On Supersymmetric and Topological Quantum Mechanical Models

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Abstract. We explain the details of two supersymmetric quantum mechanical models. Their simplicity make them solvable although they share the characteristics of more sophisticated models based on the gauge fixing of topological invariants. The first model is a supersymmetric quantum mechanical system defined on a punctured plane and leads to topological observables which we compute. The absences of a ground state and of a mass gap are special features of this system. The second model is the supersymmetric description of spin-one particles moving in D-dimensional space-time. We show that it is a topological model in a space with two more dimensions.

1 Introduction

During the last years, Topological Quantum Field Theories have emerged as possible realizations of general coordinates invariant symmetries [1][2].

One of the special features of these theories is their ability to produce space-time metric independent correlations functions, although they are defined from a local action.

In Topological Quantum Field Theories, an important symmetry operator which is at disposal is the BRST operator Q, such that the Hamiltonian is $H = \frac{1}{2}[Q, \overline{Q}]$. Q and $\overline{Q}$ can be often understood as 'twisted" deformations of $N = 2$ supersymmetry generators.

An attractive scheme is to introduce Topological Quantum Field Theories by the path integral quantization of topological terms. The techniques relies on the BRST formalism. More precisely, one can often start from a topological term, expressed as the integral over a manifold of a Lagrangian locally equal to a pure divergency which is a function of a set of given fields. Such a "classical" action is for instance a characteristic number, or any given invariant depending only of the topology of field configurations and/ or the space over which the fields are defined. No classical dynamics is generated. However, the existence of a gauge symmetry of the Lagrangian, namely the group of arbitrary infinitesimal deformations of fields, permits the quantization of the theory through the general formalism of BRST invariant gauge fixing. Our present knowledge makes this construction quite generic, provided one gets the intuition of (i) which manifold should be studied, and (ii) which fields should be introduced for this purpose. Actually, it is interesting to speculate that the symmetries of nature could be fundamentally of the topological type, and that the observed gauge symmetries would be obtained by gauge-fixing the huge topological symmetry in a BRST invariant way, leaving therefore an $N=2$ supersymmetric theory of particles.

Not surprisingly, the problem of computing observables in Topological Quantum Field Theories is often technically complicated. The basic idea is the introduction of fields with positive and negative degrees of freedom (classical and ghost fields) which permit the exploration of topological properties through the computation of Green functions whose coefficient turn out to be topological invariants. Once the theory has been defined, dimensional reductions may appear as the only possible technichal way to perform realistic computations. The technicity of these computations may hide the beautiful simplicity of the idea! As an example, to compute the knot polynomials associated to the Chern-simons theory, one reduces the $3-D$ theory into $2-D$ conformal theories [3].

One usually defines the physical Hilbert space of Topological Quantum Field Theories as the cohomology of Q (states which are annihilated by Q without being the Q transformation of other states). This definition of the physical Hilbert space is perfectly suited for ordinary gauge theories. For Topological Field Theories there are doubts on the general validity of this definition. Due to properties of the vaccuum, other relevant observables than those defined by the BRST cohomology could exist. In particular, $Q-$exact observables with non vanishing mean values can exist. This is for instance the case in topological models of the type of those introduced by Witten in [1]. In other topological ones, based on first order actions like the Chern-Simons action [3], formal arguments show that the situation is similar. In all these models, one sees furthermore that a local version of of the topological BRST symmetry seems to single out the form of the supersymmetric potential [5][6].

The simplest examples of Topological Quantum Field Theories are zero-dimensional and turn out to be are $N=2$ supersymmetric quantum mechanics models. Interestingly enough, two models exist which illustrate both extrem cases: (i) the Hilbert space is made of pure topological observables and (ii) the Hilbert space is made of particle degrees of freedom. In these notes we find it interesting to detail them. Indeed they provide elementary examples showing the basic rules of the BRST invariant topological gauge fixing procedure. In particular, they address the questions of the selection of gauge functions and of the calculability of observables (which can be completely worked out in the case (i)). The example (ii) is intriguing since it might be generalized to other particle or string models with N=2 supersymmetry.

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2 Model (i): Supersymmetric quantum mechanics on a punctured plane

We wish to work with a simple topological classical Lagrangian that is a candidate to generate a topological quantum mechanics. We consider as a target space a plane from which we exclude the origin, so that one has a non trivial, although very simple topological structure defined by the winding number around the origin of the trajectories of a particle. We denote the time by the real variable t and the Euclidian time by τ, with $t = i\tau$ and τ real. The cartesian coordinates on the plane are q_i, with $i = 1, 2$. We select trajectories with periodic conditions, namely such that between the initial and final times $t = 0$ and $t = T$ the particle ends up at its starting point so an integer value of the winding number can be assigned to its trajectory.

From our understanding of the nature of a topological field theory [4], we start from a topological classical action $\mathcal{I}_{cl}[\vec{q}]$. $\mathcal{I}_{cl}[\vec{q}]$ must not depend on the time metric. This condition is satisfied if it is the integral of a locally closed form. The natural candidate is

$$\begin{aligned} \mathcal{I}_{cl}[\vec{q}] = \int f d\tau &= \int_0^T d\tau f \dot{r}(\tau) \\ &= \int_0^T d\tau \, f \frac{\epsilon^{ij} \dot{q}_i q_j}{\vec{q}^2} \end{aligned} \tag{2.1}$$

where f is a real number. This action measures the winding number of the particle times $f/2\pi$. It shares analogy with the second Chern class $\int d^4x \operatorname{tr} F \wedge F$ where F is the curvature of a Yang-Mills field. Here and in what follows the symbol $\dot{X}$ denotes $\frac{dX}{d\tau}$.

To obtain the Topological Quantum Theory associated to our space, we need to give sense to the Euclidian path integral

$$\int \mathcal{D}[\vec{q}] \exp -\mathcal{I}_{cl}[\vec{q}] \tag{2.2}$$

as well as to compute topological quantities from Green functions

$$\text{Topological information} = \int \mathcal{D}[\vec{q}]\, O \exp -\mathcal{I}_{cl}[\vec{q}] \tag{2.3}$$

where O is a well chosen composite operator.

The difficulty for realizing this objective is that our action is different from that of conventional quantum mechanics where classical degrees of freeedom exist at the classical level and quantum fluctuations occur around the solutions of equations of motion. Here the Lagrangian is locally a pure derivative, the Hamiltonian vanishes and one has no equation of motion. On the other hand, one observes that the action $\mathcal{I}_{cl}[\vec{q}]$ is invariant under the gauge symmetry

$$q(\vec{t}) \to q(\vec{t}) + \epsilon(\vec{t}) \tag{2.4}$$

where $\epsilon(t)$ is any given local shift of the particle position $q(t)$ which does not change the winding number of the trajectory. Using the BRST technique it is then possible to define the path integrals (2.2) and (2.3) by a conventional gauge fixing of the action $\mathcal{I}_{cl}[\vec{q}]$.

The BRST transformation laws associated to the symmetry (2.4) are of the simple form

$$s\vec{q} = \vec{\Psi} \quad s\vec{\Psi} = 0 \quad s\vec{\overline{\Psi}} = \vec{\tau} \quad s\vec{\tau} = 0 \tag{2.5}$$

The anticommuting fields $\vec{\Psi}(t)$ and $\vec{\overline{\Psi}}(t)$ are the topological ghosts and antighosts associated to the particle position $\vec{q}(t)$. $\vec{\tau}(t)$ is a Lagrange multiplier. s acts on field functions as a differential operator graded by the ghost number.

To get a gauge fixed action with a quadratic dependence on the velocity $\dot{\vec{q}}$, one choses a gauge function of the type $\dot{q}_i + \frac{\delta V}{\delta q_i}$, where the prepotential V is an arbitrary given function of $\vec{q}$. This yields the following gauge fixed BRST invariant action $\mathcal{I}_{gf}$ which is supersymmetric

$$\begin{aligned}\mathcal{I}_{gf}[\vec{q},\vec{\Psi},\vec{\overline{\Psi}},\vec{\tau}] &= \int_0^T d\tau \left(f\dot{\tau} - s\overline{\Psi}_i(\frac{1}{2}\tau_i - i\dot{q}_i + \frac{\delta V}{\delta q_i}) \right) \\ &= \int_0^T d\tau \Bigg(f\dot{\tau} - \frac{1}{2}\tau_i^2 + i\tau_i\left(\dot{q}_i + \frac{\delta V}{\delta q_i}\right) - \\ &\quad -i\overline{\Psi}_i\left(\dot{\Psi}_i + \frac{\delta^2 V}{\delta q_i q_j}\Psi_j\right)\Bigg)\end{aligned} \tag{2.6}$$

The BRST symmetry $s\mathcal{I}_{gf}[\vec{q},\vec{\Psi},\vec{\overline{\Psi}},\vec{\tau}] = 0$ holds true independently of the choice of the function $V(\vec{q})$ and the partition function and the mean values of BRST invariant observables

$$Z = \int \mathcal{D}[\vec{q}]\mathcal{D}[\vec{\Psi}]\mathcal{D}[\vec{\overline{\Psi}}]\mathcal{D}[\vec{\tau}] \exp -\mathcal{I}_{gf} \tag{2.7}$$

$$< O > = \int \mathcal{D}[\vec{q}]\mathcal{D}[\vec{\Psi}]\mathcal{D}[\vec{\overline{\Psi}}]\mathcal{D}[\vec{\tau}]\, O \exp -\mathcal{I}_{gf} \tag{2.8}$$

are now well defined Euclidian path integrals. To understand $\dot{q}_i + \frac{\delta V}{\delta q_i}$ as a gauge function for the quantum variable $\vec{q}$, one may interpret the result of the integration over the ghosts as a determinant. The BRST invariance of the field polynomial O allows one to prove, at least formally, the topological properties of $< O >$. On the other hand our knowledge of supersymmetric quantum mechanics tells us that this mean value may depend on the class of the function V. What happens is that in the case of topological field theories, the Euclidian path integral explores the moduli space of the equation $\dot{q}_i + \frac{\delta V}{\delta q_i} = 0$, as a result of the gauge fixing.

