Spacetime Quantum Fluctuations, Minimal Length and Einstein Equations

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Abstract

In the process of work it has been found that space-time quantum fluctuations are naturally described in terms of the deformation parameter introduced on going from the well-known quantum mechanics to that at Planck's scales and put forward in the previous works of the author. As shown, with the use of quite natural assumptions, these fluctuations must be allowed for in Einstein Equations to lead to the dependence of the latter on the above-mentioned parameter, that is insignificant and may be ignored at low energies but is of particular importance at high energies. Besides, some inferences form the obtained results are maid.

1 Introduction

The notion "space-time foam", introduced by J. A. Wheeler about 60 years ago for the description and investigation of physics at Planck's scales (Early Universe) [1],[2], is fairly settled. Despite the fact that in the last decade numerous works have been devoted to physics at Planck's scales within the scope of this notion, for example [3]–[22], by this time still their no clear understanding of the "space-time foam" as it is.
On the other hand, it is undoubtful that a quantum theory of the Early Universe should be a deformation of the well-known quantum theory.
The deformation is understood as an extension of a particular

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theory by inclusion of one or several additional parameters in such a way that the initial theory appears in the limiting transition [23].
In his works with the colleagues [24]–[32] the author has put forward one of the possible approaches to resolution of a quantum theory at Planck's scales on the basis of the density matrix deformation. This work demonstrates that space-time quantum fluctuations, in essence generating the space-time foam, may be naturally described in terms of the deformation parameter α_l introduced in [24]–[32], where l – measuring scale. Further it is shown that, with the use of quite natural assumptions, these fluctuations must be allowed for in Einstein Equations [33] to result in their dependence on the parameter α_l, insignificant and negligible at low energies (i.e. in the limit $l \to \infty$) but important at Planck's scales $l \to\propto l_P$.
Actually it is revealed that, if the metrics $g_{\mu\nu}$ is measured at some fixed energy scale $E \sim 1/l$ (as is always the case in real physics), Einstein Equations are α_l–deformed, and the known Einstein Equations [33] appear in the low-energy limit. However, this aspect may be ignored in all the known cases and the corresponding energy ranges because the scale l is very distant from l_P. Two clear illustrations of the high-energy α_l–deformation of Einstein Equations are given.
Some inferences from the results obtained are considered, in particular for the cosmological term Λ.
This work is a natural continuation of the paper [50]. In [50] it has been shown that in particular cases the General Relativity Einstein Equations may be written in the α_l–representation, i.e. they are dependent on the parameter α_l. Also, it has been demonstrated that for the indicated cases one can derive the high-energy (Planck) α_l - deformation of Einstein Equations. Then the question arises whether Einstein Equations are dependent on α_l in the most general case.
Proceeding from the present work, this question may be answered positively.

2 Quantum Fluctuations of Space-time and High-Energy Deformation

In accordance with the modern concepts, the space-time foam [2] notion forms the basis for space-time at Planck's scales (Big Bang). This object

is associated with the quantum fluctuations generated by uncertainties in measurements of the fundamental quantities, inducing uncertainties in any distance measurement. A precise description of the space-time foam is still lacking along with an adequate quantum gravity theory. But for the description of quantum fluctuations we have a number of interesting methods (for example, [34],[12]-[22]).
In what follows, we use the terms and symbols from [14]. Then for the fluctuations $\widetilde{\delta l}$ of the distance l we have the following estimate:

$$\widetilde{\delta l} \gtrsim l_P^{\gamma} l^{1-\gamma}, \tag{1}$$

where $0 \leq \gamma \leq 1$ and $l_P = (\hbar G/c^3)^{1/2}$ is the Planck length.
At the present time three principal models associated with different values of the parameter γ are considered:

A) $\gamma = 1$ that conforms to the initial (canonical) model from [2]

$$\widetilde{\delta l} \gtrsim l_P; \tag{2}$$

B) $\gamma = 2/3$ that conforms to the model [34],[14] compatible with the holographic principle [35]–[39]

$$\widetilde{\delta l} \gtrsim (l l_P^2)^{1/3} = l_P \left(\frac{l}{l_P}\right)^{1/3}; \tag{3}$$

C) $\gamma = 1/2$ - random-walk model [21] [22]

$$\widetilde{\delta l} \gtrsim (l l_P)^{1/2} = l_P \left(\frac{l}{l_P}\right)^{1/2}. \tag{4}$$

But, because of the experimental data obtained with the help of the Hubble Space Telescope [40], a random-walk model C) may be excluded from consideration (for example, see [19]) and is omitted in this work.

