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Thermodynamics and weak cosmic censorship conjecture in the Kerr-AdS black hole

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Abstract

We investigate the laws of thermodynamics and weak cosmic censorship conjecture in the normal and extended phase space of a Kerr-AdS black hole by analyzing the energy-momentum relation of the absorbed fermion dropping into the horizon. In the normal phase space, the first law, second law as well as the weak cosmic censorship conjecture are found to be valid in all the initial states of the black hole. However, in the extended phase space, although the first law and weak cosmic censorship conjecture are still valid, the second law becomes more subtle. We find that the validity or violation of the second law depends on the spin parameter, the radius of the AdS spacetime, and their variations. In addition, in the extended phase space, the configurations of the extremal and near-extremal Kerr-AdS black holes are unchanged as the fermions are absorbed since the final and initial states are the same.
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1. Introduction

Due to the pioneering work of Hawking [1], we know that a black hole is not only a celestial body, but also a thermodynamic system. Just like the ordinary thermodynamic systems, there are also four laws for the black hole thermodynamic systems [2], which are valid near the hori-

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zon. The importance of thermodynamics of black holes is that it provides fundamental relations between theories such as gravity, thermodynamics and quantum field theory.

Thermodynamics in the AdS space is more prevalent for the existence of a cosmological parameter. Initially, the cosmological parameter in the thermodynamic system is treated as a constant. However, more and more evidences have shown that this parameter may be considered as a dynamical variable [3]. Moreover, Ref. [4] put forward to regarding it as a state function. In particular, Refs. [5–7] proposed that the cosmological constant can be regarded as the pressure of the thermodynamic system, while its thermodynamic conjugate variable can be defined naturally as volume. The thermodynamic phase space thus is extended. In the extended phase space, the Smarr relation derived by the scaling method is also valid besides the first law of thermodynamics. The mass, of course, is not the internal energy but rather the enthalpy of the thermodynamic system. The emergent property in the extended phase space is that the phase structures of black holes are more abundant, and one can observe the van der Waals-like phase transition [8–13], reentrant phase transitions [14–17].

Another reason for the importance of the thermodynamics in AdS space arises from the AdS/CFT correspondence [18–21], which states that the gravity theory in d-dimensional AdS spacetime is dual to the conformal field theory in $(d-1)$-dimension. Various holographic applications have been proposed in the last two decades, for example, the holographic superconductors [22–25], holographic thermalizations [26–29], holographic Fermi/non-Fermi liquids [30–32], and so on.

For the general relativity, if singularities are naked, causality in the spacetimes may break down, and physics will lose its predictive power. Penrose conjectured that the singularity will not be naked for it is always covered by an event horizon [33], which is the so-called 'weak cosmic censorship conjecture'. Since there are no general ways to prove this conjecture, we thus should investigate each type of black holes. Wald proposed firstly a gedanken experiment to check this conjecture by throwing a particle into the Kerr-Newman black hole, and he found that the weak cosmic censorship conjecture was valid for the extremal Kerr-Newman black holes [34]. However, for the near-extremal Reissner-Nordström [35] black hole and near-extremal Kerr black hole [36], this conjecture were found to be violated. Later it was pointed out that the invalidity of the weak cosmic censorship conjecture in [35,36] was that the backreaction and self-force effects are neglected [37]. So far, the weak cosmic censorship conjecture has been checked in various black holes with and without the self-force effect by adding a particle [38–54].

As mentioned above, there are many works to investigate the thermodynamics of black holes in the extended phase space, however all of them focus only on the first law of thermodynamics. There is little work to discuss the second law of thermodynamics and the weak cosmic censorship conjecture in the extended phase space. Recently, Ref. [55] suggested that the first law, second law as well as the weak cosmic censorship conjecture can be investigated in the extended phase space under charged particles absorption. Their results showed that though the first law and the weak cosmic censorship conjecture were valid, the second law was violated for the near-extremal Reissner-Nordström-AdS black holes. Subsequently, the idea in [55] was extended to Born-Infeld-anti-de Sitter black hole [56], torus-like black hole [57] and charged fermions absorption [58–60]. In this paper, we are going to investigate the laws of thermodynamics and weak cosmic censorship conjecture in the Kerr-AdS black hole. Different from previous investigations, the topology of this black hole is axisymmetric. In fact, Refs. [61,62] have investigated the laws of thermodynamics of this black hole in the normal phase space. In order to get the first law as well as the second law, a reference energy of the particle in the asymptotic region was imposed. In this paper, we will explore whether the first law and second law may be obtained by the

energy-momentum relation of the particles without any imposition. In addition, the laws of thermodynamics and weak cosmic censorship conjecture of the rotating black holes in the extended phase space are also explored.

This paper is arranged as follows. In section 2, we briefly review the thermodynamics of the Kerr-AdS black hole, and then study the dynamics of the spinning fermions in this black hole. In section 3, we investigate the first law, the second law as well as the weak cosmic censorship conjecture in the normal phase space. The law of thermodynamics and weak cosmic censorship conjecture in the extended phase space are studied in section 4. Section 5 is devoted to our conclusions. Throughout this paper, we set the gravitational constant G and the light velocity c to be one.

2. Energy and momentum of the spinning fermions

2.1. Brief review of the Kerr-AdS black holes

Line element of the Kerr-AdS black hole in Boyer-Lindquist coordinates can be written as [63]

$$ds^2 = -\frac{\Delta}{\rho^2}\left(dt - \frac{a\sin^2\theta}{\Xi}d\varphi\right)^2 + \frac{\rho^2}{\Delta}dr^2 + \frac{\rho^2}{\Delta_\theta}d\theta^2 + \frac{\Delta_\theta \sin^2\theta}{\rho^2}\left(adt - \frac{r^2+a^2}{\Xi}d\varphi\right)^2, \tag{1}$$

with the metric functions given by

$$\Delta = (r^2+a^2)(1+r^2/l^2) - 2mr, \quad \rho^2 = r^2 + a^2\cos^2\theta, \tag{2}$$

$$\Xi = 1 - \frac{a^2}{l^2}, \quad \Delta_\theta = 1 - a^2/l^2\cos^2\theta, \tag{3}$$

in which, m, a and l are the black hole mass parameter, spin parameter and the AdS radius respectively. Cosmological constant Λ is related to the AdS radius as $\Lambda = -3/l^2$.

