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Algebraic geometry and p-adic numbers for scattering amplitude ansätze

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Abstract. Scattering amplitudes in perturbative quantum field theory exhibit a rich structure of zeros, poles and branch cuts which are best understood in complexified momentum space. It has been recently shown that by leveraging this information one can significantly simplify both analytical reconstruction and final expressions for the rational coefficients of transcendental functions appearing in phenomenologically-relevant scattering amplitudes. Inspired by these observations, we present a new algorithmic approach to the reconstruction problem based on p-adic numbers and computational algebraic geometry. For the first time, we systematically identify and classify the relevant irreducible surfaces in spinor space with five-point kinematics, and thanks to p-adic numbers – analogous to finite fields, but with a richer structure to their absolute value – we stably perform numerical evaluations close to these singular surfaces, thus completely avoiding the use of floating-point numbers. Then, we use the data thus acquired to build ansätze which respect the vanishing behavior of the numerator polynomials on the irreducible surfaces. These ansätze have fewer free parameters, and therefore reduced numerical sampling requirements. We envisage future applications to novel two-loop amplitudes.

1. Introduction

Theoretical predictions in the Standard Model are being driven towards higher loop orders and multiplicities to match the increasing precision of experimental results from the LHC. To handle the increasing algebraic complexity of loop-amplitude computations, while avoiding numerical instabilities, methods based on finite-field sampling have been introduced [1,2] and implemented in publicly available software [3–5]. These methods allow one to numerically sample rational coefficients of master integrals or transcendental functions, and "reconstruct" their analytical form for subsequent phenomenological applications. Very recently, a number of cutting-edge computations for processes at two loop with five-point kinematics including an off-shell external leg have been successfully tackled in this way [6–9].

Our present work is mainly motivated by the exponential increase in the number of free parameters in the ansätze which need to be fixed from finite-field samples as the phase-space multiplicity increases. For example, the five-point massless kinematics computation of Ref. [10] required $\mathcal{O}(10^5)$ samples, while the five-point one-mass kinematics computation of Ref. [8] required $\mathcal{O}(10^6)$ evaluations. An additional motivation comes from the need to obtain stable and fast to evaluate amplitudes for phenomenological applications. To this aim, taking inspiration from previous computations relying on spinor-helicity ansätze and constraints from singular limits in complexified momentum space [11, 12], we formulate an algorithm based on p-adic numbers and algebraic geometry to obtain ansätze with reduced sampling requirements.

1.1. Problem setup

When working in dimensional regularization, the ϵ-pole dependence of the amplitude is well understood in terms of universal factors and lower-order amplitudes [13, 14]. Therefore, instead of the amplitudes $\mathcal{A}$, we can consider the so-called finite remainders $\mathcal{R}$. Given a phase space that involves n massless particles, we can write

$$\mathcal{R} = \sum_i \mathcal{C}_i(\lambda, \tilde{\lambda})\, \mathcal{F}_i(\lambda, \tilde{\lambda})\,, \tag{1}$$

where the $\mathcal{C}_i$ and $\mathcal{F}_i$ are rational and transcendental functions respectively, and $(\lambda, \tilde{\lambda})$ denotes the set of left- and right-handed Weyl spinors, which we treat as independent. As the set of possible poles $\{\mathcal{D}\}$ is well known, the rational functions can be written as

$$\mathcal{C}_i(\lambda, \tilde{\lambda}) = \frac{\mathcal{N}_i(\lambda, \tilde{\lambda})}{\prod_j \mathcal{D}_j(\lambda, \tilde{\lambda})^{q_{ij}}}\,. \tag{2}$$

The aim is to obtain refined ansätze for the numerator polynomials $\mathcal{N}_i$. As an example application we reconsider the two-loop finite remainders for the process $q\bar{q} \to 3\gamma$ [15, 16].

2. Geometry of spinor space

Our starting point is the *polynomial ring* of spinor components

$$S_n = \mathbb{F}\big[|1\rangle, [1|, \ldots, |n\rangle, [n|\big]\,, \tag{3}$$

where we are employing spinor-helicity notation, and the spinors are understood to be taken component wise. This is the infinite set of polynomials in spinor components with coefficients in the field $\mathbb{F}$, together with the operations of addition and multiplication.