Moreover, in fact it is clear that **at Planck's scales, i.e. for**

$$l \rightarrow\propto l_P, \tag{5}$$

models A) are B) are coincident.
Using(2)–(4), we can derive the quantum fluctuations for all the primary

space-time characteristics, specifically for the time $\widetilde{\delta} t$, energy $\widetilde{\delta} E$, and metrics

$\widetilde{\delta} g_{\mu\nu}$ (formula (10) of [14]):

$$\widetilde{\delta} g_{\mu\nu} \gtrsim (l_P/l)^{\gamma}. \qquad (6)$$

It is obvious that all of them are dependent on one and the same dimensionless parameter l_P/l and on the Planck length l_P, i.e. on the fundamental constants.

Note also that in fact this parameter is introduced as a deformation parameter on going from the well-known quantum mechanics (QM) to a quantum mechanics with the fundamental length (QMFL), provided this length l_{min} is on the order of Planck's length $l_{min} \propto l_P$, as revealed by the author in the works written with his colleagues [24] –[32]. Let us recollect in short the central idea of the above-mentioned works (pp. 1267,1268 in [25]).

The main object under consideration in this case is the density matrix ρ. We assume that in QMFL the measuring procedure adopted in QM is valid being defined by ρ. Then

$$Sp[(\rho \widehat{X}^2) - Sp^2(\rho \widehat{X})] \geq l_{min}^2 > 0, \qquad (7)$$

where $\widehat{X}$ is the coordinate operator. Expression (7) gives the measuring rule used in QM. However, in the case considered here, in comparison with QM, the right part of (7) cannot be done arbitrarily near to zero since it is limited by $l_{min}^2 > 0$. A natural way for studying QMFL is to consider this theory as a deformation of QM, turning to QM at the low energy limit (during the expansion of the Universe after the Big Bang).

We will consider precisely this option. Here the following question may be formulated: how should be deformed density matrix conserving quantum-mechanical measuring rules in order to obtain self-consistent measuring procedure in QMFL? For answering to the question we will use the R-procedure. For starting let us to consider R-procedure both at the Planck's energy scale and at the low-energy one. At the minimal length scale $l \approx i l_{min}$ where i is a small quantity. Further l tends to infinity and we obtain for density matrix [24]-[32]:

$$Sp[\rho l^2] - Sp[\rho l] Sp[\rho l] \simeq l_{min}^2 \quad or \quad Sp[\rho] - Sp^2[\rho] \simeq l_{min}^2/l^2. \qquad (8)$$

Therefore:

1. When $l < \infty$, $Sp[\rho] = Sp[\rho(l)]$ and $Sp[\rho] - Sp^2[\rho] > 0$. Then, $Sp[\rho] < 1$ that corresponds to the QMFL case.

2. When $l = \infty$, $Sp[\rho]$ does not depend on l and $Sp[\rho] - Sp^2[\rho] \rightarrow 0$. Then, $Sp[\rho] = 1$ that corresponds to the QM case.

The above deformation parameter is as follows:

$$\alpha_l = l_{min}^2/l^2. \tag{9}$$

This parameter is variable within the interval

$$0 < \alpha_l \leq 1/4, \tag{10}$$

whereas the density matrix in QMFL becomes deformed and dependent on α_l: $\rho = \rho(\alpha_l)$, and we get

$$\lim_{\alpha_l \rightarrow 0} \rho(\alpha_l) \rightarrow \rho, \tag{11}$$

where ρ – known density matrix from QM.
When $l_{min} \propto l_P$, it is cleat that $\alpha_l \propto l_P^2/l^2$ and all the fluctuations above $\widetilde{\delta} l, \widetilde{\delta} g_{\mu\nu}, \widetilde{\delta} t, \widetilde{\delta} E$ may be expressed in terms of the deformation parameter α_l. For example, this is the case when the Generalized Uncertainty Principle (GUP) [41]–[48] is valid

$$\Delta x \geq \frac{\hbar}{\Delta p} + \ell^2 \frac{\Delta p}{\hbar}, \ell^2 = \lambda l_P^2, \tag{12}$$

and λ is the model-depended dimensionless numerical factor.
Then, as seen in (12), we have a minimal length on the order of the Planck length

$$l_{min} = 2\sqrt{\lambda} l_P. \tag{13}$$

Therefore, we obtain

$$(\frac{l_P}{l})^2 = \frac{1}{4\lambda}\alpha_l \tag{14}$$

and the factor $\frac{1}{4\lambda}$ is introduced into all of the formula (2)–(8) as soon as the fundamental quantities involved are expressed in terms of α_l. Specifically, the most important formula (6) in this case is of the form

$$\widetilde{\delta} g_{\mu\nu} \gtrsim (4\lambda)^{-\gamma/2} \alpha_l^{\gamma/2}. \tag{15}$$

