Anomalous Dimensions for Scalar Operators in ABJM Theory

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Declaration of Authorship

This dissertation titled, 'Anomalous Dimensions for Scalar Operators in ABJM Theory' is being submitted to the Faculty of Science, University of the Witwatersrand, Johannesburg in fulfilment of the requirements for the degree of Master of Science. I, Rocky Kreyfelt, declare that this thesis is my own unaided work. It has not been submitted for any other degree or examination in any other university.

My two original works discussed in this dissertation are:

1. R. de Mello Koch, R. Kreyfelt and N. Nokwara, "Finite N Quiver Gauge Theory," (2014), Physical Review D, Volume 89, ID 126004, arXiv:1403.7592 [hep-th].
2. R. de Mello Koch, R. Kreyfelt and S. Smith, "Heavy Operators in Superconformal Chern-Simons Theory" (2014), Physical Review D, Volume 90, ID 126009, arXiv:1410.0874 [hep-th].

These papers are references [24] and [37] in the bibliography and constitute the subject matter of chapters 6 and 7 (along with the appendices B, C and D) respectively.

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Abstract

At finite N, the number of restricted Schur polynomials is greater than, or equal to the number of generalized restricted Schur polynomials. In this dissertation we study this discrepancy and explain its origin. We conclude that, for quiver gauge theories, in general, the generalized restricted Shur polynomials correctly account for the complete set of finite N constraints and they provide a basis, while the restricted Schur polynomials only account for a subset of the finite N constraints and are thus overcomplete. We identify several situations in which the restricted Schur polynomials do in fact account for the complete set of finite N constraints. In these situations the restricted Schur polynomials and the generalized restricted Schur polynomials both provide good bases for the quiver gauge theory. Further, we demonstrate situations in which the generalized restricted Schur polynomials reduce to the restricted Schur polynomials and use these results to study the anomalous dimensions for scalar operators in ABJM theory in the $SU(2)$ sector. The operators we consider have a classical dimension that grows as N in the large N limit. Consequently, the large N limit is not captured by summing planar diagrams – non-planar contributions have to be included. We find that the mixing matrix at two-loop order is diagonalized using a double coset ansatz, reducing it to the Hamiltonian of a set of decoupled oscillators. The spectrum of anomalous dimensions, when interpreted in the dual gravity theory, shows that the energy of the fluctuations of the corresponding giant graviton is dependent on the size of the giant. The first subleading corrections to the large N limit are also considered. These subleading corrections to the dilatation operator do not commute with the leading terms, indicating that integrability probably does not survive beyond the large N limit.

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I would like to thank my supervisor Prof. Robert de Mello Koch, without whom this thesis would not be possible, for his assistance, enthusiasm, time and patience. I am grateful that he was willing to accommodate me as a student.

Many thanks to James Anthonyrajah for countless interesting and fruitful discussions.

Contents

Chapter 1

Introduction

The standard model of particle physics is a quantum field theory in $3+1$-dimensional Minkowski space-time which incorporates all known forces in nature except gravity into a single theory. The methods used to quantize fields in quantum field theory do not work for gravity. Gravity is not renormalizable, meaning that physical observables take infinite values with no way to extract sensible finite physical results. Quantum gravitational effects may only be probed at energies significantly higher than those currently achieved in experiments – this fact has confined quantum gravity exclusively to the realm of theoretical physics for the time being.

Since gravity is not renormalizable, new approaches for developing a quantum theory of gravity must be considered. String theory provides one such approach and is perhaps the most promising candidate for a theory of quantum gravity. String theory models particles not as zero dimensional points but as one dimensional strings with different modes of oscillation giving rise to different particles. Just as the path of a 0-dimensional point particle moving through space-time is represented by a 1-dimensional world line, a 1-dimensional string moving through space-time traces out a 2-dimensional surface called a string world sheet.

Much recent work in string theory has focused on the AdS/CFT correspondence[73] first conjectured by Maldacena in 1997. The AdS/CFT correspondence conjectures a duality between string theories (which contain gravity) and large N gauge theories (which do not contain gravity). The most extensively studied example of the correspondence is the duality between type IIB string theory on $AdS_5 \times S^5$ and $\mathcal{N} = 4$ supersymmetric Yang-Mills (SYM) theory (a $3+1$-dimensional gauge theory which enjoys supersymmetry and conformal invariance). Conformal invariance and supersymmetry are not properties of any fundamental physical theory known to describe nature, but supersymmetric and conformally invariant toy models are simpler and provide a practical environment for testing approaches which can then be applied to more complex models.

The AdS/CFT correspondence has proven to be useful for a number of reasons, one of which comes from the fact that the duality is a strong-weak duality. A strongly coupled string theory is dual to a weakly coupled field theory and vice versa. The AdS/CFT correspondence therefore provides a powerful computational tool as computations in a strongly coupled theory are more mathematically tractable in the weakly interacting dual theory. To this end, much work has been done to establish a 'dictionary' or map between field theory observables and the corresponding dual observables in the string theory. Of particular importance to us is the duality between the spectrum of anomalous dimensions on the CFT side and energies on the string theory side.

Integrability has proven to be a powerful tool in analysing the spectrum of anomalous dimensions in $\mathcal{N} = 4$ Super Yang-Mills theory in the planar limit[7, 8]. An interesting question is whether or not there are other large N limits that are also integrable. This question has been the focus of a number of recent studies[9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20]. At this point there is evidence that suggests certain large N limits, that are not captured by simply summing the planar diagrams, do enjoy integrability.

The studies described above have all focused on $\mathcal{N} = 4$ super Yang-Mills theory. In this dissertation we extend existing studies by exploring a large N but non-planar limit of the ABJM theory, which is an $\mathcal{N} = 6$ superconformal Chern-Simons-matter theory with gauge group $U(N) \times U(N)$ on $R^{1,2}$ and Chern-Simons levels k and $-k$. Almost all of the results that have been obtained in the planar limit of $\mathcal{N} = 4$ super Yang-Mills theory hold in an appropriately modified form for the ABJM theory[21]. Further, the technology needed to study operators with anomalous dimensions that grow as N (called "heavy operators") has been developed[22, 23, 24]. It is thus very natural to search for possible large N but non-planar limits of ABJM theory that enjoy integrability. This is the primary motivation for the study reported in this dissertation.

This dissertation is structured as follows:

- The remainder of this chapter is devoted to the study of the duality between a simple matrix model and a string theory.
- Chapter 2 and chapter 3 briefly introduce some group representation theory which provides the mathematical tools necessary to perform the computations presented in chapters 4 through 7.

- Chapters 4 and 5 review important details concerning Schur polynomials and Restricted Schur polynomials respectively, the most important of which is the action of the dilatation operator on a Restricted Schur.
- Chapter 6 and chapter 7 present the work published in [24]and [37] respectively and comprise the bulk of this dissertation.
- In chapter 8 we summarize our results and point out some interesting directions in which this study can be extended.

1.1 A Matrix Toy Model – A Simple Demonstration of Gauge/Gravity Duality

By studying a simple matrix toy model which has the same basic mathematical structure of more complex matrix field theories we can see a manifestation of the AdS/CFT correspondence. This section is dedicated to introducing a simple matrix model with the goal of illustrating the connection between this model and a string theory. We demonstrate that expectation values of operators in this matrix model can be calculated using ribbon diagrams in much the same way as Feynman diagrams are used in Quantum Field Theory. These ribbon diagrams triangulate surfaces of different topologies which correspond to the world sheets of strings moving through space-time.

1.1.1 Expectation Values and Factorisation – A Classical Limit

Recall that in quantum mechanics our system can be in any one of a collection of states. The probability of finding the system in the state i (i is a label for our states – it can be discrete, continuous or even represent a set of labels) is μ_i where:

$$1 \geq \mu_i \geq 0$$

and

$$\sum_i \mu_i = 1$$

If an observable takes the value $O_A(i)$ in state i, where A is a label for our observables, then the expectation value of O_A is:

$$\langle O_A \rangle = \sum_i \mu_i O_A(i)$$

Sometimes the expectation value of a product of operators is the product of the expectation values. This is called factorisation:

$$\begin{aligned} \langle O_{A_1} O_{A_2} \dots O_{A_p} \rangle &= \langle O_{A_1} \rangle \langle O_{A_2} \rangle \dots \langle O_{A_p} \rangle \\ \sum_i \mu_i O_{A_1}(i) O_{A_2}(i) \dots O_{A_p}(i) &= \sum_{i_1} \mu_{i_1} O_{A_1}(i_1) \sum_{i_2} \mu_{i_2} O_{A_2}(i_2) \dots \sum_{i_p} \mu_{i_p} O_{A_p}(i_p) \end{aligned}$$

The sum on the right hand side of the above equation can be expanded to give:

$$\sum_i \mu_i O_{A_1}(i) O_{A_2}(i) \dots O_{A_p}(i) = \sum_i (\mu_i)^p O_{A_1}(i) O_{A_2}(i) \dots O_{A_p}(i) + \text{crossterms}$$

The left hand side of the above equation contains no crossterms (crossterms are the terms which mix the states of the system). By equating coefficients we see that the right hand side cannot contain any crossterms either. We also obtain the set of equations:

$$\mu_j = (\mu_j)^p \qquad j = 0, 1, 2, \dots$$

The above equation will be satisfied when $\mu_i = 0$ for all values of i except one value of i which we call $i^\star$. Also, $\mu_{i^\star}$ must be equal to 1. We see that when factorisation holds the system is in a definite state with probability 1. We can extend this idea to Quantum Field Theory. If factorisation holds in QFT then only one field configuration contributes:

$$\int \mathcal{D}\phi \phi(x_1) \dots \phi(x_n) e^{iS} = \int \mathcal{D}\phi \phi(x_1) e^{iS} \dots \int \mathcal{D}\phi' \phi'(x_n) e^{iS}$$

This implies that we are in a classical limit of the theory.

1.1.2 A Matrix Toy Model in 0 Dimensions

Consider the integral:

$$I_n = \mathcal{N} \int [dM] \mathrm{Tr}(M^n) e^{-\frac{\alpha}{2}\mathrm{Tr}(M^2)}$$

Where M is an $N \times N$ Hermitian matrix and $\mathcal{N}$ is a normalisation factor. These integrals are normalised so that $I_0 = 1$. In the case where $N = 2$:

$$M = \begin{pmatrix} M_{11} & \frac{M_{12}^R + iM_{12}^I}{\sqrt{2}} \\ \frac{M_{12}^R - iM_{12}^I}{\sqrt{2}} & M_{22} \end{pmatrix}$$

M_{12}^R is the real part of M_{12} and M_{12}^I is the imaginary part of M_{12}. For a 2×2 Hermitian matrix I_0 is:

$$\begin{aligned} I_0 &= \mathcal{N} \int dM_{11} dM_{22} dM_{12}^R dM_{12}^I e^{-\frac{\alpha}{2}(M_{11}^2 + {M_{12}^R}^2 + {M_{12}^I}^2 + M_{22}^2)} \\ &= \mathcal{N} \left[\int dx e^{-\frac{\alpha}{2}x^2} \right]^4 \\ &= \mathcal{N} \left[\sqrt{\frac{2\pi}{\alpha}} \right]^4 \end{aligned}$$

Generalising this result for an $N \times N$ Hermitian matrix gives:

$$I_0 = \mathcal{N} \left[\sqrt{\frac{2\pi}{\alpha}} \right]^{N^2}$$

Next we evaluate some expectation values:

$$\langle M_{11} M_{22} \rangle = 0$$

This expectation value is zero because the integral it represents is odd. To obtain a non-zero expectation value of two matrix elements we must consider even integrals:

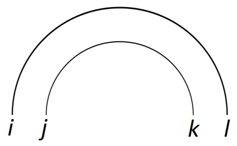
$$\begin{aligned} \langle M_{12} M_{21} \rangle &= \mathcal{N} \int [dM] \frac{1}{2} \left[\left(M_{12}^R\right)^2 + \left(M_{12}^I\right)^2 \right] e^{-\frac{\alpha}{2}(M_{11}^2 + {M_{12}^R}^2 + {M_{12}^I}^2 + M_{22}^2)} \\ &= \mathcal{N} \left[\int dx e^{-\frac{\alpha}{2}x^2} \right]^3 \int dy y^2 e^{-\frac{\alpha}{2}y^2} \\ &= \frac{1}{\alpha} \end{aligned}$$

In general, the expectation value of two matrix elements in this matrix model evaluates to:

$$\langle M_{ij} M_{kl} \rangle = \frac{\delta_{il} \delta_{jk}}{\alpha}$$

One may compute the expectation values of these operators using a graphical approach by defining some Feynman rules. For m operators, draw m pairs of labelled dots. Join the dots in all possible ways using ribbons. The ribbons may not twist and must lie above the dots. The above expectation value is represented by a ribbon diagram:

$$\langle M_{ij} M_{kl} \rangle =$$



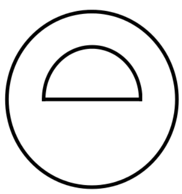
$$= \frac{\delta_{il}\delta_{jk}}{\alpha} \tag{1.1}$$

The ribbon gives a factor of $\frac{1}{a}$ and each pair of dots joined by a line produce a Kronecker delta. Using these results we can derive Feynman rules for evaluating integrals of the form of I_n. For $n = 1$ the integral is odd and evaluates to zero i.e. $I_1 = 0$. For $n = 2$:

$$\begin{aligned} I_2 &= \langle \mathrm{Tr}(M^2) \rangle \\ &= \langle \sum_i \sum_j M_{ij} M_{ji} \rangle \\ &= \sum_i \sum_j \langle M_{ij} M_{ji} \rangle \\ &= \frac{1}{\alpha} \sum_i \sum_j \delta_{ii}\delta_{jj} \\ &= \frac{1}{\alpha} N^2 \end{aligned}$$

$\langle \mathrm{Tr}(M^2) \rangle$ is independent of any indices. We must therefore draw the ribbon graph for $\langle \mathrm{Tr}(M^2) \rangle$ without using any indices. Consider the ribbon graph in (1.1) but with $i = l$ and $j = k$. By joining the pair of i-labels with another line and the pair of j-labels with another line we remove all indices from the ribbon graph. We therefore use the following diagram for $\langle \mathrm{Tr}(M^2) \rangle$:

$$\langle \mathrm{Tr}(M^2) \rangle =$$



$$= \frac{1}{\alpha} N^2$$

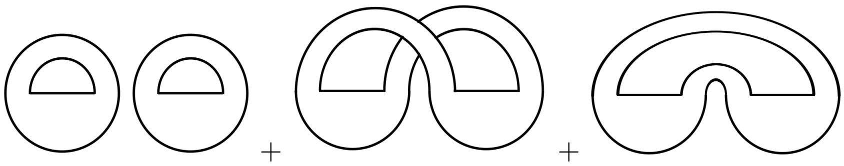
The modifications made imply that we must associate with every closed loop in this diagram, a factor of $\sum_a \delta_{aa} = N$. The above diagram has two closed loops and therefore comes with a factor of N^2.

Lets study the N independent quantity:

$$\left\langle \frac{1}{N^2}\mathrm{Tr}(M^2) \right\rangle = \frac{1}{\alpha}$$

In the limit that $N \to \infty$ factorisation holds. Consider the expectation value of four fields. There are three ribbon graphs for this expectation value:

$\langle (\mathrm{Tr}(M^2))^2 \rangle =$ (diagram) $+$ (diagram) $+$ (diagram) (1.2)



Each term in (1.2) has two ribbons and therefore comes with a factor of $\frac{1}{\alpha^2}$. The first diagram has four closed loops and therefore comes with a factor of N^4. The last two diagrams both have two closed loops and therefore come with factors of N^2. Summing these diagrams yields the expression:

$$\left\langle \left(\frac{1}{N^2}\mathrm{Tr}(M^2) \right)^2 \right\rangle = \frac{1}{\alpha^2 N^4}(N^4 + 2N^2)$$

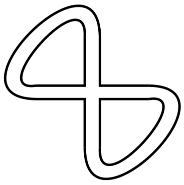
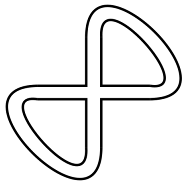
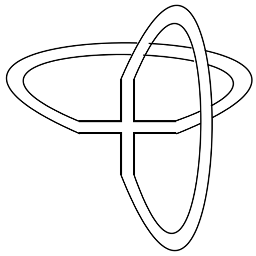
The second and third diagrams in (1.2) do not contribute in the limit that $N \to \infty$. Consequently we have

$$\left\langle \left(\frac{1}{N^2}\mathrm{Tr}(M^2) \right)^2 \right\rangle = \left\langle \frac{1}{N^2}\mathrm{Tr}(M^2) \right\rangle^2$$

Although we have only computed a single example, one can verify quite generally that factorisation holds in the large N limit, indicating that the large N limit is a classical limit of this zero dimensional theory of matrices.

1.1.3 A String Theory

In the previous section we demonstrated that factorisation holds in the large N limit of the matrix model considered and therefore the large N limit is the classical limit of some theory. In this section we will motivate that this theory is a string theory. We begin by studying $\langle \mathrm{Tr}(M^4) \rangle$. This is the expectation value of four fields. We can calculate this expectation value using ribbon diagrams by associating a vertex with $\mathrm{Tr}(M^4)$. Closing this vertex in all possible ways gives these diagrams:

$\left\langle \mathrm{Tr}\left(M^4\right)\right\rangle =$  $+$  $+$ 

The first two diagrams lie in a plane and are called planar diagrams. The third diagram is a non-planar diagram. Summing these diagrams by using the Feynman rules introduced in section 1.1.2 yields the mathematical expression:

$$\frac{1}{N^3}\langle \mathrm{Tr}(M^4)\rangle = \frac{2}{\alpha^2} + \frac{1}{N^2\alpha^2}$$

In the large N limit the second term in equation (1.1.3) vanishes. Notice that this term is the contribution from the non-planar diagram. First order corrections to the classical matrix theory are controlled by N^{-2} in the same way that $\hbar$ is the expansion parameter for quantum corrections in Quantum Field Theory. N^{-2} therefore maps to $\hbar$ in QFT. We now have a convenient way to think about this parameter N^{-2}:

$$\frac{1}{N^2} \leftrightarrow \hbar$$

This relationship draws an analogy between Quantum Field Theory and the matrix model we consider. Now consider a slightly more complicated model:

$$\begin{aligned} I_0 &= \int [dM] e^{-\frac{\alpha}{2}\mathrm{Tr}(M^2) - g\mathrm{Tr}(M^4)} \\ &= \int [dM] e^{-\frac{\alpha}{2}\mathrm{Tr}(M^2)} \left[1 - g\mathrm{Tr}(M^4) + \frac{g^2 \mathrm{Tr}(M^4)^2}{2} + ...\right] \\ &= \langle 1\rangle - g\langle \mathrm{Tr}(M^4)\rangle + \frac{g^2}{2}\langle \mathrm{Tr}(M^4)^2\rangle + .. \end{aligned}$$

These expectation values can be calculated using ribbon graphs. We only consider the N dependence of the ribbon graphs as we work to higher orders in perturbation theory:

$$N \text{ dependence} = 1 - g(N + N^3) + g^2(N^2 + N^6 + \dots) + \dots$$

Similarly:

$$
\begin{aligned}
I_2 &= \int [dM] \frac{\mathrm{Tr}(M^2)}{N^2} e^{-\frac{\alpha}{2}\mathrm{Tr}(M^2) - g\mathrm{Tr}(M^4)} \\
&= \frac{1}{N^2}\langle \mathrm{Tr}(M^2)\rangle - \frac{g}{N^2}\langle \mathrm{Tr}(M^2)\mathrm{Tr}(M^4)\rangle + \frac{g^2}{N^2}\langle \mathrm{Tr}(M^2)\mathrm{Tr}(M^4)^2\rangle + \dots
\end{aligned}
$$

Figure 1.1 shows one of the ribbon graphs which contribute to the $O(g)$ term.

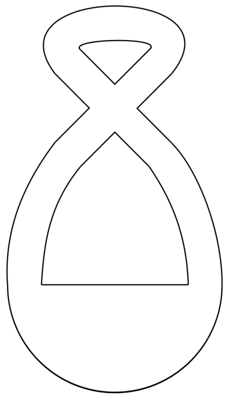


Figure 1.1: One of the ribbon graphs used to compute $\langle \mathrm{Tr}(M^2)\mathrm{Tr}(M^4)\rangle$

From the ribbon graphs of these expectation values we see that the order of the N-dependence increases as we work to higher orders in perturbation theory:

$$
N \text{ dependence} = 1 - \frac{g}{N} - gN + g^2N^2 + ...
$$

In the limit that $N \to \infty$ only the highest order diagrams contribute so that we do not obtain a sensible perturbation series. We would like to fix this problem because calculating these quantities without perturbation theory and ribbon graphs is difficult. We correct this behaviour by introducing a new parameter $\lambda = gN$ and we keep λ fixed as we take $N \to \infty$. This parameter is called the t'Hooft coupling [1]. The expansion now becomes:

$$
1 - \frac{\lambda}{N^2} - \lambda + \lambda^2 + ...
$$

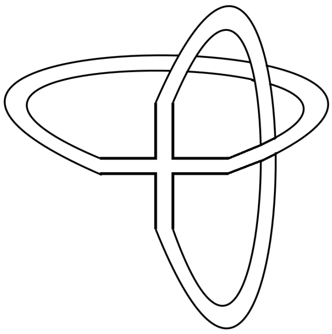
This is an expansion of the type:

$$
f_0(\lambda) + \frac{1}{N^2} f_1(\lambda) + \frac{1}{N^4} f_2(\lambda) + ...
$$

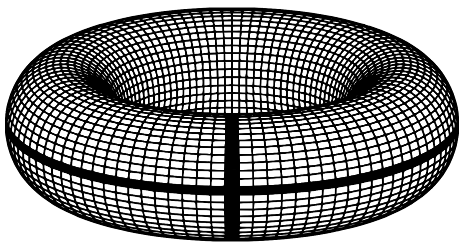
Note that we have two expansion parameters, N and λ. Using this expansion is easier than solving the theory exactly as only f_0 contributes as $N \to \infty$. There are diagrams that contribute to f_0 to all orders in g. The large N limit is therefore a classical limit, but not of the original theory.

In string theory there are also two expansion parameters. $\hbar_{\text{string}}$ controls quantum fluctuations while l_s, the string length, is the smallest distance at which we can probe space and time. This is one indication that the matrix model is a string theory.

The best indication that this matrix model is a string theory comes from considering the ribbon diagrams. These ribbon diagrams triangulate surfaces of different topologies. Figure 1.2 shows a ribbon graph and the topological surface it maps to. These topologies are naturally interpreted in a string theory as the world sheets of strings moving through space time. Figure 1.2 for example represents the world sheet for a string and anti-string which pop out of the vacuum, propagate through space time and then annihilate.



(a) A Ribbon Graph appearing in $\left\langle \text{Tr}\left(M^4\right)\right\rangle$



(b) The Ribbon Graph in figure 1.2a triangulates a torus

Figure 1.2: A Ribbon Graph and the Topological Surface it Triangulates

For this mapping of ribbon graphs to topological surfaces to be concrete there must be a way of determining the N-dependence of a ribbon graph from the surface it triangulates. The N-dependence of a topology is given by the Euler characteristic:

$$\begin{aligned} \chi &= V - E + F \\ &= 2 - 2g - h \end{aligned} \tag{1.3}$$

where V is the number of vertices, E is the number of edges, F is the number of faces, g is the number of handles and h is the number of holes. The torus in figure 1.2 has one handle and therefore has $\chi = 0$ and an N-dependence of N^0 which is the same N-dependence as figure 1.2a when using the t'Hooft coupling as an expansion parameter.

