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FROM HADRON GAS TO QUARK MATTER 1 *)

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A B S T R A C T

An analytical, non-perturbative description of a strongly interacting hadron gas is presented. Its main features are: the formulation is relativistically covariant, hadrons have finite extensions which are treated à la Van der Waals and their strong interactions are simulated by a hadronic mass spectrum generated by a bootstrap equation under the constraints of baryon number conservation. The system exhibits a singularity, which has the typical features of a phase transition gas → liquid, but which we interpret here as the transition into a quark-gluon plasma phase, which, however, cannot be described by this model. (In Part 2, a quark-gluon plasma model will be sketched and matched to the bootstrap model. The joint models are then applied to heavy ion collisions).

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1. INTRODUCTION

We wish to describe the thermodynamic properties of a hadron gas with its strong interactions leading to a phase transition - the "dissolution" of individual hadrons in a weakly interacting quark-gluon plasma. This new phase seems to be the most obvious continuation of the known hadron phase.

In Part 1 we present the hadronic aspects[1], in Part 2 below the quark-gluon aspects and the nature of the transition between these two, as well as some phenomenological predictions for relativistic heavy ion collisions[2].

For the description of the strongly interacting hadron gas we require:

- Lorentz covariant formulation of thermodynamics;
- conservation of the total four momentum of the system;
- conservation of the baryon number;
- kinetic and "chemical" equilibrium between all constituents (pions, nucleons and antinucleons as well as all their resonances and bound states);
- a finite "natural volume" for each constituent to be used à la Van der Waals. This natural volume is the volume due to internal dynamics in the absence of external forces.

2. NOTATION

$\hbar = c = k = 1$, mass units MeV, GeV; energy, inverse temperature and volume are generalized to four vectors[3]:

$$
\begin{aligned}
E &\rightarrow p^\mu = (p^0, \vec{p}) = m u^\mu ; \quad u_\mu u^\mu = 1 \\
\frac{1}{T} &\rightarrow \beta^\mu = (\beta^0, \vec{\beta}) = \frac{1}{T} v^\mu ; \quad v_\mu v^\mu = 1 \\
V &\rightarrow V^\mu = (V_0, \vec{V}) = V w^\mu ; \quad w_\mu w^\mu = 1
\end{aligned}
\tag{1}
$$

where u^μ, v^μ, w^μ are the four velocities of the total mass, of the thermometer and of the volume, respectively. Usually $\langle u^\mu \rangle = v^\mu = w^\mu$.

3. THERMODYNAMICS

The usual level density of a system enclosed in a volume V and having energy E and baryon number b becomes in covariant notation[3)]

$$\sigma(E,V,b)\,dE \longrightarrow \sigma(p,V,b)\,d^4p \tag{2}$$

Given $\sigma(p,V,b)$, we can calculate the grand canonical partition function:

$$Z(T,V,\lambda) := \sum_{b=-\infty}^{\infty} \lambda^b \int \sigma(p,V,b)\, e^{-\beta_\mu p^\mu} d^4p \tag{3}$$

where λ is the fugacity for baryon conservation.

In the present model, four momentum and baryon number are the only conserved quantities; further conservation laws, including non-Abelian ones[4)], can be incorporated.

From $\ell n Z$ all relevant thermodynamic quantities can be found as usual by differentiation. Thus the theoretical problem is to find $\sigma(p,V,b)$.

4. THE DENSITY OF STATES $\sigma(p,V,b)$

We postulate the following ansatz:

$$\sigma(p,V,b) = \sum_{N=0}^{\infty} \frac{1}{N!} \int \delta^4\Big(p-\sum_{i=1}^{N} p_i\Big) \sum_{\{b_i\}} \delta_K\Big(b-\sum_{i=1}^{N} b_i\Big) \times$$
$$\times \prod_{i=1}^{N} \frac{2\Delta_\mu p^\mu}{(2\pi)^3}\, \tau(p_i^2, b_i)\, d^4p_i \tag{4}$$

In this expression the complete set of contributing states is subdivided into any number N of subsets corresponding to any partition of the total four momentum p and the total baryon number b. These subsets - the constituents, called clusters from now on - have an internal density of states $\tau(p^2,b)$ (mass spectrum). The following natural properties are incorporated in the above density of states:

i) four momentum conservation $[\delta^4(p - \Sigma p_i)]$;

ii) baryon number conservation $[\delta_K(b - \Sigma b_i)]$;

iii) unlimited (as far as is allowed by i) and ii)) creation and absorption of particles (sum over N);

iv) kinetic and "chemical" equilibrium between all possible constituents (pions, nucleons and their clusters) which are counted in the mass spectrum $\tau(p^2,b)$;

v) a Van der Waals treatment of the volume: $\Delta = V - \Sigma V_i$ is the "available volume" after subtracting from V the natural volumes of the constituents (clusters).

