

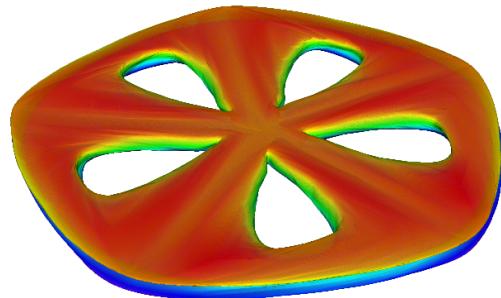
# Superstring measures in genus 5

Abel Stern

Friday 10th August, 2012

Bachelor thesis

Supervisor: Petr Dunin-Barkowski



KdV Institute for Mathematics

Faculty of Science

University of Amsterdam



## Abstract

By comparing expressions for degenerated curves, we show that combining two semi-modular forms from the two different ansätze for the chiral superstring measure in genus 5 yields a form that is not contained in either ansatz. We use this form to construct a modified ansatz for genus 5. By calculating the resulting two-point function for genus 4 and the cosmological constant in genus 5 we show that for our modified ansatz, both of them vanish as required. Thus, we solve the problem posed in [1]. Last, we show that from the currently known forms we cannot construct an ansatz for genus 6 that satisfies all requirements.

## Data

Title: Superstring measures in genus 5

Author: Abel Stern, abel.stern@gmail.com, 5948983

Supervisor: Petr Dunin-Barkowski

Second assessor: prof. dr. Gerard van der Geer

End date: Friday 10th August, 2012

Korteweg de Vries Institute for Mathematics

University of Amsterdam

Science Park 904, 1098 XH Amsterdam

<http://www.science.uva.nl/math>

# Contents

<b>1</b>	<b>Introduction</b>	<b>2</b>
<b>2</b>	<b>Definitions</b>	<b>5</b>
2.1	The symplectic group . . . . .	5
2.2	The Abel-Jacobi map and the period matrix . . . . .	5
2.3	The semi-modular forms from OPSMY and Grushevsky . . . . .	6
<b>3</b>	<b>Research</b>	<b>9</b>
3.1	Degeneration . . . . .	9
3.1.1	The expansion of $G_5^{(5)}$ . . . . .	10
3.1.2	The expansion of $\vartheta_5^{(5)}$ . . . . .	13
3.1.3	The final expression . . . . .	14
3.2	The trace of $f^{(5)}$ . . . . .	15
3.3	The difference between $f^{(5)}$ and $J^{(5)}$ . . . . .	17
3.4	The two-point function in genus 4 . . . . .	18
3.5	The situation in genus 6 . . . . .	21
3.6	Conclusion . . . . .	22
<b>4</b>	<b>Popular summary</b>	<b>23</b>

# Chapter 1

## Introduction

In perturbative string theory scattering amplitudes can be represented as integrals over the moduli space of Riemann surfaces  $\mathcal{M}_g$  with respect to a certain measure. Therefore, this *string measure* is one of its main ingredients. In the case of NSR superstring theory, for every genus a set of measures is needed, which correspond to theta characteristics: vectors containing a zero or one for each out of the  $2g$  non-contractable cycles, up to homotopy, on the Riemann surface.

Constructing these superstring measures directly has proven to be exceedingly difficult, and it took many years until in 2002 the genus 2 measure was found, in the papers [2] by E. D'Hoker and D. Phong. Another approach has surfaced recently, where instead of explicit calculation, ansätze were made based on supposed requirements for the measure.

It has been conjectured [3] that the NSR measures  $d\mu[e]$  can be written as a product of the Mumford measure for the critical bosonic string  $d\mu$  and for each characteristic a semi-modular form  $\Xi[e]$  of weight 8 on the Siegel upper half-space:

$$d\mu[e] = \Xi[e]d\mu. \quad (1.1)$$

In genus  $g \leq 3$  it is known that this is in fact the case, but in general for higher genera it is not known a priori whether a suitable form can be constructed, even if the form is only defined on the subspace of period matrices inside the Siegel upper half-space. The closure of the subspace of period matrices is called the Jacobian locus and it has nonzero codimension from genus 4 on. All definitions can be found in section 2.3.

The conditions to which the measures, if the above conjecture holds, must conform are the following:

- a) The forms  $\Xi[e]$  are semi-modular forms of weight 8 when restricted to the Jacobian locus.
- b) When the Riemann surface degenerates to a disjunct union of lower-genus surfaces, the forms factorize into a product of lower-genus forms. That is,  $\Xi_{e_1 \times e_2}^{(g_1+g_2)} \begin{pmatrix} \tau^{(g_1)} & 0 \\ 0 & \tau^{(g_2)} \end{pmatrix} = \Xi_{e_1}^{(g_1)}(\tau^{(g_1)}) \Xi_{e_2}^{(g_2)}(\tau^{(g_2)})$ .
- c) The trace (the cosmological constant) should vanish, i.e.  $\sum_e \Xi[e] = 0$ . Also, the trace of the  $1, \dots, 3$ -point functions  $\sum_e A_n[e]$  should vanish, cf. [4].

d) In genus 1 and 2 the ansatz should conform to the known answers.

Two sets of ansätze were proposed: one in terms of theta series for 16-dimension self-dual lattices by M. Oura, C. Poor, R. Salvati Manni and D. Yuen (OPSMY) in [5], and one in terms of summations of powers of products of ordinary theta constants, which was originally suggested by S.L. Cacciatori, F. Dalla Piazza and B. van Geemen in [6] and written in its final and elegant form by S. Grushevsky in [7]. These have been shown to coincide for genus  $g \leq 4$  and, in fact, to be the unique measures constructed in the above way that conform to all requirements for these genera.

As shown by S. Grushevsky and R. Salvati Manni in [8], the genus 5 cosmological constant did not vanish for these ansätze. This problem with the cosmological constant was solved by OPSMY and Grushevsky by modifying the genus 5 ansätze. However, it was shown by M. Matone and R. Volpato in [1] that the genus 4 two-point function obtained by degeneration from the modified genus 5 ansätze does not vanish, contrary to the above requirements.

A natural question, then, became whether these ansätze do in fact coincide for genus  $g = 5$  and if not, what can be done by combining their building blocks.

The paper [9] compares the semi-modular forms  $G_p^{(g)}$  and  $\vartheta_p^{(g)}$ , from which the Grushevsky and OPSMY ansätze were constructed. For all but one  $p$  (where  $0 \leq p \leq 7$ ) it was shown that  $\vartheta_p^{(g)}$  was expressible through the  $G_i^{(g)}$ , for *all* genera. For genus 5 and above, however, the question remained open whether  $G_5^{(g)}$  and  $\vartheta_5^{(g)}$  agree on the Jacobian locus.

**Results** In the present paper (at the end of section 3.1) we show that in fact, for genus  $g \geq 5$ , on the Jacobian locus,  $G_g^{(g)}$  and  $\vartheta_5^{(g)}$  do *not* agree. We use the fact that  $\vartheta_5^{(5)} - G_5^{(5)}$  is nonzero on the Jacobian locus to present a modified genus 5 ansatz,

$$\tilde{\Xi} := \Xi_{OPSMY}^{(5)} - \frac{222647008}{217} \left( \vartheta_6^{(5)} - \vartheta_7^{(5)} \right) + \frac{77245568}{17} \left( \vartheta_5^{(5)} - G_5^{(5)} \right). \quad (1.2)$$

We prove the vanishing of both the genus 5 cosmological constant and the genus 4 two-point function, obtained from degeneration, for this modified ansatz. Then, we look at the situation in genus 6. We show that it is not possible to construct a genus 6 ansatz from the currently known forms that satisfies all properties. To be precise, condition c) cannot be satisfied.

**Structure of the present paper** The paper is organized as follows: in section 2.3 we define the semi-modular forms used in the OPSMY and Grushevsky ansätze and list the known relations between those sets of forms. In section 3.1 we expand  $\vartheta_5^{(5)} - G_5^{(5)}$  in a perturbative series by contracting one handle of the curves and show that this series does not vanish on the entire Jacobian locus, which means  $\vartheta_5^{(5)} - G_5^{(5)}$  is nonzero there. In section 3.2 we calculate the trace (the summation  $\sum_e f[e]$  over even characteristics) of this function. We need this to prove that the cosmological constant for our modified ansatz

in genus 5 vanishes. In section 3.3 we compare  $\vartheta_5^{(5)} - G_5^{(5)}$  with other semi-modular forms to show it is not equal to one of the already known forms. In section 3.4 we look at the two-point function in genus 4 obtained by degenerating the genus 5 ansatz  $\Xi_{OPSMY}^{(5)} + c(\vartheta_6^{(5)} - \vartheta_7^{(5)}) + d(\vartheta_5^{(5)} - G_5^{(5)})$ , by the method used in [1]. We show that this, together with the condition of vanishing genus 5 cosmological constant leads to our main formula (3.76): a unique ansatz built from the known semi-modular forms in genus 5. In section 3.5 we discuss the factorization property for any genus 6 ansatz implied by our proposed modification for genus 5. We show that it cannot be satisfied using only the known forms. Finally, in section 3.6 we briefly summarize our results.

# Chapter 2

## Definitions

In this chapter, we define the relevant concepts and introduce the semi-modular forms  $G_p^{(g)}$  and  $\vartheta_k^{(g)}$ .

### 2.1 The symplectic group

A  $2n \times 2n$  matrix  $A$  containing entries in a ring  $R$  is called *symplectic* if  $A^t M A = M$ , where  $M = \begin{bmatrix} 0 & I \\ -I & 0 \end{bmatrix}$ . This implies, among other things, that  $A$  has determinant 1 and is invertible. The set of all such matrices  $A$  form a group under ordinary matrix multiplication: the *symplectic group*  $\mathrm{Sp}(2n, R)$ .

