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Topological classification of critical points for hairy black holes in Lovelock gravity

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Abstract In various fields of mathematical research, the Brouwer degree is a potent tool for topological analysis. By using the Brouwer degree defined in one-dimensional space, we interpret the equation of state for temperature in black hole thermodynamics, $T = T(V, x_i)$, as a spinodal curve, with its derivative defining a new function f. The sign of the slope of f indicates the topological charge of the black hole's critical points, and the total topological charge can be deduced from the asymptotic behavior of the function f. We analyze a spherical hairy black hole within the framework of Lovelock gravity, paying particular attention to the topological structure of black hole thermodynamics under Gauss–Bonnet gravity. Here, the sign of the scalar hair parameter influences the topological classification of uncharged black holes. When exploring the thermodynamic topological properties of hairy black holes under cubic Lovelock gravity, we find that the spherical hairy black hole reproduces the thermodynamic topological classification results seen under Gauss–Bonnet gravity.

1 Introduction

Topology, a powerful mathematical tool for studying physical systems, has been widely applied across various fields of theoretical physics, including particle physics and condensed matter physics. In gravitational systems, topology has been employed to examine the light rings (LRs) of ultracompact objects (UCOs) [1], revealing that the locations of LRs correspond to the zeros of a specific vector. It was found that these objects must have at least two LRs, with one being stable. This method was later applied to black holes in 1+3 dimensional asymptotically flat spacetimes [2], it was discovered that, for each sense of rotation, there is at least one standard LR outside the horizon. Building on this foundation, reference [3] utilizes topological methods in progressive AdS and dS spacetimes, establishing that these spacetimes must contain at least one standard photon sphere. If the topological charge vanishes, the spacetime transitions into a naked singularity.

Topological methods have been widely applied in various gravitational systems, and they are naturally used to study black hole thermodynamics. Motivated by this, Wei et al. [4,5] proposed the generalized off-shell free energy method and the temperature method to explore the global and local properties of black hole thermodynamics. The former approach utilizes Duan's ϕ-mapping topological current theory [6] and the residue theorem [7] to study various black holes, yielding intriguing results [8–27]. This approach has been extended to the study of non-Boltzmann statistics [28,29]. For the latter approach, in addition to applying topological current theory to classify the critical points of black holes [30–37], an alternative, straightforward method has been developed for conducting topological analysis of black hole thermodynamic critical points [38]. This method relies on an important concept in topology, namely the Brouwer degree.

Assume $n \geq 1$ is an integer, and let Ω be a bounded open set in $\mathbb{R}^n$, i.e., $\Omega \subset \mathbb{R}^n$. Consider a continuous mapping $f : \Omega \to \mathbb{R}^n$. Suppose $z \notin f(\partial\Omega)$, (where $\partial\Omega$ denotes the

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boundary of Ω) is a regular value of f. Then the preimage $f^{-1}(z) = \{x_1, ..., x_m\}$, consisting of points $x_n \in \Omega$ is finite. Consequently, $f(x_n) = z$. Then the Brouwer degree of the mapping was defined by [39]

$$d(f, \Omega, z) := \sum_{x \in f^{-1}(z)} \text{sgn}\left(J_f(x)\right), \tag{1}$$

the expression $J_f(x) = \det(f'(x))$ represents the Jacobian determinant of f at the point x, and sgn: $\mathbb{R} \to \{-1, 0, 1\}$ denotes the defined sign function

$$\text{sgn}(x) = \begin{cases} -1, & x < 0 \\ 0, & x = 0 \\ 1, & x > 0 \end{cases}, \tag{2}$$

this quantity is used to determine the positivity or negativity of the determinant of the Jacobian matrix $\det(f'(x))$, independent of the choice of the regular value z, and remains invariant under continuous deformation of the mapping. In the one-dimensional case, let $f : \Omega = [a, b] \to \mathbb{R}$ be a continuously differentiable function with $f(a) \neq 0$ and $f(b) \neq 0$. Let $A = f^{-1}(0)$ be the set of zeros of f. If for all $x \in A$, $f'(x) \neq 0$, i.e., 0 is a regular value of f, then A is a finite set. According to the definition of the Brouwer degree, we have

$$d(f, \Omega, 0) := \sum_{x \in A} \text{sgn}\, f'(x). \tag{3}$$

By applying the method for calculating topological charges at critical points [5], each critical point can be assigned a topological charge Q_n based on sgn $f'(x)$, such that $Q_n =$ sgn $f'(x)$. The sum of all these charges Q_t represents the Brouwer degree

$$Q_t = \sum_n Q_n. \tag{4}$$

Additionally, we can use a straightforward formula to calculate the total topological charge directly, without needing to determine the topological charge at each individual critical point. This formula is [40]

$$Q_t = \frac{1}{2}(\text{sgn}\, f(b) - \text{sgn}\, f(a)) = \sum_{x \in A} \text{sgn}\, f'(x), \tag{5}$$

this equation allows for the direct determination of the total topological charge through the analysis of the asymptotic behavior of f. It is important to note that Eq. (5) is valid only for continuously differentiable functions with non-zero boundary values [40].

