SUPERFIELD ALGEBRAIC STRUCTURES WITH GRASSMANN-VALUED STRUCTURE CONSTANTS*

J.A. de Azcárraga
Depto. de Física Teórica and IFIC, Centro Mixto Universidad de Valencia-CSIC, E-46100 Burjasot (Valencia) Spain

J. Lukierski
Institute for Theoretical Physics, University of Wrocław, PL-50205 Wrocław, Poland

Abstract

Generalized algebras and superalgebras whose generators and structure constants take values in a Grassmann algebra are introduced. They arise when the superfield formalism is used to describe equal time (super)algebras.

1. Introduction

It is well known that the procedure to obtain a current algebra structure (as, e.g., Gell-Mann current algebra) from a Lagrangian field theory follows essentially two steps (see, e.g., (1)). First, one finds the expression for the currents associated with the different parameters of the inner group of transformations. These expressions, as well as the value of the divergences of the currents, can be read directly from the Lagrangian density by means of the Gell-Mann-Lévy identities (2). Secondly, one uses the canonical commutation/anticommutation relations to obtain, for instance, the equal time commutators of the time components of the currents by taking advantage of the fact that the charge densities are bilinears in the original fields and their conjugate momenta. The important point here is that

* Lecture delivered (by J.A.) at the **XVI** International Colloquium on Group Theoretical Methods in Physics, Varna, Bulgaria, June 1987.

these commutators make no reference to the detailed structure of the Lagrangian; they are solely determined by the structure of the transformations which generate the currents. As a result, the commutation relations of the original algebra are reproduced through the closure of the algebra of charge densities. (Also, the algebra may be enlarged by considering more general bilinears).

The situation is however different when trying to apply the same philosophy to supersymmetric field theories. These theories present a highly constrained structure: apart from the usual constraints for the fermionic fields, the relations between the auxiliary and the dynamical fields, inherent to any supersymmetric theory, prevent the existence of simple commutation relations of the type

$$[\phi(x,\theta), \phi(x',\theta')]_{x^0=x^{0\prime}} = 0 \quad , \quad [\Pi(x,\theta), \Pi(x',\theta')]_{x^0=x^{0\prime}} = 0$$

$$[\phi(x,\theta), \Pi(x',\theta')]_{x^0=x^{0\prime}} = i\hbar\, \delta(\vec{x}-\vec{x}')\, \delta(\theta-\theta') \qquad (1.1)$$

where $\Pi(x,\theta) = \partial\mathcal{L}/\partial\dot{\phi}(x,\theta)$. Thus, the canonical quantization formalism cannot be directly applied in superspace and, one way or another, the constraints have to be taken into account to achieve the quantization of superfields (3). The net result is that eqs. (1.1) are modified and, although relations equivalent to the Gell-Mann-Lévy identities can still be defined in superspace (5,6), there does not exist the canonical structure in Dirac deltas (product of $\delta(\vec{x} - \vec{x}')$ and $\delta(\theta - \theta') = (\theta - \theta')$) which is essential for the second part of the above construction to hold.

This fact provides a motivation to look for more general algebraic structures involving superfields (see also (5,7) for a further discussion).

2. Closed superfield algebraic structures

Let us write the expansion of a Bose (internal) supercurrent J^a in the form

$$J^a(\vec{\theta}) = J^a(\theta_1 \ldots \theta_N) = j^a + j^a_{(i)}\theta_i + \ldots + j^a_{(i_1 \ldots i_k)}\theta_{i_1} \ldots \theta_{i_k} + \ldots + \tilde{j}^a \theta_1 \ldots \theta_N \quad (2.1)$$

were $i,j = 1\ldots N$ and a is the internal index. We shall omit the space-time indices. For $D = 1+0$ supersymmetry, where superfields depend only on t, this corresponds to considering (2.1) at a given time; for $D > 1$ the $j^a_{(i_1 \ldots i_k)}$ may be understood as the result of smearing out the components $j^{ak}_{(i_1 \ldots i_k)}(t, \vec{x})$ with a test function $f(t,\vec{x})$ (if $f = \delta(t)$, then (2.1) is the superfield expansion at fixed time of a local supercurrent integrated over the (D-1) spatial coordinates). Allowing also for the presence of Fermi operators $S^a(\theta_1 \ldots \theta_N)$, the most general closed algebraic structure is of the form

$$[J^a(\vec{\theta}), J^b(\vec{\theta}')] = \int d^N\eta \, [f^{ab}_c(\vec{\theta}, \vec{\theta}', \vec{\eta}) \, J^c(\vec{\eta}) + h^{ab}_\alpha(\vec{\theta}, \vec{\theta}', \vec{\eta}) \, S^\alpha(\vec{\eta})] \quad (2.2a)$$

