Basics of Apparent horizons in black hole physics

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Abstract. [1]Event Horizon, a null hypersurface defining the boundary of the black hole region of a spacetime, is not particularly useful for evolving black holes since it is non-local in time. Instead, one uses the more tangible concept of Apparent Horizon for dynamical black holes out there in the sky that do all sorts of things: evolve, merge and feed on the environment. Event Horizon, being a gauge-independent, global property of the total spacetime is easy to define and locate in the stationary case; on the other hand, Apparent Horizon depends on the embedding of the surface in spacetime and hence it is somewhat tricky to define. But for numerical simulations in General Relativity, locating the Apparent Horizon helps one to excise the black hole region and the singularity to have a stable computation. Moreover, for stationary solutions cross-sections of these horizons match. Here we give a detailed pedagogical exposition of the subject and work out the non-trivial case of a slowly moving and spinning black hole.

1. Introduction

Stationary (Kerr) and static (Schwarzschild) black hole solutions of General Relativity have rather dull lives: stationary ones do the same thing, static ones do nothing as observed by an observer outside the black hole. While these vacuum solutions obtained in an isolated universe serve as our starting point for a more physical and detailed understanding of actual astrophysical black holes, the latter are almost never isolated: the black holes out in the sky have accretion disks, companion stars, neutron stars or black holes. Black holes feed on their environment and grow; in fact they are the most dynamical parts of the vacuum. As the first LIGO/VIRGO gravitational wave detection showed [1], black holes can grow feeding on other black holes: cannibalistic behavior of these objects-highly curved vacua-could explain the existence of intermediate mass black holes.

As the astrophysical black holes evolve, concepts such as the Event Horizon defined easily for eternal black holes are not clearly adequate for us, the transient observers. Recall that the Event Horizon ($\mathcal{H}$) of a stationary black hole is a co-dimension one null hypersurface in the totality of the spacetime defined as the *boundary of the black hole region from which time-like or light-like geodesics cannot reach future null infinity* [2]. Stated in a different way: it is the boundary of the region which is not in the causal past of the future null-infinity. This says that the Event Horizon is a global property of the totality of events which is all of the spacetime. Therefore, one cannot locate the Event Horizon with local experiments in a finite interval of time. In this respect, it is apt to say that the Event Horizon Telescope detected the environment

[1] This paper is written for Prof. Tekin Dereli's 70th Birthday Festschrift.

of the black hole from which one can see at best the cross-section of the Event Horizon, not the Event Horizon itself.

For dynamical black holes one invents the more useful concept of the ‘‘Apparent Horizon” [3], a co-dimension two spatial surface (hence local in time), which, unfortunately, in general does not carry geometric invariant data as the Event Horizon but it contains sufficient information regarding the possible formation of an Event Horizon in the future that it pays to describe it in detail. In numerical relativity computations, detection of a black hole region is best done with Apparent Horizons. Within the context of General Relativity, existence of an Apparent Horizon implies the appearance of a future Event Horizon outside of it. Therefore, one can excise the region inside the Apparent Horizon (that also includes the singularity) for the stability of the computation since nothing will come out of that region in classical physics. For modified gravity theories, an Apparent Horizon need not be inside the Event Horizon (See the discussion and references in [4]).

Our task in this work is to give a detailed definition of the Apparent Horizon and some related concepts and apply it to slowly rotating and moving initial data which was recently given in [5]. The layout of this work is follow: in section II we introduce the necessary tools for the defining equation of an Apparent Horizon as a co-dimension two spatial hypersurface in n dimensions and use the ADM decomposition of the metric to arrive at an equation in local coordinates, in section III we consider a conformally flat initial data for $n = 1 + 3$ dimensions for which the momentum constraints can be solved exactly following the Bowen-York construction [6]; and we solve the Hamiltonian constraint for slowly moving and spinning initial data and compute the properties of the Apparent Horizon. In the Appendix we expound upon some technical points alluded to in the text.

2. Derivation of the apparent horizon equation

As stated above, the Event Horizon of a black hole, as a null hypersurface, cannot be determined locally: one has to know the total spacetime to define it. On the other hand, the Apparent Horizon can be determined locally in time. For this purpose, we need to define a congruence of null geodesics and its expansion. Our notations will be similar to those of the excellent lecture notes [2, 7]. We invite the reader to see [9] for detailed information about the apparent horizons.

As shown in Figure 1, we have an n dimensional spacetime manifold $\mathscr{M}$, with a co-dimension one spacelike hypersurface Σ, that is $\dim\Sigma = n-1$; and we introduce a co-dimension two subspace $\mathcal{S}$, $\dim\mathcal{S} = n-2$. Let n^μ be a timelike unit vector orthogonal to Σ:

$$n^\mu n_\mu = -1, \tag{1}$$

and s^μ be a spacelike unit vector orthogonal to $\mathcal{S}$

$$s^\mu s_\mu = 1. \tag{2}$$

We impose the condition that n and s-vectors are perpendicular to each other

$$n^\mu s_\mu = 0. \tag{3}$$

Instead of these two vectors, one can also work with the ingoing null vector k^μ and the outgoing null vector ℓ^μ, defined respectively as follows (see Figure 1)

$$k^\mu := \frac{1}{2}\left(n^\mu - s^\mu\right), \qquad \ell^\mu := n^\mu + s^\mu. \tag{4}$$

Let g denote the spacetime metric, then the induced metric on the hypersurface Σ is

$$\gamma_{\mu\nu} = g_{\mu\nu} + n_\mu n_\nu, \tag{5}$$

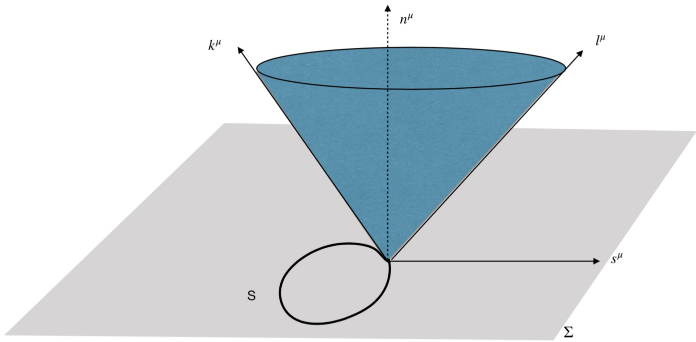

Figure 1. The unit vectors n^μ, s^μ together with the ingoing null vector k^μ and outgoing null vector ℓ^μ are shown.