Matter near the event horizon will be dragged inevitably by the gravitational field with an azimuthal angular velocity defined by

$$\bar{\Omega} = \frac{a\Xi\big(\Delta_\theta(r^2+a^2) - \Delta\big)}{\Delta_\theta(r^2+a^2)^2 - \Delta a^2\sin^2\theta}. \tag{4}$$

The angular velocities at the event horizon and infinity can be readily obtained from Eq. (4) as

$$\bar{\Omega}_h = \frac{a\Xi}{r_h^2+a^2}, \quad \bar{\Omega}_\infty = -\frac{a}{l^2}, \tag{5}$$

where r_h is position of the event horizon. It should be noted that the angular velocity at infinity does not vanish, which is distinct from that in the asymptotic flat case. Thus, the net angular velocity that contributes to the thermodynamics is given by [4]

$$\Omega_h = \bar{\Omega}_h - \bar{\Omega}_\infty = \frac{a\Xi}{r_h^2+a^2} + \frac{a}{l^2}. \tag{6}$$

Hawking temperature and Bekenstein-Hawking entropy of the Kerr-AdS black holes are

$$T_h = \frac{r_h}{4\pi(r_h^2+a^2)}\left(1 + \frac{a^2}{l^2} + 3\frac{r_h^2}{l^2} - \frac{a^2}{r_h^2}\right), \quad S = \frac{\pi(r_h^2+a^2)}{\Xi}. \tag{7}$$

The black hole mass M and angular momentum J can be expressed by the mass parameter m and spin parameter a as $M = \frac{m}{\Xi^2}$, $J = \frac{am}{\Xi^2}$, respectively. Therefore, we get the first law of thermodynamics

$$dM = T_h dS + \Omega_h dJ. \tag{8}$$

In Eq. (8), the cosmological parameter is treated as a constant, and the resulting phase space of the thermodynamics is called 'normal phase space'. Recent studies have shown that the normal phase space can be extended by treating the cosmological parameter as an extensive variable, leading to an 'extended phase space' [5–7]. Thus, the first law of thermodynamics in the extended phase space can be written as [64,65]

$$dM = T_h dS + \Omega_h dJ + V dP, \tag{9}$$

in which P and V are respectively the pressure and volume of the system, defined by

$$P = \frac{3}{8\pi l^2}, \quad V = \frac{2\pi\left(a^2 + r_h^2\right)\left(a^2 l^2 - a^2 r_h^2 + 2l^2 r_h^2\right)}{3l^2 \Xi^2 r_h}. \tag{10}$$

Based on the first law in the extended phase space, the phase structures of the Kerr-AdS black holes have been investigated extensively. It was shown that in the $P - V$ plane, there were phase transitions between the small black holes and large black holes [64,65], which is similar to the van der Waals phase transition.

2.2. Motion of the spinning fermions in the Kerr-AdS black holes

When particles are absorbed by spherical symmetric black holes, it has been confirmed that they satisfy the first law of thermodynamics as the energy-momentum relations of particles are employed. We will adopt the similar strategy to investigate the laws of thermodynamics of Kerr-AdS black holes, when the absorbed particles are spinning fermions. The Dirac equation now is $i\gamma^\mu\left(\partial_\mu + \Omega_\mu\right)\psi - \frac{u}{\hbar}\psi = 0$, in which u is the rest mass of the fermion, while $\Omega_\mu = \frac{i}{2}\Gamma_\mu^{\alpha\beta}\prod_{\alpha\beta}$, $\prod_{\alpha\beta} = \frac{i}{4}\left[\gamma^\alpha, \gamma^\beta\right]$, and Dirac matrices γ^μ satisfy $\{\gamma^\mu, \gamma^\nu\} = 2g^{\mu\nu}$.

In order to investigate the motion of the fermions in the rotating black holes, one should carry out the dragging coordinate transformation with the definition $d\phi = d\varphi - \bar{\Omega}dt$. In this case, the metric in Eq. (1) becomes

$$ds^2 = -F(r)dt^2 + \frac{1}{G(r)}dr^2 + H(r)d\theta^2 + K(r)d\phi^2, \tag{11}$$

in which

$$F(r) = \frac{\Delta\Delta_\theta\rho^2}{\Delta_\theta(r^2 + a^2)^2 - \Delta a^2 \sin^2\theta}, \quad G(r) = \frac{\Delta}{\rho^2}, \tag{12}$$

$$H(r) = \frac{\rho^2}{\Delta_\theta}, \quad K(r) = \frac{\sin^2\theta}{\rho^2\Xi^2}\left((r^2 + a^2)^2\Delta_\theta - \Delta a^2 \sin^2\theta\right). \tag{13}$$

From Eq. (11), we know that the event horizon and infinite red-shift surface coincide. This coincidence is important to investigate the tunneling radiation of the particles, for in this case Landau's condition of the coordinate clock synchronization is satisfied.

To solve the Dirac equation in Eq. (11), we choose γ^μ matrices as [66]

$$\begin{aligned}\gamma^t &= \frac{1}{\sqrt{F}}\begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix},\ \gamma^r = \sqrt{G}\begin{pmatrix} 0 & \sigma^3 \\ \sigma^3 & 0 \end{pmatrix}, \\ \gamma^\theta &= \frac{1}{\sqrt{H}}\begin{pmatrix} 0 & \sigma^1 \\ \sigma^1 & -1 \end{pmatrix},\ \gamma^\phi = \frac{1}{\sqrt{K}}\begin{pmatrix} 0 & \sigma^2 \\ \sigma^2 & 0 \end{pmatrix},\end{aligned} \tag{14}$$

in which σ^i are the Pauli matrices. For a fermion with spin 1/2, the wave function have spin up and down state. In this paper, we are interested in the spin up state since the spin down case is similar. The wave function for the spin up state is supposed to be

$$\psi = \begin{pmatrix} C\,(t,r,\theta,\phi) \\ 0 \\ D\,(t,r,\theta,\phi) \\ 0 \end{pmatrix} \exp\left(\frac{i}{\hbar} I\,(t,r,\theta,\phi)\right). \tag{15}$$

Inserting Eq. (15) and Eq. (14) into the Dirac equation, and applying the WKB approximation, we are led to

$$-\left(\frac{iC}{\sqrt{F}}\partial_t I + D\sqrt{G}\partial_r I\right) + uC = 0, \quad \left(\frac{iD}{\sqrt{F}}\partial_t I - C\sqrt{G}\partial_r I\right) + uD = 0, \tag{16}$$

$$-D\left(\frac{1}{\sqrt{H}}\partial_\theta I + \frac{i}{\sqrt{K}}\partial_\phi I\right) = 0, \quad -C\left(\frac{1}{\sqrt{H}}\partial_\theta I + \frac{i}{\sqrt{K}}\partial_\phi I\right) = 0, \tag{17}$$

in which we only take the leading order of $\hbar$. In order to solve these equations, one should separate the quantity I. According to the symmetries of the spacetime, the quantity I in the dragging coordinate system can be rewritten as [66]

$$I = -(\omega - j\Omega_h)t + I\,(r) + L\phi + \Theta(\theta), \tag{18}$$

where ω and j are respectively fermion's energy and angular momentum measured by the observer at the infinity. Substituting Eq. (18) into Eqs. (16) and (17), we obtain

$$C\left(u + \frac{i}{\sqrt{F}}(\omega - j\Omega_h)\right) - D\sqrt{G}\partial_r I = 0, \quad -C\sqrt{G}\partial_r I + D\left(u - \frac{i}{\sqrt{F}}(\omega - j\Omega_h)\right) = 0. \tag{19}$$

Eqs. (19) have non-trivial solutions for C and D only if the determinant of the coefficient matrix vanishes. From this condition, we can get $\partial_r I$, and further the radial momentum $p^r = g^{rr}\partial_r I$,

$$p^r = \pm\sqrt{\frac{G}{F}}\sqrt{(\omega - j\Omega_h)^2 + \mu^2 F}. \tag{20}$$

The laws of thermodynamics should be studied near horizon region. Thus, Eq. (20) is simplified as

$$\omega = \frac{\rho_h^2}{a^2 + r_h^2}|p_h^r| + j\Omega_h, \tag{21}$$

in which the lower index h represents near horizon region, with $\rho_h^2 = a^2\cos^2\theta + r_h^2$. From Eq. (21), we know that when $\omega < j\Omega_h$, the supperradiation occurs and the energy of the black hole flows out of the horizon, since the sign in front of $|p_h^r|$ is negative. In this paper, we assume that the supperradiation does not occur, which implies $\omega > j\Omega_h$ and the sign in front of $|p_h^r|$ is positive.