The question of finding a symmetry principle which would select the prepotential $V(\vec{q})$ leading to interesting topological information was investigated in

[5]. The idea is to ask for the invariance of the action under a symmetry which is more restrictive than the topologocal BRST symmetry, namely a local version of it, for which the parameter becomes an affine function of the time, with arbitrary infinitesimal coefficients. One requires

$$\delta_l \mathcal{I}_{gf}[\vec{q}, \vec{\Psi}, \vec{\bar{\Psi}}, \vec{\tau}] = 0 \tag{2.9}$$

where the "local" BRST transformations δ_l are

$$\delta l\vec{q} = \eta(t)\vec{\Psi} \quad \delta_l\vec{\Psi} = 0 \quad \delta_l\vec{\bar{\Psi}} = \eta(t)\vec{\tau} - \dot{\eta}(t)\vec{q} \quad \delta_l\vec{\tau} = \dot{\eta}(t)\vec{\Psi} \tag{2.10}$$

and $\eta(t) = a+bt$ where a and b are constant anticommuting parameters. The idea of local BRST symmetry was considered in [9] for the sake of interpreting higher order cocycles which occurs when solving the anomaly consistency conditions, and has been shown to play a role in topological field theories in [6].

Imposing this local symmetry implies that V satisfies the constraint [5]

$$\frac{\delta V}{\delta q_i} + q_j \frac{\delta^2 V}{\delta q_i\, \delta q_j} = 0 \tag{2.11}$$

This constraint is solved for $V(\vec{q}) = f\tau$ where τ is the angle such that $q_1 + iq_2 = |\vec{q}| \exp i\tau$ and f is a number [5][1]. By putting this value of $\dot{q}_i + \frac{\delta V}{\delta q_i}$ in (2.6) and eliminating the Lagrange multiplier τ by its equation of motion we obtain

$$\mathcal{I}_{gf}[\vec{q}, \vec{\Psi}, \vec{\bar{\Psi}}] = \int_0^T d\tau \left(\frac{1}{2}\dot{q}_i^2 + \frac{f^2}{2\vec{q}^2} - \bar{\Psi}_i \left(\dot{\Psi}_i \pm f \frac{\delta^2 \tau}{\delta q_i \delta q_j} \Psi_j \right) \right) \tag{2.12}$$

Notice that

$$\begin{aligned} \frac{\delta^2 \tau}{\delta q_i \delta q_j} &= \frac{1}{\vec{q}^2} \begin{pmatrix} -\sin 2\tau & \cos 2\tau \\ \cos 2\tau & \sin 2\tau \end{pmatrix}_{ij} \\ &= \frac{1}{\vec{q}^2} \begin{pmatrix} \cos\tau & -\sin\tau \\ \sin\tau & \cos\tau \end{pmatrix} \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} \cos\tau & \sin\tau \\ -\sin\tau & \cos\tau \end{pmatrix}_{ij} \end{aligned} \tag{2.13}$$

The superconformal potential $1/\vec{q}^2$ has been already studied in [7][10]. We shall shortly compute the observables which seems interesting to us from the topological point of view in the canonical quantization formalism. We will show that a very specific supersymmetry breaking mechanism occurs and implies the existence of non vanishing Q exact observables which are metric independent as well as of a fractional Witten indexD.

We believe that the signal that the theory truly carries some topological information is the existence of an interesting instanton structure. Let us remember that, from our gauge fixing in the Euclidian time region, we have obtained an

[1]For the case of one variable x we would obtain $V = \log x$, with quite similar properties of the supersymmetric system, but the geometrical interpretation would be less clear and no meaningful observable exists

action whose bosonic part is the square of the gauge function. It follows that the solutions to the Euclidien equations of motion can be written as

$$\dot{q}_i + \frac{\epsilon^{ij} q_j}{\vec{q}^2} = 0 \tag{2.14}$$

$$\dot{\Psi}_i \pm f \frac{\delta^2 \tau}{\delta q_i \delta q_j} \Psi_j = 0 \tag{2.15}$$

If we introduce $z = q_1 + iq_2$ and $\Psi_z = \Psi_1 + i\Psi_2$, with $sz = \Psi_z$, we can write these equations as

$$\dot{z} + \frac{i}{z^*} = 0 \tag{2.16}$$

$$\dot{\Psi}_z - \frac{i}{(z^*)^2} \Psi_z^* = 0 \tag{2.17}$$

Assuming periodic boundary conditions, the solutions for $\vec{q}$ are circles described at constant velocities and indexed by an integer n

$$z^{(n)} = \sqrt{\frac{T}{2n\pi}} \exp -i\frac{2nt}{T} \qquad n \in Z \tag{2.18}$$

while for the ghost

$$\Psi_z^{(n)} = \eta \exp i\frac{2nt}{T} \tag{2.19}$$

where η is a constant fermion. The Euclidian energy and angular momentum of the action evalueted for these field configurations vanish for all valuese of n.

Due to the existence of these degenerate zero modes of the action we expect that BRST invariant observables should exist and that their mean values should be non zero as well as energy and time reparametrization independent. The corresponding numbers should to be expressable as a series over an integer related to the one which label the instanton solutions. This is the conjecture that we shall now verify.

To compute observables in the canonical, we will formalism. We do a Wick rotation to recover the real Minkowski time t by setting $\tau = it$, and change the quantum mechanichal variables into operators. The Hamiltonian associated to the action $\mathcal{I}_{gf}$ is

$$H = \frac{1}{2}\varphi^2 + \frac{f^2}{2\vec{q}^2} - f\overline{\Psi}_i \frac{\delta^2 \tau}{\delta q_i \delta q_j} \Psi_j \tag{2.20}$$

where the quantization rules are (remember that $q_i = (x, y)$ stands for the cartesian coordinates on the plane)

$$[p_i, q_j] = -i\delta_{ij} \qquad [\overline{\Psi}_i, \Psi_j]_+ = \delta_{ij}$$

$$[\Psi_i, \Psi_j]_+ = [\overline{\Psi}_i, \overline{\Psi}_j]_+ = [\Psi_i, p_j] = [\overline{\Psi}_i, p_j] = [\Psi_i, q_j] = [\overline{\Psi}_i, q_j] = 0 \quad (2.21)$$

By construction H can be written as

$$H = \frac{1}{2}\{Q, \overline{Q}\} \quad (2.22)$$

with

$$Q = \Psi_i(p_i + if\frac{\delta\tau}{\delta q_i}) \qquad \overline{Q} = \Psi_i(p_i - if\frac{\delta\tau}{\delta q_i}) \quad (2.23)$$

Following [10], we use the following matricial representation for the ghost and antighost operator

$$\Psi_1 = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix} \qquad \Psi_2 = \begin{pmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \quad (2.24)$$

One has $\overline{\Psi} = \Psi^\dagger$ and $p_i = -i\delta/\delta q_i$. In this representation

$$H = \begin{pmatrix} H_0 & 0 & 0 & 0 \\ 0 & H_{11} & H_{12} & 0 \\ 0 & H_{21} & H_{22} & 0 \\ 0 & 0 & 0 & H_2 \end{pmatrix} \quad (2.25)$$

where

$$H_0 = H_2 = -\frac{1}{2r}\frac{\delta}{\delta r}r\frac{\delta}{\delta r} - \frac{1}{2r^2}\frac{\delta^2}{\delta\tau^2} + \frac{f^2}{2r^2} \quad (2.26)$$

and

$$\begin{aligned} \begin{pmatrix} H_{11} & H_{12} \\ H_{21} & H_{22} \end{pmatrix} &= H_0\delta_{ij} + f\frac{\delta^2\tau}{\delta q_i \delta q_j} \\ &= R_{-\tau}\Bigg(-\frac{1}{2r}\frac{\delta}{\delta r}r\frac{\delta}{\delta r} + \frac{f^2 + 1 - \frac{\delta^2}{\delta\tau^2}}{2r^2} - \\ &\quad -\frac{1}{r^2}\begin{pmatrix} 0 & f + i\frac{\delta}{\delta\tau} \\ f - i\frac{\delta}{\delta\tau}\tau & 0 \end{pmatrix}\Bigg)R_\tau \end{aligned} \quad (2.27)$$

where r and τ are the polar coordinates on the plane and

$$R_\tau = \begin{pmatrix} \cos\tau & \sin\tau \\ -\sin\tau & \cos\tau \end{pmatrix} \quad (2.28)$$

The spectrum of H is straightforward to derive in this representation. One uses the usual strategy based on the fact that if an eigenstate of H has energy E, its Q and $\overline{Q}$ transforms are either zero or an eigenstate of H with the same energy.