It is natural to consider subsets of this infinite set of polynomials. In particular, it is interesting to consider those subsets, called *ideals*, formed by all polynomial linear combinations of a given set of starting polynomials, which are dubbed ideal generators. The most fundamental ideal in S_n is the momentum conservation ideal, dubbed J_{Λ_n} and defined as

$$J_{\Lambda_n} = \Big\langle \sum_{i=1}^{n} |i\rangle[i| \Big\rangle_{S_n}\,. \tag{4}$$

Here it is expressed in terms of a single tensor generator, which is a shorthand for four generators in components. The subscript indicates the ring over which the ideal is defined. Note that different generating sets may correspond to the same ideal. A variety of operations involving ideals, such as equality checks, can be done via Gröbner basis techniques.

The geometric concept corresponding to the algebraic one of an ideal is that of a *variety*. Given an ideal J in S_n, the associated variety, denoted as $V(J)$, is defined as the set of phase-space points $(\lambda, \tilde{\lambda}) \in \mathbb{F}^{4n}$ which set the generators of J to zero. For instance, $V(J_{\Lambda_n})$ is the set of points in n-point phase space which satisfy momentum conservation. Similarly, given a variety U we can associate to it an ideal $I(U)$, defined as the set of all polynomials vanishing on U. Two natural concepts to associate to ideals and varieties are dimension and codimension. The latter is defined as the complement of dimension w.r.t. the dimension of the full space $V(\langle 0\rangle)$. The dimension of $V(J_{\Lambda_n})$ is $4n-4$, while its codimension in S_n is 4.

Since we wish for momentum conservation to always be respected, it is convenient to introduce the following quotient ring

$$R_n = S_n/J_{\Lambda_n}\,, \tag{5}$$

where two polynomials are identified as equivalent if they differ by a member of J_{Λ_n}; that is, in R_n all polynomials in J_{Λ_n} are considered as re-writings of zero. Geometrically, all varieties associated to ideals of R_n are sub-varieties of $V(J_{\Lambda_n})$. In the quotient ring $V(\langle 0\rangle_{R_n}) \sim V(J_{\Lambda_n})$, thus the codimension of $V(J_{\Lambda_n})$ in R_n is 0.

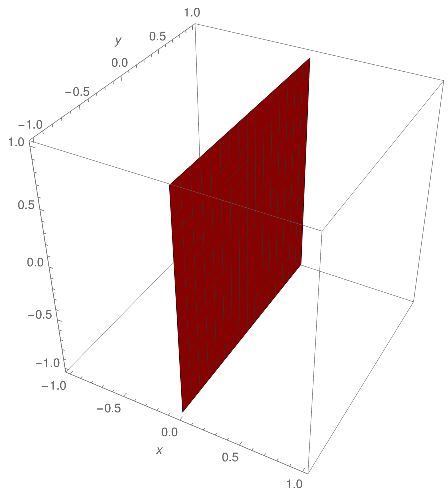

Figure 1. The irreducible variety $V(\langle x\rangle)$.

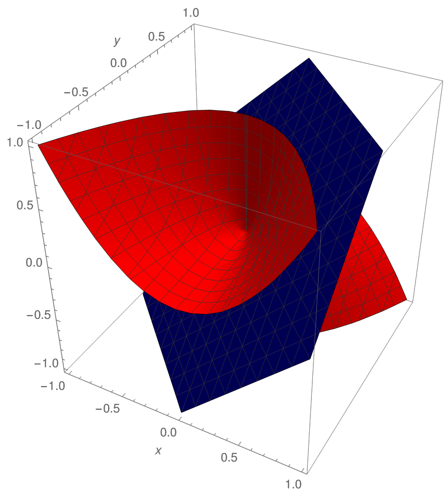

Figure 2. A reducible variety given by the union of $V(\langle x+y-z\rangle)$ in blue and $V(\langle x^2+yz\rangle)$ in red.