Chapter 2

A Brief Introduction to Group Representation Theory

Group representation theory is the most important branch of mathematics used in this work. Irreducible representations of the symmetric group label the Schur Polynomials and Restricted Schur Polynomials we study. The goal of this section is to develop a basic working knowledge of aspects of group representation theory used in this work. A more detailed discussion of the contents of this chapter can be found in chapter 3 of [2]. We begin with a general mathematical discussion of group representation theory before discussing group representation theory of the symmetric group in more detail.

2.1 The Group Axioms

A group is a set $\mathcal{G}$ of elements with a group operation, $\cdot$, which obey the following axioms:

1. Closure: $a \cdot b \in \mathcal{G} \quad \forall a, b \in \mathcal{G}$
2. Associativity: $(a \cdot b) \cdot c = a \cdot (b \cdot c) \;\; \forall a, b, c \in \mathcal{G}$
3. Existence of the identity: There exists an element of the group, e, called the identity for which $a \cdot e = e \cdot a = a \quad \forall a \in \mathcal{G}$.
4. Existence of an inverse: For every $a \in \mathcal{G}$ there exists an inverse element denoted by a^{-1} such that $a \cdot a^{-1} = a^{-1} \cdot a = e$

The number of elements in a group is called the order of the group and will be denoted by $[\mathcal{G}]$. The only group we will study in detail is the symmetric group which is of finite order.

A subgroup $\mathcal{H} \subset \mathcal{G}$ is a subset of $\mathcal{G}$ which obeys the group axioms. Of course, any subset of $\mathcal{G}$ must contain the identity element.

2.2 Group Representation Theory

A matrix representation of dimension n of a group $\mathcal{G}$ is a homomorphism mapping elements of $\mathcal{G}$ to elements of the general linear group $GL(n, \mathbb{C})$:

$$D : \mathcal{G} \to GL(n, \mathbb{C})$$

Since D is a homomorphism, group structure is preserved with the group operation $\cdot$ replaced by matrix multiplication:

$$D\left(g_1 \cdot g_2\right) = D\left(g_1\right) D\left(g_2\right) \qquad \forall\ g_1, g_2 \in \mathcal{G} \tag{2.1}$$

Equation (2.1) ensures that the identity element of $\mathcal{G}$ is mapped to the identity matrix and that inverses are preserved:

$$D(g^{-1}) = D(g)^{-1}$$

The vector space in which the matrices $D(g)$ act is called the carrier space of the representation.

2.2.1 Schur's Lemma and the Fundamental Orthogonality Relation

Schur's Lemma (see page 100 of [2]) states :

$$B\Gamma_R(g) = \Gamma_S(g)B \qquad \forall\ g \in \mathcal{G} \qquad \Rightarrow B = \lambda \delta_{RS} \mathbb{1} \tag{2.2}$$

In this equation $B : U_R \to U_S$ where U_R is the carrier space of the irreducible representation R and U_S is the carrier space of the irreducible representation S. Vectors which are elements of different carrier spaces are orthogonal:

$$\mathbf{v}_r \cdot \mathbf{v}_s = 0 \qquad \forall\ \mathbf{v}_r \in U_R,\ \mathbf{v}_s \in U_S$$

The matrix elements of different irreducible representations are orthogonal (see chapters $3-13$ and $3-15$ of [2] for an in depth discussion of reducibility and orthogonality of representations respectively). Our demonstration of this orthogonality begins with the following choice for B:

$$B = \sum_{g\in\mathcal{G}} \Gamma_S(g) A \Gamma_R(g^{-1}) \tag{2.3}$$

where $A : U_R \to U_S$. It is simple to see that the above choice for B is a map from U_R to U_S since $\Gamma_R(g) : U_R \to U_R\ \forall g \in \mathcal{G}$ and $\Gamma_S(g) : U_S \to U_S\ \forall g \in \mathcal{G}$. Next we check if this choice for B satisfies Schur's lemma. In the equation which follows $h \in \mathcal{G}$:

$$\begin{aligned}\Gamma_S(h)B &= \Gamma_S(h) \sum_{g\in\mathcal{G}} \Gamma_S(g) A \Gamma_R(g^{-1}) \\ &= \sum_{g\in\mathcal{G}} \Gamma_S(h \cdot g) A \Gamma_R(g^{-1})\end{aligned}$$

where in the second line we have used the homomorphic property of group representations. Since h and g are both elements of $\mathcal{G}$, $h \cdot g = g' \in \mathcal{G}$ by closure. Hence we may change the sum over g to an equivalent sum over g'. This step is only valid for finite groups. Using $g^{-1} = (h^{-1} \cdot g')^{-1} = (g')^{-1} \cdot h$ to express the above equation in terms of g' and h we confirm that our choice for B satisfies Schur's lemma:

$$\begin{aligned}\Gamma_S(h)B &= \sum_{g'\in\mathcal{G}} \Gamma_S(g') A \Gamma_R((g')^{-1} \cdot h) \\ &= \sum_{g'\in\mathcal{G}} \Gamma_S(g') A \Gamma_R((g')^{-1}) \Gamma_R(h) \\ &= B\Gamma_R(h)\end{aligned}$$

In the above equation it was not necessary to specify the exact form of A. Schur's lemma is satisfied by any $A : U_R \to U_S$. We make a particularly simple choice for A. Pick $A = |k\rangle \langle\gamma|$, where Roman letters label orthonormal basis states spanning U_S and Greek letters label orthonormal basis states spanning U_R. Using this convention we may write the representation matrices in equation (2.3) as:

$$\Gamma_S(g) = \sum_{i,j} \Gamma_S(g)_{ij} |i\rangle \langle j|$$

$$\Gamma_R(g^{-1}) = \sum_{\alpha,\beta} \Gamma_R(g^{-1})_{\alpha\beta} |\alpha\rangle \langle\beta|$$

Substituting for B in equation (2.2) demonstrates the orthogonality between matrix elements of different irreducible representations:

$$\begin{aligned}
\lambda\delta_{RS}\mathbb{1} &= \sum_{g\in\mathcal{G}} \Gamma_S(g) A \Gamma_R(g^{-1}) \\
&= \sum_{g\in\mathcal{G}} \sum_{i,j} \Gamma_S(g)_{ij} |i\rangle \langle j|k\rangle \langle\gamma| \sum_{\alpha,\beta} \Gamma_R(g^{-1})_{\alpha\beta} |\alpha\rangle \langle\beta| \\
&= \sum_{g\in\mathcal{G}} \sum_{i,j} \delta_{jk}\Gamma_S(g)_{ij} \sum_{\alpha,\beta} \delta_{\gamma\alpha}\Gamma_R(g^{-1})_{\alpha\beta} |i\rangle \langle\beta| \\
&= \sum_{g\in\mathcal{G}} \sum_{i} \Gamma_S(g)_{ik} \sum_{\beta} \Gamma_R(g^{-1})_{\gamma\beta} |i\rangle \langle\beta| \\
\implies \lambda\delta_{RS}\langle j|\alpha\rangle &= \sum_{g\in\mathcal{G}} \sum_{i} \Gamma_S(g)_{ik} \sum_{\beta} \Gamma_R(g^{-1})_{\gamma\beta} \langle j|i\rangle\langle\beta|\alpha\rangle \\
\lambda\delta_{RS}\delta_{j\alpha} &= \sum_{g\in\mathcal{G}} \sum_{i} \delta_{ji}\Gamma_S(g)_{ik} \sum_{\beta} \delta_{\beta\alpha}\Gamma_R(g^{-1})_{\gamma\beta} \\
&= \sum_{g\in\mathcal{G}} \Gamma_S(g)_{jk}\Gamma_R(g^{-1})_{\gamma\alpha}
\end{aligned}$$

The right hand side of this equation depends on k and γ which means λ must also depend on k and γ (the dependence on k and γ comes from our choice for A). To determine λ we set $R = S$ and trace over $j = \alpha$. When $R = S$ both Roman and Greek indices represent basis states from the same carrier space, U_R, and therefore the sum over j runs from 1 to $\dim_R$ where $\dim_R$ is the dimension of the carrier space of irreducible representation R:

$$\begin{aligned}
\lambda\dim_R &= \sum_{g\in\mathcal{G}} \sum_{j} \langle\gamma|\Gamma_R(g^{-1})|j\rangle\langle j|\Gamma_R(g)|k\rangle \\
&= \sum_{g\in\mathcal{G}} \langle\gamma|\Gamma_R(g\cdot g^{-1})|k\rangle \\
&= [\mathcal{G}]\, \delta_{\gamma k}
\end{aligned}$$

We have derived a property of the matrix elements of irreducible representations known as the fundamental orthogonality relation:

$$\sum_{g\in\mathcal{G}} \Gamma_S(g)_{jk}\Gamma_R(g^{-1})_{\gamma\alpha} = \frac{[\mathcal{G}]}{\dim_R}\delta_{RS}\delta_{\gamma k}\delta_{j\alpha} \tag{2.4}$$

2.3 Characters

The character of an irreducible representation $\Gamma_R(g)$ is $\chi_R(g) = \mathrm{Tr}(\Gamma_R(g))$. The study of characters in this section will ultimately allow us to determine the number of irreducible representations of a group. First note that characters have the following properties:

1. Equivalent representations have the same character. Two representations are said to be equivalent if they can be related to each other by a change of basis. If the representation $\Gamma(g)$ is given by the matrix D in some basis and the matrix D' in some other basis where $D' = SDS^{-1}$ then, by using cyclicity of the trace we find $\mathrm{Tr}(D) = \mathrm{Tr}(D')$.
2. Conjugate elements have the same character. Recall that two elements of a group, g and g' are conjugate with respect to h if $g' = h\cdot g\cdot h^{-1}$. Since a representation is a homomorphism we have $\Gamma(g') = \Gamma(h)\Gamma(g)\Gamma(h^{-1})$. Again, by cycling the trace we have $\chi(g') = \chi(g)$.
3. If a representation is unitary then

$$\begin{aligned}\chi(g^{-1}) &= \mathrm{Tr}(\Gamma(g)^{-1})\\ &= \mathrm{Tr}(\Gamma(g)^{\dagger})\\ &= \chi(g)^*\end{aligned}$$

By tracing equation (2.4) over $j = k$ and $\gamma = \alpha$ we obtain an orthogonality relation for characters:

$$\frac{1}{[\mathcal{G}]}\sum_{g\in\mathcal{G}} \chi_S(g)\chi_R(g^{-1}) = \delta_{RS}$$

Since all representations of a finite group are unitary-equivalent (see appendix A for a proof) we may write:

$$\frac{1}{[\mathcal{G}]}\sum_{g\in\mathcal{G}} \chi_S(g)\chi_R(g)^* = \delta_{RS} \tag{2.5}$$

Since conjugate group elements have the same character we may write equation (2.5) as a sum over conjugacy classes. Let c_i be the number of elements in the i^{th} conjugacy class and suppose there are c conjugacy classes in total. The above sum over $g \in \mathcal{G}$ can be written as a sum over conjugacy classes:

$$\frac{1}{[\mathcal{G}]} \sum_i c_i \chi_S^i {\chi_R^i}^* = \delta_{RS} \tag{2.6}$$

We may think of this expression as the inner product between r c-dimensional, orthogonal vectors $\sqrt{c_i}\chi_S^i$ where r is the number of irreducible representations. Since there can be at most c such vectors we conclude that $c \leq r$. Characters of different conjugacy classes are also orthogonal, as demonstrated by equation $(3-178)$ on page 110 of [2]:

$$\frac{1}{[\mathcal{G}]} \sum_R \sqrt{c_i}\chi_R^i \sqrt{c_j} {\chi_R^j}^* = \delta_{ij} \tag{2.7}$$

This expression demonstrates the orthogonality of c r-dimensional vectors. There can be at most r such vectors. Hence, $c \leq r$. The only way that $c \leq r$ and $c \leq r$ is if $r = c$. We must therefore conclude that the number of irreducible representations of a group is equal to the number of conjugacy classes. This point will be vital when constructing representations of the symmetric group.

Chapter 3

Group Representation theory of the Symmetric Group

The symmetric group, S_n, is the set of all possible permutations of n objects. The order of the symmetric group is $[S_n] = n!$. Throughout this text we use cycle notation to represent permutations. In cycle notation, the group element (1 2 3) permutes the first three elements in a set so that the third element in the set moves to the second position, the second element in the set moves to the first position and the first element in the set moves to the third position i.e. $3 \to 2 \to 1 \to 3$. Here are some examples of permutations in S_4:

$$\begin{aligned}
(1)(2)(3)(4) &: \{1,2,3,4\} \to \{1,2,3,4\} \\
(1\ 2\ 3)(4) &: \{1,2,3,4\} \to \{2,3,1,4\} \\
(1\ 3)(2\ 4) &: \{1,2,3,4\} \to \{3,4,1,2\} \\
(1\ 2\ 3\ 4) &: \{1,2,3,4\} \to \{2,3,4,1\} \\
(1\ 2\ 3)(3\ 4) &: \{1,2,3,4\} \to \{2,4,1,3\}
\end{aligned}$$

Some of these elements have a different cycle structure. By cycle structure we mean the number of 1-,2-,3- and 4-cycles which appear when each element is written in terms of disjoint cycles. Disjoint cycles are cycles which act on different elements and therefore disjoint cycles commute. (1 2 3)(4), (1)(2)(3)(4) and (1 3)(2 4) are all disjoint cycles. The identity element (1)(2)(3)(4) is built from four 1-cycles whereas (1 2 3)(4) is built from a single 3-cycle and a single 1-cycle. 1-cycles indicate which elements are kept inert.

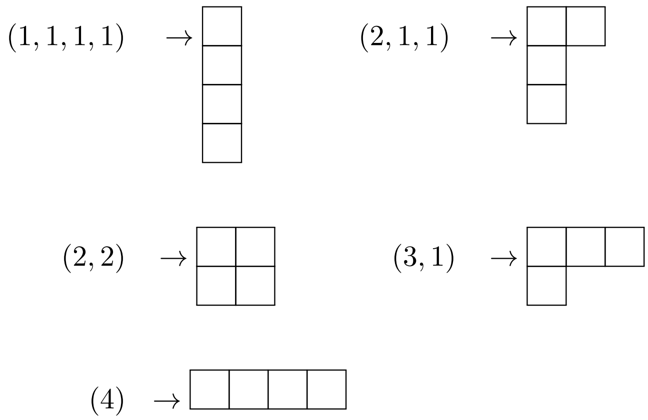
(1 2 3)(4) is an element of the S_3 subgroup of S_4 which leaves the element in the fourth slot inert whereas (1)(2 3 4) is an element of the S_3 subgroup which leaves the element in the first slot inert. Now consider the third example: (1 3)(2 4). This element is built from two disjoint 2-cycles (also called transpositions). Let us now shift our attention to the final example studied above: (1 2 3)(3 4). It is clear that this element is not disjoint, but any element of the symmetric group can be written in terms of disjoint cycles. Since no element remains in the same slot we see that we may write (1 2 3)(3 4) as a single 4-cycle which takes $4 \to 2 \to 1 \to 3 \to 4$ and therefore (1 2 3)(3 4) = (3 1 2 4). Conjugating elements in S_n preserves cycle structure: group elements with the same cycle structure are related by conjugation. As an example consider conjugation of (1 3)(2 4) by $\sigma = (1\ 2\ 3)(4)$:

$$\sigma(1\ 3)(2\ 4)\sigma^{-1} = (2\ 1)(3\ 4) \tag{3.1}$$

Equation 4.3 illustrates the affect of conjugation by σ on an element of S_4: simply permute the numbers by the action of σ while keeping the cycle structure fixed. Note that the sum of the cycle lengths of any element of S_n when written in terms of disjoint cycles is equal to n. Hence cycle structures and therefore conjugacy classes are in one-to-one correspondence with partitions of n. Partitions of n are a natural way to label irreducible representations of S_n. We use Young diagrams to describe partitions of n.

3.1 Young Diagrams

Young diagrams are rows of boxes which describe partitions and are therefore in one-to-one correspondence with conjugacy classes and irreducible representations of the symmetric group. A partition of n is represented by a Young diagram with n boxes. To avoid over-counting a row must be as long or shorter than the row above it and all the rows must be left aligned. Young diagrams are best understood by studying a few examples. S_4 has five conjugacy classes: $(1, 1, 1, 1)$ (which represents four one-cycles - the identity is the only element belonging to this conjugacy class), $(2, 1, 1)$ (which represents a single two-cycle and two one-cycles), $(2, 2)$,$(3, 1)$ and finally (4) (which represents a single four cycle). Each of these conjugacy classes is represented by a Young diagram:



Young diagrams provide convenient labels for the irreducible representations of the symmetric group. They can be used to determine the dimension of an irreducible representation and provide a means for constructing an orthonormal basis for the carrier space of irreducible representations of S_n. Recall that a representation $\Gamma(g)$ is said to be reducible if it can be brought to the same block diagonal form $\forall g \in \mathcal{G}$. This occurs because the carrier space of $\Gamma(g)$ may be divided into invariant subspaces. A subspace is invariant if it is closed under the action of $\Gamma(g)$. In this case the representation can be written as a direct sum of irreducible representations:

$$\Gamma(g) = \bigoplus_R \Gamma_R(g)^{\oplus a_R} \tag{3.2}$$

where a_R is the multiplicity of irreducible representation R (the number of times the irreducible representation R occurs). Every representation is either irreducible or a direct sum of irreducible representations. An irreducible representation $\Gamma_R(g)$ cannot be brought to the same block diagonal form $\forall g \in \mathcal{G}$. This does not rule out the existence of a transformation S such that $S\Gamma_R(h)S^{-1}$ is block diagonal $\forall h \in \mathcal{H}$ where $\mathcal{H}$ is a subgroup of $\mathcal{G}$. This process of restricting an irreducible representation to a subgroup is known as subduction. In general we must assume that the subduced representation is reducible and can be written in block diagonal form with irreducible representations of the subgroup appearing on the diagonal. When restricting to subgroups of the symmetric group, the irreducible representations which are subduced are determined by the Littlewood-Richardson rule. Suppose we restrict an irreducible representation of S_{n+m} labelled by a Young diagram R with $n+m$ boxes to a general $S_n \times S_m$ subgroup of S_{n+m}. We will write an irreducible representation of $S_n \times S_m$ as $S \times T$ where S is a Young diagram with n boxes which labels an irreducible representation of S_n and T is a Young diagram with m boxes which labels an irreducible representation of S_m. Upon restriction from S_{n+m} to an $S_n \times S_m$ subgroup, Γ_R subduces irreducible representations of $S_n \times S_m$ as equation $(7-1)$ on page 183 of [2] demonstrates:

$$\Gamma_R = \bigoplus_{S,T} (\Gamma_{S\times T})^{\oplus f_{RST}} \tag{3.3}$$

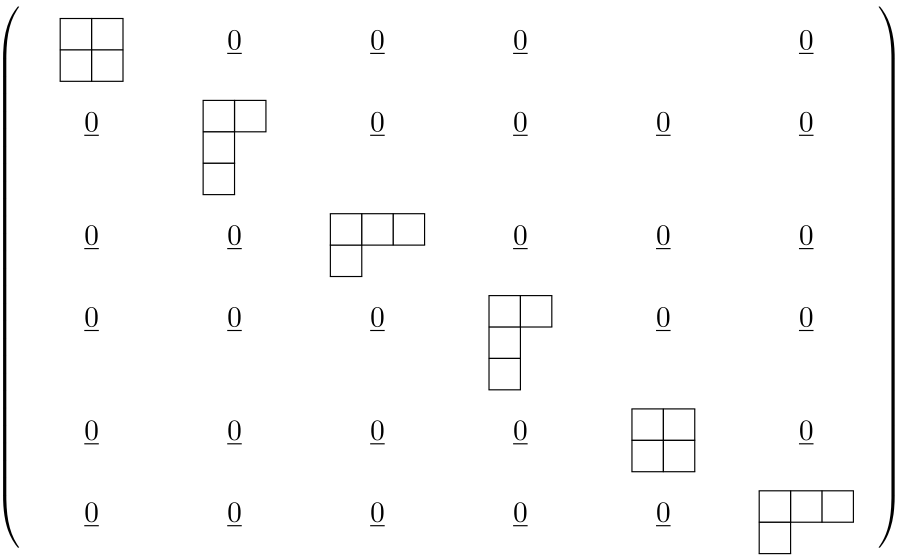
where f_{RST} are the Littlewood-Richardson numbers. f_{RST} counts how many times the irreducible representations of S_n and S_m labelled by the Young diagrams S and T respectively are subduced when an irreducible representation of S_{n+m} labelled by R is restricted to a $S_n \times S_m$ subgroup of S_{n+m}. The Littlewood-Richardson numbers are determined by the Littlewood-Richardson rule. As a simple example of the Littlewood-Richardson rule consider what happens when we restrict an irreducible representation of S_m labelled by a Young diagram R to a $S_{m-1} \times S_1$ subgroup. R is a Young diagram with m boxes. The only representation of S_1 is the trivial one dimensional irreducible representation labelled by the Young diagram $\yng(1)$. This representation is equal to one. The Littlewood-Richardson rule states that the only irreducible representations of S_{m-1} which are subduced are those which can be obtained from the Young diagram R by removing a single box to leave a valid Young diagram. As an example consider the S_6 irreducible representation $\yng(3,2,1)$. When we restrict to an $S_5 \times S_1$ subgroup of S_6 the following irreducible representations of $S_5 \times S_1$ are subduced:

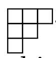
$$\yng(3,2,1) = \yng(2,2,1) \times \yng(1) \oplus \yng(3,1,1) \times \yng(1) \oplus \yng(3,2) \times \yng(1) \tag{3.4}$$

$\yng(3,2,1)$ is brought to block diagonal form with the above S_5 irreducible representations on the diagonal:

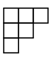
$$\begin{pmatrix} \yng(2,2,1) & \underline{0} & \underline{0} \\ \underline{0} & \yng(3,1,1) & \underline{0} \\ \underline{0} & \underline{0} & \yng(3,2) \end{pmatrix}$$

If we now restrict to an $S_4 \times {S_1}^2$ subgroup, each of these S_5 irreducible representations are diagonalised with S_4 irreducible representations on the diagonal according to the Littlewood-Richardson rule:



We now see multiplicities appearing since each S_4 irreducible representation appearing above is subduced twice. This process of successive subduction can be continued until we have restricted to the ${S_1}^6$ subgroup of S_6. The resulting matrix is block diagonal with single boxes on the diagonal. Clearly, this matrix is the identity matrix in the original irreducible representation. This must be so since the only element of S_6 belonging to the ${S_1}^6$ subgroup is the identity element (1)(2)(3)(4)(5)(6) and this element is mapped to the identity matrix. By counting the number of single boxes on the diagonal one can determine the dimension of the irreducible representation labelled by . Young diagrams encode in them the dimension of the irreducible representations which they label. Of course, performing successive subductions is not practical when working with Young diagrams with many boxes. Fortunately the dimension of a Young diagram R with n boxes may be determined in a more practical way by using the equation:

$$\dim_R = \frac{n!}{\mathrm{hooks}_R} \tag{3.5}$$

where hooks_R is the product of the hook lengths of all the boxes in R. The hook length of a box is simply the number of boxes below the box (in the same column), plus the number of boxes to the right of the box (in the same row) plus one for the box itself. The hook lengths for each box in  are filled in below:

5	3	1
3	1	
1		

Hence,

$$\dim_{(3,2,1)} = \frac{6!}{5 \times 3 \times 3} = 16$$

When restricting an irreducible representation R of S_{n+m} to an $S_n \times S_m$ subgroup we first restrict to an $S_n \times (S_1)^m$ subgroup and then induce a representation of S_m from the m boxes removed from the Young diagram R. We arrange the m boxes into irreducible representations of S_m being careful to preserve the same row and column relations between the boxes i.e. the boxes appearing in the same row or column in R appear in the same row or column in the induced irreducible representations of S_m. Symmetrica [1] is a programme which we have found useful for checking and performing calculations related to the representation theory of the symmetric group.