States are labelled by their non negative energy E, angular momentum n and fermion number α, that is ghost number. We denote them as $|E,n,\alpha>$. For each value E and n, one has four states labelled by $\alpha = 1,2,3,4$. The states with $\alpha = 1$ and $\alpha = 4$ are respectively anihilated by Q and $\overline{Q}$. This is due to the fact that states $|\phi>$ which are BRST invariant, $Q|\phi>=0$, are such that

$$|\phi>= \begin{pmatrix} |E,n> \\ 0 \\ 0 \\ 0 \end{pmatrix} \qquad \overline{Q}|\phi>= \begin{pmatrix} 0 \\ (p_1 - if\frac{\delta\tau}{\delta q_1})|E,n> \\ -(p_2 - if\frac{\delta\tau}{q_2})|E,n> \\ 0 \end{pmatrix}$$

$$H|\phi>= \begin{pmatrix} H_0|E,n> \\ 0 \\ 0 \\ 0 \end{pmatrix} \tag{2.29}$$

One has similar relations for states $|\overline{\phi}>$ satisfying $\overline{Q}|\overline{\phi}>=0$.

Let us define $g_{E,n} =< r,\tau|E,n>$. This function is the solution of the equation

$$< r,\tau|H_0|E,n>= \left(-\frac{1}{2r}\frac{\delta}{\delta r}r\frac{\delta}{\delta r} - \frac{1}{2r^2}\frac{\delta^2}{\delta\tau^2} + \frac{f^2}{2r^2}\right) g_{E,n} = E g_{E,n} \tag{2.30}$$

$g_{E,n}$ is also the solution of the ghost number 2 equation $< r,\tau|H_2|E,n>= E|E,n>$. Its knowledge is sufficient to get the full spectrum for $E \neq 0$. One has indeed

$$|E,n,1>= \begin{pmatrix} |E,n> \\ 0 \\ 0 \\ 0 \end{pmatrix} \qquad |E,n,2>= \frac{1}{\sqrt{E}}\overline{Q}|E,n,1>$$

$$|E,n,4>= \begin{pmatrix} 0 \\ 0 \\ 0 \\ |E,n> \end{pmatrix} \qquad |E,n,3>= \frac{1}{\sqrt{E}}\overline{Q}|E,n,4> \tag{2.31}$$

The diagonalization of the part with ghost number one of the Hamiltonian (2.27) amounts to solve the equations

$$\left(-\frac{1}{2r}\frac{\delta}{\delta r}r\frac{\delta}{\delta r} + \frac{f^2+1-\frac{\delta^2}{\delta\tau^2} \pm 2\sqrt{f^2-\frac{\delta^2}{\delta\tau^2}}}{2r^2}\right) g_{E,n,\pm} = E g_{E,n,\pm} \tag{2.32}$$

which are of the same type as (2.30).

To solve (2.30) and (2.32) we set

$$\begin{aligned} g_{E,n} &= \frac{1}{\sqrt{2\pi}} \exp in\tau \; f_{E,n}(r) \qquad n \in Z \\ g_{E,n,\pm} &= \frac{1}{\sqrt{2\pi}} \exp in\tau \; f_{E,n,\pm}(r) \qquad n \in Z \end{aligned} \tag{2.33}$$

For $E \neq 0$, $f_{E,n}(r)$ and $f_{E,n,\pm}$ are expressable as a Bessel function $J_\nu(\sqrt{2}Er)$ of order ν, with

$$f_{E,n}(r) = \frac{1}{\sqrt{2}} J_{\sqrt{n^2+f^2}}(\sqrt{2}Er) \tag{2.34}$$

and

$$f_{E,n,\pm}(r) = \frac{1}{\sqrt{2}} \exp in\tau J_{\sqrt{f^2+1+n^2\pm 2\sqrt{f^2+n^2}}}(\sqrt{2}Er) \tag{2.35}$$

These states are normalizable as plane waves in one dimension. This is a consequence of the continuity of the spectrum in the radial direction. They build an appropriate basis of stationary solutions since, with the normalization factor which is explicit in (2.34), one has $\sum_n \int_{E>0} dE|E,n><E,n| = 1$. On the other hand, for $E = 0$, the Schrödinger equations (2.30) and (2.32) have no admissible normalizable solution. Thus we have a continuum spectrum, bounded from below, with a spin degeneracy equal to 4 and an infinite degeneracy in the angular momentum quantum number n. The peculiarity of this spectrum is that there is no ground state, since we have states with energy as little as we want, but we cannot have $E = 0$. This is a consequence of the conformal property of the potential $\frac{1}{|\vec{q}|^2}$.

Since we cannot reach the energy zero which woud be the only Q and $\overline{Q}$ invariant state, we conclude that supersymmetry is broken.

It is useful for what follows to redefine the ghost and antighost operators into

$$\begin{pmatrix}\Psi_r\\ \Psi_\tau\end{pmatrix} = \begin{pmatrix}\cos\tau & \sin\tau\\ -\sin\tau & \cos\tau\end{pmatrix}\begin{pmatrix}\Psi_1\\ \Psi_2\end{pmatrix} \quad \begin{pmatrix}\overline{\Psi}_r\\ \overline{\Psi}_\tau\end{pmatrix} = \begin{pmatrix}\cos\tau & \sin\tau\\ -\sin\tau & \cos\tau\end{pmatrix}\begin{pmatrix}\overline{\Psi}_1\\ \overline{\Psi}_2\end{pmatrix} \tag{2.36}$$

These rotated ghost operators satisfy similar anticommutation rotations as the Ψ_i and $\overline{\Psi}_i$. On the other hand, notice that

$$[\frac{\delta}{\delta\tau}, \begin{pmatrix}\overline{\Psi}_r\\ \overline{\Psi}_\tau\end{pmatrix}]_+ = \begin{pmatrix}\overline{\Psi}_\tau\\ -\overline{\Psi}_r\end{pmatrix} \qquad [\frac{\delta}{\delta\tau}, \begin{pmatrix}\Psi_r\\ \Psi_\tau\end{pmatrix}] = \begin{pmatrix}\Psi_\tau\\ -\Psi_r\end{pmatrix} \tag{2.37}$$

$$[\frac{\delta}{\delta r}, \begin{pmatrix}\overline{\Psi}_r\\ \overline{\Psi}_\tau\end{pmatrix}] = [\frac{\delta}{\delta r}, \begin{pmatrix}\Psi_r\\ \Psi_\tau\end{pmatrix}]_+ = 0 \tag{2.38}$$

One has the following expression of Q and $\overline{Q}$ which will be used shortly

$$Q = -i\Psi_r\frac{\delta}{\delta r} - i\frac{1}{r}\Psi_\tau(\frac{\delta}{\delta\tau} - f) \qquad \overline{Q} = -i\overline{\Psi}_r\frac{\delta}{\delta r} - i\frac{1}{r}\overline{\Psi}_\tau(\frac{\delta}{\delta\tau} + f) \tag{2.39}$$

These expressions in curved coordinates could be obtained from the general formalism of [11]].

We now turn to tomputation of BRST Invariant Observables. We have just seen that supersymmetry is broken in a very special way. This opens the possibility of having non vanishing BRST-exact Green functions which are topological in the sense that they are scale independent, that is independent of time, or energy, rescalings.

From dimensional arguments the candidates for such commutators are

$$O_\tau = [Q, r\overline{\Psi}_\tau]_+ = [\overline{Q}, r\Psi_\tau]_+^\dagger \qquad O_r = [Q, r\overline{\Psi}_r]_+ = [\overline{Q}, r\Psi_r]_+^\dagger \tag{2.40}$$

The mean values of these operators between normalized states are

$$\frac{< E, n|[Q, r\overline{\Psi}_\tau]_+|E, n >}{< E, n|E, n >} = n + if \tag{2.41}$$

and

$$\frac{< E, n|[Q, r\overline{\Psi}_r]_+|E, n >}{< E, n|E, n >} = \lim_{L\to\infty} \frac{L^2 J^2_{\sqrt{n^2+f^2}}(L)}{\int_0^L dr J_{\sqrt{n^2+f^2}}(r)} \tag{2.42}$$

The last quantity is bounded but ill-defined, so we reject it. We get therefore that for any normalized state $|\phi_n >= \int dE\rho(E)|E, n >$ with a given angular momentum n, the expectation value of $[Q, r\overline{\Psi}_\tau]_+$ is

$$< \phi_n|[Q, r\overline{\Psi}_\tau]_+|\phi_n >= n + if \tag{2.43}$$

indepently of the weighting function ρ.