Comments:

- points iii) and iv) represent the interaction: if $\tau(m^2,b)$ contains all participating elementary particles (here π,N) and all their resonances and bound states, then the interaction is perfectly taken into account (as far as thermodynamics goes and apart from long-range and short-range, strongly repulsive forces). For details on this crucial point, see Refs. 5 - 8).

- point v) says that the volume is reduced à la Van der Waals; the usual factor 4 in front of ΣV_i is left out, since our particles, the clusters, are considered incompressible but deformable and having natural cluster volumes V_i, which, in a summary way, represent short-range repulsive forces. If V is given, then

$$\Delta = V - \sum_{i=1}^{N} V_i$$

will be a function of V and N. We shall instead consider Δ as the free parameter and then the external, total volume is a function of Δ and N, so that in the grand canonical ensemble it will become an expectation value $< V >$.

- equation (4) states: the density of states of extended particles in the volume V is identical to that of point particles in the available volume Δ.

- the integration measure reduces in the rest frame of Δ to the usual one:

$$\frac{2\Delta_\mu p^\mu}{(2\pi)^3}\tau(p^2,b)\,d^4p \longrightarrow \frac{\Delta_R}{(2\pi)^3}\tau(m^2,b)\,dm^2 d^3p \tag{5}$$

- as we shall see that $\tau(m^2,b)$ grows exponentially, Bose-Einstein and Fermi-Dirac statistics can be neglected (for temperatures $\gtrsim$ 50 MeV).

5. BACK TO THERMODYNAMICS

As explained, Eq. (4) implies:

$$\sigma(p,V,b) \equiv \sigma_{pt}(p,\Delta,b) \tag{6}$$

where the subscript pt denotes "point particles". The double Laplace transform (3) of this density obeys therefore:

$$Z(T,\langle V\rangle,\lambda) \equiv Z_{pt}(T,\Delta,\lambda) \tag{7}$$

which permits us to calculate everything for fictitious point particles in Δ and afterwards obtain the correct quantities by eliminating Δ in favour of $\langle V \rangle$.

Assume now $\tau(p^2,b)$ and therefore $\sigma(p,V,b)$ to be known. Then

$$Z_{pt}(T,\Delta,\lambda) = \sum_{N=0}^{\infty}\frac{1}{N!}\int \delta^4\Big(p-\sum_{i=1}^{N}p_i\Big)\,e^{-\beta\cdot p}\,d^4p \;\times$$
$$\times \sum_{b=-\infty}^{\infty}\lambda^b \sum_{\{b_i\}}\delta_K\Big(b-\sum_{i=1}^{N}b_i\Big)\prod_{i=1}^{N}\frac{2\Delta\cdot p_i}{(2\pi)^3}\tau(p_i^2,b_i)\,d^4p_i \tag{7}$$

The δ functions permit us to do the d^4p integration and λ summation; the integrand thereafter splits into N independent, identical integrals and the sum yields an exponential function; thus, taking its logarithm:

$$\ln Z_{pt}(T,\Delta,\lambda) = \ln Z(T,\langle V\rangle,\lambda) =: Z_1(T,\Delta,\lambda)$$

$$Z_1(T,\Delta,\lambda) := \int \frac{2\Delta_\mu p^\mu}{(2\pi)^3}\,\tau(p^2,\lambda)\,e^{-\beta_\mu p^\mu}\,d^4p$$

where (8)

$$\tau(p^2,\lambda) := \sum_{b=-\infty}^{\infty} \lambda^b\,\tau(p^2,b)$$

All the interaction is contained in $\tau(p^2,\lambda)$. If it were a simple $\delta_0(p^2 - m^2)$, Z_1 would be the usual "one-particle partition function" of the ideal gas; here it is the "one-cluster partition function".

Note that the simple result (8) is due to keeping Δ fixed as external parameter; there is always room for more particles, the volume V grows with N and N can go to ∞ ; had we instead considered V as external parameter, the sum over N would have had to break off when the box V was full and we would not have obtained an exponential function.

Now we face two questions:

- what is $\tau(p^2,\lambda)$?
- what is the relation $\Delta \leftrightarrows \langle V\rangle$?

Both will be answered by our dynamical hypothesis: statistical bootstrap.

6. STATISTICAL BOOTSTRAP

The idea is rather old[9)] and has undergone some development making it clearer, more consistent and, perhaps, more convincing. For details the reader is referred to Ref. 7) and in particular to Ref. 8) and the references therein.

