

Causal set actions in various dimensions

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Abstract. Causal set theory is a discrete approach to quantum gravity. The idea is that space-time is not a smooth manifold but a discrete causal set. This means that differential structure can not be assumed but has to be replaced by discrete alternatives. In this paper a pattern for a discrete version of the scalar D'Alembertian operator in various dimensions is proposed and tested. Corrections to this operator in curved space-time are proportional to the curvature scalar, and the proportionality constant is calculated for various dimensions. These corrections can be used to define a Causal Set action which is then calculated for various dimensions.

1. Introduction

In Causal set theory the smooth space-time manifold of ordinary gravity is replaced by a fundamentally discrete description. The only information needed are the causality relation between the discrete points and a fundamental volume associated with each point. These information suffice to reconstruct the whole manifold. For an introduction to Causal sets the papers [1] and [2] can be recommended. If a fundamentally discrete description is to keep Lorentz symmetry we have to give up on locality, as Sorkin showed in his paper [3]. Although this seems a grave problem, there would be no contradiction between this nonlocality and the continuum physics that has been tested so far so long as the nonlocality can be confined to unobserved scales.

2. Discrete Operators

In his paper [3] Sorkin pointed out that in discrete space-times either locality or Lorentz invariance have to be sacrificed. The obvious discrete replacement for differential operators is some sum over nearest neighbours of every point. But in causal sets only a causal definition of nearest neighbour can make sense. So the nearest neighbours of a point are those which are linked to it¹. In a causal set that arises from the Poisson process of sprinkling into a continuum space-time [3], most of the points linked and to the past to any given point lie very close to its past light cone and are in a sense quite far away from it. In a causal set sprinkled into infinite Minkowski space-time there is an infinite number of links for any point. For a causal set which is a sprinkling into Minkowski space-time of dimension d , and a real scalar field ϕ on it Sorkin proposed that the discrete operator to replace the d'Alembertian should have the form:

$$B_1\phi(x) = \frac{1}{l^2} \left(\alpha\phi(x) + \beta \sum_i^j C_i \sum_{y \in L_i} \phi(y) \right).$$

¹ Linked meaning that one is in the past of the other and there exists no point causally between these two.

where L_0 is the “zeroth layer,” the set of elements linked to and in the past of x , L_1 is the set of past elements where one other point lies causally between them and x and so forth. C_i are some coefficients to be determined, C_0 is set to be 1 and j is some finite rather small number. This operator is nonlocal on the fundamental discreteness scale. As Sorkin elaborates in his paper [3] these sums actually fluctuate more strongly with growing sprinkling density $\rho = \frac{1}{l^d}$. Because infinite density is the continuum limit, this makes it necessary to introduce the nonlocality scale $k = \frac{1}{l_k}$ where $m_{EW} \ll k \ll \frac{1}{l}$ with m_{EW} being the scale at which current experiments probe space-time². Introducing this intermediate scale k corresponds to smearing out the sum over layers of the causal set to a sum over a region around a layer, which suppresses the fluctuations. The nonlocal discrete operator in d dimensions is

$$B_k\phi(x) = \frac{\epsilon^2}{l^2} \left(\alpha\phi(x) + \beta\epsilon^d \sum_{y \prec x} f_d(n(x, y), \epsilon)\phi(y) \right),$$

where $n(x, y)$ is the number of elements of the causal set that lie between y and x . The parameter ϵ is a measure of the nonlocality and is given by $\epsilon = \frac{l}{l_k}$, and $f_d(n, \epsilon)$ is

$$f_d(n, \epsilon) = \sum_{i=0}^j C_i \left(\frac{\epsilon^d}{1 - \epsilon^d} \right)^i \frac{n!}{(n - i)!i!}$$

with C_i dimension dependent constants and $C_0 = 1$.