When applying the Brouwer degree definition to the topological analysis of thermodynamic critical points, it is necessary to assign specific physical meaning to the function f and its zeros. The equation that determines the critical points is

$$(\partial_S T)_{P,x^i} = 0, \quad \left(\partial_{S,S} T\right)_{P,x^i} = 0, \tag{6}$$

where $(\partial_S T)_{P,x^i}$ and $\left(\partial_{S,S} T\right)_{P,x^i}$ represent the first and second derivatives with respect to S, respectively, under constant P and x^i. By eliminating the pressure from (6), we obtain a new temperature function known as the spinodal curve $T^s = T(S, x^i)$ [41]. Consequently, the Eq. (6) is transformed into

$$\left(\partial_S T^s\right)_{x^i} = 0, \quad \left(\partial_{S,S} T^s\right)_{x^i} = 0, \tag{7}$$

now let $f = (\partial_S T^s)_{x^i}$. According to this definition, it is not difficult to see that the zero of f corresponds precisely to the critical point of the thermodynamic system. We can assign topological charges to each critical point based on the characteristics of the zeros of f. The total topological charge is the sum of the topological charges of all critical points. In other words, the spinodal curve plays a crucial role in the topological analysis of critical points. To illustrate this more clearly, we explore black hole thermodynamics within a higher-order gravity model-specifically, charged hairy black holes under Lovelock gravity.

We organize the remainder of the paper as follows: In Sect. 2, we begin by reviewing the thermodynamic properties of charged hairy black holes in the context of Gauss–Bonnet gravity. We then use the Brouwer degree to calculate the topological charge of both charged and uncharged black holes. In Sect. 3, we extend the analysis from Sect. 2 to the third-order Lovelock gravity framework. Finally, Sect. 4 presents the conclusions of this paper.

2 Hairy black holes in Gauss–Bonnet gravity

We begin by briefly introducing the thermodynamic properties of general Lovelock gravity, where the action in d-dimensional spacetime is given by [42]

$$\mathcal{I} = \frac{1}{16\pi G} \int d^d x \sqrt{-g} \left(\sum_{k=0}^{k_{\max}} \mathcal{L}^{(k)} - 4\pi G F_{\mu\nu} F^{\mu\nu} \right), \tag{8}$$

where $k_{\max} = \left[\frac{d-1}{2}\right]$ such that the brackets represent the integer part of $(d-1)/2$, $F^{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$, $\mathcal{L}^{(k)}$ is the Lagrangian densities

$$\mathcal{L}^{(k)} = \frac{1}{2^k} \delta^{(k)} \left(a_k R^{(k)} + b_k \phi^{d-4k} S^{(k)} \right), \tag{9}$$

with $\delta^{(k)} = (2k)!\delta^{\mu_1}_{[\alpha_1}\delta^{\nu_1}_{\beta_1}\cdots\delta^{\mu_k}_{\alpha_k}\delta^{\nu_k}_{\beta_k]}$ is the generalized Kronecker tensor. Here the tensors $R^{(k)}$ and $S^{(k)}$ are

$$R^{(k)} = \prod_r^k R^{\alpha_r\beta_r}_{\mu_r\nu_r}, \quad S^{(k)} = \prod_r^k S^{\alpha_r\beta_r}{}_{\mu_r\nu_r}. \tag{10}$$

We focus on the charged AdS hairy black hole solution, with the metric given by

$$\begin{aligned} ds^2 &= -f dt^2 + f^{-1}dr^2 + r^2 d\Sigma^2_{(\sigma)d-2}, \\ F &= \frac{Q}{r^{d-2}} dt \wedge dr, \end{aligned} \tag{11}$$

here, $d\Sigma_{(\sigma)d-2}$ represents the line element on a $d-2$ dimensional spatial hypersurface with constant scalar curvature equal to $(d-2)(d-3)\sigma$. The values $\sigma = -1, 0, +1$ correspond to hyperbolic, planar, and spherical curvatures, respectively. This leads to the following polynomial equation

$$\sum_{k=0}^{k_{\max}} \alpha_k \left(\frac{\sigma - f}{r^2}\right)^k = \frac{16\pi G M}{(d-2)\Sigma^{\sigma}_{d-2} r^{d-1}} + \frac{H}{r^d} - \frac{8\pi G}{(d-2)(d-3)} \frac{Q^2}{r^{2d-4}}, \tag{12}$$

where

$$\begin{aligned} \alpha_0 &= \frac{a_0}{(d-1)(d-2)}, \quad \alpha_1 = a_1, \\ \alpha_k &= a_k \prod_{n=3}^{2k}(d-n) \quad \text{when } k \geq 2. \end{aligned} \tag{13}$$

In Eq. (12), H represents the introduced scalar hair term

$$H = \sum_{k=0}^{k_{\max}} \frac{(d-3)!}{(d-2(k+1))!} b_k \sigma^k N^{d-2k}. \tag{14}$$

The relationship between N and the scalar field ϕ is given by $\phi = \frac{N}{r}$. To satisfy the equations of motion of the scalar field, N must meet the following constraints [42]:

$$\begin{aligned} \sum_{k=1}^{k_{\max}} k b_k \frac{(d-1)!}{(d-2k-1)!} \sigma^{k-1} N^{2-2k} &= 0, \\ \sum_{k=0}^{k_{\max}} b_k \frac{(d-1)!\left(d(d-1)+4k^2\right)}{(d-2k-1)!} \sigma^k N^{-2k} &= 0. \end{aligned} \tag{15}$$

Since there are two equations with a single unknown (N), one of the equations imposes a restriction on the permissible coupling constants, b_k. In what follows we consider $b_k = 0$ for $k \geq 3$ in an ensemble.