In what follows we assume that a minimal length in a theory - l_{min} is existent no matter how it is introduced, from GUP (12) or in some other way. Then the parameter α_l (9) is quite naturally brought about from (7), (8).
With the use of this "coordinate system" the above-mentioned models A) and B) of the space-time quantum fluctuations may be "unified" as follows:

I. The minimal length l_{min} , similar to cases A) and B), is introduced at Planck's level

$$l_{min} \propto l_P.$$

II. In both cases fluctuations of the fundamental quantities may be expressed in terms of the parameter α_l.

III. The principal difference between A) and B) resides in the fact that in the second case a minimal fluctuation of the length is dependent on the measuring scale l, $(\widetilde{\delta}^{min} l) = (\widetilde{\delta}^{min} l)[l]$, whereas in the first case it is completely determined by the minimal length $\widetilde{\delta}^{min} \approx l_{min}$, being absolute in its character.

IV. As noted above, in the high-energy limit, i.e. for

$$l \rightarrow l_{min}, \tag{16}$$

both models are coincident.

3 Quantum Fluctuations and Einstein Equations

Thus, from the preceding section it follows that in any case we have minimal fluctuations $\widetilde{\delta}^{min}$ (dependent on the measuring scale l or on the energy $E \sim 1/l$) for all the fundamental physical quantities $l, t, E, g_{\mu\nu}, ...$, expressed in terms of the parameter α_l. Specifically, we have

$$(\widetilde{\delta}^{min} g_{\mu\nu})[l] = (\widetilde{\delta}^{min} g_{\mu\nu})[\alpha_l] \propto \alpha_l^{\gamma/2}. \tag{17}$$

Next we make the only natural assumption
if the metric $g_{\mu\nu}$ in General Relativity (GR) is measured at the scale l or, that is the same, on the scale of the energies $E \sim 1/l$, variation of the metric $\delta g_{\mu\nu}$ is governed by its fluctuation $(\widetilde{\delta} g_{\mu\nu})[l]$

and hence it is dependent on l or, actually, on α_l

$$\delta g_{\mu\nu} = (\delta g_{\mu\nu})[l] = (\delta g_{\mu\nu})[\alpha_l].$$

In particular, it can't be arbitrary small as its lower limit is the fixed value

$$(\widetilde{\delta}^{min} g_{\mu\nu})[\alpha_l] > 0.$$

That means

$$(\delta^{min} g_{\mu\nu})[\alpha_l] = \kappa \alpha_l^{\gamma/2}, \tag{18}$$

where $\kappa > 0$ – some model-dependent factor.
Obviously, we have

$$\lim_{l\to\infty} (\delta g_{\mu\nu})[l] = \lim_{\alpha_l\to 0} (\delta g_{\mu\nu})[\alpha_l] \to 0. \tag{19}$$

From this it follows immediately that in this case variation of the action of δS_G in General Relativity [33] is also dependent on α_l

$$\delta S_G = (\delta S_G)[\alpha_l] \tag{20}$$

and hence $G_{\mu\nu} \equiv R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu}$ is dependent on α_l too:

$$G_{\mu\nu}^{[\alpha_l]} \equiv G_{\mu\nu}[\alpha_l]. \tag{21}$$

Then the knowns **Einstein tensor**

$$\lim_{l\to\infty} G_{\mu\nu}^{[\alpha_l]} = \lim_{\alpha_l\to 0} G_{\mu\nu}^{[\alpha_l]} \equiv G_{\mu\nu} \tag{22}$$

and **Einstein Equations** in the vacuum

$$\lim_{l\to\infty} G_{\mu\nu}^{[\alpha_l]} = \lim_{\alpha_l\to 0} G_{\mu\nu}^{[\alpha_l]} \equiv G_{\mu\nu} = 0 \tag{23}$$

are brought about in the low-energy limit.
Naturally, the right side of Einstein Equations [33] should be dependent on α_l as

$$(8\pi T_{\mu\nu} - \Lambda g_{\mu\nu})^{[\alpha_l]} \equiv (8\pi T_{\mu\nu} - \Lambda g_{\mu\nu})[\alpha_l]. \tag{24}$$