while the induced metric on the subspace $\mathcal{S}$ reads

$$q_{\mu\nu} = \gamma_{\mu\nu} - s_\mu s_\nu = g_{\mu\nu} + n_\mu n_\nu - s_\mu s_\nu, \tag{6}$$

where μ, ν run over the spacetime directions. The important concept here is the *extrinsic curvature* of both of these surfaces. For the hypersurface Σ, we have

$$K_{\mu\nu} := -\gamma_{\mu\sigma}\gamma_{\nu\rho}\nabla^\sigma n^\rho, \tag{7}$$

where ∇_μ denotes the covariant derivative compatible with the spacetime metric, $\nabla_\mu g_{\nu\rho} = 0$. From a more geometric vantage point, our definition is as follows: given two vectors (X, Y) on the tangent space at the point p, that is $T_p\Sigma$, and n being the unit normal to Σ, then the extrinsic curvature of Σ is defined as $K(X,Y) := -\gamma(\nabla_X n, Y)$. So in local coordinates, one can take $X = \partial_\mu$, $Y = \partial_\nu$ to get $K_{\mu\nu} := K(\partial_\mu, \partial_\nu) = -\gamma(\nabla_{\partial_\mu} n, \partial_\nu)$ which matches (7). The minus sign is a convention. Equivalently, one has

$$K_{\mu\nu} = -\nabla_\mu n_\nu - n_\mu n^\sigma \nabla_\sigma n_\nu. \tag{8}$$

Similarly, we define the extrinsic curvature of the $(n-2)$-dimensional space $\mathcal{S}$ as

$$k_{\mu\nu} := -q_{\mu\sigma} q_{\nu\rho} \nabla^\sigma s^\rho, \tag{9}$$

and using the definition of the induced metric (6) one obtains

$$k_{\mu\nu} = -\nabla_\mu s_\nu - n_\nu n^\sigma \nabla_\mu s_\sigma - n_\mu n^\sigma \nabla_\sigma s_\nu - n_\mu n_\nu n_\sigma n_\rho \nabla^\sigma s^\rho + s_\mu s^\sigma \nabla_\sigma s_\nu + s_\mu n_\nu s_\sigma n_\rho \nabla^\sigma s^\rho. \tag{10}$$

One defines the *expansion* of the out-going null geodesic congruence as

$$\Theta_{(\ell)} := q^{\mu\nu} \nabla_\mu \ell_\nu, \tag{11}$$

which is the divergence of the null geodesic congruence along its propagation in the outgoing null direction. Using (4) and the extrinsic curvatures of the hypersurface and the surface, we can recast the expansion of the null geodesic congruence as

$$\Theta_{(\ell)} = K + k + (n^\mu n^\nu - s^\mu s^\nu)(K_{\mu\nu} + k_{\mu\nu}), \tag{12}$$

where $K = g^{\mu\nu} K_{\mu\nu}$ and $k = g^{\mu\nu} k_{\mu\nu}$. Since $n^\mu n^\nu K_{\mu\nu} = 0 = n^\mu n^\nu k_{\mu\nu} = s^\mu s^\nu k_{\mu\nu}$, $\Theta_{(\ell)}$ reduces to the following neat equation as

$$\Theta_{(\ell)} = -K - k + s^\mu s^\nu K_{\mu\nu}. \tag{13}$$

Equivalently one has

$$\Theta_{(\ell)} = -q^{\mu\nu}\Big(K_{\mu\nu} + k_{\mu\nu}\Big), \tag{14}$$

or

$$\Theta_{(\ell)} = -q^{ij}\Big(K_{ij} + k_{ij}\Big), \tag{15}$$

where the i, j indices run over coordinates on the hypersurface Σ.

The expansion $\Theta_{(\ell)}$ is employed to define the very important concept of a *trapped surface*. An outer trapped surface on Σ is a *closed* (that is compact without a boundary) co-dimension two surface such that for outgoing null geodesics orthogonal to the surface, one has $\Theta_{(\ell)} < 0$ *everywhere* on the surface. The subset of Σ that contains the trapped surfaces is called the *trapped region* $\mathcal{T}$, a co-dimension one surface. Finally, Apparent Horizon is the boundary of the trapped region (an obviously spatial surface) which we shall denote by $\mathcal{S} := \partial\mathcal{T}$. By definition Apparent Horizon is a marginally outer trapped surface (MOTS) and satisfies the Apparent Horizon equation:

$$\Theta_{(\ell)} = -K - k + s^{\mu}s^{\nu}K_{\mu\nu} = 0. \tag{16}$$

It is clear that for the case of time-symmetric initial data ($K_{\mu\nu} = 0$), the Apparent Horizon becomes a minimal surface since $k = 0$.

Now that we have defined the Apparent Horizon, given a metric in some coordinates, to proceed we need to lay out in detail how (16) is expressed in terms of the metric functions. For this purpose we choose the ADM decomposition of the metric [8].

Let $N = N(t, x^i)$ be the lapse function and $N^i = N^i(t, x^j)$ be the shift vector, then the line-element reads

$$ds^2 = (N_iN^i - N^2)dt^2 + 2N_i dt\, dx^i + \gamma_{ij}dx^i\, dx^j, \tag{17}$$

or in components one has

$$g_{00} = N_iN^i - N^2, \qquad g_{0i} = N_i, \qquad g_{ij} = \gamma_{ij}, \tag{18}$$

with the inverses given as

$$g^{00} = -N^{-2}, \qquad g^{0i} = N^iN^{-2}, \qquad g^{ij} = \gamma^{ij} - N^iN^jN^{-2}. \tag{19}$$

Using the definition (5), one has

$$g_{ij} = \gamma_{ij} - n_in_j = \gamma_{ij}, \tag{20}$$

hence $n_i = 0$. Similarly the relation $g^{ij} = \gamma^{ij} - n^in^j = \gamma^{ij} - N^iN^jN^{-2}$ yields $n^i = \pm N^i/N$. Since n^μ is a timelike vector, using $n_i = 0$, one has $n_0 = \pm N$ and choosing for $N > 0$, we choose the plus sign for the future-directed time-like vector to arrive at

$$n^\mu = \left(\frac{1}{N}, -\frac{N^i}{N}\right), \qquad n_\mu = (-N, \vec{0}). \tag{21}$$

We can work out the additional relations between the spacetime metric g and the metric of the hypersurface γ as follows

$$g_{00} = \gamma_{00} - n_0n_0 = N_iN^i - N^2, \tag{22}$$

which yields $\gamma_{00} = N_iN^i$. And similarly

$$g_{0i} = \gamma_{0i} - n_0n_i = N_i \tag{23}$$

yields $\gamma_{0i} = N_i$; and from the inverse metric relations, one obtains $\gamma^{0\mu} = 0$.

Similar computations for the co-dimension 2 spatial subspace $\mathcal{S}$, after using the condition $n_\mu s^\mu = 0$, yield $s^0 = 0$ and $q^{0\mu} = 0$.

Now we go back to (11) and express it for the apparent horizon as

$$q^{ij}\left(\nabla_i n_j + \nabla_i s_j\right) = 0. \tag{24}$$

From (8), one has $K_{ij} = -\nabla_i n_j = -N\Gamma^0_{ij}$ and we obtain[2]

$$q^{ij}\left(K_{ij} - \partial_i s_j + \Gamma^0_{ij} s_0 + \Gamma^k_{ij} s_k\right) = 0. \tag{25}$$

We denote the Christoffel connection of the induced metric γ as ${}^\Sigma\Gamma^k_{ij}$. Then substituting the corresponding components of the Christoffel connection one has

$$\Gamma^k_{ij} = {}^\Sigma\Gamma^k_{ij} + \frac{N^k}{N} K_{ij}, \tag{26}$$

and we arrive at

$$q^{ij}\left(K_{ij} - D_i s_j\right) = 0, \tag{27}$$

where D_i denotes the covariant derivative compatible with the spatial metric γ, $D_i \gamma_{jk} = 0$.

Before we start working out an example, let us note that there is another simple expression of the $\theta_{(\ell)}$ in (11) and hence equation (27). One can show that (see section A of the Appendix for the proof) Lie-dragging the metric on $\mathcal{S}$ along ℓ yields exactly the expansion: namely, one has

$$\theta_{(\ell)} = q^{\mu\nu}\nabla_\mu l_\nu = \frac{1}{2} q^{\mu\nu}\mathcal{L}_\ell q_{\mu\nu}, \tag{28}$$

where $\mathcal{L}_\ell$ denotes the Lie-derivative along the vector ℓ. In section B of the Appendix, $\theta_{(\ell)}$ is derived from the minimization of the area along the outing null direction which also yields a complementary physical picture.