### 2.2 The Abel-Jacobi map and the period matrix

Let  $C$  be a complex manifold of genus  $g$ . This means, or implies, that there are  $2g$  classes of loops in  $C$  generating the first homology group. It also implies that the space of holomorphic differentials on  $C$  has dimension  $g$ ; we pick a basis  $v_i$  with  $1 \leq i \leq g$ . Now, let us take a representative  $\gamma_i$  from each of the  $2g$  classes of loops mentioned above. Define a lattice  $L$  in  $\mathbb{C}^g$  by the following basis:  $l_i := \left( \int_{\gamma_i} v_1, \dots, \int_{\gamma_i} v_g \right)$ . Then, picking an arbitrary base point  $p_0 \in C$ , we define the Abel-Jacobi map:  $A(p) := \left( \int_{p_0}^p v_1, \dots, \int_{p_0}^p v_g \right) \bmod L$ .

In a similar construction, we can divide the set of loops  $\gamma_i$  in two sets with  $g$  elements,  $A$  and  $B$ , by requiring that  $A_i \cap B_j = \delta_{ij}$ ,  $A_i \cap A_j = B_i \cap B_j = \emptyset$ ,  $i \neq j$ . Then, choose a basis  $\omega_i$  for the space of holomorphic differentials by requiring that  $\int_{B_i} \omega_j = \delta_{ij}$ . This leaves us with a symmetric  $g \times g$  matrix determined by the manifold  $C$ , as follows:  $\tau_{ij} := \int_{A_i} \omega_j$ . This matrix  $\tau$  is called the *period matrix* and is symmetric, with positive definite imaginary part.

## 2.3 The semi-modular forms from OPSMY and Grushevsky

The superstring ansätze are composed of linear combinations of semi-modular forms of weight 8 on the Jacobian locus. Here, we will define the relevant concepts.

Let  $\mathcal{H}_g$  be the Siegel upper half-space, i.e. the set of complex symmetric  $g \times g$ -matrices for which the imaginary part is positive definite. Let  $\mathrm{Sp}(2g, \mathbb{Z})$  be the symplectic group of degree  $2g$  over  $\mathbb{Z}$ , here called the *modular group*  $\Gamma_g$ . The modular group acts on the Siegel upper half-space through modular transformations, defined as follows: let  $\gamma = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \Gamma_g$ . Then,

$$\gamma \circ \tau := (A\tau + B)(C\tau + D)^{-1}, \quad \tau \in \mathcal{H}_g \quad (2.1)$$

Hence we can also define an action on functions on the Siegel upper half-space. The action is defined as follows, for a given  $k$ :

$$(f|_k \gamma)(\tau) := \det(C\tau + D)^{-k} f(\gamma \circ \tau). \quad (2.2)$$

Theta characteristics are elements of  $\mathbb{F}_2^{(2g)}$  which we will write as  $e$  or as  $[\delta]_\epsilon$ , where  $\delta, \epsilon \in \mathbb{F}_2^g$ ; see the introduction. We will often regard theta characteristics as vectors in  $\mathbb{C}^{2g}$ , sending the unit of  $\mathbb{F}_2$  to 0 and the other element to 1.

The theta characteristics are called even (resp. odd) if  $\sum_i \delta_i \epsilon_i$  is even (resp. odd).

There is a natural set of subgroups of the modular group, corresponding to the theta characteristics: let  $\gamma_e \in \Gamma_g$  be such that  $\begin{bmatrix} \mathrm{diag}(A^T C) \\ \mathrm{diag}(B^T D) \end{bmatrix} = e$ , and let  $\Gamma(1, 2)_g$  be the subgroup of the modular group for which the diagonals of  $AB^T$  and  $CD^T$  contain only even elements. Then, define  $\Gamma[e]_g := \gamma_e \Gamma(1, 2)_g \gamma_e^{-1}$ . This definition does not depend on the particular choice of  $\gamma_e$ , because any two such elements  $\gamma_e$  and  $\tilde{\gamma}_e$  are conjugated by an element of  $\Gamma(1, 2)_g$ .

A holomorphic function  $f$  on the Siegel upper half-space is called a semi-modular form of weight  $k$  if the following holds:

$$\forall \gamma \in \Gamma[e]_g, \quad (f|_k \gamma) = f. \quad (2.3)$$

Let  $C$  be a Riemann surface of genus  $g$ . Let us pick a basis for the homology group  $H_1(C, \mathbb{Z})$ . Then we have the period matrix  $\tau \in \mathcal{H}_g$  of  $C$ ; for details, we refer to [10]. The period matrix induces a map  $\mathcal{M}_g \rightarrow \mathcal{H}_g / \Gamma_g$ , where  $\mathcal{M}_g$  is the moduli space of Riemann surfaces of genus  $g$ . We will write  $\omega_i$  for the  $i$ th holomorphic differential in the basis corresponding to the period matrix. Also, we use the Abel-Jacobi map  $A$ , constructed from the same basis mentioned above, and we will write  $A_{pq} := A(p) - A(q)$ .

The OPSMY ansatz from [5] is constructed using lattice theta series, defined as follows for any lattice  $\Lambda \subset \mathbb{R}^n$ :

$$\vartheta_\Lambda^{(g)}(\tau) := \sum_{p_1, \dots, p_g \in \Lambda} e^{\pi i \sum_{i,j} \tau_{ij} p_i \cdot p_j} \quad (2.4)$$

The theta series of self-dual  $8n$ -dimensional lattices provide us with semi-modular forms of weight  $4n$ , which are in addition modular with respect to the entire group  $\Gamma_g$  if the lattice is even.

There are 8 self-dual lattices of dimension 16, the theta series of which we will write in shorthand as follows, in line with [9],

Notation	Lattice	Glueing vectors
$\vartheta_0$	$\mathbb{Z}^{16}$	-
$\vartheta_1$	$\mathbb{Z}^8 \oplus E_8$	-
$\vartheta_2$	$\mathbb{Z}^4 \oplus D_{12}^+$	$(0^4, \frac{1}{2}^{12})$
$\vartheta_3$	$\mathbb{Z}^2 \oplus (E_7 \oplus E_7)^+$	$(\frac{1}{4}^6, -\frac{3}{4}^2, \frac{1}{4}^6, -\frac{3}{4}^2)$
$\vartheta_4$	$\mathbb{Z} \oplus A_{15}^+$	$(\frac{1}{4}^{12}, -\frac{3}{4}^2), (\frac{1}{2}^8, -\frac{1}{2}^8), (\frac{3}{4}^8, -\frac{1}{4}^{12})$
$\vartheta_5$	$(D_8 \oplus D_8)^+$	$(\frac{1}{2}^8, 0^7, 1)$
$\vartheta_6$	$E_8 \oplus E_8$	-
$\vartheta_7$	$D_{16}^+$	$(\frac{1}{2}^{16})$

The Grushevsky ansatz, from [7], is instead built using Riemann theta functions, defined as follows for a theta characteristic  $e = [\delta]_\epsilon$ , here regarded as a vector in  $\mathbb{C}^{2g}$ ,

$$\theta_{[\delta]}(z, \tau) := \sum_{n \in \mathbb{Z}^g} \exp \left\{ i\pi \left( n + \frac{1}{2}\delta \right)^t \tau \left( n + \frac{1}{2}\delta \right) + 2\pi i \left( n + \frac{1}{2}\delta \right)^t \left( z + \frac{1}{2}\epsilon \right) \right\}. \quad (2.5)$$

Riemann theta functions for  $z = 0$  are called Riemann theta constants. The Riemann theta constants of odd characteristics are zero for any  $\tau \in \mathcal{H}_g$ . We will write  $\theta_e := \theta_{[\delta]}(0, \tau)$ .

The semi-modular forms used in [7] are defined as follows. Let  $V \subset \mathbb{F}_2^{(2g)}$  be a set of characteristics in genus  $g$ . Then, we define

$$P(V) := \prod_{e \in V} \theta_e. \quad (2.6)$$

Now, define  $\mathcal{A}_p^{(g)}$  to be the set of all  $p$ -dimensional subspaces of  $\mathbb{F}_2^{(2g)}$ . Then, we define the Grushevsky forms  $\{G_p^{(g)}, 0 \leq p \leq g \in \mathbb{Z}\}$  as follows:

$$G_p^{(g)} := \sum_{V \in \mathcal{A}_p^{(g)}} P(V)^{2^{4-p}}. \quad (2.7)$$

Note that this notation differs from that in [9] as follows:

$$G_p^{(g)} = \left( 2^{\frac{p(p-1)}{2}} \prod_{i=1}^p (2^i - 1) \right) \sum_{\substack{e_1, \dots, e_p \in \mathbb{F}_2^{(2g)} \\ e_1, \dots, e_p \text{ lin.ind.}}} \left( \prod_{e \in \text{span}\{e_1, \dots, e_p\}} \theta_e \right)^{2^{4-p}} \quad (2.8)$$

(taken to be 1 for  $p = 0$ ).

From [9] we have several equalities between lattice theta series and Riemann theta constants, in our notation less elegantly written

$$G_p^{(g)} = \sum_{k=0}^p (-1)^{k+p} \cdot 2^{\frac{k(k+2(g-p)+1)}{2}} \cdot \left( \prod_{i=1}^k (2^i - 1) \prod_{i=1}^{p-k} (2^i - 1) \right)^{-1} \vartheta_k^{(g)}, \quad p = 0, \dots, 4 \quad (2.9)$$

where  $\prod_{i=1}^k (2^i - 1)$  is taken to be 1 for  $k = 0$ .

We will throughout the paper denote

$$f^{(g)} := \vartheta_5^{(g)} - G_g^{(g)} \quad (2.10)$$

$$J^{(g)} := \vartheta_6^{(g)} - \vartheta_7^{(g)}. \quad (2.11)$$

It was shown in [9] that  $f^{(g)}$  vanishes on the Jacobian locus  $\mathcal{J}_g$  for  $g \leq 4$ . In the present paper we show that  $f^{(5)}$  does not vanish on  $\mathcal{J}_5$ .

# Chapter 3

## Research

In this chapter, we present the actual research done in the accompanying paper.

### 3.1 Degeneration

The conjecture which we in this section investigate and disprove, is whether  $G_5^{(5)}$  and  $\vartheta_5^{(5)}$  agree on the Jacobian locus  $\mathcal{J}_5$ .

If it were to be the case that  $G_5^{(5)} = \vartheta_5^{(5)}$  on  $\mathcal{J}_5$ , then also on the closure, and in particular on genus 4 degenerations. But we show that the latter is not the case.