The mass, temperature, and electric potential of the black hole can be described in terms of the event horizon radius

$$\begin{aligned} M &= \frac{(d-2)\Sigma^{\sigma}_{d-2}}{16\pi G} \sum_{k=0}^{k_{\max}} \alpha_k \sigma^k r_+^{d-2k-1} - \frac{(d-2)\Sigma^{\sigma}_{d-2} H}{16\pi G r_+} \\ &\quad + \frac{\Sigma^{\sigma}_{d-2} Q^2}{2(d-3) r_+^{d-3}}, \\ T &= \frac{f'(r_+)}{4\pi} = \frac{1}{4\pi r_+ D(r_+)} \\ &\quad \times \left[\sum_k \sigma \alpha_k (d-2k-1) \left(\frac{\sigma}{r_+^2}\right)^{k-1} \right. \\ &\quad \left. + \frac{H}{r_+^{d-2}} - \frac{8\pi G Q^2}{(d-2) r_+^{2(d-3)}} \right], \\ \Phi &= \frac{\Sigma^{\sigma}_{d-2} Q}{(d-3) r_+^{d-3}}, \end{aligned} \tag{16}$$

where

$$D(r_+) = \sum_{k=1}^{k_{\max}} k \alpha_k \left(\sigma r_+^{-2}\right)^{k-1}. \tag{17}$$

In Lovelock gravity, black holes no longer adhere to the area law; instead, the entropy is determined using Wald's method [43]. In this case, the black hole's entropy not only includes the standard Lovelock black hole entropy but also accounts for the contribution from the scalar hair. Through precise calculations, the hairy contribution to the entropy is given by

$$S = \frac{\Sigma^{(\sigma)}_{d-2}}{4G} \left[\sum_{k=1}^{k_{\max}} \frac{(d-2) k \sigma^{k-1} \alpha_k}{d-2k} r_+^{d-2k} - \frac{d}{2\sigma(d-4)} H \right]. \tag{18}$$

the mass, entropy, temperature, charge and and other quantities of the black hole obey the extended first law

$$\delta M = T\delta S + \Phi \delta Q + \sum_k \Psi^{(k)} \delta\alpha_k + \sum_k \mathcal{K}^{(k)} \delta b_k, \tag{19}$$

here α_k and b_k are considered thermodynamic quantities, while $\Psi^{(k)}$ and $\mathcal{K}^{(k)}$ are their respective conjugate thermodynamic potentials.

Considering the negative cosmological constant as the thermodynamic pressure

$$P = -\frac{\Lambda}{8\pi} = \frac{(d-1)(d-2)}{16\pi L^2}, \; V = \frac{\Sigma^{(\sigma)}_{d-2} r_+^{d-1}}{d-1}, \tag{20}$$

here V is the volume. We first discuss the impact of scalar hair on the thermodynamic topological classification of black holes in Gauss–Bonnet gravity (i.e., second-order Lovelock gravity). The polynomial in this case is given by:

$$\alpha_0 + \frac{\sigma - f}{r^2} + \alpha_2 \left(\frac{\sigma - f}{r^2}\right)^2 - \frac{h}{r^d} - \frac{16\pi M}{(d-2)\Sigma_{d-2}^{(\sigma)} r^{d-1}} + \frac{8\pi}{(d-2)(d-3)} \frac{Q^2}{r^{2d-4}} = 0. \tag{21}$$

Introducing the dimensionless thermodynamic variables

$$r_+ = v\sqrt{\alpha_2}, \quad T = \frac{t}{\sqrt{\alpha_2}(d-2)}, \quad Q = \frac{q}{\sqrt{2}} \alpha_2^{\frac{d-3}{2}}, \quad \frac{(d-1)(d-2)\alpha_0}{16\pi} = \frac{p}{4\alpha_2}, H = \frac{4\pi h}{d-2} \alpha_2^{\frac{d-2}{2}}. \tag{22}$$

And the the dimensionless counterpart of Helmholtz free energy was found as $g = \frac{\alpha_2^{\frac{3-d}{2}}}{\Sigma_{d-2}^{(\sigma)}} G$, where $G = M - TS$ characterizes the canonical ensemble. Then the dimensionless equation of state reads

$$t = \frac{1}{4\pi v^{1+2d}\left(v^2 + 2\sigma\right)} \left[-4\pi q^2 v^8 + 4\pi v^{4+d}\left(h + p v^d\right) + (d-2) v^{2d} \sigma \left((d-3)v^2 + (d-5)\sigma\right)\right]. \tag{23}$$

2.1 Charged hairy black holes

Using the spherical horizon ($\sigma = 1$) as an example. With the dimensionless state Eq. (23) already derived, the condition for finding the critical points is now as follows

$$(\partial_v t)_{p,q,h} = 0, \quad \left(\partial_{v,v} t\right)_{p,q,h} = 0. \tag{24}$$