3. Apparent horizon detection

3.1. The equation in explicit form

From now on we shall work in $n = 1 + 3$ dimensions. Assume now that the local coordinates on Σ are denoted as (r, θ, ϕ) and that the location of the Apparent Horizon depends both on θ and ϕ. The equation to be solved is

$$q^{ij}\left(\partial_i s_j -{}^\Sigma \Gamma^k_{ij} s_k - K_{ij}\right) = 0. \tag{29}$$

Assume that the surface $\mathcal{S}$ can be parameterized as a level set such that

$$\Phi(r, \theta, \phi) := r - h(\theta, \phi) = 0, \tag{30}$$

with h being a sufficiently differentiable function of its arguments. Since s^i is normal to the surface, one has $s_i \sim \partial_i \Phi$; and because it is a normal vector one can express $s_i := \lambda \partial_i \Phi$, which yields

$$s_i = \lambda\Big(1, -\partial_\theta h, -\partial_\phi h\Big). \tag{31}$$

To solve the equation defining the apparent horizon together with the constraint equations, let us take the metric on Σ to be conformally flat as in [6]

$$\gamma_{ij} = \psi^4 \begin{pmatrix} 1 & 0 & 0 \\ 0 & r^2 & 0 \\ 0 & 0 & r^2 \sin^2\theta \end{pmatrix}, \tag{32}$$

[2] Note that, explicitly, we have $K_{ij} = \frac{1}{2N}\left(D_i N_j + D_j N_i - \partial_t \gamma_{ij}\right)$.

then one has

$$s^i = \lambda \left(\gamma^{rr}, -\gamma^{\theta\theta}\partial_\theta h, -\gamma^{\phi\phi}\partial_\phi h\right), \tag{33}$$

with the normalization factor given as

$$\lambda = \left(\gamma^{rr} + \gamma^{\theta\theta}(\partial_\theta h)^2 + \gamma^{\phi\phi}(\partial_\phi h)^2\right)^{-1/2}. \tag{34}$$

As should be clear at this stage, the Apparent Horizon equation will be a rather complicated non-linear partial differential equation with little hope to yield an exact analytical solution. Let us further assume (following [6]) $\gamma^{ij}K_{ij} = K = 0$, which is called the maximal slicing gauge. Then (29) reads more explicitly as

$$\gamma^{ij}\partial_i m_j - \gamma^{ij}\Gamma^k_{ij}m_k - \lambda^2 m^i m^j \partial_i m_j + \lambda^2 m^i m^j m_k \Gamma^k_{ij} + \lambda m^i m^j K_{ij} = 0, \tag{35}$$

where we defined $m_i := \partial_i \Phi$. After working out each piece in a somewhat tedious computation, one arrives at

$$\begin{aligned}
&-\gamma^{\theta\theta}\partial^2_\theta h - \gamma^{\phi\phi}\partial^2_\phi h - \frac{1}{2}\Big((\gamma^{rr})^2\partial_r\gamma_{rr} - \gamma^{\theta\theta}\gamma^{rr}\partial_r\gamma_{\theta\theta} - \gamma^{\phi\phi}\gamma^{rr}\partial_r\gamma_{\phi\phi} + \partial_\theta h\gamma^{\phi\phi}\gamma^{\theta\theta}\partial_\theta\gamma_{\phi\phi}\Big)\\
&+\lambda^2\Big((\gamma^{\theta\theta})^2(\partial_\theta h)^2\partial^2_\theta h + (\gamma^{\phi\phi})^2(\partial_\phi h)^2\partial^2_\phi h + 2\gamma^{\phi\phi}\gamma^{\theta\theta}\partial_\phi h\partial_\theta h\partial_\theta\partial_\phi h\Big)\\
&+\frac{\lambda^2}{2}\Big((\gamma^{rr})^3\partial_r\gamma_{rr} + (\gamma^{\theta\theta})^2\gamma^{rr}(\partial_\theta h)^2\partial_r\gamma_{\theta\theta} + (\gamma^{\phi\phi})^2\gamma^{rr}(\partial_\phi h)^2\partial_r\gamma_{\phi\phi}\\
&\qquad - (\partial_\phi h)^2\partial_\theta h(\gamma^{\phi\phi})^2\gamma^{\theta\theta}\partial_\theta\gamma_{\phi\phi}\Big)\\
&+\lambda\Big((\gamma^{rr})^2K_{rr} + (\gamma^{\theta\theta})^2(\partial_\theta h)^2K_{\theta\theta} + (\gamma^{\phi\phi})^2(\partial_\phi h)^2K_{\phi\phi} - 2\gamma^{rr}\gamma^{\theta\theta}\partial_\theta hK_{r\theta}\\
&\qquad - 2\gamma^{rr}\gamma^{\phi\phi}\partial_\phi hK_{r\phi} + 2\gamma^{\theta\theta}\gamma^{\phi\phi}\partial_\theta h\partial_\phi hK_{\theta\phi}\Big) = 0.
\end{aligned} \tag{36}$$

Given the metric γ_{ij} and the extrinsic curvature K_{ij}, one can find numerical solutions of this equation up to the desired accuracy. Our goal here is to find approximate analytical solutions to some physically reasonable initial data which must satisfy the Hamiltonian and the momentum constraints on the hypersurface Σ which we discuss next.

3.2. 1+3 form of Einstein equations

The Hamiltonian and the momentum constraints on the hypersurface Σ follow from Einstein's equations as

$$\begin{aligned}
&-{}^\Sigma R - K^2 + K_{ij}K^{ij} - 2\kappa T_{nn} = 0,\\
&2D_k K^k_i - 2D_i K - 2\kappa T_{ni} = 0.
\end{aligned} \tag{37}$$

We chosen $K = 0$ and consider the vacuum case with $T_{\mu\nu} = 0$. Of course this initial data evolves in time and the remaining parts of the Einstein equations written as a dynamical system are given as

$$\frac{\partial}{\partial t}\gamma_{ij} = -2NK_{ij} + D_iN_j + D_jN_i, \tag{38}$$

$$\frac{\partial}{\partial t}K_{ij} = -N\left(R_{ij} - {}^\Sigma R_{ij} - KK_{ij} + 2K_{ik}K^k_j\right) + \mathscr{L}_{\vec{N}}K_{ij} - D_iD_jN, \tag{39}$$

where $\mathscr{L}_{\vec{N}}$ is the Lie derivative along the shift vector N^i. Derivation of these well-known equations can be found in many textbooks, see our derivation in [10].

3.3. Conformally flat Bowen-York type data

For a conformally flat hypersurface Σ ($\gamma_{\text{ij}} = \psi^4 f_{ij}$ with f being the flat metric in some coordinates), the constraint equations (37) (together with the "maximal slicing" condition $K = 0$) reduce to a non-linear elliptic equation and an easily solvable linear equation, respectively given as

$$\hat{D}_i \hat{D}^i \psi = -\frac{1}{8}\psi^{-7}\hat{K}^2_{ij}, \tag{40}$$

$$\hat{D}^i \hat{K}_{ij} = 0, \tag{41}$$

with $\hat{D}_i f_{jk} = 0$ and $K_{ij} = \psi^{-2}\hat{K}_{ij}$.