Given a continuum space-time, a point x in it and a scalar field, ϕ on it of compact support, a continuum version of the nonlocal operator can be derived as the expected value, in the sprinkling process, of the discrete operator applied to the induced scalar field on the sprinkled causal set:

$$\square_l\phi(x) := \frac{\alpha}{l^2}\phi(x) + \frac{\beta}{l^{d+2}} \int_{J^-(x)} d^d y \sqrt{-g(y)}\phi(y) \sum_i^j C_i \left(\frac{V_{\diamond d}(x, y)}{l^d} \right)^i \frac{e^{-\frac{V_{\diamond d}(x, y)}{l^d}}}{i!} \quad (1)$$

where $V_{\diamond d}(x, y)$ is the space-time volume of the causal interval between x and y . We will first concentrate on the case where the space-time is Minkowski space-time in which case $\sqrt{-g} = 1$ and the volume of the causal interval depends only on $x - y$. We will choose coordinates with x at the origin.

Then, when $l_k \neq l$, the nonlocal analog to equation (1), *i.e.* the expected value of discrete operator (2), can be shown to be

$$\square_k\phi(x) = \alpha k^2\phi(x) + \beta k^{d+2} \int_{J^-(x)} d^d y \phi(x - y) \sum_i^j C_i \left(k^d V_{\diamond d}(x - y) \right)^i \frac{e^{-k^d V_{\diamond d}(x - y)}}{i!}$$

Looking at the sum under the integral we observe that

$$k^d V_{\diamond d}(x - y) e^{-k^d V_{\diamond d}} \propto k \frac{\partial}{\partial k} e^{-k^d V_{\diamond d}}$$

So we can use this derivative to generate the sum over layers. For that we simply introduce an Operator $\hat{\mathcal{O}}$ such that

$$\hat{\mathcal{O}} e^{-k^d V_{\diamond d}(x - y)} = \sum_i^j C_i \left(k^d V_{\diamond d}(x - y) \right)^i \frac{e^{-k^d V_{\diamond d}(x - y)}}{i!}$$

² The relation $m_{EW} \ll k$ is necessary for this theory not to contradict experimental data. But as $\frac{1}{l}$ is of the same order of magnitude as the Planck scale there is enough space between these constraints for k

The continuum nonlocal operator then can be written as

$$\square_k \phi(0) = \alpha k^2 \phi(0) + \beta k^{d+2} \hat{O} \int_{J^-(0)} d^d x \phi(x) e^{-k^d V_{\circ d}(x)}. \quad (2)$$

\hat{O} is a polynomial in $k \frac{\partial}{\partial k} = H$, the homogeneity operator, generating the layer structure. We now turn to finding the correct coefficients that will make $\square_k \phi(0)$ approximately equal to the continuum D'Alembertian.

To actually perform the integrals we use light cone coordinate $v := \frac{1}{\sqrt{2}}(t+r)$ and $u := \frac{1}{\sqrt{2}}(t-r)$ and do a Taylor expansion of the scalar field to second order in x . This can be interpreted as a restriction on the test-functions to be slowly varying on the length-scales examined here.

2.1. 3d operator explicit treatment

To make the approach applied clearer we will show how this is done in 3 dimensions. The volume of a 3 dimensional Alexandrov neighbourhood is calculated in [4] as $V_{\circ 3} := \frac{\pi}{12} \tau^3 = \frac{\pi}{\sqrt{2} \cdot 3} (u \cdot v)^{\frac{3}{2}}$. The operator in 3d is $\hat{O} = \frac{1}{8}(H + 2)(H + 4)$ [5]. This gives a structure with three layers:

$$\begin{aligned} \square_k \phi(0) := & \alpha k^2 \phi(0) + \beta k^5 \int_{-\infty}^0 dv \int_v^0 du \int_0^{2\pi} d\varphi \frac{(v-u)}{\sqrt{2}} \phi(x) \\ & \times \left(1 - \frac{27}{8} k^3 \frac{\pi}{\sqrt{2} \cdot 3} (u \cdot v)^{\frac{3}{2}} + \frac{9}{8} k^6 \left(\frac{\pi}{\sqrt{2} \cdot 3} \right)^2 (u \cdot v)^3 \right) e^{-k^3 \cdot \frac{\pi}{\sqrt{2} \cdot 3} (u \cdot v)^{\frac{3}{2}}} \end{aligned} \quad (3)$$

After Taylor expanding, integrating over φ and plugging in u and v the field $\phi(x)$ is:

$$\phi(x) = 2\pi\phi + \sqrt{2}\pi(u+v)\partial_t\phi + \frac{\pi}{2}(u+v)^2\partial_t^2\phi + \frac{\pi}{4}(v-u)^2(\partial_x^2 + \partial_y^2)\phi$$