By eliminating the parameter p, a new temperature function can be obtained, which is the spinodal curve $t^s = t(v, q, h)$ that we are looking for

$$t^s = \frac{1}{2\pi v^{1+2d}\left(v^2 + 6\right)} \left[4\pi(2-d) q^2 v^8 + 2\pi d h v^{4+d} + 2(5-d)(2-d) v^{2d} + (3-d)(2-d) v^{2d+2}\right], \tag{25}$$

the condition (24) then becomes

$$\left(\frac{\partial t^s}{\partial v}\right)_{q,h} = 0. \tag{26}$$

When we set $f = \left(\frac{\partial t^s}{\partial v}\right)_{q,h}$, the expression for f is obtained as follows

$$\begin{aligned} f = \frac{1}{2\pi v^{2+2d}\left(v^2+6\right)^2} \Big[& 2\pi d h (1-d) v^{6+d} \\ & + 12\pi d h (3-d) v^{4+d} \\ & + 4\pi q^2 (2-d)(5-2d) v^{10} + 24\pi q^2 (2-d)(7-2d) v^8 \\ & - (2-d)(3-d) v^{4+2d} - 12(2-d) v^{2+2d} \\ & - 12(2-d)(5-d) v^{2d} \Big]. \end{aligned} \tag{27}$$

Let us first discuss the charged case. It is evident that this function is continuously differentiable. We find that for any $d \geq 5$ and $q > 0$, the following asymptotic behavior can be obtained

$$\begin{aligned} f\left(v \to 0^+\right) &\sim \frac{q^2(2-d)(7-2d)}{v^{2d-6}\left(v^2+6\right)^2} \to +\infty, \\ f(v \to +\infty) &\sim -\frac{(2-d)(3-d)v^2}{2\pi\left(v^2+6\right)^2} \to 0^-. \end{aligned} \tag{28}$$

From the above expression, it can be seen that f has a nonzero boundary. By solving the equation for the topological charge, Eq. (5), the total topological charge of a charged hairy black hole in this system can be determined as follows

$$\begin{aligned} Q_t &= \frac{1}{2}\left[\text{sgn}\, f(v \to +\infty) - \text{sgn}\, f\left(v \to 0^+\right)\right] \\ &= \frac{1}{2}(-1-1) = -1. \end{aligned} \tag{29}$$

We observe that the result mentioned above is independent of the black hole charge q, the hairy parameter h, and the dimension d. This implies that the hairy parameter does not affect the topological charge of charged black holes with spherical event horizon geometry, indicating that they belong to the same topological class.

To determine the topological charge of each critical point under various parameters, we would like to analyze the spinodal curve and the behavior of the function f. As an example, for $d = 5$, Fig. 1 illustrates the number of critical points for different charges. It shows that up to three critical points can exist within a suitable range of parameters q and h.

For larger q, we find only one critical point. Let $d = 5, q = 0.4, h = 0.1$. In Fig. 2a, the colored solid lines represent isobaric curves, and the gray dashed line represents the spinodal curve. It can be clearly seen that the critical point CP_1 is exactly the extremum point of the spinodal curve (the extremum point of the critical temperature). According to the analysis in [30,31,34], the topological charge of the conventional critical point CP_1, which serves as an annihilation

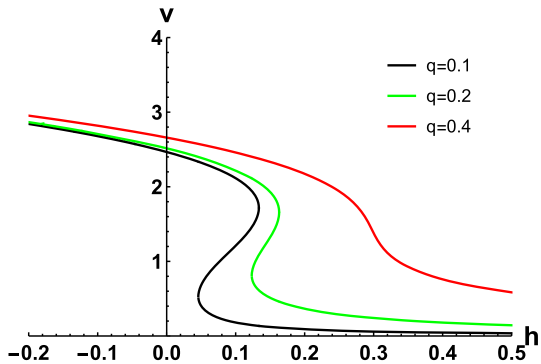
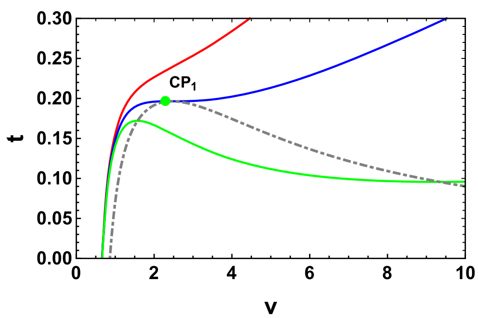
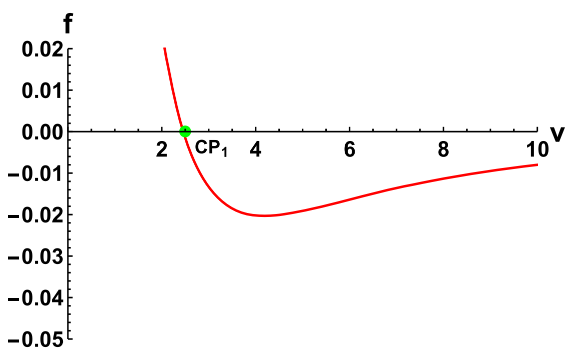


Fig. 1 The number of critical points for charged hairy black holes under Gauss–Bonnet gravity with $d = 5$



(a) The isobaric curves (colored solid lines) and the spinodal curve (gray dashed line), with the pressure of the isobaric curves increasing from bottom to top.



(b) The graph of $f(v)$ as a function of v, where the green dot represents the zero piont with negative topological charge.

Fig. 2 The topological charge with only one critical point, $Q_{CP_1} = -1$. Set $d = 5$, $q = 0.4$, $h = 0.1$

point, is -1. This corresponds to the zero point of the function f with a negative slope at CP_1, as illustrated in Fig. 2b. Therefore, we infer that the sign of the slope at the zero point of the function f represents the topological charge of the black hole critical point, i.e., $Q_{CP} = \text{sgn}\, f'(CP)$.