Bowen and York [6] gave the following 7-parameter $(p_i, c, \mathcal{J}_i)$ solution to (41) on $\mathbb{R}^3$ whose origin is removed:

$$\begin{aligned}\hat{K}_{ij} = &\frac{3}{2r^2}\Big(p_i n_j + p_j n_i + (n_i n_j - f_{ij}) p \cdot n\Big) + \epsilon\frac{3c^2}{2r^4}\Big(p_i n_j + p_j n_i + (f_{ij} - 5 n_i n_j) p \cdot n\Big) \\ &+ \frac{3}{r^3}\mathcal{J}^l n^k \Big(\varepsilon_{kil} n_j + \varepsilon_{kjl} n_i\Big),\end{aligned} \tag{42}$$

where $r > 0$ is the radial coordinate, n^i is the unit normal on a sphere of radius r (not related to the unit normal to Σ); $\epsilon = \pm 1$ and $p \cdot n = p^k n_k$. At this stage, one should note that the physical meaning of the parameters $(p_i, a, \mathcal{J}_i)$ is not clear; secondly, linearity of (41) means that each bracketed term solves the equation separately. For the sake of simplicity, we shall choose $a = 0$ in what follows.

Here we follow [5]. We shall need the following expression for the right-hand side of (40)

$$\hat{K}_{ij}\hat{K}^{ij} = \frac{9}{2r^4}\left(p^2 + 2(\vec{p}\cdot\vec{n})^2\right) + \frac{18}{r^5}\left(\vec{J}\times\vec{n}\right)\cdot\vec{p} + \frac{18}{r^6}\left(\vec{J}\times\vec{n}\right)\cdot\left(\vec{J}\times\vec{n}\right). \tag{43}$$

Inserting this expression to (40), one arrives at the complicated Hamiltonian constraint which can only be solved exactly after making several assumptions. We shall not go into that discussion which was given in [11] in some detail.

3.4. Conserved quantities

To understand the physical meaning of the parameters in the solution, we shall assume that the spacetime is asymptotically flat, hence the conformal factor behaves as

$$\psi(r) = 1 + \mathcal{O}(1/r), \qquad \text{as } r \to \infty. \tag{44}$$

Then one has the conserved *total momentum* associated to Σ easily written as a boundary integral on a sphere at spatial infinity:

$$P_i = \frac{1}{8\pi}\int_{S^2_\infty} dS\, n^j\, K_{ij} = \frac{1}{8\pi}\int_{S^2_\infty} dS\, n^j\, \hat{K}_{ij}. \tag{45}$$

Observe, from the second equality, that only the leading term in the conformal factor is relevant for this and the following computation. The total conserved *total angular momentum* can also be found easily as

$$J_i = \frac{1}{8\pi}\varepsilon_{ijk}\int_{S^2_\infty} dS\, n_l\, x^j K^{kl} = \frac{1}{8\pi}\varepsilon_{ijk}\int_{S^2_\infty} dS\, n_l\, x^j \hat{K}^{kl}. \tag{46}$$

Given (45) and (46), it is straightforward to compute the integrals for the extrinsic curvature (42) which at the end yield $P_i = p_i$ and $J_i = \mathcal{J}_i$. So for the computation of these two quantities, let us note once again that, the full form of the conformal factor is not needed; one only needs to know its behavior at infinity, that is the $\mathcal{O}(1)$ term.

From these two conserved quantities, one can see that physically the assumed extrinsic curvature (42) belongs to a a self-gravitating system (a curved vacuum) with non-zero momentum and angular momentum. To compute the total mass-energy, the ADM energy, of the system, the $\mathcal{O}(1)$ term of the conformal factor is not sufficient. For that computation we keep the next order term and assume

$$\psi(r) = 1 + \frac{E}{2r} + \mathcal{O}(1/r^2) \qquad \text{as } \; r \to \infty. \tag{47}$$

Then defining the deviation from the background as $h_{ij} := (\psi^4 - 1)\delta_{ij}$, the ADM energy simplifies as

$$E_{ADM} = \frac{1}{16\pi}\int_{S^2_\infty} dS\, n_i \left(\partial_j h^{ij} - \partial_i h^j_j\right) = -\frac{1}{2\pi}\int_{S^2_\infty} dS\, n^i\, \partial_i \psi, \tag{48}$$

whose explicit evaluation for (47) yields $E_{ADM} = E$, which of course at this stage is almost a tautology: we have to find the constant E by solving the Hamiltonian constraint.

3.5. Approximate solution of the Hamiltonian constraint for a boosted slowly rotating gravitating system

To solve the elliptic equation (40) using (43), let us take $\hat{k}$ to be the direction of the conserved angular momentum and choose $\vec{p}$ to be in the xz plane (this is just a choice of the orientation of the coordinates and no generality is lost)

$$\vec{J} = J\hat{k}, \qquad \vec{p} = p\sin\theta_0 \hat{i} + p\cos\theta_0 \hat{k}, \tag{49}$$

with θ_0 a fixed, conserved angle. Then the Hamiltonian constraint (40) becomes

$$\hat{D}_i\hat{D}^i\psi = \psi^{-7}\left(\frac{9Jp}{4r^5}c_1\sin\theta\sin\phi - \frac{9J^2}{4r^6}\sin^2\theta - \frac{9p^2}{16r^4}(1 + 2(c_1\sin\theta\cos\phi + c_2\cos\theta)^2)\right), \tag{50}$$

where $c_1 := \sin\theta_0, c_2 := \cos\theta_0$.

Needless to say, an exact solution of this equation is hopeless, therefore we shall search for the lowest order perturbative solution assuming an expansion in terms of the momentum and spin which corresponds to a curved 3-surface with a small linear and small angular momentum. In [12] the slowly spinning case with no linear momentum was solved in the leading order; and in [13] slowly moving without spin was solved and in [5], both motions were considered at the leading order. We now present this solution.

A cursory inspection on the right-hand side suggests that one should have a double series of the form

$$\psi(r,\theta,\phi) := \psi^{(0)} + J^2\psi^{(J)} + p^2\psi^{(p)} + Jp\psi^{(Jp)} + \mathcal{O}(p^4, J^4, p^2J^2), \tag{51}$$

where the functions on the right-hand side depend on (r,θ,ϕ). At the zeroth order, one has the usual Laplace equation

$$\hat{D}_i\hat{D}^i\psi^{(0)} = 0, \tag{52}$$

which needs boundary conditions to be uniquely solved. The following boundary conditions as employed by [13] are apt for the problem at hand: at spatial infinity one demands

$$\lim_{r\to\infty}\psi(r) = 1, \qquad \psi(r) > 0 \tag{53}$$

and near the origin one has

$$\lim_{r\to\ 0} \psi(r) = \psi^{(0)}, \tag{54}$$

where $\psi^{(0)}$ might have a singularity at the origin. In fact the zeroth order solution satisfying these boundary conditions reads

$$\psi^{(0)} = 1 + \frac{a}{r}. \tag{55}$$

The equations at the next order are

$$\hat{D}_i \hat{D}^i \psi^{(J)} = -\frac{9}{4}\sin^2\theta \frac{r}{(r+a)^7}, \tag{56}$$

$$\hat{D}_i \hat{D}^i \psi^{(Jp)} = \frac{9}{4} c_1 \sin\theta \sin\phi \frac{r^2}{(r+a)^7}, \tag{57}$$

$$\hat{D}_i \hat{D}^i \psi^{(p)} = -\frac{9}{16}\left(1 + 2(c_1 \sin\theta\cos\phi + c_2\cos\theta)^2\right)\frac{r^3}{(r+a)^7}. \tag{58}$$