Therefore, Einstein Equations with a nonzero right side are of the following form:

$$\lim_{\alpha_l\to 0} G_{\mu\nu}^{[\alpha_l]} = \lim_{\alpha_l\to 0} (8\pi T_{\mu\nu} - \Lambda g_{\mu\nu})^{[\alpha_l]}. \tag{25}$$

Of course, at low energies, i.e. for

$$l \gg l_P \tag{26}$$

or,that is the same with a very high accuracy, for

$$\alpha_l \approx 0, \tag{27}$$

the function of α_l may be disregarded and in this case, with a very high accuracy, we can obtain the well-known Einstein Equations

$$G_{\mu\nu}^{[\alpha_l]} \approx G_{\mu\nu} = (8\pi T_{\mu\nu} - \Lambda g_{\mu\nu}) \approx (8\pi T_{\mu\nu} - \Lambda g_{\mu\nu})^{[\alpha_l]}.$$

All the scales (energy), at which Einstein Equations have been studied until the present time, satisfied (26),(27), being far away from the Planck scale $l_P \propto 10^{-33}cm$, and in fact had no α_l–dependence.
But on going to the high-energy limit

$$l \dashrightarrow 2l_{min} \propto l_P; \alpha_l \dashrightarrow 1/4 \tag{28}$$

there appears a nontrivial α_l–deformation of Einstein Equations, later referred to as $\alpha-$ deformation

$$G_{\mu\nu}^{[\alpha_l]} = (8\pi T_{\mu\nu} - \Lambda g_{\mu\nu})^{[\alpha_l]}. \tag{29}$$

Note that from [25] (practically from formula (7),(8)) we have found: with the canonical measuring procedure (7), the minimal length l_{min} **is unattainable** and **a minimal measurable length**, denoted as l_{min}^{measur}, is the quantity

$$l_{min}^{measur} = 2l_{min} \tag{30}$$

in accordance with (28).
Consider two examples of the $\alpha-$ deformation of Einstein Equations.

E1.Phenomenological Markov's Model [49].
This example is taken from Section 3 of [50].
Let us dwell on the work [49], where it is assumed that "by the universal decree of nature a quantity of the material density ϱ is always bounded by

its upper value given by the expression that is composed of fundamental constants" ([49], p.214):

$$\varrho \leq \varrho_P = \frac{c^5}{G^2\hbar}, \tag{31}$$

with ϱ_P as "Planck's density".

It is clearly seen that, proceeding from the involvement of the fundamental length on the order of the Planck's $l_{min} \sim l_P$, one can obtain ϱ_P (31) up to a constant. Indeed, within the scope of GUP (12) (but not necessarily) we have $l_{min} \propto l_P$ and then, as it has been shown in [26], (12) may be generalized to the corresponding relation of the pair "energy - time" as follows:

$$\Delta t \geq \frac{\hbar}{\Delta E} + \lambda t_p^2 \frac{\Delta E}{\hbar}. \tag{32}$$

This directly suggests the existence of the "minimal time" $t_{min} \propto t_P$ and of the "maximal energy" corresponding to this minimal time $E_{max} \sim E_P$. Clearly, this maximal energy is associated with some "maximal mass" M_{max}:

$$E_{max} = M_{max}c^2, M_{max} \sim M_P. \tag{33}$$

Whence, considering that the existence of a minimal three-dimensional volume $V_{min} = l_{min}^3 \sim V_P = l_P^3$ naturally follows from the existence of $l_{min} \sim l_P$, we immediately arrive at the "maximal density" ϱ_P (31) but only within the factor determined by λ

$$\frac{M_{max}}{V_{min}} = \varrho_{max} \sim \varrho_P. \tag{34}$$

Actually, the quantity

$$\wp_\varrho = \varrho/\varrho_P \leq 1 \tag{35}$$

in [49] is the deformation parameter as it is used to construct the deformation of Einstein's equation ([49],formula (2)):

$$R_\mu^\nu - \frac{1}{2}R\delta_\mu^\nu = \frac{8\pi G}{c^4}T_\mu^\nu(1 - \wp_\varrho^2)^n - \Lambda\wp_\varrho^{2n}\delta_\mu^\nu, \tag{36}$$

where $n \geq 1/2$, T_μ^ν–energy-momentum tensor, Λ– cosmological constant. The case of the parameter $\wp_\varrho \ll 1$ or $\varrho \ll \varrho_P$ correlates with the classical Einstein equation, and the case when $\wp_\varrho = 1$ – with the de Sitter Universe. In this way (36) may be considered as $\wp_\varrho$-deformation of the General Relativity.