To achieve this, we express  $G_5^{(5)}$  and  $\vartheta_5^{(5)}$  for genus 5 surfaces that degenerate by pinching one of the handles, in the way used in [8], originally from [11], and show that these expressions do not agree on  $\mathcal{J}_5$ .

More precisely, we will take a 1-parameter family of Riemann surfaces  $C_s \subset \mathcal{M}_5$ , with parameter  $s$ , which degenerates to a genus 4 surface  $C$  with two indistinguishable marked points  $p$  and  $q$ , inside the boundary divisor  $\delta_0 \subset \overline{\mathcal{M}_5}$ . The points  $p$  and  $q$  are the endpoints of the cusps that used to be the now-pinched handle.

Namely, following [8], we take such a family of surfaces that their period matrices  $\tau_s$  have the following form:

$$\tau_s = \begin{pmatrix} \lambda & z \\ z^t & \tau \end{pmatrix} = \begin{pmatrix} \ln s + c_1 + c_2 s & A_{pq}^t + \frac{1}{4}s(\omega(p) - \omega(q))^t \\ A_{pq} + \frac{1}{4}s(\omega(p) - \omega(q)) & \tau_0 + s\sigma \end{pmatrix} \quad (3.1)$$

for some constants  $c_1$  and  $c_2$ , where  $\tau_0$  is the period matrix of  $C_0$  and

$$\sigma_{ij} := \frac{1}{4} (\omega_i(p) - \omega_i(q)) (\omega_j(p) - \omega_j(q)), \quad i, j \leq 4.$$

Define, for legibility,

$$q := e^{2\pi i \lambda}. \quad (3.2)$$

Now, if we obtain the Fourier-Jacobi expansions of  $G_5^{(5)}$  and  $\vartheta_5^{(5)}$ , we can use this to express the forms evaluated in  $\tau_s$  as series in  $s$ . That is, for any  $z$  function  $f$  on  $\mathcal{J}_5$  that is holomorphic on a neighbourhood of the curve  $\{\tau_s\} \subset \mathcal{J}_5$ , if

$$f(\tau_s) = f_0(\tau) + qf_1(\tau, z) + O(q^2) \quad (3.3)$$

we have

$$f(\tau_s) = f_0(\tau_0) + s \left( \sum_{i \leq j}^4 \frac{\partial f_0(\tau)}{\partial \tau_{ij}} \sigma_{ij}(p, q) + f_1(\tau, z) \right) + O(s^2). \quad (3.4)$$

We will express the first terms above in a Taylor series. We take for a local chart  $x$  the parameter  $u = x(p) - x(u)$  near  $u = 0$  and calculate

$$\sigma_{ij}(p, q) = S_{ij} + O(u^4) \quad (3.5)$$

$$S_{ij} := \frac{u^2}{4} \frac{\partial \omega_i(p)}{\partial x} \frac{\partial \omega_j(p)}{\partial x} + \frac{u^3}{2} \frac{\partial^2 \omega_i(p)}{\partial x^2} \frac{\partial \omega_j(p)}{\partial x} \quad (3.6)$$

and therefore, if  $\frac{\partial f_1}{\partial z_i}$  and  $\frac{\partial f_1^3}{\partial z_i \partial z_j \partial z_k}$  vanish,

$$f(\tau_s(p, q)) = f_0(\tau_0) + s \sum_{i \leq j}^4 \left( u^2 \frac{\partial^2 f_1}{\partial z_i \partial z_j} \omega_i(p) \omega_j(p) + \frac{\partial f_0}{\partial \tau_{ij}} S_{ij} + O(u^4) \right) + O(s^2). \quad (3.7)$$

These series for  $G_5^{(5)}$  and  $\vartheta_5^{(5)}$ , then, can finally be shown to disagree, by an argument used in [8].

### 3.1.1 The expansion of $G_5^{(5)}$

To determine the degeneration of  $G_5^{(5)}$  and  $\vartheta_5^{(5)}$  we will here take the Fourier-Jacobi expansion of  $G_5^{(5)}$ , obtaining the analogue of (3.3). That is, we will express  $G_5^{(5)}(\tau_s)$  in the limit  $\lambda \rightarrow \infty$ . Also, we will calculate  $\frac{\partial^2 h_1}{\partial z_i \partial z_j}$  where  $h_1$  stands for the  $q$ -linear term in the Fourier-Jacobi expansion of  $G_5^{(5)}$ .

#### Expanding $P(V)^{\frac{1}{2}}$

First, we will calculate the Fourier-Jacobi expansion of the summands  $P(V)^{\frac{1}{2}}$  for  $V \in \mathcal{A}_5^{(5)}$ . Let  $\delta^1$  stand for the first entry in the vector  $\delta \in \mathbb{F}_2^g$ . Let  $\pi$  be the projection from  $\mathbb{F}_2^{(2g)}$  to  $\mathbb{F}_2^{(2g-2)}$  by deleting the first coordinates  $(\delta_e^1, \epsilon_e^1)$  of  $\delta$  and  $\epsilon$ . We will write  $[\tilde{\delta}_e] := \tilde{e} := \pi(e)$ . We will use the known formulae for the Fourier-Jacobi expansion of theta constants, which look as follows:

$$\theta \begin{bmatrix} 0 & \tilde{\delta}_e \\ \epsilon_e^1 & \tilde{\epsilon}_e \end{bmatrix} \begin{pmatrix} \lambda & z^t \\ z & \tau \end{pmatrix} = \theta_{\tilde{e}} + 2 \sum_{l=1}^{\infty} e^{\pi i (l^2 \lambda + l \epsilon_e^1)} \theta_{\tilde{e}}(l \vec{z}, \tau) \quad (3.8)$$

$$\theta \begin{bmatrix} 1 & \tilde{\delta}_e \\ \epsilon_e^1 & \tilde{\epsilon}_e \end{bmatrix} \begin{pmatrix} \lambda & z^t \\ z & \tau \end{pmatrix} = e^{\pi i (\frac{1}{4} \lambda + \frac{1}{2} \epsilon_e^1)} \theta_{\tilde{e}}(\frac{1}{2} \vec{z}, \tau) + 2 \sum_{l=1}^{\infty} e^{\pi i ((l + \frac{1}{2})^2 \lambda + (l + \frac{1}{2}) \epsilon_e^1)} \theta_{\tilde{e}} \left( \left( l + \frac{1}{2} \right) \vec{z}, \tau \right). \quad (3.9)$$

As each component of the characteristics contained in  $V$  can be either 0 or 1, and  $P(V)^{\frac{1}{2}}$  vanishes if  $V$  contains any odd characteristics, we can distinguish three kinds of subspaces  $V$ :

1. First, we consider subspaces containing only characteristics of the form  $e = \begin{bmatrix} 0 & \tilde{\delta}_e \\ \epsilon_e^1 & \tilde{\epsilon}_e \end{bmatrix}$ .

Thus, expanding  $P(V_1)$  for  $V_1$  of this type, using (3.8), we get

$$\begin{aligned} P(V_1) &= \prod_{e \in V_1} \theta_{\tilde{e}} + 2q^{1/2} \sum_{e \in V_1} e^{\pi i \epsilon_e^1} \theta_{\tilde{e}}(\tau, z) \prod_{\substack{v \in V_1 \\ v \neq e}} \theta_{\tilde{v}} \\ &\quad + 2q \sum_{\substack{e_1, e_2 \in V_1 \\ e_1 \neq e_2}} e^{\pi i (\epsilon_{e_1}^1 + \epsilon_{e_2}^1)} \theta_{\tilde{e}_1}(\tau, z) \theta_{\tilde{e}_2}(\tau, z) \prod_{\substack{v \in V_1 \\ v \neq e_1 \\ v \neq e_2}} \theta_{\tilde{v}} + O(q^2). \end{aligned} \quad (3.10)$$

For such  $V_1$ , the image  $\pi(V_1)$  is totally isotropic, and therefore the space  $\pi(V_1)$  has maximal dimension 4. Because additionally the kernel of  $\pi$  has a maximal dimension of 1 (only  $\epsilon_e^1$  can be picked freely), the  $\tilde{e}$  are necessarily pairwise equal, the corresponding pairs of  $e$  differing only in their  $\epsilon_e^1$ . We denote by  $e^*$  the characteristic which equals  $e \in V_1$  except in the component  $\epsilon_e^1$ . The above consideration shows that  $e^*$  is contained in  $V_1$ . Unless  $e_1 = e_2^*$ , then, each term  $e_1, e_2$  in the summation in the third term from (3.10) will be canceled by a  $e_1^*, e_2$  term. Combining these facts, we can rewrite the above formula as follows:

$$P(V_1) = \prod_{\tilde{e} \in \pi(V_1)} \theta_{\tilde{e}}^2 - 4q \sum_{\tilde{e} \in \pi(V_1)} \theta_{\tilde{e}}^2(\tau, z) \prod_{\substack{\tilde{v} \in \pi(V_1) \\ \tilde{v} \neq \tilde{e}}} \theta_{\tilde{v}}^2 + O(q^2). \quad (3.11)$$

Expanding the square root then easily yields

$$P(V_1)^{\frac{1}{2}} = \prod_{\tilde{e} \in \pi(V_1)} \theta_{\tilde{e}} - 2q \sum_{\tilde{e} \in \pi(V_1)} \frac{\theta_{\tilde{e}}^2(\tau, z)}{\theta_{\tilde{e}}^2(\tau, 0)} \prod_{\tilde{v} \in \pi(V_1)} \theta_{\tilde{v}} + O(q^2). \quad (3.12)$$

Finally, we use the heat equation for the theta functions, where  $\delta_{ij}$  is the Kronecker delta,

$$\frac{\partial^2 \theta_e}{\partial z_i \partial z_j} = 2\pi i (1 + \delta_{ij}) \frac{\partial \theta_e}{\partial \tau_{ij}} \quad (3.13)$$

to obtain

$$\left. \frac{\partial^2 P(V_1)^{\frac{1}{2}}}{\partial z_i \partial z_j} \right|_{z=0} = -4\pi i (1 + \delta_{ij}) \left( \sum_{\tilde{e} \in \pi(V_1)} \frac{\partial \theta_{\tilde{e}}}{\partial \tau_{ij}} \prod_{\tilde{v} \neq \tilde{e}} \theta_{\tilde{v}} \right) + O(q^2). \quad (3.14)$$

Note that  $P(V_1)$  is an even function of  $z$  and thus the odd partial derivatives vanish (up to  $O(q^2)$ ).