According to Fig. 1, if the charge is sufficiently small, there can be up to three critical points. Next, we calculate the topological charge of each critical point in this case, taking

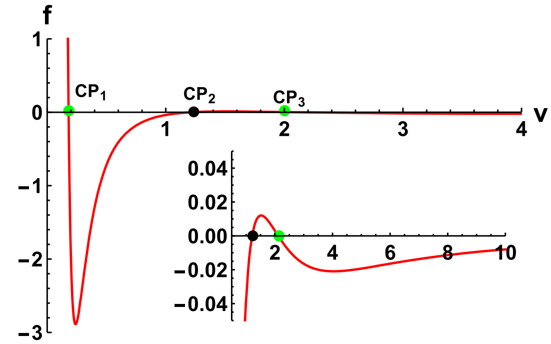


Fig. 3 The graph of the function $f(v)$ as it varies with v. The black dot represents the zero point with positive topological charge. $Q_{CP_1} = -1$, $Q_{CP_2} = 1$ and $Q_{CP_2} = -1$. Set $d = 5$, $q = h = 0.1$

$q = h = 0.1$. The behavior of the function f is shown in Fig. 3. It is evident that the topological charges of these critical points are $Q_{CP_1} = \text{sgn}\, f'(CP_1) = -1$, $Q_{CP_2} = \text{sgn}\, f'(CP_2) = 1$, $Q_{CP_3} = \text{sgn}\, f'(CP_3) = -1$. Thus, the topological charges of adjacent critical points are opposite, and the sum of all topological charges is $Q_t = Q_{CP_1} + Q_{CP_2} + Q_{CP_3} = -1$, which is the sum of the topological charges of each zero (critical point). This result is consistent with the result obtained from Eq. (29).

It is worth noting that when we set $d = 5$ and $q = 0.1$, and consider the variation of the hairy parameter h, that is, we can regard h as a "time evolution factor", as shown in Fig. 4. When $h < h_1$, the system has only one critical point, CP_3, with a topological charge of -1. As h increases and precisely reaches h_1, in addition to the original critical point CP_3, we also observe another critical point, $\hat{\text{CP}}$, whose position can be accurately calculated using Eq. (24). It is found that this point is exactly at the critical point where $f' = 0$, so the critical point $\hat{\text{CP}}$ has a topological charge of 0. With further increase of h, we find that not only does CP_3 exist, but $\hat{\text{CP}}$ also produces two new critical points, CP_2 and CP_1, where CP_2 has a topological charge of $+1$, and CP_1 has a topological charge of -1. As h continues to increase, CP_2 and CP_3 get closer, while CP_2 and CP_1 move further apart. When h increases to $h = h_2$, we observe that CP_2 and CP_3 merge into a single critical point, $\hat{\text{CP}}$, which corresponds exactly to another critical point where $f' = 0$, so the topological charge of $\hat{\text{CP}}$ is also zero. When $h > h_2$, $\hat{\text{CP}}$ also disappears, leaving only CP_1. This means that the critical points creat and annihilate in pairs does not affect the total value of the topological charge.

2.2 Un charged hairy black holes

Next we discuss the uncharged case, when $q = 0$, then the spinodal curve in (25) becomes

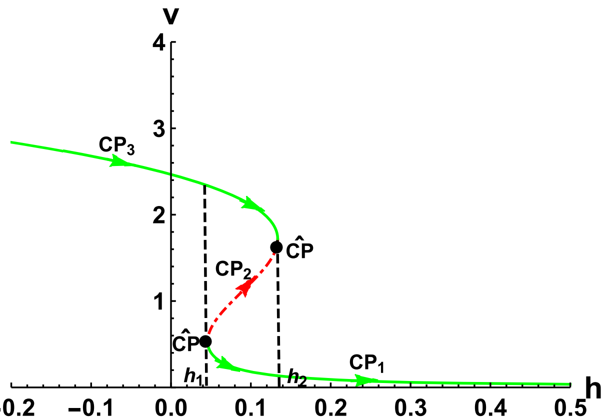
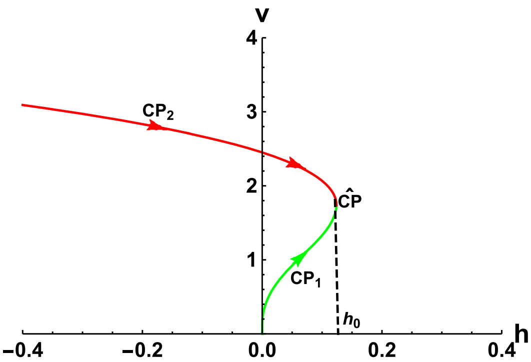
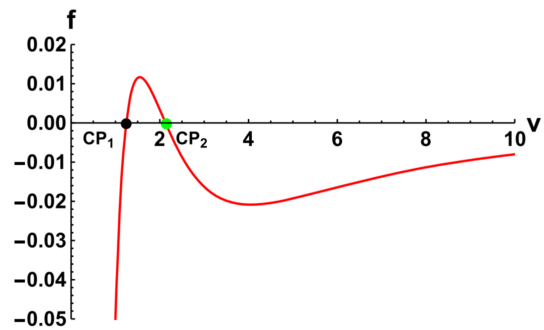


Fig. 4 The critical points creat and annihilate in pairs for charged hairy black holes in Gauss–Bonnet gravity. Assume $d = 5, q = 0.1$



(a) The critical points annihilate in pairs for uncharged hairy black holes in Gauss-Bonnet gravity.