If we now sum over all values of n, what remains is the topological number

$$< O_\tau >= \sum_n < \phi_n|[Q, r\overline{\Psi}_\tau]_+|\phi_n >= \sum_n n + if \sum_n 1 \tag{2.44}$$

¿From a topological point of view, our result mean that there are two observables, organized in a complex form, in the cohomology of the punctured plane. The summation over the index n, that is the angular momentum, could have expected from the formal argument that in the path integral one gets a single finite contribution from each instanton solution to the mean value of a topological observable, so that

$$\text{Topological information} = \int \mathcal{D}[\vec{q}]\, O_f \exp -\mathcal{I}_{cl}[\vec{q}] \sim \sum_n f(n) \tag{2.45}$$

Our computation shows the existence of a BRST invariant observable with non zero mean value which is *Q-closed.* The supersymmetry breaking mechanism made possible by our potential choice (on the basis of local BRST symmetry) is responsible of this situation. With other potentials than the one that we have chosen , either supersymmetry would be unbroken, or a mass gap would occur. In the previous case all Q-exact observable would vanish; in the latter case they could be nonzero but they would be scale dependent.

As another topological observable of the theory, we may consider the Witten index [12] [13]. The idea is that although there is no normalizable vacuum in the theory, we can consider the trace

$$\Delta = \mathrm{Tr}(-)^F \exp -\beta H \tag{2.46}$$

where the trace means a sum over angular momentum as well as over all energy including energy zero, and $(-)^F$ is the ghost or fermion number operator. The result should be finite because, although the state with energy zero is not normalizable, it contributes only over a domain of integration with zero measure. Indeed, since supersymmetric compensations occur for $E \neq 0$ and provided one uses a BRST symmetry preserving regularization, the full contribition to Δ should come from the domain of integration concentrated at $E \sim 0$, while the topological nature of the theory should warranty that Δ is non zero and independent on β.

By using the suitably normalized eigenfunctions of the Hamiltonian, eqs.(2.34) and (2.35), one can write the index Δ as follows

$$\begin{aligned} \Delta &= \sum_n \int_0^\infty dE \exp -\beta E \int r dr \frac{1}{2}\Big(2J^2_{\sqrt{n^2+f^2}}(\sqrt{2E}r) - \\ &\quad - J^2_{\sqrt{f^2+1+n^2+\sqrt{f^2+n^2}}}(\sqrt{2E}r) - J^2_{\sqrt{f^2+1+n^2-\sqrt{f^2+n^2}}}(\sqrt{2E}r)\Big) \end{aligned} \tag{2.47}$$

To compute this double integral one needs a regularisation. Following for instance [13], we can use a dimensional regularization . Thus we change dr into $r^\epsilon dr$. Then, the analytic comtinuation of the result when $\epsilon \to 0$ is

$$\begin{aligned} \Delta &= \sum_n \frac{1}{2}\Big(2\sqrt{f^2+n^2} - \sqrt{f^2+1+n^2+\sqrt{f^2+n^2}} - \\ &\quad - \sqrt{f^2+1+n^2-\sqrt{f^2+n^2}}\Big) \end{aligned} \tag{2.48}$$

As announced this result is independent on β. As a series, it diverges logarimically as $\sum 1/n$ which is presumably the consequence of the conformal invariance of the potential. We see that the contribution of each topological sector is n dependant.

Let us now summarize what we understood from this model. We have shown an example for which the requirement of local BRST symmetry for topological quantum mechanics results in selecting a superconformal quantum mechanichal system. As a result, the spectrum of the theory has no ground state and a supersymmetry breaking mechanism occurs, without the the presence of a dimensionful parameter. Our goal was to understand the mechanism which provide topological observables. We observed that the special properties of the potential allows the computation of energy independant quantities although they are of mean values of BRST exact observables between non zero energy states. These quantities deserve to be called topological and they get a contribution from the whole spectrum of the theory. We have also singled out the Witten index, in a computation which includes a contribution from the non normalizable state of zero energy. The generalization of these observations to quantum field theory is an interesting open question.

3 Model (ii): The supersymmetric Lagrangian for spin-one particles

Supersymmetric quantum mechanics can be used to describe the dynamics of spinning point particles. The use of anticommuting variables to describe spinning particles was introduced in [14]. Then, it was found that local supersymmetry of rank $2S$ on the worldline is necessary to describe consistently a particle of spin S. The resulting constrained system [17] [18] requires a careful gauge-fixing of the einbein and the gravitini. One obtains eventually a tractable Lagrangian formulation [19], [20]. (There are many references on the subject, of which we quote very few) as well as to compute a certain number of topological invariants of the target space [12].

Using these facts, we will now point out an example showing that topological quantum theories may exhibit a phase with a Hilbert space made of particle degrees of freedom. We will interpret local supersymmetry on the worldline as a residue of a more fundamental topological symmetry, defined in a target-space with two extra dimensions. One of the coordinates is eventually identified as the einbein on the worldline. Other fields must be introduced to enforce the topological BRST invariance. They can be eliminated by their equations of motion and decouple from the physical sector. To obtain in a natural way a nowhere vanishing einbein, we use a disconnected higher dimensional target-space where the hyperplane $\{e = 0\}$ is a priori extracted. Thus, one introduces some topology before any gauge-fixing. Two disconnected topological sectors exist, $\{e > 0\}$ and $\{e < 0\}$, which correspond to the prescription $\pm i\epsilon$ for the propagators. It is fundamental that the gauge functions be compatible with the topology of space: they must induce a potential which rejects the trajectories from the hyperplane $\{e = 0\}$.

We will first review the supersymmetric description of a relativistic spinning particle in a Riemannian space-time. Then we will consider the case of $N = 2$ supersymmetry and show a link between the supersymmetric description of scalar or spin-one particles and topological quantum mechanics in a higher dimensional target-space. Finally, we will verify that the constraints of the theory identify its physical content and illustrate the result by computing the deviation of the trajectories from geodesics due to the interactions between geometry and spin.

Consider a spin-S particle in a D-dimensional space-time. Classically, it follows a worldline whose coordinates $X^\mu(\tau)$ are parametrized by a real number τ. If the particle is massive, a natural choice of this parameter is the proper-time. The idea originating from [14] is to describe the spin of the particle by assigning to each value of τ a vector with anticommuting coordinates $\Psi_i^\mu(\tau)$ where the vector index μ runs between 1 and D and i between 1 and $2S$. Indeed, in the case of a flat space-time and spin one-half, the Lagrangian density introduced in [14] is

$$\mathcal{L} = \frac{1}{2}(\dot{X}^2(\tau) - \Psi^\mu(\tau)\dot{\Psi}_\mu(\tau)) \tag{3.1}$$

where the dot $\dot{}$ means ∂_τ, τ being a parametrization of the worldline. Upon canonical quantization $\Psi^\mu(\tau)$ is replaced by a τ-independent operator $\hat{\Psi}^\mu$ which satisfies anticommutation relations

$$\{\hat{\Psi}^\mu, \hat{\Psi}_\nu\}_+ = 2\delta^\mu_\nu \tag{3.2}$$

The Hamiltonian is

$$H = \frac{1}{2}p^2 = \frac{1}{2}Q^2 \tag{3.3}$$

with $Q = p_\mu \hat{\Psi}^\mu$. Due to (3.2) the $\hat{\Psi}$'s can be represented by Dirac matrices and Q is the free Dirac operator. Q commutes with H and it makes sense to consider the restriction of the Hilbert space to the set of states $|\varphi>$ satisfying

$$Q|\varphi>= 0 \tag{3.4}$$

By definition of Q, this equation means that the $|\varphi>$ are the states of a massless spin one-half particle. The extension to the case of a massive particle implies the introduction of an additional Grassmannian variable Ψ^{D+1} and the generalization of $\mathcal{L}$ to

$$\mathcal{L} = \frac{1}{2}(\dot{X}^2(\tau) - \Psi^\mu(\tau)\dot{\Psi}_\mu(\tau) - \Psi^{D+1}(\tau)\dot{\Psi}^{D+1}(\tau) + m^2) \tag{3.5}$$

(Formally, $\dot{X}^{D+1} \to m$), so that

$$H = \frac{1}{2}(p^2 - m^2) = \frac{1}{2}Q^2 \tag{3.6}$$

with

$$Q = p_\mu \hat{\Psi}^\mu + m\hat{\Psi}^{D+1} \tag{3.7}$$

and one has in addition to (3.2)

$$\{\hat{\Psi}^{D+1}, \hat{\Psi}^{D+1}\}_+ = -2 \qquad \{\hat{\Psi}^\mu, \hat{\Psi}^{D+1}\}_+ = 0 \tag{3.8}$$

The condition (3.4) is now the free Dirac equation for a spin one-half particle of mass m, multiplied by $\hat{\Psi}^{D+1}$. The generalization to the case of an arbitrary spin is obtained by duplicating $2S$ times the components of Ψ, $\Psi^\mu \to \Psi^\mu_i, 1 \le i \le 2S$, as can be seen by constructing the representations of $SO(D)$ by suitable tensor products of spin one-half representations [21] [22].

