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Correlation Functions in Integrable Theories

From weak to strong coupling

RAUL PEREIRA

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Abstract
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The discovery of integrability in planar N=4 super Yang-Mills and ABJM has enabled a precise study of AdS/CFT. In the past decade integrability has been successfully applied to the spectrum of anomalous dimensions, which can now be obtained at any value of the coupling. However, in order to solve conformal field theories one also needs to understand their structure constants. Recently, there has been great progress in this direction with the all-loop proposal of Basso, Komatsu and Vieira. But there is still much to understand, as it is not yet possible to use that formalism to find structure constants of short operators at strong coupling. It is important to study wrapping corrections and resum them as the TBA did for the spectrum. It is also crucial to obtain perturbative data that can be used to check if the all-loop proposal is correct or if there are new structures that need to be unveiled.

In this thesis we compute several structure constants of short operators at strong coupling, including the structure constant of Konishi with half-BPS operators. Still at strong coupling, we find a relation between the building blocks of superstring amplitudes and the tensor structures allowed by conformal symmetry. We also consider the case of extremal correlation functions and the relation of their poles to mixing with double-trace operators.

We also study three-point functions at weak coupling. We take the OPE limit in a four-point function of half-BPS operators in order to shed some light on the structure of five-loop wrapping corrections of the Hexagon form factors. Finally, we take the first steps in the generalization of the Hexagon programme to other theories. We find the non-extremal setup in ABJM and the residual symmetry that it preserves, which we use to fix the two-particle form factor and constrain the four-particle hexagon. Finally, we find that the Watson equations hint at a dressing phase that needs to be further investigated.

Keywords: Superstring theory, AdS/CFT correspondence, Integrable field theories, Hexagon form factors

Raul Pereira, Department of Physics and Astronomy, Theoretical Physics, Box 516, Uppsala University, SE-751 20 Uppsala, Sweden.

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List of papers

This thesis is based on the following papers, which are referred to in the text by their Roman numerals.

I T. Bargheer, J. A. Minahan and R. Pereira, *Computing Three-Point Functions for Short Operators*, JHEP **03** (2014) 096

II J. A. Minahan and R. Pereira, *Three-point correlators from string amplitudes: Mixing and Regge spins* JHEP **04** (2015) 134

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Contents

1. Introduction

Quantum field theory is a theoretical framework which has been extremely successful in the description of particle physics, both with the $SU(2) \times U(1)$ gauge theory for electroweak interactions as well as with quantum chromodynamics for the strong interactions. But the usefulness of quantum field theories goes well beyond the realm of the standard model, as they can provide an effective description for relativistic theories at low energies. Some of these effective models have also proved useful in the description of phenomena in condensed matter physics.

However, despite all the applications of quantum field theories, there is still much we do not understand. In most situations, all one can hope to do analytically is perturbation theory, but that will clearly be insufficient if the theory happens to be strongly coupled. Furthermore, general relativity predicts the existence of black holes, which contain spacetime singularities behind the event horizon. This indicates that our understanding is incomplete and we need a quantum theory of gravity in such high curvature regimes. There have been many attempts to incorporate gravity effects in the framework of quantum field theory, but they have been unsuccessful.

It turns out that string theory is a mathematical framework with the ability to tackle both these problems. Instead of considering point-like particles, the fundamental object is a single one-dimensional object sweeping out a two-dimensional worldsheet in spacetime, and particles are simply different excitation modes of the same string. In a theory of strings one can also consider higher-dimensional extended objects called D-branes, where open strings end [1]. In order to have fermions, we need to consider a supersymmetric worldsheet theory, which happens to be consistent only in ten dimensions.

Interactions are obtained through joining and splitting of strings, so the quantum theory introduces a genus expansion. This can be seen as a first hint at duality, as we can also obtain a genus expansion in quantum field theories. It was shown by 't Hooft that in a gauge theory with N colours, Feynman diagrams with the same topology also have the same power of N [2]. It then looks like classical strings are just a planar limit of gauge theories, while quantum strings correspond to gauge theories at small finite N.