The basic postulate of statistical bootstrap is that the mass spectrum $\tau(m^2,b)$, containing all the "particles": elementary, bound states and resonances (clusters) is generated by the same interactions which we see at work if we consider our thermodynamical system. Therefore if we were to compress this system until it reaches its natural volume $V_c(m,b)$, then it would itself be almost a cluster appearing in the mass spectrum $\tau(m^2,b)$. Since $\sigma(p,\Delta,b)$ and $\tau(p^2,b)$ are both densities of states (with respect to different measures: d^4p and dm^2) we postulate

$$\sigma(p,\Delta,b)\Big|_{\langle V\rangle \to V_c(m,b)} \;\hat{=}\; \tau(p^2,b) \tag{9}$$

where $\hat{=}$ means "corresponds to" (in some way to be specified). As $\sigma(p,\Delta,b)$ is [see (4)] the sum over N of N - fold convolutions of τ, the above "bootstrap postulate" will yield a highly non-linear integral equation for τ.

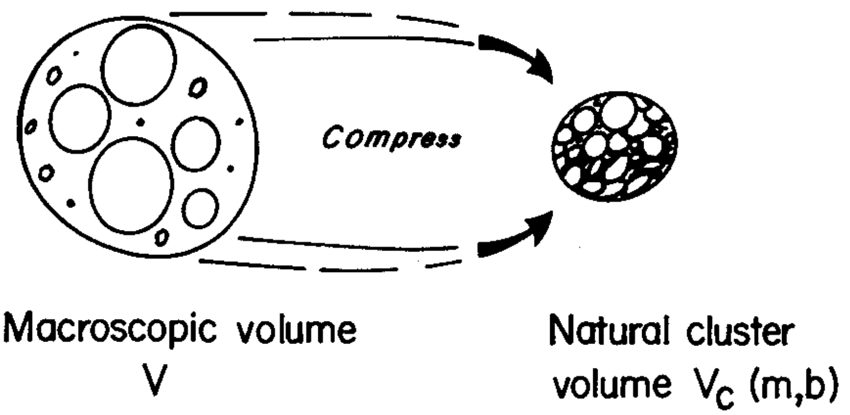

Fig.1 : The bootstrap idea: a macroscopic system compressed to the "natural cluster volume" becomes itself almost a cluster consisting of clusters.

There is, however, one important difference between the macroscopic system - even if compressed to the natural cluster volume - and the clusters making up the system: the macroscopic system is enclosed in a fixed, externally given volume, while the cluster chooses its own natural cluster volume and carries it with it; the natural cluster volume is therefore a four vector V_c^μ parallel to the cluster's four momentum

$$V_c^\mu(m,b) = A(m,b)\cdot\frac{p^\mu}{m} \quad ; \quad m = \sqrt{p_\mu p^\mu} \tag{10}$$

where the scalar function A(m,b) depends on the dynamics: it expresses how the cluster chooses its volume as a function of its mass and baryon number. As the bootstrap is to represent the dynamics, it not only should determine $\tau(p^2,b)$ but also A(m,b). We settle the last question first by requiring, in addition to the bootstrap postulate (9), the postulate of "uniform packing": for any partition of a cluster into N subclusters, we postulate

$$V_c^\mu(m,b) = \sum_{i=1}^{N} K\, V_c^\mu(m_i,b_i) \tag{11}$$

where K is some constant; $K = 1$ is dense packing, $K < 1$ superdense, $K > 1$ dilute. Together with (10) this yields:

$$A(m,b)\frac{p^\mu}{m} = \sum_{i=1}^{N} K\, A(m_i,b_i)\frac{p_i^\mu}{m_i} \tag{12}$$

and since for any partition $p = \sum_{i=1}^{N} p_i$,

$$\sum_{i=1}^{N}\left\{\frac{A(m,b)}{m} - K\,\frac{A(m_i,b_i)}{m_i}\right\} p_i^\mu = 0 \qquad \left[\forall\, N, p_i^\mu\right] \tag{13}$$

whence $K = 1$ and $A(m,b)/m$ = const (independent of b). Thus we obtain dense packing and a volume proportional to the mass:

$$V_c^\mu(m,b) = \mathrm{Const}\cdot p^\mu =: \frac{p^\mu}{4B} \xrightarrow[\text{rest frame}]{} \frac{m}{4B} \tag{14}$$

by which the free parameter B is defined; $4B$ is the constant energy density of all clusters, which cannot be found from the bootstrap hypothesis; it has to be fixed via outside information. Tentatively we identify B with the quark bag constant[10] $B = (145\ \mathrm{MeV})^4$, thereby interpreting at the same time our clusters as quark-gluon bags.

The bootstrap postulate (9) requires that τ should obey the equation resulting from replacing σ in Eq. (4) by some expression containing τ linearly and by taking into account the volume condition (10), (14).