3. Thermodynamics and weak cosmic censorship conjecture in the normal phase space

We will employ the energy-momentum relation derived in Eq. (21) to study the first, the second laws of thermodynamics and the weak cosmic censorship conjecture in the normal phase space of Kerr-AdS black hole in this section.

3.1. Laws of thermodynamics in the normal phase space

We assume that the absorbed fermion has energy ω and angular momentum j. As it drops into the black hole, the internal energy M and angular momentum J of the black hole would change according to

$$\omega = dM, \quad j = dJ. \tag{22}$$

Due to the first law of thermodynamics, Eq. (21) can be rewritten as

$$dM = j\Omega_h + \frac{\rho_h^2}{a^2 + r_h^2} p_h^r. \tag{23}$$

In the normal phase space, the initial state of the black hole is represented by (m, a, r_h), while the final state is represented by $(m + dm, a + da, r_h + dr_h)$. That is, the absorbed fermions will change the configurations of the black holes. However, the horizon is always determined by the function $\Delta(r)$, i.e., $\Delta(r_h + dr_h) = 0$ always holds. Thus, the change of the horizon should satisfy

$$d\Delta_h = \frac{\partial \Delta_h}{\partial m} dm + \frac{\partial \Delta_h}{\partial a} da + \frac{\partial \Delta_h}{\partial r_h} dr_h = 0. \tag{24}$$

In addition, by considering $M = \frac{m}{\Xi^2}$, Eq. (23) can be rewritten as

$$\frac{\partial}{\partial a}\frac{m}{\Xi^2} da + \frac{\partial}{\partial m}\frac{m}{\Xi^2} dm - \left(\frac{a\Xi}{a^2 + r_h^2} + \frac{a}{l^2}\right)\left(\frac{\partial}{\partial a}\frac{am}{\Xi^2} da + \frac{\partial}{\partial m}\frac{am}{\Xi^2} dm\right) - \frac{\rho_h^2}{a^2 + r_h^2} p_h^r = 0. \tag{25}$$

After solving Eq. (25), we reach

$$dm = \frac{al^2 m\left(l^2 - 3r_h^2\right) da}{l^2 r_h^2\left(l^2 - a^2\right)} + \frac{\rho_h^2\left[a^6 p_h^r + a^4 p_h^r\left(r_h^2 - 2l^2\right) + a^2 l^2 p_h^r\left(l^2 - 2r_h^2\right) + l^4 p_h^r r_h^2\right]}{\left(a^2 + r_h^2\right)\left[l^2 r_h^2\left(l^2 - a^2\right)\right]}. \tag{26}$$

Inserting Eq. (26) into Eq. (24), we arrive at

$$dr_h = \frac{al^2 da}{a^2 r_h - l^2 r_h} + \frac{ar_h da}{a^2 - l^2} + \frac{2p_h^r(a - l)(a + l)\left(a^2 + r_h^2\right)\rho_h^2}{\left(a^2 + r_h^2\right)\left[a^2 l^2 - r^2\left(a^2 + l^2\right) - 3r_h^4\right]}. \tag{27}$$

Based on the above formula, one can readily obtain the variation of entropy by making use of Eq. (7), which is

$$dS = \frac{\partial S}{\partial a} da + \frac{\partial S}{\partial r_h} dr_h = \frac{4\pi l^2 r_h\left(a^2 + r_h^2\right)\rho_h^2 p_h^r}{(a^2 + r_h^2)[r_h^2\left(a^2 + l^2\right) - a^2 l^2 + 3r_h^4]}. \tag{28}$$

With the help of Eqs. (7) and (28), we can get the following relation

$$T_h dS = \frac{\rho_h^2}{a^2 + r_h^2} p_h^r. \tag{29}$$

Therefore, the internal energy in Eq. (23) can be rewritten as

$$dM = T_h dS + \Omega_h dJ, \tag{30}$$

which is consistent with Eq. (8). Therefore, we see that as a spinning fermion drops into the Kerr-AdS black hole, the first law of thermodynamics is valid in the normal phase space.

For the extremal black holes, the inner horizon and outer horizon coincide and the temperature vanishes at the horizon. From Eq. (7), we can get the radius of the extremal black hole as

$$r_{extreme} = \frac{\sqrt{-a^2 + \sqrt{a^4 + 14a^2l^2 + l^4} - l^2}}{\sqrt{6}}. \tag{31}$$

The second law of thermodynamics for the extremal black holes is meaningless for the temperature vanishes. In fact, from Eq. (29), we know that the variation of the entropy is divergent in this case. For the non-extremal black hole, the temperature is larger than zero for $r > r_{extreme}$, thus dS in Eq. (28) is positive, implying that the second law of thermodynamics is also valid in the normal phase space for the non-extremal Kerr-AdS black holes.

3.2. Weak cosmic censorship conjecture in the normal phase space

The validity of the weak cosmic censorship can be tested by computing the minimum value of the function $\Delta(r)$ after the absorption of fermions. For the Kerr-AdS black holes, we label the radial position of the minimum of $\Delta(r)$ as r_l. At r_l, the following relations always satisfy

$$\Delta(r)|_{r=r_l} \equiv \Delta_l = \lambda \leq 0, \quad \partial_r \Delta(r)|_{r=r_l} \equiv \Delta_l' = 0. \tag{32}$$

For the extremal black holes, $\lambda = 0$, the horizon and the location of the minimum value are coincident. For the near-extremal black holes, λ is a small quantity, the location of the minimum value is located between the inner horizon and outer horizon. As a fermion with mass ω and angular momentum j drops into the black hole, the mass and angular momentum of the black hole will increase as $M + \omega$, $J + j$ respectively. Correspondingly, the location of the minimum value and event horizon change into $r_l + dr_l$ and $r_h + dr_h$, respectively. Thus, at $r_l + dr_l$, we have

$$\Delta(r_l + dr_l) = \Delta_l + d\Delta_l = \lambda + \left(\frac{\partial \Delta_l}{\partial m} dm + \frac{\partial \Delta_l}{\partial a} da\right), \tag{33}$$

where we have used Eq. (32). For the case of $\Delta(r_l + dr_l) > 0$, there is no horizon while for the case of $\Delta(r_l + dr_l) \leq 0$, there are always horizons. In the following we will focus on finding the final form of Eq. (33). For simplicity, we consider the extremal black holes, since the horizon is located at r_l and $\Delta_l = 0$. In this case, Eq. (26) can be adopted, thus we reach

$$\begin{aligned} d\Delta_l = & \frac{-2ada\left(l^2 r_l \left(a^2 - r_l(3m + r_l)\right) + a^2 r_l^3 + l^4(m - r_l)\right)}{l^2 r_l \left(l^2 - a^2\right)} \\ & - \frac{2\left(a^2 - l^2\right)^2 \left(a^2 + r_l^2\right)\left(a^2 \cos^2\theta + r_l^2\right)}{l^2 r_l \left(l^2 - a^2\right)\left(a^2 + r_l^2\right)}. \end{aligned} \tag{34}$$