To understand the constraint (3.4), it is in fact necessary to promote the global supersymmetry of the action, corresponding to the commutation of H and Q, into a local supersymmetry. Indeed, when time flows, the state of the particle must evolve from a solution of the Dirac equation to another solution of this equation, without any possibility to collapse in an unphysical state (out of $Ker(Q)$). A natural way to reach such a unitarity requirement is to impose the supersymmetry independently for all values of τ, that is, to gauge the supersymmetry on the worldline. In this way, the condition (3.4) appears as the definition of physical states in a gauge theory with generator Q which ensures unitarity, like the transversality condition of gauge bosons in ordinary Yang-Mills theory. For consistency, the diffeomorphism invariance on the worldline must be also imposed since the commutator of two supersymmetry transformations contains a diffeomorphism. One thus introduces gauge fields for these symmetries, the einbein $e(\tau)$ and the (anticommuting) gravitino $\alpha(\tau)$. By minimal coupling on the worldline, (3.5) is thus generalized to the following Lagrangian which is locally supersymmetric and reparametrization invariant, up to a pure derivative with respect to τ :

$$\mathcal{L} = \frac{1}{2}\left(e^{-1}\dot{X}^2 - \Psi(\dot{\Psi} + \alpha e^{-1}\dot{X}) - \Psi^{D+1}(\dot{\Psi}^{D+1} + m\alpha) + em^2\right) \tag{3.9}$$

(we will now omit the vector and spin indices). Formally, $\dot{X}^{D+1} \rightarrow me$. The transformation laws of e and α are those of one-dimensional supergravity of rank $2S$.

The gauge-fixing $e(\tau) = 1$ and $\alpha(\tau) = 0$ identifies (2.5) and (2.9), up to Faddeev-Popov ghost terms. These ghost terms have a supersymmetric form $b\dot{c}+\beta\dot{\gamma}$. They decouple effectively, since their effect is to multiply all the amplitudes by a ratio of determinants, independent of the metric in space-time. This gauge-fixing is however inconsistent because it is too strong, since the Lagrangian is gauge invariant only up to boundary terms. Therefore, given a general gauge transformation, one must put restrictions on its parameters to get the invariance of the action, and there are not enough degrees of freedom in the symmetry to enforce the gauge $e(\tau) = 1$ and $\alpha(\tau) = 0$. One can at most set $e(\tau) = e_0$ and $\alpha(\tau) = \alpha_0$, letting the constants $e_0 > 0$ and α_0 free, that is, doing an ordinary integration over e_0 and α_0 in the path integral after the gauge-fixing [19]. This yields the following partition function for the theory

$$Z = \int_0^\infty de_0 \int d\alpha_0 \int [dX(\tau)][d\Psi(\tau)] \exp - \int_0^1 d\tau \mathcal{L}_0 \tag{3.10}$$

with

$$\mathcal{L}_0 = \frac{1}{2}\left({e_0}^{-1}\dot{X}^2 + e_0 m^2 - \Psi(\dot{\Psi} + \alpha_0 {e_0}^{-1}\dot{X}) - \Psi^{D+1}(\dot{\Psi}^{D+1} + m\alpha_0)\right) \tag{3.11}$$

Using the Lagrangian (3.5) instead of (3.11) implies that one misses crucial spin-orbit interactions described by the Grassmannian integration over the constant α_0 which induces the fermionic constraint $\int d\tau(\Psi\dot{X} + me_0\Psi^{D+1}) = 0$. The use of (3.5) leads indeed to a spin-zero particle propagator while (2.11) leads to the expected spin one-half propagator. One gets the $\pm i\epsilon$ propagators depending on the choice of the integration domain $\{e_0 > 0\}$ or $\{e_0 < 0\}$. Notice that the e-dependence of the Lagrangian (3.5) gives a negligible weight in the path integral (3.10) to the trajectories with points near the hyperplane $\{e_0 = 0\}$. The integration over e_0 and α_0 has a simple interpretation in Hamiltonian formalism. The Hamiltonian associated to (3.11) is

$$H \;=\; \frac{e_0}{2}(p^2 - m^2) + \frac{\alpha_0}{2}(p_\mu \hat{\Psi}^\mu + m\hat{\Psi}^{D+1})$$

$$= \frac{e_0}{2}(p^2 - m^2) + \frac{\alpha_0}{2}Q \tag{3.12}$$

The constants e_0 and α_0 are thus Lagrange multipliers which force the particle to satisfy the Klein-Gordon equation and the Dirac equation (or its higher spin generalizations $Q_i|\varphi >= 0$). Observe that in Lagrangian formalism, the Klein-Gordon equation is not a consequence of the Dirac equation, due to the anticommutativity of Grassmann variables, and the two constraints $Q|\varphi >= 0$ and $H|\varphi >= 0$ must be used separately. Therefore, we have a theory where the Hamiltonian is a sum of constraints, which leads to known technical difficulties [17][18]. In Lagrangian formalism, supergravity on the worldline and its correct gauge-fixing take care of all details [19].

The above description is valid for a flat space-time. It can be generalized to the case where the particle moves in a curved space-time and/or couples to an external electromagnetic field, by minimal coupling in the target-space. The compatibility between the worldline diffeomorphism invariance and local supersymmetry with reparametrization invariance in the target-space for a general metric $g_{\mu\nu}$ is however possible only for $N \leq 2$ [21]. This phenomenon is possibly related to the limited number of consistent supergravities [23].

We will now consider the case $N = 2$ and show n the link of the theory with a topological model.

The $N = 2$ supersymmetric Lagrangian with a general background metric $g_{\mu\nu}$ is

$$\begin{aligned} \mathcal{L}_{SUSY} &= \frac{1}{2e} g_{\mu\nu}\dot{X}^\mu \dot{X}^\nu - \overline{\Psi}^\mu (g_{\mu\nu}\dot{\Psi}^\nu + e\Gamma_{\mu\nu\rho}\dot{X}^\nu \Psi^\rho) + e^{-1} g_{\mu\nu}\dot{X}^\mu (\overline{\Psi}^\nu \alpha + \overline{\alpha}\Psi^\nu) \\ &+ \frac{em^2}{2} - \overline{\Psi}^{D+1}\dot{\Psi}^{D+1} + m(\overline{\Psi}^{D+1}\alpha + \overline{\alpha}\Psi^{D+1}) \\ &- e^{-1}\overline{\alpha}\alpha\overline{\Psi}\Psi + \frac{e}{2}R_{\mu\nu\rho\sigma}\overline{\Psi}^\mu \Psi^\nu \overline{\Psi}^\rho \Psi^\sigma \end{aligned} \tag{3.13}$$

where Ψ and $\overline{\Psi}$ are independent Grassmannian coordinates. (Compare with [21]). The Lagrangian (3.13) has two local supersymmetries, with generators Q and $\overline{Q}$. An $O(2)$ symmetry between Ψ and $\overline{\Psi}$ can be enforced by introducing a single gauge field $f(\tau)$ and adding a term $f\overline{\Psi}\Psi$. However, no new information is provided, since one increases the symmetry by one generator, which is compensated by the introduction of the additional degree of freedom carried by f. The latter can indeed be gauge-fixed to zero and one recovers (3.13). Moreover, in view of identifying Ψ and $\overline{\Psi}$ as ghosts and antighosts, one wishes to freeze the symmetry between these two fields. We thus ignore the possibility of gauging the $O(2)$ symmetry. We will check shortly that the Hilbert space associated to the Lagrangian (3.1) contains spin-one particles.