With this replacing $\phi(x)$ the other integrations in (3) can be solved. Leading to:

$$\square_k \phi(0) = \alpha k^2 \phi(0) + \beta \left(-k^2 - \left(\frac{3\sqrt{2}}{\pi} \right)^{\frac{2}{3}} \Gamma\left(\frac{5}{3}\right) (-\partial_t^2 + \partial_x^2 + \partial_y^2) \right)$$

There are no corrections to this if the integration is performed over the whole light cone. However for simulations on small finite causal sets we need to consider them (see 3.1 below for the corrections in the 3d case). Fixing $\beta = -\frac{\left(\frac{\pi}{3\sqrt{2}}\right)^{2/3}}{\Gamma\left(\frac{5}{3}\right)}$ and $\alpha = -\frac{\left(\frac{\pi}{3\sqrt{2}}\right)^{2/3}}{\Gamma\left(\frac{5}{3}\right)}$ the expression approximates the continuum d'Alembertian. The right operator is then

$$B_k \phi(0) = \frac{\epsilon^2}{l^2} \left(\alpha \phi(0) + \beta \epsilon^3 \sum_{y < 0} f(n(0, y), \epsilon) \phi(y) \right),$$

where α and β are to be taken as above and

$$f(n, \epsilon) = (1 - \epsilon^3)^n \left(1 - \frac{27\epsilon^3 n}{8(1 - \epsilon^3)} + \frac{9\epsilon^6 n(n-1)}{8(1 - \epsilon^3)^2} \right)$$

where $n(x, y)$ is the number of points lying causally between x and y .

2.2. Other dimensions

The principal course of action in other dimensions stays the same. First of all one needs to find the operator \hat{O} . The form of the operator follows a simple pattern, being

$$\hat{O} = \frac{(H + 2)(H + 4) \dots (H + d + 2)}{2^{\frac{d}{2}+1}(\frac{d}{2} + 1)!} \tag{4}$$

for d even, and \hat{O} for $d = 2n + 1$ is the same as for $d = 2n$. Up to the normalisation factor $\frac{1}{2^{\frac{d}{2}+1}(\frac{d}{2}+1)!}$ this is the so called Double Pochhammer triangle [6], which is well known in mathematics.

For the dimensions examined in this paper the resulting coefficients C_i are given in table (1). The case of one dimension -time only- does not follow the pattern and is just given for completeness, it is the usual discretization of the second derivative. The 2d case was examined in an earlier paper[7]. There is no simple closed form for the Coefficients as they are given as a

Table 1. Homogeneity operators and the coefficients arising from them.

Operator \hat{O}	Coefficients for discrete expression				
	C_1	C_2	C_3	C_4	C_5
1D	1	-2	1		
2D $\frac{1}{8}(H + 2)(H + 4)$	1	-2	1		
3D $\frac{1}{8}(H + 2)(H + 4)$	1	$-\frac{27}{8}$	$+\frac{9}{4}$		
4D $\frac{1}{48}(H + 2)(H + 4)(H + 6)$	1	-9	16	-8	
5D $\frac{1}{48}(H + 2)(H + 4)(H + 6)$	1	$-\frac{215}{16}$	$\frac{225}{8}$	$-\frac{125}{8}$	
6D $\frac{1}{384}(H + 2)(H + 4)(H + 6)(H + 8)$	1	-34	141	-189	81
7D $\frac{1}{384}(H + 2)(H + 4)(H + 6)(H + 8)$	1	$-\frac{6307}{128}$	$\frac{14749}{64}$	$-\frac{10633}{32}$	$\frac{2401}{16}$

super-position of the double Pochhammer triangle with numbers which arise from the derivatives action on the exponential, namely

$$H^n e^{-k^d V} = \left(k \frac{\partial}{\partial k}\right)^n e^{-k^d V} = d^n \sum_i^n (-1)^i i^{n-i} (k^d V)^i \tag{5}$$

The volume of the Alexandrov neighbourhood is dimension dependent, for those dimensions examined the volumes are given in table 2, taken from [4].