2. Next, we consider subspaces containing both characteristics of the form  $e = \begin{bmatrix} 0 & \tilde{\delta} \\ 0 & \tilde{\epsilon} \end{bmatrix}$  and of the form  $e = \begin{bmatrix} 1 & \tilde{\delta} \\ 0 & \tilde{\epsilon} \end{bmatrix}$ .

For these subspaces  $V_2$  (as well as for those under 3) below), every basis vector will appear in exactly half of the characteristics because  $V_2$  is a vector space over  $\mathbb{F}_2$ . Thus, if there is least one  $e$  such that  $\delta_e^{(1)} = 1$  we have exactly 16  $e$  such that  $\delta_e^{(1)} = 1$ , and for the other ones  $\delta_e^{(1)} = 0$ . Therefore, using (3.8) and (3.9) to expand all theta constants, we have

$$P(V_2) = 2^{16}q^2 \prod_{\substack{e_1 \in V_2 \\ \delta_{e_1}^{(1)} = 0}} \theta_{\tilde{e}_1}(\tau, 0) \prod_{\substack{e_2 \in V_2 \\ \delta_{e_2}^{(1)} = 1}} \theta_{\tilde{e}_2}(\tau, \frac{z}{2}) + O(q^2) \quad (3.15)$$

Similar to case 1) above, the  $\tilde{e}$  are pairwise equal and the corresponding pairs of  $e$  differ only in  $\delta_e^{(1)}$ . Thus, we end up with

$$P(V_2)^{\frac{1}{2}} = 2^8q \sqrt{\prod_{\tilde{e} \in \pi(V_2)} \theta_{\tilde{e}}(\tau, 0) \theta_{\tilde{e}}(\tau, \frac{z}{2})} + O(q^2). \quad (3.16)$$

Also, again using the theta heat equation, after a short calculation we find

$$\begin{aligned} \frac{\partial^2 P(V_2)^{\frac{1}{2}}}{\partial z_i \partial z_j} \Big|_{z=0} &= 2^4 \sum_{\tilde{e} \in \pi(V_2)} \frac{\partial^2 \theta_{\tilde{e}}}{\partial z_i \partial z_j} \prod_{\tilde{v} \neq \tilde{e}} \theta_{\tilde{v}} + O(q^2) \\ &= 32\pi i(1 + \delta_{ij}) \sum_{\tilde{e} \in \pi(V_2)} \frac{\partial \theta_{\tilde{e}}}{\partial \tau_{ij}} \prod_{\tilde{v} \neq \tilde{e}} \theta_{\tilde{v}} + O(q^2). \end{aligned} \quad (3.17)$$

Note that  $P(V_2)^{\frac{1}{2}}$  is an even function of  $z$  and thus the odd partial derivatives vanish (up to  $O(q^2)$ ).

3. Last, we consider subspaces containing, in addition to characteristics contained in subspaces from case 2) above, characteristics of the form  $e = \begin{bmatrix} 0 & \tilde{\delta} \\ 1 & \tilde{\epsilon} \end{bmatrix}$ . These do not have the simple pairings observed above, but we can still expand the theta constants and obtain the similar expression below, but it cannot be simplified as easily. This, however, will turn out not to be necessary for our purposes. The 16 factors of  $e^{\pi i \epsilon_e^1}$  together yield 1, and we end up with

$$P(V_3)^{\frac{1}{2}} = 2^8q \sqrt{\prod_{\substack{e_1 \in V_3 \\ \delta_{e_1}^{(1)} = 0}} \theta_{\tilde{e}_1}(\tau, 0) \prod_{\substack{e_2 \in V_3 \\ \delta_{e_2}^{(1)} = 1}} \theta_{\tilde{e}_2}(\tau, \frac{z}{2})} + O(q^2). \quad (3.18)$$

For any genus  $g$  there will be at least  $2^{g-2}$  odd characteristics in  $\pi(V_3)$  when  $V_3$  is of this type. Therefore, we have

$$\frac{\partial P(V_3)^{\frac{1}{2}}}{\partial z_i} \Big|_{z=0} = \frac{\partial^2 P(V_3)^{\frac{1}{2}}}{\partial z_i \partial z_j} \Big|_{z=0} = \frac{\partial^3 P(V_3)^{\frac{1}{2}}}{\partial z_i \partial z_j \partial z_k} \Big|_{z=0} = 0 \quad (3.19)$$

up to  $O(q^2)$ .

### The expression for $G_5^{(5)}$

Let  $\mathcal{V}_*$  be the subset of  $\mathcal{A}_5^{(5)}$  containing all subspaces from case 3) above. Now, combining the results from the previous subsection,

$$\begin{aligned} G_5^{(5)} = \sum_{V \in \mathcal{A}_5^{(5)}} P(V)^{\frac{1}{2}} &= \sum_{V \in \mathcal{A}_4^{(4)}} \prod_{e \in V} \theta_e + q \left( 2^8 \sqrt{\prod_{e \in V} \theta_e \cdot \theta_e(\tau, \frac{z}{2})} - 2 \sum_{e \in V} \frac{\theta_e^2(\tau, z)}{\theta_e^2} \prod_{v \in V} \theta_v \right) \\ &\quad + \sum_{V_3 \in \mathcal{V}_*} 2^8 q \sqrt{\prod_{\substack{e_1 \in V_3 \\ \delta_{e_1}^{(1)}=0}} \theta_{\tilde{e}_1} \prod_{\substack{e_2 \in V_3 \\ \delta_{e_2}^{(1)}=1}} \theta_{\tilde{e}_2}(\tau, \frac{z}{2})} + O(q^2). \end{aligned} \quad (3.20)$$

Because  $\pi(V)$ , for  $V \notin \mathcal{V}_*$ , is a totally isotropic element of  $\mathcal{A}_4^{(4)}$ , and in fact the image  $\mathcal{A}_5^{(5)} \setminus \mathcal{V}_*$  under  $\pi$  is the set of all 4-dimensional totally isotropic elements of  $\mathcal{A}_4^{(4)}$ , we can write the following:

$$\begin{aligned} G_5^{(5)} = G_4^{(4)} + 2^8 q \left( \sum_{V \in \mathcal{A}_4^{(4)}} \left( \sqrt{\prod_{e \in V} \theta_e \cdot \theta_e(\tau, \frac{z}{2})} - 2^{-7} \sum_{e \in V} \frac{\theta_e^2(\tau, z)}{\theta_e^2} \prod_{v \in V} \theta_v \right) \right. \\ \left. + \sum_{V_3 \in \mathcal{V}_*} \sqrt{\prod_{\substack{e_1 \in V_3 \\ \delta_{e_1}^{(1)}=0}} \theta_{\tilde{e}_1} \prod_{\substack{e_2 \in V_3 \\ \delta_{e_2}^{(1)}=1}} \theta_{\tilde{e}_2}(\tau, \frac{z}{2})} \right) + O(q^2). \end{aligned} \quad (3.21)$$

Also, this gives us

$$\left. \frac{\partial^2 G_5^{(5)}}{\partial z_i \partial z_j} \right|_{z=0} = 28\pi i (1 + \delta_{ij}) q \sum_{V \in \mathcal{A}_4^{(4)}} \sum_{e \in V} \frac{\partial \theta_e}{\partial \tau_{ij}} \prod_{v \neq e} \theta_v + O(q^2) = 28\pi i (1 + \delta_{ij}) q \frac{\partial G_4^{(4)}}{\partial \tau_{ij}} + O(q^2). \quad (3.22)$$

And finally, as the contribution from all  $V_3 \in \mathcal{V}_*$  will vanish in  $z = 0$  because  $\pi(V_3)$  contains odd characteristics, we can see that

$$G_5^{(5)} \Big|_{z=0} = (1 + 224 q) G_4^{(4)} + O(q^2). \quad (3.23)$$

Note that, because the first terms from the expansion of  $G_1^{(1)}(\lambda)$  are  $1 + 224 q$ , this is consistent with the factorization property for  $G_g^{(g)}$ .

#### 3.1.2 The expansion of $\vartheta_5^{(5)}$

We will now do the same for  $\vartheta_5^{(5)}$  as done above for  $G_5^{(5)}$ , that is, take the Fourier-Jacobi expansion and calculate the  $z_i, z_j$  derivatives of the first terms.

Note that as  $\vartheta_5(\tau)^{(g)} := \sum_{p_1, \dots, p_g \in \Lambda_5} e^{\pi i (p_k \cdot p_l) \tau_{kl}}$ , we can write

$$\vartheta_5 \begin{pmatrix} \lambda & z^t \\ z & \tau \end{pmatrix} = \sum_{p_1, \dots, p_5 \in \Lambda_5} e^{\pi i p_1 \cdot p_1 \lambda} e^{2\pi i \sum_i p_1 p_i z_i} e^{\pi i \sum_{i,j>1}^5 p_i p_j \tau_{ij}}. \quad (3.24)$$

The first term in the  $q$ -expansion is easy to obtain, and we will obtain the  $q$ -linear term as in [1] by writing

$$F^{(g)}(\tau, z) := \sum_{p_1, \dots, p_g \in (D_8 \oplus D_8)^+} e^{\pi i \sum_{i,j=1}^g p_i p_j \tau_{ij}} \sum_{\tilde{p} \cdot \tilde{p}=2} e^{2\pi i \sum_{i=1}^g \tilde{p} p_i z_i} \quad (3.25)$$

Clearly, the norm 2 vectors are  $(\dots, \pm 1, \dots, \pm 1, \dots, 0^8)$  and  $(0^8, \dots, \pm 1, \dots, \pm 1, \dots)$ , where  $\dots$  denotes a possibly empty sequence of zeroes. There are  $2 \cdot 4 \cdot \binom{8}{2} = 224$  of those.