We cannot simply put $V = V_c$ and $\Delta = 0$, because now, when each cluster carries its own, dynamically determined volume, Δ loses its original meaning. Therefore, in Eq. (4) we tentatively replace

$$\begin{aligned} \sigma(p,V_c,b) &\Longrightarrow \frac{2V_c(m,b)\cdot p}{(2\pi)^3}\,\tau(p^2,b) = \frac{2m^2}{(2\pi)^3\,4B}\,\tau(p^2,b) \\ \frac{2\Delta\cdot p_i}{(2\pi)^3}\,\tau(p_i^2,b_i) &\Longrightarrow \frac{2V_c(m_i,b_i)\cdot p_i}{(2\pi)^3} = \frac{2m_i^2}{(2\pi)^3\,4B}\,\tau(p_i^2,b_i) \end{aligned} \tag{15}$$

Next we argue that the explicit factors m^2 and m_i^2 arise from the dynamics and therefore must be absorbed into $\tau(p_i^2,b_i)$ as dimensionless factors (m_i^2/m_0^2). Thus, in Eq. (4), we replace

$$\sigma(p,V_c,b) \Longrightarrow \frac{2m_0^2}{(2\pi)^3 4B}\tau(p^2,b) = H\tau(p^2,b)$$

$$\frac{2\Delta\cdot p_i}{(2\pi)^3}\tau(p_i^2,b_i) \Longrightarrow \frac{2m_0^2}{(2\pi)^3 4B}\tau(p_i^2,b_i) = H\tau(p_i^2,b_i) \tag{16}$$

$$H := \frac{2m_0^2}{(2\pi)^3 4B}$$

where either H or m_0 may be taken as a new free parameter of the model, to be fixed later (if m_0 is taken, then it should be of the order of the "elementary masses" appearing in the system, e.g., somewhere between m_π and m_N in a model using pions and nucleons as elementary input). Finally: if clusters consist of clusters which consist of clusters, which ....., this should end at some "elementary" particles (where what we consider as elementary is fixed by convention). The bootstrap equation (BE) reads then

$$H\tau(p^2,b) = Hg_b\delta_0(p^2-\bar{m}_b^2) +$$

$$+\sum_{N=2}^{\infty}\frac{1}{N!}\int\delta^4\Big(p-\sum_{i=1}^{N}p_i\Big)\sum_{\{b_i\}}\delta_K\Big(b-\sum_{i=1}^{N}b_i\Big)\prod_{i=1}^{N}H\tau(p_i^2,b_i)d^4p_i \tag{17}$$

In words: the cluster with mass $\sqrt{p^2}$ and baryon number b is either elementary (mass: $\bar{m}_b$, spin isospin multiplicity: g_b) or it is composed of any number $N \geq 2$ of subclusters having the same internal composite structure described by this equation. The bar over $\bar{m}_b$ indicates that one has to take the mass, which the "elementary particle" will have effectively when bound in a large cluster: $\bar{m} = m - \langle E_{bind} \rangle$ (e.g., $\bar{m}_N \simeq 925$ MeV). That this must be so, becomes obvious if one imagines Eq. (17) solved by iteration (the iteration solution exists and is the physical solution): then $H\tau(p,b)$ becomes in the end a complicated function of p^2,b, all $\bar{m}_b$ and all g_b. In other words: in the end a cluster consists of the "elementary particles"; as these are all bound into the cluster, their mass should be the effective mass, not the free mass m.

Clearly, the bootstrap equation (17) has not been derived; we have made it more or less plausible and state it as a postulate. For more motivation see Ref. 7).

7. SOLUTION OF THE BOOTSTRAP EQUATION

We solve the BE by the same double Laplace transformation which we used before (Eq. (3)): define

$$\varphi(\beta,\lambda) := \int e^{-\beta_\mu p^\mu} \sum_{b=-\infty}^{\infty} \lambda^b H g_b \delta_0(p^2 - \bar{m}_b^2)\, d^4p = $$

$$= 2\pi H T \sum_{b=-\infty}^{\infty} \lambda^b g_b \bar{m}_b K_1\left(\frac{\bar{m}_b}{T}\right) \tag{18}$$

$$\Phi(\beta,\lambda) := \int e^{-\beta_\mu p^\mu} \sum_{b=-\infty}^{\infty} \lambda^b H\tau(p^2,b)\, d^4p$$

Once the set of "elementary particles" $\{\bar{m}_b, g_b\}$ is given, $\varphi(\beta,\lambda)$ is a known function, while $\Phi(\beta,\lambda)$ is unknown. Applying the double Laplace transformation to the BE, we obtain

$$\Phi(\beta,\lambda) = \varphi(\beta,\lambda) + \exp\Phi(\beta,\lambda) - \Phi(\beta,\lambda) - 1 \tag{19}$$

This implicit equation for ϕ in terms of φ can be solved without regard to the actual $\beta - \lambda$ dependence. Writing

$$G(\varphi) := \phi(\beta,\lambda)$$
$$\varphi = 2G - e^{G} + 1 \tag{20}$$

we can draw the curve $\varphi(G)$ and then invert it graphically to obtain $G(\varphi) = \phi(\beta,\lambda)$.