The pre-factors α and β are also given in table 2. They follow a clear pattern. For odd dimensions α is $-1/\Gamma(\frac{d+2}{d}) \cdot f_d^{2/d}$ where f_d is the factor from the Alexandrov volume in light cone coordinates. In even dimensions there is an additional factor of 2 so $\alpha = -2/\Gamma(\frac{d+2}{d}) \cdot f_d^{2/d}$. Additionally β can be calculated from α as

$$\beta = -\frac{\alpha}{\hat{O} \int dV e^{-k^d V}} \tag{6}$$

Here we will briefly list the complete expressions for the discrete operators in 4,5,6 and 7 dimensions.

³ In 4-d by coincidence the factor comes out to be $\frac{-2}{\Gamma(\frac{6}{4})} (\frac{\pi}{6})^{2/4} = -\frac{4}{\sqrt{6}}$

Table 2. Volumes of d- dimensional Alexandrov neighbourhoods and the prefactors alpha and beta

	2d	3d	4d
Volume	$u \cdot v$	$\frac{\pi}{3\sqrt{2}}(u \cdot v)^{3/2}$	$\frac{\pi}{6}(u \cdot v)^2$
α	$-2 = -\frac{2}{\Gamma(\frac{4}{2})}$	$-\frac{1}{\Gamma(\frac{5}{3})} \left(\frac{\pi}{3\sqrt{2}}\right)^{2/3}$	$-\frac{4}{\sqrt{6}} = -\frac{2}{\Gamma(\frac{6}{4})} \left(\frac{\pi}{6}\right)^{1/2}$
β	$4 = \frac{4}{\Gamma(\frac{4}{2})}$	$-\frac{1}{\Gamma(\frac{5}{3})} \left(\frac{\pi}{3\sqrt{2}}\right)^{2/3}$	$\frac{4}{\sqrt{6}} = \frac{2}{\Gamma(\frac{6}{4})} \left(\frac{\pi}{6}\right)^{1/2}$

	5d	6d	7d
Volume	$\frac{\pi^2}{20\sqrt{2}}(u \cdot v)^{5/2}$	$\frac{\pi^2}{45}(u \cdot v)^3$	$\frac{\pi^2}{168\sqrt{2}}(u \cdot v)^{7/2}$
α	$-\frac{1}{\Gamma(\frac{7}{5})} \left(\frac{\pi^2}{20\sqrt{2}}\right)^{2/5}$	$-\frac{2}{\Gamma(\frac{8}{6})} \left(\frac{\pi^2}{45}\right)^{2/6}$	$-\frac{1}{\Gamma(\frac{9}{7})} \left(\frac{\pi^2}{168\sqrt{2}}\right)^{2/7}$
β	$-\frac{3}{8\Gamma(\frac{7}{5})} \left(\frac{\pi^2}{20\sqrt{2}}\right)^{2/5}$	$\frac{4}{5\Gamma(\frac{8}{6})} \left(\frac{\pi^2}{45}\right)^{2/6}$	$-\frac{1}{8\Gamma(\frac{9}{7})} \left(\frac{\pi^2}{168\sqrt{2}}\right)^{2/7}$

2.2.1. 4 d operator

$$B_k\phi(0) = \frac{\epsilon^2}{l^2} \left(\alpha\phi(0) + \beta\epsilon^4 \sum_{y \prec x} f_4(n(x, y), \epsilon)\phi(y) \right),$$

where

$$\alpha = \frac{4}{\sqrt{6}} \quad \text{and} \quad \beta = \frac{4}{\sqrt{6}}$$

and

$$f_4(n, \epsilon) = (1 - \epsilon^4)^n \left(1 - \frac{9\epsilon^4 n}{(1 - \epsilon^4)} + \frac{8\epsilon^8 n(n - 1)}{(1 - \epsilon^4)^2} - \frac{4\epsilon^{12} n(n - 1)(n - 2)}{3(1 - \epsilon^4)^3} \right)$$

