SLAC-PUB-345
September 1967

STRONG INTERACTION SUM RULES FOR PION-HADRON SCATTERING*

Frederick J. Gilman

and

Haim Harari†

Stanford Linear Accelerator Center
Stanford University, Stanford, California

(to be submitted to Physical Review)

---

* Work supported by the U. S. Atomic Energy Commission

† On leave from the Weizmann Institute, Rehovoth, Israel.

ABSTRACT

Using Regge high energy behavior, the chiral algebra of charges, and pion pole dominance of the divergence of the axial-vector current, all the strong interaction sum rules which hold for elastic pion-hadron scattering amplitudes at $t=0$ are derived. These include charge algebra sum rules as well as superconvergence relations. We distinguish between "pure" $t=0$ sum rules (Class I) and "extrapolated" sum rules (Class II) and relate them to the evenness or oddness of the helicity flip in the t-channel. Using the explicit crossing relations for the relevant helicity amplitudes, the connection of Class I superconvergence relations to the charge algebra sum rules is established, and the algebraic structure of Class I sum rules is then discussed in terms of representations of the $SU(2) \times SU(2)$ chiral algebra of charges, for particle states moving with infinite momentum. The properties of the mass operator in $SU(2) \times SU(2)$ are analyzed and it is shown that even in the presence of $SU(2) \times SU(2)$ symmetry breaking the $(\text{mass})^2$ values of all the (mixed) eigenstates of an irreducible representation of the algebra are predicted to be equal. Since these eigenstates can be determined from the matrix elements of the vector and axial-vector charges, a large number of non-trivial mass relations are obtained. Class II superconvergence relations, sum rules for $t\neq 0$, and sum rules for the derivative with respect to t of the scattering amplitude at $t=0$ are briefly discussed. Many applications of the strong interaction sum rules are presented including a model consisting of $I=0$ and $I=1$ scalar, pseudoscalar, vector, and axial-vector mesons. The predictions of the model as well as those of the other sum rules are derived and found to be in satisfactory agreement with experiment.

I. Introduction

A large number of sum rules involving amplitudes for strong interaction scattering processes have been derived in the last few years using various theoretical ideas including ordinary dispersion relations, the algebra of currents, Regge pole theory and pole dominance of the weak and electromagnetic currents. The procedure followed in most cases is to use experimental information or theoretical prejudices in deciding which amplitudes may satisfy unsubtracted dispersion relations, to write down such relations for the amplitudes at threshold and to derive low energy theorems for the relevant amplitudes at the chosen low energy points. The connection between the sum rules and our experimental knowledge is often made by a "saturation assumption" which asserts that in most cases the dispersion integrals are dominated by the contributions of a few low-lying single particle states. It is this assumption which enables us to utilize sum rules for experimentally unrealistic processes such as π-ρ or π-N^* scattering and to obtain new dynamical relations among strong coupling constants and masses. The saturation hypothesis is, in most cases, the weakest link in the long chain of assumptions that we use in deriving sets of strong interaction sum rules. Moreover, in some specific cases we may apriori expect saturation to provide us only with very crude approximations. This does not mean, however, that we should abandon it completely as a powerful tool for studying the various sum rules, particularly in view of the fact that explicit scattering data for most of the relevant processes do not and will not exist in the foreseeable future.

The strong interaction sum rules that we discuss stem from two main sources: Some are typical current algebra sum rules[1] in which the low energy theorems are provided by the use of current commutators and the partially conserved axial vector current hypothesis (PCAC). Other sum rules follow from the so-called superconvergence relations[2] which state that if a scattering amplitude A(s,t) obeys an unsubtracted dispersion relation and satisfies:

$$\lim_{s \to \infty} sA(s,t) = 0 \qquad (1)$$

then

$$\int_{-\infty}^{+\infty} \mathrm{Im}\, A(s,t)ds = 0 \qquad (2)$$

Some of the most interesting questions that have recently been raised involve the problem of saturating a continuously infinite set of strong interaction sum rules (e.g. for all values of the invariant momentum transfer of a given amplitude) by finite or infinite discrete sets of single particle states[3]. Although we will briefly touch on this point, we would like to address ourselves in this paper to a less ambitious problem which has not been fully analyzed before, and which is a necessary step in understanding many aspects of the strong interaction sum rules. We refer to the general question of strong interaction sum rules for forward (t=0) amplitudes, their algebraic structure, self-consistency, agreement with experiment and connection to the algebra of weak and electromagnetic charges.

In particular, we discuss the following questions: What is the general connection between the superconvergence relations for forward amplitudes and the sum rules derived from PCAC, vector dominance and the algebra of charges?

What are all the possible t=0 sum rules which can be derived for a general strong interaction scattering process, using the theoretical tools mentioned in the opening paragraph?

What, if any, is the significance of the large number of "SU(6) results" or "higher symmetry results" which were obtained by imposing specific saturation assumptions on various t=0 superconvergence relations?

What can we say about the possible algebraic structure of t=0 strong interaction sum rules?

Which sets of t=0 sum rules can be saturated by which single-particle states without leading to internal inconsistencies?

How good is the agreement with experiment of the various sum rules and saturation assumptions?

What happens in the neighborhood of t=0?

What dynamical information can we obtain from the strong interaction sum rules, with respect to the mass spectrum and coupling constants of the hadrons?

We will generally adopt the approach of considering the saturated sum rules as sets of equations in the masses and coupling constants of the intermediate states, and will study the possible solutions of such sets of equations[4].

In Section II we formulate and discuss the assumptions which are used throughout the paper. The special importance of the I=2 amplitudes is discussed in detail. The necessary kinematics, including some of the explicit crossing relations for the relevant helicity amplitudes are presented in Section III. In Section IV we derive all t=0 strong interaction sum rules which follow from our assumptions for π-x scattering (where x is any hadron) and discuss the connection between the superconvergence relations and the PCAC and charge-algebra sum rules. We distinguish between "pure" (class I) t=0 sum rules and "extrapolated" (class II) sum rules. In Section V we proceed to discuss the algebraic structure of the class I sum rules in terms of the representations of the $SU(2) \times SU(2)$ chiral algebra of charges. In particular we analyze the properties of the mass operator in $SU(2) \times SU(2)$ and present a few examples in which we illustrate our general results. Section VI deals with sum rules for $t \neq 0$. We briefly discuss sum rules for $t \neq 0$, Class II superconvergence relations and the sum rules obtained by taking the derivative with respect to t of scattering amplitudes at t=0. In Section VII we present the analysis of a model consisting of I=0 and I=1 scalar, pseudoscalar, vector and axial vector non-strange mesons and discuss its possible relation to the experimental situation. A few other applications of complete sets of t=0 sum rules are discussed in Section VIII. In Section IX we summarize our results and outline some of the many related problems which remain open.

II. General Assumptions

The assumptions used throughout this paper fall into four categories: (a) The algebra of charges; (b) Pole dominance of currents; (c) High energy behavior; (d) Saturation. In this section we explicitly state our assumptions and discuss the theoretical and experimental evidence supporting them.

A. The Algebra of Charges

We assume that the isotopic spin vector and axial vector charges (i = 1,2,3):

$$Q^i = \int V_o^i(\vec{x},t)d^3x \qquad (3)$$

$$Q_5^i = \int A_o^i(\vec{x},t)d^3x \qquad (4)$$

obey, at equal times, the commutation relations of the chiral SU(2) × SU(2) algebra[5]:

$$[Q^i, Q^j] = i\epsilon_{ijk} Q^k \tag{5}$$

$$[Q_5^i, Q^j] = i\epsilon_{ijk} Q_5^k \tag{6}$$

$$[Q_5^i, Q_5^j] = i\epsilon_{ijk} Q^k \tag{7}$$

This suggestion is strongly supported by the success of the Adler-Weisberger calculation[1] of g_A as well as by a few other successful applications[6] of the commutator (7). The isospin charges Q_i are conserved by the strong interactions:

$$\frac{d}{dt} Q^i(t) = 0 \tag{8}$$

What can we say about the time derivatives of the axial charges? We define:

$$\frac{d}{dt} Q_5^i(t) \equiv D^i(t) \tag{9}$$

Taking the time derivative of both sides of Eq. (6) we learn that $D^i(t)$ is a vector in isospin space:

$$[D^i, Q^j] = i\epsilon_{ijk} D^k \tag{10}$$

Differentiating (7) with respect to t gives:

$$[D^i, Q_5^j] = [D^j, Q_5^i] \tag{11}$$

The commutator $[D, Q_5]$ must therefore be symmetric in isospin and may include only I=0 and I=2 parts. Most of the Lagrangian models that have so far been proposed as possible underlying structures for the SU(2) × SU(2) charge algebra (including the σ-model[7] and the free quark model[8]) indicate that the I=2 part of the $[D,Q_5]$ commutator is absent. This assumption may be based on grounds of simplicity, but it could also be intuitively related to the striking absence of any evidence for I=2 currents, particles or resonances. <u>We therefore assume</u>:

$$[D^i(t),Q_5^j(t)] = \delta_{ij}S(t) \,, \tag{12}$$

where S(t) is an isoscalar quantity, and thus satisfies:

$$[S(t),Q^i] = 0 \,. \tag{13}$$

S(t) corresponds to the integrated σ-density in the σ model and is propro-tional to a scalar charge $\int\psi^\dagger\beta\psi d^3x$ in the free quark model. Could the commutator (12) vanish? It turns out that as long as $Q_5(t)$ is not a conserved charge ($D(t) \neq 0$) the commutator (12) does not vanish and, furthermore:

$$[S(t),Q_5^i(t)] = D^i(t) \tag{14}$$

This can be easily deduced from the Jacobi identity:

$$[[D^i,Q_5^j],Q_5^k] + [[Q_5^j,Q_5^k],D^i] + [[Q_5^k,D^i],Q_5^j] = 0 \tag{15}$$

which leads directly to Eq. (14). Eqs. (10)-(14) demonstrate that the four operators $D^i(t)(i=1,2,3)$, $S(t)$ transform into each other under commutation with the generators of $SU(2) \times SU(2)$ and therefore belong to a 4 dimentional $(\frac{1}{2},\frac{1}{2})$ representation of the algebra[9].

B. Pole Dominance

The commutation relations of the weak and electromagnetic currents lead to predictions for strong interaction parameters, when supplemented by pole dominance assumptions such as PCAC and vector meson dominance.

Since most of our analysis deals with pion-hadron scattering we will mainly use PCAC, namely: we assume[10] that the pion pole dominates the matrix elements of the divergence of the axial current at $q^2 = 0$. Inserting this divergence between single-nucleon states then leads to the Goldberger-Treiman relation[11]:

$$f_\pi = \frac{\sqrt{2}\, g_A m_N}{g_{\pi N}} \tag{16}$$

where $g_A = 1.18$, m_N is the nucleon mass, $\frac{g_{\pi N}^2}{4\pi} = 14.6$ and f_π is the decay constant of the charged pion. The expression for the pion decay rate is:

$$\Gamma_\pi = \frac{1}{8\pi} (G \cos \theta_C)^2 m_\pi m_\mu^2 \left[1 - \frac{m_\mu^2}{m_\pi^2}\right]^2 f_\pi^2 \tag{17}$$

where $G = 1.02 \times 10^{-5} m_N^{-2}$ is the weak interaction coupling constant, θ_C is the Cabibbo angle and m_π and m_μ are the masses of the π and μ. The experimental π^+ lifetime[12] gives:

$$f_\pi = 135 \text{ MeV} \tag{18}$$

C. High Energy Behavior

The success of Regge pole theory in explaining the energy dependence of many scattering processes at high energies leads us to believe that it can serve as a reliable criterion for the convergence of various sum rules. According to Regge theory (or any other theory based on the dynamical importance of t-channel exchanges at high energy and small momentum transfer) the energy dependence of a strong interaction scattering amplitude is essentially determined by its t-channel quantum numbers. In the next section we construct t-channel helicity amplitudes $\bar{f}^{t(I)}_{\lambda_c\lambda_d,\lambda_a\lambda_b}(s,t)$ for the process d + b $\to$ c + a (viewed in the s-channel) which are free from kinematic singularities. As $s \to \infty$ these amplitudes satisfy:

$$\bar{f}^{t(I)}_{\lambda_c \lambda_d, \lambda_a \lambda_b}(s,t) \propto s^{\alpha_I(t)-\Delta} \tag{19}$$

where I is the isotopic spin in the t-channel, λ_i are t-channel helicities, $\alpha_I(t)$ is the position of the leading trajectory having the appropriate t-channel quantum numbers and:

$$\Delta = \max\left\{|\lambda_c - \lambda_d|, |\lambda_a - \lambda_b|\right\} \tag{20}$$

The two components which determine the high energy behavior (Eq. (19)) are of different character. The parameter Δ represents the kinematic structure of the helicity amplitudes and will be discussed in detail in Section III, while $\alpha_I(t)$ represents the dynamical information of the Regge model. Since in this paper we deal mainly with t=0 amplitudes we have to make specific assumptions only with respect to $\alpha_I(0)$. Moreover, the convergence of all the sum rules discussed here is determined by whether $\alpha_I(0) = 1$, $0 \leq \alpha_I(0) < 1$ or $\alpha_I(0) < 0$ and does not depend on the precise numerical value of α. We will assume that, for the leading trajectories which couple to the π-π system: $\alpha_0(0) = 1$, $\alpha_1(0) < 1$, $\alpha_2(0) < 0$.

The first of these assumptions, $\alpha_0(0) = 1$, means that total cross-sections are finite as $s \to \infty$. In Regge language it corresponds to an intercept $\alpha_0(0) = 1$ for the Pomeranchuk trajectory, but it could also follow, independent of Regge theory, from a diffraction type picture for elastic processes or merely from the statement of maximal strength of the strong interactions[13].

The assumption $\alpha_1(0) < 1$ follows in Regge theory from the fact that all I=1 trajectories have t=0 intercepts which are smaller than unity[14]. The same statement follows, however, from the Pomeranchuk theorem on the asymptotic charge independence of cross sections and would probably be correct in almost any other reasonable theory. $\alpha_1(0) < 1$ is also strongly supported by the energy dependence of the forward amplitude for $\pi^- p \to \pi^0 n$.[15]

The assumption[2] $\alpha_2(0) < 0$ is the most crucial assumption that we make here. Within the framework of Regge theory we simply observe that an I=2 meson has never been seen and it is probably safe to assume that below 1.5 - 2 BeV there is no such state with an appreciable coupling to the π-π system. Using the slope of the known trajectories as a guide, we can then deduce that even if an I=2 particle is found at a higher mass value, its trajectory intercepts t=0 well below $\alpha = 0$. It is amusing that the quark model leads to the same conclusion, at least for forward elastic amplitudes[16]. If the forward scattering amplitude at high energy is given by a sum over quark-quark scattering amplitudes, the highest isospin that can be exchanged is I=1 (which is the highest possible isospin in the $q\bar{q}$ system) and therefore all I=2 t-channel amplitudes should vanish, corresponding to $\alpha_2(0) < 0$.

It is very hard to perform a direct experimental test of this assumption. The only simple process which can be measured relatively easily and which corresponds to a pure I=2 exchange is $\pi^- p \to \pi^+ N^{*-}$. In order to "measure" $\alpha_2(0)$ one has to find the cross section for this process at small values of t over a wide energy range (e.g. 5-15 BeV). The $\pi^+ N^{*-}$ final state is seen in the $\pi^+\pi^- n$ events which are overwhelmingly dominated by $\rho^0 n$ production. The present available data[17] are mostly at low energies (≤ 5 BeV) and indicate that the forward $\pi^+ N^{*-}$ production rate is very much smaller than that of any $\pi p \rightarrow \pi N^*$ cross section which can proceed via I=1 exchange and that it is, in fact, consistent with zero. The smallness of the I=2 t-channel amplitude is encouraging in the sense that it demonstrates the absence of important I=2 trajectories. It is, however, discouraging from the experimental point of view, since it is very difficult to measure the energy dependence of a cross section for a process which cannot be separated from its background[18]. Other I=2 exchange processes which are even more difficult to analyze are $\pi^+ d \rightarrow \pi^- p N^{*++}$ and $\bar{p}p \rightarrow \overline{N^{*-}} N^{*-}$. A possible experimental test which might be feasible but has never been done is simply to count the number of high energy π^-'s emitted at 0° from a $\pi^+ p$ interaction at various beam energies.

There are at least two <u>indirect</u> ways of testing the assumption $\alpha_2(0) < 0$. One of them is to pursue the (Regge theory or quark model) line of reasoning that led to this assumption and to suggest that all double charge exchange processes exhibit a similar energy dependence, except for the exchange of an N^{*++} (the only known doubly-charged resonance). In particular, we could propose that the full 27 representation of SU(3) has $\alpha(0) < 0$ or that the 10 and $\overline{10}$ meson representations have $\alpha(0) < 0$. Such assumptions can be tested by studying the high energy behavior of the forward amplitudes for $\pi^- p \rightarrow K^+ \Sigma^-$, $K^- p \rightarrow \pi^+ \Sigma^-$, $K^- p \rightarrow K^+ \Xi^-$, $\bar{p}p \rightarrow \overline{\Sigma^-} \Sigma^-$, the total cross-section combination[19] $[\sigma_t(K^+p) - \sigma_t(K^-p) + \sigma_t(K^-n) - \sigma_t(K^+n) + \sigma_t(\pi^-p) - \sigma_t(\pi^+p)]$ or the backward scattering amplitudes for $K^- p \rightarrow K^- p$, $K^- p \rightarrow \pi^- \Sigma^+$, etc. In all of these cases the same picture appears[20]: The relevant amplitudes are very small and in many cases still consistent with zero. At any given energy it is certainly justified to neglect the double charge exchange amplitude with respect to the single charge exchange. The energy dependence of these small amplitudes cannot, however, be determined and one cannot rule out the possibility of a tiny but persistent I=2 exchange contribution.

Another indirect way of testing the $\alpha_2(0) < 0$ assumption is, of course, to use it in deriving as many theoretical predictions as possible and to try to find whether or not it leads to inconsistencies or contradictions with experiment. A large number of interesting results[21] have been derived from this assumption, so far, and many more are presented in this paper. None of them lead to contradictions, while most of them can definitely be considered as successful predictions. We therefore believe that this assumption is valid, but suggest that some experimental effort be directed into verifying it by one or more of the tests suggested here.

D. The Saturation Assumption

In most cases we will assume that the strong interaction sum rules at t=0 are saturated by the s-channel contributions of the known single-particle states having the appropriate quantum numbers. In a few cases we will have theoretical or experimental reasons to omit some of these states, while in other cases we will study the possible necessity of a sizeable contribution from so far undiscovered states. It is clear that this flexibility makes it difficult to prove that a given sum rule is incorrect, and that it reduces the significance of those predictions which turn out to be very successful. This is, however, a price that we have to pay if we want to study a large number of sum rules and saturation schemes, without having a complete theory. At this point we only remark that we certainly do not exploit this freedom too much (we do not "invent" two new particles for every new sum rule) and that every one of our basic assumptions plays a role in deriving a large number of sum rules, so that the failure of one of these assumptions would almost certainly lead either to a highly artificial saturation scheme or simply to contradictions that cannot be reconciled.

We close this section by restating our assumptions:

1. $SU(2) \times SU(2)$ chiral algebra of charges.
2. For $D^i = d/dt\, Q_5^i$, $[D^i, Q_5^j] = \delta_{ij} S$ where S is an isoscalar.
3. PCAC.
4. $s^{\alpha_I(0)-\Delta}$ behavior for $s \to \infty$ and $t=0$ for an amplitude with helicity flip Δ and isospin I in the t-channel.
5. $\alpha_0(0) = 1$, $\alpha_1(0) < 1$, $\alpha_2(0) < 0$.

III. Helicity Amplitudes and Crossing Matrices at t=0

The strong interaction sum rules for pion-hadron scattering which follow from the algebra of charges and PCAC are most easily expressed in terms of s-channel helicity amplitudes. On the other hand, the superconvergence sum rules are most naturally written for t-channel helicity amplitudes whose energy dependence for large s and fixed t is given by Eq. (19). In order to have a unified treatment of both families of sum rules it is necessary to evaluate the relevant kinematic relations, including the helicity crossing matrices at t=0. In this section we present some of the relations which will be required for our various applications. For completeness we also include a review of some basic results concerning the asymptotic behavior of the t-channel helicity amplitudes.

A. Asymptotic Behavior of t-channel Helicity Amplitudes

In order to study the asymptotic behavior of scattering amplitudes for particles with spin we consider the t-channel helicity amplitudes[22]

$f^t_{\lambda_c\lambda_d,\lambda_a\lambda_b}(s,t)$ for the process a + b → c + d where $t = -(p_a + p_b)^2$, $s = -(p_a - p_c)^2$. The amplitude $f^t_{\lambda_c\lambda_d,\lambda_a\lambda_b}(s,t)$ has the t-channel partial wave expansion:

$$f^t_{\lambda_c\lambda_d,\lambda_a\lambda_b}(s,t) = \sum_J (2J+1) F^J_{\lambda_c\lambda_d,\lambda_a\lambda_b}(t)\, d^J_{\lambda\mu}(\theta_t) \tag{21}$$

where $\lambda = \lambda_a - \lambda_b$, $\mu = \lambda_c - \lambda_d$ and θ_t is the scattering angle in the t-channel between particles a and c. The differential cross-section is given by:

$$\frac{d\sigma(ab \to cd)}{d\Omega} = \frac{1}{64\pi^2} \frac{|\vec{p}_{cd}|}{|\vec{p}_{ab}|} \cdot \frac{1}{s} \left| f^t_{\lambda_c\lambda_d,\lambda_a\lambda_b}(s,t) \right|^2 \tag{22}$$

where $\vec{p}_{cd}$, $\vec{p}_{ab}$ are the final and initial three-momenta in the center of mass system. We notice[23,24,25,26] that $\cos\theta_t$ is an analytic function of s and that every term in the partial wave expansion (21) contains the factor $d^J_{\lambda\mu}(\theta_t)$ which can be expressed as $[(\cos\frac{1}{2}\theta_t)^{|\lambda+\mu|}(\sin\frac{1}{2}\theta_t)^{|\lambda-\mu|}]$ times a Jacobi polynomial in $\cos\theta_t$. We may thus divide $f^t_{\lambda_c\lambda_d,\lambda_a\lambda_b}(s,t)$ by $[(\cos\frac{1}{2}\theta_t)^{|\lambda+\mu|}(\sin\frac{1}{2}\theta_t)^{|\lambda-\mu|}]$ <u>without introducing additional singularities in s</u>, obtaining a new set of amplitudes:

$$\bar{f}^t_{\lambda_c\lambda_d,\lambda_a\lambda_b}(s,t) = \frac{f^t_{\lambda_c\lambda_d,\lambda_a\lambda_b}(s,t)}{(\cos\frac{1}{2}\theta_t)^{|\lambda+\mu|}(\sin\frac{1}{2}\theta_t)^{|\lambda-\mu|}} \tag{23}$$

We are interested in the large s behavior of the amplitudes defined in Eq. (23). In order to study it we notice that, for large s and fixed t:

$$[(\cos\tfrac{1}{2}\theta_t)^{|\lambda+\mu|}(\sin\tfrac{1}{2}\theta_t)^{|\lambda-\mu|}] \xrightarrow[s\to\infty]{} s^{\Delta} \tag{24}$$

where $\Delta = \max\{|\lambda|, |\mu|\}$. If the amplitude $f^t_{\lambda_c\lambda_d,\lambda_a\lambda_b}(s,t)$ is asymptotically proportional to $s^{\alpha(t)}$ we then find[26]:

$$\bar{f}^{t}_{\lambda_c\lambda_d,\lambda_a\lambda_b}(s,t) \xrightarrow[s\to\infty]{} \beta(t)s^{\alpha(t)-\Delta} \tag{25}$$

While α depends on the dynamics of the process (and can be determined by measuring $\frac{d\sigma}{d\Omega}$), the factor Δ is clearly of kinematic origin and is independent of the particular dynamical model that we use. Given the asymptotic behavior of Eq. (25), some of the kinematic-singularity-free amplitudes defined in Eq. (23) have, for $s\to\infty$, a sufficiently rapid fall-off in s to satisfy the superconvergence relation (Eq. (2)) discussed in the Introduction[2].