These are linear equations whose solutions can be found with the help of the following spherical harmonics :

$$Y_0^0(\theta,\phi) = \frac{1}{\sqrt{4\pi}}, \qquad Y_1^0(\theta,\phi) = \sqrt{\frac{3}{4\pi}}\cos\theta, \qquad Y_2^0(\theta,\phi) = \sqrt{\frac{5}{16\pi}}(3\cos^2\theta - 1),$$
$$Y_1^{-1}(\theta,\phi) = \sqrt{\frac{3}{4\pi}}\sin\theta\sin\phi, \quad Y_2^1(\theta,\phi) = \sqrt{\frac{15}{4\pi}}\sin\theta\cos\theta\cos\phi, \quad Y_1^1(\theta,\phi) = \sqrt{\frac{3}{4\pi}}\sin\theta\cos\phi.$$

Then a close inspection of (56) suggests that the proper ansatz for $\psi^{(J)}$ should be of the form

$$\psi^{(J)}(r,\theta,\phi) = \psi_0^{(J)}(r) Y_0^0(\theta,\phi) + \psi_1^{(J)}(r) Y_2^0(\theta,\phi),$$

from which the solution obeying the boundary conditions (53, 54) can be found to be

$$\psi^{(J)}(r,\theta,\phi) = \frac{(a^4 + 5a^3 r + 10a^2 r^2 + 5ar^3 + r^4)}{40a^3(a+r)^5} - \frac{r^2}{40a(a+r)^5}(3\cos^2\theta - 1). \tag{59}$$

To solve (57) one should take

$$\psi^{(Jp)}(r,\theta,\phi) = \psi_0^{(Jp)}(r) Y_0^0(\theta,\phi) + \psi_1^{(Jp)}(r) Y_1^{-1}(\theta,\phi),$$

for which the solution obeying the boundary conditions is

$$\psi^{(Jp)}(r,\theta,\phi) = -\frac{c_1 r\,(a^2 + 5ar + 10r^2)}{80a(a+r)^5}\sin\theta\sin\phi. \tag{60}$$

The $\psi^{(p)}$ equation (58) is similar albeit slightly more complicated: the proper ansatz reads

$$\psi^{(p)} = \psi_0^{(p)}(r) Y_0^0(\theta,\phi) + \psi_1^{(p)}(r) Y_1^1(\theta,\phi)^2 + \psi_2^{(p)}(r) Y_2^1(\theta,\phi) + \psi_3^{(p)}(r) Y_1^0(\theta,\phi)^2,$$

from which four equations follow whose solutions are as follows:

$$\begin{aligned}\psi_0^{(p)}(r) \quad &= \quad -\frac{\sqrt{\pi}\,(84a^6 + 378a^5 r + 653a^4 r^2 + 514a^3 r^3 + 142a^2 r^4 - 35ar^5 - 25r^6)}{80ar^2(a+r)^5} \\ &\quad -\frac{21\sqrt{\pi}a}{20r^3}\log\frac{a}{a+r},\end{aligned} \tag{61}$$

and

$$\psi_1^{(p)}(r) = \frac{\pi c_1^2 \left(84a^5 + 378a^4 r + 658a^3 r^2 + 539a^2 r^3 + 192ar^4 + 15r^5\right)}{40r^2(a+r)^5} + \frac{21\pi a c_1^2}{10r^3} \log \frac{a}{r+a}. \quad (62)$$

$\psi_2^{(p)}(r)$ can be obtained from (62) with the replacement $c_1^2 \to \sqrt{\frac{3}{5\pi}} c_1 c_2$ and $\psi_3^{(p)}(r)$ can be obtained from (62) with the replacement $c_1^2 \to c_2^2$. All these pieces can be combined to get $\psi^{(p)}$ at this stage, but a depiction of the final result is redundant since all the parts are given above and the final expression is cumbersome. We have now all the information at our disposal to compute the relevant quantities defined on Σ including the location of the Apparent Horizon.

First let us revisit the ADM energy computation which we started above: We need the dominant terms up to and including $\mathcal{O}(\frac{1}{r})$ in $\psi(r, \theta, \phi)$. A quick power series expansion yields

$$\psi(r) = 1 + \frac{a}{r} + \frac{J^2}{40a^3 r} + \frac{5p^2}{32ar} + \mathcal{O}(\frac{1}{r^2}), \quad (63)$$

in which the Jp term appears at $\mathcal{O}(\frac{1}{r^2})$ and therefore makes no contribution to the energy. Then from (47), the ADM energy of the solution follows as

$$E_{\mathrm{ADM}} = 2a + \frac{J^2}{20a^3} + \frac{5p^2}{16a}. \quad (64)$$

So one can immediately see that for vanishing spin and vanishing linear momentum (that is the case of the Schwarschild black hole written in the isotropic coordinates) the constant a is related to the mass of the Schwarschild black hole mass as $a = M/2$.

3.6. Apparent Horizion area and the irreducible mass

While studying the efficient processes of extracting energy from rotating black holes, Christodoulou [14] realized[3] that there is an irreducible mass M_{irr} which is related to the area A_{EH} of a *section* of the event horizon via

$$M_{\mathrm{irr}} := \sqrt{\frac{A_{\mathrm{EH}}}{16\pi}}. \quad (65)$$

For a moving, rotating black hole, the total energy was obtained in [14] as

$$E^2 = M_{\mathrm{irr}}^2 + p^2 + \frac{J^2}{4M_{\mathrm{irr}}^2}, \quad (66)$$

in which the physical meaning of each part is clear.

Since we have a dynamical, evolving system, we have at our disposal the area of the Apparent Horizon, not a section of the Event Horizon. But, following [13], a good approximation to M_{irr} can be given with the help of the area of the Apparent Horizon via

$$M_{\mathrm{irr}} := \sqrt{\frac{A_{\mathrm{AH}}}{16\pi}}. \quad (67)$$

[3] Note that after Hawking's area theorem [15] which came later than Christodoulou's observation, it became clear that there must be an irreducible mass at the classical level.

As we shall see, this definition yields the correct expression for the energy of our system obtained from an expansion of (66). But first we need to find the location of the Apparent Horizon, namely solve (36) up to the accuracy we have been working with. That area is given simply as

$$A = \int_0^{2\pi} d\phi \int_0^{\pi} d\theta \sqrt{\det q}, \tag{68}$$

which yields the following *exact* form:

$$A_{\rm AH} = \int_0^{2\pi} d\phi \int_0^{\pi} d\theta \, \sin\theta \, \psi^4 \, h^2 \left(1 + \frac{1}{h^2}\left(\partial_\theta h\right)^2 + \frac{1}{h^2 \sin^2\theta}\left(\partial_\phi h\right)^2\right)^{1/2}. \tag{69}$$

Hence to get the area, all we need is to find the location of the Apparent Horizon up to first order in the spin and momentum. This suggests the following ansatz:

$$h(\theta, \phi) = h^0 + p h^p + J h^J + \mathcal{O}(p^2, J^2, Jp), \tag{70}$$

where

$$\partial_r h = 0, \qquad \partial_r h^0 = 0 = \partial_\theta h^0 = \partial_\phi h^0. \tag{71}$$

Ignoring the terms such as $(\partial_\theta h)^2$, $(\partial_\phi h)^2$ and $\partial_\theta h \partial_\phi h$, (36) reduces to

$$\partial_\theta^2 h + \frac{1}{\sin^2\theta}\partial_\phi^2 h + \cot\theta \partial_\theta h - 2r - 4r^2 \frac{\partial_r \psi}{\psi} + \frac{6J}{\psi^4 r^2}\partial_\phi h - \frac{3p}{\psi^4}\Big(c_1 \sin\theta\cos\phi + c_2\cos\theta\Big) = 0. \tag{72}$$

At the zeroth order, $\mathcal{O}(p^0, J^0)$, it yields

$$1 + 2r\frac{\partial_r \psi}{\psi} = 0, \tag{73}$$

with $\psi = 1 + \frac{a}{r}$; setting $r = h$, one finds

$$h^0 = a. \tag{74}$$

This solution identifies the parameter a as the location of the apparent horizon at the lowest, dominant, order. For example, for the Schwarzschild black hole $h = 2M$ (as noted above) would be the exact solution for which the Event Horizon and the Apparent horizon coincide in these conformally flat, isotropic coordinates.