2.2.2. 5 d operator

$$B_k\phi(0) = \frac{\epsilon^2}{l^2} \left(\alpha\phi(0) + \beta\epsilon^5 \sum_{y \prec x} f_5(n(x, y), \epsilon)\phi(y) \right),$$

where

$$\alpha = \frac{\pi^{4/5}}{2 \cdot 5^{2/5} \Gamma(\frac{7}{5})} \quad \text{and} \quad \beta = -\frac{3\pi^{4/5}}{16 \cdot 5^{2/5} \Gamma(\frac{7}{5})}$$

and

$$f_5(n, \epsilon) = (1 - \epsilon^5)^n \left(1 - \frac{215\epsilon^5 n}{16(1 - \epsilon^5)} + \frac{225\epsilon^{10} n(n - 1)}{16(1 - \epsilon^5)^2} - \frac{125\epsilon^{15} n(n - 1)(n - 2)}{24(1 - \epsilon^5)^3} \right)$$

2.2.3. 6 d operator

$$B_k\phi(0) = \frac{\epsilon^2}{l^2} \left(\alpha\phi(0) + \beta\epsilon^6 \sum_{y \prec x} f_6(n(x, y), \epsilon)\phi(y) \right),$$

where

$$\alpha = \frac{2 \left(\frac{\pi}{3}\right)^{2/3}}{5^{1/3}\Gamma\left(\frac{4}{3}\right)} \quad \text{and} \quad \beta = \frac{4 \left(\frac{\pi}{3}\right)^{2/3}}{5 \cdot 5^{1/3}\Gamma\left(\frac{4}{3}\right)}$$

and

$$f_6(n, \epsilon) = (1 - \epsilon^6)^n \left(1 - \frac{34\epsilon^6 n}{(1 - \epsilon^6)} + \frac{141\epsilon^{12} n(n-1)}{2(1 - \epsilon^6)^2} - \frac{63\epsilon^{18} n(n-1)(n-2)}{2(1 - \epsilon^6)^3} + \frac{27\epsilon^{24} n(n-1)(n-2)(n-3)}{8(1 - \epsilon^6)^4} \right)$$

2.2.4. 7 d operator

$$B_k\phi(0) = \frac{\epsilon^2}{l^2} \left(\alpha\phi(0) + \beta\epsilon^7 \sum_{y \prec x} f_7(n(x, y), \epsilon)\phi(y) \right),$$

where

$$\alpha = \frac{\pi^{6/7}}{2 (21)^{2/7}\Gamma\left(\frac{9}{7}\right)} \quad \text{and} \quad \beta = -\frac{\pi^{6/7}}{16 (21)^{2/7}\Gamma\left(\frac{9}{7}\right)}$$

and

$$f_7(n, \epsilon) = (1 - \epsilon^7)^n \left(1 - \frac{6307\epsilon^7 n}{128(1 - \epsilon^7)} + \frac{14749\epsilon^{14} n(n-1)}{256(1 - \epsilon^7)^2} - \frac{10633\epsilon^{21} n(n-1)(n-2)}{192(1 - \epsilon^7)^3} + \frac{2401\epsilon^{28} n(n-1)(n-2)(n-3)}{384(1 - \epsilon^7)^4} \right)$$

3. Simulations

To test if these discrete operators give the correct results they have to be applied in simulated causal sets. The best way to simulate a causal set which is in some sense manifold like is to sprinkle the Causal set into a preexisting manifold. For more on this see [8]. As one can only simulate finite Causal Sets the corrections resulting from the finite size L will have significant effects.

3.1. Corrections

The size of the Causal Set grows with L^d , as does the number of sprinkled points necessary to get sensible results. So for this paper only simulations in 3 d were done. Corrections appear for a finite cut-off L in the integral (3).

$$I_{gen} = \int_{-L}^0 dv \int_v^0 du \int_0^{2\pi} d\varphi \frac{1}{\sqrt{2}} (v - u)\phi(x) e^{-k^3 V_{\circ 3}(x)}.$$