(b) The graph of function f as a function v.

Fig. 5 The topological charges with two critical point, $Q_{CP_1} = 1$ and $Q_{CP_2} = -1$. Set $d = 5, q = 0, h = 0.09$

$$t^s = \frac{1}{2\pi v^{1+d}\left(v^2+6\right)}\Big[2\pi d h v^4 + 2(5-d)(2-d)v^d + (3-d)(2-d)v^{2+2}\Big], \tag{30}$$

the function $f = \left(\frac{\partial t^s}{\partial v}\right)_h$ becomes

$$f = \frac{1}{2\pi v^{2+d}\left(v^2+6\right)^2}\Big[2\pi d h(1-d)v^6 + 12\pi d h(3-d)v^4 - (2-d)(3-d)v^{4+d} - 12(2-d)v^{2+d} - 12(2-d)(5-d)v^d\Big]. \tag{31}$$

Similar to the analysis of the charged case, Eq. (31) is continuously differentiable, and when $d \geq 5$ and $q = 0$, there is

$$f\left(v \to 0^+\right) \sim -\frac{dh(d-3)v^2}{v^d\left(v^2+6\right)^2} \to \begin{cases} +\infty & \text{when } h < 0 \\ -\infty & \text{when } h > 0 \end{cases},$$
$$f(v \to +\infty) \sim -\frac{(2-d)(3-d)v^2}{2\pi\left(v^2+6\right)^2} \to 0^-. \tag{32}$$

We find that the asymptotic result of the function f is related to the positive and negative value of h, so the total topological number (topological charge) of the black hole in this case can be divided into two cases, i.e

$$Q_{\text{total}} = \frac{1}{2}\left[\operatorname{sgn} f(v \to +\infty) - \operatorname{sgn} f\left(v \to 0^+\right)\right] = \begin{cases} -1 & \text{when } h < 0 \\ 0 & \text{when } h > 0. \end{cases} \tag{33}$$

The above results indicate that the total topological number of the black hole in the uncharged case depends on the sign of the hairy parameter h. Taking $d = 5$ as an example, when $h < 0$, as shown in Fig. 5a, there is only one critical point CP_2, so the total topological charge is -1, which can be classified in the same category as the charged black hole. When $0 < h \ll h_0$, we find that not only does the critical point CP_2 exist, but a new critical point CP_1 also appears. At this point, the two critical points are far apart, with the topological charge of CP_1 being +1. Therefore, the total topological charge is 0, placing this case in a different topological class from the charged black hole. As h increases, CP_2 and CP_1 move infinitely closer together and merge into a single critical point $\hat{\text{CP}}$, which is the annihilation point of the critical point pair $\hat{\text{CP}}$, with a topological charge of 0. The graph of the function f for $0 < h < h_0$ is shown in Fig. 5b. When $h = 0.09$, the two critical points in Fig. 5 have opposite topological charges, so the total topological charge is $Q_t = Q_{CP_1} + Q_{CP_2} = 0$. For $h > h_0$, no critical points are found, so there is no topological charge, which can be understood as a topological charge of 0, consistent with the result in (33).

3 Hairy black holes in cubic Lovelock gravity

In this section, we consider the thermodynamic topological properties of U(1) charged hairy black holes under cubic Lovelock gravity. Let's consider the following polynomials

$$\alpha_3\left(\frac{\sigma-f}{r^2}\right)^3+\alpha_2\left(\frac{\sigma-f}{r^2}\right)^2+\left(\frac{\sigma-f}{r^2}\right)+\alpha_0 = \frac{16\pi GM}{(d-2)\Sigma_{d-2}^{(\sigma)}r^{d-1}}+\frac{H}{r^d}-\frac{8\pi G}{(d-2)(d-3)}\frac{Q^2}{r^{2d-4}}. \tag{34}$$

Dimensionless parameters are also constructed

$$r_+=v\alpha_3^{1/4},\alpha=\frac{\alpha_2}{\sqrt{\alpha_3}},\quad T=\frac{t\alpha_3^{-1/4}}{d-2},\quad H=\frac{4\pi h}{d-2}\alpha_3^{\frac{d-2}{4}}, \quad Q=\frac{q}{\sqrt{2}}\alpha_3^{\frac{d-3}{4}},m=\frac{16\pi M}{(d-2)\Sigma_{d-2}^{(\kappa)}\alpha_3^{\frac{d-3}{4}}}. \tag{35}$$

So the equation of state is

$$t=\frac{v^{-1-2d}}{4\pi\left(3+v^4+2v^2\alpha\right)}\Big[-4\pi q^2v^{10}+14v^{2d}-9dv^{2d} +d^2v^{2d}+4h\pi v^{6+d}+6v^{4+2d}-5dv^{4+2d}+d^2v^{4+2d} +4p\pi v^{6+2d}\alpha+10v^{2+2d}-7dv^{2+2d}\alpha+d^2v^{2+2d}\alpha\Big]. \tag{36}$$

After eliminating p, the spinodal curve is

$$t=\frac{1}{2\pi v^{1+2d}\left(15+v^4+6v^2\alpha\right)}\Big[4\pi(2-d)q^2v^{10} +2\pi dhv^{6+d}+2(5-d)(2-d)v^{2d+2} +3(7-d)(2-d)v^{2d}\alpha+(3-d)(2-d)v^{2d+4}\Big]. \tag{37}$$