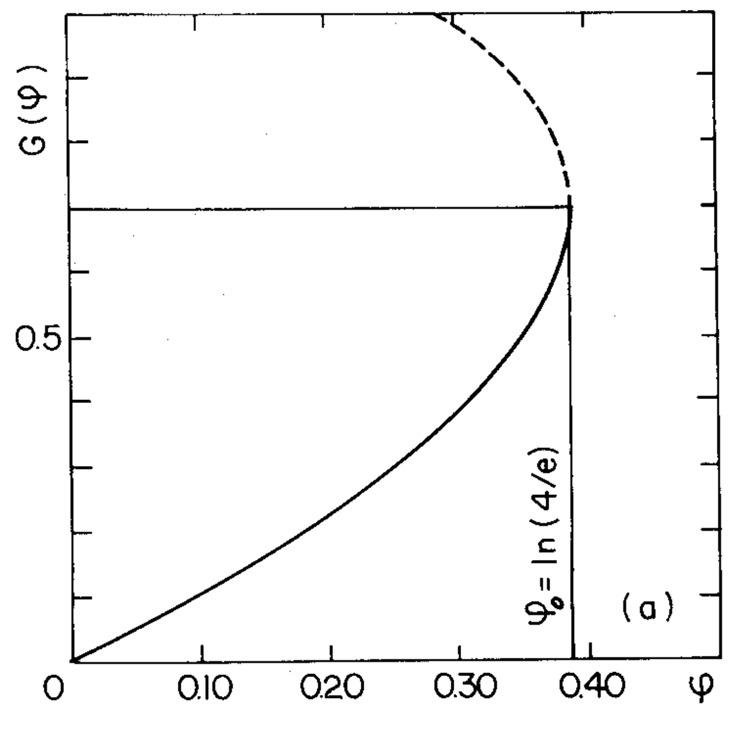

Fig. 2 : Bootstrap function $G(\varphi)$ - the dashed line represents the unphysical branch. The root singularity is at $\varphi_0 = \ell n(4/e) = 0.3863$.

(see Fig. 2). $G(\varphi)$ has a square root singularity[11] at $\varphi = \varphi_0 = \ell n(4/e)$; beyond that value, $G(\varphi)$ becomes complex. Apart from this graphical solution, other forms of solutions are known:

$$G(\varphi) = \sum_{n=1}^{\infty} s_n \varphi^n = \sum_{n=0}^{\infty} w_n \sqrt{\varphi_0 - \varphi}^{\,n} = \quad \{\text{integral representation}\} \tag{21}$$

The power expansion in φ^n was first given in 1870 (yes: eighteen hundred and seventy[12]) and rediscovered in 1973 [13]; the expansion in terms of $\sqrt{\varphi_0 - \varphi}^{\,n}$ has been used in our numerical work (12 terms yield a solution within computer accuracy) and the integral representation will be published elsewhere[14].

We consider $\phi(\beta,\lambda) \equiv G(\varphi)$ to be a known function of $\varphi(\beta,\lambda)$. Consequently, $\tau(m^2,b)$ is also in principle known. From the singularity of $\varphi = \varphi_0$ it follows[14] that $\tau(m^2,b)$ grows, for $m \gg m_N b$, exponentially $\sim m^{-3}\exp(m/T_0)$. In some weaker form this has been known for a long time[7,9,11,13,15,16].

8. ONCE MORE BACK TO THERMODYNAMICS

Having answered the two questions left open at the end of Section 5, we now have the full information to write down $\ell n Z_{pt}$ of Eq. (8). Fortunately we do not need to know $\tau(p^2,\lambda)$ explicitly; the formal similarity between Eq. (8) and Eq. (18) immediately yields a relation between ℓnZ and ϕ (go to rest frame of Δ and β):

$$\ell n Z_{pt}(T,\Delta,\lambda) = -\frac{2\Delta}{(2\pi)^3 H}\frac{\partial}{\partial\beta}\Phi(\beta,\lambda) + C(\beta,\lambda)$$

$$C(\beta,\lambda) := \sum_{b=-\infty}^{\infty} \lambda^b \int \frac{2\Delta\cdot p}{(2\pi)^3}\left\{\delta_0(p^2 - m_b^2) - \delta_0(p^2 - \bar{m}_b^2)\right\} e^{-\beta\cdot p} d^4p \tag{22}$$

As is obvious from the last line, $C(\beta,\lambda)$ corrects the partition function by replacing the one-particle contribution of the bound masses $\bar{m}_b$ by those of the free masses m_b with the effect that all unbound, free particles in our system now have the free mass, while all those bound in clusters still have the bound mass. We have included this correction in all our numerical work, though it is almost always negligible; but we drop it in the rest of this paper (it might be important in other contexts).