As it has been noted before, the existence of a maximal density directly, up to a constant, follows from the existence of a fundamental length (31). It is clear that the corresponding deformation parameter $\wp_\varrho$ (35 may be obtained from the deformation parameter α_x (9). In fact, since $\alpha_x = l_{min}^2/x^2$, we have

$$\alpha_x^{3/2} = \frac{l_{min}^3}{x^3} \sim \frac{V_{min}}{V}, \tag{37}$$

where V is the three-dimensional volume associated with the linear dimension x.
As α_x may be represented in the form [24]–[32]:

$$\alpha_x = E^2/E_{max}^2, \tag{38}$$

$E_{max} \sim E_P$, and $V_{min} \sim V_P = l_P^3$, then from (33)–(35),(37),(38) we get

$$\wp_\varrho \sim \frac{E/V}{E_{max}/V_{min}} = \frac{\varrho}{\varrho_{max}} = \alpha_x^2. \tag{39}$$

Of course, the proportionality factor in (39) is model dependent. Specifically, if QMFL is related to GUP, this factor is depending on λ (12). But the deformation parameters $\wp_\varrho$ and α are differing considerably: the limiting value $\wp_\varrho = 1$ is obviously associated with singularity, whereas originally (by the approach involving the density matrix deformation [25]–[27],[32]) no consideration has been given to the deformation parameter $\alpha = 1$ associated with singularity,(formula (30))).
So, $\wp_\varrho$-deformation of the General Relativity [49] may be interpreted as α-deformation.

E2.Spherically-symmetric horizon spaces [51].
As shown in [51], the Einstein Equation for horizon in this case may be written as a thermodynamic identity (the first principle of thermodynamics): ([51], formula (119))

$$\underbrace{\frac{\hbar c f'(a)}{4\pi}}_{k_B T} \underbrace{\frac{c^3}{G\hbar} d\left(\frac{1}{4} 4\pi a^2\right)}_{dS} \underbrace{- \frac{1}{2}\frac{c^4 da}{G}}_{-dE} = \underbrace{P d\left(\frac{4\pi}{3} a^3\right)}_{P\,dV}, \tag{40}$$

where a static, spherically symmetric horizon in space-time is described by the metric

$$ds^2 = -f(r)c^2dt^2 + f^{-1}(r)dr^2 + r^2 d\Omega^2, \tag{41}$$

and the horizon location will be given by simple zero of the function $f(r)$ ($f(a) = 0$, $f'(a) \neq 0$) at $r = a$.(Here $r = a$ is the radius of a sphere.) And $P = T_r^r$ is the trace of the momentum-energy tensor and radial pressure.
In Sections 5 and 6 of [50] first the Einstein Equations on horizon (40) have been written in terms of the parameter α_a, next the high-energy ($\alpha_a \rightarrow 1/4$), α_a - deformation of these equations has been derived in two different cases: equilibrium and nonequilibrium thermodynamics.
The latter case is distinguished from the first one by the dynamic cosmological term dependent on α_a, appearing with the corresponding factor in the right side of high-energy deformed (40) as follows:

$$\Lambda \equiv \Lambda[\alpha_a]. \tag{42}$$

4 Comments and Conclusion

In this way we can conclude that

C1) with inclusion of the space-time quantum fluctuations (e.g., in the form of (2) or (3), we can naturally assume that in the most general case of Einstein Equations there is a dependence on the small dimensionless parameter α_l, infinitesimal at normal energies to be neglected but important at high energies which are on the order of the Planck energy.

C2) The parameter α_l is a deformation parameter on going from the well-known quantum theory to a quantum theory of the Early Universe (Planck's scales)and hence the above-mentioned dependence of Einstein Equations on this parameter may be considered as α_l – deformation of the General Relativity whose well-known, i.e. canonical, Einstein Equations are brought about in the low-energy limiting transition.