The s-channel helicity amplitudes $f^{s}_{\lambda'_c\lambda'_a,\lambda'_d\lambda'_b}(s,t)$ are defined in analogy to those in the t-channel, only now we define: $\lambda' = \lambda'_d - \lambda'_b$, $\mu' = \lambda'_c - \lambda'_a$ and θ_s is the scattering angle between particles d and c. The amplitudes $\bar{f}^{s}_{\lambda'_c\lambda'_a,\lambda'_d\lambda'_b}(s,t)$ are then defined by:

$$\bar{f}^{s}_{\lambda'_c\lambda'_a,\lambda'_d\lambda'_b}(s,t) = \frac{f^{s}_{\lambda'_c\lambda'_a,\lambda'_d\lambda'_b}(s,t)}{(\cos\frac{1}{2}\theta_s)^{|\lambda'+\mu'|}(\sin\frac{1}{2}\theta_s)^{|\lambda'-\mu'|}} \tag{26}$$

The factor $[(\cos\frac{1}{2}\theta_s)^{|\lambda'+\mu'|}(\sin\frac{1}{2}\theta_s)^{|\lambda'-\mu'|}]$ can again be seen to be present in every term of the partial wave expansion of $f^{s}_{\lambda'_c\lambda'_a,\lambda'_d\lambda'_b}(s,t)$, or alternately, since angular momentum conservation leads to the vanishing of $f^{s}_{\lambda'_c\lambda'_a,\lambda'_d\lambda'_b}(s,t)$ at $\theta_s = 0^{\circ}$ (unless $\lambda' = \mu'$) and $\theta_s = 180^{\circ}$ (unless $\lambda' = -\mu'$), one finds that exactly such a factor must be present in $f^{s}_{\lambda'_c\lambda'_a,\lambda'_d\lambda'_b}(s,t)$. The amplitudes defined in Eq. (26) are free from kinematic singularities in t and <u>non-vanishing</u> at $\theta_s = 0$ and 180°. The transformation between the helicity amplitudes in the t-channel and s-channel is given by[27]:

$$f^t_{\lambda_c\lambda_d,\lambda_a\lambda_b} = \sum_{\lambda'_a\lambda'_b\lambda'_c\lambda'_d} d^{J_a}_{\lambda'_a\lambda_a}(X_a)d^{J_b}_{\lambda'_b\lambda_b}(X_b)d^{J_c}_{\lambda'_c\lambda_c}(X_c)d^{J_d}_{\lambda'_d\lambda_d}(X_d)f^s_{\lambda'_c\lambda'_a,\lambda'_d\lambda'_b}(s,t) \qquad (27)$$

where J_i and $\lambda_i(\lambda'_i)$ are the spin and t-channel (s-channel) helicity of particle i. The X_i's are functions of s,t and the particle masses and are given explicitly in Reference 27.

B. Forward (t=0) Elastic Pion-Hadron Scattering

We now proceed to the specific case of forward (t=0) elastic scattering of massless pions[28] on a hadron with mass M and spin J. We will use Eqs. (23)-(27) for the case in which a,b are pions ($m_\pi = 0$), c,d are identical hadrons and $t \to 0$ in the physical region for s-channel scattering. In this particular case the kinematics simplifies greatly and we find:

$$\cos\frac{\theta_s}{2} \to 1 \qquad\qquad \cos\theta_s \to 1$$

$$\sin\frac{\theta_s}{2} \to \frac{\sqrt{s}}{s-M^2}\sqrt{-t} \qquad\qquad \sin\theta_s \to \frac{2\sqrt{s}}{s-M^2}\sqrt{-t} \qquad (28)$$

$$\cos\theta_t \to \frac{s-M^2}{M\sqrt{-t}} \qquad\qquad \sin\theta_t \to \frac{s-M^2}{M\sqrt{-t}}$$

The angles X_i needed in Eq. (27) for crossing the helicity amplitudes have the following behavior (up to linear terms in $\sqrt{-t}$)[27]:

$$X_c \to \frac{\pi}{2} - \frac{\sqrt{-t}}{2M}\,\frac{s+M^2}{s-M^2}$$

$$X_d \to \frac{\pi}{2} + \frac{\sqrt{-t}}{2M}\,\frac{s+M^2}{s-M^2} \qquad (29)$$

Parity conservation implies[22]:

$$f^s_{\lambda_c'0,\lambda_d'0}(s,t) = (-1)^{\lambda_d'-\lambda_c'} f^s_{-\lambda_c'0,-\lambda_d'0}(s,t) \qquad (30)$$

while time reversal invariance gives:

$$f^s_{\lambda_c'0,\lambda_d'0}(s,t) = (-1)^{\lambda_d'-\lambda_c'} f^s_{\lambda_d'0,\lambda_c'0}(s,t) \qquad (31)$$

At t=0 it proves convenient to define:

$$\nu = \tfrac{1}{2}(s - M^2) \qquad (32)$$

Under crossing from the s-channel to the u-channel $\nu \longleftrightarrow -\nu$ and the amplitudes defined in Eq. (23) have the simple property:

$$\bar{f}^{t(I)}_{\lambda_c\lambda_d,00}(-\nu) = (-1)^{|\lambda_c-\lambda_d|+I} \bar{f}^{t(I)}_{\lambda_c\lambda_d,00}(\nu) \qquad (33)$$

where I is the total isotopic spin in the t-channel. From Eqs. (23) and (28) we also note that the amplitudes $\bar{f}^t(\nu,t)$ which are free from kinematic singularities in ν are obtained from the amplitudes $f^t(\nu,t)$ by dividing by:

$$(\cos \tfrac{1}{2}\theta_t)^{|\lambda+\mu|} (\sin \tfrac{1}{2}\theta_t)^{|\lambda-\mu|} = (\tfrac{1}{2} \sin \theta_t)^{|\mu|} \propto \nu^{|\mu|} = \nu^{\Delta} ,$$

since $\lambda = 0$ for pion-hadron scattering.

If an amplitude $g(\nu)$ is even under crossing and proportional to ν^β for large ν it will, at best:

(a) Satisfy a once-subtracted dispersion relation in ν if $\beta \geq 0$.

(b) Satisfy an unsubtracted dispersion relation if $0 > \beta \geq -2$.

(c) Satisfy a superconvergence relation of the form:

$$\int_0^\infty \nu^N \mathrm{Im}\, f(\nu) d\nu = 0 \qquad (34)$$

where N is an odd integer, if $-(N + 1) > \beta \geq -(N + 3)$.

An odd amplitude $(h(\nu) = -h(-\nu))$ which is proportional to ν^β for large ν will, at most:

(a) Satisfy a once-subtracted dispersion relation in ν if $\beta \geq 1$.

(b) Satisfy an unsubtracted dispersion relation if $1 > \beta \geq -1$.

(c) Satisfy a superconvergence relation (Eq. (34)) with even N if: $-(N + 1) > \beta \geq -(N + 3)$.

Assuming the asymptotic behavior given by Eq. (19) and using the crossing relation Eq. (33) we conclude:

1. If $\alpha_2(0) < 0$ all the I=2 t-channel helicity amplitudes may satisfy unsubtracted dispersion relations in ν at t=0. All I=2 t-channel helicity amplitudes for pion-hadron scattering having $\Delta = |\lambda_c - \lambda_d| \geq 1$ obey superconvergence relations of the form:

$$\int_0^\infty \nu^{\Delta-1} \mathrm{Im}\, \bar{f}^{t(2)}(\nu) d\nu = 0 \qquad (35)$$

2. If $\alpha_1(0) < 1$ all the I=1 t-channel helicity amplitudes for pion-hadron elastic scattering obey unsubtracted dispersion relations. Amplitudes having $\Delta \geq 2$ obey superconvergence relations of the form:

$$\int_0^\infty \nu^{\Delta-2} \mathrm{Im}\, \bar{f}^{t(1)}(\nu) d\nu = 0 \qquad (36)$$

3. If $\alpha_0(0) = 1$ all I=0, $\Delta \geq 1$ t-channel helicity amplitudes obey unsubtracted dispersion relations. All amplitudes with $\Delta \geq 3$ obey superconvergence relations of the form:

$$\int_0^\infty \nu^{\Delta-3} \mathrm{Im}\, \bar{f}^{t(0)}(\nu) d\nu = 0 \tag{37}$$

C. Helicity Crossing Relations for Superconvergent Pion-Hadron Amplitudes

We are now fully equipped for writing down the crossing relations of those t-channel helicity amplitudes for forward pion-hadron elastic scattering, which satisfy superconvergence relations of the form (35)-(37). In particular, we discuss the cases where the target hadron has spin $J = \frac{1}{2}$, 1, $\frac{3}{2}$ and 2 and consider all amplitudes having $\Delta \geq 1$. For J = 0 we obviously have $\Delta = 0$ and no superconvergence relations exist.

1. $0 + \frac{1}{2} \rightarrow 0 + \frac{1}{2}$

There are two independent s-channel amplitudes which we choose as $f^s_{\frac{1}{2}0,\frac{1}{2}0}$ and $f^s_{-\frac{1}{2}0,\frac{1}{2}0}$ and two independent t-channel amplitudes: $f^t_{\frac{1}{2}\frac{1}{2},00}$ (with $\Delta = 0$) and $f^t_{-\frac{1}{2}\frac{1}{2},00}$ (with $\Delta = 1$). Only $(f^t_{-\frac{1}{2}\frac{1}{2},00}(\nu,t)/(\nu\sqrt{-t}))^{29}$ may be superconvergent for I=2. As $t \rightarrow 0$:

$$f^t_{-\frac{1}{2}\frac{1}{2},00} \xrightarrow[t \rightarrow 0]{} \frac{\sqrt{-t}}{s - M^2} \left[\frac{s + M^2}{2M} \bar{f}^s_{\frac{1}{2}0,\frac{1}{2}0} + \sqrt{s}\, \bar{f}^s_{-\frac{1}{2}0,\frac{1}{2}0}\right] \tag{38}$$

In terms of the usual invariant amplitudes for π-N scattering:

$$\left[\frac{f^t_{-\frac{1}{2}\frac{1}{2},00}(\nu,t)}{\nu\sqrt{-t}}\right]_{t=0} = \frac{1}{2M^2} B(\nu,0) \tag{39}$$

where B is defined by[30]:

$$T(\nu,t) = \bar{u}(p_d)[A(\nu,t) - i\,\gamma\cdot(\frac{p_a + p_b}{2})B(\nu,t)]u(p_c) \qquad (40)$$

2. 0 + 1 → 0 + 1

There are four independent helicity amplitudes in each channel. We choose them to be $f^s_{10,10}$; $f^s_{00,00}$; $f^s_{10,00}$ and $f^s_{10,-10}$ in the s-channel and $f^t_{11,00}$, $f^t_{00,00}$ (with $\Delta = 0$), $f^t_{01,00}$ (with $\Delta = 1$), and $f^t_{-11,00}$ (with $\Delta = 2$) in the t-channel. $[f^t_{01,00}(\nu,t)/(\nu\sqrt{-t})]_{t=0}$ is superconvergent for I=2 and satisfies:

$$f^t_{01,00} \xrightarrow[t\to 0]{} \frac{\sqrt{-t}}{2\sqrt{2}\,M}\left[\frac{s+M^2}{s-M^2}\bar{f}^s_{10,10} + \frac{s+M^2}{s-M^2}\bar{f}^s_{00,00} - \frac{2M\sqrt{2s}}{s-M^2}\bar{f}^s_{10,00}\right] \qquad (41)$$

$[f^t_{-11,00}(\nu,0)/\nu^2]$[29] is superconvergent for I=1,2 and obeys:

$$f^t_{-11,00} \xrightarrow[t\to 0]{} \tfrac{1}{2}\,(\bar{f}^s_{10,10} - \bar{f}^s_{00,00}) \qquad (42)$$

The relation between the $\Delta \geq 1$ t-channel helicity amplitudes and the perturbative invariant amplitudes is:

$$\left[\frac{f^t_{01,00}(\nu,t)}{\nu\sqrt{-t}}\right]_{t=0} = \frac{\sqrt{2}}{8M}\left[\frac{2\nu}{M^2}A(\nu,0) + B(\nu,0)\right]$$

$$\frac{1}{\nu^2}f^t_{-11,00}(\nu,0) = -\frac{1}{2M^2}A(\nu,0) \qquad (43)$$

where A, B are defined by[2]:

$$T(\nu,t) = (\epsilon_c\cdot P)(\epsilon_d\cdot P)A(\nu,t) + \frac{1}{2}\left[(\epsilon_c\cdot P)(\epsilon_d\cdot Q) + (\epsilon_d\cdot P)(\epsilon_c\cdot Q)\right] B(\nu,t) +$$

$$+ (\epsilon_c\cdot Q)(\epsilon_d\cdot Q)C_1(\nu,t) + (\epsilon_c\cdot\epsilon_d)C_2(\nu,t), \qquad (44)$$

ϵ_c, ϵ_d are the polarizations of the spin-one particles and $P = \frac{1}{2}(p_a + p_b)$, $Q = \frac{1}{2}(p_c + p_d)$.

3. $0 + \frac{3}{2} \to 0 + \frac{3}{2}$

There are six independent amplitudes. In the s-channel we choose $f^s_{\frac{3}{2}0,\frac{3}{2}0}$; $f^s_{\frac{1}{2}0,\frac{3}{2}0}$; $f^s_{-\frac{1}{2}0,\frac{3}{2}0}$; $f^s_{-\frac{3}{2}0,\frac{3}{2}0}$; $f^s_{\frac{1}{2}0,\frac{1}{2}0}$ and $f^s_{-\frac{1}{2}0,\frac{1}{2}0}$. In the t-channel:

$f^t_{\frac{3}{2}\frac{3}{2},00}$ and $f^t_{\frac{1}{2}\frac{1}{2},00}$ (with $\Delta = 0$); $f^t_{\frac{1}{2}\frac{3}{2},00}$ and $f^t_{-\frac{1}{2}\frac{1}{2},00}$ (with $\Delta = 1$);

$f^t_{-\frac{1}{2}\frac{3}{2},00}$ (with $\Delta = 2$), and $f^t_{-\frac{3}{2}\frac{3}{2},00}$ (with $\Delta = 3$). The $\Delta = 1$ amplitudes

$[f^t_{\frac{1}{2}\frac{3}{2},00}(\nu,t)/(\nu\sqrt{-t})]_{t=0}$ and $[f^t_{-\frac{1}{2}\frac{1}{2},00}(\nu,t)/(\nu\sqrt{-t})]_{t=0}$ obey superconvergence relations for I=2 and have the s-channel expansions:

$$f^t_{\frac{1}{2}\frac{3}{2},00} \xrightarrow[t\to 0]{} \frac{\sqrt{-t}}{s - M^2}\left[\frac{\sqrt{3}\,(s + M^2)}{8M}\,\bar{f}^s_{\frac{3}{2}0,\frac{3}{2}0} + \frac{\sqrt{s}}{2}\,\bar{f}^s_{\frac{1}{2}0,\frac{3}{2}0} + \right.$$

$$\left. + \frac{3\sqrt{3}\,(s + M^2)}{8M}\,\bar{f}^s_{\frac{1}{2}0,\frac{1}{2}0} + \frac{\sqrt{3s}}{4}\,\bar{f}^s_{-\frac{1}{2}0,\frac{1}{2}0}\right] \qquad (45)$$

and

$$f^t_{-\frac{1}{2}\frac{1}{2},00} \xrightarrow[t\to 0]{} \frac{\sqrt{-t}}{s - M^2}\left[\frac{3(s + M^2)}{8M}\,\bar{f}^s_{\frac{3}{2}0,\frac{3}{2}0} + \frac{\sqrt{3s}}{2}\,\bar{f}^s_{\frac{1}{2}0,\frac{3}{2}0} + \right.$$

$$\left. + \frac{5(s + M^2)}{8M}\,\bar{f}_{\frac{1}{2}0,\frac{1}{2}0} + \frac{\sqrt{s}}{4}\,\bar{f}^s_{-\frac{1}{2}0,\frac{1}{2}0}\right] \qquad (46)$$

$[f^t_{-\frac{1}{2}\frac{3}{2},00}(\nu,0)/\nu^2]$ is superconvergent for I=1,2 and obeys:

$$f^t_{-\frac{1}{2}\frac{3}{2},00} \xrightarrow[t\to 0]{} \frac{\sqrt{3}}{4}\left[\bar{f}^s_{\frac{3}{2}0,\frac{3}{2}0} - \bar{f}^s_{\frac{1}{2}0,\frac{1}{2}0}\right] \tag{47}$$

Lastly $[f^t_{-\frac{3}{2}\frac{3}{2},00}(\nu,t)/(\nu^3\sqrt{-t})]_{t=0}$ [29] is superconvergent for I=0,1,2 and has the s-channel expansion:

$$f^t_{-\frac{3}{2}\frac{3}{2},00} \xrightarrow[t\to 0]{} \frac{\sqrt{-t}}{s-M^2}\left[\frac{3(s+M^2)}{8M}\bar{f}^s_{\frac{3}{2}0,\frac{3}{2}0} - \frac{3(s+M^2)}{8M}\bar{f}^s_{\frac{1}{2}0,\frac{1}{2}0} + \frac{\sqrt{3s}}{2}\bar{f}^s_{\frac{1}{2}0,\frac{3}{2}0} - \frac{3\sqrt{s}}{4}\bar{f}^s_{-\frac{1}{2}0,\frac{1}{2}0}\right] \tag{48}$$

4. $0 + 2 \to 0 + 2$

In this case there are nine independent s-channel amplitudes, which we take to be $f^s_{20,20}$; $f^s_{10,20}$; $f^s_{00,20}$; $f^s_{-10,20}$; $f^s_{-20,20}$; $f^s_{10,10}$; $f^s_{00,10}$; $f^s_{-10,10}$, and $f^s_{00,00}$. Our corresponding t-channel amplitudes are $f^t_{22,00}$, $f^t_{11,00}$ and $f^t_{00,00}$ (with $\Delta = 0$); $f^t_{12,00}$ and $f^t_{01,00}$ ($\Delta = 1$); $f^t_{02,00}$ and $f^t_{-1+1,00}$ ($\Delta = 2$); $f^t_{-12,00}$ ($\Delta = 3$) and $f^t_{-22,00}$ ($\Delta = 4$).

The possible superconvergence relations are as follows:

$[f^t_{12,00}(\nu,t)/(\nu\sqrt{-t})]_{t=0}$ and $[f^t_{01,00}(\nu,t)/(\nu\sqrt{-t})]_{t=0}$ satisfy superconvergence relations for I=2 and have the s-channel helicity decompositions:

$$f^t_{12,00} \xrightarrow[t\to 0]{} \frac{\sqrt{-t}}{s-M^2}\left[\frac{s+M^2}{8M}\bar{f}^s_{20,20} + \frac{s+M^2}{2M}\bar{f}^s_{10,10} + \frac{3(s+M^2)}{8M}\bar{f}^s_{00,00} + \frac{\sqrt{s}}{4}\bar{f}^s_{10,20} + \frac{\sqrt{6\,s}}{4}\bar{f}^s_{00,10}\right] \tag{49}$$

and

$$f^t_{01,00} \xrightarrow[t \to 0]{} \frac{\sqrt{-t}}{s - M^2} \left[\frac{\sqrt{6}(s + M^2)}{8M} \bar{f}^s_{20,20} + \frac{\sqrt{6}(s + M^2)}{4M} \bar{f}^s_{10,10} \right.$$

$$\left. + \frac{\sqrt{6}(s + M^2)}{8M} \bar{f}^s_{00,00} + \frac{\sqrt{6\, s}}{4} \bar{f}^s_{10,20} + \frac{\sqrt{s}}{2} \bar{f}^s_{00,10} \right] \tag{50}$$

$[f^t_{-11,00}(\nu,0)/\nu^2]$ and $[f^t_{02,00}(\nu,0)/\nu^2]$ satisfy superconvergence relations for I=1,2. Their expansion in terms of s-channel helicity amplitudes is:

$$f^t_{-11,00} \xrightarrow[t \to 0]{} \tfrac{1}{2} (\bar{f}^s_{20,20} - \bar{f}^s_{10,10}) \tag{51}$$

and

$$f^t_{02,00} \xrightarrow[t \to 0]{} \frac{\sqrt{6}}{8} (\bar{f}^s_{20,20} - \bar{f}^s_{00,00}) \tag{52}$$

$[f^t_{-12,00}(\nu,t)/(\nu^3\sqrt{-t})]_{t=0}$ is superconvergent for I=0,1,2 and has the s-channel decomposition:

$$f^t_{-12,00} \xrightarrow[t \to 0]{} \frac{\sqrt{-t}}{s - M^2} \left[\frac{3(s + M^2)}{8M} \bar{f}^s_{20,20} - \frac{3(s + M^2)}{8M} \bar{f}^s_{00,00} \right.$$

$$\left. + \frac{3\sqrt{s}}{4} \bar{f}^s_{10,20} - \frac{\sqrt{6\, s}}{4} \bar{f}^s_{00,10} \right] \tag{53}$$

Finally, $[f^t_{-22,00}(\nu,0)/\nu^4]$[29] superconverges for I=0,1,2 and is given by:

$$f^t_{-22,00} \xrightarrow[t \to 0]{} \frac{1}{8} \bar{f}^s_{20,20} - \frac{1}{2} \bar{f}^s_{10,10} + \frac{3}{8} \bar{f}^s_{00,00} \tag{54}$$

The asymptotic behavior, crossing properties and type of non-trivial dispersion or superconvergence relation holding for the kinematic-singularity-free amplitudes discussed above are summarized in Table I.