Thus, once the "elementary particles" $\{m_b, g_b\}$ and the constants B and H are fixed, a specific model is defined and $\ell n Z_{pt}$ is a known function. Also the relation between $\langle V \rangle$ and Δ is now fixed: since $V_c^\mu = p^\mu/4B$, we have

$$\langle V^\mu \rangle = \Delta^\mu + \frac{\langle p^\mu \rangle}{4B} \xrightarrow[\text{rest frame}]{} \Delta + \frac{\langle E \rangle}{4B} \tag{23}$$

Finally we recall Eq. (7), which enables us to calculate physical quantities for a system of extended particles.

9. PHYSICAL PROPERTIES OF OUR SYSTEM

As an example we calculate the energy density as a function of β and λ:

$$\langle E \rangle = -\frac{\partial}{\partial\beta} \ln Z(\beta, \langle V \rangle, \lambda) = -\frac{\partial}{\partial\beta} \ln Z_{pt}(\beta, \Delta, \lambda) \tag{24}$$

as $\ell n Z_{pt}$ is linear in Δ, the last term is equal to $\Delta . \varepsilon_{pt}(\beta,\lambda)$; hence, after using (23)

$$\frac{\langle E \rangle}{\langle V \rangle} =: \mathcal{E}(\beta,\lambda) = \frac{\varepsilon_{pt}(\beta,\lambda)}{1 + \varepsilon_{pt}(\beta,\lambda)/4B} \tag{25}$$

where

$$\varepsilon_{pt}(\beta,\lambda) = \frac{2}{(2\pi)^3 H}\frac{\partial^2}{\partial\beta^2}\Phi(\beta,\lambda) \tag{25 cont.}$$

Similarly one obtains the baryon number density

$$\nu(\beta,\lambda) := \frac{\langle b\rangle}{\langle V\rangle} = \frac{\nu_{pt}(\beta,\lambda)}{1+\varepsilon_{pt}(\beta,\lambda)/4B}$$

$$\nu_{pt}(\beta,\lambda) := \frac{1}{\Delta}\lambda\frac{\partial}{\partial\lambda}\ln Z_{pt}(\beta,\Delta,\lambda) = -\frac{2}{(2\pi)^3 H}\lambda\frac{\partial}{\partial\lambda}\frac{\partial}{\partial\beta}\Phi(\beta,\lambda) \tag{26}$$

and the pressure

$$P(\beta,\lambda) = \frac{P_{pt}(\beta,\lambda)}{1+\varepsilon_{pt}(\beta,\lambda)/4B}$$

$$P_{pt}(\beta,\lambda) := \frac{T}{\Delta}\ln Z_{pt}(\beta,\Delta,\lambda) = -\frac{2T}{(2\pi)^3 H}\frac{\partial}{\partial\beta}\Phi(\beta,\lambda) \tag{27}$$

$$\langle V\rangle = \Delta\cdot\left(1+\mathcal{E}_{pt}(\beta,\lambda)/4B\right)$$

$$\Delta = \langle V\rangle\cdot\left(1-\mathcal{E}(\beta,\lambda)/4B\right) \tag{28}$$

As all the point-particle quantities involve derivatives of $\phi(\beta,\lambda)$, they become singular at $\varphi = \varphi_0$; e.g.,

$$\frac{\partial}{\partial\beta}\Phi(\beta,\lambda) = \frac{dG}{d\varphi}\frac{\partial\varphi}{\partial\beta} \tag{29}$$

and $dG/d\varphi \sim (\varphi_0-\varphi)^{-\frac{1}{2}}$ (see Fig. 2). Therefore $\varphi \to \varphi_0$ implies point-particle infinities. Consider first

$$\varphi(\beta,\lambda) = \varphi_0 = \ln(4/e) \tag{30}$$

This defines a curve in the β-λ plane. Its position depends, of course, on the actually given form of $\varphi(\beta,\lambda)$, i.e., on the set of "elementary" particles $\{\bar{m}_b, g_b\}$ and the value of the constant H [Eq. (16)]. In the case of three elementary pions $(\pi^+\ \pi^0\ \pi^-)$ and four elementary nucleons (spin $\otimes$ isospin) and four antinucleons, we have from Eq. (18)

$$\varphi(\beta,\lambda) = 2\pi HT\left\{3m_\pi K_1\left(\frac{m_\pi}{T}\right) + 4\left(\lambda+\frac{1}{\lambda}\right)\bar{m}_N K_1\left(\frac{\bar{m}_N}{T}\right)\right\} \tag{31}$$

and the condition (30), written in T and $\mu = T\ell n\lambda$ yields the curve shown in Fig. 3, the "critical curve". For $\mu = 0$ the curve ends at $T = T_0 = 0.190$ GeV, where T_0, the "limiting temperature of hadronic matter", is the same as that appearing in the mass spectrum[7,9,15,16] $\tau(m^2,b) \sim m^{-3}\exp(m/T_0)$ (for $m \gg bm_N$).