3.2 Young Tableau

In the previous section we saw that the dimension of an irreducible representation of S_n labelled by a Young diagram R is equal to the number of possible ways of removing all the boxes from R (at each step a valid Young diagram remains) so that we are only left with a single box. A standard Young tableau is a Young diagram with numbered boxes indicating a valid order in which boxes may be removed. The box labelled 1 is to be removed first, the box labelled 2 is to be removed second and so forth. The number of Young tableaux with the same shape as a Young diagram R is therefore equal to the dimension of the irreducible representation labelled by R, hence Young tableaux of shape R may be used to label a set of orthonormal basis vectors spanning the carrier space of irreducible representation R. The Young tableaux for $(3,1)$ are listed below:

$$\begin{array}{|c|c|c|}\hline 4 & 3 & 2 \\ \hline 1 \\ \cline{1-1}\end{array} \qquad \begin{array}{|c|c|c|}\hline 4 & 3 & 1 \\ \hline 2 \\ \cline{1-1}\end{array} \qquad \begin{array}{|c|c|c|}\hline 4 & 2 & 1 \\ \hline 3 \\ \cline{1-1}\end{array}$$

There are 3 tableau, each of which label a unique subduction chain when restricting from S_4 to $(S_1)^4$. By using equation (3.5) we may check that $(3,1)$ is indeed a three dimensional irreducible representation:

$$\dim_{(3,1)} = \frac{4!}{4 \times 2} = 3$$

[1] http://www.algorithm.uni-bayreuth.de/en/research/SYMMETRICA/

The matrix elements of $\sigma \in S_n$ in irreducible representation R are (i and j label tableau):

$$\Gamma_R(\sigma)_{ij} = \langle R_i | \, \Gamma_R(\sigma) \, | R_j \rangle$$

with

$$\langle R_i | R_j \rangle = \delta_{ij}$$

Young tableaux are more than just a convenient way of labelling orthonormal basis states spanning the carrier space of an irreducible representation of the symmetric group - by defining an ordering for Young tableaux and a convenient action for permutations acting on a Young tableau we can explicitly construct the representation matrices. We order tableaux by looking at the rows in which the box labelled 1 appears. The tableau with the box labelled 1 in the lowest position is assigned the lowest value. If more than one tableau contains the box labelled 1 in the same row then we look at the position of the box labelled 2 and so forth. As an example consider the application of this rule to Young tableau of shape $\begin{array}{|c|c|c|}\hline \; & \; & \; \\ \hline \multicolumn{1}{|c|}{\;} \\ \cline{1-1}\end{array}$:

$$\left| \begin{array}{|c|c|c|}\hline \; & \; & \; \\ \hline \multicolumn{1}{|c|}{\;} \\ \cline{1-1}\end{array} ; 1 \right\rangle = \left| \begin{array}{|c|c|c|}\hline 4 & 3 & 2 \\ \hline \multicolumn{1}{|c|}{1} \\ \cline{1-1}\end{array} \right\rangle \qquad \left| \begin{array}{|c|c|c|}\hline \; & \; & \; \\ \hline \multicolumn{1}{|c|}{\;} \\ \cline{1-1}\end{array} ; 2 \right\rangle = \left| \begin{array}{|c|c|c|}\hline 4 & 3 & 1 \\ \hline \multicolumn{1}{|c|}{2} \\ \cline{1-1}\end{array} \right\rangle \qquad \left| \begin{array}{|c|c|c|}\hline \; & \; & \; \\ \hline \multicolumn{1}{|c|}{\;} \\ \cline{1-1}\end{array} ; 3 \right\rangle = \left| \begin{array}{|c|c|c|}\hline 4 & 2 & 1 \\ \hline \multicolumn{1}{|c|}{3} \\ \cline{1-1}\end{array} \right\rangle$$

The action of a permutation on a Young tableau is defined for neighbouring two-cycles (cycles of the form $(i, i+1)$).

$$\begin{aligned} \Gamma_R((i, i+1)) \left| R' \right\rangle &= \text{no swap}_{i,i+1} \left| R' \right\rangle + \text{swap}_{i,i+1} \left| R'_{i,i+1} \right\rangle \\ &= \frac{1}{c_i - c_{i+1}} \left| R' \right\rangle + \sqrt{1 - \frac{1}{(c_i - c_{i+1})^2}} \left| R'_{i,i+1} \right\rangle \end{aligned} \tag{3.6}$$

When $(i, i+1)$ acts on a tableau, the effect is to create the same state multiplied by a 'no swap' factor + the same state but with i and $i+1$ swapped multiplied by a 'swap' factor. Only valid tableau are kept when swapping. In equation (3.6), R' is a Young tableau with shape R and $R'_{i,i+1}$ is the same tableau but with i and $i+1$ swapped. $c_j = K - \mathcal{R}_j + \mathcal{C}_j$ is the 'factor' of the box with label j in row $\mathcal{R}_j$ and column $\mathcal{C}_j$. K is some integer. The swap and no swap factors depend on the quantity $c_i - c_{i+1}$ and are therefore independent of K. It is sufficient to define this action for a neighbouring two-cycle since any permutation can be expressed as a product of neighbouring two-cycles. It is important that equation (3.6) preserves the shape of a tableau because Γ_R is an irreducible representation and therefore the carrier space of R, $\{|R, i\rangle;\ i = 1, \ldots, \dim_R\}$, is closed.

This action simplifies significantly when using the displaced corners approximation [12]. In this limit we consider Young diagrams with $p \sim O(1)$ rows with well separated corners each containing $O(N)$ boxes. The difference in row lengths between any two rows is $O(N)$ and we take the limit as N tends to infinity. There are two cases we need to consider. When the blocks labelled with i and $i+1$ are in the same row then they must label adjacent boxes. Any other tableaux would not be valid. In this case $c_i - c_{i+1} = 1$ and $R'_{i,i+1}$ is not a valid tableau so that the action of $(i, i+1)$ on R' simplifies to:

$$\Gamma_R((i, i+1))\,|\text{same row state}\rangle = |\text{same row state}\rangle$$

It is also possible that the boxes labelled with i and $i+1$ are in two different rows. Since the difference in row lengths between any two rows is $O(N)$ we know that $c_i - c_{i+1} \sim O(N)$. In the large N limit, where N tends to infinity, $\frac{1}{c_i - c_{i+1}}$ tends to zero and therefore:

$$\Gamma_R((i, i+1))\,|\text{different row state}\rangle = |\text{different row state}_{i,i+1}\rangle$$

where $|\text{different row state}_{i,i+1}\rangle$ is identical to $|\text{different row state}\rangle$ but i and $i+1$ are swapped. This approximation greatly simplifies the action of a permutation on a state. For a more in depth discussion on the displaced corners approximation see [12].

Chapter 4

Gauge Invariant Trace Operators of a Single Matrix Field

Trace operators built from a number of matrix fields, or combinations of different matrix fields are naturally parametrised using elements of the symmetric group. For a simple example, consider the trace operators which can be built from 3 matrices. The methods used in the following example hold for trace operators built from any number of the same matrix fields. These operators are $\mathrm{Tr}(Z^3)$, $\mathrm{Tr}(Z^2)\mathrm{Tr}(Z)$ and $\mathrm{Tr}(Z)^3$. These operators can be described using elements of the symmetric group S_3. This insight is clarified when one notices that each trace operator is obtained by a specific S_3 permutation of the Z field indices. Also, S_3 elements belonging to the same conjugacy class produce the same trace operator. These statements will be proven in the text that follows, but first a note on some notation is in order:

A general trace operator built from three $N \times N$ Z matrices with $N > 3$ will be denoted by:

$$\mathcal{O} = \mathrm{Tr}(\sigma Z^{\otimes 3}) \tag{4.1}$$

where $\sigma \in S_3$.

We use the following notation to denote a tensor basis:

$$|i\rangle = |i_1\rangle \otimes |i_2\rangle \otimes |i_3\rangle = |i_1, i_2, i_3\rangle$$

In this tensor basis $|i_1\rangle$ is in slot 1, $|i_2\rangle$ is in slot 2 and so forth. $\sigma \in S_3$ acts on a tensor product of vectors by shuffling the vectors amoung the available slots:

$$\begin{aligned}
\langle j|\,\sigma\,|i\rangle =& \langle j_1|\otimes\langle j_2|\otimes\langle j_3|\,\sigma\,|i_1\rangle\otimes|i_2\rangle\otimes|i_3\rangle \\
=& \langle j_1,j_2,j_3|\,\sigma\,|i_1,i_2,i_3\rangle \\
=& \langle j_1,j_2,j_3|\,\big|i_{\sigma(1)},i_{\sigma(2)},i_{\sigma(3)}\big\rangle \\
=& \delta^{j_1}_{i_{\sigma(1)}}\delta^{j_2}_{i_{\sigma(2)}}\delta^{j_3}_{i_{\sigma(3)}}
\end{aligned}$$

We now have the mathematical tools necessary to study the operator $\mathrm{Tr}(\sigma Z^{\otimes 3})$:

$$\begin{aligned}
\mathrm{Tr}(\sigma Z^{\otimes 3}) =& \sum_{i,j}\langle i|\,\sigma\,|j\rangle\,\langle j|\,Z^{\otimes 3}\,|i\rangle \\
=& \sum_{i,j}\delta^{i_1}_{j_{\sigma(1)}}\delta^{i_2}_{j_{\sigma(2)}}\delta^{i_3}_{j_{\sigma(3)}}Z^{j_1}_{i_1}Z^{j_2}_{i_2}Z^{j_3}_{i_3} \\
=& Z^{j_1}_{j_{\sigma(1)}}Z^{j_2}_{j_{\sigma(2)}}Z^{j_3}_{j_{\sigma(3)}}
\end{aligned} \tag{4.2}$$

S_3 has $3! = 6$ elements: (1)(2)(3), (12)(3), (13)(2), (1)(23), (123) and (132). All three trace operators which can be built from three Z matrix fields can be constructed using these elements of S_3:

For $\sigma = (1)(2)(3)$:

$$\mathrm{Tr}(\sigma Z^{\otimes 3}) = \mathrm{Tr}(Z)^3$$

For $\sigma = (12)(3)$ or, $\sigma = (13)(2)$ or $\sigma = (1)(23)$ it is simple to see that:

$$\mathrm{Tr}(\sigma Z^{\otimes 3}) = \mathrm{Tr}(Z^2)\mathrm{Tr}(Z)$$

For $\sigma = (123)$ or $\sigma = (132)$ we obtain the trace operator $\mathrm{Tr}(Z^3)$. We demonstrate this explicitly by using $\sigma = (123)$:

$$\begin{aligned}
\mathrm{Tr}(\sigma Z^{\otimes 3}) &= Z^{j_1}_{j_{\sigma(1)}}Z^{j_2}_{j_{\sigma(2)}}Z^{j_3}_{j_{\sigma(3)}} \\
&= Z^{j_1}_{j_2}Z^{j_2}_{j_3}Z^{j_3}_{j_1} \\
&= \mathrm{Tr}(Z^3)
\end{aligned}$$

An important concept to grasp from these examples is that elements of the symmetric group which belong to the same conjugacy class give the same trace operators. These trace operators are gauge invariant. Suppose we perform a gauge transformation on Z:

$$Z \to U^\dagger Z U$$

where $U \in U(N)$ and therefore $UU^\dagger = \mathbb{1}$ where $\mathbb{1}$ is the identity element of the unitary group. Cyclicity of the trace ensures that trace operators are invariant. The three trace operators we have constructed above are gauge invariant:

$$\begin{aligned}\mathrm{Tr}(Z^3) \to &\mathrm{Tr}(U^\dagger Z U U^\dagger Z U U^\dagger Z U)\\ =&\mathrm{Tr}(U U^\dagger Z U U^\dagger Z U U^\dagger Z)\\ =&\mathrm{Tr}(\mathbb{1} Z \mathbb{1} Z \mathbb{1} Z)\\ =&\mathrm{Tr}(Z^3)\end{aligned}$$

$$\begin{aligned}\mathrm{Tr}(Z)^3 \to &\mathrm{Tr}(U^\dagger Z U)^3\\ =&\mathrm{Tr}(U U^\dagger Z)^3\\ =&\mathrm{Tr}(Z)^3\end{aligned}$$

$$\begin{aligned}\mathrm{Tr}(Z^2)\mathrm{Tr}(Z) \to &\mathrm{Tr}(U^\dagger Z U U^\dagger Z U)\mathrm{Tr}(U^\dagger Z U)\\ =&\mathrm{Tr}(U U^\dagger Z U U^\dagger Z)\mathrm{Tr}(U U^\dagger Z)\\ =&\mathrm{Tr}(Z^2)\mathrm{Tr}(Z)\end{aligned}$$

I now study the most general trace operator built from n Z fields. In the following equation σ and ρ are both elements of S_n :

$$\begin{aligned}\mathcal{O} &= \mathrm{Tr}(\sigma Z^{\otimes n})\\ &= \prod_{m=1}^{m=n} Z^{i_m}_{i_{\sigma(m)}}\\ &= \prod_{m=1}^{m=n} Z^{i_{\rho(m)}}_{i_{\sigma\rho(m)}}\end{aligned} \tag{4.3}$$

where in the third line we are simply shuffling the order in which the matrix elements are multiplied. This statement holds $\forall \rho \in S_n$. Picking $\rho = \sigma^{-1}$ gives:

$$\mathcal{O} = \prod_{m=1}^{m=n} Z^{i_{\sigma^{-1}(m)}}_{i_m}$$

We see that acting with $\sigma \in S_n$ on the lower indices is identical to acting with $\sigma^{-1} \in S_n$ on the upper indices. This will be useful when we study the action of the dilatation operator on Schur Polynomials. Equation (4.3) may therefore be written as:

$$\begin{aligned}\mathcal{O} &= \prod_{m=1}^{m=n} Z^{i_m}_{i_{\rho^{-1}\sigma\rho(m)}} \\ &= \mathrm{Tr}(\rho^{-1}\sigma\rho Z^{\otimes n})\end{aligned}$$

This equation demonstrates that trace operators built out of m Z-fields are invariant under conjugation by elements in S_m (swapping top and bottom indices by the same permutation). Hence all permutations belonging to the same conjugacy class produce the same trace operator. So every trace operator is in one-to-one correspondence with conjugacy classes of S_m and therefore with Young diagrams with m boxes.

4.1 Finite N Relations

In the previous section we considered trace operators built from matrix fields Z where the rank of the matrix N was larger than or equal to the number fields, n, used to construct trace operators. We found that we were able to construct independent trace operators. This is not the case when $N < n$. As an example consider the previous example with three Z fields where the Z fields are square 2×2 matrices. By working in an eigenbasis of Z we obtain the following expressions for the three trace operators (of course, we can work with diagonal Z matrices without a loss of generality):

$$\mathrm{Tr}(Z^3) = z_1^3 + z_2^3$$

$$\mathrm{Tr}(Z^2)\mathrm{Tr}(Z) = z_1^3 + z_1^2 z_2 + z_2^2 z_1 + z_2^3$$

$$\mathrm{Tr}(Z)^3 = z_1^3 + 3z_1^2 z_2 + 3z_2^2 z_1 + z_2^3$$

where z_1 and z_2 are the eigenvalues of Z. These three equations are not independent since:

$$z_1^3 + 3z_1^2 z_2 + 3z_2^2 z_1 + z_2^3 = 3\left(z_1^3 + z_1^2 z_2 + z_2^2 z_1 + z_2^3\right) - 2\left(z_1^3 + z_2^3\right)$$

We learn that if the rank of Z is less than the number of matrices our trace operators are built from then not all trace operators are independent. In this specific example there are only two independent trace operators because of the constraint imposed by the above equation. Written in terms of trace operators this constraint is:

$$\mathrm{Tr}(Z^3) = \frac{1}{2}\left[3\mathrm{Tr}(Z^2)\mathrm{Tr}(Z) - \mathrm{Tr}(Z)^3\right] \tag{4.4}$$

In the next section we will see that finite N constraints are nicely accounted for by Schur Polynomials.

4.2 Schur Polynomials

The Schur polynomial is defined as [41]:

$$\chi_R(Z) = \frac{1}{n!} \sum_{\sigma \in S_n} \chi_R(\sigma) Z^{i_1}_{i_{\sigma(1)}} Z^{i_2}_{i_{\sigma(2)}} \dots Z^{i_{n-1}}_{i_{\sigma(n-1)}} Z^{i_n}_{i_{\sigma(n)}} \tag{4.5}$$

$\chi_R(\sigma)$ is the trace of the matrix representing $\sigma \in S_n$ in the irreducible representation R. R is a Young diagram with n boxes. The number of Schurs is therefore equal to the number of Young diagrams with n boxes and as a consequence, equal to the number of independent trace operators built with n fields. The complete set of Schur polynomials with n boxes form a basis for trace operators built from n Z-fields. In the above definition of the Schur polynomial the indices i_k where $k = 1, 2 \dots n$ take integer values $i_k = 1, 2 \dots m$, hence, Z is a $m \times m$ matrix. The lower indices of the Z matrix elements are permuted by the action of σ. Repeated indices are summed according to the Einstein summation convention.

Schurs are more than just a basis for labelling trace operators. Each Schur polynomial labels a state dual to a system of giant gravitons (without strings attached) [41]. The number of giants is equal to the number of rows in the Young diagram label. Rows with equal length correspond to giants with incident branes, whereas well separated rows correspond to giants which are far apart. In the next chapter we study restricted Schur polynomials. Restricted Schurs label giant graviton states with strings attached. The strings correspond to the Y-fields in the restricted Schurs.

Chapter 5

Restricted Schur Polynomials

The restricted Schur polynomial is defined as:

$$\chi_{R,(r,s)jk}(Z,Y) = \frac{1}{n!m!} \sum_{\sigma \in S_{n+m}} \chi_{R,(r,s)jk}(\sigma) Y^{i_1}_{i_{\sigma(1)}} \dots Y^{i_m}_{i_{\sigma(m)}} Z^{i_{m+1}}_{i_{\sigma(m+1)}} \dots Z^{i_{m+n}}_{i_{\sigma(m+n)}}$$

R is an irreducible representation of S_{n+m} and is labelled by a Young diagram with $n+m$ boxes. r and s are irreducible representations of S_n and S_m respectively. There are two differences between Schur polynomials and restricted Schurs. The first difference is that restricted Schurs are built from more than one matrix. In this definition the restricted Schur is built from two matrices, Z and Y. The second difference is the restricted character $\chi_{R,(r,s)jk}(\sigma)$ [56]. As before, χ is the trace of the matrix representing $\sigma \in S_{n+m}$ in the irreducible representation R. The difference is that the trace is taken over the indices belonging to the subspace carrying an irreducible representation of $S_n \times S_m \in S_{n+m}$. The specific irreducible representation of $S_n \times S_m$ we consider is denoted by (r, s). The indices j and k are multiplicity indices. When restricting from S_{n+m} to $S_n \times S_m \in S_{n+m}$ a specific irreducible representation can be subduced multiple times, with each copy carried by a different subspace of the carrier space of R.

5.1 The Restricted Trace

Restricted Schur polynomials are difficult to construct because one needs to evaluate the restricted character. The method we will use to evaluate the restricted trace involves the construction of a symmetric group operator $P_{R\to(r,s)jk}$ which obeys:

$$\Gamma_{(r,s)j}(\sigma) P_{R\to(r,s)jk} = P_{R\to(r,s)jk} \Gamma_{(r,s)k}(\sigma) \qquad \sigma \in S_n \times S_m$$

$$\Gamma_{(r,s)m}(\sigma) P_{R\to(r,s)jk} = 0 = P_{R\to(r,s)jk} \Gamma_{(r,s)n}(\sigma) \quad \sigma \in S_n \times S_m \quad m \neq j, \quad n \neq k$$

With use of the operator $P_{R\to(r,s)jk}$ the restricted character can be written as:

$$\chi_{R,(r,s)jk}(\sigma) = \mathrm{Tr}\left(P_{R\to(r,s)jk}\Gamma_R(\sigma)\right)$$

The operator $P_{R\to(r,s)jk}$ is difficult to construct. The construction we describe in the subsequent sections is useful for Young diagrams R which have p long rows or p long columns where $p \sim O(1)$. In the rest of this section we proceed by constructing the operator $P_{R\to(r,s)jk}$. The construction we review was developed in [12].

5.2 From S_{n+m} to $S_n \times (S_1)^m$

Recall, for some irreducible representation $\Gamma_R(g)$ where $g \in G$ one can restrict Γ_R to a subgroup of G, $H \subset G$. $\Gamma_R(h)$ where $h \in H$ is in general a reducible representation of H and can be written as a direct sum of irreducible representations of H. Which irreducible representations of H enter into this direct sum? We will answer this question specifically for representations of the symmetric group. If one restricts an irreducible representation of S_{n+m} to a subgroup $S_{n+m-1} \times S_1$, the irreducible representations of S_{n+m-1} entering into the direct sum are those which can be formed from the Young diagram labelling R by removing a single box and leaving a valid Young diagram. S_1 has only one irreducible representation, the identity representation and hence does not enter into the decomposition of R. By repeating this process m times, at each step leaving a valid Young diagram, we write the irreducible representation R of S_{n+m} as a direct sum of irreducible representations of $S_n \times (S_1)^m$. R takes a block diagonal form with Young diagrams which can be formed from R by removing m boxes (at each step leaving a valid Young diagram) on the diagonal. In general, a certain irreducible representation of S_n can appear more than once by removing the m boxes in a different order. Each of these copies of the same irreducible representation are carried by a different subspace of the space carrying R. To deal with this multiplicity we use partially labelled Young diagrams.

5.3 Partially Labelled Young Diagrams

We account for the multiplicity of S_n irreducible representations when subducing the irreducible representation R of S_{n+m} by labelling the m boxes to be removed with numbers from 1 to m. The box labelled by 1 is to be removed first, the box labelled by 2 is to be removed second and so forth, until the m^{th} box is removed last. A Young diagram labelled in this way is called a partially labelled Young diagram. Specifying the order in which the m boxes are to be removed from R resolves the multiplicity.