At $\mathcal{O}(p)$ we have an inhomogeneous, linear Helmholtz equation on a sphere (S^2),

$$\left(\partial_\theta^2 + \frac{1}{\sin^2\theta}\partial_\phi^2 + \cot\theta\partial_\theta - 1\right) h^p = \frac{3}{16}\Big(c_1\sin\theta\cos\phi + c_2\cos\theta\Big), \tag{75}$$

while at $\mathcal{O}(J)$, we have a homogeneous one:

$$\left(\partial_\theta^2 + \frac{1}{\sin^2\theta}\partial_\phi^2 + \cot\theta\partial_\theta - 1\right) h^J = 0. \tag{76}$$

Therefore, we have to find *everywhere* finite solutions of the following equation

$$\left(\vec{\nabla}_{S^2}^2 + k\right) f\left(\theta, \phi\right) = g\left(\theta, \phi\right), \tag{77}$$

where $\vec{\nabla}^2_{S^2}$ is the Laplacian on S^2 given as

$$\vec{\nabla}^2_{S^2} := \partial^2_\theta + \cot\theta\partial_\theta + \frac{1}{\sin^2\theta}\partial^2_\phi. \tag{78}$$

One can employ the Green's function technique to solve this problem. For the Helmholtz operator on the sphere, the Green function $G(\hat{x}, \hat{x}')$ is defined as

$$\left(\vec{\nabla}^2_{S^2} + \kappa(\kappa+1)\right) G(\hat{x}, \hat{x}') = \delta^{(2)}(\hat{x} - \hat{x}'), \tag{79}$$

which can be found as an infinite series expansion (for example, see [16])

$$G(\hat{x}, \hat{x}') = \frac{1}{4\sin\pi\kappa}\sum_{n=0}^{\infty}\frac{1}{(n!)^2}\frac{\Gamma(n-\kappa)}{\Gamma(-\kappa)}\frac{\Gamma(n+\kappa+1)}{\Gamma(\kappa+1)}\left(\frac{1+\hat{x}\cdot\hat{x'}}{2}\right)^n, \tag{80}$$

where $\hat{x} = \sin\theta\cos\phi\hat{i} + \sin\theta\sin\phi\hat{j} + \cos\theta\hat{k}$ and $\hat{x}'$ is a similar expression with some other θ and ϕ. Employing this Green's function with $\kappa = \frac{-1+i\sqrt{3}}{2}$, one finds the first non-trivial correction to the location of the Apparent Horizon as

$$h^p = -\frac{1}{16}\left(c_1\sin\theta\cos\phi + c_2\cos\theta\right), \tag{81}$$

and $h^J = 0$. Therefore the apparent horizon is perturbed from the zeroth order expansion to

$$r = h(\theta, \phi) = a - \frac{p}{16}\Big(\sin\theta_0\sin\theta\cos\phi + \cos\theta_0\cos\theta\Big), \tag{82}$$

where, recall that, θ_0 is the angle between the linear momentum and the spin vectors. So the magnitude of the spin vector is irrelevant at this order for the location of the Apparent Horizon, but the angle it makes with the momentum vector is relevant. In Figure 2, we plotted and example of how the shape of the horizon looks like. There is a dimple on the sphere whose size depends on the ratio p/a which we took to be large to see the dimple. In the limit $\theta_0 = 0$, h reduces to the form given in [13].

Let us now evaluate the area of the Apparent Horizon from (69) which at the end yields

$$A_{\text{AH}} = 64\pi a^2 + 4\pi p^2 + \frac{11\pi J^2}{5a^2}. \tag{83}$$

Thus the irreducible mass M_{irr} turns out to be

$$M_{\text{irr}} = 2a + \frac{p^2}{16a} + \frac{11J^2}{320a^3}, \tag{84}$$

so comparing with the energy, E_{ADM}, we have

$$E_{\text{ADM}} = M_{\text{irr}} + \frac{p^2}{2M_{\text{irr}}} + \frac{J^2}{8M^3_{\text{irr}}}, \tag{85}$$

which matches the result (66) of Christodoulou at this order.

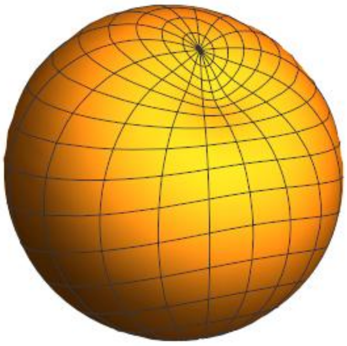

Figure 2. Shape of the Apparent Horizon when the angle between $\vec{p}$ and $\vec{J}$ is 45 degrees; and $p/a = 8\sqrt{2}$ which is outside the validity of the approximation we have worked with.

4. Conclusions

We have presented a step-by-step construction of the Apparent Horizon equation which is of extreme importance in black hole physics; and described in detail how it correctly yields the expected results, such as the irreducible mass, for a slowly moving and spinning black hole. For stationary black holes cross-sections of the event Horizon and the Apparent Horizon coincide. This exposition is of a pedagogical nature with details given in the Appendix including the derivation of the null Raychaudhuri equation which we have not used in the text, but added for more insight for the expansion of a null geodesic. We have skipped some interesting issues such as: numerically solving the case with no symmetry; multi black hole initial data; the proof that when the dominant energy condition is satisfied, the topology of the Apparent Horizon is that of S^2. For other nice expositions regarding horizons and related concepts see [7, 17, 18].

Acknowledgments

The work of E.A. is partially supported by the TUBITAK Grant No. 120F253. This work was written to celebrate the 70th birthday of Tekin Dereli who has been extremely influential in gravity research in Turkey.

Appendix A. An equivalent definition of the expansion $\Theta_{(l)}$

Here we give a proof of the second equality in (28): we have

$$\Theta_{(\ell)} = q^{\mu\nu}\nabla_\mu \ell_\nu = \frac{1}{2} q^{\mu\nu}\mathcal{L}_\ell q_{\mu\nu}, \tag{A.1}$$

where the first equality is identical to the definition of the $\Theta_{(\ell)}$. Starting from $q^{\mu\nu}\mathcal{L}_\ell q_{\mu\nu}$, one can easily arrive at the expansion $\Theta_{(\ell)}$. The construction is as follows:

$$q^{\mu\nu}\mathcal{L}_\ell q_{\mu\nu} = q^{\mu\nu}\left(\ell^\sigma \nabla_\sigma q_{\mu\nu} + q_{\sigma\nu}\nabla_\mu \ell^\sigma + q_{\sigma\mu}\nabla_\nu \ell^\sigma\right). \tag{A.2}$$