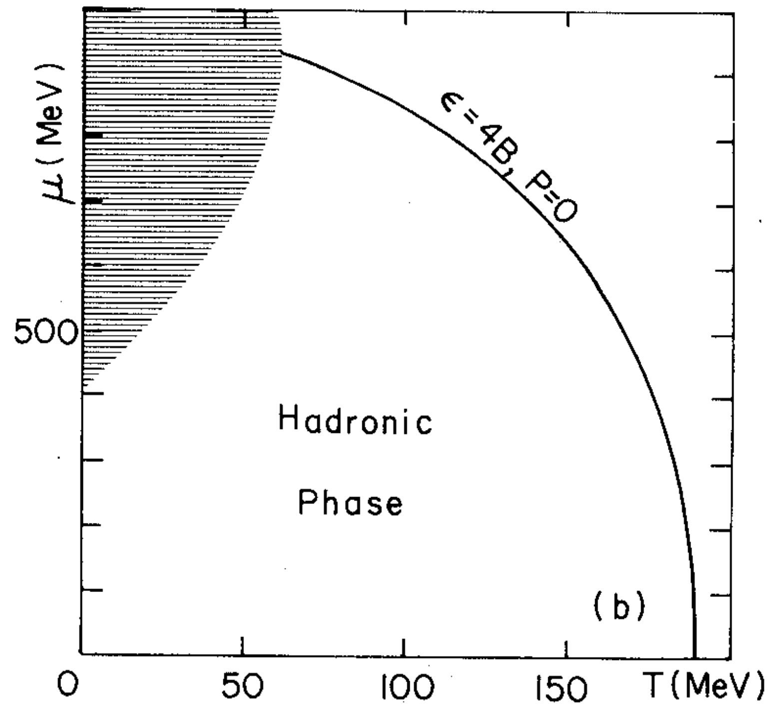

Fig.3 : The critical curve corresponding to $\varphi(T,\mu) = \varphi_0$ in the $\mu - T$ plane. Beyond it the usual hadronic world ceases to exist. In the shaded region our theory is not valid because we neglected Bose-Einstein and Fermi-Dirac statistics.

Our system consists, for small T and μ, of nucleons and nuclei. For increasing T, pion creation sets in and finally also baryon-antibaryon pair creation, K-hyperon associated production, etc. If the latter is to be taken into account, the input set of "elementary particles" must be enlarged. This hardly changes the position of the critical curve and the equations of state of hadron matter, since T_0 is of the order of the pion mass, while the other particles have larger masses and give little contribution to $\varphi(\beta,\lambda)$. More precisely: each new conserved quantum number (strangeness, charm, ...) gives rise to another λ; hence the singularity is defined by $\varphi(\beta,\lambda_1,\lambda_2,\lambda_3 \ldots \lambda_n) = \varphi_0$ as a hypersurface in an $n+1$ dimensional space. Since, however, in normal physical situations only the baryon number is different from zero, we have to consider only the intersection of this hypersurface with the T-μ_{baryon} plane. That is the curve which was said to be little different from the one shown in Fig. 3.

The value of the constant H in Eq. (16) has been chosen to yield $T_0 = 0.19$ GeV (this apparently large value of T_0 is necessary to yield a maximal average decay temperature of the order of 0.16 GeV; see part 2 below). Thus

$$
\begin{aligned}
H &= 0.724\ \text{GeV}^{-2} \\
m_0 &= 0.398\ \text{GeV}\ [\text{when}\ \ B = (145\ \text{MeV})^4]
\end{aligned}
\qquad (32)
$$

where the value of m_0 lies, as expected, between m_π and m_N $[(m_\pi m_N)^{\frac{1}{2}} = 0.36\ \text{GeV}]$.

Taking the critical curve of Fig. 3 as representative, we ask: what does our system do when it approaches the critical curve? As the point particle quantities ε_{pt}, ν_{pt}, P_{pt} diverge, one sees easily (by comparing degrees of divergence when $\varphi \to \varphi_0$) that

$$
\begin{aligned}
&\varepsilon(\beta^*,\lambda^*) = 4B \\
&\nu(\beta^*,\lambda^*) = \nu_{crit}(\beta^*,\lambda^*) \neq 0 \\
&P(\beta^*,\lambda^*) = 0 \\
&\Delta(\beta^*,\lambda^*) = 0 \quad \text{if} \quad \langle V\rangle \neq 0 \\
&\langle V(\beta^*,\lambda^*)\rangle = \infty \quad \text{if} \quad \Delta \neq 0
\end{aligned}
\qquad (33)
$$

where β^*,λ^* are the values along the critical curve.