2.1. Irreducible varieties and prime ideals

An important concept associated to varieties is that of reducibility. Mathematically, a variety U is said to be irreducible if

$$U = U_1 \cup U_2 \;\;\Rightarrow\;\; U_1 = U \;\text{ or }\; U_2 = U\,, \tag{6}$$

that is, if it cannot be written as an irredundant union of two simpler varieties U_1 and U_2. Algebraically, irreducible varieties correspond to prime ideals, which are defined as follows

$$J \text{ is prime if } ab \in J \;\;\Rightarrow\;\; a \in J \;\text{ or }\; b \in J\,. \tag{7}$$

Figures 1 and 2 provide explicit examples, although to allow for a graphical visualization we need to consider the simpler polynomial ring $\mathbb{F}[x, y, z]$ instead of S_n. On the left-hand side, the variety $V(\langle x\rangle)$ is irreducible: it is just the plane $x = 0$. On the right-hand side, the variety $V\big(\big\langle(x+y-z)(x^2+yz)\big\rangle\big)$ is the union of two simpler varieties: the plane $x+y-z=0$, in blue, and the conical surface $x^2+yz=0$, in red.

2.2. Irreducible singular varieties in spinor space

For our study of the rational functions $\mathcal{C}_i$ in the five-point two-loop remainders of $q\bar{q} \to 3\gamma$ it suffices to consider poles of the form

$$\big\{\mathcal{D}\big\} = \big\{\langle ij\rangle, \langle i|j+k|i] \;\forall\; i \neq j \neq k \in (1,\ldots,5)\big\}\,. \tag{8}$$

The associated varieties $\big\langle\mathcal{D}_j\big\rangle_{R_5}$ are of codimension one. There are 35 such varieties, and they are all irreducible, which implies that the least common denominator $\prod_j \mathcal{D}_j$ of Eq. (2) is unique.

Contrary to codimension-one varieties generated by a single denominator factor in R_5, varieties of codimension two generated by a pair of poles are not all irreducible. For instance, let us consider the following ideal of codimension two

$$J_1 = \big\langle\langle 12\rangle, \langle 23\rangle\big\rangle_{R_5}\,, \tag{9}$$

and ask ourselves whether it is prime. By writing down a Schouten identity between its generators it is easy to find a product such that Eq. (7) is violated

$$|3\rangle\langle 12\rangle + |1\rangle\langle 23\rangle = |2\rangle\langle 13\rangle \tag{10}$$
$$\Rightarrow \quad |2\rangle\langle 13\rangle \in J_1 \quad \text{and} \quad |2\rangle \notin J_1\,,\ \langle 13\rangle \notin J_1\,. \tag{11}$$

Then, clearly, J_1 cannot be prime. In fact, it admits the following decomposition

$$J_1 = P_1 \cap P_2 \cap P_3\,, \tag{12}$$

where the three prime ideals are given by

$$P_1 = \big\langle\, |2\rangle \big\rangle_{R_5}\,,\quad P_2 = \big\langle \langle 12\rangle, \langle 23\rangle, \langle 13\rangle, [45] \big\rangle_{R_5}\,,\quad P_3 = \big\langle \langle ij\rangle \;\; \forall\, i \neq j \big\rangle_{R_5}\,. \tag{13}$$

The second split $P_2 \cap P_3$ is a peculiarity of 5-point phase space, as it follows from $s_{123} = s_{45}$. Note also that union of varieties corresponds to intersection of ideals, hence Eq. (13) implies, according to the definition of Eq. (6), that $V(J_1)$ is a reducible variety. Physically, we can think of P_1 as a "soft" ideal, and of P_2, P_3 as "collinear" ideals.

Starting from the poles of Eq. (8), we identify 10 distinct prime ideals at codimension two in R_5, up to permutations and parity (i.e. the symmetries of J_{Λ_5}). Accounting for their multiplicities under these symmetries we find 317 distinct irreducible varieties. For example, there are 10 varieties analogous to P_1, 20 to P_2 and only 2 to P_3.

3. p-adic numbers

To make practical use of Section 2, we need to be able to generate phase-space points close to irreducible surfaces. This is impossible over finite fields, and potentially unstable over the usual floating-point numbers, which leads us to introducing p-adic numbers.