Now the first terms of the series in  $q$  will be:

$$\vartheta_5^{(5)} \begin{pmatrix} \lambda & z^t \\ z & \tau \end{pmatrix} = \vartheta_5^{(4)}(\tau) + q F^{(4)}(\tau, z) + O(q^2). \quad (3.26)$$

Now we will express the  $z_i z_j$ -derivatives of  $F^{(4)}$ , the  $q$ -linear term from (3.26), as done above for  $G_5^{(5)}$ . Because the norm 2 vectors are the same as those from  $D_8$ , we can use the fact that

$$\sum_{\tilde{p} \in (D_8 \oplus D_8)^+ : \tilde{p} \cdot \tilde{p}=2} (p_i \cdot \tilde{p})(p_j \cdot \tilde{p}) = 28 p_i \cdot p_j, \quad (3.27)$$

which is mentioned and used in [1]. We then obtain

$$\begin{aligned} \frac{\partial^2 F^{(4)}}{\partial z_i \partial z_j} \Big|_{z=0} &= \sum_{p_1, \dots, p_4 \in \Lambda_5} e^{\pi i \sum_{i,j=1}^g p_i p_j \tau_{ij}} \sum_{\tilde{p} \cdot \tilde{p}=2} (2\pi i)^2 (\tilde{p} p_i)(\tilde{p} p_j) \\ &= 28 \cdot 2\pi i (1 + \delta_{ij}) \frac{\partial F^{(4)}}{\partial \tau_{ij}} \Big|_{z=0} = 28 \cdot 2\pi i (1 + \delta_{ij}) \frac{\partial \vartheta_5^{(4)}}{\partial \tau_{ij}}. \end{aligned} \quad (3.28)$$

### 3.1.3 The final expression

Let now, for brevity,  $f^{(g)}$ ,  $f_0^{(g)}$  and  $f_1^{(g)}$  be defined by

$$f^{(g)} := \vartheta_5^{(g)} - G_g^{(g)} \quad (3.29)$$

$$f^{(g)} = f_0^{(g)} + q f_1^{(g)} + O(q^2). \quad (3.30)$$

We now develop  $f^{(5)}$  as a function of  $s$ . Applying formula (3.4) to  $f^{(5)}$  and noting that  $f_0^{(5)} = f^{(4)}$ , we have

$$f^{(5)}(\tau_s) = f^{(4)}(\tau_0) + s \left( f_1^{(5)}(\tau_0, z) + \sum_{i \leq j} \frac{\partial f^{(4)}}{\partial \tau_{ij}} \sigma_{ij}(p, q) \right) + O(s^2). \quad (3.31)$$

Now, we expand this using (3.7), letting  $u := x(p) - x(q)$  for a local chart  $x$ . For brevity we write

$$S_{ij} := \frac{u^2}{4} \frac{\partial \omega_i(p)}{\partial x} \frac{\partial \omega_j(p)}{\partial x} + \frac{u^3}{2} \frac{\partial^2 \omega_i(p)}{\partial x^2} \frac{\partial \omega_j(p)}{\partial x}. \quad (3.32)$$

Remember that  $\sigma_{ij}(p, q) = S_{ij} + O(u^4)$ . Then,

$$f^{(5)}(\tau_s) = f^{(4)}(\tau_0) + s \sum_{i \leq j} \left( u^2 \frac{\partial^2 f_1^{(5)}}{\partial z_i \partial z_j} \omega_i(p) \omega_j(p) + \frac{\partial f^{(4)}}{\partial \tau_{ij}} S_{ij} + O(u^4) \right) + O(s^2). \quad (3.33)$$

By (3.22) and (3.28) we know that  $\frac{\partial^2 f_1^{(5)}}{\partial z_i \partial z_j} = 28\pi i(1 + \delta_{ij}) \frac{\partial f^{(4)}}{\partial \tau_{ij}}$ . This leaves us with

$$f^{(5)}(\tau_s) = f^{(4)}(\tau_0) + s \sum_{i \leq j} \frac{\partial f^{(4)}}{\partial \tau_{ij}} (28\pi i(1 + \delta_{ij}) u^2 \omega_i(p) \omega_j(p) + S_{ij} + O(u^4)) + O(s^2). \quad (3.34)$$

Now, let  $J^{(g)} := \vartheta_6^{(g)} - \vartheta_7^{(g)}$ . Because  $f^{(4)} = \frac{3}{7} J^{(4)}$ , from [9], we can rewrite the above as follows:

$$f^{(5)}(\tau_s) = \frac{3}{7} J^{(4)}(\tau_0) + \frac{3s}{7} \sum_{i \leq j} \frac{\partial J^{(4)}}{\partial \tau_{ij}} (28\pi i(1 + \delta_{ij}) u^2 \omega_i(p) \omega_j(p) + S_{ij} + O(u^4)) + O(s^2). \quad (3.35)$$

In [8, p. 16-17] Grushevsky and Salvati Manno obtain a similar expression for the degeneration of  $J^{(5)}$ , differing only in the numerical coefficients. They show that the  $\omega_i(p) \omega_j(q)$  term vanishes and that  $\sum_{i \leq j} \frac{\partial J^{(4)}}{\partial \tau_{ij}} S_{ij}$  cannot vanish everywhere due to the fact that  $J^{(4)}$  is the Schottky form. We refer to [8] for details. This shows that  $f^{(5)}(\tau_s)$  does not vanish everywhere. Thus, the above leads to the conclusion

$$\vartheta_5^{(5)} \neq G_5^{(5)} \quad (3.36)$$

when restricted to  $\mathcal{J}_5$ , as promised.  $\square$

## 3.2 The trace of $f^{(5)}$

Here we will look at the trace of  $f^{(5)}$ , defined as  $\sum_e f^{(5)}[e]$ , because it occurs in the cosmological constant and is thus of interest for the genus 5 measure.

The definition of a term  $f^{(5)}[e]$  is as follows: for any semi-modular form  $f$  and for  $\gamma_e = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$  such that  $\begin{bmatrix} \text{diag}(A^T C) \\ \text{diag}(B^T D) \end{bmatrix} = e$ , we have  $f[e] := (f|_{\gamma_e})$ . Because  $f$  is a semi-modular form,  $f[e]$  does not depend on the particular choice of  $\gamma_e$ .

In [8] Grushevsky and Salvati calculate the traces of the forms  $G_p^{(g)}$ . They use a different notation: their  $S_i$  is  $2^{-i} \sum_e G_i[e]$ . Here we present their formulae. Note that they are only valid for the  $G_p^{(g)}$  with  $p \leq g$ , because the others vanish identically.

$$2^{-5} \sum_e 3720 G_5^{(g)}[e] = 2^{-3} \sum_e (2^{2g-6} - 1) G_3^{(g)}[e] - 2^{-4} \sum_e 90 G_4^{(g)}[e] \quad (3.37)$$

$$2^{-4} \sum_e 840 G_4^{(g)}[e] = 2^{-2} \sum_e (2^{2g-4} - 1) G_2^{(g)}[e] - 2^{-3} \sum_e 42 G_3^{(g)}[e] \quad (3.38)$$

$$2^{-3} \sum_e 168 G_3^{(g)}[e] = 2^{-1} \sum_e (2^{2g-2} - 1) G_1^{(g)}[e] - 2^{-2} \sum_e 18 G_2^{(g)}[e] \quad (3.39)$$

$$2^{-2} \sum_e 24 G_2^{(g)}[e] = \sum_e (2^{2g} - 1) G_0^{(g)}[e] - 2^{-1} \sum_e 6 G_1^{(g)}[e] \quad (3.40)$$

Because  $G_0^{(5)}[e] = \theta_e^{16}$  and  $G_1^{(5)}[e] = \theta_e^8 \sum_{e_1 \neq 0} \theta_{e+e_1}^8$ , we see that  $\sum_e G_0^{(5)}[e] = \sum_e \theta_e^{16} = \vartheta_7$ , and  $\sum_e G_1^{(5)}[e] = (\sum_e \theta_e^8)^2 - \sum_e \theta_e^{16} = \vartheta_6 - \vartheta_7$ . Therefore, we have

$$\sum_e G_5^{(5)}[e] = \frac{32}{217} \left( 950 \vartheta_6^{(5)} - 733 \vartheta_7^{(5)} \right) \quad (3.41)$$

$$\sum_e G_4^{(4)}[e] = -\frac{16}{7} \left( 22 \vartheta_6^{(4)} - 29 \vartheta_7^{(4)} \right). \quad (3.42)$$

From [1, p. 28] we learn that

$$\sum_e \vartheta_5^{(g)}[e] = 2^{g-1} \left( \vartheta_6^{(5)} + \vartheta_7^{(5)} \right). \quad (3.43)$$

Combining the above facts, we obtain the following expressions for the genus 4 and genus 5 trace of  $f^{(g)}$ :

$$\sum_e f^{(4)}[e] = -\frac{2^3 \cdot 3 \cdot 17}{7} J^{(4)} \quad (3.44)$$

$$\sum_e f^{(5)}[e] = \frac{2^4 \cdot 3^2 \cdot 11 \cdot 17}{7 \cdot 31} J^{(5)}. \quad (3.45)$$

In genus 4 there are  $2^3(2^4 + 1)$  even characteristics. In genus 5, there are  $2^4(2^5 + 1)$  even characteristics. Because  $J^{(g)}$  is a modular form with respect to the entire modular group  $\Gamma_g$ , its trace is simply the number of even characteristics times  $J^{(g)}$ . Note that  $\frac{\sum_e f^{(5)}[e]}{\sum_e f^{(4)}[e]} \neq \frac{\sum_e J^5[e]}{\sum_e J^4[e]}$ . This fact will be used in Section 3.4 to obtain both a vanishing cosmological constant in genus 5 *and* a vanishing two-point function in genus 4; in [1] it was shown that it is impossible to do this using only the OPSMY forms while conforming to the other requirements for the measure.

*Remark.* Note that if  $f^{(5)}$  were to vanish on  $\mathcal{J}_5$ , this would imply that the trace would vanish as well. Since  $J^{(5)}$  is not everywhere zero on  $\mathcal{J}_5$ , see [8], this gives a second, less explicit, proof of the nonvanishing of  $f^{(5)}$ .

