{2}\left(l_1-l_{-1}\right),\qquad \psi\left(l_0\right)=\frac{i}{2}\left(l_1+l_{-1}\right),\tag{58}$$

which in terms of the generators (50) is given by

$$\psi\left(J_1\right)=-J_2\,,\qquad \psi\left(J_2\right)=J_1\,,\qquad \psi\left(J_3\right)=J_3\,.\tag{59}$$

It has the following property on the generators $\{l_{-1},l_0,l_1\}$: $X^{\#}=(\psi(X))^{\dagger}$ (as well as $X^{\dagger}=(\psi(X))^{\#}$), transforming one conjugation into the other[7]. Furthermore, $\psi^2=\phi_{\mathrm{III}}$ and $\phi_{\mathrm{I}}=\psi\circ\phi_{\mathrm{II}}\circ\psi^{-1}$, which finally yields the relation

$$\sqrt{\phi_{\mathrm{I}}}=\psi\circ\sqrt{\phi_{\mathrm{II}}}\circ\psi^{-1}.\tag{60}$$

This is the required isomorphism (see [74] theorem 3, equation (20), p 222) between the real forms (I) and (II) restricted to the subalgebra $\mathrm{span}\{l_{-1},l_0,l_1\}$. The point is that it cannot be extended to the whole Witt algebra since ψ fails to be its automorphism. Therefore, (I) and (II) are not isomorphic as real forms of the algebra (44). We can treat the remaining cases in an analogous manner because the number of candidates for the Witt algebra automorphisms is very limited, cf (45), all transforming l_0 into $\pm l_0$.

Similar techniques can be used to analyze real forms of the Λ-BMS$_3\equiv\mathfrak{W}\oplus\mathfrak{W}$ algebra, as well as BMS$_4$, which is a semidirect product of Λ-BMS$_3$ with supertranslations. In the first case, the group of involutive automorphisms is $\mathbb{Z}_2^{\times5}$, therefore one may expect many non-isomorphic real forms, including the ones considered in the present paper.

Finally, one can notice that all four real forms of the Witt algebra can be extended to the Virasoro algebra provided the conjugation of the central element c will be the same as for the semi-simple element l_0, i.e. c would be hermitian for the forms (II) and (IV) and anti-hermitian for (I) and (III). This is important for exploring unitary representations. Further extensions to Neveu-Schwarz or Ramond supercharges are also possible.

[7] One also finds that $X^*=(\psi(X))^*$ and $X^{\ddagger}=(\psi(X))^{\ddagger}$.

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