As an example, consider a partially labelled Young diagram with $n = 3 = m$:

$$\begin{array}{|c|c|c|}\hline \; & \; & 1 \\ \hline \; & 2 \\ \cline{1-2} 3 \\ \cline{1-1}\end{array}$$

There are two possible tableau if this diagram is fully labelled:

$$\begin{array}{|c|c|c|}\hline 6 & 5 & 1 \\ \hline 4 & 2 \\ \cline{1-2} 3 \\ \cline{1-1}\end{array} \qquad \begin{array}{|c|c|c|}\hline 6 & 4 & 1 \\ \hline 5 & 2 \\ \cline{1-2} 3 \\ \cline{1-1}\end{array}$$

Each of these tableau labels a basis vector in the carrier space of the S_6 irreducible representation with the same shape as the above diagrams. A partially labelled Young diagram therefore represents a set of states in the Young-Yamanouchi basis. Mathematically we can write:

$$\left|\begin{array}{|c|c|c|}\hline 6 & 5 & 1 \\ \hline 4 & 2 \\ \cline{1-2} 3 \\ \cline{1-1}\end{array}\right\rangle, \left|\begin{array}{|c|c|c|}\hline 6 & 4 & 1 \\ \hline 5 & 2 \\ \cline{1-2} 3 \\ \cline{1-1}\end{array}\right\rangle \in \left|\begin{array}{|c|c|c|}\hline \; & \; & 1 \\ \hline \; & 2 \\ \cline{1-2} 3 \\ \cline{1-1}\end{array}\right\rangle$$

If we consider $\sigma \in S_3$ such that the action of σ on the partially labelled tableau leaves the labels $1, 2$ and 3 unchanged, then:

$$\Gamma_R(\sigma)\left|\begin{array}{|c|c|c|}\hline \; & \; & 1 \\ \hline \; & 2 \\ \cline{1-2} 3 \\ \cline{1-1}\end{array}\right\rangle \in \left|\begin{array}{|c|c|c|}\hline \; & \; & 1 \\ \hline \; & 2 \\ \cline{1-2} 3 \\ \cline{1-1}\end{array}\right\rangle$$

Clearly, the subspace represented by the partially labelled Young diagram is invariant under the action of σ and is therefore the carrier space for an irreducible representation of σ, labelled by the Young diagram $\begin{array}{|c|c|}\hline \; & \; \\ \hline \; \\ \cline{1-1}\end{array}$. This Young diagram is formed by removing the boxes labelled $1, 2, 3$. We could remove the same boxes in a different order and be left with the same irreducible representation of S_3. This is the multiplicity that appears when we restrict an irreducible representation of S_{n+m} to $S_n \times (S_1)^m$. Since each possible partially labelled Young diagram represents a different subspace of the carrier space of the irreducible representation of S_{n+m} which is subduced, we can isolate the subspace which carries a certain copy of  by specifying the order in which the boxes are to be removed.

5.4 Young Diagrams with p rows and O(N) Columns

We will not consider the most general Young diagram. We look at Young diagrams which have p rows with $m + n \sim O(N)$ boxes in total. Each row also has $O(N)$ boxes. Set $m = \alpha N$ with $\alpha \ll 1$. With this kind of Young diagram it is possible to remove all m boxes from any single row and still be left with a valid Young diagram. The total number of partially labelled Young diagrams is p^m. Assign to each box that is to be removed from the Young diagram a p-dimensional vector $\vec{v}(i)$ with $i = 1, 2, \dots m$. We denote the components of these vectors by $\vec{v}(i)_n$ where $n = 1, 2, \dots p$. If box i is to be pulled from row j then define the components of vector $\vec{v}(i)$ as:

$$\vec{v}(i)_n = \delta_{nj}$$

The right hand side of this equation is independent of i so that different boxes can be labelled by the same vector as long as they are in the same row. Taking the tensor product of these m vectors yields a tensor of the form:

$$\vec{v}(1) \otimes \vec{v}(2) \otimes \cdots \otimes \vec{v}(m-1) \otimes \vec{v}(m)$$

$\vec{v}(1)$ is the vector associated with the first box to be removed, $\vec{v}(2)$ is the vector associated with the second box to be removed and so forth. All the tensors of this form label the possible ways of removing m boxes from the Young diagram. Call the p^m dimensional vector space in which these tensors live, $V_p^{\otimes m}$. The action of $\sigma \in S_m$ on an element of $V_p^{\otimes m}$ is:

$$\sigma \cdot (\vec{v}(1) \otimes \vec{v}(2) \otimes \cdots \otimes \vec{v}(m)) = \vec{v}(\sigma(1)) \otimes \vec{v}(\sigma(2)) \otimes \cdots \otimes \vec{v}(\sigma(m))$$

σ will move the vector in slot i to slot $\sigma(i)$. The action of $U \in U(p)$ on $V_p^{\otimes m}$ is defined as:

$$U \cdot (\vec{v}(1) \otimes \vec{v}(2) \otimes \cdots \otimes \vec{v}(m)) = D(U)\vec{v}(1) \otimes D(U)\vec{v}(2) \otimes \cdots \otimes D(U)\vec{v}(m)$$

Where $D(U)$ is the $p \times p$ unitary matrix representing $U \in U(p)$. The action of U on $V_p^{\otimes m}$ commutes with the action of σ on $V_p^{\otimes m}$:

$$\begin{aligned} U \cdot (\sigma \cdot (\vec{v}(1) \otimes \cdots \otimes \vec{v}(m))) &= U \cdot (\vec{v}(\sigma(1)) \otimes \cdots \otimes \vec{v}(\sigma(m))) \\ &= D(U)\vec{v}(\sigma(1)) \otimes \cdots \otimes D(U)\vec{v}(\sigma(m)) \\ &= \sigma \cdot (D(U)\vec{v}(1) \otimes \cdots \otimes D(U)\vec{v}(m)) \\ &= \sigma \cdot (U \cdot (\vec{v}(1) \otimes \cdots \otimes \vec{v}(m))) \end{aligned}$$

Since the group actions commute, Schur-Weyl duality implies that [3]:

$$V_p^{\otimes m} = \bigoplus_s V_s^{U(p)} \otimes V_s^{S_m}$$

$V_s^{U(p)}$ is the carrier space of the $U(p)$ irreducible representation labelled by a Young diagram s and $V_s^{S_m}$ is the carrier space of the S_m irreducible representation labelled by s. As a consequence of Schur-Weyl duality we can write the dimension of $V_p^{\otimes m}$ in terms of the dimension of $V_s^{U(p)}$ and $V_s^{S_m}$.

$$p^m = \sum_s \dim(V_s^{U(p)})\dim(V_s^{S_m})$$

The same label, s, labels the $U(p)$ states and the S_m states. Thus, by identifying $U(p)$ states we have identified S_m states. We can therefore use $U(p)$ group representation theory to construct the symmetric group projection operators. The next step in this procedure is to create a dictionary for translating between the old labels, R,r,s,j,k and the new $U(p)$ labels. s is a Young diagram which labels a $U(p)$ irreducible representation and r still labels an irreducible representation of S_n. The final label labels states in the carrier space of the $U(p)$ irreducible representation s. The Δ weight of this Gelfand-Tsetlin pattern tells us how boxes were removed from R to obtain r. For example, a $U(3)$ irreducible representation with $\Delta = (2, 1, 0)$ implies that 2 boxes were removed from the first row of R, 1 box was removed from the second row of R and no boxes were removed from the third row of R. Every Gelfand-Tsetlin pattern can be put into a one-to-one correspondence with semi-standard Young tableau. This correspondence is key in understanding how the labels $R, (r, s)$ translate into the new $U(p)$ labels.

5.5 U(p) Representation Theory - Gelfand-Tsetlin Patterns

In this section I review a labelling for basis states of $u(p)$ irreducible representations. This labelling was first introduced by Gelfand and Tsetlin [4]. An irreducible representation for $GL(p, C)$ is given by the weight $\mathbf{m}$. The weight is a sequence of p integers:

$$\mathbf{m} = (m_{1p}, m_{2p}, \ldots, m_{pp})$$

The allowed sequences are constrained by $m_{1p} \geq m_{2p} \geq \cdots \geq m_{pp}$. Upon restriction to a $GL(p-1, C)$ the irreducible representation given by $\mathbf{m}$ generally subduces many irreducible representations of $GL(p-1, C)$ with highest weights:

$$\mathbf{m}' = (m_{1,p-1}, m_{2,p-1}, \ldots, m_{p-1,p-1})$$

The allowed weights after restriction are constrained by the betweenness condition:

$$m_{1p} \geq m_{1,p-1} \geq m_{2p} \geq m_{2,p-1} \geq \cdots \geq m_{p-1,p} \geq m_{p-1,p-1} \geq m_{pp}$$

After every restriction there are a number of possible irreducible representations which are subduced. These are the irreducible representations with weights satisfying the betweenness condition. We can continue restricting to subgroups until we reach $GL(1, C)$. A Gelfand-Tsetlin pattern is a triangular array of weights. Each weight corresponds to a particular irreducible representation. The weights are subject to the betweenness condition. Thus, a Gelfand-Tsetlin pattern gives the sequence of irreducible representations as we restrict from $GL(p, C)$ to $GL(p-1, C)$ to ... to $GL(1, C)$. The number of valid Gelfand-Tsetlin patterns with $\mathbf{m}$ as their first line is equal to the dimension of the irreducible representation represented by $\mathbf{m}$. Each Gelfand-Tsetlin pattern is therefore in a one-to-one correspondence with the basis states of the carrier space of $\mathbf{m}$ and can be used to label the basis states. The following is a Gelfand-Tsetlin pattern:

$$M = \begin{bmatrix} m_{1p} & & m_{2p} & & \ldots & & m_{p-1,p} & & m_{pp} \\ & m_{1,p-1} & & m_{2,p-1} & & \ldots & & m_{p-1,p-1} & \\ & & \ldots & & \ldots & & \ldots & & \\ & & & m_{12} & & m_{22} & & & \\ & & & & m_{11} & & & & \end{bmatrix}$$

5.6 Σ Weights and Δ Weights

The Σ weight is defined as:

$$\Sigma(M) = (\sigma_p(M), \sigma_{p-1}(M), \ldots, \sigma_2(M), \sigma_1(M))$$

where $\sigma_l(M)$ is the sum of the entries in row l of the Gelfand-Tsetlin pattern M:

$$\sigma_l(M) = \sum_{i=1}^{l} m_{il}$$

$\Sigma(M)$ is not unique. It is possible that two different Gelfand-Tsetlin patterns have the same sigma weight. The number of Gelfand-Tsetlin patterns in the carrier space which have the same weight is called the inner multiplicity of the state and is denoted by $I(\Sigma)$. The delta weight of a Gelfand-Tsetlin pattern M is defined as:

$$\begin{aligned}\Delta(M) &= (\sigma_p(M) - \sigma_{p-1}(M), \sigma_{p-1}(M) - \sigma_{p-2}(M), \ldots, \sigma_2(M) - \sigma_1(M), \sigma_1(M)) \\ &= (\delta_p(M), \delta_{p-1}(M), \ldots, \delta_1(M))\end{aligned}$$

Clearly, $I(\Sigma) = I(\Delta)$.

5.7 Projection Operator

We have identified a complete set of orthonormal states which span the carrier space of the irreducible representation $(r,s)j$ of $S_n \times S_m \subset S_{n+m}$. We use these states to construct the projection operator. This construction was first done in [12]:

$$P_{R\to(r,s)jk} = \sum_{\alpha=1}^{d_s} |s, M^j, \alpha\rangle\langle s, M^k, \alpha| \otimes \mathbf{I}_r$$

α is a multiplicity label for the $U(p)$ states. The sum over α runs from 1 to the dimension of the S_m irreducible representation s. These limits on the sum over α come from the use of Schur-Weyl duality. The multiplicity label for the $U(p)$ states is organised by s which is an irreducible representation of the symmetric group. M^i and M^j are labels for $U(p)$ states with the same delta weight. i and j therefore take values from $1, 2, \ldots, I(\Delta(M))$.

5.8 Dilatation Operator

In this section we quote some key ideas from [12] which are used in chapter 7. This section will elucidate some of the methods used in 7 to calculate the action of the dilatation operator on restricted Schurs as well as highlighting some key differences between the action of the dilatation operator in $\mathcal{N}=4$ SYM and Chern-Simons theory. The dilatation operator to one loop in the $SU(2)$ sector of $\mathcal{N}=4$ SYM is [5]:

$$D = -g_{YM}^2 \mathrm{Tr}\left([Y,Z][\partial_Y,\partial_Z]\right)$$

The action of the dilatation operator on a restricted Schur polynomial is:

$$\begin{aligned} D\chi_{R,(r,s)jk} =& \frac{g_{YM}^2}{(n-1)!(m-1)!} \sum_{\psi\in S_{n+m}} \mathrm{Tr}_{(r,s)jk}(\Gamma_R((1,m+1)\psi) - \psi(1,m+1))\times \\ & \times \delta^{i_1}_{i_{\psi(1)}} Y^{i_2}_{\psi(2)} \dots Y^{i_m}_{i_{\psi(m)}} (YZ-ZY)^{i_m+1}_{i_{\psi(m+1)}} Z^{i_{m+2}}_{i_{\psi(m+2)}} \dots Z^{i_{m+n}}_{i_{\psi(m+n)}} \end{aligned}$$

The dilatation operator removes a Z and a Y field and inserts a Y and a Z field. The Kronecker delta appearing above restricts the sum over group elements of S_{n+m} by setting $\psi(1)=1$. Hence, the sum runs only over group elements of S_{n+m} which leave 1 inert. One can express S_{n+m} in terms of the subgroup which leaves 1 inert:

$$S_{n+m} = S_{n+m-1} \bigcup (1,2)S_{n+m-1} \bigcup (1,3)S_{n+m-1} \bigcup \cdots \bigcup (1,m+n)S_{n+m-1} \bigcup S_{n+m-1}$$

We can therefore write the sum over S_{n+m} as a sum over S_{n+m-1} and its cosets. This result follows from the reduction rule for restricted Schur Polynomials ([6] and appendix C of [56]):

$$\begin{aligned} D\chi_{R,(r,s)jk} =& \frac{g_{YM}^2}{(n-1)!(m-1)!} \sum_{\psi\in S_{n+m-1}} \sum_{R'} c_{RR'} \mathrm{Tr}_{(r,s)jk}(\Gamma_R((1,m+1))\Gamma_{R'}(\psi) \\ & - \Gamma_{R'}(\psi)\Gamma_R((1,m+1))) Y^{i_2}_{\psi(2)} \dots Y^{i_m}_{i_{\psi(m)}} (YZ-ZY)^{i_m+1}_{i_{\psi(m+1)}} Z^{i_{m+2}}_{i_{\psi(m+2)}} \dots Z^{i_{m+n}}_{i_{\psi(m+n)}} \end{aligned}$$

The sum over R' is a sum over all the Young diagrams formed by removing a single box from R. $c_{RR'}$ is the factor associated with the removed box. For convenience we use the following notation:

$$\mathrm{Tr}(\sigma Z^{\otimes n} Y^{\otimes m}) = Z^{i_1}_{i_{\sigma(1)}} \dots Z^{i_n}_{i_{\sigma(n)}} Y^{i_{n+1}}_{i_{\sigma(n+1)}} \dots Y^{i_{n+m}}_{i_{\sigma(n+m)}}$$

We will now use two identities (bear in mind that $\psi(1)=1$):

$$Y^{i_2}_{\psi(2)} \dots Y^{i_m}_{i_{\psi(m)}} (YZ-ZY)^{i_m+1}_{i_{\psi(m+1)}} Z^{i_{m+2}}_{i_{\psi(m+2)}} \dots Z^{i_{m+n}}_{i_{\psi(m+n)}} = \mathrm{Tr}(((1,m+1)\psi) - \psi(1,m+1)\sigma Z^{\otimes n} Y^{\otimes m})$$

and [25]

$$\mathrm{Tr}\left(\sigma Z^{\otimes n} Y^{\otimes m}\right) = \sum_{T,(t,u)lq} \frac{d_T n! m!}{d_t d_u (n+m)!} \mathrm{Tr}_{(t,u)lq}\left(\Gamma_T(\sigma^{-1})\right) \chi_{T,(t,u)ql}(Z,Y) \tag{5.1}$$

Equation (5.1) shows that restricted Schur polynomials are a basis for expressing trace operators of two matrix fields. Using these identities in the expression for the action of the dilatation operator on the restricted Schur we obtain:

$$D\chi_{R,(r,s)jk}(Z,Y) = \sum_{T,(t,u)lq} M_{R,(r,s)jk;T,(t,u)lq} \chi_{T,(t,u)ql}(Z,Y)$$

where

$$M_{R,(r,s)jk;T,(t,u)lq} = g_{YM}^2 \sum_{\psi \in S_{n+m-1}} \sum_{R'} \frac{c_{RR'} d_T nm}{d_t d_u (n+m)!} \mathrm{Tr}_{(r,s)jk}\left([\Gamma_R((1,m+1)), \Gamma_{R'}(\psi)]\right)$$
$$\times \mathrm{Tr}_{(t,u)lq}\left(\left[\Gamma_{T'}(\psi^{-1}), \Gamma_T((1,m+1))\right]\right)$$

Use of the fundamental orthogonality relation to evaluate the sum over ψ yields a result which depends on the trace of a product of commutators. This result was first obtained in [10]:

$$M_{R,(r,s)jk;T,(t,u)lq} = -g_{YM}^2 \sum_{R'} \frac{c_{RR'} d_T nm}{d_{R'} d_t d_u (n+m)} \mathrm{Tr}\Big(\left[\Gamma_R((1,m+1)), P_{R\to(r,s)jk}\right] I_{R'T'} \times$$
$$\times \left[\Gamma_T((1,m+1)), P_{T\to(t,u)ql}\right] I_{T'R'}\Big)$$

Note that the dilatation operator has a similar form in the sense that it is also written as the trace of a product of commutators.

5.8.1 Intertwiners

The objects $I_{R'T'}$ and $I_{T'R'}$ appearing above are intertwiners. In this section we explain their explicit form. Consider some representation of the symmetric group, $\Gamma(\sigma)$ for $\sigma \in S_n$ acting on a carrier space $V^{\otimes n}$. $\Gamma(\sigma)$ is completely reducible and can be written as the direct sum of irreducible representations whose dimensions sum to the dimension of $\Gamma(\sigma)$. In matrix form these irreducible representations appear on the diagonal of $\Gamma(\sigma)$ Suppose R and S are two of the irreducible representations on the diagonal. The form of $\Gamma(\sigma)$ is:

$$\Gamma(\sigma) = \begin{pmatrix} \Gamma_R(\sigma) & 0 & \dots \\ 0 & \Gamma_S(\sigma) & \dots \\ \dots & \dots & \dots \end{pmatrix}$$

Upon restriction to a S_{n-1} subgroup of S_n, all the irreducible representations on the diagonal are in general reducible and can be written as a direct sum of irreducible representations of S_{n-1}. Suppose R subduces two irreducible representations, R'_1 and R'_2 and S also subduces two irreducible representations, S'_1 and S'_2. $\Gamma(\sigma)$ now takes the form:

$$\Gamma(\sigma) = \begin{pmatrix} \Gamma_{R'_1}(\sigma) & 0 & 0 & 0 & \dots \\ 0 & \Gamma_{R'_2}(\sigma) & 0 & 0 & \dots \\ 0 & 0 & \Gamma_{S'_1}(\sigma) & 0 & \dots \\ 0 & 0 & 0 & \Gamma_{S'_2}(\sigma) & \dots \\ \dots & \dots & \dots & \dots & \dots \end{pmatrix}$$

If $R'_1 = S'_1$ then by application of the fundamental orthogonality relation we get:

$$\sum_{\sigma \in S_{n-1}} \begin{bmatrix} \Gamma_{R'_1}(\sigma) & 0 & 0 & 0 & \dots \\ 0 & 0 & 0 & 0 & \dots \\ 0 & 0 & 0 & 0 & \dots \\ 0 & 0 & 0 & 0 & \dots \\ \dots & \dots & \dots & \dots & \dots \end{bmatrix}_{ij} \begin{bmatrix} 0 & 0 & 0 & 0 & \dots \\ 0 & 0 & 0 & 0 & \dots \\ 0 & 0 & \Gamma_{S'_1}(\sigma) & 0 & \dots \\ 0 & 0 & 0 & 0 & \dots \\ \dots & \dots & \dots & \dots & \dots \end{bmatrix}_{ab}$$

$$= \frac{(n-1)!}{d_{R'_1}} \delta_{R'_1 S'_1} \begin{bmatrix} 0 & 0 & \mathbf{1} & 0 & \dots \\ 0 & 0 & 0 & 0 & \dots \\ 0 & 0 & 0 & 0 & \dots \\ 0 & 0 & 0 & 0 & \dots \\ \dots & \dots & \dots & \dots & \dots \end{bmatrix}_{ib} \begin{bmatrix} 0 & 0 & 0 & 0 & \dots \\ 0 & 0 & 0 & 0 & \dots \\ \mathbf{1} & 0 & 0 & 0 & \dots \\ 0 & 0 & 0 & 0 & \dots \\ \dots & \dots & \dots & \dots & \dots \end{bmatrix}_{aj}$$

$$= \frac{(n-1)!}{d_{R'_1}} \delta_{R'_1 S'_1} \left(I_{R'_1 S'_1}\right)_{ib} \left(I_{S'_1 R'_1}\right)_{aj}$$

The form of the intertwiners has been made explicit in the above example.

Chapter 6

Finite N Quiver Gauge Theory

Our focus in this chapter is on free gauge theories whose structure is elegantly summarized in a quiver. By a quiver we mean a set of nodes (or vertices) connected by directed arrows, that is, a quiver is a directed graph. The gauge group of the quiver gauge theory is a direct product of groups, one associated to each node of the quiver so that there is a gauge field associated to each node of the quiver. We are interested in the case that each node corresponds to a unitary group $U(N_a)$. Although our arguments carry over to a general quiver gauge theory, we will mostly focus on quivers with two nodes, which corresponds to studying a $U(N_1) \times U(N_2)$ gauge group. For each directed arrow there is a bifundamental scalar. An arrow stretching from node a to node b gives a field that transforms in the fundamental representation of $U(N_a)$, in the antifundamental of $U(N_b)$ and is a singlet of $U(N_c)$, $c \neq a, b$.

Our primary interest is in the finite N physics of these theories. A natural basis for the local gauge invariant operators of the theory is provided by taking traces of products of fields. At finite N, not all trace structures are independent. As a simple example, consider a scalar field Z which is an $N \times N$ matrix transforming in the adjoint representation of $U(N)$. A complete set of operators built using three fields is given by $\{\mathrm{Tr}(Z^3), \mathrm{Tr}(Z^2)\mathrm{Tr}(Z), \mathrm{Tr}(Z)^3\}$, when $N > 2$. For $N = 2$ this set is overcomplete because we have the identity

$$\mathrm{Tr}(Z^3) = \frac{1}{2}\left[3\mathrm{Tr}(Z^2)\mathrm{Tr}(Z) - \mathrm{Tr}(Z)^3\right] \tag{6.1}$$

It is a highly non-trivial problem to write a basis of local operators that is not over complete at finite N. This problem has been solved for multimatrix models with $U(N)$ gauge group in [41, 42, 43, 44, 45, 46, 47, 15, 17] and for single matrix models with $SO(N)$ or $Sp(N)$ gauge groups in [48, 49, 50]. The result of these studies is a basis of local operators that also diagonalizes the free field two point function. These bases have been useful for exploring giant gravitons[51, 52, 53, 30, 54, 55, 56, 57, 58, 59, 60] and new background geometries[61, 62, 63, 64, 65, 66, 67, 68, 69, 70, 71, 72] in AdS/CFT[73], as well as for the computations of anomalous dimensions in large N but non-planar limits[74, 11, 12, 13, 14, 19, 18]. Elements in the basis are labelled by Young diagrams. The finite N relations are encoded in the statement that operators labeled by Young diagrams with more than N rows vanish. To illustrate this point note that a basis for operators built using a single field are the Schur polynomials. For $N = 2$ the constraint (6.1) is the statement

$$\chi_{\tiny\begin{array}{|c|}\hline\\\hline\\\hline\\\hline\end{array}}(Z) = \frac{1}{6}\left(\mathrm{Tr}(Z)^3 - 3\mathrm{Tr}(Z^2)\mathrm{Tr}(Z) + 2\mathrm{Tr}(Z^3)\right) = 0 \tag{6.2}$$

For quiver gauge theories, there are two distinct approaches that have been developed to study the finite N physics[22, 23][1]. In the remainder of this introductory section, we will review these two approaches with the goal of exhibiting a tension between them. The primary goal of this chapter is to clarify the origin of this tension and to explain how it is resolved.