### 3.3 The difference between $f^{(5)}$ and $J^{(5)}$

Now that we know that  $f^{(5)}$  does not vanish everywhere on  $\mathcal{J}_5$ , a natural question which arises is whether this form is linearly independent from the already known modular forms on  $\mathcal{J}_5$ . By the factorization property for both the Grushevsky and OPSMY basis, we can eliminate all but one candidate. Because (from [9])  $f^{(4)} = \frac{3}{7}J^{(4)}$ , we see that

$$f^{(5)} \begin{pmatrix} \lambda & 0 \\ 0 & \tilde{\tau} \end{pmatrix} = \vartheta_5^{(1)} f^{(4)} = \frac{3}{7} \vartheta_5^{(1)} J^{(4)}. \quad (3.46)$$

Because  $J^{(4)}$  vanishes on  $\mathcal{J}_4$ , the only form that factorizes similarly is  $J^{(5)}$ : there are no other linear combinations of lattice theta series for which the restriction to  $\mathcal{J}_1 \times \mathcal{J}_4 \subset \mathcal{J}_5$  vanishes, and the other functions from the Grushevsky basis (i.e.,  $G_p^{(g)}$  for  $p < 5$ ) can be expressed in terms of the lattice theta series in every genus.

We will prove by a simple argument that  $f^{(5)}$  and  $J^{(5)}$  cannot coincide on the Jacobian locus  $\mathcal{J}_5$ . Because

$$\sum_e f^{(5)}[e] = \frac{3 \cdot 17}{7 \cdot 31} \sum_e J^{(5)}[e], \quad (3.47)$$

if  $f^{(5)}$  is a multiple of  $J^{(5)}$  it must be equal to  $\frac{3 \cdot 17}{7 \cdot 31} J^{(5)}$ . Looking at the degeneration found in section 3.1,

$$\begin{aligned} f^{(5)} = f^{(4)} + \frac{3}{7} s \sum_{i \leq j} \frac{\partial J^{(4)}}{\partial \tau_{ij}} & \left( 28u^2(1 + \delta_{ij})\omega_i(q)\omega_j(q) + u^2 \frac{1}{4} \frac{\partial \omega_i}{\partial x}(q) \frac{\partial \omega_j}{\partial x}(q) \right. \\ & \left. + \frac{1}{2} u^3 \frac{\partial^2 \omega_i}{\partial x^2}(q) \frac{\partial \omega_j}{\partial x}(q) + O(u^4) \right) + O(s^2), \end{aligned} \quad (3.48)$$

we can compare it with the very similar expression found in [8] for the first terms in  $u$  in the  $s$ -linear term when taking the same degeneration for  $J^{(5)}$ ,

$$\begin{aligned} J^{(5)} = J^{(4)} + s \sum_{i \leq j} \frac{\partial J^{(4)}}{\partial \tau_{ij}} & \left( 30u^2(1 + \delta_{ij})\omega_i(q)\omega_j(q) + u^2 \frac{1}{4} \frac{\partial \omega_i}{\partial x}(q) \frac{\partial \omega_j}{\partial x}(q) \right. \\ & \left. + \frac{1}{2} u^3 \frac{\partial^2 \omega_i}{\partial x^2}(q) \frac{\partial \omega_j}{\partial x}(q) + O(u^4) \right) + O(s^2). \end{aligned} \quad (3.49)$$

Because

$$\sum_{i \leq j} \frac{\partial J^{(4)}}{\partial \tau_{ij}} u^2(1 + \delta_{ij})\omega_i(q)\omega_j(q) = 0 \quad (3.50)$$

$$f^{(4)} = \frac{3}{7} J^{(4)}, \quad (3.51)$$

formula (3.48) differs from formula (3.49) by a factor of  $\frac{3}{7}$ . Together, this implies

$$\frac{7 \cdot 31}{3 \cdot 17} f^{(5)} \neq J^{(5)} \quad (3.52)$$

on  $\mathcal{J}_5$  and therefore,  $f^{(5)}$  cannot be a multiple of  $J^{(5)}$  everywhere on  $\mathcal{J}_5$ .

### 3.4 The two-point function in genus 4

Matone and Volpato show in [1] that it is not possible to make a genus 5 measure from the OPSMY forms that satisfies all requirements. To be precise, the degeneration to genus 4 yields a nonvanishing two-point function if the genus 5 cosmological constant is made to vanish, i.e. requirement c) from the introduction is not satisfied. Therefore, one may ask whether by combining these forms with  $G_5^{(5)}$  one can construct a measure that does satisfy these properties. The answer is yes.

In order to obtain the genus 4 two-point function from the genus 5 measure, we follow the procedure set by [1]. That is, consider  $X_{NS}[(\delta, \epsilon)] := \frac{1}{2} \left( \tilde{\Xi}^{(g+1)} \begin{bmatrix} \delta & 0 \\ \epsilon & 0 \end{bmatrix} + \tilde{\Xi}^{(g+1)} \begin{bmatrix} \delta & 0 \\ \epsilon & 1 \end{bmatrix} \right)$  and contract one handle from a family of curves, where then the term linear in the perturbation parameter will be the two-point function. As the argument from [1] is quite detailed, we will just look at what happens with the terms  $c_J J^{(5)} + c_f f^{(5)}$  which we would like to add to the measure, instead of  $-B_5 J^{(5)}$  as originally proposed, where  $B_5$  is the coefficient of  $J^{(5)}$  in the cosmological constant from the 'plain' OPSMY ansatz. From the degeneration in the limit  $s \rightarrow 0$ , we obtain a surface with two marked points  $a$  and  $b$ , where the handle was pinched. Now, let  $\nu_*^2(c) = \partial_i \theta_*(0) \omega_i(c)$  for an odd theta characteristic  $*$  and define

$$E(a, b) := \frac{\theta_*(A_{ab})}{\nu_*(a)\nu_*(b)} \quad (3.53)$$

which is the prime form, see [11]. Let  $A_2[e](a, b)$  be the two-point function. We will have up to a factor independent of  $e$ , in some choice of local coordinates,

$$X_{NS}[e] = sE(a, b)^2 A_2[e](a, b) + O(s^2), \quad (3.54)$$

from [1]. For the OPSMY part of the ansatz we will stick to the notation from Matone and Volpato, that is, we will write  $\Theta_k$  for the lattice theta series, with a different numbering of lattices for  $k \leq 5$ , so that it is easier to compare the formulae. Here we present a translation diagram:

[1] notation	Lattice	Our notation	[1] notation	Lattice	Our notation
$\Theta_0$	$(D_8 \oplus D_8)^+$	$\vartheta_5$	$\Theta_4$	$\mathbb{Z}^8 \oplus E_8$	$\vartheta_1$
$\Theta_1$	$\mathbb{Z} \oplus A_{15}^+$	$\vartheta_4$	$\Theta_5$	$\mathbb{Z}^{16}$	$\vartheta_0$
$\Theta_2$	$\mathbb{Z}^2 \oplus (E_7 \oplus E_7)^+$	$\vartheta_3$	$\Theta_6$	$E_8 \oplus E_8$	$\vartheta_6$
$\Theta_3$	$\mathbb{Z}^4 \oplus D_{12}^+$	$\vartheta_2$	$\Theta_7$	$D_{16}^+$	$\vartheta_7$

Let  $N_k$  be the number of norm two vectors in the lattice corresponding to  $\Theta_k$ . Let  $c_k^g$  be the coefficient of  $\Theta_k$  in the OPSMY ansatz for genus  $g$ , where the same normalization as in [7] is used ( $c_k^g$  is  $2^{4g}$  times the coefficients from [5]) for easier comparison.

We have, for the OPSMY ansatz, from [1],

$$X_{NS}[e](s, \Omega, z) = \sum_{k=0}^7 c_k^5 (1 + N_k s + O(s^2)) \Theta_k^{(4)}[e](\Omega). \quad (3.55)$$

We will write

$$X_{NS}[e](s, \tau, z) = T_0[e](\tau, z) + s T_1[e](\tau, z) + O(s^2). \quad (3.56)$$

Note that  $E_8 \oplus E_8$  and  $D_{16}^+$  contain 480 norm 2 vectors and  $(D_8 \oplus D_8)^+$  contains 224 of them. Also, the  $s$ -linear term from  $G_5^{(5)}$ , formula (3.23), equals  $244 G_4^{(4)}$  in  $z = 0$ . Therefore, we have

$$T_0[e](\tau, 0) = \sum_{k=0}^5 c_k^5 \Theta_k^{(4)}[e] + c J^{(4)} + c_f f^{(4)} = \left( c_J - \frac{2^5 \cdot 3}{7} \right) J^{(4)} + c_f f^{(4)} \quad (3.57)$$

$$T_1[e](\tau, 0) = 128 \Xi_{OPSMY}^{(4)}[e](\tau) + \left( 480 c_J - \frac{720 \cdot 2^5 \cdot 3}{7} \right) J^{(4)} + 224 c_f f^{(4)} \quad (3.58)$$

As  $s \rightarrow 0$ , we get

$$X_{NS}[e] = s \sum_{i,j}^4 2\pi i E(a, b)^2 \omega_i(a) \omega_j(b) (1 + \delta_{ij}) \left( \left( c_J - \frac{2^5 \cdot 3}{7} \right) \frac{\partial J^{(4)}}{\partial \tau_{ij}} + c_f \frac{\partial f^{(4)}}{\partial \tau_{ij}} \right) + s T_1^{(4)}[e](\tau, A_{ab}) + O(s^2). \quad (3.59)$$

Calculating  $T_1[e](\tau, A_{ab})$  from  $T_1[e](\tau, 0)$  can be done using the fact that  $T_1[e]$  is a section of  $|2\Theta|$ , because it is composed of quasiperiodic forms. Here  $\Theta$  is the divisor of  $\theta_0(z)$ . Matone and Volpato prove that from that fact it follows that

$$T_1[e](\tau, A_{ab}) = E(a, b)^2 \left( T_1[e](\tau, 0) \omega(a, b) + \frac{1}{2} \sum_{i,j}^4 \partial_i \partial_j T_1[e](\tau, 0) \omega_i(a) \omega_j(b) \right). \quad (3.60)$$