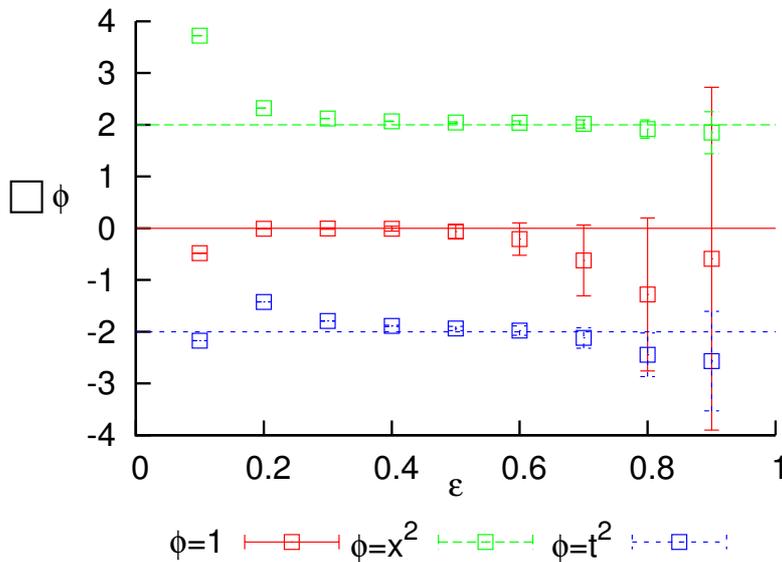


Figure 1. Simulation results

Solving this and using the values for α and β from 2 leads to

$$\begin{aligned} \square_{k_{3d}}\phi(0) = & \square\phi(0) - \left(\frac{3\sqrt{2}}{\pi}\right)^{\frac{1}{3}} \frac{\sqrt{2} \partial_t\phi(0)}{kL^2 \Gamma(\frac{5}{3})} + \left(\frac{3\sqrt{2}}{\pi}\right)^{\frac{1}{3}} \frac{\partial_t^2\phi(0)}{kL \Gamma(\frac{5}{3})} \\ & - \left(\frac{3\sqrt{2}}{\pi}\right)^{\frac{1}{3}} \frac{3 (\partial_x^2 + \partial_y^2)}{2 kL \Gamma(\frac{5}{3})}\phi(0) + \mathcal{O}\left(\frac{1}{k^2L^3}\right) \end{aligned}$$

neglecting terms in $\Gamma\left(\frac{2}{3}, \frac{k^3L^3\pi}{3\sqrt{2}}\right)$ and $e^{-\frac{k^3L^3\pi}{3\sqrt{2}}}$ as these fall off very fast with L .

3.2. Simulation results

To test if the operator and the corrections give the right result simulations were done. The functions which were tested were $\phi = 1$ to test that a constant gives the right result, $\phi = x^2$ to test that functions of space work correctly and $\phi = t^2$ to test functions of time. The functions were chosen because the results if the d'Alembertian is applied to them are constants, so that the value does not depend on the causal set. Every point in the plot 1 corresponds to 100 000 causal sets with 1000 points. For these sprinkled sets L grows like $l \cdot N^{1/3}$. The corrections due to the finite size of the causal set are of order one. In the plot of the simulated data the variable used is $\epsilon = k \cdot l$. It is obvious that the best results are achieved with $\epsilon \approx 0.5$. This is to be expected, as for epsilon close to one the nonlocality is small so the fluctuations in the results are still large. For very small ϵ the nonlocality becomes very large. The errors here are due to the relatively small size of the sprinkled sets as the layers now are thick and would reach points that are not in the small sprinkled set. These simulations show that the discrete operator in three dimensions works.

4. Curvature Scalar R

As Dionigi Benincasa shows in his paper [9] in curved space-time there is a connection between these discrete operators and the curvature scalar. The relevant relation is:

$$\square_k \phi(0) = \square \phi(0) + aR\phi(0) + b\frac{1}{k}R^2\phi(0) + \dots \quad (7)$$

In his paper he finds that the constant a is $-\frac{1}{2}$ in 2 and 4 dimensions. The next step is then to calculate these coefficients for 3,5,6 and 7 dimensions. They are found to be $\frac{1}{2}$, $\frac{1}{2}$, $-\frac{1}{2}$ and $\frac{1}{2}$ so odd and even dimensions only differ by a sign.