For concreteness, consider a quiver gauge theory with gauge group $U(N_1) \times U(N_2)$ and assume that $N_1 > N_2$. We will use Roman indices for the $U(N_1)$ gauge group and Greek indices for the $U(N_2)$ gauge group. Consider the problem of building gauge invariant operators using the bifundamentals $(A^I)^a_\alpha$ and $(B^{I\dagger})^\alpha_a$, where $I = 1, 2$. It is clear that any gauge invariant operator must be a product of traces of an alternating product of As and $B^\dagger$s. This motivates the products

$$\phi^{IJ}{}^a_b = (A^I)^a_\alpha (B^{J\dagger})^\alpha_b \tag{6.3}$$

which transform in the adjoint of $U(N_1)$. Any gauge invariant single trace operator is the trace of a unique (up to cyclic permutations) product of ϕ^{IJ} fields. Thus, we can use the restricted Schur polynomials[44] to build a basis for the local operators of the quiver[22]. The Young diagrams labelling these operators are cut off to have no more than N_1 rows. If we had instead chosen to work with the fields

$$\psi^{JI}{}^\alpha_\beta = (B^{J\dagger})^\alpha_a (A^I)^a_\beta \tag{6.4}$$

we would have constructed restricted Schur polynomials that have Young diagram labels cut off to have no more than N_2 rows. These cut offs are different and they do not give the same number of gauge invariant operators, so there is a puzzle. To see how this is resolved, restrict attention to a single field ϕ^{11} in which case our operators are the Schur polynomials $\chi_R(\phi^{11})$. For $R \vdash d$ we obtain a Schur polynomial of degree d. Recall that the degree d Schur polynomials in N variables are a linear basis for the space of homogeneous

[1]For earlier work, focusing on essentially single matrix dynamics, see [26, 27, 28, 29]

degree d symmetric polynomials in N variables[40]. Thus these Schur polynomials are functions of the N_1 eigenvalues λ_i of ϕ^{11}. Concretely, we can write the Schur polynomial as a sum of monomials

$$\chi_R(\lambda_1, \lambda_2, \ldots, \lambda_N) = \sum_T \lambda^T = \sum_T \lambda_1^{t_1} \cdots \lambda_n^{t_n} \tag{6.5}$$

where the summation is over all semistandard Young tableaux T of shape R. The powers of the eigenvalues t_i counts the number of times the number i appears in T. We have not yet considered the eigenvalues of

$$\phi^{11} = A^1 (B^1)^\dagger \tag{6.6}$$

$(B^1)^\dagger$ is an $N_2 \times N_1$ matrix, while A^1 is an $N_1 \times N_2$ matrix. These matrices are not square, so they don't admit an eigendecomposition. There is however the notion of a singular value decomposition (SVD) which can be applied[39]. The SVD decomposition of $(B^1)^\dagger$ is

$$(B^1)^\dagger = U_B \Sigma_B V_B^\dagger \tag{6.7}$$

where U_B is an $N_2 \times N_2$ unitary matrix, $V_B^\dagger$ is an $N_1 \times N_1$ unitary matrix and Σ_B is an $N_2 \times N_1$ rectangular matrix with non-zero singular values on its diagonal. Since $(B^1)\dagger$ has (at most) N_2 non-zero singular values, the generic matrix $(B^1)^\dagger$ has a null space of dimension $N_1 - N_2$. (Non-generic $(B^1)^\dagger$ can have an even larger null-space.) Of course, ϕ^{11} and $(B^1)^\dagger$ share the same null space, so that ϕ^{11} has at least $N_1 - N_2$ zero eigenvalues.

Recall that a semistandard Young tableau is column strict, that is, the entries weakly increase along each row and strictly increase down each column. This implies that if R has more than N_2 rows every term in $\chi_R(\phi^{11})$ is a product of at least N_2+1 distinct eigenvalues. Since only N_2 of these can be non-zero, it follows that $\chi_R(\phi^{11})$ actually vanishes as soon as R has more than N_2 rows. This proves that the Schur polynomials $\chi_R(\phi^{11})$ and $\chi_R(\psi^{11})$ are both cut off such that R must have at most N_2 rows. A very simple generalization of this reasoning allows us to conclude that we can construct restricted Schur polynomials using either ψ^{IJ} or ϕ^{IJ}. The finite N constraints are encoded in the statements that operators labelled by Young diagrams with more than[2] $\min(N_1, N_2)$ rows vanish. This implies in particular that the number of gauge invariant operators that can be constructed will depend only on the smallest of N_1 and N_2. We will call this the restricted Schur basis.

A second approach to the finite N physics entails working with the field A^I and $(B^I)^\dagger$ directly[23]. In this case, we organize the $U(N_1)$ indices using Young diagrams that have no more than N_1 rows and we organize the $U(N_2)$ indices using Young diagrams that have no more than N_2 rows. Thus, each operator is labeled by two types of Young diagrams

[2] $\min(N_1, N_2)$ is equal to the smallest of N_1 or N_2.

that have distinct cut offs. In this case both N_1 and N_2 enter. This dependence is genuine and one finds, for example, that the number of operators that can be constructed depend on both N_1 and N_2. This is the generalized restricted Schur basis[23].

At infinite N, the counting of restricted Schur polynomials and generalized restricted Schur polynomials agree. At finite N there are more restricted Schur polynomials than there are generalized restricted Schur polynomials. This means that either the restricted Schur polynomials are over complete or the generalized restricted Schur polynomials are under complete. We will show in what follows that the restricted Schur polynomials are over complete, for a subtle reason that is peculiar to quiver gauge theories, as we now explain. Given a collection of fields $\{A^I, (B^J)^\dagger\}$we can form the fields ϕ^{IJ}. The number n_{IJ} of each type of field is not unique and it depends on the details of how we pair the A^Is and the $(B^J)^\dagger$s. To get the complete set of restricted Schur polynomials, we need to consider each possible pairing with its collections of fields described by the numbers $\{n_{IJ}\}$. For a given pairing $\{n_{IJ}\}$, the restricted Schur polynomials do give the correct finite N constraints. There are however extra genuinely new conditions that can be written which involve fields that come from different pairings, pairing $\{n_{IJ}\}$ and pairing $\{n'_{IJ}\}$ say. The restricted Schur polynomials do not respect these additional constraints and are thus over complete. The generalized restricted Schur basis correctly accounts for the complete set of finite N trace relations. This is an important general lesson: at finite N the physics of quiver gauge theories is not correctly captured by contracting fields to construct adjoints of specific gauge groups and then building operators from these adjoints. The adjoints retain knowledge that they are constructed from more basic bifundamental fields in the form of extra finite N relations. To correctly account for the complete set of finite N relations it seems easiest to work directly with the original bifundamental fields and hence the generalized restricted Schur polynomial basis.

There are exceptions to this general lesson: in certain subsectors of the theory and in specific limits, some of which we identify below, the restricted Schur polynomials do provide a complete basis and do account for all finite N relations. In these cases, it may be simpler to use the restricted Schur polynomials rather than the generalized restricted Schur polynomials.

In section 6.1 we will outline in detail, using a specific example, the origin and form of the new constraints. There are situations in which the restricted Schur polynomials do capture the complete set of finite N constraints and are consequently not overcomplete. In these situations one may use either basis, as dictated by the problem being considered. In section 6.2 we will identify and describe these situations. Section 6.3 considers the computation of some simple correlators which provide further useful and independent insight into the finite N physics. Finally in section 6.4 we compare the structure of the restricted Schur polynomials and the generalized restricted Schur polynomials, with the goal of explaining why it may be simpler to use the restricted Schur polynomials rather than the generalized restricted Schur polynomials for certain computations. Section 6.4 also demonstrates situations in which the generalized restricted Schur polynomials reduce to the restricted Schur polynomials.

In what follows we will talk of a Young diagram r that has m boxes or of a Young diagram r that is a partition of m or even more simply, $r \vdash m$.

6.1 New Finite N Relations

The number of generalized restricted Schur polynomials $\mathcal{N}_g(n_1, n_2, m_1, m_2)$ that can be built in a theory with gauge group $U(N_1) \times U(N_2)$, using n_1 copies of the field A^1, n_2 copies of A^2, m_1 copies of $(B^1)^\dagger$ and m_2 copies of $(B^2)^\dagger$ is given by ($l(R)$ is the length of the first column in R and $l(S)$ is the length of the first column in S)[23]

$$\sum_{\substack{R, S \vdash n_1+n_2 \\ l(R) \leq N_1 \;\; l(S) \leq N_2}} \sum_{\substack{r_1 \vdash n_1 \\ r_2 \vdash n_2}} \sum_{\substack{s_1 \vdash m_1 \\ s_2 \vdash m_2}} g(r_1, r_2, R) g(r_1, r_2, S) g(s_1, s_2, R) g(s_1, s_2, S) \tag{6.8}$$

where we have $n_1+n_2 = m_1+m_2$ and where $g(\cdot, \cdot, \cdot)$ is a Littlewood-Richardson coefficient. The finite N relations are accounted for by restricting the above sum so that R has no more than N_1 rows and S has no more than N_2 rows.

Consider now the counting for the restricted Schur polynomial. The first step in the construction of the resticted Schur polynomials entails pairing the As and $B^\dagger$s to produce n_{IJ} copies of ϕ^{IJ}. There is one Young diagram for each of these ϕ^{IJ} fields. The number of restricted Schur polynomials is now given by ($N_- \equiv \min(N_1, N_2)$)

$$\mathcal{N}_r(n_1, n_2, m_1, m_2) = \sum_{\{n_{IJ}\}} \mathcal{N}_{\{n^{IJ}\}} \tag{6.9}$$

where the above sum is a sum over all possible distinct ways of pairing, that is it is a sum over all possible distinct sets $\{n_{IJ}\}$ and [38]

$$\mathcal{N}_{\{n^{IJ}\}} = \sum_{\substack{R \vdash n_1+n_2 \\ l(R) \leq N_-}} \sum_{r_{IJ} \vdash n_{IJ}} (g(r_{11}, r_{12}, r_{21}, r_{22}; R))^2 \tag{6.10}$$

In general, (6.8) and (6.9) do not agree. The goal of this section is to explain the origin of the discrepancy.[3]

To make the discussion concrete, we will focus on a specific example. Consider $n_1 = 3$, $n_2 = 1$, $m_1 = m_2 = 2$, and take $N_1, N_2 > 4$ so that there are no finite N constraints. In this case, a simple application of (6.8) gives $\mathcal{N}_g(3,1,2,2) = 28$ generalized restricted Schur polynomials. For the number of restricted Schur polynomials, we need to consider two cases

$$\begin{array}{ll} \text{Case I}: & n_{11}=2 \quad n_{12}=1 \quad n_{21}=0 \quad n_{22}=1 \\ \text{Case II}: & n_{11}=1 \quad n_{12}=2 \quad n_{21}=1 \quad n_{22}=0 \end{array} \tag{6.11}$$

For these cases (6.9) gives $\mathcal{N}_I$=14, $\mathcal{N}_{II}$=14, so that in total $\mathcal{N}_r(3,1,2,2) = 28$. In the next section, we prove that the number of restricted Schur polynomials and generalized restricted Schur polynomials always agree in the absence of finite N constraints.

We will see that it is $\mathcal{N}_r(3,1,2,2)$ that does not correctly count the number of gauge invariant operators at finite N. Since this is one of the main points of our discussion, we will give the complete details on how equation (6.9) is applied. Towards this end, we have summarized the labels for the relevant restricted Schur polynomials in Appendix B. Consider next the case that $N_1 = N_2 = 2$. A simple application of (6.8) gives $\mathcal{N}_g(3,1,2,2) = 13$ generalized restricted Schur polynomials. Next, consider the complete set of possible restricted Schur polynomial labels given in Appendix B. For Case I, the operators given in (B.1), (B.2) and (B.3) vanish so that we have 8 operators. For Case II, the operators given in (B.9), (B.10) and (B.11) vanish so that we have 8 operators. This gives a total of $\mathcal{N}_r(3,1,2,2) = 16$ restricted Schur polynomials, which shows a clear discrepancy between (6.8) and (6.9).

To explore the origin of this discrepancy, we have developed a numerical algorithm to determine the number and precise form of the finite N constraints. Consider first the case of a single $N \times N$ matrix Z. For $N = 2$ we know one of the finite N constraints is given by (6.1). If we choose a random 2×2 matrix Z and form the vector

$$\vec{v} = \begin{bmatrix} \mathrm{Tr}(Z^3) \\ \mathrm{Tr}(Z^2)\mathrm{Tr}(Z) \\ \mathrm{Tr}(Z)^3 \end{bmatrix} \tag{6.12}$$

it will point in a random direction depending on the specific matrix Z. However, we know

[3] The Littlewood-Richardson number has three indices $g(r,s,t)$. The number $g(r,s,t)$ gives the number of times irrep t of GL_N appears in the tensor product of GL_N representations r and s. By $g(r_1, r_2, ..., r_n; R)$ we mean the number of times R appears in the tensor product of r_1 with r_2 with r_3 with ... with r_n. We could write this as $\sum_{s_i} g(r_1, r_2, s_1) g(s_1, r_3, s_2) \cdots g(s_{n-1}, r_n, R)$.

that it must lie in a two dimensional subspace of the three dimensional space it belongs to because, thanks to (6.1) we know that

$$\vec{v}\cdot\vec{u}=0 \qquad \vec{u}=\begin{bmatrix} 2\\ -3\\ 1\end{bmatrix} \tag{6.13}$$

Now imagine preparing an ensemble of random matrices $Z^{(i)}$, $i=1,...,k$. This ensemble of $Z^{(i)}$ can be used to construct an ensemble $\vec{v}^{(i)}$ using (6.12) and then we can form the matrix

$$M=\frac{1}{k}\sum_{i=1}^{k} v^{(i)T}v^{(i)} \tag{6.14}$$

Since the $\vec{v}^{(i)}$ are all orthogonal to $\vec{u}$, but otherwise explore the orthogonal two dimensional subspace, we know that M will have a single null vector, which is $\vec{u}$ itself.

The logic clearly generalizes to multimatrix models. We collect the complete set of multitrace structures into a vector $\vec{v}$. By preparing an ensemble of random matrices, we can prepare an ensemble of random vectors $\vec{v}^{(i)}$ and construct the matrix M as in (6.14). Each null vector of M then corresponds to a finite N constraint. In this way the finite N constraints are recovered from the null vectors of M.

For Case I described above, we find a total of 14 multitrace structures are possible. Setting $N_1=N_2=2$ we find that M has a total of 6 null vectors. Thus, there are 6 finite N constraints leaving 8 independent multitrace operators, in perfect agreement with the number of restricted Schur polynomials. For Case II we again find a total of 14 multitrace structures are possible and again, for $N_1=N_2=2$ we find that M has 6 null vectors. Thus, there are 6 finite N constraints leaving 8 independent multitrace operators, again in perfect agreement with the number of restricted Schur polynomials. If we now form the complete set of gauge invariant operators that we can construct using $n_1=3$, $n_2=1$ and $m_1=m_2=2$, we find a total of 28 multitrace structures are possible, given by the operators of Case I and Case II above. In this case M has a total of 15 null vectors, leaving a total of 13 independent multitrace operators, in perfect agreement with the number of generalized restricted Schur polynomials. At this point the origin of the discrepancy is clear. The construction of restricted Schur polynomials starts by breaking the complete space of gauge invariant operators up into two sets, Case I and Case II above. By searching for the finite N constraints within the operators of Case I and Case II separately, we have discovered 12 constraints. This is 3 short of the complete set of 15 constraints discovered when searching in the complete set of gauge invariant operators. Clearly there are some finite N constraints that mix operators from Case I and operators from Case II, and these constraints are not captured in the restricted Schur construction of [22].

To summarize the conclusion of our discussion, the generalized restricted Shur polynomials correctly account for the complete set of finite N constraints and they provide a basis, while the restricted Schur polynomials only account for a subset of the finite N constraints and are thus overcomplete.

6.2 Situations Without New Finite N Relations

As our discussion in the chapter introduction suggests, in the absence of finite N constraints we expect that both the generalized restricted Schur polynomials and the restricted Schur polynomials provide good bases. This implies, in particular, that in the absence of finite N constraints the number of restricted Schur polynomials is equal to the number of generalized restricted Schur polynomials. This is indeed the case as we now explain. For concreteness we again consider a $U(N_1)\times U(N_2)$ model, building our operators from the fields $(A^I)^a_\alpha$ and $(B^{I\dagger})^\alpha_a$, where $I=1,2$. Thus, we can form four adjoint fields ϕ^{IJ} and our restricted Schur polynomials are labeled by 5 Young diagrams, one Young diagram r_{IJ} for each field ϕ^{IJ} and one which organizes the complete set of fields. According to [38, 17] the number of restricted Schur polynomials at $N=\infty$ is given by expanding

$$Z_r(t_{11},t_{12},t_{21},t_{22}) = \sum_{n_1,n_2,m_1,m_2}\sum_{a,b,c,d}\delta_{a+b,n_1}\delta_{c+d,n_2}\delta_{a+c,m_1}\delta_{b+d,m_2}\times \tag{6.15}$$

$$\times\mathcal{N}_r(n_1,n_2,m_1,m_2)t_{11}^a t_{12}^b t_{21}^c t_{22}^d$$

$$= \prod_{k=1}^{\infty}\frac{1}{1-t_{11}^k-t_{12}^k-t_{21}^k-t_{22}^k} \tag{6.16}$$

The coefficient of $t_{11}^{n_{11}}t_{12}^{n_{12}}t_{21}^{n_{21}}t_{22}^{n_{22}}$ tells us the number of restricted Schur polynomials that can be built using n_{11} ϕ^{11} fields, n_{12} ϕ^{12} fields, n_{21} ϕ^{21} fields and n_{22} ϕ^{22} fields. The number of generalized restricted Schur polynomials at $N=\infty$ is given by expanding[23]

$$Z_g(t_{a_1},t_{a_2},t_{b_1},t_{b_2}) = \sum_{n_1,n_2,m_1,m_2}\mathcal{N}_g(n_1,n_2,m_1,m_2)t_{a_1}^{n_1}t_{a_2}^{n_2}t_{b_1}^{m_1}t_{b_2}^{m_2}$$

$$= \prod_{k=1}^{\infty}\frac{1}{1-(t_{a_1}t_{b_1})^k-(t_{a_1}t_{b_2})^k-(t_{a_2}t_{b_1})^k-(t_{a_2}t_{b_2})^k} \tag{6.17}$$

The coefficient of $t_{a_1}^{n_1}t_{a_2}^{n_2}t_{b_1}^{m_1}t_{b_2}^{m_2}$ tells us how many generalized restricted Schur polynomials can be built using n_1 A_1 fields, n_2 A_2 fields, m_1 $B_1^\dagger$ fields and m_2 $B_2^\dagger$ fields. We can clearly transform (6.16) into (6.17) by setting $t_{ij}=t_{a_i}t_{b_j}$ which proves that in the absence of finite N constraints the number of restricted Schur polynomials is equal to the number of generalized restricted Schur polynomials. This change of variables provides important insight into how to relate the counting of restricted Schur polynomials and generalized restricted Schur polynomials, even when finite N constraints play a role, as we will see.

6.2.1 A single n_{IJ} sector

Consider next the case that one of n_1, n_2, m_1, m_2 is equal to zero. In this case there is only one possible value for the n_{IJ} so that, according to our discussion above the restricted Schur polynomials correctly account for all finite N constraints and we therefore expect the number of restricted Schur polynomials matches the number of generalized restricted Schur polynomials. For concreteness, consider the case that $n_1 = 0$. In this case, the Young diagram appearing in (6.8) is the Young diagram with no boxes, which we denote as $\cdot$. Consequently,

$$g(r_1, r_2, R) = g(\cdot, r_2, R) = \delta_{r_2,R} \qquad g(r_1, r_2, S) = g(\cdot, r_2, S) = \delta_{r_2,S}$$

so that the number of generalized restricted Schur polynomials (6.8) becomes

$$\begin{aligned}&\sum_{R,S\vdash n_2}\ \sum_{l(R)\le N_1}\ \sum_{l(S)\le N_2}\ \sum_{r_2\vdash n_2}\ \sum_{s_1\vdash m_1}\ \sum_{s_2\vdash m_2} \delta_{r_2,R}\ \delta_{r_2,S} g(s_1, s_2, R) g(s_1, s_2, S) \\ &= \sum_{R\vdash n_1}\ \sum_{l(R)\le N_-}\ \sum_{s_1\vdash m_1}\ \sum_{s_2\vdash m_2} g(s_1, s_2, R) g(s_1, s_2, R)\end{aligned} \tag{6.18}$$

To count the number of restricted Schur polynomials, note that now $r_{11} = \cdot$, $r_{12} = \cdot$, $n_{21} = m_1$ and $n_{22} = m_2$ so that (6.9) becomes

$$\sum_{R\vdash n_2}\ \sum_{l(R)\le N_-}\ \sum_{r_{21}\vdash m_1}\ \sum_{r_{22}\vdash m_2} (g(r_{21}, r_{22}; R))^2 \tag{6.19}$$

This demonstrates an exact match between the number of restricted Schur polynomials and the number of generalized restricted Schur polynomials as we predicted. We will recover this result, by showing that in this case the generalized restricted Schur polynomials reduce to the restricted Schur polynomials in section 6.4.