$$[J^a(\vec{\theta}), S^\alpha(\vec{\theta}')] = \int d^N\eta \, [f^{a\alpha}_\beta(\vec{\theta}, \vec{\theta}', \vec{\eta}) \, S^\beta(\vec{\eta}) + h^{a\alpha}_c(\vec{\theta}, \vec{\theta}', \vec{\eta}) \, J^c(\vec{\eta})] \quad (2.2b)$$

$$\{S^\alpha(\vec{\theta}), S^\beta(\vec{\theta}')\} = \int d^N\eta \, [f^{\alpha\beta}_c(\vec{\theta}, \vec{\theta}', \vec{\eta}) \, J^c(\vec{\eta}) + h^{\alpha\beta}_\gamma(\vec{\theta}, \vec{\theta}', \vec{\eta}) \, S^\gamma(\vec{\eta}) \quad , \quad (2.2c)$$

where the integration over $\vec{\eta}$ is understood in sense of Berezin (8) and the three different types of f's (h's), which have an even (odd) number of Greek indices, are even (odd) functions of the Grassmann variables if N is even; if N is odd, their grading is the opposite.

The position of the greek and latin indices should be noticed: for instance, $f^{a\alpha}{}_{\beta}$ and $f^{\alpha\beta}{}_{c}$ have the same parity, but they multiply generators with opposite gradings in (2.2b) and (2.2c) respectively. The f's and the h's satisfy the obvious relation

$$f^{uv}{}_{w}(\vec{\theta},\vec{\theta}',\vec{\eta}) = -(-)^{\deg u.\deg v} f^{vu}{}_{w}(\vec{\theta}',\vec{\theta},\vec{\eta}) \qquad (2.3)$$

where deg u is the grading of the index u = a, b ... or α, β ... (deg u = 0(1) for latin (Greek) indices). Notice that with $J^a(\theta) = \Pi_i(\theta)\lambda^a_{ij}\Phi_j(\theta)$ and canonical commutators we would get the commutator (2.2a) with S = 0 and f $(\vec{\theta},\vec{\theta}',\vec{\eta}) = (\vec{\theta}-\vec{\eta}).(\vec{\eta}-\vec{\theta}).\ F^{ab}_{c}$ but, as already mentioned, quantized superfields do not follow the canonical formalism. As a result, the r.h.s. of the commutators of the charge components is not reduced to a product of Dirac deltas in the Grassmann variables with the structure constants F^{ab}_{c} of the inner algebra.

Assuming irreducibility for J^a and S^a, the graded Jacobi identities lead to the following identities for the structure functions $f^{uv}{}_w$, $h^{uv}{}_w$

$$\int d^N\eta\, [f^{ab}{}_{d}(\vec{\theta},\vec{\theta}',\vec{\eta})\, f^{dc}{}_{e}(\vec{\eta},\vec{\theta}'',\vec{\eta}') + h^{ab}{}_{\epsilon}(\vec{\theta},\vec{\theta}',\vec{\eta})\, h^{\epsilon c}{}_{e}(\vec{\eta},\vec{\theta}'',\vec{\eta}')] + \mathrm{cycl}\,\{(a\vec{\theta}),(b\vec{\theta}'),(c\vec{\theta}'')\} = 0$$

$$\int d^N\eta\, [f^{ab}{}_{d}(\vec{\theta},\vec{\theta}',\vec{\eta})\, h^{dc}{}_{\beta}(\vec{\eta},\vec{\theta}'',\vec{\eta}') + h^{ab}{}_{\epsilon}(\vec{\theta},\vec{\theta}',\vec{\eta})\, f^{\epsilon c}{}_{\beta}(\vec{\eta},\vec{\theta}'',\vec{\eta}')] + \mathrm{cycl}\,\{(a\vec{\theta}),(b\vec{\theta}'),(c\vec{\theta}'')\} = 0 \quad ; \qquad (2.4a)$$

$$\int d^N\eta\, [f^{\alpha b}{}_{u}(\vec{\theta},\vec{\theta}',\vec{\eta})\, f^{uc}{}_{\gamma}(\vec{\eta},\vec{\theta}'',\vec{\eta}') + h^{\alpha b}{}_{u}(\vec{\theta},\vec{\theta}',\vec{\eta})\, h^{uc}{}_{\gamma}(\vec{\eta},\vec{\theta}'',\vec{\eta}')] + \mathrm{cycl}\,\{(\alpha\vec{\theta}),(b\vec{\theta}'),(c\vec{\theta}'')\} = 0$$