4.1. Calculations

The continuum version of the discrete operator is defined in equation (2) to be

$$\square_k \phi(0) = \alpha k^2 \phi(0) + \beta k^{d+2} \int_{J^-(0)} d^d x \sqrt{-g} \phi(x) f(V) e^{-k^d V_{\circ d}(x)} .$$

This expression is valid in flat space, so $\sqrt{-g} = 1$. To simplify the calculations we now take the frame to be Riemannian normal coordinates centred at the origin, as in [10]. The function $f(V)$ is given by $\hat{O}e^{-k^d V_{\circ d}(x)} = f(V)e^{-k^d V_{\circ d}(x)}$. So it could be rewritten as function $f(k^d V)$ which will be used later. If space is curved there are corrections to $\sqrt{-g}$ and to V , to lowest order in the curvature these expressions are:

$$\sqrt{-g} = 1 - \frac{1}{6} R_{\mu\nu} x^\mu x^\nu \quad (8)$$

and

$$V_{\circ d} = V_{\circ 0 d} \left(1 - \frac{d}{24(d+1)(d+2)} R \tau^2 + \frac{d}{24(d+1)} R_{\mu\nu} T^\mu T^\nu + \dots \right) . \quad (9)$$

The last expression is taken from Solodukhin and Gibbons [4]. T^μ is a time-like vector, pointing along the unique geodesic that connects the origin and x . Plugging this into equation (7) and doing a little algebra we can see that actually the corrections are just proportional to the variation of \square_k and with an Operator approach similar to the one used before we can get that in d dimensions

$$\delta \square_k \phi(0) = \frac{(-1)^{d+1}}{2} R . \quad (10)$$

For more details on this calculations please see [5], [10].

4.2. Causal Set Actions

Now using (10) we can generate the causal set actions by just applying the operator \square_k on constant fields ϕ with the value $(-1)^{d+1} 2$. (For more detail compare to [9] This leads to the actions:

$$\frac{1}{\hbar\alpha} S^{(d)}[\mathcal{C}] = N + \frac{\beta}{\alpha} \sum_i^j C_i N_i \quad \text{in } d \text{ dimension}$$

leading to:

$$\begin{aligned} \frac{1}{\hbar} S^{(2)}[\mathcal{C}] &= N - 2N_1 + 4N_2 - 2N_3 && \text{in 2-d} \\ \frac{1}{\hbar} S^{(3)}[\mathcal{C}] &= N + N_1 - \frac{27}{8}N_2 + \frac{9}{4}N_3 && \text{in 3-d} \\ \frac{1}{\hbar} S^{(4)}[\mathcal{C}] &= N - N_1 + 9N_2 - 16N_3 + 8N_4 && \text{in 4-d} \\ \frac{1}{\hbar} S^{(5)}[\mathcal{C}] &= N + \frac{3}{8}N_1 - \frac{645}{128}N_2 + \frac{675}{64}N_3 - \frac{375}{64}N_4 && \text{in 5-d} \\ \frac{1}{\hbar} S^{(6)}[\mathcal{C}] &= N - \frac{3}{5}N_1 + \frac{68}{5}N_2 - \frac{282}{5}N_3 + \frac{378}{5}N_4 - \frac{243}{5}N_5 && \text{in 6-d} \\ \frac{1}{\hbar} S^{(7)}[\mathcal{C}] &= N + \frac{1}{8}N_1 - \frac{6307}{1024}N_2 + \frac{14749}{512}N_3 - \frac{10633}{256}N_4 + \frac{2401}{128}N_5 && \text{in 7-d} \end{aligned}$$

with $N_i = \sum_{L_i}$ the number of causal set points in the i -th layer.

5. Conclusion

This shows that the method of finding a discrete operator devised in [7] works for dimensions higher and lower than 4. Thus it is necessary to be extremely careful not to break Lorentz invariance. This means we have to apply the generated operator \hat{O} before solving the integral. Ordering things in this way makes the introduction of a cut off unnecessary, and therefore keeps Lorentz invariance.

The rule to find the operator \hat{O} is

$$\hat{O} = \frac{(H+2)(H+4)\dots(H+d+2)}{2^{\frac{d}{2}+1}(\frac{d}{2}+1)!} \quad (11)$$

for d even, and \hat{O} for $d = 2n + 1$ is the same as for $d = 2n$.

It was possible to evaluate the proportionality constant for the Ricci Scalar in all dimensions evaluated here and it seems fair to guess that the so found pattern continues in all other dimensions. The coefficient to the Ricci Scalar is $-\frac{1}{2}$ in even and $\frac{1}{2}$ in odd dimensions. This gives a possibility to define gravity actions on the causal set.

Another interesting question about this operator pattern is how it is connected to the Greens functions on 2 and 4 dimensional Causal sets found by Steven Johnston [11], and what kind of Greens functions in other dimensions.

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