The first term on the right hand side automatically vanishes. To be able to see this explicitly, we express the metric $q_{\mu\nu}$ in terms of the spacetime metric $g_{\mu\nu}$

$$q^{\mu\nu}\ell^\sigma\nabla_\sigma q_{\mu\nu} = q^{\mu\nu}\ell^\sigma\nabla_\sigma\left(g_{\mu\nu} + k_\mu \ell_\nu + k_\nu \ell_\mu\right). \tag{A.3}$$

Since $\nabla_\sigma g_{\mu\nu} = 0$, the non vanishing terms are

$$q^{\mu\nu}\ell^\sigma\nabla_\sigma q_{\mu\nu} = q^{\mu\nu}\ell^\sigma\left(k_\mu\nabla_\sigma l_\nu + \ell_\nu\nabla_\sigma k_\mu + k_\nu\nabla_\sigma\ell_\mu + \ell_\mu\nabla_\sigma k_\nu\right), \tag{A.4}$$

where $q^{\mu\nu}k_\mu = 0 = q^{\mu\nu}\ell_\mu$, and so one gets

$$q^{\mu\nu}l^\sigma\nabla_\sigma q_{\mu\nu} = 0. \tag{A.5}$$

Now let us evaluate the second and third terms in (A.2) (which contribute equally). We can write

$$q^{\mu\nu}q_{\sigma\nu}\nabla_\mu\ell^\sigma = q^{\mu\nu}\left(g_{\sigma\nu} + k_\sigma\ell_\nu + k_\nu\ell_\sigma\right)\nabla_\mu\ell^\sigma. \tag{A.6}$$

Using $q^{\mu\nu}k_\mu = 0 = q^{\mu\nu}\ell_\mu$ again, the last expression reduces to the following

$$q^{\mu\nu}q_{\sigma\nu}\nabla_\mu\ell^\sigma = q^{\mu\nu}g_{\sigma\nu}\nabla_\mu\ell^\sigma = q^{\mu\nu}\nabla_\mu\ell_\nu. \tag{A.7}$$

Then (A.2) becomes

$$q^{\mu\nu}\mathcal{L}_\ell q_{\mu\nu} = 2q^{\mu\nu}\nabla_\mu\ell_\nu, \tag{A.8}$$

and one ends up with

$$\frac{1}{2}q^{\mu\nu}\mathcal{L}_\ell q_{\mu\nu} = q^{\mu\nu}\nabla_\mu\ell_\nu = \Theta_{(\ell)}, \tag{A.9}$$

which is the expression we wanted to prove.

Appendix B. Derivative of the area along the null vector field ℓ^μ

To gain a better physical insight to the expansion of the null geodesic congruence, let us show that when one takes the derivative of the area of the cross section along the null vector field ℓ^μ, the expansion $\Theta_{(\ell)}$ can be directly obtained as the integrand. On the surface $\mathcal{S}$, let us start with the area formula

$$A = \int dS\sqrt{q}, \tag{B.1}$$

of which the derivative along ℓ yields

$$\ell^\mu\partial_\mu A = \int dS\,\ell^\mu\partial_\mu\sqrt{q}, = \frac{1}{2}\int dS\,\sqrt{q}\,\ell^\mu q^{ab}\partial_\mu q_{ab}. \tag{B.2}$$

Equivalently one can express the result in terms of the Lie derivative along the vector field ℓ^μ using

$$q^{ab}\mathcal{L}_\ell q_{ab} = q^{ab}\ell^\mu\partial_\mu q_{ab} + 2q^{ab}q_{\mu b}\partial_a\ell^\mu. \tag{B.3}$$

Since the null vectors ℓ^μ and k^μ are the elements of the compliment of the subspace $\mathcal{S}$, one has $k^a = 0 = \ell^a$. By definition (6) we obtain

$$q^a_\mu = \delta^a_\mu + k_\mu\ell^a + k^a\ell_\mu = \delta^a_\mu, \tag{B.4}$$

and similarly

$$q^{a\mu} = g^{a\mu} + k^\mu\ell^a + k^a\ell^\mu = g^{a\mu}. \tag{B.5}$$

So that $q^{ab}q_{\mu b} = \delta^a_\mu$; and the last term in (B.3) becomes

$$q^{ab}q_{\mu b}\partial_a\ell^\mu = \delta^a_\mu\partial_a\ell^\mu = \partial_a\ell^a = 0. \tag{B.6}$$

Then $q^{ab}\mathcal{L}_\ell q_{ab}$ reduces to

$$q^{ab}\mathcal{L}_\ell q_{ab} = q^{ab}\ell^\mu\partial_\mu q_{ab}, \tag{B.7}$$

which can be related to $\ell^\mu \partial_\mu \sqrt{q}$ via

$$\ell^\mu \partial_\mu \sqrt{q} = \frac{1}{2} \sqrt{q}\, q^{ab} \mathcal{L}_\ell q_{ab}. \tag{B.8}$$

Now we can rewrite (B.2) in terms of Lie derivative

$$\ell^\mu \partial_\mu A = \int dS\, \sqrt{q}\, \frac{1}{2}\, q^{ab} \mathcal{L}_\ell q_{ab}. \tag{B.9}$$

In order to show the appearance of the expansion $\Theta_{(\ell)}$ explicitly, we should use the spacetime coordinates, recall that we have $\Theta_{(l)} = q^{\mu\nu} \mathcal{L}_\ell q_{\mu\nu}/2$, instead of the coordinates on the co-dimension two surface $\mathcal{S}$. It is straightforward to write

$$q^a_\mu q^b_v \mathcal{L}_\ell q_{ab} = \delta^a_\mu \delta^b_v \mathcal{L}_\ell q_{ab} = \mathcal{L}_\ell q_{\mu\nu}. \tag{B.10}$$

Multiplying this with $q^\mu_\sigma q^\nu_\rho$ one obtains

$$q^\mu_\sigma q^\nu_\rho q^a_\mu q^b_v \mathcal{L}_\ell q_{ab} = q^\mu_\sigma q^\nu_\rho \mathcal{L}_\ell q_{\mu\nu}, \tag{B.11}$$

which in terms of Kronecker delta functions reads

$$q^\mu_\sigma q^\nu_\rho \delta^a_\mu \delta^b_v \mathcal{L}_\ell q_{ab} = q^\mu_\sigma q^\nu_\rho \mathcal{L}_\ell q_{\mu\nu}, \tag{B.12}$$

and yields

$$q^a_\sigma q^b_\rho \mathcal{L}_\ell q_{ab} = q^\mu_\sigma q^\nu_\rho \mathcal{L}_\ell q_{\mu\nu}. \tag{B.13}$$

We multiply the last identity with $q^{\sigma\rho}$. Then we find the identity

$$q^{\sigma\rho} \delta^a_\sigma \delta^b_\rho \mathcal{L}_\ell q_{ab} = q^{\sigma\rho} q^\mu_\sigma q^\nu_\rho \mathcal{L}_\ell q_{\mu\nu}, \tag{B.14}$$

and so one arrives at

$$q^{ab} \mathcal{L}_\ell q_{ab} = q^{\sigma\rho} q^\mu_\sigma q^\nu_\rho \mathcal{L}_\ell q_{\mu\nu}, \tag{B.15}$$

where $q^{\sigma\rho} q^\mu_\sigma q^\nu_\rho = q^{\mu\nu}$. Finally we end up with

$$q^{ab} \mathcal{L}_\ell q_{ab} = q^{\mu\nu} \mathcal{L}_\ell q_{\mu\nu}. \tag{B.16}$$

This proves that the expansion $\Theta_{(\ell)}$ appears in the change of the area along the vector field ℓ^μ. The final expression is therefore

$$\ell^\mu \partial_\mu A = \int dS\, \sqrt{q}\, \frac{1}{2}\, q^{\mu\nu} \mathcal{L}_\ell q_{\mu\nu} = \int dS\, \sqrt{q}\, \frac{1}{2}\, q^{ab} \mathcal{L}_\ell q_{ab} = \int dS\, \sqrt{q}\, \Theta_{(l)}. \tag{B.17}$$

So setting $\Theta_{(\ell)} = 0$ to define the Apparent Horizon boils down to setting $\ell^\mu \partial_\mu A = 0$.