IV. Strong Interaction Sum Rules for Forward Pion-Hadron Scattering

We now proceed to write down the complete set of t=0 strong interaction sum rules for elastic pion-hadron scattering, using the assumptions of Section II.

A. Charge Algebra Sum Rules

The t=0 charge algebra sum rules are derived from the commutators:

$$[Q_5^+, Q_5^-] = 2Q^z \tag{55}$$

$$[D^+, Q_5^+] = 0 \tag{56}$$

which follow from Eqs. (7) and (12), respectively. If we insert these commutators between hadron states moving with equal and infinite momenta and having identical helicities λ, we obtain sum rules for the amplitudes $f^s_{\lambda 0,\lambda 0}(\nu,0)$. In particular the commutator (55), supplemented by PCAC, leads to the (generalized) Adler-Weisberger[1] sum rule (for an I=1 hadron):

$$\frac{2}{\pi}\int_0^\infty \frac{d\nu}{\nu^2}\,\mathrm{Im}\,\bar{f}^{s(1)}_{\lambda 0,\lambda 0}(\nu,0) = \frac{8}{f_\pi^2} \tag{57}$$

where f_π is defined by Eqs. (16)-(18). Eq. (57) can be interpreted as a forward dispersion relation for the amplitude for scattering of a massless pion on a hadron with spin J and s-channel helicity λ, evaluated at threshold and supplemented by a low energy theorem which states: <u>The threshold value for $f^{s(1)}_{\lambda 0,\lambda 0}$ is independent of λ and equals $8m_\pi/f_\pi^2$.</u> This λ-independence becomes obvious when we recall that at threshold only s-wave scattering contributes, and as long as ℓ=0 it is clear that the scattering is independent of the polarization of the target. The actual magnitude of the threshold amplitude is, of course, determined by PCAC and the commutator (55).

Eq. (57) represents J+1 (J+$\frac{1}{2}$) independent sum rules for the scattering of pions on a target having integer (half-integer) spin J, with total isotopic spin I=1 in the t-channel. The convergence of the integral in Eq. (57) is guaranteed if $\alpha_1(0) < 1$.

A second set of forward sum rules is obtained for scattering with total isospin I=2 in the t-channel. The commutator (56), supplemented by PCAC, leads to:

$$\int_0^\infty \frac{d\nu}{\nu} \text{ Im } \bar{f}^{s(2)}_{\lambda 0,\lambda 0}(\nu,0) = 0 \tag{58}$$

Here, again, the sum rule can be interpreted as a statement that $\bar{f}^{s(2)}_{\lambda 0,\lambda 0}$ satisfies an unsubtracted dispersion relation in ν and that its value at threshold vanishes[31]. (In other words, the s-wave scattering length for π-hadron scattering with I=2 in the t-channel vanishes.) The convergence of Eq. (58) follows from the assumption $\alpha_2(0) < 0$.

We do not obtain any sum rules for I=0 exchange amplitudes of the form $f^{s(0)}_{\lambda 0,\lambda 0}$, since at least one subtraction is then required in the dispersion relations[32].

B. Superconvergence Sum Rules: Class I and Class II

From the analysis of Section III where the relations of the t-channel to s-channel helicity amplitudes were given for the scattering of massless pions on spin $J \leq 2$ targets, we see explicitly that the amplitudes satisfying superconvergence relations fall into two categories. Those that correspond to even $\Delta = \max\left\{|\lambda|, |\mu|\right\}$ are related to s-channel helicity amplitudes which contribute to the forward (t=0) scattering[33], while those that correspond to odd values of Δ are proportional to $\sqrt{-t}$ as $t \to 0$. The coefficient of $\sqrt{-t}$ generally involves helicity amplitudes which, in principle, can only be obtained by extrapolation from $t \neq 0$ to the forward direction[34]. We find it convenient to divide the superconvergence sum rules for amplitudes at t=0 into two classes[4]: Class I sum rules are those involving "pure" t=0 amplitudes and the Class II sum rules will be those in which we are forced to extrapolate to t=0, and the sum rules are therefore not directly related to the physical forward scattering amplitude[35].

The superconvergence relations for pion-hadron scattering are not necessarily independent of the charge algebra sum rules[36]. The crossing relations between the t-channel and s-channel helicity amplitudes which we have presented in Section III-C enable us to rewrite the superconvergence relations on the t-channel helicity amplitudes in terms of integrals over the s-channel amplitudes $\bar{f}^{s}_{\lambda_c 0, \lambda_d 0}(\nu, 0)$, some of which appear in the charge algebra sum rules. When we do so, we find the following general results[4]:

(1) All t-channel helicity amplitudes having even values of Δ are, at t=0, linear combinations of s-channel amplitudes of the form $f^{s}_{\lambda 0, \lambda 0}(\nu, 0)$ and therefore correspond to Class I sum rules[35]. Moreover, for every even $\Delta \neq 0$ the sum of the coefficients of the various $f^{s}_{\lambda 0, \lambda 0}$ components vanishes.

(2) Given a t-channel helicity amplitude for pion-hadron scattering with $\Delta = 2$ the (Class I) superconvergence relations that it satisfies (for I=1,2) are linear combinations of differences of charge algebra sum rules.

(3) All Class II superconvergence relations are on amplitudes corresponding to odd values of Δ and are not related to the charge algebra sum rules.

(4) Whenever a t-channel helicity amplitude satisfies more than one superconvergence relation (and therefore has $\Delta \geq 3$), all relations of the form

$$\int_0^{\infty} \nu^N \operatorname{Im} f(\nu) d\nu = 0 \qquad (59)$$

with $N \geq 2$ are, at $t=0$, linear combinations of other superconvergence relations with smaller values of Δ. For example, for the scattering of pions on spin 2 hadrons the superconvergence relation on the amplitude $[f^{t(1)}_{-22,00}(\nu,0)/\nu^4]$:

$$\int_0^\infty \nu^2 \mathrm{Im}[f^{t(1)}_{-22,00}(\nu,0)/\nu^4]\, d\nu = 0 \tag{60}$$

is a linear combination of sum rules on $[f^{t(1)}_{02,00}(\nu,0)/\nu^2]$ and $[f^{t(1)}_{-11,00}(\nu,0)/\nu^2]$:

$$\begin{aligned} &\int_0^\infty \mathrm{Im}\,[f^{t(1)}_{02,00}(\nu,0)/\nu^2]d\nu = 0 \\ &\int_0^\infty \mathrm{Im}\,[f^{t(1)}_{-11,00}(\nu,0)/\nu^2]d\nu = 0. \end{aligned} \tag{61}$$

This can be easily verified from the crossing relations of Section III-C.

The general proof of these statements is straightforward. In order to prove the results (1) and (3) we consider the crossing relation Eq. (27) for pion-hadron (spin J) elastic scattering:

$$f^t_{\lambda_c\lambda_d,00} = \sum_{\lambda_c',\lambda_d'} d^J_{\lambda_c'\lambda_c}(X_c)d^J_{\lambda_d'\lambda_d}(X_d)f^s_{\lambda_c'0,\lambda_d'0} \tag{62}$$

As $t \to 0$, $X_d \to \pi - X_c \to \frac{\pi}{2} + \frac{\sqrt{-t}}{2M}\frac{s+M^2}{s-M^2}$ and $f^s_{\lambda_c'0,\lambda_d'0} \to 0$ unless $\lambda_c' = \lambda_d' = \lambda'$. On using parity conservation to restrict the summation over λ' to values of $\lambda' \geq 0$, we have

$$\begin{aligned} f^t_{\lambda_c\lambda_d,00} \xrightarrow[t\to 0]{} \sum_{\lambda'\geq 0} &[d^J_{\lambda'\lambda_c}(X_c)d^J_{\lambda'\lambda_d}(\pi-X_c) \\ &+ d^J_{-\lambda'\lambda_c}(X_c)d^J_{-\lambda'\lambda_d}(\pi-X_c)]f^s_{\lambda'0,\lambda'0} \end{aligned} \tag{63}$$

Now, using $d^J_{M'M}(X) = (-1)^{J+M'} d^J_{M'-M}(\pi-X)$, we find

$$f^t_{\lambda_c\lambda_d,00} \xrightarrow[t \to 0]{} \sum_{\lambda' \geq 0} d^J_{\lambda'\lambda_c}(\tfrac{\pi}{2}) d^J_{\lambda'\lambda_d}(\tfrac{\pi}{2})[1+(-1)^{\Delta}] f^s_{\lambda'0,\lambda'0} + O(\sqrt{-t}) \tag{64}$$

where $\Delta = |\mu| = |\lambda_c - \lambda_d|$. Thus for even values of Δ, $f^t_{\lambda_c\lambda_d,00}$ is a linear combination of $f^s_{\lambda'0,\lambda'0}$ amplitudes with coefficients $2d^J_{\lambda'\lambda_c}(\frac{\pi}{2})\, d^J_{\lambda'\lambda_d}(\frac{\pi}{2})$. The sum of these coefficients vanishes since for even $\Delta > 0$:

$$\sum_{\lambda' \geq 0} d^J_{\lambda'\lambda_c}(\tfrac{\pi}{2})\, d^J_{\lambda'\lambda_d}(\tfrac{\pi}{2}) = 0. \tag{65}$$

Eq. (64) also shows generally that only even Δ amplitudes contribute to the scattering at t=0 and therefore satisfy Class I superconvergence relations, while odd Δ amplitudes vanish like $\sqrt{-t}$ as $t \to 0$ and satisfy Class II superconvergence relations.

The truth of assertion (2) that $\Delta = 2$ amplitudes for pion-hadron scattering satisfy superconvergence relations which are linear combinations of differences of charge algebra sum rules is easily established by comparing the superconvergence relations for $\Delta = 2$, I = 1 and 2 amplitudes given in Section III-B with the charge algebra sum rules given in IV-A and using Eq. (64) to relate one to the other at t=0.

Statement (4) is more difficult to prove. For hadrons with spin $J \leq 2$ it can be verified explicitly from the relation of t-channel to s-channel helicity amplitudes given in Section III-C. In general, the required relation(s) on t-channel helicity amplitudes at t=0 needed to prove (4) can be established by considering an s-channel amplitude $f^s_{\lambda_c'0,\lambda_d'0}$, with $\lambda_c' \neq \lambda_d'$ so that it vanishes as $t \to 0$, and relating it by the inverse of the crossing relation (27) to a linear conbination of t-channel amplitudes which must also vanish in the forward direction[37].

C. Class I Sum Rules as Threshold Relations on Forward Amplitudes

We have just seen that Class I superconvergence relations on amplitudes with $\Delta = 2$ are related to differences of charge algebra sum rules, which in turn can be interpreted as forward dispersion relations evaluated at threshold and supplemented by a low energy theorem. Reversing the line of argument, it is apparent that Class I superconvergence relations for forward amplitudes could have been "discovered" as follows: we write unsubtracted dispersion relations for the forward pion-hadron scattering amplitude[38] evaluated at the threshold value of $\nu(=\nu_0)$:

$$\bar{f}^{s}_{\lambda 0,\lambda 0}(\nu=\nu_0, t=0) = \frac{1}{\pi} \int_{-\infty}^{+\infty} \frac{d\nu'}{\nu'-\nu_0} \,\mathrm{Im}\, \bar{f}^{s}_{\lambda 0,\lambda 0}(\nu', t=0) \tag{66}$$

Now, at threshold $\bar{f}^{s}_{\lambda 0,\lambda 0}$ is independent of λ and essentially equal to the s-wave scattering length, while $\mathrm{Im}\,\bar{f}^{s}_{\lambda 0,\lambda 0}$ has the expansion[39]

$$\mathrm{Im}\,\bar{f}^{s}_{\lambda 0,\lambda 0}(\nu, t=0) = q(a_\lambda + b_\lambda q^2 + c_\lambda q^4 + \ldots), \tag{67}$$

where q is the center-of-mass three-momentum satisfying:

$$q^2 = \frac{(\nu+\nu_0)}{s}\,(\nu-\nu_0) \tag{68}$$

and $a_\lambda, b_\lambda, c_\lambda, \ldots$ are constants. a_λ is independent of λ and is proportional to the s-wave cross-section at threshold[40]. If we form the amplitude

$$g(\nu) = \frac{1}{\nu-\nu_0} [\bar{f}^s_{\lambda_1 0,\lambda_1 0}(\nu,t=0) - \bar{f}^s_{\lambda_2 0,\lambda_2 0}(\nu,t=0)], \tag{69}$$

where λ_1 and λ_2 are two different hadron helicities, then $g(\nu)$ is free of kinematic singularities since the only place such a singularity might occur is threshold and we are assured that Im $g(\nu)$ is finite there by Eqs. (67)-(68) and the fact that $a_{\lambda_1} = a_{\lambda_2}$. The amplitude $g(\nu)$ then satisfies the superconvergence relation

$$\frac{1}{\pi} \int_{-\infty}^{+\infty} d\nu \, \mathrm{Im}\, g(\nu) = 0, \tag{70}$$

obtained by subtracting the two versions of Eq. (66) with helicity λ_1 and λ_2, respectively. In the case of massless pion-hadron scattering $\nu_0 = 0$ and the algebra of currents predicts the actual value of the s-wave scattering length $\bar{f}^s_{\lambda 0,\lambda 0}(\nu=\nu_0,t=0)$. Note, however, that the superconvergence sum rule (70) is independent of the particular value of a_λ and is correct for on-mass-shell pions.

For hadrons of spin $J \geq 2$ there exist combinations of forward amplitudes such that not only do the a_λ terms in Eq. (67) cancel, but also the b_λ terms which depend on the s-wave effective range and the p-wave scattering lengths (and are λ dependent). Such a combination of amplitudes is divisible by $q^4 \propto (\nu-\nu_0)^2$ and still free from kinematic singularities. An example of this occurs in pion-hadron (spin 2) scattering where[41]

$$f^t_{-22,00}(\nu,0) = \frac{1}{8}\bar{f}^s_{20,20}(\nu,0) - \frac{1}{2}\bar{f}^s_{10,10}(\nu,0) + \frac{3}{8}\bar{f}^s_{00,00}(\nu,0) \tag{71}$$

is divisible by $(\nu-\nu_0)^2$.

Thus by careful enough analysis of s-channel forward amplitudes and their relations at threshold one can derive all the Class I t=0 superconvergence relations considered in this paper by starting from unsubtracted dispersion relations for the purely forward scattering amplitudes and forming appropriate differences of them. Combinations of sum rules which eliminate the s-wave parameters lead to the superconvergence relations on amplitudes with $\Delta = 2$, while eliminating p-wave, d-wave,... parameters leads to the superconvergence relations on amplitudes with $\Delta = 4,6, \ldots.$

D. Counting of Independent Sum Rules for Pion-Hadron Scattering

From the preceding analysis it is evident that the following superconvergence relations are obtained for the scattering of pions on $J \leq 2$ targets:

(1) $\Delta = 1$, $I = 2$ t-channel amplitudes satisfy the Class II superconvergence relation[29],

$$\int_0^\infty [\mathrm{Im}\, f^{t(2)}_{(\lambda-1)\lambda,00}(\nu,t)/(\nu\sqrt{-t})]_{t=0}\, d\nu = 0. \tag{72}$$

(2) $\Delta = 2$, $I = 1$ and $I = 2$ t-channel amplitudes obey the Class I superconvergence relations

$$\int_0^\infty \mathrm{Im}[f^{t(1)}_{(\lambda-2)\lambda,00}(\nu,0)/\nu^2]\, d\nu = 0 \tag{73}$$

and

$$\int_0^\infty \nu \mathrm{Im}[f^{t(2)}_{(\lambda-2)\lambda,00}(\nu,0)/\nu^2]\, d\nu = 0 \tag{74}$$

which are differences of the charge algebra sum rules (57) and (58), respectively.

(3) $\Delta = 3$, I = 0, 1 and 2 t-channel amplitudes obey the independent Class II superconvergence relations

$$\int_0^\infty \mathrm{Im}[f^{t(0)}_{(\lambda-3)\lambda,00}(\nu,t)/(\nu^3\sqrt{-t})]_{t=0}\, d\nu = 0, \tag{75}$$

$$\int_0^\infty \nu\ \mathrm{Im}[f^{t(1)}_{(\lambda-3)\lambda,00}(\nu,t)/(\nu^3\sqrt{-t})]_{t=0} d\nu = 0, \tag{76}$$

and

$$\int_0^\infty \mathrm{Im}[f^{t(2)}_{(\lambda-3)\lambda,00}(\nu,t)/(\nu^3\sqrt{-t})]_{t=0}\, d\nu = 0, \tag{77}$$

as well as the relation for I=2,

$$\int_0^\infty \nu^2 \mathrm{Im}[f^{t(2)}_{(\lambda-3)\lambda,00}(\nu,t)/(\nu^3\sqrt{-t})]_{t=0} d\nu = 0, \tag{78}$$

which is a linear combination of $\Delta = 1$, I = 2 sum rules.

(4) $\Delta = 4$, I = 0, 1, and 2 t-channel amplitudes obey independent Class I superconvergence relations of the form:

$$\int_0^\infty \nu\ \mathrm{Im}[f^{t(0)}_{(\lambda-4)\lambda,00}(\nu,0)/\nu^4]\ d\nu = 0, \tag{79}$$

$$\int_0^\infty \mathrm{Im}\ [f^{t(1)}_{(\lambda-4)\lambda,00}(\nu,0)/\nu^4]\ d\nu = 0 \tag{80}$$

and

$$\int_0^\infty \nu \, \mathrm{Im}[f^{t(2)}_{(\lambda-4)\lambda,00}(\nu,0)/\nu^4] \, d\nu = 0, \tag{81}$$

while the relations

$$\int_0^\infty \nu^2 \mathrm{Im}[f^{t(1)}_{(\lambda-4)\lambda,00}(\nu,0)/\nu^4] \, d\nu = 0 \tag{82}$$

and

$$\int_0^\infty \nu^3 \mathrm{Im}[f^{t(2)}_{(\lambda-4)\lambda,00}(\nu,0)/\nu^4] \, d\nu = 0 \tag{83}$$

are linear combinations of $\Delta = 2$ sum rules.

The case of $\pi\rho$ scattering furnishes a good example of the counting of independent sum rules for a given process[4]. There are two charge algebra sum rules arising from the commutator $[Q_5^+,Q_5^-] = 2Q^z$ for the cases of target ρ-meson helicity 0 and 1, respectively. Likewise, there are two sum rules arising from the commutator $[D^+,Q_5^+] = 0$. With respect to superconvergence relations, there is one t-channel amplitude with $\Delta = 1$, $[f^{t(2)}_{01,00}(\nu,t)/(\nu\sqrt{-t})]_{t=0}$, which satisfies a Class II superconvergence relation, Eq. (72), and which is essentially the superconvergence relation on the $B^{(2)}(\nu,0)$ amplitude discussed in Refs. 2 and 4. The only $\Delta = 2$ amplitude, $f^t_{-11,00}(\nu,0)$ obeys the Class I superconvergence relations, Eqs. (73) and (74), which are differences of the charge algebra sum rules. Thus we arrive at five independent sum rules[4]: two arising from the commutator $[Q_5^+,Q_5^-] = 2Q^z$, two arising from $[D^+,Q_5^+] = 0$, and one Class II superconvergence relation on the $B^{(2)}(\nu,0)$ amplitude of Ref. 2.

This way of counting already includes the two Class I superconvergence relations which could have been obtained either as differences of the charge algebra sum rules or directly from the asymptotic behavior of the $\Delta = 2$ amplitudes. The counting of independent sum rules at t=0 for other pion-hadron processes is just as simple as in the $\pi\rho$ case, and is summarized in Table II.

V. Representations of the Algebra of Charges and Class I Superconvergence Relations

We have shown that a large class of superconvergence relations for pion-hadron scattering can be expressed as linear combinations of charge algebra sum rules. The algebraic structure of such relations can be analyzed in terms of the infinite momentum classification[42] of particles into representations of chiral $SU(2) \times SU(2)$. In particular, we find that, with the exception of a few "accidents", the only superconvergence relations which lead to the so-called "SU(6)-results"[43] are those which can be expressed in terms of charge algebra sum rules. On the other hand, we have already pointed out that the Class II superconvergence relations are not related to charge algebra sum rules, and we will show in Section VI that, in general, there is no reason for them to lead, in some saturation limit, to "SU(6) results".

In this section we consider sets of "pure" t=0 sum rules (both charge algebra sum rules and Class I superconvergence relations) and show that they can be properly described in terms of infinite momentum representations of $SU(2) \times SU(2)$. We also show that our assumptions with respect to I=2 t-channel amplitudes lead to relations among the masses of the particles which are assumed to saturate the sum rules.