From [9], we have  $f^{(4)} = \frac{3}{7} J^{(4)}$  which is the Schottky form and vanishes on  $\mathcal{J}_4$ . Thus we have  $T_1[e](\tau, 0) = 128 \Xi_{OPSMY}^{(4)}$  on the Jacobian locus. Then, we get

$$A_2[e](a, b) = 128 \Xi^{(4)}[e](\tau) \omega(a, b) + \sum_{i,j}^4 \omega_i(a) \omega_j(b) \left( 2\pi i (1 + \delta_{ij}) \left( \left( c_J - \frac{2^5 \cdot 3}{7} \right) \frac{\partial J^{(4)}}{\partial \tau_{ij}} + c_f \frac{\partial f^{(4)}}{\partial \tau_{ij}} \right) + \frac{1}{2} \partial_i \partial_j T_1^{(4)}[e](\tau, 0) \right). \quad (3.61)$$

Denoting by  $f_1^{(5)}$  the  $s$ -linear term from the  $s$ -expansion of  $f^{(5)}$ , and using the functions

$$F_k^{(g)}(\tau, z) := \sum_{p_1, \dots, p_g \in \Lambda_k} e^{\pi i \sum_{i,j=1}^g p_i p_j \tau_{ij}} \sum_{\tilde{p} \cdot \tilde{p}=2} e^{2\pi i \sum_{i=1}^g \tilde{p} p_i z_i} \quad (3.62)$$

we end up with the modified formula

$$\partial_i \partial_j T_1^{(4)}[e](\tau, 0) = \partial_i \partial_j \left( \sum_{k=0}^5 c_k^5 F_k^{(4)}[e](\tau, 0) + c_J (F_6^{(4)} - F_7^{(4)}) + c_f f_1^{(5)} \right). \quad (3.63)$$

Here, Matone and Volpato introduce the coefficients  $s_k^g$  and  $t_k^g$ , defined by the following formula:

$$\partial_i \partial_j c_k^{g+1} F_k^{(g)}[e](\tau, 0) = 2\pi i (1 + \delta_{ij}) \partial_i \partial_j s_k^g \Theta_k^{(g)}[e] - t_k^g \Theta_k^{(g)} \partial_i \partial_j \log \theta[e](\tau, 0). \quad (3.64)$$

Continuing the process from [1], and noting that  $f_1^{(5)}$  has the property that  $\frac{\partial^2 f_1^{(5)}}{\partial z_i \partial z_j} = 28(2\pi i)(1 + \delta_{ij}) \frac{\partial f^{(4)}}{\partial \tau_{ij}}$  (see formulae (3.22) and (3.28)), we then get

$$\begin{aligned} \partial_i \partial_j T_1^{(4)}[e](\tau, 0) &= 2\pi i (1 + \delta_{ij}) \frac{\partial}{\partial \tau_{ij}} \left( \sum_{k=0}^5 s_k^4 \Theta_k^{(4)}[e](\tau) + 60c_J J^{(4)} + 28c_f f^{(4)} \right) \\ &\quad - \left( \sum_{k=0}^5 t_k^4 \Theta_k^{(4)}[e](\tau) \right) \partial_i \partial_j \log \theta[e](\tau, 0). \end{aligned} \quad (3.65)$$

And further following the calculations from [1] the first term in big brackets can be written as

$$\sum_{k=0}^5 s_k^4 \Theta_k^{(4)}[e](\tau) + 60c_J J^{(4)} + 28c_f f^{(4)} = 32\Xi^{(4)}[e](\tau) + \left( 60c_J + \frac{3 \cdot 28}{7} c_f - \frac{152 \cdot 2^5 \cdot 3}{7} \right) J^{(4)}. \quad (3.66)$$

So, having carried the modified  $\tilde{\Xi}$  through the degeneration, we end up with a slightly different two-point function,

$$\begin{aligned} A_2[e](a, b) &= 128\Xi^{(4)}[e](\tau) \omega(a, b) + \sum_{i,j}^4 \omega_i(a) \omega_j(b) \left[ -128\Xi^{(4)}[e](\tau) \partial_i \partial_j \log \theta[e](\tau, 0) \right. \\ &\quad \left. + 2\pi i (1 + \delta_{ij}) \frac{\partial}{\partial \tau_{ij}} \left( 16\Xi^{(4)}[e](\tau) + \left( (30 + 1)c_J + (6 + 1)c_f - \frac{(76 + 1) \cdot 2^5 \cdot 3}{7} \right) J^{(4)} \right) \right] \end{aligned} \quad (3.67)$$

The last step of the procedure from [1] is to sum over even characteristics. This procedure yields, finally,

$$\sum_e A_2[e](a, b) \bar{\mathcal{J}}_4 D \sum_{i,j}^4 \omega_i(a) \omega_j(b) 2\pi i (1 + \delta_{ij}) \frac{\partial J^{(4)}}{\partial \tau_{ij}} \quad (3.68)$$

$$D = 2^3(2^4 + 1) \left( 16B_4 - 8D_4 - 77 \frac{2^5 \cdot 3}{7} + 31c_J + 7c_f \right) \quad (3.69)$$

So, to make  $\sum_e A_2[e](a, b)$  vanish, we would need

$$31c_J + 7c_f = 77 \frac{2^5 \cdot 3}{7} + 8 \frac{2^7 \cdot 3}{7 \cdot 17} - \frac{2^6 \cdot 3^3 \cdot 5 \cdot 11}{7 \cdot 17}. \quad (3.70)$$

The genus 5 cosmological constant from the 'plain' OPSMY ansatz, that is, without the  $-B_5 J^{(5)}$  part, equals (again, see [1]),

$$\sum_e \sum_{k=0}^5 c_k^5 \Theta_k[e] = -2^4(2^5 + 1) \frac{2^5 \cdot 17}{7 \cdot 11} J^{(5)}. \quad (3.71)$$

From Section 3.2, we have for the trace of  $f^{(5)}$ :

$$\sum_e f^{(5)}[e] = \frac{2^4 \cdot 3^2 \cdot 11 \cdot 17}{7 \cdot 31} J^{(5)}. \quad (3.72)$$

Because  $E_8 \oplus E_8$  and  $D_{16}^+$  are even lattices, they are invariant under modular transformations and therefore

$$\sum_e J^{(5)}[e] = 2^4(2^5 + 1) J^{(5)}. \quad (3.73)$$

Thus, to make the genus 5 cosmological constant vanish we would need

$$2^4(2^5 + 1)c_J + \frac{2^4 \cdot 3^2 \cdot 11 \cdot 17}{7 \cdot 31} c_f = 2^4(2^5 + 1) \frac{2^5 \cdot 17}{7 \cdot 11}. \quad (3.74)$$

Combining the above linear equations (3.70) and (3.74), we find the solution

$$c_J = -\frac{222647008}{217}, c_f = \frac{77245568}{17}. \quad (3.75)$$

Hence we present our main formula:

$$\tilde{\Xi} := \Xi_{OPSMY}^{(5)} - \frac{222647008}{217} J^{(5)} + \frac{77245568}{17} f^{(5)}$$

(3.76)

and the above amounts to proving our main result:

**Theorem 3.4.1.**  $\tilde{\Xi}$  is the unique linear combination of known semi-modular forms of weight 8 that yields both a vanishing genus 5 cosmological constant and a vanishing genus 4 two-point function.

## 3.5 The situation in genus 6

Here we take a brief look at the current state of the ansätze in genus 6 and the possibility of improving it using our findings.

Let  $\tilde{\Xi}_e^{(6)}$  be the Grushevsky ansatz for genus 6 (see [7, Th.22]). Then, define

$$\tilde{\Xi}_e^{(6)} := \Xi_e^{(g)} + k_6 f^{(6)} + l_6 J^{(6)}. \quad (3.77)$$

For genus 6, the factorization condition gives

$$\begin{aligned}\tilde{\Xi}_e^{(6)} \begin{pmatrix} \lambda & 0 \\ 0 & \tilde{\tau} \end{pmatrix} &= \Xi_e^{(5)} \Xi_e^{(1)} + k_6 \left( \vartheta_5^{(1)} \theta_5^{(5)} - G_1^{(1)} G_5^{(5)} \right) + l_6 \left( \vartheta_6^{(1)} \vartheta_6^{(5)} - \vartheta_7^{(1)} \vartheta_7^{(5)} \right) \\ &\stackrel{?}{=} \Xi_e^{(5)} \Xi_e^{(1)} + \Xi_e^{(1)} \left( k_5 f^{(5)} + l_5 J^{(5)} \right)\end{aligned}\quad (3.78)$$

and as  $G_1^{(1)} = \vartheta_5^{(1)}$ ,  $\Xi^{(1)} = \frac{1}{2} \left( G_0^{(1)} - G_1^{(1)} \right)$  and  $\vartheta_6^{(1)} = \vartheta_7^{(1)} = \sum_e G_0^{(1)}[e]$ , this implies

$$k_6 G_1^{(1)}[e] + l_6 \sum_{e'} G_0^{(1)}[e'] = \frac{1}{2} (k_5 + l_5) \left( G_0^{(1)}[e] - G_1^{(1)}[e] \right) \quad (3.79)$$

and that implies  $k_6 = l_6 = k_5 + l_5 = 0$ . By theorem 3.4.1 and equation (3.76) we have  $k_5 + l_5 \neq 0$ ; so if we want both the genus 4 two-point function and the genus 5 cosmological constant to vanish, this cannot work.

We conclude that to satisfy the factorization constraint in genus 6 while using the proposed modification in genus 5, one needs a new form that degenerates in a way that solves the above problem.

## 3.6 Conclusion

We have solved the problems posed in [9] and [1]: to compare the remaining two forms from the OPSMY and Grushevsky ansätze and to use them to make vanish both the cosmological constant in genus 5 and the two-point function in genus 4.

More precisely, we have shown that combining the two not previously compared forms from OPSMY and Grushevsky yields a form that cannot be expressed through the others. We have used this form to construct a slightly modified version of the OPSMY ansatz for genus 5, which does not only have a vanishing cosmological constant in genus 5 but also a vanishing two-point function in genus 4, as obtained from degeneration.

We have looked at the behaviour of this form in genus 6. We found that there is no way to satisfy the factorization property using our modified genus 5 ansatz and a genus 6 ansatz constructed solely from the currently known semi-modular forms of weight 8.

Thus, there are two possibilities: either there are more semi-modular forms to be found, perhaps forms that, like the higher genus forms from Grushevsky, live only on the Jacobian locus. Or, it may be that the conjecture that the NSR measures can always be expressed in terms of semi-modular forms is wrong, and breaks down at genus 6.