The constant energy density of our clusters was, independently of m and b, always 4B. Hence the first line suggests that on the critical curve the whole hadron system has condensed into one giant cluster. This is also witnessed by the vanishing of the pressure; indeed, one can explicitly see that for any given external volume $\langle V \rangle$ the number N of "particles" (clusters) contained in it goes to zero on the critical curve: Eq. (8) can be written

$$Z_{pt}(\beta,\Delta,\lambda) = Z_{pt}^{(\xi)}(\beta,\Delta,\lambda,\xi)\Big|_{\xi=1} := \sum_{N=0}^{\infty} \frac{1}{N!}\left(\xi Z_1\right)^N \Big|_{\xi=1} \tag{34}$$

Hence, with (7) and (22)

$$\langle N \rangle = \xi \frac{\partial}{\partial \xi} \ln Z_{pt}^{(\xi)} \Big|_{\xi=1} = Z_1 = - \frac{2\Delta}{(2\pi)^3 H} \frac{\partial}{\partial \beta} \Phi(\beta,\lambda) \tag{35}$$

and, with (28)

$$\frac{\langle N \rangle}{\langle V \rangle} = - \frac{2}{(2\pi)^3 H} \frac{\partial}{\partial \beta} \Phi(\beta,\lambda) \Big/ \left(1 + \varepsilon_{pt}/4B\right) \xrightarrow[\text{crit.}]{} 0 \tag{36}$$

because ε_{pt} contains second derivatives of ϕ.

Note that from (27) and (36) it follows that

$$P \langle V \rangle = \langle N \rangle T \tag{37}$$

that is: our hadron gas obeys the ideal gas equation if $< N >$ is the number of clusters; of course, $< N >$ is not a constant (as for an ideal gas), but a function of β, λ.

The critical curve limits the hadron gas phase; by approaching it, all hadrons dissolve into a giant cluster, which we might call "hadron liquid", but which we would prefer to identify with a quark-gluon plasma. Indeed, as the energy density along the critical curve is constant $(= 4B)$, the critical curve can be attained and, if the energy density becomes $> 4B$, we enter into a region which cannot be described without making assumptions about the inner structure and dynamics of the "elementary particles" $\{\bar{m}_b, g_b\}$ - here pion and nucleon - entering into the input function $\varphi(\beta,\lambda)$. Considering pions and nucleons as quark-gluon bags leads naturally to the above interpretation. We discuss these points and applications of our theory to relativistic heavy ion collisions in Part 2 of these lectures.

REFERENCES

1) R. Hagedorn and J. Rafelski, CERN preprint TH.2922 (August 1980), to be published in Phys. Lett. B.

2) J. Rafelski, H.Th. Elze and R. Hagedorn, CERN preprint TH.2912 (August 1980) (invited paper at the 5th European $\bar{p}$ Symposium, June 1980, Bressanone, Italy).

3) B. Touschek, Nuovo Cimento B58 (1968) 295.

4) K. Redlich and L. Turko, Z. Physik C, Particles and Fields 5 (1980) 201, and lecture at this conference.

5) E. Beth and G.E. Uhlenbeck, Physica 4 (1937) 915.

6) L. Landau and E. Lifshitz, Statistical Physics §77, in "Course of Theoretical Physics", Vol. 5 (Pergamon Press, Oxford).

7) R. Hagedorn, Cargèse Lectures in Physics, Vol. 6, Gordon and Breach (New York 1973) and references therein.

8) R. Hagedorn, I. Montvay and J. Rafelski, Lecture in Erice Workshop, "Hadronic Matter at Extreme Energy Density", eds. N. Cabibbo and L. Sertorio, Plenum Press (New York 1980), p. 49.

9) R. Hagedorn, Suppl. Nuovo Cimento 3 (1965) 147.

10) K. Johnson, Acta Phys. Polon. B6 (1975) 865.

11) W. Nahm, Nucl. Phys. B45 (1972) 525.

12) E. Schröder, ZS. für Math. und Physik 15 (1870) 361.

13) J. Yellin, Nucl. Phys. B52 (1973) 583.

14) R. Hagedorn and J. Rafelski, Phys. Rep. (in preparation).

15) S. Frautschi, Phys. Rev. D3 (1971) 2821.

16) R. Hagedorn and I. Montvay, Nucl. Phys. B59 (1973) 45.

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