Appendix C. Null Raychaudhuri equation

The form of (28) already suggests that one can define a tensor whose trace is the expansion. Here we explore this tensor and obtain an expression for the change of the null expansion along the null direction ℓ as well as the null Raychaudhuri equation [2]. So let us introduce the *deformation tensor* $\Theta_{\mu\nu}$ as

$$\Theta_{\mu\nu} := \frac{1}{2} q^\sigma_\mu q^\rho_\nu \mathcal{L}_\ell q_{\sigma\rho}, \tag{C.1}$$

such that

$$\Theta_{(\ell)} = g^{\mu\nu}\Theta_{\mu\nu} = q^{\mu\nu}\nabla_\mu \ell_\nu. \tag{C.2}$$

Carrying out the Lie-derivative in (C.1), one has

$$\Theta_{\mu\nu} = \nabla_\mu \ell_\nu - \omega_\mu \ell_\nu + \ell_\mu k^\sigma \nabla_\sigma \ell_\nu, \tag{C.3}$$

with

$$\omega_\mu := -k^\sigma \nabla_\mu \ell_\sigma - k^\sigma k^\rho \ell_\mu \nabla_\sigma \ell_\rho \tag{C.4}$$

which is called the rotation one form.

In what follows, we will make use of the Ricci identity

$$\nabla_\mu \nabla_\nu \ell^\mu - \nabla_\nu \nabla_\mu \ell^\mu = R_{\nu\lambda}\ell^\lambda. \tag{C.5}$$

From (C.2), one has

$$\Theta_{(\ell)} = \nabla_\mu \ell^\mu + k^\nu \ell^\mu \nabla_\mu \ell_\nu. \tag{C.6}$$

Here we assume that ℓ is a geodesic null vector but not necessarily affinely parameterized so that

$$\ell^\mu \nabla_\mu \ell_\nu = \kappa \ell_\nu, \tag{C.7}$$

where κ is a function on spacetime. Using $k^\nu \ell_\nu = -1$, one has

$$\Theta_{(l)} = \nabla_\mu \ell^\mu - \kappa. \tag{C.8}$$

So we have the following two equations:

$$\nabla_\mu \ell^\mu = \Theta_{(\ell)} + \kappa, \qquad \nabla_\mu \ell_\nu = \Theta_{\mu\nu} + \omega_\mu \ell_\nu - \ell_\mu k^\sigma \nabla_\sigma \ell_\nu. \tag{C.9}$$

Substituting these in (C.5), one has

$$\nabla_\mu \left(\Theta_\nu{}^\mu + \omega_\nu \ell^\mu - \ell_\nu k^\sigma \nabla_\sigma \ell^\mu\right) - \nabla_\nu (\Theta_{(\ell)} + \kappa) = R_{\nu\lambda}\ell^\lambda, \tag{C.10}$$

which more explicitly becomes

$$\nabla_\mu \Theta_\nu{}^\mu + \ell^\mu \nabla_\mu \omega_\nu + \omega_\nu \nabla_\mu \ell^\mu - k^\sigma \nabla_\sigma \ell^\mu \nabla_\mu \ell_\nu - \ell_\nu \nabla_\mu (k^\sigma \nabla_\sigma \ell^\mu) - \nabla_\nu (\Theta_{(\ell)} + \kappa) = R_{\nu\lambda}\ell^\lambda. \tag{C.11}$$

We use the expressions (C.9) one more time and reexpress the third and the fourth terms to obtain

$$\begin{aligned} &\nabla_\mu \Theta_\nu{}^\mu + \ell^\mu \nabla_\mu \omega_\nu + \omega_\nu (\Theta_{(\ell)} + \kappa) - k^\sigma \nabla_\sigma \ell^\mu \left(\Theta_{\mu\nu} + \omega_\mu \ell_\nu - \ell_\mu k^\gamma \nabla_\gamma \ell_\nu\right) \\ &\qquad -\ell_\nu \nabla_\mu (k^\sigma \nabla_\sigma \ell^\mu) - \nabla_\nu (\Theta_{(l)} + \kappa) = R_{\nu\lambda}\ell^\lambda. \end{aligned} \tag{C.12}$$

Since $\ell_\mu \nabla_\sigma \ell^\mu = 0$, the last term on the first line automatically vanishes. Contracting the final expression with ℓ^ν and using the fact that it is a null vector, one arrives at

$$\ell^\nu \nabla_\mu \Theta_\nu{}^\mu + \ell^\nu \ell^\mu \nabla_\mu \omega_\nu + \ell^\nu \omega_\nu (\Theta_{(\ell)} + \kappa) - \ell^\nu \Theta_{\mu\nu} k^\sigma \nabla_\sigma \ell^\mu - \ell^\nu \nabla_\nu (\Theta_{(\ell)} + \kappa) = R_{\nu\lambda}\ell^\lambda \ell^\nu. \tag{C.13}$$

It is easy to show that the contraction of the null vector ℓ^ν and the deformation tensor identically vanishes, $\ell^\nu \Theta_\nu{}^\mu = 0$. Then one has

$$\ell^\nu \nabla_\mu \Theta_\nu{}^\mu = -\Theta_{\mu\nu}\Theta^{\mu\nu}, \tag{C.14}$$

and also

$$\ell^\nu \ell^\mu \nabla_\mu \omega_\nu = \ell^\mu \nabla_\mu \kappa - \kappa^2. \tag{C.15}$$

Inserting these expressions in (C.13) we get

$$-\Theta_{\mu\nu}\Theta^{\mu\nu} + \kappa\Theta_{(l)} - \ell^\nu \nabla_\nu \Theta_{(l)} = R_{\nu\lambda}\ell^\lambda \ell^\nu. \tag{C.16}$$

The first term $\Theta_{\mu\nu}\Theta^{\mu\nu}$ can be written in terms of the trace free shear tensor $\sigma_{\mu\nu}$

$$\sigma_{\mu\nu} := \Theta_{\mu\nu} - \frac{1}{n-2} q_{\mu\nu}\Theta_{(\ell)} \tag{C.17}$$

as follows

$$\Theta_{\mu\nu}\Theta^{\mu\nu} = \sigma_{\mu\nu}\sigma^{\mu\nu} + \frac{1}{n-2}\Theta^2_{(\ell)}, \tag{C.18}$$

where $\sigma_{\mu\nu}\sigma^{\mu\nu} = \sigma_{ab}\sigma^{ab}$. Then (C.16) becomes

$$\ell^\mu \nabla_\mu \Theta_{(\ell)} = \nabla_\ell \Theta_{(\ell)} = \kappa\Theta_{(l)} - R_{\mu\nu}\ell^\mu \ell^\nu - \sigma_{ab}\sigma^{ab} - \frac{1}{n-2}\Theta^2_{(\ell)}. \tag{C.19}$$

The null vector field ℓ^μ is oriented in the future direction. Therefore the last equation, known as the null Raychaudhuri equation, is an evolution equation for the expansion $\Theta_{(\ell)}$.

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