Then the continuous differentiable function is constructed

$$f=\frac{1}{2\pi v^{2+2d}\left(15+v^4+6v^2\alpha\right)^2}\Big[30(5-d)dh\pi v^{6+d} +2dh\pi(1-d)v^{10+d}-12(-3+d)\alpha dh\pi v^{8+d} +60(-2+d)(-9+2d)\pi q^2v^{10} +4(-2+d)(-5+2d)\pi q^2v^{14} +24(-7+2d)\alpha(-2+d)\pi q^2v^{12} -45(-7+d)(-2+d)v^{2d} -(-3+d)(-2+d)v^{2d+8}+12\alpha(-2+d)v^{2d+6} -6\left(5-5d+2(-5+d)\alpha^2\right)(-2+d)v^{2d+4} +12\alpha(19-2d)(-2+d)v^{2d+2}\Big]. \tag{38}$$

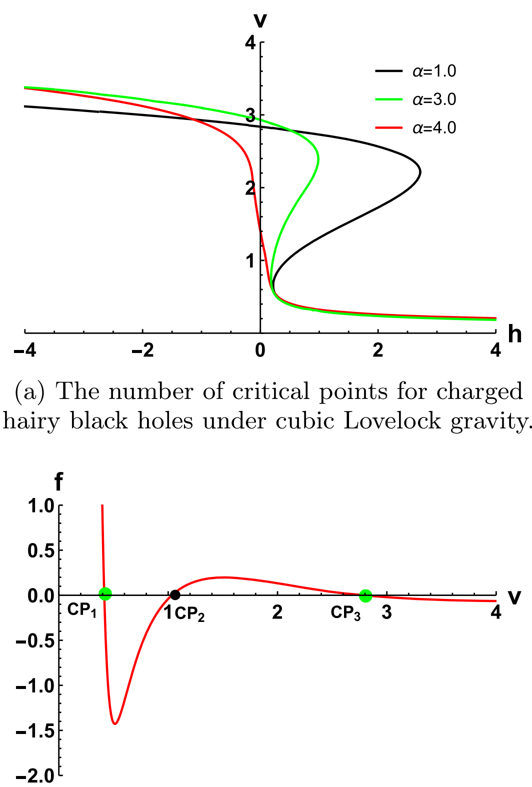
Similar to the analysis in the previous section. When we consider that the charge is non-zero, we find that for any dimension $d\geq 7$, we get the following asymptotic case

$$f\left(v\to 0^+\right)\sim\frac{(-2+d)(-9+2d)q^2}{v^{2d-8}\left(15+v^4+6v^2\alpha\right)^2}\to+\infty, \quad f(v\to+\infty)\sim-\frac{(-3+d)(-2+d)v^6}{\left(15+v^4+6v^2\alpha\right)^2}\to 0^-. \tag{39}$$

The above formula also does not depend on any black hole parameters, and the total topological charge is

$$Q_{\text{total}}=\frac{1}{2}\left[\operatorname{sgn}f(v\to+\infty)-\operatorname{sgn}f\left(v\to 0^+\right)\right] =\frac{1}{2}(-1-1)=-1. \tag{40}$$

The topological charge of a charged hairy black hole in Lovelock gravity for $d\geq 7$ is the same as that of a charged hairy black hole in Gauss–Bonnet gravity for $d\geq 5$, and they can be classified into the same category. The distribution trajectory of the critical volume with respect to the hairy param-



(a) The number of critical points for charged hairy black holes under cubic Lovelock gravity.

(b) The graph of function f as a function v.

Fig. 6 The topological charges with three critical point, $Q_{CP_1}=-1$, $Q_{CP_2}=1$ and $Q_{CP_2}=-1$. Set $d=7$, $q=0.1$, $h=0.5$ and $\alpha=1$

eter h, and the plot of the function f are shown in Fig. 6. For a certain range of h up to three critical points may exist. Similar to the case in Gauss–Bonnet gravity, if α is sufficiently small, three critical points can exist for sufficiently small charges. Therefore, the topological charge associated with the critical points is given by $Q_t = Q_{CP_1} + Q_{CP_2} + Q_{CP_3} = -1$.

Next, we discuss the case without charge. In this case, the corresponding spinodal curve is

$$
\begin{aligned}
t^s =& \frac{1}{2\pi v^{1+2d}\left(15+v^4+6v^2\alpha\right)} \\
&\Big[2\pi d h v^{6+d} + 3(7-d)(2-d)v^{2d} \\
&+2(5-d)(2-d)v^{2d+2}\alpha + (3-d)(2-d)v^{2d+4}\Big].
\end{aligned} \tag{41}
$$

So the function (38) becomes

$$
\begin{aligned}
f =& \frac{1}{2\pi v^{2+2d}\left(15+v^4+6v^2\alpha\right)^2}\Big[30(5-d)dh\pi v^{6+d} \\
&+ 2dh\pi(1-d)v^{10+d} - 12(-3+d)\alpha dh\pi v^{8+d} \\
&- 45(-7+d)(-2+d)v^{2d} - (-3+d)(-2+d)v^{2d+8} \\
&- 6\left(5-5d+2(-5+d)\alpha^2\right)(-2+d)v^{2d+4} \\
&+12\alpha(19-2d)(-2+d)v^{2d+2} + 12\alpha(-2+d)v^{2d+6}\Big].
\end{aligned} \tag{42}
$$