A. Saturated Charge Algebra Sum Rules and the Infinite Momentum Classification

The axial charges Q_5^i are generators of the chiral $SU(2) \times SU(2)$ algebra and therefore have non-vanishing matrix elements only between states belonging to the same irreducible representation (IR) of the algebra. We have good reasons to believe that at $p_z \to \infty$ the $SU(2) \times SU(2)$ classification of single particle states is the simplest[44], and that the charge algebra sum rules are approximately saturated by a relatively small number of such states. In particular, if one assumes that the $SU(2) \times SU(2)$ classification of particles at infinite momentum is the one implied by a simple SU(6)-type picture, one finds that the complete set of intermediate states allowed to contribute to sum rules of the form (57) is given by the 35 mesons or the 56 baryons, as long as the target hadron belongs to one of these groups of states[45]. Alternately, we find that if, without referring explicitly to the representations, we saturate a set of Adler-Weisberger type sum rules by a set of states which "happens" to include only members of the 35 mesons or 56 baryons, and then solve the equations for the matrix elements of Q_5 (or for the pionic coupling constants), we obtain "SU(6) results" such as $g_A = \frac{5}{3}$, $4g^2_{\rho\pi\pi} = m^2_\rho g^2_{\omega\rho\pi}$, etc. A typical example is the case of sum rules for π-ρ scattering. We have two sum rules of the form (57):

$$\frac{2}{\pi} \int_0^\infty \frac{d\nu}{\nu^2} \operatorname{Im} \bar{f}^{s(1)}_{10,10}(\nu,0) = \frac{8}{f_\pi^2} \tag{84}$$

$$\frac{2}{\pi} \int_0^\infty \frac{d\nu}{\nu^2} \operatorname{Im} \bar{f}^{s(1)}_{00,00}(\nu,0) = \frac{8}{f_\pi^2} \tag{85}$$

If we now assume that the only s-channel resonances which contribute to the sum rules are the Y=0, G = -1 pseudoscalar and vector mesons (π,ω,φ) we obtain the equations[46]:

$$g^2_{\omega\rho\pi} + g^2_{\varphi\rho\pi} = \frac{8}{f^2_\pi} \tag{86}$$

$$\frac{4g^2_{\rho\pi\pi}}{m^2_\rho} = \frac{8}{f^2_\pi} \tag{87}$$

If, furthermore, we identify φ as the particular SU(3) octet-singlet mixture which does not couple to the $\pi\rho$ system[47], we may omit its contribution to Eq. (86) and obtain the usual "SU(6) result":

$$g^2_{\omega\rho\pi} = \frac{4g^2_{\rho\pi\pi}}{m^2_\rho} = \frac{8}{f^2_\pi} \tag{88}$$

From the experimental point of view this result is of little interest, since it is clear that additional states have appreciable couplings to the $\pi\rho$ system, and our saturation assumption is inadequate. Algebraically we have demonstrated here that "solving" the equations obtained from the saturated sum rules leads to the same results as classifying the states into the IR's of SU(2) × SU(2) at infinite momentum. For the π,ρ,ω and φ states this classification is the following: For $\lambda = 1$, ρ and ω are in $(\frac{1}{2},\frac{1}{2})$ and φ is in $(0,0)$. For $\lambda = 0$, ρ and π are in $(1,0) \pm (0,1)$, ω and φ are each in a $(0,0)$.

What happens if we make the same saturation assumptions for sum rules of the type (58)? For $\pi\rho$ scattering we have:

$$\int_0^\infty \frac{d\nu}{\nu}\ \mathrm{Im}\ \bar{f}^{s(2)}_{10,10}(\nu,0) = 0 \tag{89}$$

$$\int_0^\infty \frac{d\nu}{\nu}\ \mathrm{Im}\ \bar{f}^{s(2)}_{00,00}(\nu,0) = 0. \tag{90}$$

Saturating these with π and ω leads to:

$$(m_\omega^2 - m_\rho^2)\ g^2_{\omega\rho\pi} = 0 \tag{91}$$

$$4(m_\pi^2 - m_\rho^2)\ \frac{g^2_{\rho\pi\pi}}{m_\rho^2} = 0 \tag{92}$$

Since $g^2_{\omega\rho\pi}$, $g^2_{\rho\pi\pi} \neq 0$, Eqs. (91) and (92) lead to:

$$m_\omega^2 = m_\rho^2 \tag{93}$$

$$m_\pi^2 = m_\rho^2 \tag{94}$$

We conclude that <u>in this saturation limit, the sum rules for I=2 t-channel amplitudes predict that all the intermediate state masses are equal to the target mass</u>. In order to understand this last result we must study the properties of the mass operator in chiral SU(2) × SU(2), to which we now turn.

B. Particle Masses and Chiral SU(2) × SU(2)

In Section II-A we have assumed that the commutator $[D^i, Q_5^j]$ does not have an I=2 piece and have shown that this is sufficient to prove that the operators $D^i(i=1,2,3)$ and $S = [D^+, Q_5^-]$ transform according to the $(\frac{1}{2},\frac{1}{2})$

representation of SU(2) × SU(2). We now consider single particle states $|\alpha\rangle, |\beta\rangle$ which for $p_z \to \infty$ belong to the same (arbitrary) irreducible representation of SU(2) × SU(2). Replacing the time derivative of Q_5 by its commutator with the total Hamiltonian of the system we find[4,48]:

$$\lim_{p_z \to \infty} p_z (\alpha|D^i|\beta) = -\frac{i}{2}(m_\beta^2 - m_\alpha^2)(\alpha|Q_5^i|\beta). \tag{95}$$

Since, by assumption, $|\alpha\rangle, |\beta\rangle$ are in the same IR and D^i belongs to the $(\frac{1}{2},\frac{1}{2})$ representation:

$$(\alpha|D^i|\beta) = 0. \tag{96}$$

Furthermore, in general:

$$(\alpha|Q_5^i|\beta) \neq 0. \tag{97}$$

We therefore conclude[49]:

$$m_\beta^2 = m_\alpha^2 . \tag{98}$$

<u>The (mass)^2 values of all particles belonging to an (unmixed) IR of SU(2) × SU(2) are therefore predicted to be equal in spite of the fact that the axial charge is not conserved and does not commute with the energy density[4].</u> The case of the π-ρ sum rules, saturated by π and ω intermediate states, which we have discussed in the previous section, can now be easily understood. We have already noted that the saturation assumption is equivalent to assigning ρ and ω to the $(\frac{1}{2},\frac{1}{2})$ IR for helicity $\lambda = 1$ and ρ and π to the $(1,0) \pm (0,1)$ IR's for $\lambda = 0$. Eqs. (95)-(98) then explain[50] why we found that $m_\omega^2 = m_\rho^2 = m_\pi^2$.

We have indicated that the pure representation case, although interesting, is of no particular relevance to the real world in which most (if not all) single particle states correspond at $p_z \to \infty$ to mixtures of IR's of the chiral algebra of charges. We therefore have to study the case in which $|\alpha >$ and $|\beta >$ are eigenstates of the same IR, but describe linear combinations of single particle states with known mixing coefficients. We consider a set of n particle states $|X_i >$ having the same isotopic spin I, which belong to various known mixtures of n irreducible representations of SU(2) × SU(2) and assume that the eigenstates $|\alpha_i >$ of these representations are given by:

$$|\alpha_i > = \sum_{j=1}^{n} a_{ij} |X_j > \qquad (99)$$
$$|X_i > = \sum_{j=1}^{n} a_{ji} |\alpha_j >$$

In addition we have m particle states $|Y_j >$ with isospin[51] $I' \neq I$, described in terms of SU(2) × SU(2) eigenstates $|\beta_i >$:

$$|\beta_i > = \sum_{j=1}^{m} b_{ij} |Y_j > \qquad (100)$$
$$|Y_i > = \sum_{j=1}^{m} b_{ji} |\beta_j >$$

We now prove the following theorem: If $|\alpha_s >$ and $|\beta_t >$ are in the same IR:

$$\sum_{j=1}^{n} a_{sj}^2 m^2(X_j) = \sum_{k=1}^{m} b_{tk}^2 m^2(Y_k), \tag{101}$$

i.e., the weighted averages of the $(\text{mass})^2$ values of all states in the same IR of $SU(2) \times SU(2)$ are equal even in the case when all single particle states correspond to mixtures of the IR's of the chiral charge algebra. The proof of this statement is straightforward:

$$0 = \lim_{p_z \to \infty} 2ip_z(\alpha_s|D|\beta_t) = \lim_{p_z \to \infty} 2ip_z \sum_{j=1}^{n} \sum_{k=1}^{m} a_{sj} b_{tk} < X_j|D|Y_k > =$$

$$= \sum_{j=1}^{n} \sum_{k=1}^{m} a_{sj} b_{tk} [m^2(Y_k) - m^2(X_j)](X_j|Q_5|Y_k) = \tag{102}$$

$$= \sum_{j,\ell=1}^{n} \sum_{i,k=1}^{m} a_{sj} a_{\ell j} b_{tk} b_{ik} [m^2(Y_k) - m^2(X_j)](\alpha_\ell|Q_5|\beta_i) =$$

$$= \sum_{\ell=1}^{n} \sum_{k,i=1}^{m} \left\{ \sum_{j=1}^{n} a_{sj} a_{\ell j} \right\} b_{tk} b_{ik} m^2(Y_k)(\alpha_\ell|Q_5|\beta_i) -$$

$$- \sum_{j,\ell=1}^{n} \sum_{i=1}^{m} \left\{ \sum_{k=1}^{m} b_{tk} b_{ik} \right\} a_{sj} a_{\ell j} m^2(X_j)(\alpha_\ell|Q_5|\beta_i) =$$

$$= \sum_{k,i=1}^{m} b_{tk} b_{ik} m^2(Y_k)(\alpha_s|Q_5|\beta_i) - \sum_{j,\ell=1}^{n} a_{sj} a_{\ell j} m^2(X_j)(\alpha_\ell|Q_5|\beta_t)$$

Since $(\alpha_s|Q_5|\beta_i) = \delta_{it}(\alpha_s|Q_5|\beta_t)$ and $(\alpha_\ell|Q_5|\beta_t) = \delta_{s\ell}(\alpha_s|Q_5|\beta_t)$ we obtain:

$$0 = (\alpha_s|Q_5|\beta_t)\left\{\sum_{k=1}^{m} b_{tk}^2 m^2(Y_k) - \sum_{j=1}^{n} a_{sj}^2 m^2(X_j)\right\} , \tag{103}$$

and Eq. (101) follows.

The significance of this result stems from the fact that the mixing coefficients a_{ij}, b_{ij} are, in principle, completely determined by the weak and electromagnetic transitions among the physical states, and Eq. (101) then provides us with non-trivial relations between the particle masses. This situation is similar to the usual SU(3) picture in which, for example, the ω-φ mixture which belongs to the octet can be determined from the decays $\rho \to \pi\pi$ and $\varphi \to K\bar{K}$ and is then predicted to satisfy the Gell-Mann-Okubo mass formula. The SU(2) × SU(2) "mass formula" is, however, much simpler since it simply states that all $(\text{mass})^2$ values are equal for a given IR.

The part of the Hamiltonian which breaks SU(2) × SU(2) is essentially proportional to the isoscalar operator S. This follows directly from the commutation relation (14) of Section II. Had we allowed an isotopic spin 2 term in the commutator $[D,Q_5]$ we would, in principle, find additional symmetry breaking terms which transform according to higher representations of the type (k,k). In particular a (1,1) symmetry breaking term _might_ connect two states belonging to the _same_ IR of SU(2) × SU(2) and thus _would_ contribute a "diagonal" mass splitting term within the representation. _It is the assumption on the absence of I=2 terms in $[Q_5,D]$, which led to the general mass formula Eq. (101)._

C. Representation Mixing at Infinite Momentum: π-ρ Scattering

The saturation assumptions used in Section V-A were highly unrealistic, since all the relevant single particle states were assumed to correspond to "pure" (unmixed) IR's of SU(2) $\times$ SU(2) at $p_z \to \infty$. Unfortunately, the real world is more complicated and the non-conservation of the axial charge breaks SU(2) $\times$ SU(2) symmetry and induces large amounts of representation mixing for the lowest lying baryons and mesons, even for $p_z \to \infty$ where, presumably, the classification is the simplest[42]. That the low-lying states are strongly mixed can be seen from many different experimental facts, some of which are: the importance of many $I = \frac{1}{2}$ N^*'s in the Adler-Weisberger sum rule, the (factor 2) discrepancy between Eq. (87) and experiment, the prediction $G_A = \frac{5}{3}$, the wrong predicted width for[52] $\Gamma(N^* \to N\pi)$ and (assuming $[D^+, Q_5^+] = 0$) the π-ρ mass difference, N-N* mass difference, etc. The "art" of determining the mixing coefficients for various single particle states from various weak, electromagnetic and pionic matrix elements still involves much guessing, especially in view of the absence of experimental data for most of the relevant transitions. A few successful mixing schemes for various sets of particles have been proposed, however, and we would like to discuss here one of them which we have first introduced in a previous paper[4]. We refer to the saturation of all t=0 sum rules for π-ρ scattering by the π,ω and A_1 intermediate states. In this section we will mainly be interested in the algebraic aspects of this idea and in Section VII we will return to some of the more phenomenological points.

We again consider the four sum rules (84),(85),(89), and (90) but now assume that the dispersion integrals are dominated by the contributions of

π,ω, and A_1 intermediate states. With the exception of the φ and A_2 these are all the known single-particle states which couple to the π-ρ system. At this point we neglect the (small) contribution of φ and A_2 and we will return to them later, when we discuss (in Section VII) the sensitivity of our assumptions. There are two independent $A_1\rho\pi$ couplings and we choose them as the longitudinal coupling $g_L(p_\mu - \frac{(p\cdot q)q_\mu}{q^2})(q_\lambda - \frac{(p\cdot q)p_\lambda}{p^2})e_\lambda e_\mu'$ and the transverse coupling $\frac{g_T}{m_{A_1}^2}\epsilon^{\lambda\alpha\beta\gamma}\epsilon^{\mu\alpha'\beta'\gamma}p_\alpha q_\beta p_{\alpha'} q_{\beta'} e_\lambda e_\mu'$ where p(q) and e(e') are the momentum and polarization of the $A_1(\rho)$. The saturated sum rules read:

$$g_{\omega\rho\pi}^2 + \frac{\nu_{A_1}^2}{m_{A_1}^4} g_T^2 = \frac{8}{f_\pi^2} \tag{104}$$

$$\frac{4g_{\rho\pi\pi}^2}{m_\rho^2} + \frac{\nu_{A_1}^2}{m_\rho^2 m_{A_1}^2} g_L^2 = \frac{8}{f_\pi^2} \tag{105}$$

$$\nu_\omega g_{\omega\rho\pi}^2 - \frac{\nu_{A_1}^3}{m_{A_1}^4} g_T^2 = 0 \tag{106}$$

$$4\nu_\pi g_{\rho\pi\pi}^2 + \frac{\nu_{A_1}^3}{m_{A_1}^2} g_L^2 = 0 \tag{107}$$

where $\nu_x = \frac{1}{2}(m_x^2 - m_\rho^2)$.

We first consider the two "longitudinal" sum rules, whose saturated versions are Eqs. (105),(107). If the A_1 were not coupled to $\pi\rho$, Eq. (105) would collapse into its "unmixed" version, Eq. (87). However, in the present model the A_1 contributes part of the sum rule and we can parametrize its contribution by defining a mixing angle ψ satisfying:

$$\frac{4g^2_{\rho\pi\pi}}{m^2_\rho} = \frac{8}{f^2_\pi}\cos^2\psi$$

$$\frac{\nu^2_{A_1}}{m^2_\rho m^2_{A_1}}\, g^2_L = \frac{8}{f^2_\pi}\sin^2\psi \tag{108}$$

The "unmixed" case corresponds to $\psi = 0$. Substituting this into the I=2 longitudinal sum rule we obtain:

$$\nu_\pi \cos^2\psi + \nu_{A_1}\sin^2\psi = 0 \tag{109}$$

or:

$$m^2_\pi \cos^2\psi + m^2_{A_1}\sin^2\psi = m^2_\rho \tag{110}$$

The experimental width $\Gamma(\rho \rightarrow \pi\pi)$ enables us to determine ψ, thereby predicting g_L and obtaining one new relation between the π,ρ and A_1 masses.

Instead of obtaining Eqs. (108)-(110) directly from the saturated sum rules, we could have assumed instead that, while the $\lambda = 0$ component of the ρ is still in the [(1,0)+(0,1)] IR (as it was in the "unmixed" case), the π and A_1 are mixed and the [(1,0)-(0,1)] eigenstate is given by $\cos\psi|\pi^i\rangle + \sin\psi|A^i_1\rangle$. This would immediately lead to the same set of equations including (108) and (110). The actual value of ψ as determined from the ρ-width is around 45°, and its precise determination depends on the particular values used for $\Gamma(\rho \rightarrow \pi\pi)$ and m_{ρ}, as well as on whether or not the external pion is taken to be massless[53].

For $\psi = 45^{\circ}$ we find[4,54]:

$$g_L = \frac{2m_{\rho} m_{A_1}}{f_{\pi} \nu_{A_1}} \tag{111}$$

$$\Gamma(A_1 \rightarrow \rho\pi, \text{ longitudinal}) = \frac{\nu_{A_1}^3}{3\pi f_{\pi}^2 m_{A_1}^3} = 110 \text{ MeV} \tag{112}$$

$$m_{A_1}^2 = 2m_{\rho}^2 - m_{\pi}^2 = (1070 \text{ MeV})^2, \tag{113}$$

in excellent agreement with the experimental A_1 mass[55] (1080 MeV). The experimental width of the A_1 is 130 ± 40 MeV. Eq. (112) then implies that the transverse $A_1\rho\pi$ coupling g_T is consistent with zero, and in any case, does not contribute more than 60 MeV to the total width.

We now turn to the saturated "transverse" sum rules (Eqs. (104) and (106)) and, again, parametrize the ω and A_1 contributions:

$$\begin{aligned} g_{\omega\rho\pi}^2 &= \frac{8}{f_{\pi}^2} \cos^2\chi \\ \frac{\nu_{A_1}^2}{m_{A_1}^4} g_T^2 &= \frac{8}{f_{\pi}^2} \sin^2\chi \end{aligned} \tag{114}$$

Eq. (106) then leads to:

$$\nu_{\omega}\cos^2\chi - \nu_{A_1}\sin^2\chi = 0 \tag{115}$$

We immediately notice that the "unmixed" case (χ=0) is, in this case, consistent with the data since it leads to

$$g_{\omega\rho\pi} = 21 \text{ BeV}^{-1} \tag{116}$$

$$g_T = 0 \tag{117}$$

$$m_{\omega} = m_{\rho} \tag{118}$$

In order to study the possible effect of the small ω-ρ mass difference on Eqs. (114),(115) we notice that for $m_{\omega} = 0.78$, $m_{\rho} = 0.76$, $m_{A_1} = 1.08$ (in BeV) Eq. (115) leads to $\chi \sim 10^{\circ}$ and:

$$g_{\omega\rho\pi} = 20 \text{ BeV}^{-1} \tag{119}$$

$$g_T = \frac{2\sqrt{2}\, m_{A_1}^2}{f_{\pi}\, \nu_{A_1}} \sin\chi \tag{120}$$

$$\Gamma(A_1 \rightarrow \rho\pi, \text{ transverse}) = 20 \text{ MeV} \tag{121}$$

We therefore find that the total width for $A_1 \rightarrow \rho\pi$ is between[54] 70-140 MeV, $g_{\omega\rho\pi} = 20\text{-}21 \text{ BeV}^{-1}$ (to be compared with $g_{\omega\rho\pi} = (17 \pm 3) \text{ BeV}^{-1}$ as obtained from the Gell-Mann-Sharp-Wagner model[56] for $\omega \rightarrow \pi\gamma$) and g_T/g_L is consistent with zero and smaller than 0.2. This last ratio is consistent with the present inconclusive data on the decay $A_1 \rightarrow \rho\pi$.

It is extremely interesting to note that we predict that the A_1 does not decay to $\rho\pi$ predominantly via the s-wave coupling and that, in fact, the d-wave may dominate[57].

Thus, the addition of the A_1 contribution to the four π-ρ t=0 charge algebra sum rules has a dramatic effect on the agreement of the saturated sum rules with experiment.

D. Symmetry Results and Class I Superconvergence Relations

In the previous sections we have shown that (1) Charge algebra sum rules for the amplitudes $\bar{f}^{s}_{\lambda 0,\lambda 0}$ lead to the so-called "SU(6)-results"[43] in the (unrealistic) case in which they are all saturated by the 35 mesons or 56 baryons, (2) The contributing additional states can be analyzed in terms of the IR's of SU(2) $\times$ SU(2) and (3) The strongest Class I t=0 superconvergence relation for any even-Δ t-channel helicity amplitude $[f^{t}_{(\lambda-\Delta)\lambda,00}(\nu,0)/\nu^{\Delta}]$ is a linear combination of charge algebra sum rules. It is now evident that this particular class of t=0 superconvergence relations should, <u>in the limit of zero external pion mass</u>, lead to "SU(6) results" when saturated by the 35 mesons or 56 baryons[58]. It is also clear that these superconvergence relations will be consistent with the charge algebra for any other (more realistic) saturation assumption that we may wish to suggest. Note, however, that the overall scale of the predictions, which in the charge algebra sum rules was supplied by the $[Q_5^+,Q_5^-]$ commutator and PCAC is lost when we restrict ourselves to the superconvergence relations alone.

For pion-hadron scattering on targets with spin $J \leq 2$ there is one case in which a Class I superconvergence relation cannot be written as a linear combination of charge algebra sum rules. We refer to the sum rules (79)-(81) for amplitudes of the type $[f^t_{(\lambda-4)\lambda,00}(\nu,0)/\nu^4]$. The amplitude itself corresponds, at t=0, to a linear combination of s-channel helicity amplitudes $f^s_{\lambda 0,\lambda 0}(\nu,0)$, and the sum rules (82)-(83) that it obeys <u>are</u> combinations of charge algebra sum rules. The sum rules (79)-(81) can be obtained by making a zero-energy subtraction in the dispersion relations for $\bar{f}^s_{\lambda 0,\lambda 0}$ and selecting a particular combination of the once-subtracted dispersion relations in which the subtraction constants cancel.

The assumption $m_\pi^{ext} = 0$ is natural when we deal with strong interaction sum rules which are derived from <u>PCAC</u> and the algebra of charges. In principle, it is totally unnecessary when we discuss only superconvergence relations, which are independent of PCAC. We find, however, that our algebraic understanding of the saturated t=0 superconvergence sum rules stems from their relation to the charge algebra sum rules and from our ability to analyze these in terms of the representations of SU(2) × SU(2). This algebraic structure as well as the self-consistency of the saturation by an arbitrary set of states are valid only for $m_\pi^{ext} = 0$. We are therefore led to believe that, to the extent that the Class I superconvergence relations have the algebraic structure of the SU(6), SU(3) × SU(3) or SU(2) × SU(2) type, it emerges from the related charge algebra sum rules and is valid only for massless pions[59]. The more practical problem of whether neglecting the pion mass results in any major effect on the comparison with the data, will be discussed in Section VII.