6.2.2 One finite rank

Finally, consider the case that one of the ranks of the two gauge groups goes to infinity. For concreteness, we will take $N_2 \to \infty$. The counting of restricted Schur polynomials is

$$Z_r(t_{11}, t_{12}, t_{21}, t_{22}) = \sum_{r_{11}, r_{12}, r_{21}, r_{22}, R, l(R)\le N_1} (g(r_{11}, r_{12}, r_{21}, r_{22}; R))^2 t_{11}^{|r_{11}|} t_{12}^{|r_{12}|} t_{21}^{|r_{21}|} t_{22}^{|r_{22}|} \tag{6.20}$$

A simple change of variables gives

$$Z_r = \sum_{r_{11}, r_{12}, r_{21}, r_{22}, R, l(R)\le N_1} (g(r_{11}, r_{12}, r_{21}, r_{22}; R))^2 (t_{a_1} t_{b_1})^{|r_{11}|} (t_{a_1} t_{b_2})^{|r_{12}|} (t_{a_2} t_{b_1})^{|r_{21}|} (t_{a_2} r_{b_2})^{|r_{22}|}$$

Employing the identities

$$
\begin{aligned}
g(r_{11},r_{12},r_{21},r_{22};R) &= \sum_{r\vdash n_1}\sum_{s\vdash n_2} g(r_{11},r_{12},r)g(r_{21},r_{22},s)g(r,s,R) \\
&= \sum_{t\vdash m_1}\sum_{u\vdash m_2} g(r_{11},r_{21},t)g(r_{12},r_{22},u)g(t,u,R) \qquad (6.21)
\end{aligned}
$$

we find

$$
\begin{aligned}
Z_r &= \sum_{r,s,t,u}\sum_{R,l(R)\le N_1} g(r,s,R)g(t,u,R)t_{a_1}^{n_1}t_{a_2}^{n_2}t_{b_1}^{m_1}t_{b_2}^{m_2} \\
&\times \sum_{r_{11},r_{12},r_{21},r_{22}} g(r_{11},r_{12},r)g(r_{21},r_{22},s)g(r_{11},r_{21},t)g(r_{12},r_{22},u) \qquad (6.22)
\end{aligned}
$$

We have used $n_1 = |r_{11}| + |r_{12}|$, $n_2 = |r_{21}| + |r_{22}|$, $m_1 = |r_{11}| + |r_{21}|$ and $m_2 = |r_{12}| + |r_{22}|$ in writing this expression. We will now compute the sum

$$
S = \sum_{r_{11},r_{12},r_{21},r_{22}} g(r_{11},r_{12},r)g(r_{21},r_{22},s)g(r_{11},r_{21},t)g(r_{12},r_{22},u) \qquad (6.23)
$$

In the sum above, the number of rows in the r_{IJ} is not restricted. Indeed, to capture the finite N constraints, it is enough to cut the number of rows of R off as we have done in (6.22). Making use of the identity ($r \vdash n$, $s \vdash m$, $t \vdash n+m$)

$$
g(r,s,t) = \frac{1}{n!m!}\sum_{\sigma_1\in S_n}\sum_{\sigma_2\in S_m} \chi_r(\sigma_1)\chi_s(\sigma_2)\chi_t(\sigma_1\circ\sigma_2) \qquad (6.24)
$$

and the formula

$$
\sum_{R\vdash n}\chi_R(\sigma)\chi_R(\tau) = \sum_{\gamma\in S_n}\delta(\gamma\sigma\gamma^{-1}\tau^{-1}) \qquad (6.25)
$$

we can write S as

$$
\begin{aligned}
S &= \sum_{n_{i1}+n_{i2}=n_i}\sum_{n_{1i}+n_{2i}=m_i}\sum_{\psi_1\in S_{n_{11}}}\sum_{\psi_2\in S_{n_{21}}}\sum_{\tau_1\in S_{n_{12}}}\sum_{\tau_2\in S_{n_{22}}}\frac{1}{n_{11}!n_{12}!n_{21}!n_{22}!} \\
&\quad\times\chi_r(\psi_1\circ\tau_1)\chi_s(\psi_2\circ\tau_2)\chi_t(\psi_1\circ\psi_2)\chi_u(\tau_1\circ\tau_2) \\
&= \sum_{\sigma_1\in S_{n_1}}\sum_{\sigma_2\in S_{n_2}}\sum_{\rho_1\in S_{m_1}}\sum_{\rho_2\in S_{m_2}}\sum_{\gamma\in S_{n_1+n_2}}\frac{1}{n_1!n_2!m_1!m_2!} \\
&\quad\times\delta(\sigma_1\circ\sigma_2(\rho_1\circ\rho_2)^{-1})\chi_r(\sigma_1)\chi_s(\sigma_2)\chi_t(\rho_1)\chi_u(\rho_2) \\
&= \sum_{\sigma_1\in S_{n_1}}\sum_{\sigma_2\in S_{n_2}}\sum_{\rho_1\in S_{m_1}}\sum_{\rho_2\in S_{m_2}}\sum_{S\vdash n_1+n_2}\frac{1}{n_1!n_2!m_1!m_2!} \\
&\quad\times\chi_S(\sigma_1\circ\sigma_2)\chi_S(\rho_1\circ\rho_2)\chi_r(\sigma_1)\chi_s(\sigma_2)\chi_t(\rho_1)\chi_u(\rho_2) \\
&= \sum_{S\vdash n_1+n_2} g(r,s,S)g(t,u,S) \qquad (6.26)
\end{aligned}
$$

Plugging this back into (6.22) we find

$$
\begin{aligned}
Z_r &= \sum_{r,s,t,u}\sum_{R,l(R)\le N_1}\sum_S g(r,s,S)g(t,u,S)g(r,s,R)g(t,u,R)t_{a_1}^{n_1}t_{a_2}^{n_2}t_{b_1}^{m_1}t_{b_2}^{m_2} \\
&= Z_g \qquad (6.27)
\end{aligned}
$$

proving the equality. See Appendix C for a non-trivial example demonstrating this equality.

6.3 Correlators

In this section we will compute correlation functions of restricted Schur polynomials. There are two things this will teach us. First, we can confirm that the correct cut off on the number of rows of our Young diagram labels is the smallest of N_1 and N_2. Second, we want to point out that operators from different n_{IJ} sectors are not orthogonal, which corrects a statement in [22].

The operators we study were given in[22]

$$O_{R,\{r\}\alpha\beta} = \frac{1}{\prod_{IJ} n_{IJ}!} \sum_{\sigma\in S_{n_1+n_2}} \mathrm{Tr}_{\{r\}\alpha\beta}\left(\Gamma_R(\sigma)\right)\mathrm{Tr}(\sigma(\phi^{11})^{\otimes n_{11}}(\phi^{12})^{\otimes n_{12}}(\phi^{21})^{\otimes n_{21}}(\phi^{22})^{\otimes n_{22}}) \tag{6.28}$$

The irrep R will in general be a reducible representation of the $S_{n_{11}} \times S_{n_{12}} \times S_{n_{21}} \times S_{n_{22}}$ subgroup of $S_{n_1+n_2}$. One of the $S_{n_{11}} \times S_{n_{12}} \times S_{n_{21}} \times S_{n_{22}}$ irreps that R subduces is $\{r\}$. $\{r\}$ may be subduced more than once from R. α and β label these copies. In the above formula, $\mathrm{Tr}_{\{r\}}$ is an instruction to trace only over the $\{r\}$ subspace of the carrier space of R. More precisely, we trace the row label over the α copy of $\{r\}$ and the column label over the β copy of $\{r\}$. For simplicity we will set $n_2 = 0$. The two point function

$$\langle O_{R,\{r\}\alpha\beta} O^\dagger_{S,\{s\}\gamma\delta}\rangle = \delta_{RS}\delta_{\{r\},\{s\}}\delta_{\alpha\gamma}\delta_{\beta\delta}\frac{\mathrm{hooks}_R f_R(N_1) f_R(N_2)}{\mathrm{hooks}_{r_{11}}\mathrm{hooks}_{r_{12}}} \tag{6.29}$$

follows immediately after using the results of [22]. When the right hand side of this correlator vanishes, the operator itself vanishes. Thus, by determining where the right hand side of this correlation function vanishes, we learn how the rows of the Young diagram labels should be restricted to obtain non-zero operators. Towards this end, recall that $f_R(N)$ is a product of the factors of the Young diagram, one for each box, where the box in row i and column j has factor $N - i + j$. Consequently $f_R(N)$ vanishes whenever R has more than N rows. Studying (6.29) we see that R can have no more than N_- rows where N_- is the smallest of N_1 and N_2. This is precisely the conclusion we reached in the chapter introduction. By studying two point functions, one can in general conclude that for gauge group $U(N_1) \times U(N_2) \times \cdots \times U(N_p)$, all Young diagram labels must have no more than N_- rows, where N_- is the smallest of $N_1, N_2, ..., N_p$[75].

To consider the case of general n_1, n_2, m_1, m_2, it proves convenient to use the operators

$$\begin{aligned} O_{R,\{r\}\alpha\beta} &= \mathrm{Tr}(P_{R,\{r\}\alpha\beta}A^{\otimes n}\,\tau\, B^{\dagger\otimes n}) \\ &= \frac{1}{n_{11}!n_{22}!n_{12}!n_{21}!}\sum_{\sigma\in S_n}\mathrm{Tr}_{\{r\}}\left(\Gamma_R(\sigma)\right)\prod_{i=1}^{n_1}(A_1)^{a_i}_{\alpha_i}\prod_{j=1+n_1}^{n}(A_2)^{a_j}_{\alpha_j}(\tau)^{\alpha_1\cdots\alpha_n}_{\beta_1\cdots\beta_n}\times \\ &\times\prod_{i=1}^{n_{11}}(B_1^\dagger)^{\beta_i}_{a_{\sigma(i)}}\prod_{i=1+n_{11}}^{n_1}(B_2^\dagger)^{\beta_i}_{a_{\sigma(i)}}\prod_{i=1+n_1}^{n_1+n_{21}}(B_1^\dagger)^{\beta_i}_{a_{\sigma(i)}}\prod_{i=1+n_1+n_{21}}^{n}(B_2^\dagger)^{\beta_i}_{a_{\sigma(i)}} \end{aligned} \tag{6.30}$$

where τ is an element of the group algebra, constructed to obey

$$\mathrm{Tr}(\tau\rho^{-1}\tau\sigma^{-1}) = \delta(\rho^{-1}\sigma^{-1}) \tag{6.31}$$

The two point function is[22]

$$\langle \mathcal{O}_{R,\{r\}\alpha\beta}\mathcal{O}^{\dagger}_{S,\{s\}\gamma\delta}\rangle = n_{11}!n_{12}!n_{21}!n_{22}!\mathrm{Tr}(P_{R,\{r\}\alpha\beta}P_{S,\{s\}\gamma\delta})\,.$$

Thus the two point function in the subspace of operators with fixed n_{IJ} is diagonal. However, even after fixing n_I, m_J, we can change the n_{IJ}. Projectors corresponding to different n_{IJ} will not in general be orthogonal. The identity (6.31) also does not help. Operators from different n_{IJ} sectors are not orthogonal, which is again an indication that the restricted Schur basis for quiver gauge theories is, in general, overcomplete. Note however that the operators constructed in [23] are a complete basis and they do diagonalize the two point function.

6.4 Polynomial Structure

The key general lesson of this chapter is that at finite N, the physics of quiver gauge theories is not correctly captured by contracting fields to construct adjoints of specific gauge groups. The fact that the adjoints are constructed from more basic bifundamental fields is reflected in extra finite N relations. To correctly account for all finite N relations it seems easiest to work directly with the original bifundamental fields and hence the generalized restricted Schur polynomial basis. In section 6.2 we have proved that there are exceptions to this general lesson: in certain subsectors and in specific limits, the restricted Schur polynomials correctly account for all finite N relations and hence do provide a suitable basis. In these cases, it may be simpler to use the restricted Schur polynomials rather than the generalized restricted Schur polynomials, as we explain in this section. Finally, we show that when there is a single n_{IJ} sector the generalized restricted Schur polynomials reduce to the restricted Schur polynomials constructed in [22].

The restricted Schur polynomial (6.28) can be written as

$$\begin{aligned} O_{R,\{r\}\alpha\beta} = \frac{1}{\prod_{IJ} n_{IJ}!} \qquad & \sum_{\sigma\in S_{n_1+n_2}} \sum_a \langle R,\{s\},\alpha,a|\Gamma_R(\sigma)|R,\{s\},\beta,a\rangle \times \\ & \times \mathrm{Tr}(\sigma(\phi^{11})^{\otimes n_{11}}(\phi^{12})^{\otimes n_{12}}(\phi^{21})^{\otimes n_{21}}(\phi^{22})^{\otimes n_{22}}) \end{aligned} \tag{6.32}$$

Above we have explicitly written the restricted trace using the states $|R,\{s\},\gamma,a\rangle$. These states span a subspace of the carrier space of representation R of $S_{n_1+n_2}$. The subspace carries a representation $\{s\}$ of the subgroup $S_{n_{11}}\times S_{n_{12}}\times S_{n_{21}}\times S_{n_{22}}$. Since $\{s\}$ will in general be subduced more than once, we need the multiplicity label γ. Finally, index a indexes states in the basis that spans the subspace. The key technical challenge is then to develop a good enough working knowledge of the states $|R, r, \gamma, a\rangle$, that one can carry out computations using the restricted Schur polynomials. The group theoretic quantity

$$\sum_a \langle R, \{r\}, \alpha, a|\Gamma_R(\sigma)|R, \{r\}, \beta, a\rangle \tag{6.33}$$

is the *restricted character* introduced in [56].

Using the same notation, the generalized restricted Schur polynomials can be written as

$$\begin{aligned} O_{R,S;\{t\},\{r\};\alpha\beta\gamma\delta} &= \frac{1}{\prod_{IJ} n_{IJ}!} \sum_{\sigma,\rho\in S_{n_1+n_2}} \sum_{a,b} \langle R, \{t\}, \alpha, b|\Gamma_R(\sigma)|R, \{r\}, \beta, a\rangle \\ &\times \langle S, \{r\}, \gamma, a|\Gamma_S(\rho)|S, \{t\}, \delta, b\rangle \mathrm{Tr}\left(\sigma A_1^{\otimes n_1} A_2^{\otimes n_2} \rho (B_1^\dagger)^{\otimes m_1} (B_2^\dagger)^{\otimes m_2}\right) \end{aligned}$$

Notice that four collections of states have been introduced: $|R, \{t\}, \alpha, b\rangle$, $|R, \{r\}, \beta, a\rangle$, $|S, \{t\}, \alpha, b\rangle$ and $|S, \{r\}, \beta, a\rangle$. The label $\{r\}$ specifies an irrep of $S_{n_1} \times S_{n_2}$ and $\{t\}$ specifies an irrep of $S_{m_1} \times S_{m_2}$. The collections of states introduced provide a basis for the advertised carrier spaces, within the carrier space of R and S, which are both irreps of $S_{n_1+n_2}$. Greek labels are multiplicity labels. a labels states within the basis of $\{r\}$ and b labels states within the basis of $\{t\}$. The group theoretic quantity

$$\sum_{a,b} \langle R, \{t\}, \alpha, b|\Gamma_R(\sigma)|R, \{r\}, \beta, a\rangle \langle S, \{r\}, \gamma, a|\Gamma_S(\rho)|S, \{t\}, \delta, b\rangle \tag{6.34}$$

is the *quiver character* introduced in [23].

From a group theory point of view restricted characters seem to be simpler quantities than quiver characters. Efficient methods have been developed in [12] to work with restricted characters. These methods are extended to Quiver characters in chapter 7.

Finally, consider the situation for which (say) $m_2 = 0$ so that there is a single n_{IJ} sector. In this case we find the generalized restricted Schur polynomial reduces to the restricted Schur polynomial

$$\begin{aligned} O_{R,S;\{t\}\{S\};\alpha\delta} &= \frac{\delta_{RS}}{\prod_{IJ} n_{IJ}!} \sum_{\sigma,\rho\in S_{n_1+n_2}} \sum_{a,b} \langle S, \{t\}, \alpha, b|\Gamma_S(\sigma)|S, \{S\}, a\rangle \\ &\times \langle S, \{S\}, a|\Gamma_S(\rho)|S, \{t\}, \delta, b\rangle \mathrm{Tr}\left(\rho A_1^{\otimes n_1} A_2^{\otimes n_2} \sigma (B_1^\dagger)^{\otimes n_1+n_2}\right) \\ &= \frac{\delta_{RS}}{\prod_{IJ} n_{IJ}!} \sum_{\sigma,\rho\in S_{n_1+n_2}} \sum_{a,b} \langle S, \{t\}, \alpha, b|\Gamma_S(\sigma\rho)|S, \{t\}, \delta, b\rangle \\ &\times \mathrm{Tr}\left(\rho A_1^{\otimes n_1} A_2^{\otimes n_2} \sigma(\sigma^{-1} (B_1^\dagger)^{\otimes n_1+n_2} \sigma)\right) \\ &= \frac{\delta_{RS}}{\prod_{IJ} n_{IJ}!} \sum_{\sigma,\rho\in S_{n_1+n_2}} \sum_{a,b} \langle S, \{t\}, \alpha, b|\Gamma_S(\sigma\rho)|S, \{t\}, \delta, b\rangle \end{aligned}$$

$$
\begin{aligned}
&\times \mathrm{Tr}\left(\sigma\rho A_1^{\otimes n_1} A_2^{\otimes n_2} (B_1^\dagger)^{\otimes n_1+n_2})\right) \\
&= \frac{\delta_{RS}(n_1+n_2)!}{\prod_{IJ} n_{IJ}!} \sum_{\sigma\in S_{n_1+n_2}} \sum_{a,b} \langle S,\{t\},\alpha,b|\Gamma_S(\sigma)|S,\{t\},\delta,b\rangle \times \\
&\times \mathrm{Tr}\left(\sigma(\phi^{11})^{\otimes n_1}(\phi^{22})^{\otimes n_2})\right) \\
&= \frac{\delta_{RS}(n_1+n_2)!}{\prod_{IJ} n_{IJ}!} O_{S,\{t\},\alpha\delta}
\end{aligned} \tag{6.35}
$$

In the above computation $\{t\}$ specifies an irreducible representation of $S_{n_1} \times S_{n_2}$

Chapter 7

Heavy Operators in Superconformal Chern-Simons Theory

In this chapter we confine attention to the $SU(2)$ sector of ABJM theory and work at two loops. In this case, relying on results of [24], we are able to give a simple description, which employs restricted Schur polynomials. Concretely, [24] proved that a basis for the operators in this sector of the theory is provided by restricted Schur polynomials in the adjoints (of one of the $U(N)$ factors) constructed out of the bifundamental scalars fields. The delicate point, resolved in [24], involves demonstrating that the finite N constraints are correctly accounted for. Our polynomials employ two adjoints, called ϕ_{11} and ϕ_{12} below. The number of ϕ_{11} fields is n_{11} and the number of ϕ_{12} fields is n_{12}. As we show in section 7.1, the structure of the one loop dilatation operator for ABJM theory differs from that of $\mathcal{N} = 4$ super Yang-Mills theory. The operators we consider are labeled by Young diagrams with $O(1)$ rows or columns and a total of $O(N)$ boxes. For these operators we can employ the displaced corners approximation of [12]. This requires $n_{12} \gg n_{11}$. In this approximation, the leading terms in the dilatation operator are diagonalized using a double coset ansatz[14] and the results of spring field theory[13]. The dilatation operator reduces to a set of decoupled oscillators. There are subleading terms of size $\frac{n_{11}}{n_{12}}$ relative to the leading contribution, which represent corrections to the large N limit. These subleading terms are not diagonalized by the ansatz of [14], so that a careful treatment of these terms would indicate whether the large N but non-planar integrability is a property only of the large N limit. Our study shows that these subleading terms do not commute with the leading order, so that they are not diagonalized by the ansatz of [14]. Although this does not prove that the system is not integrable, it does suggest that the integrability we have found is only a property of the large N limit. Given similar results obtained in the planar limit of the theory[32, 33], this is not surprising.

There are a number of further works related to our study, with relevant background. In particular, [34] lays the foundation for the description of membranes in ABJM using a group theoretic perspective. See also [55, 56] for background from the $\mathcal{N} = 4$ super Yang-Mills theory which is relevant for our study.

7.1 $SU(2)$ Dilatation Operator in Adjoint Variables

We are studying an $\mathcal{N} = 6$ Chern-Simons gauge theory with $U(N) \times U(N)$ gauge group. The generalized restricted Schur polynomials, introduced and studied in [23] provide a basis for the local operators of any quiver gauge theory with gauge group built from unitary group factors. In constructing our local operators we will use scalar fields A_1, A_2 both transforming in the $(N, \bar{N})$ of $U(N) \times U(N)$, as well as $B_1^\dagger, B_2^\dagger$ which transform in the $(\bar{N}, N)$. Given these transformation properties, it is clear that the fields

$$\phi_{11}{}^a_b = A_1{}^a_\alpha B_{1b}^{\dagger\alpha}\,, \qquad \phi_{12}{}^a_b = A_1{}^a_\alpha B_{2b}^{\dagger\alpha}\,,$$

$$\phi_{21}{}^a_b = A_2{}^a_\alpha B_{1b}^{\dagger\alpha}\,, \qquad \phi_{22}{}^a_b = A_2{}^a_\alpha B_{2b}^{\dagger\alpha}\,.$$

transform in the adjoint of the first $U(N)$ and as a singlet of the second. In general, the description of the theory in terms of these adjoint fields does not correctly capture the finite N physics. Indeed, as explained in [24], the constraints on local operators at finite N arising from the fact that the adjoints are $N \times N$ matrices is a subset of the full set of constraints, arising because both A_I and $B_I^\dagger$ are $N \times N$ matrices. However, if we restrict to the so called $SU(2)$ sector in which only ϕ_{11} and ϕ_{12} are used, the finite N constraints resulting from the description employing adjoint scalars ϕ_{11} and ϕ_{12} agree with the constraints obtained from the original variables. The description employing adjoints has the advantage that the restricted Schur polynomials of [22] provides a suitable basis, and the technology to work with these operators is well developed (see for example [12]). The restricted Schur polynomials we use are

$$\chi_{R,\{r\},\alpha\beta}(\phi_{11},\phi_{12}) = \frac{1}{n_{11}!n_{12}!}\sum_{\sigma\in S_{m_1+m_2}} \mathrm{Tr}_{\{r\},\alpha\beta}(\Gamma_R(\sigma))Tr\Big(\sigma(\phi_{11})^{\otimes n_{11}}(\phi_{12})^{\otimes n_{12}}\Big) \quad (7.1)$$

where we are considering an operator constructed using n_{11} ϕ_{11} fields and n_{12} ϕ_{12} fields. $\{r\}$ denotes an irreducible representation of $S_{n_{11}} \times S_{n_{12}} \subset S_{n_{11}+n_{12}}$. It is useful to think of $\{r\}$ as a pair of Young diagrams, one with n_{11} boxes and one with n_{12} boxes. The irreducible representation $\{r\}$ may appear more than once upon restricting the representation R of $S_{n_{11}+n_{12}}$ to the $S_{n_{11}} \times S_{n_{12}}$ subgroup. The multiplicity labels α, β distinguish between these different copies. The trace $\mathrm{Tr}_{\{r\},\alpha\beta}(\Gamma_R(\sigma))$ is an instruction to trace only over the $\{r\}$ subspace within the carrier space of R. Further, row indices are traced over the α copy of $\{r\}$ while the column indices are traced over the β copy. To implement the restricted trace we introduce intertwining operators $P_{R,\{r\},\alpha\beta}$ defined so that

$$\mathrm{Tr}_R\Big(P_{R,\{r\},\alpha\beta}\Gamma_R(\sigma)\Big) = \mathrm{Tr}_{\{r\},\alpha\beta}\Big(\Gamma_R(\sigma)\Big) \tag{7.2}$$

where the trace on the LHS now runs over the full carrier space of R. Our conventions for the action of the symmetric group in the space $V^{\otimes n_{11}+n_{12}}$ on which the multilinear operators $(\phi_{11})^{\otimes m_1}(\phi_{12})^{\otimes m_2}$ act are as follows

$$(\sigma)^I_J = \delta^{i_1}_{j_{\sigma(1)}} \cdots \delta^{i_{n_{11}+n_{12}}}_{j_{\sigma(n_{11}+n_{12})}} \tag{7.3}$$

The two point function of these operators is[22]

$$\langle \chi_{R,\{r\},\alpha\beta}(\phi_{11},\phi_{12})\chi_{S,\{s\},\gamma\delta}(\phi_{11},\phi_{12})^\dagger\rangle = \delta_{RS}\delta_{r_{11}s_{11}}\delta_{r_{12}s_{12}}\delta_{\alpha\gamma}\delta_{\beta\delta}\frac{f_R^2 \mathrm{hooks}_R}{\mathrm{hooks}_{r_{11}}\mathrm{hooks}_{r_{12}}} \tag{7.4}$$

We will need this result below.