It is also true that any theory of closed string must contain a massless spin-two excitation, and it can be shown that the low energy limit of string theory is a supersymmetric quantum field theory of gravity. Finally, the extended nature of string interactions indicates that the theory should be free of UV divergences, so string theory naturally becomes a strong candidate for a theory of quantum gravity.

There is an important open/closed string duality, which claims that the presence of D-branes can be viewed in two very different ways [3, 4]. When closed strings propagate in the presence of a D-brane, we have to consider all possible interactions where closed strings split into open strings which can later fuse back into closed strings. We then obtain an integral over the moduli of Riemann surfaces which can have an arbitrary number of holes. It is however possible to replace holes in the worldsheet with operator insertions, so we can also see the D-brane as a source of closed strings. This background of closed strings can in turn be mapped to a deformation of the background geometry in which strings propagate.

Let us now take the low energy limit of a stack of d-dimensional branes. Only the massless string modes survive in this limit, and the string action decomposes into two parts, a brane action for the open strings and a bulk action for the closed strings, which do not interact in this limit. The bulk action corresponds to ten-dimensional supergravity, while the open strings are described by a d-dimensional quantum field theory. On the other hand, we can instead consider the presence of the branes as effectively deforming the background geometry. At infinity the low energy excitations correspond both to massless strings far from the horizon, which are described by ten-dimension supergravity, but also by stringy excitations close to the horizon. This is simply a manifestation of the redshift effect for particles in strong gravitational potentials. If we consider stacks of D3, M2 or M5 branes, the near-horizon geometries are of the form $AdS_{d+1} \times \mathcal{M}$, with $\mathcal{M}$ some compact manifold, and the low energy worldvolume theories become conformal. We are then forced to conclude that the d-dimensional conformal field theories are dual to superstring theories in AdS_{d+1} backgrounds [5–7].

This duality is the most explicit realization of the holographic principle, which in fact precedes these discoveries. Going back to the black hole solutions of general relativity, it was found that the charges describing the black hole obey thermodynamic-like relations. Using semiclassical methods Hawking showed that they emit a thermal radiation, and also that the entropy of the black hole is proportional to the area of its event horizon [8]. This was the first indication that the degrees of freedom of gravity in d dimensions can be mapped to the degrees of freedom of boundary theories in one lower dimension [9].

Two of the most well studied examples of AdS/CFT are given by the propagation of type IIB strings in $AdS_5 \times S^5$ and type IIA strings in $AdS_4 \times \mathbb{CP}^3$, whose dual conformal field theories are $\mathcal{N} = 4$ SYM in four dimensions and ABJM in three dimensions. These theories are supersymmetric, but in the planar limit they are conjectured to become integrable since there is an infinite tower of hidden symmetries. Integrability can be found explicitely both in the two-dimensional worldsheet theory and in the spin-chain one obtains as a representation of single-trace operators [10, 11]. There are many important consequences of integrability on two-dimensional theories. It is possible to show that particle production or annihilation is forbidden, and the presence of an infinite number of conserved charges also fixes outgoing momenta to be a permutation of the incoming momenta. Finally, another key implication of integrability is that the S-matrix factorizes into two-to-two scatterings. This means that any scattering process is determined by the dispersion relation and the two-body S-matrix. In the past decade these features have been explored thoroughly, leading to non-perturbative frameworks for the computation of the spectrum of $\mathcal{N} = 4$ SYM and ABJM [12, 13]. In recent years there has been strong evidence that integrability can also be used to obtain the structure constants of $\mathcal{N} = 4$ SYM at any value of the coupling [14], and we can hope that the same will happen for ABJM and other integrable theories appearing in the context of AdS/CFT.

Note that these are weak/strong dualities so they have strong implications both for quantum field theories and for string theory. When the conformal field theory is weakly coupled, the string tension goes to zero and we have a quantum regime of string theory. This is extremely helpful as it gives an understanding of quantum gravity from simple perturbative calculations in gauge theories. Meanwhile, semi-classical strings are dual to strongly coupled quantum field theories, so the duality provides a very powerful tool for the study of strong interactions. Notice however that it is usually very difficult to ascertain the validity of such a duality, since one has to perform all orders perturbative computations to match observables in the two theories. If that is not possible one can also match protected observables which still provide evidence for the validity of the conjecture.