VI. CLASS II AND NON-FORWARD SUM RULES

A. Class II Superconvergence Relations

In Section IV the connection of Class I superconvergence relations to sum rules arising from the algebra of charges was established, and in the preceding section we have used the chiral SU(2) $\times$ SU(2) algebra of the vector and axial-vector charges to elucidate the algebraic structure of Class I superconvergence relations. As Class II superconvergence relations are not sum rules on purely forward amplitudes in the s-channel and therefore not related directly to charge algebra sum rules, it is clear that the chiral SU(2) $\times$ SU(2) algebra will not be directly useful in analyzing the algebraic structure of Class II superconvergence relations. However, in a few cases the Class II superconvergence relations happen to be satisfied when saturated using the "SU(6) results"[43] for masses and coupling constants obtained from specific saturation schemes for Class I superconvergence relations. In other cases Class II superconvergence relations are either inconsistent with "SU(6) results"[60] or help in fixing certain mixing angles between SU(2) $\times$ SU(2) representations (often forcing them to be zero) which are left free by Class I and charge algebra sum rules.

A good example of this is again provided by $\pi\rho$ scattering where the t-channel amplitude $[f^{t(2)}_{01,00}(\nu,t)/(\nu\sqrt{-t})]_{t=0}$, which is essentially the $B^{(2)}(\nu,0)$ amplitude of Ref. 2, satisfies a superconvergence relation,

$$\int_0^\infty [f^{t(2)}_{01,00}(\nu,t)/(\nu\sqrt{-t})]_{t=0}\, d\nu = 0, \tag{122}$$

which is in Class II. We recall from Section III that the relation of $[f^{t(2)}_{01,00}(\nu,t)/(\nu\sqrt{-t})]_{t=0}$ to s-channel helicity amplitudes is (recall $\nu = (s - M^2)/2$):

$$[f^{t(2)}_{01,00}(\nu,t)/(\nu\sqrt{-t})]_{t=0} = \frac{\sqrt{2}}{4M\nu^2}\left[\frac{\nu + M^2}{M}\,\bar{f}^{s(2)}_{10,10} + \frac{\nu + M^2}{M}\,\bar{f}^{s(2)}_{00,00} - \sqrt{4\nu + 2M^2}\,\bar{f}^{s(2)}_{10,00}\right]. \tag{123}$$

From the preceding section, we know that saturating the charge algebra sum rules (and Class I superconvergence relations) with only the π and ω states yields the "SU(6) results", $4g^2_{\rho\pi\pi}/m^2_\rho = g^2_{\omega\rho\pi} = 8/f^2_\pi$ and $m^2_\rho = m^2_\omega = m^2_\pi$. For I=2 in the t-channel, the π and ω, <u>with equal masses</u>, then contribute with equal magnitude but opposite sign to the first and second terms, respectively, on the right hand side of Eq. (123). The last term on the right hand side receives no contribution in this approximation, since the ω intermediate state contributes only to transverse (helicity $\lambda = \pm 1$) and the pion intermediate state contributes only to longitudinal (helicity $\lambda = 0$) $\pi\rho$ scattering, but not to both. Thus the superconvergence relation, Eq. (122), will be satisfied by the "SU(6) results" obtained from the charge algebra sum rules.

If we expand the saturation scheme to include the π, ω, and A_1 mesons (see Section V), the last term in Eq. (123) no longer vanishes in general, as the A_1 intermediate state in $\pi\rho$ scattering contributes in principle to both the transverse and longitudinal amplitudes, and therefore to the amplitude $f^s_{10,00}$ which involves a cross-term between transverse and longitudinal scattering. The explicit form of the saturated version of

Eq. (122) is:

$$m_\omega^2 g_{\omega\rho\pi}^2 - 4g_{\rho\pi\pi}^2 - \frac{\nu_{A_1}^2}{m_{A_1}^2} (g_L - g_T)^2 = 0, \tag{124}$$

However, when we "saturate" the charge algebra sum rules by the π, ω, and A_1, we find that Eq. (122) can be consistently saturated by the same states only if the last term in Eq. (123) is forced to be zero, so that either the transverse or longitudinal $A_1\rho\pi$ coupling must vanish. As the charge algebra sum rules and $\Gamma(\rho \to \pi\pi)$ give a non-vanishing value for the longitudinal coupling (see Section V), the transverse $A_1\rho\pi$ coupling must vanish (corresponding to $\chi=0$).

We also know that "SU(6) results" are consistent with the Class II superconvergence relation on the amplitude $[f^{t(2)}_{\frac{1}{2}\frac{1}{2},00}(\nu,t)/(\nu\sqrt{-t})]_{t=0}$ for $\pi\Sigma$ scattering[61]. However, for πN^* elastic scattering "SU(6) results" are inconsistent with some of the Class II sum rules[60]. The $\pi\rho$ and $\pi\Sigma$ cases thus appear to be "lucky accidents" due to low spins, mass degeneracy of the states assumed to saturate the sum rules, and the vanishing of certain amplitudes containing cross-terms between different helicity states when very simple saturation schemes are used. The inconsistency of the Class II superconvergence relations for $\pi N^* \to \pi N^*$ just provides an explicit example of what we expected, namely: as Class II superconvergence relations are not related to sum rules on purely forward amplitudes, we do not expect any general connection between them and the "SU(6) results" obtained from the charge algebra sum rules on forward amplitudes.

B. Sum Rules for "Small" $t\neq 0$ and df/dt

In this paper we have been interested in sum rules at t=0, in large part because the charge algebra sum rules are on forward amplitudes and we can gain some insight into the algebraic structure of at least Class I superconvergence relations from the charge algebra. Moreover, the t=0 charge algebra sum rules are the only inhomogeneous sum rules that we have and they fix the overall scale of the coupling constants and axial transitions. The Class II superconvergence relations on non-forward amplitudes extrapolated to t=0, were the only $t\neq 0$ information that we have discussed so far. However, in addition to these, one can obtain additional sum rules by considering values of $t\neq 0$, or by taking the derivative with respect to t of superconvergence relations and evaluating the result at t=0.

For small values of t, say $|t| \sim m_\pi^2$, we expect that the superconvergence relations are saturated in approximately the same way as at t=0, since neither the contribution of the low lying states nor the high energy continuum are drastically changed. However, the saturation of sum rules obtained by taking the derivative with respect to t of sum rules at t=0 can be entirely changed from that of the original t=0 sum rules because of the important contribution of the large slope of the forward peak at high energy. As an example, consider again $\pi\rho$ scattering and the superconvergence relation (Class I) on the amplitude $[f^{t(1)}_{-11,00}(\nu,t)/\nu^2]$, which is essentially the amplitude $A^{(1)}(\nu,t)$ of Ref. 2. Let us assume saturation of the superconvergence relation by π, ω, and A_1, as given in Section V, except that we take an additional contribution of the typical form ae^{10t} (t negative in the physical region and in units of BeV^2) arising from the forward peak above the low energy region, and which we have neglected in our saturation by π, ω, and A_1. We normalize the high energy contribution at t=0 to 10% of the largest contribution (from the ω) to the sum rule, i.e., we assume we have made a 10% error in neglecting the high energy contribution in our saturation scheme. (See Table III) This 10% estimate is probably realistic for the region above, say $\sqrt{s}$ = 2 BeV and, in fact, in the Adler-Weisberger sum rule for πN scattering the contribution from above 2 BeV is of this size relative to the largest single particle terms (N, N^*).

From Table III we see that at $t = -m_\pi^2$ there is very little change in the saturation of the sum rule, but when we consider the sum rule obtained by taking the derivative with respect to t at t=0, the high energy slope in t yields a major contribution to the resulting sum rule, while the pion, a major contributor at t=0, yields a vanishing contribution. The sum rule for the second derivative at t=0 "loses" the contributions of all three states π, ω, A_1 and only $J \geq 2$ states and high energy terms contribute. It is therefore, apriori, more dangerous to saturate derivative sum rules at t=0 by low lying resonances and to neglect high energy contributions and intermediate energy (high spin) s-channel resonances. We believe that within the framework of an analysis which ignores the high energy contributions, we can make a careful selection of the sum rules that are best saturated by a few states. We should remember that: (1) Derivative sum rules are apriori worse canditates, since they tend to enhance high energy contributions. (2) Sum rules for "small" ($\sim m_\pi^2$) values of t are saturated as well as those at t=0 and could replace them (but not be added, since that would be effectively equivalent to using derivative sum rules!).

(3) Sum rules at larger values of $|t|$ (say, $|t| \sim m_\rho^2$) could lead to strong enhancement of high spin intermediate energy states for which $|t| \sim m_\rho^2$ is already in the unphysical region, $|\cos\theta| > 1$ and the absolute magnitude of the contributions of a spin J partial wave grows like $|\cos\theta|^J$. The immediate neighborhood of t=0 seems to be the most appropriate place for applying the saturation assumption, as long as we have no information on the scattering at high energy.

VII. A MODEL FOR THE LOW-LYING MESONS

In this section we discuss a simple model for the low-lying mesons. We assume the existence of a certain set of mesons, write all t=0 charge algebra and superconvergence relations for the scattering of pions on these mesons, and assume that all sum rules are approximately saturated by the same set of mesons. Had we chosen these mesons to be π, η, ρ, and ω we would end up with the "SU(6) results" for the 35-meson representation. However, in view of the well-known importance of additional states (such as the A_1 in the $\pi\rho$ sum rules) we now propose to extend the set of mesons to include 0^+ and 1^+ mesons, thereby improving the agreement with experiment.

A. The Model

We consider only non-strange mesons with isotopic spin 1 or 0 and assume that we have scalar, pseudoscalar, vector and axial-vector mesons, with one isotriplet and one isosinglet for every spin-parity. The quantum numbers of these particles are summarized in Table IV. At this stage we ignore the interesting possibility of having two isosinglets for each value of J^P, but we will return to it later.

The total number of t=0 charge algebra and superconvergence relations for the scattering of (massless) pions on any of the particles of Table IV can be easily read off from Table II: For π-π scattering there are two independent sum rules and only the σ and ρ can contribute as s-channel resonances; for π-δ scattering we have two sum rules and the contributing states are X^o and D; there are four "pure" t=0 π-ρ sum rules (plus one of Class II) and the contributing states are π, ω, and A_1; finally, we have four more "pure" t=0 sum rules (and one of Class II) for π-A_1 scattering with contributions from σ, ρ, and D. There are no sum rules for the scattering of pions on any of the isosinglet states of our model.

We now list the 12 sum rules which can be derived from the charge algebra. Four of them can be alternately derived as Class I superconvergence relations. In the saturation limit the sum rules read[62]:

$$\frac{4g_{\rho\pi\pi}^2}{m_\rho^2} + g_{\sigma\pi\pi}^2 = \frac{8}{f_\pi^2} \tag{125}$$

$$(m_\rho^2 - m_\pi^2)\frac{4g_{\rho\pi\pi}^2}{m_\rho^2} - (m_\sigma^2 - m_\pi^2)\, g_{\sigma\pi\pi}^2 = 0 \tag{126}$$

$$g_{\omega\rho\pi}^2 + \frac{(m_A^2 - m_\rho^2)^2}{4m_A^4}\, g_T^2 = \frac{8}{f_\pi^2} \tag{127}$$

$$(m_\omega^2 - m_\rho^2)\, g_{\omega\rho\pi}^2 - \frac{(m_A^2 - m_\rho^2)^3}{4m_A^4}\, g_T^2 = 0 \tag{128}$$

$$\frac{4g_{\rho\pi\pi}^2}{m_\rho^2} + \frac{(m_A^2 - m_\rho^2)^2}{4m_\rho^2 m_A^2} g_L^2 = \frac{8}{f_\pi^2} \tag{129}$$

$$4(m_\pi^2 - m_\rho^2)\ g_{\rho\pi\pi}^2 + \frac{(m_A^2 - m_\rho^2)^3}{4m_A^2} g_L^2 = 0 \tag{130}$$

$$g_{DA\pi}^2 + \frac{(m_\rho^2 - m_A^2)^2}{4m_A^4} g_T^2 = \frac{8}{f_\pi^2} \tag{131}$$

$$(m_D^2 - m_A^2)\ g_{DA\pi}^2 - \frac{(m_\rho^2 - m_A^2)^3}{4m_A^4} g_T^2 = 0 \tag{132}$$

$$\frac{4g_{\sigma A\pi}^2}{m_A^2} + \frac{(m_\rho^2 - m_A^2)^2}{4m_A^2 m_\rho^2} g_L^2 = \frac{8}{f_\pi^2} \tag{133}$$

$$4(m_\sigma^2 - m_A^2)\ g_{\sigma A\pi}^2 - \frac{(m_\rho^2 - m_A^2)^3}{4m_\rho^2} g_L^2 = 0 \tag{134}$$

$$\frac{4g_{D\delta\pi}^2}{m_D^2} + g_{x\delta\pi}^2 = \frac{8}{f_\pi^2} \tag{135}$$

$$(m_D^2 - m_\delta^2)\frac{4g_{D\delta\pi}^2}{m_D^2} + (m_x^2 - m_\delta^2)\ g_{x\delta\pi}^2 = 0 \tag{136}$$

We have 11 independent equations (as it turns out that there is a linear relation among Eqs. (125),(126),(129),(130),(133), and (134)) in 17 unknowns: The eight particle masses and the 9 coupling constants $g_{\rho\pi\pi}$, $g_{\sigma\pi\pi}$, $g_{\omega\rho\pi}$, g_T, g_L, $g_{DA\pi}$, $g_{\sigma A\pi}$ $g_{D\delta\pi}$, $g_{x\delta\pi}$. We can, therefore, express all the masses and coupling constants in terms of six free parameters which we choose to be m_π, m_x, m_ρ, m_ω and two mixing angles ψ and ζ which we define by:

$$\frac{4g^2_{\rho\pi\pi}}{m^2_\rho} = \frac{8}{f^2_\pi} \cos^2\psi \tag{137}$$

$$g^2_{x\delta\pi} = \frac{8}{f^2_\pi} \cos^2\zeta \tag{138}$$

Since the experimental value for $\Gamma(\rho \rightarrow \pi\pi)$ leads to[53] $\psi = 45^\circ$ we will use this value in solving the set of equations (125)-(136). We find:

$$m_\sigma = m_\rho \tag{139}$$

$$m^2_{A_1} = 2m^2_\rho - m^2_\pi \tag{140}$$

$$g^2_{\sigma\pi\pi} = \frac{4}{f^2_\pi} \tag{141}$$

$$g^2_L = \frac{16m^2_\rho(2m^2_\rho - m^2_\pi)}{f^2_\pi(m^2_\rho - m^2_\pi)^2} \tag{142}$$

$$g^2_{\sigma A\pi} = \frac{m^2_A}{f^2_\pi} \tag{143}$$

$$g^2_{\omega\rho\pi} = \frac{8}{f^2_\pi} \frac{m^2_\rho - m^2_\pi}{m^2_\omega - m^2_\pi} \tag{144}$$

$$g_T^2 = \frac{32}{f_\pi^2} \frac{(2m_\rho^2 - m_\pi^2)^2 (m_\omega^2 - m_\rho^2)}{(m_\rho^2 - m_\pi^2)^2 (m_\omega^2 - m_\pi^2)} \tag{145}$$

$$g_{DA\pi}^2 = \frac{8}{f_\pi^2} \frac{m_\rho^2 - m_\pi^2}{m_\omega^2 - m_\pi^2} \tag{146}$$

$$m_D^2 = 3m_\rho^2 - m_\omega^2 - m_\pi^2 \tag{147}$$

$$g_{D\delta\pi}^2 = \frac{2}{f_\pi^2} (3m_\rho^2 - m_\omega^2 - m_\pi^2) \sin^2\zeta \tag{148}$$

$$m_\delta^2 = m_X^2 \cos^2\zeta + (3m_\rho^2 - m_\omega^2 - m_\pi^2)\sin^2\zeta \tag{149}$$

Using the experimental values $m_\pi^2 = 0.02\ \mathrm{BeV}^2$, $m_X^2 = 0.92\ \mathrm{BeV}^2$, $m_\rho^2 \sim 0.58$, $m_\omega^2 = 0.61$ we predict[63]:

$$m_\sigma \sim 765\ \mathrm{MeV} \tag{150}$$

$$m_{A_1} \sim 1070\ \mathrm{MeV} \tag{151}$$

$$\Gamma(\sigma \to \pi\pi) \sim 650\ \mathrm{MeV} \tag{152}$$

$$\Gamma(A_1 \to \rho\pi;\ \text{longitudinal}) \sim 110\ \mathrm{MeV} \tag{153}$$

$$\Gamma(A_1 \to \sigma\pi) \sim 55\ \mathrm{MeV} \tag{154}$$

$$g_{\omega\rho\pi} \sim 20\ \mathrm{BeV}^{-1} \tag{155}$$

$$\Gamma(A_1 \to \rho\pi;\ \text{transverse}) \sim 20\ \text{MeV} \tag{156}$$

$$g_{DA\pi} \sim 20\ \text{BeV}^{-1} \tag{157}$$

$$m_D \sim 1060\ \text{MeV} \tag{158}$$

$$g^2_{D\delta\pi} \sim 110\ \sin^2\zeta \tag{159}$$

$$m_\delta \sim 960\sqrt{1 + 0.21\ \sin^2\zeta}\ \text{MeV} \tag{160}$$

B. Comparison with Experiment and Sensitivity Tests

Three questions are immediately raised in view of the predictions (150)-(160). (1) What can we say about particles such as σ, δ and D which are not yet identified experimentally? (2) How good is the agreement with experiment of those predictions which are related to known particles? (3) How sensitive are the results to approximations such as neglecting m_π^{ext}, ignoring the contributions of higher mesons (especially the $J^P = 2^+$ mesons f^o and A_2), and ignoring the contributions of other isosinglet states, especially the φ and η? We now present a long list of comments which are related to these questions and in which we try to evaluate the reliability of our model-calculation.

1. The σ-meson ($J^P = 0^+$, $I^{CG} = 0^{++}$) has not been found experimentally Many theorists and experimentalists have expressed "proofs", arguments or hopes that it does or does not exist, at masses between 300 and 800 MeV and with widths of 200 to 600 MeV[64]. The "missing" contributions to the π-π Adler sum rules was one possible piece of evidence for the existence

of a large s-wave π-π interaction[65]. We find that, in addition, the π-A_1 sum rules would be extremely hard to understand in the absence of a strong p-wave π-A_1 interaction with $J^P = 0^+$, $I^{CG} = 0^{++}$ and which lies in the same energy region as the strong π-π contribution. This may hint that both these interactions are actually related to a (very wide) π-π resonance which is also a π-A_1 "bound" state. On the other hand, it is conceivable that the general similarity between the properties of A_1 and π (especially if they dominate the axial current and its divergence, respectively) might lead to similar effects in π-π and π-A_1 scattering without explicitly demanding that the σ "particle" exists. In any event, our predicted mass value and width make it extremely difficult to find the σ, especially if its production cross section in πp reactions is small. The $\pi^+\pi^-$ decay mode will be "buried" under the huge ρ-meson peak, while the $\pi^0\pi^0$ mode is extremely difficult to detect, and could be easily confused with $\pi^0\gamma$ decays of the ρ or ω, if the σ production cross section is a few percent of that of the ρ.

2. The predicted width of the σ is somewhat sensitive to the assumed mass of the external pion. Using our value for $g_{\sigma\pi\pi}$ but taking both pions on the mass shell we find:

$$\Gamma(\sigma \to \pi\pi) \simeq 570 \text{ MeV} \tag{161}$$

3. The predicted appreciable decay $A_1 \to \pi\sigma$ makes the process $\pi + N \to A_1 + N \to \pi + \sigma + N$ a likely candidate for producing a large number of σ's. The decays $A_1^\pm \to \pi^\pm\rho^0 \to \pi^\pm\pi^+\pi^-$ should be very similar to $A_1^\pm \to \pi^\pm\sigma \to \pi^\pm\pi^+\pi^-$ and $A_1^\pm \to \pi^\pm\sigma \to \pi^\pm\pi^0\pi^0$ cannot be detected. On the other hand, $A_1^0 \to \pi^0\sigma$ is allowed while $A_1^0 \to \pi^0\rho^0$ is forbidden by charge conjugation (or isospin). A $\pi^+\pi^-$ enhancement around the ρ mass in events for

which the $\pi^+\pi^-\pi^0$ system is in the A_1 peak would indicate a $\pi^0\sigma$ decay. Alternately, if such a situation actually occurs, and if the possibility of a σ-meson is ignored, we would have an apparent $\pi^0\rho^0$ decay together with $\pi^+\rho^-$ and $\pi^-\rho^+$ and we would conclude that we see a new isoscalar $\pi\rho$ state in the A_1 mass region, while the A_1^0 itself would not be seen. Whether this has in fact happened we do not know, but we would like to point out that while no evidence for the existence of the A_1^0 has been published so far (although numerous experiments found the $A_1^\pm$ peaks) another neutral meson, known as H, decaying into $\pi^\pm\rho^\mp$ and $\pi^0\rho^0$ has been found by two groups in two different experiments around 980 MeV[66] (i.e. 100 MeV below the A_1 mass). It might be interesting to study the (admittedly speculative) possiblity that the H is the A_1^0 and its $\pi^0\rho^0$ decay mode is actually $\pi^0\sigma$.

4. The relation between $\Gamma(A_1 \to \pi\rho)$ and the coupling constants g_L and g_T involves the fifth power of the three-momentum of the pion in the A_1 rest frame. This momentum is changed by about 10% between $m_\pi^{ext} = 0$ and 140 MeV, so that the effect on $\Gamma(A_1 \to \rho\pi)$ is to reduce it by a factor of 1.6. Since PCAC is used in deriving the magnitude of g_T and g_L we consider this sensitive dependence of Γ on m_π as a measure of the ambiguity introduced by neglecting m_π. The proper way to state our results for $\Gamma(A \to \rho\pi)$ would then be to say that it may have any value between 70 and 140 MeV, depending on the precise values taken for m_ρ, m_{A_1}, and m_π^{ext}.