# Chapter 4

## Popular summary

It is a disturbing fact that string theory, although it first arose as early as 1969 in the study of what we now call quantum chromodynamics, leaves many questions wide open. That is, many advances have been made in mathematics due to the interest in this area, but the physical implications of the various kinds of string theory are not yet clear.

Lucky for aspiring mathematical physicists, this means there is a lot of work to be done in relatively unexplored areas. Whether string theory correctly describes the physical world around us is still very much unknown and might even be unlikely, seeing the lack of supersymmetric partners observed in the Large Hadron Collider at CERN. Still, it is very much possible that they will only be observed at much higher energies. Such concerns, however, must not hold us back from advancing in this field.

The present thesis focuses on perturbative supersymmetric string theory, and on a very specific problem in that area. 'Perturbative' means, in this context, that the probability of interactions, which is after all what physics is ultimately all about, is calculated by summing over the probability of all possible constituent processes. To be precise, we expand everything in powers of the string coupling constant, thereby effectively assuming that it is small, and then compute terms order-by-order, where terms then correspond to different genera.

In a diagram, we might put it like this:

$$= \text{ (circle) } + \text{ (torus) } + \text{ (double torus) } + \dots$$

There is nothing on the left-hand side, because we are concerned with the probability of the following event: we start out with a vacuum, wait a bit, and then we still have a vacuum. This is of course the simplest case possible. On the right hand side we see all the ways in which this may be brought about : it may be that a string pops up from nothing, and then vanishes again, as in the first picture (the subpictures represent processes, with time flowing from left to right inside them) or perhaps this string will split in two before coming together again and then vanishing - or after coming together it might split again for one last time, or... There is an infinite number of possibilities.

The way to neatly categorize all possibilities is to split the pictures of processes by their *genus*, loosely the number of holes in the surface. Then, we can by a known procedure assign for each genus a parameter to each possible surface of that genus, and

integrate over the space of these parameters, to count all possibilities. The integration here is as always the natural extension of the idea of summation in a space that contains uncountably many elements.

What we are looking for, then, is the so-called *superstring measure*, something that we can integrate over the space of all these parameters representing actual processes, and that of course should be so defined that it gives physically relevant results.

As it turned out to be very hard to *calculate* the superstring measure, mathematical physicists have started *guessing* it. That is less ridiculous than it sounds: the guesses are very much educated, because we know a great deal of properties that the superstring measure should possess, and it turns out there are preciously few objects possessing these properties.

This program of guessing the superstring measure has yielded some promising results a few years ago, but more recently it came to a grinding halt. Two proposals had been put forward: the so-called OPSMY and Grushevsky ansätze (i.e. proposals). It had been shown that they coincided, and were in fact unique, for genera 1 through 4, but in genus 5 (a picture of a genus 5 surface is provided on the cover) scientists had not been able<sup>1</sup> to compare them. Even more problematically, a paper was published in which it was shown that the two proposals did not actually satisfy all the properties of the superstring measure for genus 5, so they had to be modified.

Therefore, the present thesis set out to compare the OPSMY and Grushevsky proposals in genus 5, and if they would differ, to try to combine their building blocks to fix the problems that had popped up.

Slightly unexpectedly, we found by applying a rather lengthy calculation adapted from a recent paper that the proposals *did* differ, and that the difference was very well suited to fix the proposals to indeed satisfy all the properties of the superstring measure in genus 5.

Sadly, it also turned out that we currently cannot construct the superstring measure in genus 6 and above. Some new, more general approach would be needed for that, and might be as complicated as calculating it directly.

So, returning to the general setting, perturbative supersymmetric string theory has still not yielded a clear physical model. It may be that all the complications that arise now will be solved by turning to non-perturbative models, where there is no integration needed, but it is conceptually unclear how they might be defined - or perhaps, it turns out that the more interesting facts will only be found after a succesful generalization is made to the currently speculative M-theory, of which all this would just be a specific case. But as always in mathematical physics, because of the dense relations between areas of mathematics and the unerring interest of mathematicians in all that can be thought, each little step might someday make someone very happy.

---

<sup>1</sup>It must be said that here, we are talking about complicated infinite sums depending on a matrix that lives in a space with 24 real dimensions, and is not flat but has a very complicated shape. It is conceptually nontrivial to see whether two such things are or are not the same.

# Acknowledgements

We are very grateful to Gerard van der Geer for his numerous comments and helpful remarks. Our work is much indebted to previous work by M. Matone, R. Volpato, R. Salvati Manni, S. Grushevsky and A. Morozov. We would like to extend our gratitude to Alexei Morozov for enabling us to present our work at the EINSTEIN conference at SISSA and to Laura San Giorgi for her diligent reading.

# Bibliography

- [1] M. Matone and R. Volpato, “Getting superstring amplitudes by degenerating Riemann surfaces,” 2010. arXiv:1003.3452v2 [hep-th].
- [2] E.D’Hoker and D.Phong, “Two Loop Superstrings. I. Main Formulas,” *Phys. Lett.*, no. B529, pp. 241–255, 2002. hep-th/0110247.  
E.D’Hoker and D.Phong, “Two Loop Superstrings. II. The Chiral Measure on Moduli Space,” *Nucl.Phys.*, no. B636, pp. 3–60, 2002. hep-th/0110283.  
E.D’Hoker and D.Phong, “Two Loop Superstrings. III. Slice Independence and Absence of Ambiguities,” *Nucl.Phys.*, no. B636, pp. 61–79, 2002. hep-th/0111016.  
E.D’Hoker and D.Phong, “Two Loop Superstrings. IV. The Cosmological Constant and Modular Forms,” *Nucl.Phys.*, no. B639, pp. 129–181, 2002. hep-th/0111040.  
E.D’Hoker and D.Phong, “Asyzygies, Modular Forms and the Superstring Measure. I,” *Nucl.Phys.*, no. B710, pp. 58–82, 2005. hep-th/0411159.  
E.D’Hoker and D.Phong, “Asyzygies, Modular Forms and the Superstring Measure. II,” *Nucl.Phys.*, no. B710, pp. 83–116, 2005. hep-th/0411182.  
E.D’Hoker and D.Phong, “Two Loop Superstrings. V. Gauge Slice Dependence of the N-Point Function,” *Nucl.Phys.*, no. B715, pp. 91–119, 2005. hep-th/0501196.  
E.D’Hoker and D.Phong, “Two Loop Superstrings. VI. Non-Renormalization Theorems and the 4-Point Functions,” *Nucl.Phys.*, no. B715, pp. 3–90, 2005. hep-th/0501197.  
E.D’Hoker and D.Phong, “Two Loop Superstrings. VII. Cohomology of Chiral Amplitudes,” *Nucl.Phys.*, no. B804, pp. 421–506, 2005. arXiv: 0711.4314.
- [3] A. Belavin and V. Knizhnik, “Algebraic geometry and the geometry of quantum strings,” *Phys. Lett.*, no. B168, p. 201, 1986.  
A. Belavin and V. Knizhnik, “Complex geometry and the theory of quantum strings,” *Sov. Phys.*, no. 64, p. 214, 1986.  
V. Knizhnik, “Multiloop amplitudes in the theory of quantum strings and complex geometry,” *Sov. Phys.*, no. 64, p. 214, 1986.  
A. Belavin, V. Knizhnik, A. Morozov, and A. Perelomov, “Two and three loop amplitudes in the bosonic string theory,” *Phys. Lett.*, no. B177, p. 324, 1986.  
G. Moore, “Modular forms and two-loop string physics,” *Phys. Lett.*, no. B176, p. 369, 1986.

L. Alvarez-Gaume, G. Moore, P. Nelson, C. Vafa, and J. Bost, “Bosonization in arbitrary genus,” *Phys. Lett.*, no. B178, p. 41, 1986.

A. Morozov, “Explicit formulae for one, two, three and four loop string amplitudes,” *Phys. Lett.*, no. B184, p. 171, 1987.

A. Morozov, “Analytical anomaly and heterotic string in the formalism of continual integration,” *Phys. Lett.*, no. B184, p. 177, 1987.

E. D’Hoker and D. Phong, “Superholomorphic anomalies and supermoduli space,” *Nucl. Phys.*, no. B292, p. 317, 1987.

E. D’Hoker and D. Phong, “The geometry of string perturbation theory,” *Rev. Mod. Phys.*, no. 60, p. 917, 1987.

[4] E. Martinec, “Non-Renormalization Theorems and Fermionic String Finiteness,” *Phys. Lett.*, no. 171B, p. 189, 1986.

E. Martinec, “Conformal Field Theory on a (Super) Riemann Surface,” *Nucl. Phys.*, no. B281, p. 157, 1987.

[5] M. Oura, C. Poor, R. S. Manni, and D. Yuen, “Modular Forms of weight 8 for  $\Gamma_g(1, 2)$ ,” arXiv:0811.2259.

[6] S. Cacciatori, F. Piazza, and B. Geemen, “Modular Forms and Three Loop Superstring Amplitudes,” *Nucl. Phys.*, no. B800, pp. 565–590, 2008. arXiv:0801.2543.

S. Cacciatori, F. Piazza, and B. Geemen, “Genus Four Superstring Measures,” *Lett. Math. Phys.*, no. 85, pp. 185–193, 2008. arXiv:0804.0457.

[7] S. Grushevsky, “Superstring scattering amplitudes in higher genus,” *Commun. Math. Phys.* 287, p. 749, 2009. arXiv:0803.3469 [SPIRES].

[8] S. Grushevsky and R. S. Manni, “The superstring cosmological constant and the schottky form in genus 5,” 2011. arXiv:0809.1391v5 [math.AG].

[9] P. Dunin-Barkowski, A. Morozov, and A. Sleptsov, “Lattice theta constants vs Riemann theta constants and NSR superstring measures,” *Journal of High Energy Physics, Issue 10*, p. 72, 2009. arXiv:0908.2113v3 [hep-th].

[10] P. Griffiths and J. Harris, “Principles of Algebraic Geometry,” 1994.

[11] J. Fay, “Theta functions on Riemann surfaces,” *Lecture Notes in Mathematics*, no. 352, 1973.