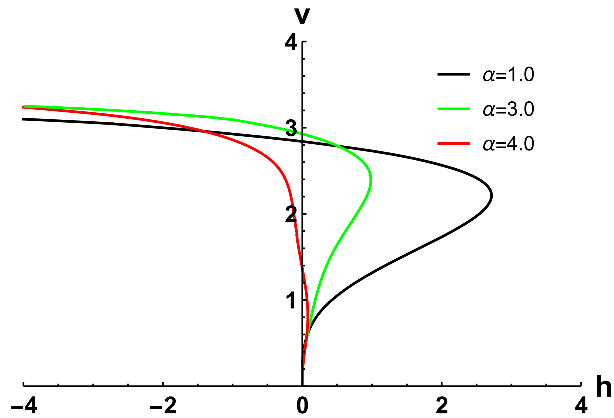
The above formula is a continuously differentiable function for any $d \geq 7$ we have

$$
\begin{aligned}
f\left(v\to 0^+\right) &\sim -\frac{(d-5)dh}{v^{d-4}\left(15+v^4+6v^2\alpha\right)^2} \\
&\to \begin{cases} +\infty & \text{when } h<0 \\ -\infty & \text{when } h>0 \end{cases}, \\
f(v\to+\infty) &\sim -\frac{(-3+d)(-2+d)v^6}{\left(15+v^4+6v^2\alpha\right)^2} \to 0^-,
\end{aligned} \tag{43}
$$

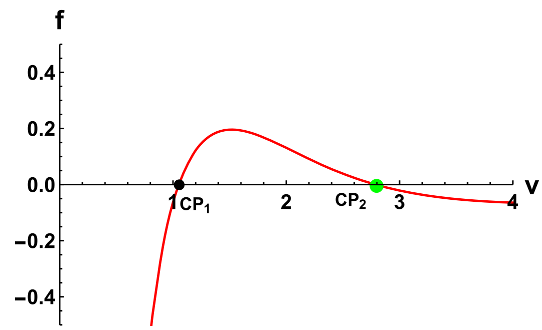
then we get the topological number of uncharged black holes

$$
\begin{aligned}
Q_{\text{total}} &= \frac{1}{2}\left[\operatorname{sgn} f(v\to+\infty) - \operatorname{sgn} f\left(v\to 0^+\right)\right] \\
&= \begin{cases} -1 & \text{when } h<0 \\ 0 & \text{when } h>0. \end{cases}
\end{aligned} \tag{44}
$$

We observe that the topological number of an uncharged black hole in cubic Lovelock gravity is the same as that of an uncharged black hole in Gauss–Bonnet gravity. In Fig. 7a, we find that an increase in the parameter α reduces the range of h, where there can be at most two critical points. Figure 7b shows that when two critical points exist, the topological charges are $Q_{CP_1} = 1$ and $Q_{CP_2} = -1$, respectively. Therefore, the total topological charge is $Q_t = Q_{CP_1} + Q_{CP_2} = 0$.



(a) The number of critical points of uncharged hairy black holes under cubic Lovelock gravity.



(b) The graph of function f as a function v.

Fig. 7 The topological charges with two critical point, $Q_{CP_1} = 1$, $Q_{CP_2} = 1$. Set $d = 7$, $q = 0$, $h = 0.5$ and $\alpha = 1$

4 Conclusion

We employ the definition of the Brouwer degree by assigning specific physical meanings to the continuous mapping function f and its zeros, and we associate a topological quantity with the thermodynamic critical points of black holes based on spinodal curve. In this approach, the topological charge can be determined without the need to precisely locate the exact value of the critical points; the zeros of the function f alone can identify these critical points. Moreover, if the slope of f at these zeros is positive, it indicates that the critical point has a topological charge of +1. Conversely, if the slope is negative, it indicates a topological charge of −1. Additionally, the total topological charge of the system can be computed based on the asymptotic behavior of the function f.

We examined the thermodynamic topological properties of hairy black holes in Gauss–Bonnet gravity, with a particular focus on the case where $\sigma = 1$. For the charged scenario, hairy black holes exhibit a topological charge of -1 across all dimensions, indicating that they belong to the same topological class. However, in the uncharged scenario, the sign of the hairy parameter h plays a crucial role in identifing the thermodynamic topological charge of black holes. When $h > 0$, the black holes have a total topological charge of -1, aligning it with the same topological class as charged hairy black holes. Conversely, when $h < 0$ the total topological charge becomes 0, placing it in aheid different topological class from charged hairy black holes. Furthermore, when analyzing the thermodynamics of hairy black holes within cubic Lovelock gravity, we discovered that cubic Lovelock spherical black holes reproduce the thermodynamic topological classification results of Gauss–Bonnet spherical black holes in dimensions $d \geq 5$. The charged cases mirror the results observed for charged Lovelock spherical black holes in dimensions $d \geq 7$, as detailed in reference [38]. The topological number of uncharged, negatively hairy black holes corresponds to that of uncharged, hairless black holes in dimensions $d > 7$ from [38].

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Code Availability Statement My manuscript has no associated code/software. [Authors' comment: Code generated during and/or analysed during the current study is available from the corresponding author on reasonable request.]

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