5. The total width of the A_1 is therefore predicted to be between 110 and 200 MeV in satisfactory agreement with the experimental value $\Gamma(A_1) = 130 \pm 40$. The detailed branching ratios for longitudinal and transverse $A_1 \to \rho\pi$ decay are not known experimentally.

6. The D meson should be the isoscalar companion of the A_1 and would correspond to some octet-singlet mixture in SU(3). Since we predict $m_D \sim m_{A_1}$ we are at least not very far from whatever predictions SU(3) would yield. So far the only reasonable candidate for these quantum numbers ($J^P = 1^+$, $I^{CG} = 0^{++}$) is the D(1280).

7. In order to test our prediction for $g_{\omega\rho\pi}$ we have used the Gell-Mann-Sharp-Wagner model and computed $\Gamma(\omega \to 3\pi)$ and $\Gamma(\omega \to \pi\gamma)$. We find[56]:

$$\begin{aligned} \Gamma(\omega \to 3\pi) &= 14 \pm 3 \text{ MeV} \\ \Gamma(\omega \to \pi + \gamma) &= 1.5 \pm 0.3 \text{ MeV} \end{aligned} \tag{162}$$

to be compared with 10.7 ± 1.5 MeV and 1.15 ± 0.25 MeV, respectively[12].

8. Our value for m_δ (Eq. (160)) is between 960 and 1050 MeV, probably closer to 960. We suggest that the state δ be identified with the observed narrow peak at 960 MeV in the missing mass spectrometer experiments. It is possible that the observed I=1 s-wave $K\bar{K}$ interaction just above the $K\bar{K}$ threshold is related to the same state (which is then a bound state of the $K\bar{K}$ system).

9. The possible contributions of additional isoscalar states have been neglected in our model. Had we included them, we would be forced to use additional experimental numbers as input, since the number of unknowns would increase without adding any new sum rules. The φ, for example, would then appear in Eqs. (127),(128) and the experimental value for $\Gamma(\varphi \to \rho\pi)$ would be an additional input quantity. The contribution of the φ meson to the left hand side of the sum rule (127) is $g^2_{\varphi\rho\pi}$, corresponding to a width for $\varphi \to \rho\pi$ of

$$\Gamma(\varphi \to \rho\pi) = \frac{g^2_{\varphi\rho\pi}\, q^3}{4\pi} \quad ,$$

where q is the three-momentum of the ρ in the rest system of the decaying φ meson. The experimental upper limit[12] on $\Gamma(\varphi \rightarrow \pi^+\pi^-\pi^0)$, including $\Gamma(\varphi \rightarrow \rho\pi)$,

$$\Gamma(\varphi \rightarrow \rho\pi) < .5 \text{ MeV},$$

gives $g^2_{\varphi\rho\pi} < 1/\text{BeV}^2$, which is less than 0.3% of the right hand side of Eq. (84), $8/f^2_\pi$. The total effect of including the φ would therefore be very small and all the predictions would remain essentially unchanged. The η-meson might contribute to the π-δ sum rules. We would then take $\Gamma(\delta \rightarrow \pi\eta)$ from experiment and find that the contribution of the η to the sum rule (135) is of the order of 1% of the right hand side, even if the δ decays only to $\pi\eta$, as long as the δ is identified with the narrow ($\Gamma \sim 5$ MeV) peak at 960 MeV. A possible candidate for a second I=0 scalar meson is the $J^P = 0^+$, $I^{CG} = 0^{++}$ $K\bar{K}$ state at 1060 MeV[12]. Since this state does not decay into two pions, its contributions to the $\pi\pi$ sum rules may be safely neglected. The overall picture that we find here is that it is probable that additional isoscalar states exist, but it is also likely that they do not couple to non-strange mesons and therefore do not have appreciable contributions to our sum rules.

10. The neglected contributions of higher spin mesons can be estimated directly from experiment in a few cases. The f^0 contributes less than 10% to the $\pi\pi$ sum rule, Eq. (125), while the $J^P = 3^-$ Regge recurrence of the ρ may also add a few percent to the left hand side. If we assume that both a sequence of I=0 $\pi\pi$ states with $J^P = 0^+, 2^+, 4^+ \ldots$ and another sequence of I=1 states, with $J^P = 1^-, 3^-, 5^- \ldots$ exist, their contributions will all have the same sign in Eq. (125), thereby decreasing our predicted value for $\Gamma(\sigma \rightarrow \pi\pi)$[67], while in Eq. (126) their contributions

will have opposite signs and will tend to cancel each other to a large extent, leading to a small change in the predicted σ mass.

The situation for the A_2 contribution is similar. It contributes to the sum rules (127),(128). In order to evaluate its contribution to Eq. (127) we write the $A_2^0\rho^+\pi^-$ coupling as $g_{A_2\rho\pi}p_\pi^\alpha \epsilon^{\beta\mu\sigma\tau} p_\pi^\sigma p_{A_2}^\tau e^{\alpha\beta} e^\mu$ where p_π and p_{A_2} are the pion and A_2 momenta, $e^{\alpha\beta}$ is the A_2 polarization tensor, and e^μ the ρ polarization vector. The width for $A_2 \to \rho\pi$ is then

$$\Gamma(A_2 \to \rho\pi) = \frac{g_{A_2\rho\pi}^2 q^5}{20\pi} \quad ,$$

where q is the ρ three-momentum in the decaying A_2's rest system, and the A_2 contribution to the left hand side of Eq. (127) is $\frac{1}{2} g_{A_2\rho\pi}^2 \nu_{A_2}^2 / M_{A_2}^2$ where $\nu_{A_2} = (M_{A_2}^2 - M_\rho^2)/2$. The experimental value for the width, $\Gamma(A_2 \to \rho\pi) = 80$ MeV then yields $\frac{1}{2} g_{A_2}^2 \nu_{A_2}^2 / M_{A_2}^2 = 38/\text{BeV}^2$, which is less than 9% of the $8/f_\pi^2$ on the right hand side of Eq. (127).

If, again, we assume that a sequence of high spin I=0 and I=1 states, each contributing a small amount, should be added to these two sum rules, the total effect would probably be mainly to decrease $g_{\omega\rho\pi}$ (in the right direction!) without modifying much elsewhere. The same characteristic situation is relevant to all sum rules on I=1 targets (see also the π-Σ sum rules in the next section). All I=0 and I=1 s-channel resonances contribute with the same sign to the sum rules for t-channel isospin I=1. The saturation assumptions therefore usually lead to overestimates for the coupling constants. For t-channel I=2 sum rules the s-channel isoscalar and isovector states contribute with opposite signs and cancel each other to a large extent. This may explain the success of most of our predictions for masses, since those follow only from the I=2 sum rules.

C. Algebraic Interpretation

Our model includes 8 particles. All of them have helicity $\lambda = 0$ components, while the four spin-one states have $\lambda = 1$ as well. We first consider the $\lambda = 1$ components of ρ, ω, A_1 and D. There are only two ways to accommodate these states: We may have two $(\frac{1}{2},\frac{1}{2})$ representations or two (0,0) IR's, a (1,0) and a (0,1). Any other combination will either include additional states or violate charge conjugation (which requires equal amounts of (1,0) and (0,1)). The second possibility immediately implies $g_{\omega\rho\pi} = g_{A_1 D\pi} = 0$ and $m_\rho = m_{A_1}$, and is therefore physically uninteresting (although formally it is a solution of our set of equations corresponding to $\psi = 90^\circ$ in Eq. (137) and $\Gamma(\rho \to \pi\pi) = 0$). We are therefore immediately led to two $(\frac{1}{2},\frac{1}{2})$ IR's. One mass formula then follows:

$$m_D^2 + m_\omega^2 = m_{A_1}^2 + m_\rho^2 \tag{163}$$

This is obtained independent of any mixing angles and is, of course, consistent with Eqs. (140) and (147). Another trivial conclusion is:

$$g_{\omega\rho\pi}^2 = g_{DA\pi}^2 \tag{164}$$

which, in the language of sum rules, could be obtained by comparing Eqs. (127) and (131). At this stage there is one free parameter left and we can write:

$$g_{DA\pi}^2 = g_{\omega\rho\pi}^2 = \frac{8}{f_\pi^2} \cos^2\chi \tag{165}$$

$$\frac{(m_A^2 - m_\rho^2)^2}{4m_A^4} g_T^2 = \frac{8}{f_\pi^2} \sin^2\chi \tag{166}$$

$$m_D^2 \sin^2\chi + m_\omega^2 \cos^2\chi = m_\rho^2 \tag{167}$$

$$m_D^2 \cos^2\chi + m_\omega^2 \sin^2\chi = m_A^2 \tag{168}$$

The mixing angle χ can be calculated from any one of these equations using experimental information and is found to be very close to zero[68]. The classification of $\lambda = 1$ states is (for $\chi = 0$): ρ and ω are "purely" in a $(\frac{1}{2},\frac{1}{2})$ IR, while D and A_1 are "purely" in a different $(\frac{1}{2},\frac{1}{2})$. For $\chi \neq 0$ there is some mixing which allows the A_1 to couple to the ρ with $\lambda = 1$.

The analysis of $\lambda = 0$ states follows similar lines and we conclude that for $\psi = 45^\circ$, $\zeta = 0$: σ and $\frac{1}{\sqrt{2}}(\pi + A_1)$ are in a $(\frac{1}{2},\frac{1}{2})$; ρ is in $(1,0) + (0,1)$; $\frac{1}{\sqrt{2}}(\pi - A_1)$ in $(1,0) - (0,1)$; D and ω are each in a $(0,0)$; and δ and X^0 in a $(\frac{1}{2},\frac{1}{2})$. This immediately leads to Eqs. (139)-(149).

It is clear that including additional states (B, A_2, f, φ, η, etc.) would introduce changes in this classification and will complicate the situation tremendously <u>without</u> dramatically improving the agreement with experiment. Since the criteria of a successful model are both the agreement with experiment <u>and</u> its simplicity, we consider our model as a reasonable description of the low-lying non-strange mesons, provided that the σ, δ and D mesons are found to exist and to have properties which approximately agree with our predictions.

We have also considered an expanded version of our model in which "abnormal" 1^+ mesons (like the B meson) and the usual 2^+ mesons are included. This would correspond to the L = 0 and 1 mesons of the $q\bar{q}$ system in the quark model. We found that the complications introduced by the additional states are not compensated by the improved agreement with experiment, while we know that these additional states individually do not contribute more than 10% to any of the Adler-Weisberger sum rules. The sum rules for π-A_2 scattering are briefly discussed in the next section.

VIII. OTHER APPLICATIONS OF t=0 STRONG INTERACTION SUM RULES

We present in this section a few other applications of complete sets of t=0 sum rules. We include the application of our results to baryons, πA_2 scattering, and the extension of our methods to SU(3) $\times$ SU(3) and mesons with non-zero strangeness.

A. Sum Rules for πN and πN^* Scattering

If we saturate all "pure" t=0 charge algebra sum rules and Class I superconvergence relations for $\pi N \to \pi N$, $\pi N \to \pi N^*$, and $\pi N^* \to \pi N^*$ with the N and N* states themselves, where the N* is the $J^P = (3/2)^+$, isospin 3/2 state at 1236 MeV, we find a unique solution with $m_N = m_{N^*}$ and the "SU(6) results" for the coupling constants[4] (like $g_A = 5/3$). Algebraically this saturation assumption is equivalent to putting the helicity $\lambda = \frac{1}{2}$ N and N* in the $(1,\frac{1}{2})$ representation of chiral SU(2) $\times$ SU(2).

As we know, however, the saturation of all t=0 sum rules by one IR does not agree with experiment, nor in this case with the Class II superconvergence relations[60] for $\pi N^* \to \pi N^*$, and a consideration of the contributions to the Adler-Weisberger sum rule shows that many other states have non-negligible contributions. The mixing coefficients of these additional states can be obtained from the weak, electromagnetic, and pionic transitions between the N and N* and the various additional states[69,70,71]. A study of these transitions indicates that the $\lambda = 1/2$, $(1,\frac{1}{2})$ representation includes the "pure" N*(1236) together with a mixed isospin 1/2 state, $\cos\theta|N> + \sin\theta|X>$, where X includes components from the $P_{11}(1400)$, $D_{13}(1520)$, $S_{11}(1560)$, $F_{15}(1670)$, $D_{15}(1688)$, $S_{11}(1700)$ isospin 1/2 nucleon resonances. We immediately obtain the mass formula from our theorem in Section V,

$$\cos^2\theta\, m_N^2 + \sin^2\theta\, m_X^2 = m_{N*}^2 , \tag{169}$$

where m_X^2 is the weighted average (i.e., weighted by the squares of the mixing coefficients) of the $(\text{mass})^2$ values of the isospin 1/2 resonances. If we substitute the experimental values for m_N and m_{N*}, and[71] $\cos\theta = 0.8$, we find that $m_X = 1.64$ BeV, which is clearly in the expected mass range. As the actual mixing coefficients can be obtained only from the so far undetermined rates for the processes $N^*_{\frac{1}{2}} \to N^*(1236) + \pi$, a more detailed analysis of all the sum rules is not yet feasible.

B. Sum Rules for $\pi\Sigma$ Scattering

For $\pi\Sigma$ scattering at t=0 we have in total three sum rules: there are two charge algebra sum rules, one from the commutator $[Q_5^+,Q_5^-] = 2Q^z$ and one from the commutator $[D^+,Q_5^+] = 0$, as well as one Class II superconvergence relation on the amplitude $[f^{t(2)}_{-\frac{1}{2}\frac{1}{2},00}(\nu,t)/(\nu\sqrt{-t})]_{t=0}$. Although we do not yet have the amount of data for the pionic transitions of the Σ that we have for the nucleon, there does exist enough information to make possible at least the rough testing of these sum rules.

Let us begin by writing the sum rules in the limit of saturating them by the Λ, Σ, and Y_1^* intermediate states. In this limit the two charge algebra sum rules read[72]:

$$4g^2_{\Lambda\Sigma\pi} + 4g^2_{\Sigma\Sigma\pi} + \frac{4}{3}\,\frac{(m_{Y_1^*} + m_\Sigma)^2}{2m^2_{Y_1^*}}\, g^2_{Y_1\Sigma\pi} = \frac{8}{f^2_\pi} \tag{170}$$

and

$$\nu_\Lambda(4g^2_{\Lambda\Sigma\pi}) + \nu_\Sigma(4g^2_{\Sigma\Sigma\pi}) + \nu_{Y_1^*}\left(\frac{4}{3}\,\frac{(m_{Y_1^*} + m_\Sigma)^2}{2m^2_{Y_1^*}}\, g^2_{Y_1\Sigma\pi}\right) = 0 \tag{171}$$

where $\nu_x = \frac{1}{2}(m^2_x - m^2_\Sigma)$. The class II superconvergence relation reads

$$4\left(\frac{m_\Lambda + m_\Sigma}{2m_\Sigma}\right)^2 g^2_{\Lambda\Sigma\pi} - 4g^2_{\Sigma\Sigma\pi} - \frac{4}{3}\,\frac{(m_{Y_1^*} + m_\Sigma)^2}{2m^2_{Y_1^*}} \cdot \frac{(m^2_{Y_1^*} - 4m_\Sigma m_{Y_1^*} + m^2_\Sigma)}{4m^2_\Sigma}\, g^2_{Y_1\Sigma\pi} = 0 \tag{172}$$

It is immediately clear that the "SU(6) results",

$$12g^2_{\Lambda\Sigma\pi} = 12g^2_{\Sigma\Sigma\pi} = 9g^2_{Y_1\Sigma\pi} = \frac{8}{f^2_\pi} \tag{173}$$

and

$$m_\Sigma = m_\Lambda = m_{Y_1^*} \quad ,$$

are a solution of both the charge algebra sum rules and the superconvergence relation. As in the pion-nucleon case, however, there is a large discrepancy with experiment or with results inferred from experiment: the values of $g^2_{\Lambda\Sigma\pi}$ and $g^2_{\Sigma\Sigma\pi}$ in Eq. (173) are roughly 50% larger than the value obtained from $g^2_{\pi NN}$ using a D/F ratio of 2/1 and the prediction $g^2_{Y_1\Sigma\pi} = 8/9f^2_\pi$ is about a factor of two larger than the experimental value $g^2_{Y_1\Sigma\pi} = 24.6\ \mathrm{BeV}^{-2}$ obtained from $\Gamma(Y_1^* \to \Sigma\pi)/\Gamma(Y_1^* \to \mathrm{all}) = 9\%$. All this is, of course, not surprising since we know that the Σ has pionic transitions to states other than to just the Λ, Σ, and Y_1^*, so we would assume that the Σ, like the nucleon, is a mixture of several IR's of SU(2) × SU(2). For example, the $Y_0^*(1405)$ is not far above the πΣ threshold and could make a large contribution to the sum rules.

The results of attempting to saturate the three sum rules with the known Y_0^* and Y_1^* resonances[12] are presented in Tables V, VI, and VII. For purposes of comparison with previous work we have written the charge algebra sum rules in a somewhat different form. In Table V we have listed the contributions of the known resonances to the Adler-Weisberger sum rule for πΣ scattering, which we have rewritten in the form

$$(g_A^\Lambda)^2 + (g_A^\Sigma)^2 + \frac{f_\pi^2}{\pi} \int_{\nu_0}^{\infty} \frac{d\nu}{\nu} \left[\sigma_T^{\pi^-\Sigma^+}(\nu) - \sigma_T^{\pi^+\Sigma^+}(\nu)\right] = 0 \qquad (174)$$

where g_A^Λ and g_A^Σ are the axial-vector coupling constants for the Λ-Σ and Σ-Σ transitions[73], and $\sigma_T^{\pi^\mp\Sigma^+}$ is the total cross section for zero-mass $\pi^\mp$ mesons on Σ^+'s. We have used the narrow resonance approximation[74] and corrected for the zero mass of the external pions by multiplying the cross section in the ℓ'th partial wave by $(q^{m_\pi^{ext}=0}/q^{m_\pi^{ext}=m_\pi})^{2\ell}$.

In Table VI we have the resonance contributions to the charge algebra sum rule which comes from the commutator $[D^+, Q_5^+] = 0$,

$$\frac{2}{\pi} \int_0^{\infty} \frac{d\nu}{\nu}\, \mathrm{Im}\, \overline{f}^{s(2)}_{\frac{1}{2}0,\frac{1}{2}0}(\nu, 0) = 0. \qquad (175)$$

We have chosen, however, to separate out the Λ and Σ pole contributions and to write the sum rule in a form analogous to the one used by Adler for pion-nucleon scattering[31],

$$\frac{2}{\pi} \int_{\nu_0}^{\infty} \frac{d\nu}{\nu} \, \mathrm{Im}\, A^{(2)}(\nu,0) = 2(m_\Lambda + m_\Sigma) g^2_{\Lambda\Sigma\pi} - 4 m_\Sigma g^2_{\Sigma\Sigma\pi}, \tag{176}$$

where $A^{(2)}(\nu,t)$ is the usual invariant amplitude[30] with I=2 in the t-channel. Finally in Table VII we present the numerical results for the Class II superconvergence relation, which we have rewritten in terms of the $B^{(2)}(\nu,t)$ invariant amplitude[30]:

$$\frac{2}{\pi} \int_{0}^{\infty} \frac{d\nu}{m_\Sigma} \, \mathrm{Im}\, B^{(2)}(\nu,0) = 0 \tag{177}$$

The numbers in Table VII are essentially those of Babu et al[61].

From the tables we see that the consistency relation and superconvergence relation on the I=2 amplitudes are roughly (and about equally well) satisfied, with the result being very dependent on the cancellation of the large Born terms, which is very sensitive to the D/F ratio.

The Adler-Weisberger sum rule for $\pi\Sigma$ scattering, Eq. (174), is not saturated[75] using the presently known Y*'s. Changing the D/F ratio has little effect in this case. However, it is not yet clear if some combination of: (1) Contributions to the sum rule from low partial wave background at low energies; (2) An increase in some of the Y* partial widths into $\pi\Sigma$; (3) Contributions from yet to be discovered Y*'s in the 1.5 to 3 MeV mass region; and (4) The contribution of, say, 10% of the right hand side of Eq. (174) from a high energy tail above 3 BeV, will not yet make the agreement of experimental data with the sum rule satisfactory. Note that, assuming that no I=2 Y*'s exist, these contributions are of the same sign and in the correct direction to improve the situation.[76] Comments (1) through (4) hold even more strongly for the consistency condition and superconvergence relation (although the signs here are not the same), and we must await better data before asserting proof of the agreement or disagreement of the $\pi\Sigma$ sum rules with experiment.

C. Sum Rules for πA_2 Scattering

As we have shown in Section IV, there are 14 independent sum rules for πA_2 scattering, with 9 of them either charge algebra sum rules or independent Class I superconvergence relations. In spite of the large number of sum rules, we find it very difficult to deduce any useful information (i.e., comparable to present experiments) using the presently known states to saturate the sum rules because: (1) The only pionic transition of the A_2 known with any certainty is $A_2 \rightarrow \rho\pi$, and it contributes less than 10% of the helicity $\lambda = 1$ generalized Adler-Wesiberger relation for πA_2 scattering; (2) The contributions to the generalized Adler-Wesiberger sum rules for a state X are proportional to Γ/ν_x^3, where $\nu_x = \frac{1}{2}(m_x^2 - m^2)$, m is the mass of the target hadron (here the A_2), and Γ is its width into $X + \pi$. Thus contributions of states with masses near the target hadron's mass are emphasized by the ν_x^3 in the denominator. For $\pi\rho$ scattering the fact that $(\nu_{A_2}/\nu_{A_1})^3 \simeq 8$ is the primary reason the contribution of the A_2 is suppressed relative to the A_1. For πA_2 scattering, however, the contributions of particles with masses near the A_2 mass are emphasized, given roughly equal widths for $A_2 \rightarrow X + \pi$ or $X \rightarrow A_2 + \pi$; (3) While superconvergence relations on amplitudes like $f^{t(1)}_{-22,00}(\nu,t)/\nu^4$ are expected to converge very rapidly for large ν because of the ν^4 in the denominator, contributions from states with masses near the A_2 mass are again very strongly emphasized. We know very little about these particular pionic transitions, particularly those to states with masses greater than the A_2 mass. Thus we find little reason to attempt to saturate all the t=0 sum rules by, say, the η, ρ, B, D, and f(1250), when we expect higher states to give very important contributions.