$$\int d^N\eta\, [f^{\alpha b}{}_{u}(\vec{\theta},\vec{\theta}',\vec{\eta})\, h^{uc}{}_{e}(\vec{\eta},\vec{\theta}'',\vec{\eta}') + h^{\alpha b}{}_{u}(\vec{\theta},\vec{\theta}',\vec{\eta})\, f^{uc}{}_{e}(\vec{\eta},\vec{\theta}'',\vec{\eta}')] + \mathrm{cycl}\,\{(\alpha\vec{\theta}),(b\vec{\theta}'),(c\vec{\theta}'')\} = 0 \quad ; \qquad (2.4b)$$

$$\int d^N\eta \, [f^{\alpha\beta}{}_d (\vec\theta,\vec\theta',\vec\eta) f^{d\gamma}{}_\varepsilon (\vec\eta,\vec\theta'',\vec\eta') + h^{\alpha\beta}{}_\epsilon (\vec\theta,\vec\theta',\vec\eta) h^{\epsilon\gamma}{}_\varepsilon (\vec\eta,\vec\theta'',\vec\eta')] + \\ + \text{cycl.} \{ (\alpha\theta),(\beta\theta'),(\gamma\theta'') \} = 0$$

$$\int d^N\eta \, [f^{\alpha\beta}{}_d (\vec\theta,\vec\theta',\vec\eta) h^{d\gamma}{}_e (\vec\eta,\vec\theta'',\vec\eta') + h^{\alpha\beta}{}_\epsilon (\vec\theta,\vec\theta',\vec\eta) f^{\epsilon\gamma}{}_e (\vec\eta,\vec\theta'',\vec\eta')] + \\ + \text{cycl.} \{ (\alpha\theta),(\beta\theta'),(\gamma\theta'') \} = 0 \quad ; \tag{2.4c}$$

$$\int d^N\eta \, [f^{\alpha\beta}{}_u (\vec\theta,\vec\theta',\vec\eta) f^{uc}{}_e (\vec\eta,\vec\theta'',\vec\eta') + h^{\alpha\beta}{}_u (\vec\theta,\vec\theta',\vec\eta) h^{uc}{}_e (\vec\eta,\vec\theta'',\vec\eta')] + \\ + \text{grad cycl.} \{ (\alpha\theta),(\beta\theta'),(c\theta'') \} = 0$$

$$\int d^N\eta \, [f^{\alpha\beta}{}_u (\vec\theta,\vec\theta',\vec\eta) h^{uc}{}_\gamma (\vec\eta,\vec\theta'',\vec\eta') + h^{\alpha\beta}{}_u (\vec\theta,\vec\theta',\vec\eta) f^{uc}{}_\gamma (\vec\eta,\vec\theta'',\vec\eta')] + \\ + \text{grad cycl.} \{ (\alpha\theta),(\beta\theta'),(c\theta'') \} = 0 \quad ; \tag{2.4d}$$

where

1) in formulae (2.4b) and (2.4d) the index u is greek or latin depending on the permutation, in such a way that the f's (h's) have always an even (odd) number of greek indices,

2) in formula (2.4d) graded cyclic means that $(\alpha\theta)$, $(\beta\theta')$ $(c\theta'')$ and $(c\theta'')$ $(\alpha\theta)$ $(\beta\theta')$ have a "+" sign and $(\beta\theta')$ $(c\theta'')$ $(\alpha\theta)$ a "-" one (β jumps over α).

It is not difficult to give a realization of the algebraic structure (2.2). It turns out that by calculating the E.T. commutators of bilinear products of free superfields the relations (2.2a-d) emerge provided that we consider products which are bilocal in the Grassmann variables, (for instance, of the form $J(\vec\theta) = \Phi(t,\theta,\bar\theta)\Phi(t,\tilde\theta,\tilde{\bar\theta})$ where we have omitted the inner indices). Therefore in superfield applications the Grassmann algebra generated by $\theta_1 \ldots \theta_N$ is described by a graded tensor product of two copies of the Grassmann algebras describing the anticommuting superspace coordinates. We shall not discuss this here and will refer to (9) instead for details.