The Lagrangian (3.13) can be conveniently rewritten in first order formalism by introducing a Lagrange multiplier $b^\mu(\tau)$. One gets the equivalent form

$$\begin{aligned} \mathcal{L}_{SUSY} \sim & -\frac{e}{2}(g_{\mu\nu}b^\mu b^\nu - m^2) + g_{\mu\nu}b^\mu(\dot{X}^\nu + e\Gamma^\nu_{\rho\sigma}\overline{\Psi}^\rho\Psi^\sigma + \overline{\Psi}^\nu\alpha + \overline{\alpha}\Psi^\nu) \\ & -\overline{\Psi}^\mu(g_{\mu\nu}\dot{\Psi}^\nu + e\partial_\rho g_{\mu\nu}\dot{X}^\nu\Psi^\rho) - \Gamma_{\nu\rho\sigma}\overline{\Psi}^\rho\Psi^\sigma(\overline{\Psi}^\nu\alpha + \overline{\alpha}\Psi^\nu) \end{aligned}$$

$$-\overline{\Psi}^{D+1}\dot{\Psi}^{D+1} + m(\overline{\Psi}^{D+1}\alpha + \overline{\alpha}\Psi^{D+1}) - \frac{e}{2}\partial_\nu\Gamma_{\mu\rho\sigma}\overline{\Psi}^\mu\Psi^\nu\overline{\Psi}^\rho\Psi^\sigma \quad (3.14)$$

(The symbol $\sim$ means that the two Lagrangians differ by a term which can be eliminated using an algebraic equation of motion, and, consequently, define the same quantum theory). For $e = 1$, $\alpha = \overline{\alpha} = 0$ and $\Psi^{D+1} = \overline{\Psi}^{D+1} = 0$, the Lagrangian (3.2) can be interpreted as the gauge-fixing of zero or of a term invariant under isotopies of the curve X [4]. In this interpretation the Ψ are topological ghosts and the $\overline{\Psi}$ are antighosts. The BRST graded differential operator s of the topological symmetry is defined by

$$\begin{aligned} sX^\mu &= \Psi^\mu \\ s\Psi^\mu &= 0 \\ s\overline{\Psi}^\mu &= b^\mu \\ sb^\mu &= 0 \end{aligned} \quad (3.15)$$

and the gauge-fixing Lagrangian is s-exact modulo a pure derivative

$$\mathcal{L}_{GF} = s(\overline{\Psi}_\mu(-\frac{1}{2}b^\mu + \dot{X}^\mu + \frac{1}{2}\Gamma^\mu_{\rho\sigma}\overline{\Psi}^\rho\Psi^\sigma)) \quad (3.16)$$

(Since $s^2 = 0$, $\mathcal{L}_{GF}$ is s-invariant.) To identify (3.1) as a topological Lagrangian, we must introduce new ingredients. We will enlarge the target-space with two additional components, and add a ghost of ghost. We will eventually identify one of the extra coordinates with the einbein e and the other one will be forced to vary in a Gaussian way around an arbitrary scale, with an arbitrary width. The gravitini α and $\overline{\alpha}$ of the effective worldline supergravity will be interpreted as ghosts of the topological symmetry. The $O(2)$ invariance corresponds to the ghost number conservation.

We consider a $(D+2)$-dimensional space-time with coordinates $X^A = (X^\mu, X^{D+1} = e, X^{D+2})$. We exclude from the space the hyperplane $\{X^{D+1} = 0\}$ which yields two separated half-spaces, characterized by the value of $sign(e)$. We wish to define a partition function through a path integration over the curves $X^A(\tau)$, with a topological action which is invariant under the BRST symmetry associated to isotopies of this curve in each half-space. In other words we wish to construct an action by consistently gauge-fixing the topological Lagrangian $sign(e)$. In a way which is analogous to the case of topological Yang-Mills symmetry, where one gauge-fixes the second Chern class $\int Tr\ F^2$ [4], we combine the pure topological symmetry, with topological ghosts $\Psi^A_{top}(\tau)$, to the diffeomorphism symmetry on the curve, with Faddeev-Popov ghost $c(\tau)$. The apparent redundancy in the number of ghost variables $\Psi^A_{top}(\tau)$ and $c(\tau)$, which exceeds the number of bosonic classical variables, is counterbalanced by the introduction of a ghost of ghosts $\Phi(\tau)$ with ghost number two. The action of the BRST differential s is defined by

$$\begin{aligned} sX^\mu &= \Psi^\mu_{top} + c\dot{X}^\mu = \Psi^\mu \\ se &= \Psi^e_{top} + c\dot{e} = 2\eta = \alpha + \dot{\Psi}^{D+1} \end{aligned}$$

$$
\begin{aligned}
sX^{D+2} &= \Psi^{D+2}_{top} + c\dot{X}^{D+2} = \Psi^{D+2} \\
s\Psi^{\mu} &= 0 \\
s\Psi^{D+2} &= 0 \\
s\Psi^{D+1} &= \Phi \\
s\alpha &= -\dot{\Phi} \\
s\Phi &= 0
\end{aligned} \tag{3.17}
$$

In agreement with the art of BRST invariant gauge-fixing, we introduce $D+2$ antighosts with ghost number (-1) and the associated Lagrange multipliers for the gauge conditions on the X^A's. We also introduce an antighost $\overline{\Phi}$ with ghost number (-2) and its fermionic partner $\overline{\eta}$ with ghost number (-1) which we will use as a fermionic Lagrange multiplier for the gauge condition in the ghost sector. In this sector the action of s is

$$
\begin{aligned}
s\overline{\Psi}^{A} &= b^{A} \\
sb^{A} &= 0 \\
s\overline{\Phi} &= \overline{\eta} \\
s\overline{\eta} &= 0
\end{aligned} \tag{3.18}
$$

The gauge-fixing Lagrangian must be written as an s-exact term

$$
\mathcal{L}^{X} + \mathcal{L}^{D+1} + \mathcal{L}^{D+2} + \mathcal{L}^{\Phi} = s\left(\overline{\Psi}^{A}(\ldots)_{A} + \overline{\Phi}(\ldots)\right) \tag{3.19}
$$

For the gauge-fixing in the X-sector, we choose

$$
\begin{aligned}
\mathcal{L}^{X} = {}& s \quad \left(-\frac{e}{2}g_{\mu\nu}\overline{\Psi}^{\nu}b^{\mu} + g_{\mu\nu}\overline{\Psi}^{\mu}(\dot{X}^{\nu} + \overline{\eta}\Psi^{\nu} + \frac{e}{2}\Gamma^{\nu}_{\rho\sigma}\overline{\Psi}^{\rho}\Psi^{\sigma})\right) \\
= {}& - \quad \frac{e}{2}g_{\mu\nu}b^{\mu}b^{\nu} + g_{\mu\nu}b^{\mu}(\dot{X}^{\nu} + e\Gamma^{\nu}_{\rho\sigma}\overline{\Psi}^{\rho}\Psi^{\sigma} + \overline{\Psi}^{\nu}\eta + \overline{\eta}\Psi^{\nu}) \\
& - \quad \overline{\Psi}^{\nu}(g_{\mu\nu}\dot{\Psi}^{\mu} + e\partial_{\rho}g_{\mu\nu}\dot{X}^{\nu}\Psi^{\rho}) - \frac{e}{2}\partial_{\nu}\Gamma_{\mu\rho\sigma}\overline{\Psi}^{\mu}\Psi^{\nu}\overline{\Psi}^{\rho}\Psi^{\sigma} - \Gamma_{\mu\rho\sigma}\overline{\Psi}^{\mu}\eta\overline{\Psi}^{\rho}\Psi^{\sigma}
\end{aligned} \tag{3.20}
$$

For the gauge-fixing in the e-sector, we choose

$$
\begin{aligned}
\mathcal{L}^{D+1} &= -s\left(\overline{\Psi}^{D+1}e(m + \frac{b^{D+1}}{2})\right) \\
&= -e\frac{(b^{D+1})^2}{2} + b^{D+1}(-me + \overline{\Psi}^{D+1}\eta) + 2m\overline{\Psi}^{D+1}\eta
\end{aligned} \tag{3.21}
$$

After elimination of the field b^{D+1},we obtain

$$
\mathcal{L}^{D+1} \sim \frac{em^2}{2} + m\overline{\Psi}^{D+1}\eta \tag{3.22}
$$

For the gauge-fixing in the X^{D+2}-sector, we choose

$$\begin{aligned}
\mathcal{L}^{D+2} &= s\left(\overline{\Psi}^{D+2}\left(-\frac{a}{2}b^{D+2}+X^{D+2}-C-\frac{1}{a}\frac{\overline{\Psi}^{D+1}\dot{\Psi}^{D+1}}{X^{D+2}-C}\right)\right) \\
&= -\frac{a}{2}(b^{D+2})^2+b^{D+2}\left(X^{D+2}-C-a\frac{\overline{\Psi}^{D+1}\dot{\Psi}^{D+1}}{X^{D+2}-C}\right) \\
&\quad -\overline{\Psi}^{D+2}\left(\Psi^{D+2}-as\left(\frac{\overline{\Psi}^{D+1}\dot{\Psi}^{D+1}}{X^{D+2}-C}\right)\right)
\end{aligned} \tag{3.23}$$

a and C are arbitrarily chosen real numbers. After elimination of the field b^{D+2}, we find

$$\begin{aligned}
\mathcal{L}^{D+2} \sim & - \overline{\Psi}^{D+1}\dot{\Psi}^{D+1}+\frac{1}{2a}(X^{D+2}-C)^2 - \\
& -\overline{\Psi}^{D+2}\left(\Psi^{D+2}-as\left(\frac{\overline{\Psi}^{D+1}\dot{\Psi}^{D+1}}{X^{D+2}-C}\right)\right)
\end{aligned} \tag{3.24}$$