A curious result which may or may not be significant is obtained if we consider only the π-A_2 sum rules of the form (57),(58) for helicity $\lambda = 2$. Among the above mentioned states the f(1250) is the only state having a $\lambda = 2$ component. If the f dominates the two sum rules we find:

$$g^2_{fA_2\pi} = \frac{8}{f^2_\pi} \tag{178}$$

$$(m^2_{A_2} - m^2_f)g^2_{fA_2\pi} = 0 \tag{179}$$

leading to:

$$m_{A_2} = m_f \tag{180}$$

Experimentally: m(f) = 1250 MeV, $m(A_2)$ = 1310 MeV.

D. SU(3) $\times$ SU(3) and Sum Rules for π-K* Scattering

Since all the known strange mesons seem to have isotopic spin $I = \frac{1}{2}$, we can only write sum rules for I=1 t-channel amplitudes for π-K or π-K* scattering. The absence of I=2 amplitudes (and sum rules) prevents us from deriving mass relations among different K*'s. In the language of the IR's of chiral SU(2) $\times$ SU(2), we find that the absence of $I \geq \frac{3}{2}$ K*'s implies that all the known K^* states are in combinations of the $(\frac{1}{2},0)$ and $(0,\frac{1}{2})$ IR's. Since every such IR contains only one isospin multiplet our general mass formula Eq. (101) is useless in this case. Only if we use the full SU(3) × SU(3) can we derive relations among the masses of various K^*'s.

We now generalize our theorem of Section V-B to the SU(3) × SU(3) case. We assume that all commutators of the form $[G_a,\dot{G}_b]$ where G is a (vector or axial) generator of the chiral SU(3) × SU(3) algebra of charges and $\dot{G}$ is its time derivative, do not contain terms belonging to any SU(3) representation larger than the octet. We can then prove: In the limit of exact SU(3) all eigenstates of a given IR of SU(3) × SU(3) have the same $(\text{mass})^2$ values, in spite of the mixing of states introduced by the breaking of SU(3) × SU(3). The proof follows the lines of the analogous theorem for SU(2) × SU(2), and intuitively it can be made plausible from the fact that the only SU(3) × SU(3) breaking, but SU(3) conserving, term belonging to an IR in which the highest SU(3) multiplet is an octet, is the SU(3) singlet of the $(3,\bar{3})$ or $(\bar{3},3)$ representation, and these representations cannot connect an arbitrary IR to itself.

This theorem is not very useful since SU(3) breaking is, in general, appreciable and it follows a non-trivial SU(3) × SU(3) pattern. However, there is one simple result that we can derive, even if SU(3) is broken. The SU(3) breaking terms presumably belong to the (8,1), (1,8), $(3,\bar{3})$ and $(\bar{3},3)$ representations[77]. None of these can connect an (8,1) to a (1,8). Furthermore, the SU(3) conserving terms belong to (1,1), $(3,\bar{3})$ and $(\bar{3},3)$ and are also unable to contribute to such a matrix element. We therefore conclude:

$$((8,1)|M^2|(1,8)) = 0 \tag{181}$$

The practical implication of Eq. (181) is the predicted equality between the $(\text{mass})^2$ values of a state belonging to $(1,8)+(8,1)$ and another state of the same hypercharge and isospin belonging to $(1,8)-(8,1)$.

We now consider the sum rules for π-K*(890) scattering. We have two sum rules of the form[78] (57) for $\lambda = 0$ and $\lambda = 1$. We consider only the $\lambda = 0$ sum rule. The possible contributing states are all K*'s having $J^P = 0^-, 1^+, 2^- \ldots$ The only known such states are the K meson and a possible $J^P = 1^+$ K_A-meson. If these are the only contributing states, the sum rule reads[78]:

$$\frac{4g^2_{K^*K\pi}}{m^2_{K^*}} + \frac{(m^2_{K_A} - m^2_{K^*})^2}{4m^2_{K^*}m^2_{K_A}} g^2 = \frac{4}{f^2_\pi} \tag{182}$$

If we now <u>assume</u> that the $\lambda = 0$ component of K*(890) is purely in $(8,1)+(1,8)$ (in analogy to the classification of the ρ given in Section VII-C) we can use the experimental value of $\Gamma(K^* \to K\pi)$ and find the mixing angle ψ_K defined by:

$$\frac{4g^2_{K^*K\pi}}{m^2_{K^*}} = \frac{4}{f^2_\pi} \cos^2\psi_K \tag{183}$$

We then predict:

$$\Gamma(K_A \to K^*\pi, \text{ longitudinal}) = \frac{1}{16\pi\ f^2_\pi} \frac{(m^2_{K_A} - m^2_{K^*})^3}{m^3_{K_A}} \sin^2\psi_K \tag{184}$$

$$m^2_{K^*} = \cos^2\psi_K\ m^2_K + \sin^2\psi_K\ m^2_{K_A} \tag{185}$$

For[12] $\Gamma(K^* \to K\pi) = 50$ MeV we find[79] $\psi_K = 50^\circ$, and:

$$m_{K_A} = 1090 \text{ MeV} \tag{186}$$

$$\Gamma(K_A \to K^*\pi, \text{ longitudinal}) = 30 \text{ MeV} \tag{187}$$

There are indications for various K^* states between 1000 and 1400 MeV, and at least one of them, $K_A(1320)$, is likely to be a 1^+ meson. Its total width is given[12] as 80 ± 20 MeV. If both the A_1 and B meson exist and have $J^P = 1^+$ they presumably have corresponding axial K^* mesons. While B cannot contribute to π-ρ scattering because of G-parity, the corresponding K^* could contribute to Eq. (182) and our predictions (186),(187) would then correspond to some weighted average of the two axial K^*-mesons[80].

IX. DISCUSSION AND SUMMARY

Our detailed analysis has mainly been concerned with sum rules for elastic pion-hadron scattering amplitudes at t=0, which were assumed to be saturated by several low-lying resonances or bound states. A few questions are immediately raised:

(1) Could we fruitfully extend our discussion to inelastic processes?

(2) Would we reach similar conclusions by analyzing other elastic meons-baryon or baryon-baryon scattering amplitudes?

(3) Is t=0 really the best place to apply the saturation assumptions, or are we losing a huge amount of relevant, accessible physical information by excluding other values of t?

(4) Could we parametrize the contribution of intermediate energy resonances in a way that will enable us to include them in the analysis without introducing an (almost) infinite number of parameters?

(5) Is it possible to include the possible high energy contributions for sum rules involving physically unrealistic processes such as π-ρ scattering?

One basic decision has to be made before we can try to answer these questions. We have to decide: are we trying to create a horribly complicated (but highly realistic) model of the world with a very large number of particles, Regge trajectories, non-resonating partial wave amplitudes, etc., which will enable us to explain an equally large number of experimental facts, or are we content, at this stage, with simplifying approximations, 20-30% errors in our predictions, but a sufficiently simple picture which indicates that at least our basic assumptions and general approach are in reasonable agreement with the existing data and present experimental trends. If we take the first point of view, namely, we decide to try and "solve the world", then we should probably extend our analysis to inelastic processes for any projectile and any target, at all values of t, with a large (or infinite) number of resonances and a suitably parametrized high energy contribution. We are convinced that all of this is technically possible. The more simple-minded attitude, which we have obviously adopted in this article, would tend to answer that the extension of our work to all of these domains would not yield a much better understanding of what we are doing, at least as long as we are dealing with reactions which cannot be carried out in the laboratory. We find it necessary, however, to give a very brief discussion of the first few steps that we would have taken, had we decided to extend our investigation and to try and guess some of the general conclusions that would emerge from such an analysis.

First of all, our analysis could be extended without difficulty to processes such as K-hadron or vector meson-hadron forward elastic scattering. In deriving the current algebra sum rules we then would have to use PCAC for the strangeness-changing axial current and vector meson dominance for the vector current. The off-mass-shell extrapolations become more questionable but all our general results remain valid, including the relation between the Class I superconvergence relations and the t=0 charge algebra sum rules, as well as the clear algebraic distinction between the Class I and Class II sum rules. In the case of K-hadron scattering we lose the I=2 t-channel amplitudes, while for vector meson-hadron scattering the current algebra sum rules involve moments of currents rather than charges, thus complicating the analysis in terms of the chiral algebra.

We can write superconvergence relations for general inelastic reactions, baryon-baryon elastic scattering, and other processes, but the current algebra information is then not applicable. The saturated superconvergence relations yield dynamical relations among coupling constants and masses, and these should probably be studied in great detail. We do not know how to to this in a systematic way, but suggest that a complete analysis of the "pure" t=0 sum rules for, say N-N and N-$\overline{N}$ elastic scattering would be a useful first step towards understanding the algebraic structure of the general superconvergence sum rules.

The question of choosing the "best" value of t for the saturation assumption is perhaps the most crucial one from the practical point of view.

Since saturation is, at best, approximate we must find clever ways of minimizing the effects of those terms that we are forced to neglect because of our ignorance. These effects are the relatively "smooth" high energy terms in the dispersion integrals and the contributions of possible resonances in the 1.5-2.5 BeV region, some of which could have a considerable effect on the sum rules. If we believe the Regge picture we will immediately conclude that at positive (timelike) values of t the high energy terms are enhanced compared to t=0, and for sufficiently large positive t we may even face a divergent sum rule. For negative (spacelike) t the convergence is improved and the high energy terms become unimportant. Thus, for example, the high energy contributions to $\Delta = 2$, $I = 1$ t-channel amplitudes for pion-hadron scattering should be the smallest for the value of t satisfying $\alpha_\rho(t) = 0$ ($t \simeq -0.6$ BeV^2). This would probably be the best place to assume saturation, if we had an "infinitely good" phase shift analysis of the low energy region. In the absence of such an analysis (which does not exist even for πN scattering, let alone π-ρ or π-N^* scattering), values of t such as $t \sim -0.6$ BeV^2 are extremely dangerous, since they are usually in the unphysical region for most of the important low energy region. In π-ρ scattering, for example, $t = -0.6$ BeV^2 is unphysical for $\sqrt{s} \leq 1.28$ BeV. This implies that the contributions of states such as ω and A_1 as well as the non-resonating s, p and d-waves for $\sqrt{s} < 1.28$ BeV grow like polynomials in $\cos\theta$, where $|\cos\theta| > 1$. In fact for the mass region of the A_1, where all kinds of unknown effects occur in π-ρ scattering, $\cos\theta \simeq -4$ at $t = -0.6$ BeV^2. All the uncertainties in the low-energy contributions are therefore grossly enhanced for sufficiently large spacelike values of t, and the saturation by a few discrete states is presumably very dangerous. Two cases in which the ambiguities for large (negative) t and low energies may not be so important are π-π and π-N scattering, in which all the relevant resonances are above threshold at t=0, and remain physical for relatively large values of t; but even in these cases we are not confident that going to large t actually improves the approximation introduced by saturation. Since the data for π-N scattering hints that most of the error introduced by the saturation originates from the resonances around 1.5-2.5 BeV and not from the high energy "tail", we believe that large negative values of t are probably less appropriate for applying the assumption of saturation by a few low-lying states. As far as the immediate neighborhood of t=0 is concerned, we have already demonstrated in Section IV that any small value of t will do, provided that we do not use a doubled number of sum rules which effectively implies using the derivative with respect to t.

The inclusion of more and more resonances in the saturation scheme would in general improve the agreement with experiment, and reduce the uncertainty with respect to various low-energy effects at the expense of adding many more parameters. We can see two alternative ways of determining these parameters: We can either write sum rules for various values of t (a procedure which becomes more reasonable and less dangerous if a sufficiently large number of states is included), or propose some "smooth" parametrization for the masses and widths of the different resonances corresponding to a given Regge trajectory in the s-channel. We believe that the second possibility may be more relevant to our present state of experimental knowledge and we consider this as a possible way of analyzing particularly simple systems such as π-π or π-ρ which couple only to a few trajectories. Some qualitative features of such a possibility have already been presented in our discussion of the sensitivity to the inclusion of additional states of the model of Section VII.

The parametrization of the high energy part for physically unfeasible processes is obvious but not very useful. The Regge prescription is probably a valid parametrization in many cases. However, every t-channel helicity amplitude with a given isospin will include at least <u>one</u> unknown residue function. Every sum rule will therefore include one additional parameter which does not appear in any other sum rule and which cannot be determined experimentally. The best we could hope to do is to compute these residue functions, assuming that the rest of the dispersion integral comes from some low-lying resonances, and then to check the consistency of various residue functions with the factorization theorem. Our present confusion with respect to the status of the low energy resonances discourages us from pursuing this line of investigation, although eventually it may prove useful.

To conclude, let us summarize the main results of our work: We have explained the connection between sum rules derived from the chiral algebra of charges and t=0 superconvergence relations. We found that some of these superconvergence relations (those having even t-channel helicity flip) are simply linear combinations of charge algebra sum rules and therefore are subject to the same algebraic analysis. We have also demonstrated that the other superconvergence relations at t=0 are not related to the algebra of charges and therefore should not (and do not) obey similar algebraic relations. This explains the "mysterious" consistency of some saturated sum rules with higher symmetry results while others seem to contradict them. We have found a simple mass formula for the infinite momentum eigenstates of the irreducible representations of chiral $SU(2) \times SU(2)$. This formula, which states that the (appropriately averaged) $(\text{mass})^2$ values of all states in a given IR are the same, becomes very powerful when the mixtures of physical states corresponding to the IR's are known. We suggest that more effort be directed into finding these mixed states either by the group-theoretical method or by solving sets of saturated "pure" t=0 sum rules. We have presented many applications of our sets of sum rules, generally with satisfactory agreement with experiment, and we believe that the overall picture obtained, strongly supports our assumptions concerning the absence of I=2 terms in the commutator $[Q_5, \frac{d}{dt} Q_5]$ and in the amplitudes for t=0 high energy pion-hadron scattering.

FOOTNOTES AND REFERENCES

1. The Adler-Weisberger sum rule is an obvious example: S. L. Adler, Phys. Rev. Letters 14, 1051 (1965); W. I. Weisberger, Phys. Rev. Letters 14, 1047 (1965).

2. V. de Alfaro, S. Fubini, G. Furlan, and C. Rossetti, Physics Letters 21, 576 (1966).

3. See for example, S. Fubini, Proceedings of the Fourth Coral Gables Conference on Symmetry Principles at High Energies, University of Miami, 1967 (W. H. Freeman and Company, San Francisco, California, 1967).

4. F. Gilman and H. Harari, Phys. Rev. Letters 18, 1150 (1967); and to be published.

5. M. Gell-Mann, Phys. Rev. 125, 1067 (1962).

6. See for example the review of R. F. Dashen, Proceedings of the XIIIth International Conference on High Energy Physics (University of California Press, Berkeley, 1967), p. 51.

7. M. Gell-Mann and M. Levy, Nuovo Cimento 16, 705 (1960).

8. M. Gell-Mann, Physics 1, 63 (1964). Also see Ref. 5, p. 1072 where another Lagrangian model is presented which has no I=2 part to the commutator $[D,Q_5]$.

9. The operators $D^i(t)$ and $S(t)$ considered here are essentially proportional to the operators v_i and u_0 discussed by M. Gell-Mann in Ref. 5.

10. J. Bernstein, S. Fubini, M. Gell-Mann, and W. Thirring, Nuovo Cimento 17, 757 (1960).

11. M. L. Goldberger and S. B. Treiman, Phys. Rev. 110, 1478 (1958).

12. A. H. Rosenfeld et al., Rev. Mod. Phys. 39, 1 (1967).

13. The possibility that $\alpha_0(0) > 1$ is of course excluded by the Froissart bound, while the possibility that $\alpha_0(0) < 1$ destroys none of our superconvergence relations or conclusions and simply makes possible additional sum rules which hold for certain amplitudes in the scattering of pions on high spin hadrons.

14. For trajectories coupling to the $\pi\pi$ system (the ρ meson) we in fact have $\alpha_1(0) \simeq 1/2$.

15. See the analysis of G. Hohler, J. Baacke, H. Schlaile, and P. Sonderegger, Phys. Letters 20, 79 (1966); F. Arbab and C. Chiu, Phys. Rev. 147, 1045 (1966), and references therein.

16. H. Pagels, Phys. Rev. Letters 18, 316 (1967).

17. See for example L. D. Jacobs, University of California Radiation Laboratory Report UCRL-16877, August 1966 and references therein.

18. This of course is the reason one presently can neither prove nor disprove the experimental existence of the Regge cuts which arise from the exchange of two Reggeized ρ mesons and possibly lie above $\alpha_2(0) = 0$, and were recently suggested by R.J.N. Phillips, Phys. Letters 24B, 342 (1967) and I. J. Muzinich, Phys. Rev. Letters 18, 381 (1967).

19. The vanishing of this combination of total cross sections is known as the weak Johnson-Treiman relation and was first derived by V. Barger and M. Rubin, Phys. Rev. 140, B1365 (1965).

20. See, for example, O. I. Dahl *et al.*, University of California Radiation Laboratory Reports UCRL - 16978 and UCRL - 17217, January 1967, submitted to Phys. Rev.

21. H. Harari, Phys. Rev. Letters 17, 1303 (1966); G. Altarelli, F. Buccella, and R. Gatto, Phys. Letters 24B, 57 (1967); P. Babu, F. J. Gilman,

and M. Suzuki, Phys. Letters 24B, 65 (1967); B. Sakita and K. C. Wali, Phys. Rev. Letters 18, 29 (1967); H. Harari, Phys. Rev. Letters 18, 319 (1967); H. Pagels, Ref. 16; V. de Alfaro *et al.*, Ref. 2; F. Gilman and H. Harari, Ref. 4. The success of these results proves that at high energies the I=2 t-channel amplitudes are very small at t=0. It does not explicitly rule out the possibility mentioned in Footnote 18. See D. Horn and C. Schmid, California Institute of Technology Report No. CALT-68-127 (unpublished); and M. Kugler, Phys. Rev., to be published.

22. M. Jacob and G. C. Wick, Ann. Phys. (N.Y.) 7, 404 (1959). Our helicity amplitudes differ from those of Jacob and Wick in that we take helicity matrix elements of the Feynman amplitude, T, defined by $S_{cd,ab} = \delta_{cd,ab} + (2\pi)^4 i\delta^{(4)}(p_c + p_d - p_a - p_b)\cdot\ (16E_aE_bE_cE_d)^{-\frac{1}{2}}T_{cd,ab}$. In this and other regards we follow the notation and conventions of Wang, Ref. 25.

23. M. Gell-Mann, M. L. Goldberger, F. E. Low, E. Marx, and F. Zachariasen, Phys. Rev. 133, B145 (1964).

24. Y. Hara, Phys. Rev. 136, B507 (1964).

25. L. L. C. Wang, Phys. Rev. 142, 1187 (1966).

26. T. L. Trueman, Phys. Rev. Letters 17, 1198 (1966).

27. T. L. Trueman and G. C. Wick, Ann. Phys. (N.Y.) 26, 322 (1964); I. Muzinich, J. Math. Phys. 5, 1481 (1964).

28. The superconvergence relations that we will consider can also be derived for massive pions, but as we are interested in the complete set of sum rules for a given process and use PCAC in deriving the current algebra sum rules, we will set $m_\pi^{ext} = 0$ in all our sum rules.

Many of our results also hold for the scattering of massive pions on hadrons, as we will note at the appropriate points in the text (see Section IV in particular).

29. The amplitude $f^t_{-\frac{1}{2}\frac{1}{2},00}(\nu,t)/(\nu\sqrt{-t})$ is essentially $\bar{f}^t_{-\frac{1}{2}\frac{1}{2},00}(\nu,t)$, but we have chosen th explicitly indicate the powers of ν and the $\sqrt{-t}$ needed to make the resulting amplitude kinematic singularity free in ν and non-vanishing as $t \to 0$. Throughout the remainder of this paper we choose for this reason to write $f^t_{-11,00}(\nu,t)/\nu^2$ rather than $\bar{f}^t_{-11,00}(\nu,t)$, $f^t_{01,00}(\nu,t)/(\nu\sqrt{-t})$ rather than $\bar{f}^t_{01,00}(\nu,t)$, etc.

30. G. Chew, M. L. Goldberger, F. E. Low, and Y. Nambu, Phys. Rev. 106, 1337 (1957).

31. This statement is essentially a generalization (for pion-hadron (spin J) scattering with I=2 in the t-channel) of the consistency condition originally discovered for pion-nucleon scattering by S. L. Adler, Phys. Rev. 137, B1022 (1965).

32. Although we do not obtain a sum rule for I=0 exchange amplitudes, we still have a low energy theorem which relates the scattering amplitude at threshold in a specific way to the matrix element of the operator S(t) (see Eq. (12)).

33. See Eqs. (42),(47),(51),(52), and (54).

34. See Eqs. (38),(41),(45),(46),(48),(49),(50), and (53).

35. The division of superconvergence relations into Classes I and II is independent of the massless pion limit, as are the general results (1) through (4) of this section. Moreover, in the expansion given in Section III of t-channel helicity amplitudes with even values of Δ (satisfying Class I superconvergence relations) in terms of the s-channel forward amplitudes, the coefficients of the

s-channel helicity amplitudes are independent of the massless pion limit.

36. V. de Alfaro, S. Fubini, G. Furlan, and C. Rossetti, Torino preprint.

37. This type of relation has recently been generally treated by E. Abers and V. Teplitz, Phys. Rev. 158, 1365 (1967), who call such a relation a generalized GGMW condition. Their methods have since been used by Y. Lin, Cornell University preprint, to prove our statement (4) quite generally.

38. This method is, of course, independent of taking massless pions and can also be used, for example, to derive the Class I superconvergence relation for nucleon-nucleon scattering where the superconverging amplitude can be constructed by considering that combination of forward amplitudes for which both the singlet and triplet s-wave scattering lengths cancel at threshold. For processes (other than pion-hadron scattering) where both $|\lambda|$ and $|\mu| \neq 0$, Class I and II superconvergence relations correspond to even and odd values of $|\lambda-\mu|$ respectively, rather than even or odd values of $\Delta = \max\{|\lambda|, |\mu|\}$. Δ coincides with $|\lambda-\mu|$ for pion-hadron scattering where the initial helicity in the t-channel, λ, is always zero.

39. This holds because for elastic scattering we expect that at threshold $\sigma_\ell \propto q^{2\ell}$, Im $f_\ell \propto q^{2\ell+1}$, and Re $f_\ell \propto q^{\ell}$.