Let us end this introduction with an outline for the rest of the thesis. In chapter 2 we review the basic concepts of conformal symmetry and then focus on its application to the superconformal field theories mentioned above, $\mathcal{N} = 4$ SYM and ABJM. We will then review superstring theory and its scattering amplitudes in chapter 3, focusing on string states from the Regge trajectory, and also at the massless and first massive levels. ln chapter 4 we introduce the flat-space approximation for short operators of $\mathcal{N} = 4$ SYM, and them move on to present the strong coupling structure constants obtained in Papers I and II. We conclude that chapter with an explanation of the relation between extremality and mixing with double-trace operators which was presented in Paper II. In chapter 5 we move on to the weak coupling side of the duality, where we take the coincidence limit in a four-point function of half-BPS operators in order to obtain their structure constant with the Konishi operator at five loops. We describe the technique of asymptotic expansions, and also elaborate on the transcendentality features of the master massless propagator integrals obtained along the way. The focus of chapters 6 and 7 is on the use of integrability to obtain non-perturbative results. In the first of these chapters we introduce the Bethe ansatz for integrable spin-chains and its application for the spectrum of $\mathcal{N} = 4$ SYM and ABJM. We also review the recent Hexagon proposal for the computation of structure constants in $\mathcal{N} = 4$ SYM, and in chapter 7 we take the first steps into generalizing such a framework for the AdS_4/CFT_3 duality.

2. Conformal Field Theories

In this chapter we review general concepts of conformal field theories and then specialize to the integrable theories relevant for the rest of this thesis. A theory with conformal symmetry is invariant under transformations that preserve angles, which is equivalent to a local scaling. It is very natural to study such symmetries, as many theories are classically conformal, such as ϕ^4 theory, classical Yang-Mills in four dimensions and even massless QCD. Conformal symmetry also plays a crucial role in condensed matter physics, as critical phenomena are described by scale-invariant systems where there is no correlation length. A characterisitic feature of such theories is the power-law behaviour of their correlation functions.

Another very important reason to study conformal symmetry is related to the renormalization group flow. It is well understood that as one varies the energy scale of a theory, the strength of its interactions changes accordingly to its β-functions. The flow leads to fixed points where the β functions vanish, corresponding to theories which are invariant under scaling. If a theory has vanishing β funtions at any loop order, then it is also conformal at the quantum level. There are many examples of such theories, but here we shall focus our attention on $\mathcal{N} = 4$ SYM in four dimensions and ABJM in three dimensions.

2.1 Conformal Symmetry

Since conformal transformations preserve angles, they must leave the metric $g_{\mu\nu}$ invariant up to a scalar factor

$$g_{\mu\nu} \frac{d\tilde{x}^\mu}{dx^\alpha} \frac{d\tilde{x}^\nu}{dx^\beta} = \Omega^2(x) g_{\alpha\beta} \,. \tag{2.1}$$

For the rest of the chapter, we will focus on spacetime dimensions higher than two. It is possible to show that the infinitesimal form of a conformal transformation is at most quadratic, more precisely of the form

$$x^\mu \to \tilde{x}^\mu = x^\mu + a^\mu + \lambda x^\mu + \omega^{\mu\nu} x_\nu + x^2 b^\mu - 2(x \cdot b) x^\mu \,. \tag{2.2}$$

We can recognize a^μ and the antisymmetric $\omega^{\mu\nu}$ as the translation and Lorentz symmetries that form the Poincaré group, present in all relativistic quantum field theories. The parameter λ corresponds to global dilations, and b^μ is the parameter of the special conformal transformations, which can be understood as a composition of translations with the inversion

$$x^\mu \to \frac{x^\mu}{x^2} \,. \tag{2.3}$$

The generators for translations, boosts, dilatations and special conformal transformations are denoted by P_μ, $L_{\mu\nu}$, D and K_μ, and their action can be expressed through the differential operators

$$\begin{aligned} P_\mu &= -i\partial_\mu \,, & L_{\mu\nu} &= i(x_\mu\partial_\nu - x_\nu\partial_\mu) \,, \\ D &= -ix^\mu\partial_\mu \,, & K_\mu &= -2ix_\mu(x\cdot\partial) + ix^2\partial_\mu \,. \end{aligned} \tag{2.4}$$