In addition, from Eq. (2) the mass of the black hole can be written as

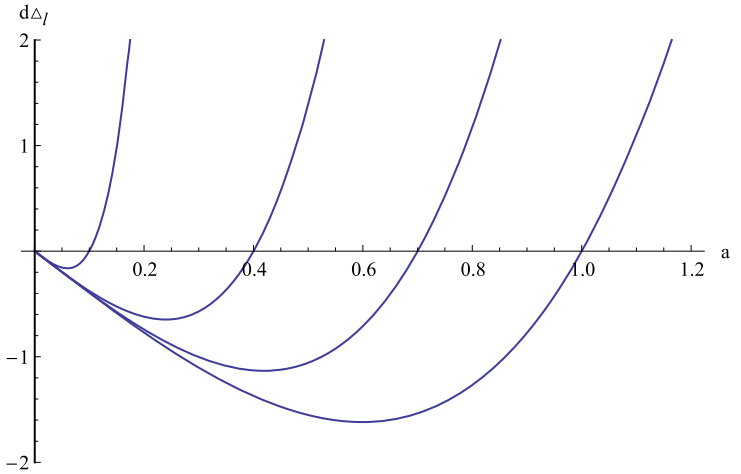

Fig. 1. The values of $d\Delta_l$ for the case of $p_h^r = 1, \theta = 0$. Curves from left to right correspond to l varying from 0.1 to 1 with step 0.3.

$$m = \frac{\left(a^2 + r_l^2\right)\left(l^2 + r_l^2\right)}{2l^2 r_l}. \tag{35}$$

Substituting Eqs. (35) and (31) into Eq. (34), we finally get

$$d\Delta_l = \frac{\sqrt{\frac{2}{3}}(a-l)(a+l)\left(a^2\left(6\cos^2\theta - 1\right) + \sqrt{a^4 + 14a^2l^2 + l^4} - l^2\right)}{l^2\sqrt{-a^2 + \sqrt{a^4 + 14a^2l^2 + l^4} - l^2}}. \tag{36}$$

Values of $d\Delta_l$ are shown in Fig. 1 for various l's. It is obvious that for a given value of l, the values of $d\Delta_l$ can be positive or negative. Take the example of $l = 0.7$, we see that for the case of $a < l$, $d\Delta_l$ is negative; while for the case of $a > l$, $d\Delta_l$ is positive. For other values of l's, we can observe similar behaviors of $d\Delta_l$. Therefore, we conclude that for the case of $a < l$, the weak cosmic censorship conjecture is valid; while for the case of $a > l$, the weak cosmic censorship conjecture is violated. Our results are consistent with those in Ref. [4], in which the authors found that the Kerr-AdS solution was valid only for $a < l$ since the singularity will be naked for $a \geq l$.[1]

4. Thermodynamics and weak cosmic censorship conjecture in the extended phase space

4.1. Laws of thermodynamics in the extended phase space

In the extended phase space, the mass is no longer related to the internal energy but rather the enthalpy of the thermodynamic system. The relation between the enthalpy M, internal energy U, pressure P and volume V are

$$M = U + PV. \tag{37}$$

[1] Please note that in higher dimensional Kerr-AdS solutions, $a > L$ is also possible. Please refer to [67].

As the fermions are absorbed by the black holes, the energy and angular momentum are supposed to be conserved. Thus, the energy and angular momentum of the spinning fermion equal to the varied energy and angular momentum of the black hole, which leads to

$$\omega = dU = d(M - PV), \quad j = dJ, \tag{38}$$

where we have used Eq. (22). In this case, Eq. (21) changes correspondingly as

$$d(M - PV) = j\Omega_h + \frac{\rho_h^2}{a^2 + r_h^2} p_h^r. \tag{39}$$

Considering $M = \frac{m}{\Xi^2}$, Eq. (39) can be rewritten as

$$\begin{aligned} &\frac{\partial}{\partial a}\frac{m}{\Xi^2}da + \frac{\partial}{\partial l}\frac{m}{\Xi^2}dl + \frac{\partial}{\partial m}\frac{m}{\Xi^2}dm - V\frac{\partial P}{\partial l}dl - PY - p_h^r X \\ &- \left(\frac{a\Xi}{a^2 + r_h^2} + \frac{a}{l^2}\right)\left(\frac{\partial}{\partial a}\frac{am}{\Xi^2}da + \frac{\partial}{\partial l}\frac{am}{\Xi^2}dl + \frac{\partial}{\partial m}\frac{am}{\Xi^2}dm\right) = 0, \end{aligned} \tag{40}$$

in which

$$Y \equiv dV = \frac{\partial V}{\partial a}da + \frac{\partial V}{\partial l}dl + \frac{\partial V}{\partial r_h}dr_h, \quad X = \frac{a^2\cos^2\theta + r_h^2}{a^2 + r_h^2}. \tag{41}$$

By solving Eq. (40), we obtain

$$dm = \frac{U_1 + U_2 + U_3 + U_4}{8\pi l^4 r_h^3 \left(l^2 - a^2\right)}, \tag{42}$$

where

$$\begin{aligned} U_1 &= 8\pi a l^4 m r_h \left(l^2 - 3r_h^2\right) da + l^3 r_h^3 \left(-8\pi r_h^3 dl + 8\pi l^3 p_h^r X + 3lY\right), \\ U_2 &= a^6 \left(4\pi l \left(r_h^2 - l^2\right) dl + r_h \left(8\pi l^2 p_h^r X + 3Y\right)\right), \\ U_3 &= a^2 l r_h \left(4\pi r_h^2 \left(l^2(8m - 5r) + r_h^3\right) dl + l\left(l^2 - 2r_h^2\right)\right), \\ U_4 &= a^4 r_h \left(r_h^2 - 2l^2\right)\left(8\pi l r_h dl + 8\pi l^2 p_h^r X + 3Y\right). \end{aligned}$$

In the extended phase space, the AdS radius is also a parameter of the black hole. Thus, the initial state should be characterized by (m, a, r_h, l). As a fermion drops into the black hole, the final configuration of the black hole becomes $(m + dm, a + da, r_h + dr_h, l + dl)$. At the initial state and final state, the horizons are determined by $\Delta(r_h) = 0$ and $\Delta(r_h + dr_h) = 0$ respectively, which implies

$$d\Delta_h = \frac{\partial \Delta_h}{\partial m}dm + \frac{\partial \Delta_h}{\partial a}da + \frac{\partial \Delta_h}{\partial r_h}dr_h + \frac{\partial \Delta_h}{\partial l}dl = 0. \tag{43}$$

Substituting Eq. (42) into Eq. (43), we can solve dr_h directly as

$$dr_h = \frac{V_1 + V_2 + V_3 + V_4}{4\pi l^2 r_h \left(l^2 - a^2\right)\left(a^2\left(l^2 - r_h^2\right) - r_h^2\left(l^2 + 3r_h^2\right)\right)}, \tag{44}$$

where

$$
\begin{aligned}
V_1 &= 4\pi a l^2 r_h^2 \left(l^4 + 4l^2 r_h^2 + 3r_h^4\right) da - l^4 r_h^3 \left(8\pi l^2 p_h^r X + 3Y\right), \\
V_2 &= a^6 \left(4\pi l \left(l^2 - r_h^2\right) dl - r_h \left(8\pi l^2 P_h^r X + 3Y\right)\right), \\
V_3 &= 4\pi a^3 l^2 \left(r_h^4 - l^4\right) da - a^2 l r_h \big(4\pi r_h^3 \left(l^2 + 3r_h^2\right) dl \\
&\quad + l \left(l^2 - 2r_h^2\right) \left(8\pi l^2 p_h^r X + 3Y\right)\big), \\
V_4 &= a^4 \left(\left(2l^2 r_h - r_h^3\right) \left(8\pi l^2 p_h^r X + 3Y\right) - 16\pi l r_h^4 dl\right).
\end{aligned} \tag{45}
$$