3. Final remarks

The superalgebra (2.2a-d) can be rewritten in various ways. For instance, one may expand $J^a(\vec{\theta}')$ (and similarly $S^a(\vec{\theta}')$) in the power series

$$J^a(\vec{\theta}') = \sum_{k=0}^{N} \frac{1}{k!} \sum_{i_1 \dots i_k} (\theta'_{i_1} - \theta_{i_1}) \dots (\theta'_{i_k} - \theta_{i_k})\, J^a_{(i_1 \dots i_k)}(\vec{\theta}) \, . \quad (3.1)$$

In this form, (2.4a-d) can be written as a finite superalgebra in terms of the superfields $\mathcal{J}^a_{(i_1 \dots i_k)}(\vec{\theta})$, $S^\alpha_{(i_1 \dots i_k)}(\vec{\theta})$. With

$$J^a_{\hat{i}}(\vec{\theta}) = \left\{ J^a_{(i_1 \dots i_k)} \text{ (k even)},\ S^\alpha_{(i_1 \dots i_k)} \text{ (k odd)} \right\}$$

$$S^a_{\hat{r}}(\vec{\theta}) = \left\{ J^a_{(i_1 \dots i_k)} \text{ (k odd)},\ S^\alpha_{(i_1 \dots i_k)} \text{ (k even)} \right\}, \quad (3.2)$$

eq.(2.2a), for example, would read

$$[J^a_{\hat{i}}(\vec{\theta}), J^b_{\hat{j}}(\vec{\theta})] = F^{ab}_{c}(\hat{i},\hat{j},\hat{k};\vec{\theta})\, J^c_{\hat{k}}(\vec{\theta}) + \\ + H^{ab}_{c}(\hat{i},\hat{j},\hat{r};\vec{\theta})\, S^\alpha_{\hat{r}}(\vec{\theta}) \, , \quad (3.3)$$

where the new structure functions F, H are obtained from the previous f, h by performing the corresponding Taylor expansions.

It is also possible to express (2.2) in terms of commutators/anticommutators (with numerical structure constants) in the "charge" components $j^a_{(i_1 \dots i_k)}$, $s^a_{(i_1 \dots i_k)}$ of $J^a(\vec{\theta})$ and $S^a(\vec{\theta})$. To this end, it is sufficient to perform the complete expansion of the f's and the h's in their Grassmann arguments to extract the relations between the charge components.

As mentioned above, the superalgebra (2.2a-d) may be realized by means of the equal time bilinears of D=1 <u>free</u> superfields (the superfields of the superharmonic oscillator; see, e.g. (4)) and their covariant derivatives. In the presence of interactions, the basic superfield commutators are modified, and these bilinears cease to form a closed algebraic system (2.2a-d). In the presence of interactions describing an asymptotically free theory, one can introduce fully bilocal products of superfields (bilocal in the Grassmann and space-time coordinates). If $D > 1$, one can consider their graded supercommutators for superspace coordinates with differences lying on the super-light cone. The postulate that such a superalgebra closes leads to the supersymmetric extension $D = 4$ Fritzsch-Gell-Mann algebra for bilocal internal symmetry currents (7).

REFERENCES

1. S.L. Adler y R.F. Dashen, Current Algebras and their Applications to Particle Physics, W.A. Benjamin, N.Y. (1968).

2. M. Gell-Mann and M.Lévy, Nuovo Cim. XVI, 53 (1960).

3. The quantization of superfields may be accomplished in various ways. The superfield commutators may be obtained as superspace Green functions for the free field equations and by functional methods (see, e.g., J. Wess and J. Bagger, Supersymmetry and Supergravity, Princeton Univ. Press (1983); S. Gates Jr., M.T. Grisaru, M. Roček and W. Siegel, Superspace, Benjamin-Cummings, (1983)); by using the Dirac quantization formalism (J. Barcelos-Neto, A. Das and W. Scherer, Phys. Rev. D34, 1108 (1986), and by a generalization to superspace of the Umezawa-Takahashi method (J.A. de Azcárraga, J. Lukierski and P. Vindel (4); Fortschr. der Phys., in press). A "canonical superspace" has been discussed recently by R.A. Marques Pereira, Class. and Quantum Grav. 4, 411 (1987).

4. J.A. de Azcárraga, J. Lukierski and P. Vindel, Mod. Phys. Lett. A1, 293 (1986).

5. J.A. de Azcárraga and J. Lukierski, Phys. Rev. D31, 912 (1985).

6. P. Vindel, unpublished.

7. J.A. de Azcárraga, J. Lukierski and P. Vindel, Phys. Lett. B195, 195 (1987).

8. F.A. Berezin, The Method of Second Quantization, Academic Press (1966); Introduction to Superanalysis (edited by A.A. Kirillov), D. Reidel (1987).

9. J.A. de Azcárraga and J. Lukierski, Trieste preprint IC/87/68, submitted to J. Math. Phys.