The dilatation operator, acting in this $SU(2)$ sector, is given by

$$D = -\left(\frac{4\pi}{k}\right)^2 : \mathrm{Tr}\Big[\left(B_2^\dagger A_1 B_1^\dagger - B_1^\dagger A_1 B_2^\dagger\right)\left(\frac{\partial}{\partial B_2^\dagger}\frac{\partial}{\partial A_1}\frac{\partial}{\partial B_1^\dagger} - \frac{\partial}{\partial B_1^\dagger}\frac{\partial}{\partial A_1}\frac{\partial}{\partial B_2^\dagger}\right)\Big] : \tag{7.5}$$

A straightforward application of the chain rule allows us to rewrite this in terms of adjoint fields as[1]

$$\begin{aligned}
&: \mathrm{Tr}\Big[\left(B_2^\dagger A_1 B_1^\dagger - B_1^\dagger A_1 B_2^\dagger\right)\left(\frac{\partial}{\partial B_2^\dagger}\frac{\partial}{\partial A_1}\frac{\partial}{\partial B_1^\dagger} - \frac{\partial}{\partial B_1^\dagger}\frac{\partial}{\partial A_1}\frac{\partial}{\partial B_2^\dagger}\right)\Big] : \\
&=: \mathrm{Tr}\Big[(\phi_{12}\phi_{11} - \phi_{11}\phi_{12})\left(\frac{\partial}{\partial\phi_{12}}\phi_{1j}\frac{\partial}{\partial\phi_{1j}}\frac{\partial}{\partial\phi_{11}} - \frac{\partial}{\partial\phi_{11}}\phi_{1j}\frac{\partial}{\partial\phi_{1j}}\frac{\partial}{\partial\phi_{12}}\right)\Big] : \\
&+N : \mathrm{Tr}\Big[(\phi_{12}\phi_{11} - \phi_{11}\phi_{12})\left(\frac{\partial}{\partial\phi_{12}}\frac{\partial}{\partial\phi_{11}} - \frac{\partial}{\partial\phi_{11}}\frac{\partial}{\partial\phi_{12}}\right)\Big] : \\
&+ : \mathrm{Tr}\left[(\phi_{12}\phi_{11} - \phi_{11}\phi_{12})\frac{\partial}{\partial\phi_{12}}\right]\mathrm{Tr}\left[\frac{\partial}{\partial\phi_{11}}\right] : \\
&- : \mathrm{Tr}\left[(\phi_{12}\phi_{11} - \phi_{11}\phi_{12})\frac{\partial}{\partial\phi_{11}}\right]\mathrm{Tr}\left[\frac{\partial}{\partial\phi_{12}}\right] :
\end{aligned} \tag{7.6}$$

We now turn to the problem of evaluating the action of the dilatation generator on the operators (7.1). The evaluation uses the technology developed in [10, 12]. The matrix

[1]For the ABJ theory with gauge group $U(N) \times U(M)$, the only change in this formula is that the factor of N in the third last line of (7.6) would be replaced by an M.

derivatives are straight forward to evaluate; in manipulating the resulting expressions the identity

$$\mathrm{Tr}(\rho\alpha\beta\phi^{\otimes n}) = \prod_{A=1}^{n} \phi^{l_{\beta^{-1}(A)}}_{l_{\alpha\rho(A)}}$$

is extremely useful. To express the result of the action of D as a linear combination of restricted Schur polynomials, a key ingredient is the identity

$$\mathrm{Tr}(\tau\phi_{11}^{\otimes n_{11}}\phi_{12}^{\otimes n_{12}}) = \sum_{R,\{r\},\alpha\beta} \frac{d_R n_{11}! n_{12}!}{d_{r_{11}} d_{r_{12}}\, n!} \chi_{R,\{r\},\alpha\beta}(\tau)\chi_{R,\{r\},\beta\alpha}$$

where the sum over R runs over all irreps of $S_{n_{11}+n_{12}}$ and $\{r\}$ is summed over all irreps of $S_{n_{11}} \times S_{n_{12}}$. This identity is derived in [25] in the context of $U(N)$ gauge theory and it applies without change to our description in terms of adjoints. We are interested in operators with a bare dimension of order N. We achieve this large dimension by taking n_{12} order N and n_{11} order $\sqrt{N}$. For these operator, not all terms in (7.6) have the same size at large N. The sizes of the different terms follow by noting that differentiating with respect to ϕ_{12} produces order N terms while differentiating with respect to ϕ_{11} produces order $\sqrt{N}$ terms. Consequently, in the first term of (7.6) the terms with $j = 2$ dominate; the terms with $j = 1$ are supressed by a relative factor of $\sqrt{N}$. We will study this first subleading contribution in this work. The second term in (7.6) also contributes at the leading order. The third and fourth terms in (7.6) are subleading, supressed by $\frac{1}{N}$ and will consequently not be considered further in our study. It would not be consistent to evaluate these terms without also including the $\frac{1}{N}$ correction to the leading terms. Finally, it is useful to express our result in terms of operators normalized so that

$$\langle \hat{O}_{R,\{r\},\alpha\beta}\hat{O}^{\dagger}_{S,\{s\},\gamma\delta}\rangle = f_R \delta_{RS}\delta_{r_{11}s_{11}}\delta_{r_{12}s_{12}}\delta_{\alpha\gamma}\delta_{\beta\delta} \qquad (7.7)$$

Clearly then

$$\hat{O}_{R,\{r\},\alpha\beta}(\phi_{11},\phi_{12}) = \sqrt{\frac{f_R \mathrm{hooks}_R}{\mathrm{hooks}_{r_{11}}\mathrm{hooks}_{r_{12}}}}\chi_{R,\{r\},\alpha\beta}(\phi_{11},\phi_{12}) \qquad (7.8)$$

The normalization in (7.7) has been chosen so that the leading contribution to the dilatation operator most closely resembles the result obtained in [10] for $\mathcal{N} = 4$ super Yang-Mills theory. Note that operators labeled by Young diagrams R with different shapes, are not normalized in the same way. Clearly, from (7.7) it follows that the ratio of their normalizations is given by the ratios of the factors of the boxes that do not agree between the two labels. For operators with a dimension of order N and number of rows (or columns) of order 1, this ratio is always equal to 1 plus $\frac{1}{N}$ corrections. Putting these ingredients together, we find

$$D\hat{O}_{R,\{r\},\alpha\beta} = \sum_{S,\{s\}\gamma\delta} \sqrt{\frac{f_S \mathrm{hooks}_S \mathrm{hooks}_{r_{11}}\mathrm{hooks}_{r_{12}}}{f_R \mathrm{hooks}_R \mathrm{hooks}_{s_{11}}\mathrm{hooks}_{s_{12}}}} M_{R,\{r\},\alpha\beta;S,\{s\},\gamma\delta}\hat{O}_{S,\{s\},\gamma\delta}$$

$$\equiv \sum_{S,\{s\}\gamma\delta} D_{R,\{r\},\alpha\beta\,;\,S,\{s\},\gamma\delta} \hat{O}_{S,\{s\},\gamma\delta} \tag{7.9}$$

where

$$\begin{aligned} M_{R,\{r\},\alpha\beta;S,\{s\},\gamma\delta} = -\left(\tfrac{4\pi}{k}\right)^2 \textstyle\sum_{R'} \frac{c_{RR'} d_S n_{11} n_{12}}{d_{s_{11}} d_{s_{12}} d_{R'}(n_{11}+n_{12})} \\ \times\Bigg[(n_{12}-1)\mathrm{Tr}_{R\oplus S}\Big[I_{S'R'}(1,n_{11}+2)\big[(1,n_{11}+1),P_{R,\{r\}\alpha\beta}\big]I_{R'S'}\big[(1,n_{11}+1),P_{S,\{s\}\gamma\delta}\big]\Big] \\ +(n_{11}-1)\mathrm{Tr}\Big[I_{S'R'}(1,2)\big[(1,n_{11}+1),P_{R,\{r\}\alpha\beta}\big]I_{R'S'}\big[(1,n_{11}+1),P_{S,\{s\}\gamma\delta}\big]\Big] \\ +N\mathrm{Tr}\Big[I_{S'R'}\big[(1,n_{11}+1),P_{R,\{r\}\alpha\beta}\big]I_{R'S'}\big[(1,n_{11}+1),P_{S,\{s\}\gamma\delta}\big]\Big] \\ +\mathrm{Tr}\Big[I_{S'R'}\left(P_{R,\{r\}\alpha\beta}-(1,n_{11}+1)P_{R,\{r\}\alpha\beta}(1,n_{11}+1)\right)I_{R'S'}\big[(1,n_{11}+1),P_{S,\{s\}\gamma\delta}\big]\Big]\Bigg] \end{aligned} \tag{7.10}$$

To obtain this result, the sum over the symmetric group appearing in (7.1) is evaluated using the fundamental orthogonality theorem of group representation theory. The sum that appears after the derivatives act is a sum over $S_{n_{11}+n_{12}-1} \subset S_{n_{11}+n_{12}}$, so that the sum is non-zero as long as one of the representations subduced by R upon restricting to $S_{n_{11}+n_{12}-1}$ agrees with one of the representations subduced by S. The sum then produces the maps $I_{S'R'}$ and $I_{R'S'}$ which map between subspaces of the carrier spaces of R and S. We have used cycle notation for elements of the symmetric group. To completely spell out our notation, note that each element of the symmetric group is in the representation inherited from the subspace it acts in. Thus, for example,

$$\mathrm{Tr}_{R\oplus S}\Big[I_{S'R'}(1,n_{11}+2)I_{R'S'}(1,n_{11}+1)\Big] = \mathrm{Tr}_{R\oplus S}\Big[I_{S'R'}\Gamma^R\left((1,n_{11}+2)\right)I_{R'S'}\Gamma^S\left((1,n_{11}+1)\right)\Big]$$

where $\Gamma^S(\sigma)$ is the matrix representing σ in irreducible representation S.

The formulas (7.9) and (7.10) are the key results of this section. These are exact in the sense that we have not used any simplifications of the large N limit to obtain this result. We now consider the eigenproblem of D which, as we explain in the next section, can be solved in a specific limit, after exploiting simplifications of large N. At large N the last line in (7.10) is subleading and will therefore be dropped in what follows[2].

7.2 Displaced Corners Approximation

It is perhaps useful to begin with a discussion of some of the intricacies inherent in the problem of diagonalizing (7.9). The key difficulty in constructing the restricted Schur

[2] The last line in (7.10) corresponds to the third and fourth terms in (7.6)

polynomials (7.1) is in the construction of the intertwining operators $P_{R,\{r\},\alpha\beta}$. To compute the two point function (7.4), after summing over the free field Wick contractions, we simply need to take a product of two of these intertwining operators and then compute their trace, which is a relativly simple computation. Indeed, the result depends only on the dimensions of the representations R and $\{r\}$ which appear. The expression in (7.9) involves computing commutators of the intertwining operators with symmetric group elements and then tracing over a product of these commutators. This is a much more sophisticated operation for which the explicit form of $P_{R,\{r\},\alpha\beta}$ is required. Fortunately there is a limit in which we can construct $P_{R,\{r\},\alpha\beta}$ in a straight forward way: this is the displaced corners limit of [12] (see also [11]). The idea is simply that for the vast majority of restricted Schur polynomials $\chi_{R,\{r\},\alpha\beta}(\phi_{11},\phi_{12})$ that can be written down, the distance between the last box in each row of R is order N. Here by the distance between boxes a and b we mean the smallest number of boxes that one needs to pass through when moving, in the Young diagram, from box a to box b. When the distance between the last box in the different rows of R is order N, the action of the symmetric group simplifies dramatically, which greatly simplifies the construction of $P_{R,\{r\},\alpha\beta}$. To guarantee this simplification it is necessary to assume in addition that $n_{12} \gg n_{11}$; for further discussion and all the details see [12]. In this dissertation we accomplish $n_{12} \gg n_{11}$ by scaling n_{12} as N and n_{11} as $\sqrt{N}$ as we take $N \to \infty$. Our results would seem to hold with n_{11} scaled as N^{α} with $\alpha < 1$, but due to the formidable technical computations needed, we have not managed to explore this important point in detail. For a Young diagram R with p rows, the maps $I_{S'R'}$ and $I_{R'S'}$ can be identified with elements of $u(p)$. The action of the symmetric group elements appearing in (7.9), on these maps, is easy to evaluate. The intertwining operators themselves take a factorized form

$$P_{R,\{r\},\alpha\beta} = p_{r_{11}\alpha\beta}\mathbf{1}_{r_{12}} \tag{7.11}$$

where $p_{r_{11}\alpha\beta}$ projects onto $S_{n_{11}}$ irrep r_{11} and $\mathbf{1}_{r_{12}}$ projects onto $S_{n_{12}}$ irrep r_{12}. The concrete construction of these intertwining operators, together with detailed examples, is given in [12].

Since we have to take $n_{12} \gg n_{11}$ we know that the terms in (7.6) with $j = 2$ will dominate. This is indeed the case: in (7.10) the terms with coefficient $n_{12} - 1$ come from the $j = 2$ term of (7.6) while the terms with coefficient $n_{11} - 1$ come from $j = 1$. In this section we will restrict our attention to large N, which implies that we should keep only the leading order in $\frac{n_{11}}{n_{12}}$. This amounts to keeping only the terms in (7.10) that have coefficient $n_{12}-1$ or coefficient N

$$D^{(0)}\hat{O}_{R,\{r\},\alpha\beta} = \sum_{S,\{s\}\gamma\delta} \sqrt{\frac{f_S \text{hooks}_S \text{hooks}_{r_{11}} \text{hooks}_{r_{12}}}{f_R \text{hooks}_R \text{hooks}_{s_{11}} \text{hooks}_{s_{12}}}} M^{(0)}_{R,\{r\},\alpha\beta;S,\{s\},\gamma\delta} \hat{O}_{S,\{s\},\gamma\delta}$$

$$\equiv \sum_{S,\{s\}\gamma\delta} D^{(0)}_{R,\{r\},\alpha\beta\,;\,S,\{s\},\gamma\delta}\hat{O}_{S,\{s\},\gamma\delta} \tag{7.12}$$

where

$$M^{(0)}_{R,\{r\},\alpha\beta;S,\{s\},\gamma\delta} = -\left(\frac{4\pi}{k}\right)^2 \sum_{R'} \frac{c_{RR'}d_S n_{11}n_{12}}{d_{s_{11}}d_{s_{12}}d_{R'}(n_{11}+n_{12})}$$
$$\times\Bigg[(n_{12}-1)\mathrm{Tr}\Big[I_{S'R'}(1,n_{11}+2)\big[(1,n_{11}+1),P_{R,\{r\}\alpha\beta}\big]I_{R'S'}\big[(1,n_{11}+1),P_{S,\{s\}\gamma\delta}\big]\Big]$$
$$+N\mathrm{Tr}\Big[I_{S'R'}\big[(1,n_{11}+1),P_{R,\{r\}\alpha\beta}\big]I_{R'S'}\big[(1,n_{11}+1),P_{S,\{s\}\gamma\delta}\big]\Big]\Bigg] \tag{7.13}$$

We will return to the term with coefficient $n_{11}-1$ in the next section. In the displaced corners approximation, using the simplifcations just outlined, we obtain

$$D^{(0)}_{R,\{r\},\alpha\beta\,;\,S,\{s\},\gamma\delta} = -\left(\frac{4\pi}{k}\right)^2\sqrt{\frac{f_S}{f_R}}\sum_{R'}\frac{c_{RR'}}{(n_{11}-1)!}(N+r_{12i})\sqrt{\mathrm{hooks}_{r_{11}}\mathrm{hooks}_{s_{11}}}\times$$
$$\Bigg[\mathrm{Tr}(E^{(1)}_{kk}p_{r_{11}\alpha\beta}E^{(1)}_{ii}p_{s_{11}\gamma\delta})\delta_{r'_{12,i};s'_{12,k}} + \mathrm{Tr}(E^{(1)}_{ii}p_{r_{11}\alpha\beta}E^{(1)}_{kk}p_{s_{11}\gamma\delta})\delta_{r'_{12,i};s'_{12,k}}$$
$$-\Big(\mathrm{Tr}(E^{(1)}_{kk}p_{r_{11}\alpha\delta})\delta_{\beta\gamma}+\mathrm{Tr}(E^{(1)}_{kk}p_{r_{11}\gamma\beta})\delta_{\alpha\delta}\Big)\delta_{R;S}\delta_{r_{11};s_{11}}\delta_{r_{12};s_{12}}\Bigg] \tag{7.14}$$

In this last formula, r_{12i} is the length of row i of Young diagram r_{12}, R' is obtained from R by dropping the last box in row i and S' is obtained from S by dropping the last box in row k. $D^{(0)}_{R,\{r\},\alpha\beta\,;\,S,\{s\},\gamma\delta}$ is diagonalized by the double coset ansatz [14].

To motivate what follows, recall that the label $\{r\} = \{r_{11}, r_{12}\}$ and that r_{12} can be obtained by removing a total of n_{11} boxes from R. Denote the number of rows in R by p. If we remove a_1 boxes from the first row, a_2 from the second and so on up to a_p from row p, then the vector $\vec{n}_{11} = (a_1, a_2, ..., a_p)$ plays an important role: in the displaced corners approximation, operators with different $\vec{n}_{11}$ do not mix at one loop [12]. Of course, we have $a_1 + a_2 + \cdots + a_p = n_{11}$. The vector $\vec{n}_{11}$ can be used to define a group H which is a product of symmetric groups

$$H = S_{a_1} \times S_{a_2} \times \cdots \times S_{a_p} \tag{7.15}$$

According to the double coset ansatz[14], each eigenfunction of the dilatation operator is in one-to-one correspondence with an element of the double coset $H \setminus S_{n_{11}}/H$. These double coset elements can also be put into correspondence with graphs whose edges are oriented and hence with open strings states that obey the Gauss Law, providing a convincing connection with the dual D-brane plus open string excited states; for background see [30, 14]. The graph has a total of p nodes and there are n_{11} oriented edges stretching between the nodes. For this reason we will refer to these operators as Gauss graph operators and to the associated oriented graphs as Gauss graphs. The Gauss graph operators are[14]

$$O_{R,r_{12}}(\sigma) = \frac{|H|}{\sqrt{n_{11}!}} \sum_{j,k} \sum_{r_{11}\vdash n_{11}} \sum_{\mu_1,\mu_2} \sqrt{d_{r_{11}}} \Gamma^{(r_{11})}_{jk}(\sigma) B^{r_{11}\to 1_H}_{j\mu_1} B^{r_{11}\to 1_H}_{k\mu_2} \hat{O}_{R,\{r\},\mu_1\mu_2} \tag{7.16}$$

where $\sigma \in H \setminus S_{n_{11}}/H$, $\Gamma^{(r_{11})}_{jk}(\sigma)$ is the matrix representing σ in the irreducible representation r_{11} of $S_{n_{11}}$ and the branching coefficients $B^{r_{11}\to 1_H}_{j\mu_1}$ resolve the projector from irreducible representation r_{11} of $S_{n_{11}}$ to the trivial representation of H

$$\frac{1}{|H|} \sum_{\gamma\in H} \Gamma^{(r_{11})}_{jk}(\sigma) = \sum_{\mu} B^{r_{11}\to 1_H}_{j\mu} B^{r_{11}\to 1_H}_{k\mu} \tag{7.17}$$

Note that these operators are not normalized. We have computed the norm of these operators in the Appendix.

The action of the dilatation operator is most easily written in terms of parameters read from the Gauss graphs. Following [31], a useful combinatoric description of a Gauss graph is obtained by dividing each string into two halves with a label for each half. Using the orientation of the string, label both the outgoing and the ingoing string endpoints with an integer $1, 2, \cdots, n_{11}$. A permutation is then determined by how the halves are joined and conversely, given a permutation, we can reconstruct the graph. A graph is not associated to a unique permutation because the strings leaving the i'th node are indistinguishable, and the strings arriving at the i'th node are indistinguishable. As a result, graphs are in one-to-one correspondence with elements of the double coset $H \setminus S_{n_{11}}/H$. Divide the integers $1, 2, \cdots, n_{11}$ into p sets, $\mathcal{S}_i$ $i = 1, 2, \cdots, p$ such that the symmetric group that is the ith factor in H permutes the elements of $\mathcal{S}_i$. In the graph corresponding to σ, the number of oriented edges stretching from node i to node j is

$$n^+_{ij}(\sigma) = \sum_{k\in\mathcal{S}_i} \sum_{l\in\mathcal{S}_j} \delta(\sigma(k), l) \tag{7.18}$$

The number of strings stretching in the opposite direction, between the same two nodes, is

$$n^-_{ij}(\sigma) = \sum_{k\in\mathcal{S}_i} \sum_{l\in\mathcal{S}_j} \delta(\sigma(l), k) \tag{7.19}$$

The total number of strings stretching between the two nodes is $n_{ij}(\sigma) = n^+_{ij}(\sigma) + n^-_{ij}(\sigma)$.

The action of the dilatation operator is naturally written in terms of an operator Δ_{ij} defined as follows: Δ_{ij} is a sum of three terms

$$\Delta_{ij} = \Delta^+_{ij} + \Delta^0_{ij} + \Delta^-_{ij} \tag{7.20}$$

To define the action of each of the above terms, we need to introduce two new Young diagrams, $(r_{12})^{\pm}_{ij}$: $(r_{12})^+_{ij}$ is the Young diagram obtained from r_{12} by removing the last box from row j and adding it to the end of row i, while $(r_{12})^-_{ij}$ is the Young diagram obtained from r_{12} by removing the last box from row i and adding to the end of row j. $R^{\pm}_{ij}$ are defined in the same way. The actions we need to define are

$$\begin{aligned}\Delta^0_{ij} O_{R,r_{12}}(\sigma) &= -(2N + r_{12i} + r_{12j}) O_{R,r_{12}}(\sigma) \\ \Delta^+_{ij} O_{R,r_{12}}(\sigma) &= \sqrt{(N + r_{12i} + r_{12j})} O_{R^+_{ij},(r_{12})^+_{ij}}(\sigma) \\ \Delta^-_{ij} O_{R,r_{12}}(\sigma) &= \sqrt{(N + r_{12i} + r_{12j})} O_{R^-_{ij},(r_{12})^-_{ij}}(\sigma)\end{aligned} \tag{7.21}$$

Recall that r_{12k} is the number of boxes in row k of Young diagram r_{12}. A computation very similar to that of [14] now shows

$$D^{(0)} O_{R,r_{12}}(\sigma_1) = -\left(\frac{4\pi}{k}\right)^2 \sum_{\gamma_1,\gamma_2 \in H} \delta(\gamma\sigma_1\gamma^{-1}\sigma_2^{-1}) \sum_{i<j} (N + r_{12,i}) n_{ij}(\sigma_1) \Delta_{ij} O_{R,r_{12}}(\sigma_2) \tag{7.22}$$

In the large N limit we can introduce continuous variables x_i defined by

$$x_i = \frac{r_{12,i} - r_{12,p}}{\sqrt{N + r_{12,p}}} \tag{7.23}$$

In terms of this continuous variable, the leading contribution to the action of the dilatation operator (7.22) becomes

$$\begin{aligned}D^{(0)} O_{R,r_{12}}(\sigma_1) &= -\left(\frac{4\pi}{k}\right)^2 \sum_{\gamma_1,\gamma_2 \in H} \delta(\gamma\sigma_1\gamma^{-1}\sigma_2^{-1}) \\ &\times \sum_{i<j} (N + r_{12,i}) n_{ij}(\sigma_1) \left(\left(\frac{d}{dx_i} - \frac{d}{dx_j} \right)^2 - \frac{(x_i - x_j)^2}{4} \right) O_{R,r_{12}}(\sigma_2)\end{aligned} \tag{7.24}$$

After diagonalizing $n_{ij}(\sigma)$ this is a sum of decoupled oscillators, which is an integrable system.