From Eq. (7), the variation of entropy can be written as

$$
dS = \frac{\partial S}{\partial a} da + \frac{\partial S}{\partial l} dl + \frac{\partial S}{\partial r_h} dr_h. \tag{46}
$$

With the help of Eqs. (7), (10), (41), and (50), we find that,

$$
T_h dS - PdV = \frac{\rho_h^2}{a^2 + r_h^2} p_h^r. \tag{47}
$$

Therefore, the internal energy in Eq. (39) can be rewritten as

$$
d(M - PV) = T_h dS + \Omega_h dJ - PdV. \tag{48}
$$

From Eq. (37), we can obtain the differential relation between the enthalpy and internal energy as $dM = dU + PdV + VdP$. Therefore, we reach

$$
dM = TdS + \Omega_h dJ + VdP, \tag{49}
$$

which is consistent with Eq. (9). Therefore, we conclude that as a fermion drops into the Kerr-AdS black hole, the first law of thermodynamics is also valid in the extended phase space.

In the normal phase space, we have known that both the first law and second law are valid as a fermion drops into the Kerr-AdS black holes. However, this is not always true for all the cases. The satisfaction of the first law of thermodynamics does not mean that the second law is also satisfied, especially in the extended phase space [55]. We will check the second law of Kerr-AdS black holes in the extended phase space in the following. By using the Eqs. (41) and (44), the variation of entropy thus can be rewritten as

$$
dS = \frac{E}{\left(a^2 - l^2\right)^2 \left(a^6 \left(r_h^2 + l^2\right) + 2a^4 r_h^4 - a^2 \left(2l^4 r_h^2 + 9l^2 r_h^4 + 3r_h^6\right) + 2l^4 r_h^4\right)}, \tag{50}
$$

in which

$$
\begin{aligned}
E = {} & 2\pi l\Big(-a^{10} \left(r_h^2 + l^2\right) dl - 4a^2 l p_h^r \left(a^2 - l^2\right)^3 \cos^2(\theta) r_h^3 + a^3 l r_h^4 \left(r_h^2 + l^2\right)^2 da \\
& + 4l^7 p_h^r r_h^5\Big) + 2\pi l\Big(a^6 \left(-r_h^4\right) \left(3\left(r_h^2 + l^2\right) dl + 4l p_h^r r_h\right) \\
& - a^4 r_h^5 \left(l^2 r_h dl + r_h^3 dl - 12l^3 p_h^r\right)\Big) + 2\pi l\Big(-3a^8 r_h^2 \left(r_h^2 + l^2\right) dl \\
& + a^7 l \left(r_h^2 + l^2\right)^2 da + 2a^5 l r_h^2 \left(r_h^2 + l^2\right)^2 da - 12a^2 l^5 p_h^r r_h^5\Big).
\end{aligned} \tag{51}
$$

From Eq. (50), we see that the variation of the entropy depends on a, l, da, dl while fixing r_h. Because we are interested in the second laws of black holes, thus $a < l$ should be satisfied, otherwise the singularity will be naked and the thermodynamic system will break down.

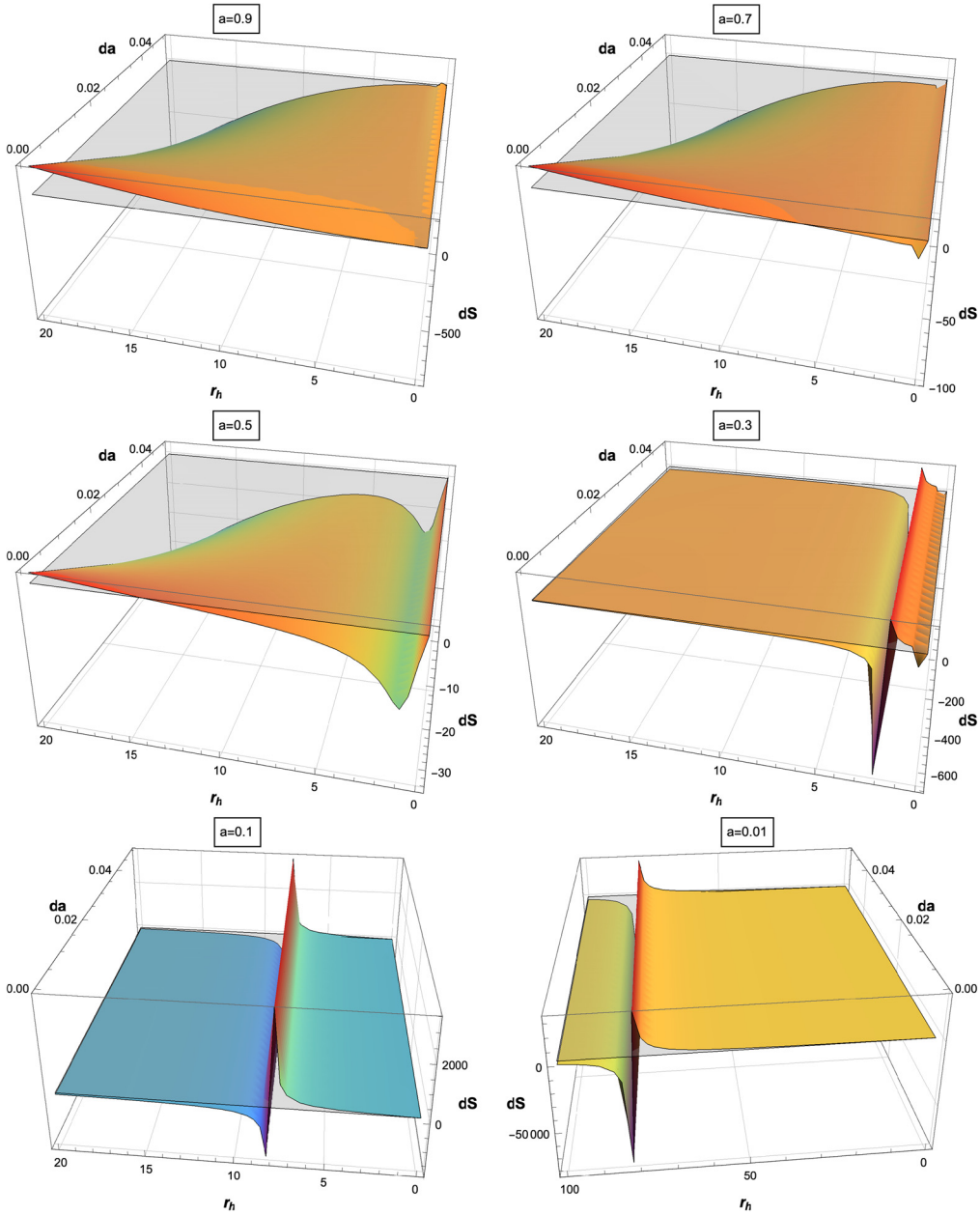

Fig. 2. Three dimensional visualizations of dS vs. da and r_h for various fixed a's. The horizontal gray plane is at $dS \equiv 0$. We can readily see that $dS > 0$ and $dS < 0$ can all exist in different parameter regimes.