The foregoing results are rather important for better understanding and investigation of the cosmological term Λ, especially in view of the Dark Energy Problem [52]–[56].
In principle, they may be used to answer the question whether $\Lambda = const$ or $\Lambda = \Lambda(t)$ is a time-variable quantity.
Despite the fact that the works taking Λ as $\Lambda(t)$, i.e. as a dynamic quantity, are numerous(for example, [57]– [60]) quite forceful arguments are given against this point of view (for example, [61]).
Indeed, according to the General Relativity, the cosmological term Λ has

been considered constant $\Lambda = const$ as, due to the Bianchi identities [33],

$$\nabla^{\mu} G_{\mu\nu} = 0. \qquad (43)$$

But in this work it has been demonstrated that, actually, Bianchi identities (43) are introduced at the low-energy limit only

$$\lim_{\alpha_l \to 0} \nabla^{\mu} G_{\mu\nu}^{[\alpha_l]} = \nabla^{\mu} G_{\mu\nu} = 0. \qquad (44)$$

Because of this, the really measured cosmological term Λ in fact is dynamic $\Lambda = \Lambda[\alpha_l(t)]$, practically invariable in the modern epoch, i.e. at low energies, due to slow variations of the deformation parameter $\alpha_l(t)$ at low energies and due to its very small value.
In the works [62]–[64] a behavior of the term Λ has been studied reasoning from $\alpha_l(t)$ on the assumption that it is dynamic, similar to the case proven in [62] GUP for the pair of conjugate variables (Λ, V), where V is the space-time volume, as with the holographic principle applied to the whole Universe [65]. The main difference of these two cases is in the leading order of expansion $\Lambda[\alpha]$ in terms of α. In the first case it is the second

$$\Lambda^{GUP}(\alpha) \propto (\alpha^2 + \eta_1 \alpha^3 + ...)\Lambda_p, \qquad (45)$$

whereas in the second case it is the first

$$\Lambda^{Hol}(\alpha) \propto (\alpha + \xi_1 \alpha^2 + ...)\Lambda_p, \qquad (46)$$

where $\Lambda_p = \Lambda_{\alpha \to 1/4}$ – cosmological term at Planck's scales.
As Λ^{Hol} is practically coincident with the experimental value of the cosmological term Λ_{exper}, a holographic model is preferable – model B) of Section 2 developed for quantum fluctuations is supported experimentally.
In conclusion, let us state some important problems of the particular concern:

A) What is the way to derive, in the most general case and in the explicit form, the high-energy ($\alpha_l \to 1/4$) α_l - representation or, that is the same, the high-energy α_l - deformation of Einstein Equations?

B) Provided the foregoing representation is derived, is it possible to have its logical series expansion in terms of α_l? Note that we must allow for the following: α_l may be considered continuous with a high accuracy only at low energies. Obviously, at high energies it is discrete as the length l

is comparable to the minimal length $l \propto l_{min}$, i.e. in fact to the Planck's length $l \propto l_P$.
As noted in point IV of Section 2, on approximation of the Planck energies, models (A) and (B) for the space-time fluctuations are practically coincident. Because of this, we can raise the following questions:

C_1) Is there some **"critical measure"** or **"critical index"** γ_{crit}:$\gamma = 2/3 < \gamma_{crit} < \gamma = 1$ – minimal bound, beginning from which models (A) and (B) are practically identical at high energies, between the coefficients $\gamma = 2/3$ and $\gamma = 1$ in formulae (3) and (3)? If such a **"critical index"** exists, what is it like? This may be of great importance for answering the question that concerns the **"phase transition"**, i.e. the minimal energies, beginning from which one should take into account the quantum-gravitational effects.

Another but similar problem:

C_2) concerns a minimal bound for α_l (denoted by $\alpha_l^{crit} = l^2_{min}/l^2_{crit}$), above which models (A) and (B) actually result in the same physics. It is clear that the problem at hand is associated with derivation of the corresponding energy: $E_{crit} \sim 1/l_{crit}$.

And, finally,
(D) it is interesting how the high-energy α_l - deformation of Einstein Equations is related to the adequate selection of a model for the space-time foam. Is it representing a set of micro worm holes(for example, [3]–[6]), micro black holes [7]– [9] or something else?
The author is planning to answer these questions, at least some of them, in his future works.

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