The variable X^{D+2} can be eliminated by its algebraic equation of motion as well as the corresponding ghosts Ψ^{D+2} and $\overline{\Psi}^{D+2}$, after some field redefinitions. X^{D+2} is concentrated in a Gaussian way around the arbitrary scale C, with an arbitrary width a. We are thus left with the propagating term for Ψ^{D+1} and $\overline{\Psi}^{D+1}$ which was missing in $\mathcal{L}^X$ and $\mathcal{L}^{D+1}$

$$\mathcal{L}^{D+2} \sim -\overline{\Psi}^{D+1}\dot{\Psi}^{D+1} \tag{3.25}$$

We finally choose the gauge-fixing in the ghost sector. To recover the full Lagrangian (3.14) and eventually identify the coordinate e as the einbein of the projection of the particle trajectory in the D-dimensional physical space-time, we need a term linear in $\overline{\eta}$ as well as another term to get rid of unwanted higher order fermionic terms. We define

$$\begin{aligned}
\mathcal{L}^{\Phi} &= s(\overline{\Phi}(m\Psi^{D+1}-\Gamma_{\nu\rho\sigma}\overline{\Psi}^{\rho}\Psi^{\sigma}\Psi^{\nu})) \qquad (3.26)\\
&= \overline{\eta}(m\Psi^{D+1}-\Gamma_{\nu\rho\sigma}\overline{\Psi}^{\rho}\Psi^{\sigma}\Psi^{\nu})+\overline{\Phi}(m\Phi-s(\Gamma_{\nu\rho\sigma}\overline{\Psi}^{\rho}\Psi^{\sigma}\Psi^{\nu}))
\end{aligned}$$

The dependence on the ghosts of ghosts Φ and $\overline{\Phi}$ is trivial: these fields decouple after a Gaussian integration. One has thus

$$\mathcal{L}^{\Phi} \sim m\overline{\eta}\Psi^{D+1}-\Gamma_{\nu\rho\sigma}\overline{\Psi}^{\rho}\Psi^{\sigma}\overline{\eta}\Psi^{\nu} \tag{3.27}$$

Adding all terms (3.20), (3.22), (3.25) and (3.27), we finally recognize that $\mathcal{L}^X+\mathcal{L}^{D+1}+\mathcal{L}^{D+2}+\mathcal{L}^{\Phi}$ is equivalent to the Lagrangian (3.2), modulo the elimination of auxiliary fields and the change of notation $(\eta,\overline{\eta})\to(\alpha,\overline{\alpha})$. We have therefore shown the announced result: the $N=2$ local supersymmetry of the Lagrangian describing spin-one particles is a residual symmetry coming from a topological model after a suitable gauge-fixing.

To verify the physical content of the model presented just above, we consider a flat space-time, and choose the gauge where the einbein and gravitini are constants over which we integrate. The Hamiltonian is

$$H = \frac{e_0}{2}(p^2 - m^2) + \overline{\alpha}_0 Q + \alpha_0 \overline{Q} \tag{3.28}$$

with

$$\begin{aligned} Q &= p_\mu \Psi^\mu + m\Psi^{D+1} \\ \overline{Q} &= p_\mu \overline{\Psi}^\mu + m\overline{\Psi}^{D+1} \end{aligned} \tag{3.29}$$

The matrices Ψ and $\overline{\Psi}$ satisfy the Clifford algebra

$$\{\Psi^A, \overline{\Psi}^B\}_+ = \eta^{AB} \quad , \quad \{\Psi^A, \Psi^B\}_+ = \{\overline{\Psi}^A, \overline{\Psi}^B\}_+ = 0 \tag{3.30}$$

for $A, B = 1, ..., D+1$. Since the underlying gauge symmetry has Q and $\overline{Q}$ as generators, the physical states satisfy

$$Q|\phi> = 0 \qquad \overline{Q}|\phi> = 0 \tag{3.31}$$

in addition to

$$(p^2 - m^2)|\phi> = 0 \tag{3.32}$$

The Ψ and $\overline{\Psi}$ are generalizations of the Pauli matrices, and it is convenient to use a Schwinger type construction, in order to exploit directly their Clifford algebra structure. One introduces a spin vacuum $|0>$ annihilated by the Ψ's. Then, the $\overline{\Psi}$'s can be identified as their adjoints and act as creation operators. In the X representation, we can write a general state as

$$\begin{aligned} |\phi> = &\left(\varphi_0 + \varphi_\mu \overline{\Psi}^\mu + \varphi_{\mu_1\mu_2}\overline{\Psi}^{\mu_1}\overline{\Psi}^{\mu_2} + \ldots + \varphi_{\mu_1\ldots\mu_D}\overline{\Psi}^{\mu_1}\ldots\overline{\Psi}^{\mu_D}\right)|0> \\ &+\overline{\Psi}^{D+1}\left(\overline{\varphi}_0 + \overline{\varphi}_\mu \overline{\Psi}^\mu + \overline{\varphi}_{\mu_1\mu_2}\overline{\Psi}^{\mu_1}\overline{\Psi}^{\mu_2} + \ldots + \overline{\varphi}_{\mu_1\ldots\mu_D}\ldots\overline{\Psi}^{\mu_D}\right)|0> \end{aligned} \tag{3.33}$$

The wave functions $\varphi_{\mu_1\ldots\mu_p}(X)$ and $\overline{\varphi}_{\mu_1\ldots\mu_p}(X)$ are antisymmetric and it is useful to consider the differential forms

$$\begin{aligned} \varphi_p &= \frac{1}{p!}dX^{\mu_1}\ldots dX^{\mu_p}\varphi_{\mu_1\ldots\mu_p}(X) \\ \overline{\varphi}_p &= \frac{1}{p!}dX^{\mu_1}\ldots dX^{\mu_p}\overline{\varphi}_{\mu_1\ldots\mu_p}(X) \end{aligned} \tag{3.34}$$

for $0 \leq p \leq D$. The constraints (3.31) can be conveniently written as

$$d\varphi_p + im\overline{\varphi}_{p+1} = 0 \tag{3.35}$$

$$d^*\overline{\varphi}_p + im\varphi_{p-1} = 0 \tag{3.36}$$

$$d\overline{\varphi}_p = 0 \tag{3.37}$$

$$d^* \varphi_p = 0 \tag{3.38}$$

Where $d = dx^\mu \partial_\mu$ and d^* is its Hodge dual. One has also

$$(d^* d + dd^*)\varphi = -m^2 \varphi \qquad (d^* d + dd^*)\overline{\varphi} = -m^2 \overline{\varphi} \tag{3.39}$$

These equations determine the independent degrees of freedom. When $m \neq 0$, they couple the two sectors of opposite chiralities. Moreover, when D is even, the first one contains $\frac{D}{2}$ forms, namely one scalar (φ_0), one vector (φ_1), ..., and one ($\frac{D}{2} - 1$)-form ($\varphi_{\frac{D}{2}-1}$). The other one has a dual structure ($\overline{\varphi}_{\frac{D}{2}+1}$,...,$\overline{\varphi}_D$). For ($\varphi_1$), the constraints (4.4) can be rewritten:

$$\begin{aligned} \partial_\mu \overline{\varphi}^{\mu\nu} + im\varphi^\nu &= 0 \\ \partial_\mu \varphi^\mu &= 0 \\ \partial_{[\mu} \varphi_{\nu]} + im\overline{\varphi}_{\mu\nu} &= 0 \end{aligned} \tag{3.40}$$

Thus the vector wave function φ_1 satisfies Proca's equations, and descibes a spin-one particle with mass m. It follows that the field equations of φ_1 and $\overline{\varphi}_1$ can be derived by minimizing Proca's Lagrangian

$$\begin{aligned} \mathcal{L}_{Proca} = &\frac{m}{2}\overline{\varphi}_{\mu\nu}\varphi^{\mu\nu} - \frac{i}{2}\varphi^{\mu\nu}(\partial_\mu \varphi_\nu - \partial_\nu \varphi_\mu) \\ &-\frac{i}{2}\overline{\varphi}^{\mu\nu}(\partial_\mu \overline{\varphi}_\nu - \partial_\nu \overline{\varphi}_\mu) + m\overline{\varphi}_\mu \varphi^\mu \end{aligned} \tag{3.41}$$

When $m = 0$, the two sectors of opposite chiralities decouple. In each sector, the independent degrees of freedom are now one 0-form A_0 (with $\varphi_1 = dA_0$), one 1-form A_1 (with $\varphi_2 = dA_1$),..., one (D-2)-form A_{D-2} (with $\varphi_{D-1} = dA_{D-2}$). The φ_p's are closed and co-closed, i.e. the A_p's satisfy Maxwell's equations and are defined up to gauge transformations. Consequently, φ_2 can be identified with the field strength of a photon. If we consider the case $D = 4$ and $m \neq 0$, the spectrum reduces to two scalars and two massive spin-one particles, and contains 8=2(1+3) degrees of freedom. For $m = 0$, we have two massless scalars and two massless vectors, so that we still have 8=2(1+1+2) independent degrees of freedom.