Since Eq. (50) consists of a bunch of parameters, for simplicity and without loss of generality, we will impose $p_h^r = 1, \theta = 0, dl = 0.01, l = 1$ in this subsection in order to see the effects of da, a and r_h to the variation of the entropy. Of course one can also set other parameters, but the general conclusions would not be changed. In Fig. 2 we show the 3-dimensional plots for the

dS in the coordinates of da and r_h while fixing various a's. We see that dS will be positive or negative for different parameter regimes. For instance of $a \equiv 0.7$ and $r_h = 20$ (the right panel of top row of Fig. 2, there exists a critical value of da such that $dS > 0$ if $da < 0.0067$ and $dS > 0$ if $da > 0.0067$. In fact, the critical values of da and r_h that will render $dS = 0$ can be easily obtained from Eq. (50) as,

$$da = \frac{a^8(r_h^2+1) + 2a^6 r_h^2(r_h^2+200r_h+1) + a^4 r_h^3(r_h^3+r_h-1200) + 1200a^2 r_h^3 - 400r_h^3}{100a^3(r_h^2+1)^2(a^2+r_h^2)}. \tag{52}$$

As a is small, for example $a \equiv 0.1$ (left panel of bottom row of Fig. 2), $dS > 0$ if $r_h < 8$ whatever da is. This means the second law of thermodynamics will always be satisfied if the black hole is near-extremal (with smaller r_h) independent of da. In the opposite limit, i.e., $r_h > 8$, the second law of thermodynamics will be violated. Therefore, in the extended phase space, the second law of thermodynamics may be valid or violated, depending on the values of parameters a, da and r_h (given that we have fixed the values of p_h^r, θ, dl and l), which is different from that in the normal phase space.

4.2. Weak cosmic censorship conjecture in the extended phase space

Like the case in the normal phase space, we are going to discuss the minimal value of the function $\Delta(r)$, since a non-positive minimum value implies the existence of horizons while a positive one does not. Different from that in the normal phase space, l now is a variable and the initial state is represented by (m, a, l). As a fermion drops into the black hole, the final state will change into $(m+dm, a+da, l+dl)$. Correspondingly, there are also shifts for the locations of the minimum value and event horizon, $r_l \to r_l + dr_l$, $r_h \to r_h + dr_h$. Thus, the shift for $\Delta(r)$ is

$$d\Delta_l = \frac{\partial \Delta_l}{\partial m}dm + \frac{\partial \Delta_l}{\partial a}da + \frac{\partial \Delta_l}{\partial l}dl = -\frac{2r_m^2\left(a^2+r_m^2\right)dl}{l^3} + 2a\left(\frac{r_m^2}{l^2}+1\right)da - 2r_m dm, \tag{53}$$

where we have used $\Delta_l' = 0$ in Eq. (32). For the extremal black holes, we have $r_h = r_l$ so that $\Delta_l = 0$. In this case, Eq. (40) can be used. Substituting m in Eq. (35) and p_h^r in Eq. (47) into Eq. (53), we obtain

$$dm = \frac{a^2\left(-2r_m^3 dl - l^3 dr_m + lr_m^2\right)dr_m}{2l^3 r_m^2} + \frac{a\left(l^2+r_m^2\right)da}{l^2 r_m} + \frac{lr_m^2\left(l^2+3r_m^2\right)dr_m - 2r_m^5 dl}{2l^3 r_m^2}. \tag{54}$$

Inserting Eq. (54) into Eq. (53), we get

$$d\Delta_l = a^2\left(\frac{1}{r_m} - \frac{r_m}{l^2}\right)dr_m + r_m\left(-\frac{3r_m^2}{l^2} - 1\right)dr_m. \tag{55}$$

For the extremal black holes, Eq. (31) is applicable. Thus, Eq. (55) can be simplified further to be $d\Delta_l = 0$. Thereofre, we see that there is no shift in Δ_l for the extremal black holes. The configuration of extremal black holes does not change, thus the extremal black holes are still extremal black holes as fermions are absorbed.

For the near-extremal black holes, Eq. (31) is not applicable at r_m since it is valid only at r_h. However, we can expand it near r_m. Replacing r_h in Eq. (54) with $r_m + \epsilon$ and expanding it to the first order of ϵ, we reach

$$dM = A + B\epsilon + O(\epsilon)^2, \tag{56}$$

in which

$$A = \frac{a^2 dr_m}{2l^2} - \frac{a^2 dr_m}{2r_m^2} + \frac{ada}{r_m} + \frac{dr_m}{2} - \frac{a^2 r_m dl}{l^3} + \frac{ar_m da}{l^2} - \frac{r_m^3 dl}{l^3} + \frac{3r_m^2 dr_m}{2l^2},$$
$$B = -\frac{a^2 dl}{l^3} + \frac{a^2 dr_m}{r_m^3} + \frac{ada}{l^2} - \frac{ada}{r_m^2} - \frac{3r_m^2 dl}{l^3} + \frac{3r_m}{l^2} dr_m.$$

Substituting Eq. (56) into Eq. (53), we obtain

$$d\Delta_l = a^2\left(\frac{1}{r_m} - \frac{r_m}{l^2}\right) dr_m + r_m\left(-\frac{3r_m^2}{l^2} - 1\right) dr_m + \frac{2\left(a^2\left(r_m^3 dl - l^3\right) dr_m + alr_m\left(l^2 - r_m^2\right) da + 3r_m^4\left(r_m dl - ldr\right)\right)}{l^3 r_m^2}\epsilon + O(\epsilon)^2. \tag{57}$$

Since at the horizon $\Delta(r_h) = 0$, one can replace r_h with $r_m + \epsilon$ and expand $\Delta(r_h)$ to the order of ϵ,

$$\Delta(r_h) = a^2\left(1 - \frac{r_m^2}{l^2}\right) - \frac{3r_m^4}{l^2} - r_m^2 = 0, \tag{58}$$

where we have employed Eq. (35). Solving Eq. (58), we obtain

$$l = \frac{r_m\sqrt{a^2 + 3r_m^2}}{\sqrt{a^2 - r_m^2}}. \tag{59}$$

Differentiating both sides of Eq. (59), we can further get

$$dl = \frac{a^4 dr_m + 6a^2 r_m^2 dr_m - 4ar_m^3 da - 3r_m^4 dr_m}{\left(a^2 - r_m^2\right)^{3/2}\sqrt{a^2 + 3r_m^2}}. \tag{60}$$

Substituting Eqs. (59) and (60) into Eq. (57), we finally arrive at $d\Delta_l = O(\epsilon)^2$. Since $O(\epsilon)^2$ is the higher order of the small quantity ϵ, thus it can be omitted. In this case, we see that the final states of the near-extremal black holes are still the near-extremal black holes, which is similar to that of the extremal black holes. Therefore, we can conclude that the weak cosmic censorship conjecture still holds for both the extremal and near-extremal black holes in the extended phase space.

5. Conclusions

We investigated the laws of thermodynamics and weak cosmic censorship conjecture by considering a spinning fermion absorbed by the Kerr-AdS black hole. The dynamics of the fermion was investigated by the Dirac equation in the dragging coordinate system. As a result, we obtained the energy-momentum relation of the spinning fermion near the event horizon, which was the basic of our investigation.

In the normal phase space, we derived firstly the first law of thermodynamics by considering the energy-momentum relation as well as the energy and angular momentum conservation. Then we investigated the second law of thermodynamics by discussing the variation of the entropy. As predicted, it was found that the variation of the entropy is always positive. The validity of the weak cosmic censorship conjecture was also checked in the normal phase space by studying the variation of the function $\Delta(r)$ at the minimal point. For the case that the initial state was a black hole, the final state was found to be a black hole as well, thus the weak cosmic censorship conjecture was valid as well in the Kerr-AdS black hole.