A p-adic number x can be expressed in terms of a series, reminiscent of a Laurent series, in a prime number p. Starting from a finite negative integer $-m$, we write

$$x = \sum_{i=-m}^{\infty} a_i p^i = a_{-m}p^{-m} + \cdots + a_{-1}p^{-1} + a_0 + a_1 p + a_2 p^2 + \cdots\,. \tag{14}$$

The subset of p-adic numbers ($\mathbb{Q}_p$) with $m = 0$ is known as the p-adic integers ($\mathbb{Z}_p$). Given $m = 0$, the first p-adic digit a_0 behaves like an element of the finite field $\mathbb{F}_p$.

The *valuation* (ν_p) of a p-adic number is defined as

$$\nu_p(x) = k \;\text{ such that }\; a_i = 0 \;\text{ for all }\; i < k\,. \tag{15}$$

In terms of the valuation, we can define the p-adic absolute value as

$$|x|_p = p^{-\nu_p(x)}\,, \tag{16}$$

where the sign in the exponent is crucial. As a consequence on this definition, the series of Eq. (14) is convergent and can thus be safely truncated.

For practical applications it is convenient to employ a "floating-point" representation, with an *exponent*, given in terms of the valuation, and a k-digits *mantissa*, in parenthesis,

$$x = p^{\nu_p(x)} \left(\sum_{i=0}^{k-1} a_i p^i + \mathcal{O}(p^k) \right) \;\text{ with }\; a_i \neq 0\,. \tag{17}$$

We note that when implementing p-adic numbers in this representation it may be convenient to allow for a variable-size mantissa, i.e. to explicitly keep track of the error term $\mathcal{O}(p^k)$.

3.1. p-adic phase-space points close to irreducible varieties

Given an irreducible surface $U \subset V(J_{\Lambda_n})$, we wish to generate a p-adic phase-space point which lies close to the singular surface U, while being on $V(J_{\Lambda_n})$, i.e. respecting momentum conservation. We label such a point with an (ϵ) superscript and a U subscript, and define it as

$$\begin{aligned} q_i(\lambda_U^{(\epsilon)}, \tilde\lambda_U^{(\epsilon)}) &= \mathcal{O}(\epsilon) \text{ where } I(U) = \langle q_1, \dots, q_l \rangle_{R_n} \text{ and } i \in \{1, \dots, l\}\,, \\ r_j(\lambda_U^{(\epsilon)}, \tilde\lambda_U^{(\epsilon)}) &= \mathcal{O}(\epsilon^k) \text{ where } J_{\Lambda_n} = \langle r_1, \dots, r_4 \rangle_{S_n} \text{ and } j \in \{1, \dots, 4\}\,. \end{aligned} \tag{18}$$

In Eq. (18) the meaning of k is the same as in Eq. (17), that is momentum conservation is respected to the full working precision. Since we wish for ϵ to be a small quantity, and we are working in $\mathbb{Q}_p$, we can take $\epsilon \sim p$. To construct such a phase-space point we can first generate a finite field point exactly on U, and then "lift" this solution to be an approximate p-adic solution with a multivariate procedure analogous to the univariate Hensel lift.

4. Ansatz construction

The first step is the determination of the orders of poles and zeros. This can be achieved by a single p-adic evaluation close to the relevant codimension one variety

$$q_{ij} = \nu_p\big(\mathcal{C}_i(\lambda_U^{(\epsilon)}, \tilde\lambda_U^{(\epsilon)})\big) \text{ with } U = V(\langle \mathcal{D}_j \rangle)\,, \tag{19}$$

as long as the varieties $V(\langle \mathcal{D}_j \rangle)$ are irreducible, which for the set of poles of Eq. (8) is always the case at five-point – but is not, for instance, at four-point.