As an application of this formalism, we study the classical behavior of spinning particles in a curved space-time. We are interested in the approximation where the trajectory of the particle is classical, while the spin effects are visible as it would be the case in a Stern-Gerlach experiment. This situation occurs if the order of magnitude of the interaction energy between the spin and the curvature, which is essentially proportional to the space-time curvature times $\hbar$ (analogously to the interaction between the the magnetic field and a magnetic moment due to the spin), is comparable to the kinematical energy of the particle. One must also measure the position of the particle on a domain much larger than its Compton wavelength. In this limit the position X^μ and momentum P_μ are ordinary numbers and the quantum Hamiltonian becomes simply a matrix built from the Ψ's and $\overline{\Psi}$'s acting in the spin-space with coefficients depending on the classical position X and momentum P. The τ-dependence of the classical dynamics of the particle can be expressed by applying Hamilton-Jacobi's method with this matricial Hamiltonian. The only quantum effects are due to the spin interaction with the space-time curvature. (In a fully classical approximation, $\hbar = 0$, and the spin effects disappear, since all the fermionic operators are proportional to $\sqrt{\hbar}$.) One can always find a basis for the spin states, which depends on the space-time position and such that the Hamiltonian is diagonal. In this basis the spin value is conserved through evolution, i.e. the spin observables are paralelly transported along the trajectory. By diagonalization in spin space, H determines independent Hamilton-Jacobi's equations for each spin degree of freedom of the particle. For the spin-one case, we expect three different trajectories corresponding to the values 1, 0 and -1 for the projection of the spin on a spatial axis in the rest frame of the particle.

We consider the case of a Schwarzschild gravitational field in four dimensional space-time ($ds^2 = (1-\frac{r_0}{r})dt^2 - (1-\frac{r_0}{r})^{-1}dr^2 - r^2(d\theta^2 + sin^2\theta\ d\phi^2)$ with $r_0 = 2GM/c^2$.) We will compute the correction, due to the spin, to Einstein's formula predicting the shift of the perihelion of a spinless point particle. For the other classical test of general relativity, i.e. the bending of light rays in a gravitational field, we will find that the wave vector of a polarized photon deviates from geodesic motions by a relative shift proportional to $\hbar$. These results are in agreement with the fact that a particle with an angular momentum interacts with the space-time curvature, as first pointed out by Papapetrou for a rotating body [25]. The advantage of a supersymmetric Hamiltonian is that it defines unambiguously the spin effects. Since we work to first non-trivial order in $\hbar$, we restore from now on the $\hbar$ dependence in the formulae. The matricial Hamilton-Jacobi's equation is obtained by replacing in the supersymmetric Hamiltonian the classical momentum p_μ by $\frac{\delta S}{\delta X^\mu}$ where $S[X^\mu, \tau]$ is the action of the classical trajectory of the particle in a given spin state, with arbitrarily chosen initial and final boundary conditions. Notice that keeping the lowest order in $\hbar$ means that we only retain the covariant derivative of the fermionic variables and not the curvature term. This yields

$$g^{\mu\nu}\frac{\delta S}{\delta X^\mu}\frac{\delta S}{\delta X^\nu} - m^2 + 2\hbar\frac{\delta S}{\delta X^\mu}\omega^\mu_{ab}\Sigma^{ab} + O(\hbar^2) = 0 \tag{3.42}$$

The space-time spin-connection ω is related to the space-time vierbein E and to Christoffel's symbol Γ

$$\omega_{\mu ab} = E^\alpha_a E^\beta_b \Gamma_{\alpha\mu\beta} \tag{3.43}$$

$$E^a_\alpha E^b_\beta \eta_{ab} = g_{\alpha\beta} \tag{3.44}$$

$$\Gamma_{\alpha\mu\beta} = \frac{1}{2}(\partial_\mu g_{\alpha\beta} + \partial_\beta g_{\alpha\mu} - \partial_\alpha g_{\beta\mu}) \tag{3.45}$$

The $\Sigma^{ab} = \frac{i}{2}\left(\overline{\Psi}^a\Psi^b - \overline{\Psi}^b\Psi^a\right)$ are the generators of the (reducible) 32-dimensional representation of the Lorentz group defined by the algebra (4.3) and acting on the states solving (4.6). If the matrix form of $\overline{\Psi}^5$ is chosen diagonal, the spin operators Σ^{ab} become block-diagonal with two independent sectors of opposite chiralities, corresponding to the eigenvalues 0 and 1 of $\overline{\Psi}^5\Psi^5$, so the 32-dimensional representation splits into two independent 16-dimensional representation, each one containing five sectors of dimensions 1,4,6,4,1 corresponding respectively to 0-forms, 1-forms, 2-forms, 3-forms, and 4-forms. As explained above, the constraints imply that only two block-sectors made of one 0-form and one 1-form sectors are independent wave-functions. The one-form sector, and the corresponding 4×4 Hamiltonian matrix, determine the dynamics of spin-one particles. Moreover, in a Schwarzschild metric with characteristic radius r_0,the motion is planar, so one can separate the variables and write

$$S = -Et + L\varphi + S_r(r) \tag{3.46}$$

The spin-dependent part of Hamilton-Jacobi's equation is obtained by the substitution

$$p_\mu \omega^\mu_{ab} \Sigma^{ab} = \frac{r_0 E}{r^2} \Sigma^{01} - \frac{2L}{r^2} \left(1 - \frac{r_0}{r}\right)^{1/2} \Sigma^{13} \tag{3.47}$$

where

$$\Sigma^{01} = \frac{1}{2} \begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad \Sigma^{13} = \frac{1}{2} \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & i \\ 0 & 0 & 0 & 0 \\ 0 & -i & 0 & 0 \end{pmatrix} \tag{3.48}$$

By inserting (3.46) and (3.47) into Hamilton-Jacobi's equation (3.42), one obtains a matricial equation for $\frac{\partial S}{\partial r}$. The diagonalization can be done easily, and one gets three possibilities S_ϵ for the classical action, indexed by $\epsilon = 0, \pm 1$

$$\frac{E^2}{c^2}\left(1 - \frac{r_0}{r}\right)^{-1} - \left(\frac{L^2}{r^2} + m^2\right) - \left(1 - \frac{r_0}{r}\right)\left(\frac{\partial S_\epsilon}{\partial r}\right) \\ +2\epsilon \frac{\hbar}{r^2} \left(L^2 \left(1 - \frac{r_0}{r}\right) - \left(\frac{r_0 E}{2c}\right)^2\right)^{1/2} = 0 \tag{3.49}$$

(We have restored the dependence in the speed of light c.) The energy E and the angular momentum L are constants of motion of the particle. The values $\epsilon = 0, \pm 1$ correspond to the three possible projections of the spin along a given spatial axis in the rest frame of the particle. The case $\epsilon = 0$ corresponds to the geodesic trajectory followed by the scalar particle. Far from the Schwarzschild horizon, we can use the standard techniques of integration of Hamilton-Jacobi's equation to determine the three possibilities for the shift of the perihelion over a quasi-periodic trajectory. This amounts to replace L in the classical formulas [26] by an effective angular momentum L_ϵ defined by

$$L_\epsilon^2 = L^2 + 2\epsilon\hbar L \sqrt{1 - \left(\frac{r_0 E}{2Lc}\right)^2} \tag{3.50}$$

(Notice that near the horizon, unitarity breaks down). In the case of a massive particle, the shift of the perihelion is thus given by:

$$\delta\phi_\epsilon = \frac{3\pi}{2}\left(\frac{mcr_0}{L_\epsilon}\right)^2 \sim \delta\phi_0\left(1-\epsilon\frac{\hbar}{L}\sqrt{1-\left(\frac{r_0E}{2Lc}\right)^2}\right) \tag{3.51}$$

For a non-relativistic Z^0 orbiting quasi-tangentially to the sun at a speed of $10^5 m/s$, which is approximately the circular velocity around the sun , we find $|\delta\phi_+ - \delta\phi_0|/\delta\phi_0 \sim \hbar/L \sim 10^{-21}$, which is much to small to be detected.

The solutions of Hamilton-Jacobi's equation are continuous when $m \to 0$. However, in this limit the interpretation of its solution S is different. The particle is a photon following the laws of the geometrical optics, S is the eikonal of the light ray, and $\frac{\delta S}{\delta X^\mu}$ is its wave-vector. The solution $\epsilon = 0$ must then be rejected. In this massless case, one finds for the deflections of the two helicities $\epsilon = \pm 1$ the following formula

$$\delta\phi_\epsilon = \frac{2r_0\omega}{cL_\epsilon} \sim \delta\phi_0\left(1-\epsilon\frac{\hbar}{2L}\sqrt{1-\left(\frac{r_0E}{2Lc}\right)^2}\right) \tag{3.52}$$

where $\omega = \frac{E}{\hbar}$. For an optical photon of wavelength $\lambda = 7\times 10^{-7}m$ (red) skimming past the sun, we find $|\delta\phi_+ - \delta\phi_0|/\delta\phi_0 \sim \frac{\hbar}{2L} = \frac{\lambda}{2R_{sun}} \sim 10^{-15}$. (Note that this ratio does not depend on $\hbar$: the gravitational field interacts classically with the two polarizations of the electromagnetic field.) However, this doubling of Einstein's rings is to small to be detected.

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