Later, we investigated the laws of thermodynamics and weak cosmic censorship conjecture in the extended phase space, which has not been reported previously as far as we know. In the extended phase space, the cosmological parameter is not a constant any more, however, the pressure of the thermodynamic system is considered as a constant. The first law thus contains the contribution from pressure and volume. From the energy-momentum relation as well as the energy and angular momentum conservation, we also derived the first law successfully. The second law, however, is more subtle since we found that it depends on the values of the spin parameter, AdS radius as well as their variations. As the value of the AdS radius is fixed, we found that the variation of entropy is different for different spin parameters. And for a fixed spin parameter, the variation of the spin parameter also affects the variation of entropy, which can be positive or negative. Therefore, the second law in the extended phase space may be valid or violated.[2] We also checked the validity of the weak cosmic censorship conjecture in the extended phase space by studying the variation of the function $\Delta(r)$ at the minimal point. For the initial states were extremal and near-extremal black holes, the final states were found to be extremal and near-extremal black holes as well. The absorbed fermion thus would not change the location of the minimum, thus the weak cosmic censorship conjecture is still valid in the extended phase space.

CRediT authorship contribution statement

The authors contributed equally to this work.

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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References

[1] S.W. Hawking, Particle creation by black holes, Commun. Math. Phys. 43 (1975) 199.

[2] In the paper [68], the authors considered that the adding particles into the black hole would change the enthalpy, rather than the internal energy as we considered in our paper. This starting difference lead to their conclusion that the second law in the extended phase space would not be violated.

[2] S.W. Hawking, Black holes and thermodynamics, Phys. Rev. D 13 (1976) 191.
[3] M. Henneaux, C. Teitelboim, Asymptotically anti-se Sitter spaces, Commun. Math. Phys. 98 (1985) 391.
[4] M.M. Caldarelli, G. Cognola, D. Klemm, Thermodynamics of Kerr-Newman-AdS black holes and conformal field theories, Class. Quantum Gravity 17 (2000) 399.
[5] B.P. Dolan, The cosmological constant and the black hole equation of state, Class. Quantum Gravity 28 (2011) 125020, arXiv:1008.5023 [gr-qc].
[6] M. Cvetic, G.W. Gibbons, D. Kubiznak, C.N. Pope, Black hole enthalpy and an entropy inequality for the thermodynamic volume, Phys. Rev. D 84 (2011) 024037, arXiv:1012.2888 [hep-th].
[7] D. Kastor, S. Ray, J. Traschen, Enthalpy and the mechanics of AdS black holes, Class. Quantum Gravity 26 (2009) 195011, arXiv:0904.2765 [hep-th].
[8] K. Bhattacharya, B.R. Majhi, S. Samanta, Van der Waals criticality in AdS black holes: a phenomenological study, Phys. Rev. D 96 (2017) 084037.
[9] D.C. Zou, Y. Liu, R.H. Yue, Behavior of quasinormal modes and van der Waals-like phase transition of charged AdS black holes in massive gravity, Eur. Phys. J. C 77 (2017) 365.
[10] X.X. Zeng, L.F. Li, Van der Waals phase transition in the framework of holography, Phys. Lett. B 764 (2017) 100.
[11] J. Xu, L.M. Cao, Y.P. Hu, P-V criticality in the extended phase space of black holes in massive gravity, Phys. Rev. D 91 (12) (2015) 124033.
[12] R.G. Cai, L.M. Cao, L. Li, R.Q. Yang, P-V criticality in the extended phase space of Gauss-Bonnet black holes in AdS space, J. High Energy Phys. 1309 (2013) 005.
[13] Y.P. Hu, H.A. Zeng, Z.M. Jiang, H. Zhang, P-V criticality in extended phase space of black holes in Einstein-Horndeski gravity, arXiv:1812.09938 [gr-qc].
[14] A. Dehyadegari, A. Sheykhi, Reentrant phase transition of Born-Infeld-AdS black holes, Phys. Rev. D 98 (2018) 024011.
[15] M. Zhang, D.C. Zou, R.H. Yue, Reentrant phase transitions and triple points of topological AdS black holes in Born-Infeld-massive gravity, Adv. High Energy Phys. 2017 (2017) 3819246.
[16] D. Kubiznak, R.B. Mann, M. Teo, Black hole chemistry: thermodynamics with Lambda, Class. Quantum Gravity 34 (2017) 063001.
[17] N. Altamirano, D. Kubiznak, R.B. Mann, Reentrant phase transitions in rotating anti-de Sitter black holes, Phys. Rev. D 88 (2013) 101502.
[18] J.M. Maldacena, The large N limit of superconformal field theories and supergravity, AIP Conf. Proc. 484 (1999) 51.
[19] S.S. Gubser, I.R. Klebanov, A.M. Polyakov, Gauge theory correlators from noncritical string theory, Phys. Lett. B 428 (1998) 105.
[20] E. Witten, Anti-de Sitter space and holography, Adv. Theor. Math. Phys. 2 (1998) 253.
[21] O. Aharony, S.S. Gubser, J.M. Maldacena, H. Ooguri, Y. Oz, Large N field theories, string theory and gravity, Phys. Rep. 323 (2000) 183.
[22] S.A. Hartnoll, C.P. Herzog, G.T. Horowitz, Building a holographic superconductor, Phys. Rev. Lett. 101 (2008) 031601.
[23] R.G. Cai, L. Li, L.F. Li, R.Q. Yang, Introduction to holographic superconductor models, Sci. China, Phys. Mech. Astron. 58 (6) (2015) 060401, arXiv:1502.00437 [hep-th].
[24] R.G. Cai, S. He, L. Li, Y.L. Zhang, Holographic entanglement entropy in insulator/superconductor transition, J. High Energy Phys. 1207 (2012) 088, arXiv:1203.6620 [hep-th].
[25] R.G. Cai, S. He, L. Li, Y.L. Zhang, Holographic entanglement entropy on P-wave superconductor phase transition, J. High Energy Phys. 1207 (2012) 027, arXiv:1204.5962 [hep-th].
[26] V. Balasubramanian, et al., Thermalization of strongly coupled field theories, Phys. Rev. Lett. 106 (2011) 191601, arXiv:1012.4753 [hep-th].
[27] X.X. Zeng, W.B. Liu, Holographic thermalization in Gauss-Bonnet gravity, Phys. Lett. B 726 (2013) 481, arXiv:1305.4841 [gr-qc].
[28] X.X. Zeng, X.M. Liu, W.B. Liu, Holographic thermalization with a chemical potential in Gauss-Bonnet gravity, J. High Energy Phys. 1403 (2014) 031, arXiv:1311.0718 [gr-qc].
[29] X.X. Zeng, X.M. Liu, W.B. Liu, Holographic thermalization in noncommutative geometry, Phys. Lett. B 744 (2015) 48, arXiv:1407.5262 [hep-th].
[30] S.S. Lee, A non-Fermi liquid from a charged black hole: a critical Fermi ball, Phys. Rev. D 79 (2009) 086006, arXiv:0809.3402 [hep-th].
[31] H. Liu, J. McGreevy, D. Vegh, Non-Fermi liquids from holography, Phys. Rev. D 83 (2011) 065029, arXiv:0903.2477 [hep-th].