Having obtained the denominator $\prod_j \mathcal{D}_j^{q_{ij}}$, we have numerical access to the numerator $\mathcal{N}_i$ as the product of the rational function $\mathcal{C}_i$ and the denominator. For each irreducible surface U_γ of codimension two, we can obtain the valuation of the numerator at a nearby point

$$\kappa_{i,\gamma} = \nu_p\big(\mathcal{N}_i(\lambda_{U_\gamma}^{(\epsilon)}, \tilde\lambda_{U_\gamma}^{(\epsilon)})\big)\,. \tag{20}$$

Then, by the Zariski-Nagata theorem [17–19], we conclude that the numerator has to belong to a particular class of polynomials, the so-called ideal symbolic powers, which are denoted by an angle-bracket exponent

$$\mathcal{N}_i \in \mathfrak{J} \quad \text{with} \quad \mathfrak{J} = \bigcap_\gamma I(U_\gamma)^{\langle \kappa_{i,\gamma} \rangle}\,. \tag{21}$$

Moreover, $\mathcal{N}_i$ needs to belong to the vector space of products of spinor brackets that have the correct mass dimension and little-group weights. Let us denote monomials in spinor brackets as

$$m_{(\alpha,\beta)} = \prod_j \prod_{i<j} \langle ij \rangle^{\alpha_{ij}} [ij]^{\beta_{ij}}\,. \tag{22}$$

We can then denote their mass dimension as $[m_{(\alpha,\beta)}]$, their k^{th} little-group weight as $\{m_{(\alpha,\beta)}\}_k$, and define the following vector space

$$\mathcal{M}_{d,\vec\phi} = \Big\{ c_{(\alpha,\beta)} m_{(\alpha,\beta)} \;:\; [m_{(\alpha,\beta)}] = d\,,\; \{m_{(\alpha,\beta)}\}_k = \phi_k\,,\; c_{(\alpha,\beta)} \in \mathbb{Q} \Big\}\,, \tag{23}$$

where we are employing Einstein summation convention over the indices (α, β). The numerator belongs to both the ideal $\mathfrak{J}$ of Eq. (21) and the appropriate vector space $\mathcal{M}_{d,\vec\phi}$ of Eq. (23), i.e.

$$\mathcal{N}_i \in \mathfrak{J} \cap \mathcal{M}_{d,\vec\phi} \quad \text{with} \quad d = [\mathcal{N}_i] \quad \text{and} \quad \phi_k = \{\mathcal{N}_i\}_k\,. \tag{24}$$

This intersection can be computed by means of polynomial reduction via Gröbner bases and standard linear algebra algorithms (e.g. for computing null-spaces and intersections of vector spaces). It may also be convenient to first compute the intersection of the spinor-brackets vector space with each ideal symbolic power, and then intersect the resulting vectors spaces, instead of first intersecting the ideals to obtain $\mathfrak{J}$.

Table 1. Number of free parameters in the ansätze.

Remainder	$\mathcal{R}^{(2,0)}_{\gamma^-\gamma^+\gamma^+}$	$\mathcal{R}^{(2,N_f)}_{\gamma^-\gamma^+\gamma^+}$	$\mathcal{R}^{(2,0)}_{\gamma^+\gamma^+\gamma^+}$	$\mathcal{R}^{(2,N_f)}_{\gamma^+\gamma^+\gamma^+}$
Old ansatz size	41301	2821	7905	1045
New ansatz size	566	20	18	6

5. Application and outlook

As an example application, we constructed refined ansätze for the two-loop finite remainders for the production of three photons at hadron colliders. Table 1 shows the reduction in the size of the ansatz for different helicity configurations in the expansion in the number of light fermion flavors N_f. The comparison is between an ansatz which only takes into account mass dimension and little-group weights, and one which also imposes the constrains from the order of vanishing of the numerator on the irreducible surfaces of codimension two. The construction of this new ansatz also requires 317 "warm-up" p-adic evaluations, while the sampling of the ansatz itself can be done over finite fields. For this computation we have made extensive use of Singular [20], through the `Python` interface of Ref. [21], and the `Mathematica` interface of Ref. [22]. We have also made use of in-house implementations of p-adic numbers with and without a variable size mantissa. For publicly available implementations we refer the reader to Refs. [23, 24].

In the future, it will be interesting to analyze the behavior of algorithms for the numerical computation of multi-loop amplitudes when the field is taken to be $\mathbb{Q}_p$. Barring severe numerical instabilities, we foresee applications to new two-loop high-multiplicity amplitudes.

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