40. For pion-hadron scattering the s-wave cross-section is independent of the polarization of the target hadron.

41. The coefficients of the various s-channel amplitudes in Eq. (71) can be directly determined by using the Clebsch-Gordan coefficients for coupling spin J=2 and orbital angular momentum L=1 to total angular

momentum 1, 2, and 3 in the s-channel for forward p-wave pion-hadron (J=2) scattering, and choosing the combination of helicity amplitudes for which these partial wave amplitudes vanish.

42. R. Dashen and M. Gell-Mann, Proceedings of the Third Coral Gables Conference on Symmetry Principles at High Energies, University of Miami, 1966 (W. H. Freeman and Company, San Francisco, California, 1966).

43. We use the term "SU(6) results" to refer to results which were originally derived using static SU(6), but are also derivable by saturating the chiral algebra of charges at infinite momentum by the 35 mesons or 56 baryons.

44. See Ref. 42 and M. Gell-Mann, Lectures given at the International School of Physics "Ettore Majoranna" in Strong and Weak Interactions, A. Zichichi, Editor (Academic Press, New York, 1966).

45. Chiral SU(2) × SU(2) is not a subgroup of SU(6) or $SU(6)_W$, but when acting on states at infinite momentum, the collinear SU(2) × SU(2) algebra, which is a common subgroup of SU(6) and $SU(6)_W$, becomes the same as the chiral algebra (see Dashen and Gell-Mann, Ref. 42).

46. We define the $\omega\rho^o\pi^o$ and $\varphi\rho^o\pi^o$ couplings as $g_{\omega\rho\pi}\epsilon^{\mu\nu\sigma\tau}e^\mu_\rho e^\nu_\omega q^\sigma p^\tau$ and $g_{\varphi\rho\pi}\epsilon^{\mu\nu\sigma\tau}e^\mu_\rho e^\nu_\varphi q^\sigma p^\tau$, where e^μ_ρ, e^ν_ω, and e^ν_φ are the ρ, ω, and φ polarization vectors and q(p) is the $\rho(\pi)$ four-momentum. The $\rho^o\pi^+\pi^-$ coupling is defined as $g_{\rho\pi\pi}(p^\mu_{\pi^+} - p^\mu_{\pi^-})e^\mu_\rho$ where p_{π^+} and p_{π^-} are the pion four-momenta and e^μ_ρ is the ρ polarization vector.

47. This also corresponds to the SU(6) classification where the φ meson is chosen as the octet-singlet combination that does not couple to $\rho\pi$.

48. The operator D was first used by S. Fubini, G. Furlan and C. Rossetti,

Nuovo Cimento 40, 1171 (1965), in deriving SU(3) mass formulae. See also V. de Alfaro *et al.*, Ref. 36.

49. We are aware of the ambiguity involved in discussing the limit of $p_z < \alpha|D^i|\beta >$ as $p_z \to \infty$ with $< \alpha|D^i|\beta > = 0$, and concluding $m_\beta^2 - m_\alpha^2 = 0$ because the left hand side of Eq. (95) is zero. However, we can formulate the same argument in terms of contributions to covariant sum rules, and derive Eq. (98) from them. We find that the procedure presented here is correct if the I=2 sum rules converge, i.e., if $\alpha_2(0) < 0$. All our results can be derived either from the covariant sum rules, or from the algebraic approach discussed in this section.

50. Actually Eqs. (95)-(98) lead directly to $m_\omega^2 = m_\rho^2$. However, it is simple to show that the operator D, being in the $(\frac{1}{2},\frac{1}{2})$ IR, does not connect the π in (1,0)-(0,1) to the ρ in (1,0)+(0,1), and the conclusion $m_\pi^2 = m_\rho^2$ then follows directly using the same argument as in Eqs. (95)-(98).

51. We take $I' \neq I$ because we will consider particles which share one IR, say (m,n), in which each isospin $m + n \geq I \geq |m-n|$ occurs once and only once.

52. See, for example, the decuplet dominance calculations of S. Fubini, G. Segre, and D. Walecka, Ann. of Phys. (N.Y.) 39, 381 (1966).

53. From $\Gamma(\rho \to \pi\pi) = 125$ MeV $= (g_{\rho\pi\pi}^2/6\pi)(q^3/m_\rho^2)$, where q is the three-momentum of either final massive pion in the ρ meson rest frame, we obtain $\cos^2\psi = 0.48$. Calculations based on the algebra of currents, PCAC, and vector dominance lead to $\cos^2\psi = \frac{1}{2}$. See M. Suzuki and K. Kawarabayashi, Phys. Rev. Letters 16, 225, 384(E)(1966); F. J. Gilman and H. J. Schnitzer, Phys. Rev. 150, 1362 (1966); J. J. Sakurai, Phys. Rev. Letters 17, 552 (1966).

54. The width given here is for the final pion being massless, $m_{A_1} = 1070$ MeV, and $m_\rho = 765$ MeV. As the width $\Gamma(A_1 \rightarrow \rho\pi$ longitudinal) $\propto g_L^2 q^5$, where q is the pion three-momentum in the A_1 rest frame, the exact width depends fairly sensitively on the masses of the A_1, ρ, and π, as well as on any assumed dependence of the coupling constant g_L on the mass of the pion. With the same A_1 and ρ masses and value of g_L as above, but with $m_\pi = 140$ MeV, the width is reduced by a factor of 1.6.

55. If we use $\cos^2\psi = \frac{1}{2}$ and neglect m_π^2/m_ρ^2, we can rewrite Eq. (113) as $m_{A_1} = \sqrt{2}\, m_\rho$. This relation was found by S. Weinberg, Phys. Rev. Letters 18, 507 (1967), who derived it from a totally different set of assumptions, but using precisely the same numerical approximations.

56. M. Gell-Mann, D. Sharp, and W. G. Wagner, Phys. Rev. Letters 8, 261 (1962). We use $\Gamma(\omega \rightarrow \pi + \gamma) = 1.2 \pm 0.3$ MeV, $f_\rho^2/4\pi = 2.5 \pm 0.4$ as determined by J. J. Sakurai, Phys. Rev. Letters 17, 1021 (1966). The quoted errors in our value for $g_{\omega\rho\pi}$ do not include the errors introduced by the model or by errors in the masses.

57. Definitive experimental evidence on this point is difficult to obtain.

58. This explains the rederivation of various "SU(6) results" by S. Fubini and G. Segre, Nuovo Cimento 45(A), 641 (1966); V. de Alfaro *et al.*, Ref. 2; H. F. Jones and M. D. Scadron, Nuovo Cimento 48(A), 545 (1967); H. F. Jones and M. D. Scadron, Nuovo Cimento, to be published; P. H. Frampton, preprint (1967); R. Oeheme and G. Venturi, preprint (1967); K. Bardakci and G. Segre, Phys. Rev., to be published.

59. Taking massless external pions also removes the difficulty found by F. E. Low, *Proceedings of the Thirteenth International Conference on High Energy Physics, Berkeley, California, 1966* (University of California Press, Berkeley, 1967), p. 244. Low remarked that if we

saturate the three superconvergence relations for $\pi\rho$ scattering with the π, ω, and φ, the only consistent solution is $g_{\rho\pi\pi} = g_{\omega\rho\pi} = g_{\varphi\rho\pi} = 0$. This would contradict the inhomogeneous Eqs. (84) and (85). In the limit $m_\pi^{ext} = 0$, however, Low's objection is not valid and a consistent solution is obtained.

60. See H. F. Jones and M. D. Scadron, Nuovo Cimento, to be published; and P. H. Frampton, preprint (1967). Our analysis of Class I and Class II superconvergence relations immediately shows which sum rules (i.e., those in Class I) one can expect from the outset to be consistent with "SU(6) results". It is exactly the Class I superconvergence relations which the above authors find are consistent, while the Class II sum rules for $\pi N^* \rightarrow \pi N^*$ are found to be inconsistent with "SU(6) results".

61. P. Babu, F. J. Gilman, and M. Suzuki, Ref. 21.

62. The $A_1^o\sigma\pi^o$ and $D\delta^o\pi^o$ couplings are defined analogously to the $\rho^o\pi^+\pi^-$ coupling, and the $DA_1^o\pi^o$ coupling analogously to the $\omega\rho^o\pi^o$ and $\varphi\rho^o\pi^o$ couplings in Ref. 46. In terms of widths: $\Gamma(\rho \rightarrow \pi\pi) = (g^2_{\rho\pi\pi}/6\pi)(q^3/m_\rho^2)$; $\Gamma(\sigma \rightarrow \pi\pi) = (3g^2_{\sigma\pi\pi}/16\pi)(m_\sigma^2 - m_\pi^2 - (m_\pi^{ext})^2)^2 q/(4m_\sigma^2)$; $\Gamma(\varphi \rightarrow \rho\pi) = (g^2_{\varphi\rho\pi}/4\pi)q^3$; $\Gamma(A_1 \rightarrow \rho\pi) = (g_T^2/6\pi)(q^5/m^2_{A_1}) + (g_L^2/12\pi)(q^5/m_\rho^2)$; $\Gamma(D \rightarrow A_1\pi) = (g^2_{DA_1\pi}/4\pi)q^3$; $\Gamma(A_1 \rightarrow \sigma\pi) = (g^2_{A_1\sigma\pi}/6\pi)(q^3/m^2_{A_1})$; $\Gamma(D \rightarrow \delta\pi) = (g^2_{D\delta\pi}/2\pi)(q^3/m_D^2)$; $\Gamma(\delta \rightarrow X\pi) = (g^2_{X\delta\pi}/8\pi)(m_\delta^2 - m_X^2 - (m_\pi^{ext})^2)^2 q/(4m_\delta^2)$; where q is the three momentum of the final pion in the decaying particles rest frame, if the decay is kinematically possible.

63. In calculating the widths below we have consistently assumed $m_\pi^{ext} = 0$. Corrections for massive pions are discussed in Section VII-B.

64. See the discussion of the experimental evidence for the existence of scalar mesons in G. Goldhaber, Proceedings of the XIIIth International Conference on High Energy Physics (University of California Press, Berkeley 1967), p. 128.

65. S. L. Adler, Phys. Rev. 140, B736 (1965).

66. The experiments where the H meson was (and was not) seen are discussed by G. Goldhaber, Proceedings of the XIIIth International Conference on High Energy Physics (University of California Press, Berkeley 1967), p. 128.

67. Note that a contribution from the f^o of 10% of the $8/f_\pi^2$ on the right hand side of Eq. (125) reduces $\Gamma(\sigma \to \pi\pi)$ by 20%, to about 460 MeV for massive pions.

68. When we consider the Class II superconvergence relation for πA_1 scattering, we find that saturation by just the σ, ρ, and D with the couplings and masses as above are inconsistent. However, introduction of the B meson ($J^P = 1^+$, $I^{CG} = 1^{-+}$) yields a unique solution to all the sum rules with $g_{\sigma A_1 \pi}$ and m_σ as before, but with $g^2_{DA_1\pi}$ reduced to $\frac{1}{2}$ its value in Eq. (146) and $m_D = m_B$ (and not related to m_{A_1}). If the D and B are to be identified with the mesons at 1280 MeV and 1220 MeV, respectively, then this also agrees better with experiment than the prediction in Eq. (158).

69. R. Gatto, L. Maiani, and G. Preparata, Phys. Rev. Letters 16, 377 (1966).

70. I. S. Gerstein and B. W. Lee, Phys. Rev. Letters 16, 1060 (1966).

71. H. Harari, Phys. Rev. Letters 16, 964 (1966), and 17, 56 (1966).

72. The $\Lambda^o\Sigma^+\pi^-$ and $\Sigma^o\Sigma^+\pi^-$ couplings used here are the pseudovector couplings, $g_{AB\pi} q_\pi^\mu \, \bar{u}(p_A)\gamma_\mu\gamma_5 u(p_B)$, while the $Y_1^{*o}\Sigma^+\pi^-$ coupling is defined as $g_{Y_1\Sigma\pi} q_\pi^\mu \, \bar{u}_\mu(p_{Y_1})u(p_\Sigma)$, with q_π the pion four-momentum and

$\bar{u}_\mu(p_{Y_1})$ the Rarita-Schwinger spinor for the Y_1^* with spin 3/2.

73. In terms of the D/F ratio and the axial-vector coupling constant $g_A = 1.18$ of the nucleon, $(g_A^\Lambda)^2 = \frac{2}{3}(D/F+D)^2 g_A^2$ and $(g_A^\Sigma)^2 = 2(F/F+D)^2 g_A^2$.

74. The narrow resonance approximation tends to overestimate the contribution of a given resonance, particularly those near threshold. For the $Y_1^*(1385)$, integrating directly over a Breit-Wigner resonance shape yields a contribution to the sum rules of 70-80% that given in the Tables. For higher mass resonances the effect is much less. The widths used for the resonances are taken from Rosenfeld et al., Ref. 12, except for the $Y_0^*(1405)$, which is from J. Kim, private communication.

75. For another attempt at saturating the $\pi\Sigma$ Adler-Wesiberger sum rule see G. Shaw, Phys. Rev. Letters 18, 1025 (1967). Our values for the resonance contributions differ somewhat from his due to different widths for the resonances in some cases, the use of $f_\pi = 135$ MeV (from the pion lifetime) instead of $\sqrt{2}\, g_A m_N/g_{\pi N}$ (the Goldberger-Treiman prediction for f_π), the correction we use for zero mass pions, and the narrow resonance approximation. All these tend to increase the value of the left hand side of the sum rule to the maximum possible, and the sum of our contributions is greater than Shaw's, but the sum rule is still only about 70% saturated.

76. See Section VII-B for a discussion of the same situation for the pion-meson (isospin 1) scattering sum rules.

77. This follows from our initial assumption above that the SU(3) breaking terms belong to representations no larger than octet.

78. Eq. (57) for isospin 1/2 targets has $4/f_\pi^2$ rather than $8/f_\pi^2$ on the right hand side.

79. Again we are calculating using massless pions. For massive pions we find $\psi_K \simeq 45^o$ and $m_{K_A} \simeq 1160$ MeV. The coupling constants are similar to the $\pi\rho$ case with $K^{*+}(890)K^o\pi$ analogous to $\rho\pi\pi$ and $K_A K^*\pi$ analogous to $A_1\rho\pi$.

80. We can also use the transverse sum rules, if we wish, to obtain the $K^*(890)K^*(890)\pi$ coupling, but the size of the errors and the lack of any experimental number to compare the result with make this mostly a formal exercise.

TABLE I

Asymptotic behavior, crossing properties and type of non-trivial dispersion or superconvergence (S.C.) relation holding for the kinematic-singularity-free amplitudes for pion-hadron scattering at t=0, where the hadron spin is $J \leq 2$. I is the t-channel isospin and $\alpha_1(0) = 1$, $\alpha_1(0) < 1$, $\alpha_2(0) < 0$. For a given spin J only the first 2J + 1 lines of the table are relevant ($\Delta \leq 2J$).

Amplitude	Δ	I = 0	I = 1	I = 2
$f^t_{\lambda\lambda,00}(\nu,0)$ $-J \leq \lambda \leq J$	0	ν^{α_0} even 1 subtraction	ν^{α_1} odd No subtraction	ν^{α_2} even No subtraction
$[f^t_{\lambda(\lambda+1),00}(\nu,t)/(\nu\sqrt{-t})]_{t=0}$ $-J+1 \leq \lambda+1 \leq J$	1	ν^{α_0-1} odd No subtraction	ν^{α_1-1} even No subtraction	ν^{α_2-1} odd S.C.(N=0)
$[f^t_{\lambda(\lambda+2),00}(\nu,0)/\nu^2]$ $-J+3 \leq \lambda+2 \leq J$	2	ν^{α_0-2} even No subtraction	ν^{α_1-2} odd S.C.(N=0)	ν^{α_2-2} even S.C.(N=1)
$[f^t_{\lambda(\lambda+3),00}(\nu,t)/(\nu^3\sqrt{-t})]_{t=0}$ $-J+3 \leq \lambda+3 \leq J$	3	ν^{α_0-3} odd S.C.(N=0)	ν^{α_1-3} even S.C.(N=1)	ν^{α_2-3} odd S.C.(N=2,0)
$[f^t_{\lambda(\lambda+4),00}(\nu,0)/\nu^4]$ $-J+4 \leq \lambda+4 \leq J$	4	ν^{α_0-4} even S.C.(N=1)	ν^{α_1-4} odd S.C.(N=2,0)	ν^{α_2-4} even S.C.(N=3,1)

TABLE II

The number of possible charge-algebra and Class I and II superconvergence relations for pion-hadron (spin J) forward elastic scattering. I is the t-channel isospin, and asterisks denote the sum rules which are linear combinations of other t=0 sum rules:

J	Charge Algebra		Class I			Class II			Total number of independent sum rules
	I=1	I=2	I=0	I=1	I=2	I=0	I=1	I=2	
0	1	1							2
$\frac{1}{2}$	1	1						1	3
1	2	2		1*	1*			1	5
$\frac{3}{2}$	2	2		1*	1*	1	1	3+1*	9
2	3	3	1	1+3*	1+3*	1	1	3+1*	14

TABLE III

Contributions of the π, ω, and A_1 intermediate states and the high energy region to the sum rule (73) for π-ρ scattering, assuming $g^2_{\rho\pi\pi} = m^2_\rho/f^2_\pi$, $g_{\rho\omega\pi} = 21\ \text{GeV}^{-1}$, $g_T = 0$, $\Gamma(A_1 \rightarrow \rho\pi) = 110$ MeV, $f(\nu,t) \propto ae^{10t}$ for large ν and normalizing a so that the high energy contribution at t=0 is 10% of that of the ω.

	π	ω	A_1	ae^{10t}
t=0	1.0	-2.0	1.0	0.20
$t = -m^2_\pi$	1.0	-2.03	0.87	0.16
$m^2_\rho \frac{d}{dt}\Big\vert_{t=0}$	0	-1.0	-4.0	-1.16
$m^4_\rho \frac{d^2}{dt^2}\Big\vert_{t=0}$	0	0	0	-6.8

TABLE IV

Quantum numbers of mesons in the model of Section VII.

Particle	J^P	I	G
π	0^-	1	-
X^o	0^-	0	+
δ	0^+	1	-
σ	0^+	0	+
ρ	1^-	1	+
ω	1^-	0	-
A_1	1^+	1	-
D	1^+	0	+

TABLE V

Contributions to the Adler-Weisberger sum rule for $\pi\Sigma$ scattering,

$$(g_A^{\Lambda})^2 + (g_A^{\Sigma})^2 + \frac{f_\pi^2}{\pi} \int_{\nu_0}^{\infty} \frac{d\nu}{\nu} \left[\sigma_T^{\pi^-\Sigma^+}(\nu) - \sigma_T^{\pi^+\Sigma^+}(\nu)\right] = 2,$$

from the bound states and resonances with known $\pi\Sigma$ couplings.

Resonance or Bound State	J^P	Γ_{tot} (MeV)	$\frac{\Gamma_{\pi\Sigma}}{\Gamma_{tot}}$ (%)	Contribution to left hand side of sum rule. D/F = 2/1	3/2
$\Lambda(1115)$	$(1/2)^+$			.41	.33
$\Sigma(1193)$	$(1/2)^+$			.31	.45
$Y_1^*(1385)$	$(3/2)^+$	40	9	.22	
$Y_0^*(1405)$	$(1/2)^-$	50	100	.33	
$Y_0^*(1520)$	$(3/2)^-$	16	51	.04	
$Y_1^*(1660)$	$(3/2)^-$	50	30	.03	
$Y_1^*(1770)$	$(5/2)^-$	89	< 1	$< .002$	
$Y_0^*(1820)$	$(5/2)^+$	83	11	.01	
$Y_1^*(1910)$	$(5/2)^+$	60	3	$< .002$	
			Sum =	1.35	1.41

TABLE VI

Contributions to the left hand side of the sum rule,

$$-2(m_\Lambda + m_\Sigma)\, g^2_{\Lambda\Sigma\pi} + 4m_\Sigma g^2_{\Sigma\Sigma\pi} + \frac{2}{\pi}\int_{\nu_0}^{\infty}\frac{d\nu}{\nu}\,\mathrm{Im}\,A^{(2)}(\nu,0) = 0$$

from the bound states and resonances with known $\pi\Sigma$ couplings.

Resonance or Bound State	J^P	Γ_{total} (MeV)	$\frac{\Gamma_{\pi\Sigma}}{\Gamma_{total}}$ (%)	Contribution (1/BeV) D/F = 2/1	3/2
$\Lambda(1115)$	$(1/2)^+$			-143	-116
$\Sigma(1193)$	$(1/2)^+$			110	160
$Y_1^*(1385)$	$(3/2)^+$	40	9	-36	
$Y_0^*(1405)$	$(1/2)^-$	50	100	+ 8	
$Y_0^*(1520)$	$(3/2)^-$	16	51	-16	
$Y_1^*(1660)$	$(3/2)^-$	50	30	+15	
$Y_1^*(1770)$	$(5/2)^-$	89	< 1	- 1	
$Y_0^*(1820)$	$(5/2)^+$	83	11	- 6	
$Y_1^*(1910)$	$(5/2)^+$	60	3	+ 1	

TABLE VII

Contributions to the left hand side of the superconvergence relation

$$\frac{2}{\pi} \int_0^\infty \frac{d\nu}{m_\Sigma} \text{ Im } B^{(2)}(\nu,0) = 0$$

from the bound states and resonances with known $\pi\Sigma$ couplings.

Resonance or Bound State	J^P	Γ_{total} (MeV)	$\frac{\Gamma_{\pi\Sigma}}{\Gamma_{total}}$ (%)	Contribution (1/BeV) D/F = 2/1	3/2
$\Lambda(1115)$	$(1/2)^+$			+140	+113
$\Sigma(1193)$	$(1/2)^+$			-110	-160
$Y_1^*(1385)$	$(3/2)^+$	40	9	+41	
$Y_0^*(1405)$	$(1/2)^-$	50	100	+ 1	
$Y_0^*(1520)$	$(3/2)^-$	16	51	+19	
$Y_1^*(1660)$	$(3/2)^-$	50	30	-18	
$Y_1^*(1770)$	$(5/2)^-$	89	< 1	< + 1	
$Y_0^*(1820)$	$(5/2)^+$	83	11	+ 3	
$Y_1^*(1910)$	$(5/2)^+$	60	3	- 2	