[32] M. Cubrovic, J. Zaanen, K. Schalm, String theory, quantum phase transitions and the emergent Fermi-liquid, Science 325 (2009) 439, arXiv:0904.1993 [hep-th].
[33] R. Penrose, Gravitational collapse: the role of general relativity, Riv. Nuovo Cimento 1 (1969) 252.
[34] R. Wald, Gedanken experiments to destroy a black hole, Ann. Phys. 82 (1974) 548.
[35] V.E. Hubeny, Overcharging a black hole and cosmic censorship, Phys. Rev. D 59 (1999) 064013, arXiv:gr-qc/9808043.
[36] T. Jacobson, T.P. Sotiriou, Over-spinning a black hole with a test body, Phys. Rev. Lett. 103 (2009) 141101, arXiv:0907.4146 [gr-qc].
[37] E. Barausse, V. Cardoso, G. Khanna, Test bodies and naked singularities: is the self-force the cosmic censor?, Phys. Rev. Lett. 105 (2010) 261102, arXiv:1008.5159 [gr-qc].
[38] M. Colleoni, L. Barack, Overspinning a Kerr black hole: the effect of self-force, Phys. Rev. D 91 (2015) 104024, arXiv:1501.07330 [gr-qc].
[39] M. Bouhmadi-Lopez, V. Cardoso, A. Nerozzi, J.V. Rocha, Black holes die hard: can one spin-up a black hole past extremality?, Phys. Rev. D 81 (2010) 084051.
[40] S. Hod, A note on black-hole physics, cosmic censorship, and the charge-mass relation of atomic nuclei, Class. Quantum Gravity 33 (2016) 037001.
[41] J. Natario, L. Queimada, R. Vicente, Test fields cannot destroy extremal black holes, Class. Quantum Gravity 33 (2016) 175002, arXiv:1601.06809 [gr-qc].
[42] G.T. Horowitz, J.E. Santos, B. Way, Evidence for an electrifying violation of cosmic censorship, Class. Quantum Gravity 33 (2016) 195007, arXiv:1604.06465 [hep-th].
[43] K. Düztaş, Overspinning BTZ black holes with test particles and fields, Phys. Rev. D 94 (2016) 124031, arXiv:1701.07241 [gr-qc].
[44] B. Gwak, Stability of horizon in warped AdS black hole via particle absorption, arXiv:1707.09128 [gr-qc].
[45] S. Gao, Y. Zhang, Destroying extremal Kerr-Newman black holes with test particles, Phys. Rev. D 87 (2013) 044028.
[46] J.V. Rocha, R. Santarelli, Flowing along the edge: spinning up black holes in AdS spacetimes with test particles, Phys. Rev. D 89 (2014) 064065.
[47] J.V. Rocha, V. Cardoso, Gravitational perturbation of the BTZ black hole induced by test particles and weak cosmic censorship in AdS spacetime, Phys. Rev. D 83 (2011) 104037.
[48] D. Chen, Weak cosmic censorship conjecture in BTZ black holes with scalar fields, arXiv:1812.03459 [gr-qc].
[49] R.M. Wald, Kerr-Newman black holes cannot be over-charged or over-spun, Int. J. Mod. Phys. D 27 (2018) 1843003.
[50] J. Sorce, R.M. Wald, Gedanken experiments to destroy a black hole. II. Kerr-Newman black holes cannot be over-charged or overspun, Phys. Rev. D 96 (2017) 104014, arXiv:1707.05862 [gr-qc].
[51] B. Ge, Y. Mo, S. Zhao, J. Zheng, Higher-dimensional charged black holes cannot be over-charged by gedanken experiments, Phys. Lett. B 783 (2018) 440, arXiv:1712.07342 [hep-th].
[52] J. An, J. Shan, H. Zhang, S. Zhao, Five-dimensional Myers-Perry black holes cannot be overspun in gedanken experiments, Phys. Rev. D 97 (10) (2018) 104007, arXiv:1711.04310 [hep-th].
[53] T. Crisford, J.E. Santos, Violating the weak cosmic censorship conjecture in four-dimensional anti-de Sitter space, Phys. Rev. Lett. 118 (2017) 181101, arXiv:1702.05490 [hep-th].
[54] B. Gwak, Cosmic censorship conjecture in Kerr-Sen black hole, Phys. Rev. D 95 (2017) 124050, arXiv:1611.09640 [gr-qc].
[55] B. Gwak, Thermodynamics with pressure and volume under charged particle absorption, J. High Energy Phys. 1711 (2017) 129, arXiv:1709.08665 [gr-qc].
[56] X.X. Zeng, H.Q. Zhang, Thermodynamics and weak cosmic censorship conjecture in Born-Infeld-anti-de Sitter black holes, arXiv:1901.04247 [hep-th].
[57] Y.W. Han, X.X. Zeng, Y. Hong, Thermodynamics and weak cosmic censorship conjecture of the torus-like black hole, Eur. Phys. J. C 79 (2019) 252.
[58] X.X. Zeng, Y.W. Han, D.Y. Chen, Thermodynamics and weak cosmic censorship conjecture of the BTZ black holes in the extended phase space, arXiv:1901.08915 [gr-qc].
[59] D. Chen, Thermodynamics and weak cosmic censorship conjecture in extended phase spaces of anti-de Sitter black holes with fermions' absorption, arXiv:1902.06489 [hep-th].
[60] Y.W. Han, M.J. Lan, X.X. Zeng, Thermodynamics and weak cosmic censorship conjecture in (2+1)-dimensional regular black hole with nonlinear electrodynamics sources, arXiv:1903.03764 [gr-qc].
[61] B. Gwak, Thermodynamics and cosmic censorship conjecture in Kerr-Newman-de Sitter black hole, Entropy 20 (11) (2018) 855.
[62] B. Gwak, B.H. Lee, Cosmic censorship of rotating anti-de Sitter black hole, J. Cosmol. Astropart. Phys. 1602 (2016) 015, arXiv:1509.06691 [gr-qc].

[63] B. Carter, Hamilton-Jacobi and Schrodinger separable solutions of Einstein's equations, Commun. Math. Phys. 10 (1968) 280.
[64] N. Altamirano, D. Kubizňák, R.B. Mann, Z. Sherkatghanad, Kerr-AdS analogue of triple point and solid/liquid/gas phase transition, Class. Quantum Gravity 31 (2014) 042001.
[65] P. Cheng, S.W. Wei, Y.X. Liu, Critical phenomena in the extended phase space of Kerr-Newman-AdS black holes, Phys. Rev. D 94 (2016) 024025, arXiv:1603.08694 [gr-qc].
[66] D.Y. Chen, Q.Q. Jiang, X.T. Zu, Hawking radiation of Dirac particles via tunnelling from rotating black holes in de Sitter spaces, Phys. Lett. B 665 (2008) 106, arXiv:0804.0131 [hep-th].
[67] B. Mcinnes, Cosmic censorship for AdS_5-Kerr, Nucl. Phys. B 950 (2020) 114845, https://doi.org/10.1016/j.nuclphysb.2019.114845, arXiv:1906.01169 [hep-th].
[68] S.Q. Hu, Y.C. Ong, D.N. Page, No evidence for violation of the second law in extended black hole thermodynamics, Phys. Rev. D 100 (10) (2019) 104022, https://doi.org/10.1103/PhysRevD.100.104022, arXiv:1906.05870 [gr-qc].