7.3 Subleading term

In this section we will consider the subleading correction contained in

$$\begin{aligned}D^{(1)} \hat{O}_{R,\{r\},\alpha\beta} &= \sum_{S,\{s\}\gamma\delta} \sqrt{\frac{f_S \text{hooks}_S \text{hooks}_{r_{11}} \text{hooks}_{r_{12}}}{f_R \text{hooks}_R \text{hooks}_{s_{11}} \text{hooks}_{s_{12}}}} M^{(1)}_{R,\{r\},\alpha\beta;S,\{s\},\gamma\delta} \hat{O}_{S,\{s\},\gamma\delta} \\ &\equiv \sum_{S,\{s\}\gamma\delta} D^{(1)}_{R,\{r\},\alpha\beta\,;\,S,\{s\},\gamma\delta} \hat{O}_{S,\{s\},\gamma\delta}\end{aligned} \tag{7.25}$$

where

$$M^{(0)}_{R,\{r\},\alpha\beta;S,\{s\},\gamma\delta} = -\left(\tfrac{4\pi}{k}\right)^2 \sum_{R'} \tfrac{c_{RR'} d_S n_{11} n_{12}}{d_{s_{11}} d_{s_{12}} d_{R'} (n_{11}+n_{12})}$$

$$\times (n_{11}-1)\mathrm{Tr}\Big[I_{S'R'}(1,2)\big[(1,n_{11}+1),P_{R,\{r\}\alpha\beta}\big]I_{R'S'}\big[(1,n_{11}+1),P_{S,\{s\}\gamma\delta}\big]\Big] \quad (7.26)$$

These terms correspond to the terms with $j=1$ in (7.6). Evaluating the above trace in the displaced corners approximation, we find

$$D^{(1)}_{R,\{r\},\alpha\beta\,;\,S,\{s\},\gamma\delta} = -\left(\frac{4\pi}{k}\right)^2\sqrt{\frac{f_S}{f_R}}\sum_{R'}\frac{c_{RR'}}{(n_{11}-2)!}\sqrt{\mathrm{hooks}_{r_{11}}\mathrm{hooks}_{s_{11}}}\times$$
$$\Bigg[\sqrt{\frac{r_{12b}}{r_{12k}}}\mathrm{Tr}(E^{(1)}_{kk}E^{(2)}_{bi}p_{r_{11}\alpha\beta}E^{(1)}_{ib}p_{s_{11}\gamma\delta})\delta_{r'_{12,b};s'_{12,k}}+\mathrm{Tr}(E^{(1)}_{id}E^{(2)}_{id}p_{r_{11}\alpha\beta}E^{(1)}_{kk}p_{s_{11}\gamma\delta})\delta_{r'_{12,i};s'_{12,k}}$$
$$-\Big(\mathrm{Tr}(E^{(1)}_{kb}E^{(2)}_{bk}p_{r_{11}\alpha\delta})\delta_{ik}\delta_{r_{11}s_{11}}\delta_{\beta\gamma}\delta_{R;S}+\sqrt{\frac{r_{12k}}{r_{12i}}}\mathrm{Tr}(E^{(2)}_{ki}p_{r_{11}\gamma\beta}E^{(1)}_{ik}p_{s_{11}\gamma\delta})\Big)\delta_{r_{12};s_{12}}\Bigg] \quad (7.27)$$

We have not managed to perform the sums needed to rewrite the action of $D^{(1)}$ on Gauss graph operators. It is however straight forward to study this problem numerically, for specific choices of n_{11} and p.

The numerical study we will discuss is focused on operators labeled by Young diagrams R that have a total of $p=3$ long rows, and $n_{11}=3$. The results of this example are rather typical. A total of 21 operators can be defined, so that the dilatation operator is a 21×21 dimensional matrix. Acting on this space, $D^{(0)}$ decomposes into a block diagonal matrix with a total of 10 blocks. Each block can be labeled by the vector $\vec{n}_{11}$. The possible blocks together with their dimension and allowed s labels are

$$\begin{array}{lll}
\vec{n}_{11}=(1,1,1) & d=6 & s=\yng(3)\ \yng(2,1)\ \yng(1,1,1)\\
\vec{n}_{11}=(2,1,0) & d=2 & s=\yng(3)\ \yng(2,1)\\
\vec{n}_{11}=(2,0,1) & d=2 & s=\yng(3)\ \yng(2,1)\\
\vec{n}_{11}=(0,2,1) & d=2 & s=\yng(3)\ \yng(2,1)\\
\vec{n}_{11}=(1,2,0) & d=2 & s=\yng(3)\ \yng(2,1)\\
\vec{n}_{11}=(0,1,2) & d=2 & s=\yng(3)\ \yng(2,1)\\
\vec{n}_{11}=(1,0,2) & d=2 & s=\yng(3)\ \yng(2,1)\\
\vec{n}_{11}=(3,0,0) & d=1 & s=\yng(3)\\
\vec{n}_{11}=(0,3,0) & d=1 & s=\yng(3)\\
\vec{n}_{11}=(0,0,3) & d=1 & s=\yng(3)
\end{array} \quad (7.28)$$

It is a simple exercise to write down the complete set of partially labelled Young diagrams[12] and write down the action of the symmetric group on these states. We need to explicitly consider all 3 ϕ_{11}-boxes as well as a single ϕ_{12} box when constructing the dilatation operator numerically. Within this space, the projectors $p_{r_{11}\gamma\beta}$ are 81×81 dimensional matrices. The only representation that carries a nontrivial multiplicity label is the $s=\yng(2,1)$ representation in the $\vec{n}_{11}=(1,1,1)$ subpace. The multiplicity free projectors can immediately be written down as

$$p_{r_{11}\,\vec{n}_{11}} = \frac{d_{r_{11}}}{3!}\sum_{\sigma\in S_3}\chi_{r_{11}}(\sigma)\Gamma^{\vec{n}_{11}}(\sigma) \tag{7.29}$$

with $\chi_{r_{11}}(\sigma)$ an S_3 character. The matrix $\Gamma^{\vec{n}_{11}}(\sigma)$ represent $\sigma \in S_3$, in the displaced corners approximation and inside the $\vec{n}_{11}$ subspace. To construct the projectors for the $s = \yng(2,1)$ representation in the $\vec{n}_{11} = (1,1,1)$ subpace, we need to resolve this subspace into two $U(3)$ states in the $\yng(2,1)$ representation. The two states are described by the Gelfand-Tsetlin patterns that have the same inner multiplicity. For our problem here, the two states are

$$\begin{bmatrix} 2 & & 1 & & 0 \\ & 1 & & 1 & \\ & & 1 & & \end{bmatrix} \qquad \begin{bmatrix} 2 & & 1 & & 0 \\ & 2 & & 0 & \\ & & 1 & & \end{bmatrix} \tag{7.30}$$

and are easily constructed using $U(3)$ Clebsch-Gordan coefficients. The detailed computation appears in Appendix C of [12].

We find that $D^{(1)}$ not diagonal in the Gauss graph basis and it does not commute with $D^{(0)}$. Further, it does not reduce to a block diagonal matrix and indeed, it mixes operators from different $\vec{n}_{11}$ sectors. This mixing is expected and has a natural interpretation in the gravity dual. Specifying $\vec{n}_{11}$ specifies how many oriented edges start and terminate at each node. Interpreting the nodes as giant gravitons and the oriented edges as open strings attached to the giant graviton system, $\vec{n}_{11}$ can only change as a result of open string splitting and joining. Thus, the mixing we see is a signal of open string splitting and joining. This interpretation is also natural given the fact that $D^{(1)}$ is a correction to the large N limit, so that we should indeed be seeing the first effects of string splitting and joining when this correction is included. Finally, a remarkable feature of $D^{(0)}$ is the appearance of the integers $n_{ij}(\sigma)$ when the diagonalization problem is solved. Numerically we find that the eigenvalues of $D^{(1)}$ are again integers suggesting there may be a nice combinatorial description of the problem, presumably exploiting the combinatorics of string splitting and joining.

Chapter 8

Discussion

In the $SU(2)$ sector of the ABJM theory we have managed to diagonalize the two loop dilatation operator by employing the double coset ansatz. This problem was already considered in [22] where the dilatation operator was already evaluated, but not diagonalized. One of the results we have reported, is precisely the solution of this diagonalization problem. The main progress achieved in this work follows from our rewriting of the dilatation operator, in terms of adjoint variables. This gives a useful organization of the dilatation operator and in particular, has allowed us to cleanly identify two terms that contribute at the leading order at large N and two that are subleading. With this organization in hand, the eigenproblem of the dilatation operator is a straight forward exercise that can be achieved using existing techniques. The leading terms are diagonalized by the double coset ansatz, reducing the problem to the diagonalization of a collection of decoupled oscillators, which is an integrable system. We find a new "conservation law": the dilation operator does not mix operators with different $\vec{n}_{11}$ quantum number. The resulting spectrum of anomalous dimensions differs from the corresponding spectrum in $\mathcal{N} = 4$ super Yang-Mills theory in an important quantitative way. In the $\mathcal{N} = 4$ super Yang-Mills theory, the frequencies of the decoupled oscillators are set by the eigenvalues of the matrix $n_{ij}(\sigma)$ which can be read straight from the permutation labeling the Gauss graph. From (7.24) we see that for ABJM the frequencies of the decoupled oscillators are set by the eigenvalues of $(1 + \frac{r_{12,i}}{N})n_{ij}(\sigma)$. Thus, the frequencies depend both on the matrix $n_{ij}(\sigma)$, determined by the Gauss graph, and on $r_{12,i}$ which are the row lengths of the Young diagram r_{12}. Each row of r_{12} corresponds to a giant graviton. The number of boxes in the i^{th} row of r_{12} determines an $\mathcal{R}$- charge which corresponds to the angular momentum of the giant graviton. Since the giant expands to a definite size by balancing a Lorentz type force (trying to expand the giant) with tension (trying to shrink the giant), the angular momentum of the giant sets the size of the giant. Consequently, our result implies that the excitation spectrum of the giant graviton picks up a dependence on the size of the giant graviton. The fact that the spectrum of the anomalous dimensions in $\mathcal{N} = 4$ super Yang-Mills theory is independent of the parameters of the Young diagram associated to the giant graviton system, matches the fact that the spectrum of small fluctuations around the giant is independent of the size of the giant[35]. This independence of the size of the giant is understood as follows[30]: as the radius of the giant increases, there is an increase in the energy of fluctuations due to blue-shifting, as well as a decrease in the energy of the states because the fluctuations now move on a bigger sphere. These two effects precisely cancel producing a size independent spectrum. For the ABJM case, our results predict that although these two effects still operate, they do not precisely cancel so that the spectrum does pick up a dependence on the size of the giant. This is consistent with the small fluctuation spectrum around a giant graviton performed in [36]. By perturbing around the near-maximal giant and the "small" giant these authors find a spectrum that is size-dependent.

In this thesis we have also given a simple formula for the normalization of the Gauss Graph operators. This will be a useful technical input when computing the effects of Gauss Graph operator mixing, at subleading orders in a large N expansion.

Finally, we have also evaluated the largest of the subleading (in $\frac{1}{N}$) terms. Although we have not managed an analytic result, a numerical study has lead to some interesting conclusions. The subleading correction does not commute with the leading order dilatation operator. Further, it allows mixing between operators with different $\vec{n}_{11}$ quantum numbers, so that it spoils the conservation law that was present at large N. This is naturally interpreted as a consequence of open string splitting and joining. The discusion of [32, 33] suggests that the failure of this conservation law may be an indication that integrability does not persist beyond the large N limit. A numerical diagonalization of this term shows that it has integer eigenvalues, suggesting that there may be a nice combinatorial description waiting to be developed.

Appendix A

Unitarity of Group Representations

All representations of a finite group are unitary equivalent with respect to the inner product (see page 92 of [2]):

$$\{\vec{u}, \vec{v}\} = \frac{1}{[\mathcal{G}]} \sum_{g \in \mathcal{G}} (T(g)\vec{u}, T(g)\vec{v})$$

A representation is unitary with respect to this inner product if $\{T(h)\vec{u}, T(h)\vec{v}\} = \{\vec{u}, \vec{v}\}$. The following proof demonstrates that this choice of inner product ensures that all representations of a finite group are unitary:

$$\begin{aligned}
\{T(h)\vec{u}, T(h)\vec{v}\} &= \frac{1}{[\mathcal{G}]} \sum_{g \in \mathcal{G}} (T(g)T(h)\vec{u}, T(g)T(h)\vec{v}) \\
&= \frac{1}{[\mathcal{G}]} \sum_{g \in \mathcal{G}} (T(gh)\vec{u}, T(gh)\vec{v}) \\
&= \frac{1}{[\mathcal{G}]} \sum_{g' \in \mathcal{G}} \big(T(g')\vec{u}, T(g')\vec{v}\big) \\
&= \{\vec{u}, \vec{v}\}
\end{aligned}$$

where $h \in \mathcal{G}$ and we have used $h \cdot g = g'$. Using a basis orthonormal with respect to this inner product ensures a unitary matrix representation.

Appendix B

Restricted Schur polynomials for $n_1 = 3$, $n_2 = 1$, $m_1 = m_2 = 2$

The construction of restricted Schur polynomials has been described in full generality in [44]. In this Appendix we will simply list the possible operators that can be defined. This is all that is needed to follow the counting arguments of section 6.1. The notation followed is to list $\chi_{R,(r_{11},r_{12},r_{21},r_{22})\alpha\beta}$ with α and β multiplicity labels. When only a single copy of representations appear there is no need for a multiplicity index and it is simply omitted.

B.1 Case I

$$\chi_{\yng(4),(\yng(2),\yng(1),\cdot,\yng(1))} \qquad \text{One operator} \tag{B.1}$$

$$\chi_{\yng(3,1),(\yng(1,1),\yng(1),\cdot,\yng(1))} \qquad \text{One operator} \tag{B.2}$$

$$\chi_{\yng(3,1),(\yng(2),\yng(1),\cdot,\yng(1))\alpha\beta} \qquad \alpha,\beta = 1,2 \qquad \text{Four operators} \tag{B.3}$$

$$\chi_{\yng(2,2),(\yng(2),\yng(1),\cdot,\yng(1))} \qquad \text{One operator} \tag{B.4}$$

$$\chi_{\yng(2,2),(\yng(2),\yng(1,1),\cdot,\yng(1))} \qquad \text{One operator} \tag{B.5}$$

$$\chi_{\yng(2,1,1),(\yng(2),\yng(1),\cdot,\yng(1))} \qquad \text{One operator} \tag{B.6}$$

$$\chi_{\ydiagram{2,1,1},(\ydiagram{1,1},\ydiagram{1},\cdot,\ydiagram{1})_{\alpha\beta}} \qquad \alpha,\beta=1,2 \qquad \text{Four operators} \tag{B.7}$$

$$\chi_{\ydiagram{1,1,1,1},(\ydiagram{1,1},\ydiagram{1},\cdot,\ydiagram{1})} \qquad \text{One operator} \tag{B.8}$$

B.2 Case II

$$\chi_{\ydiagram{4},(\ydiagram{1},\ydiagram{2},\ydiagram{1},\cdot)} \qquad \text{One operator} \tag{B.9}$$

$$\chi_{\ydiagram{3,1},(\ydiagram{1},\ydiagram{1,1},\ydiagram{1},\cdot)} \qquad \text{One operator} \tag{B.10}$$

$$\chi_{\ydiagram{3,1},(\ydiagram{1},\ydiagram{2},\ydiagram{1},\cdot)_{\alpha\beta}} \qquad \alpha,\beta=1,2 \qquad \text{Four operators} \tag{B.11}$$

$$\chi_{\ydiagram{2,2},(\ydiagram{1},\ydiagram{2},\ydiagram{1},\cdot)} \qquad \text{One operator} \tag{B.12}$$

$$\chi_{\ydiagram{2,2},(\ydiagram{1},\ydiagram{2},\ydiagram{1,1},\cdot)} \qquad \text{One operator} \tag{B.13}$$

$$\chi_{\ydiagram{2,1,1},(\ydiagram{1},\ydiagram{2},\ydiagram{1},\cdot)} \qquad \text{One operator} \tag{B.14}$$

$$\chi_{\ydiagram{2,1,1},(\ydiagram{1},\ydiagram{1,1},\ydiagram{1},\cdot)_{\alpha\beta}} \qquad \alpha,\beta=1,2 \qquad \text{Four operators} \tag{B.15}$$

$$\chi_{\ydiagram{1,1,1,1},(\ydiagram{1},\ydiagram{1,1},\ydiagram{1},\cdot)} \qquad \text{One operator} \tag{B.16}$$

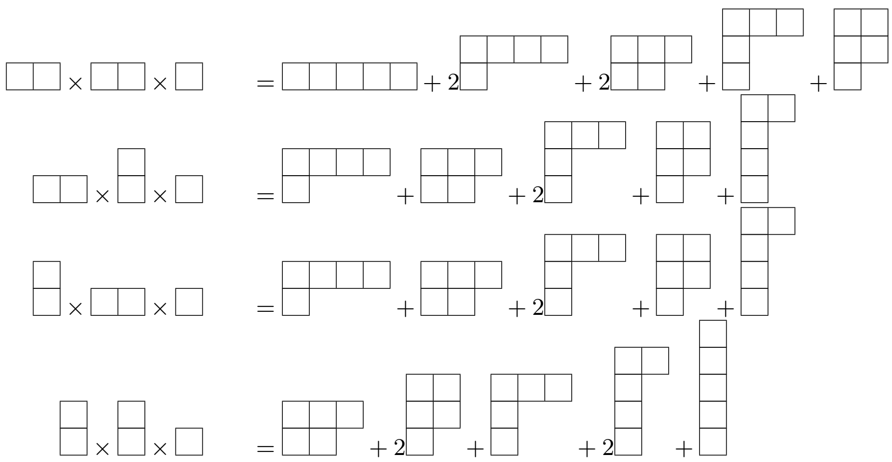
Appendix C

Counting when finite N constraints match

For the counting in this Appendix, we take $n_1 = 1$, $n_2 = 4$, $m_1 = 3$, $m_2 = 2$, $N_1 = \infty$ and $N_2 = 2$. Thus, all restricted Schur polynomial labels have at most two rows. For the generalized restricted Schur polynomials, one of the Young diagrams is unrestricted and one has at most two rows - see equation (6.8). In this example there are two $\{n_{IJ}\}$ sectors of operators:

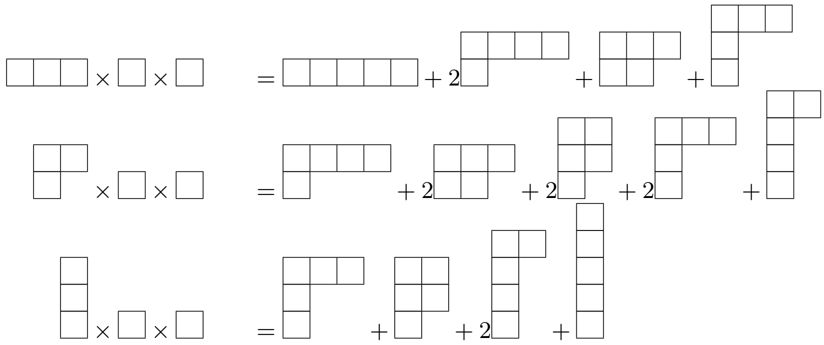
1. $\mathrm{tr}\left(\sigma\phi^{11} \otimes (\phi^{21})^{\otimes 2} \otimes (\phi^{22})^{\otimes 2}\right)$
2. $\mathrm{tr}\left(\sigma\phi^{12} \otimes (\phi^{21})^{\otimes 3} \otimes \phi^{22}\right)$

To count the restricted Schur polynomials in sector 1 we will use the Littlewood-Richardson numbers appearing in the following products



(C.1)

To count the restricted Schur polynomials in sector 2 we will use the Littlewood-Richardson numbers appearing in the following products

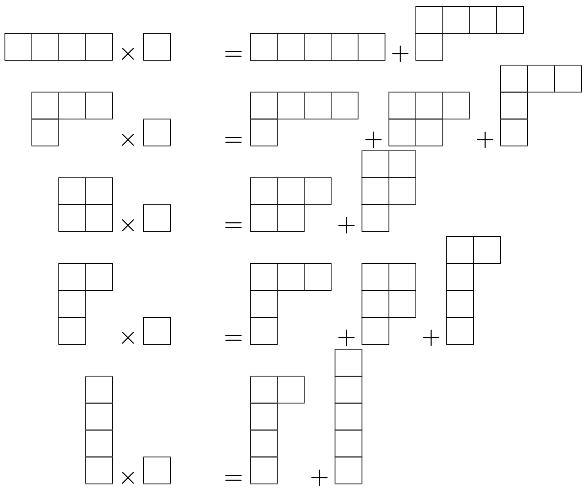


(C.2)

Restricting to Young diagrams with no more than two rows, we find

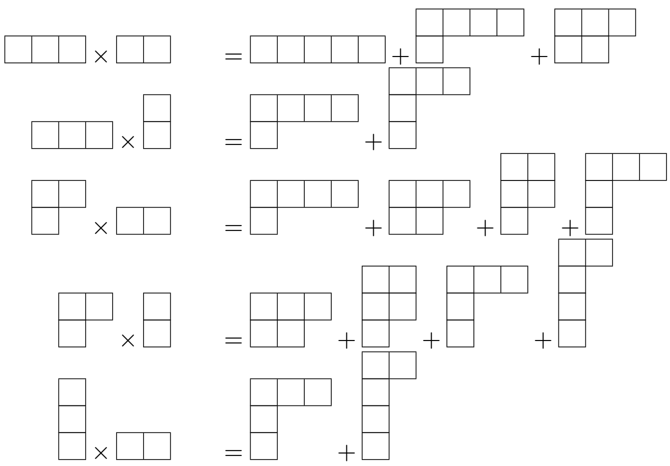
$$\begin{aligned} \mathcal{N}_{l(R)\leq 2} &= \mathcal{N}_1 + \mathcal{N}_2 \\ &= 14 + 11 \\ &= 25 \end{aligned} \tag{C.3}$$

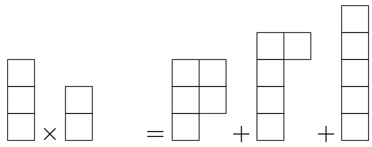
The following products appear when counting the number of generalised restricted Schur Polynomials. For $r_1 \vdash 1$ and $r_2 \vdash 4$



(C.4)

For $s_1 \vdash 3$ and $s_2 \vdash 2$





(C.5)

Using these products of Young diagrams, the number of generalised restricted Schur polynomials after restricting $l(R) \leq 2$ and leaving S unrestricted, is $\mathcal{N} = 25$ matching (C.3).

Appendix D

Normalization of the Gauss Graph Operators

The two point function of Gauss Graph Operators is

$$\langle O_{R,r}(\sigma)^\dagger O_{R,r}(\sigma)\rangle = \sum_{\gamma_1,\gamma_2\in H} \delta(\sigma^{-1}\gamma_1\sigma\gamma_2^{-1}) \tag{D.1}$$

The right hand side of the above equation is simply counting the number of solutions $\gamma_1, \gamma_2 \in H$ to

$$\sigma = \gamma_1\sigma\gamma_2^{-1} \tag{D.2}$$

Using γ_1 and γ_2 we are able to swap the endpoints of the open strings. If we swap the labels of strings that have the same start and endpoints, we leave σ unchanged and hence have a solution to (D.2). In this way, for n strings stretching from the same start point to the same endpoint, we will pick up a factor of $n!$. Denote the number of oriented line segments stretching from node i to node j by n_{ij} and the number of segments stretching from node i back to node i by n_{ii}. We have

$$\langle O_{R,r}(\sigma)^\dagger O_{R,r}(\sigma)\rangle = \prod_{i=1}^{p} n_{ii}! \prod_{k,l=1,l\neq k}^{p} n_{kl}! \tag{D.3}$$

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