Given these expressions, one can derive the commutation relations of the algebra. It is useful to organize the generators in the following way

$$\begin{aligned} J_{\mu\nu} &= L_{\mu\nu} \,, & J_{-1,\mu} &= \frac{1}{2}(P_\mu - K_\mu) \,, \\ J_{-1,0} &= D \,, & J_{0,\mu} &= \frac{1}{2}(P_\mu + K_\mu) \,, \end{aligned} \tag{2.5}$$

so that we can recognize it as $\mathfrak{so}(2,d)$, the Lie algebra associated to the Lorentz group in $\mathbb{R}^{2,d}$

$$[J_{mn}, J_{pq}] = i(\eta_{mq}J_{np} + \eta_{np}J_{mq} - \eta_{mp}J_{nq} - \eta_{nq}J_{mp}) \,. \tag{2.6}$$

This relabelling of the generators might look inocuous at first, but it will turn out to be useful later when we introduce the embedding formalism.

2.1.1 State-Operator Map and Primary Fields

In quantum field theories, one has to choose a spacetime foliation in order to quantize the theory. A standard approach is to choose equal-time surfaces, and the map between the Hilbert spaces of those surfaces is given by the action of the unitary operator e^{iHt}. Given this choice of foliation, the states are characterized by their energy and momenta. However, it turns out that in conformal field theories there is a more natural choice of foliations. We can gain some intuition by considering the map from $\mathbb{R}^d$ to $\mathbb{R}\times S^{d-1}$, where we relate the radial coordinate with the time coordinate τ on the cylinder

$$r = e^\tau \,. \tag{2.7}$$

Equal time surfaces on the cylinder correspond to a radial foliation of $\mathbb{R}^d$, so the energy spectrum on the cylinder corresponds to the spectrum of the dilatation operator on the plane. Quantization with this radial foliation is usually denoted by radial quantization, and is more useful as it naturally describes operator insertions in conformal field theories. In this case, states are described by their scaling dimensions

$$D|\Delta, l\rangle = i\Delta|\Delta, l\rangle\,. \tag{2.8}$$

Since the generators of rotations on the sphere S^{d-1} commute with the dilatation operator, we also label states by their spin on the sphere

$$L_{\mu\nu}|\Delta, l\rangle = \Sigma_{\mu\nu}|\Delta, l\rangle\,, \tag{2.9}$$

where the matrices $\Sigma_{\mu\nu}$ act on the spin indices of the operator in question. The unitary operator that maps equal-radii surfaces acts on states as $e^{iD\tau}|\Delta\rangle = r^{-\Delta}|\Delta\rangle$.

Notice that moving towards the origin of the sphere corresponds to approaching past infinity on the cylinder, so a state at a given radius is created by the insertion of local operators at the origin. We then have a state-operator correspondence, since the insertion of local operators creates a state with definite scaling dimension

$$\mathcal{O}_\Delta(0)|0\rangle \to |\Delta\rangle\,. \tag{2.10}$$

It is easy to see that P_μ and K_μ are raising and lowering operators as they shift the scaling dimension of the state by $+1$ and -1 respectively. The lowest weight of the representation, which is called a primary operator, must therefore be annihilated by K_μ. The action of the generators on a primary field is given by

$$\begin{aligned}
[K_\mu, \mathcal{O}(0)] &= 0\,,\\
[P_\mu, \mathcal{O}(0)] &= -i\partial_\mu \mathcal{O}(0)\,,\\
[D, \mathcal{O}(0)] &= -i\Delta \mathcal{O}(0)\,,\\
[M_{\mu\nu}, \mathcal{O}(0)] &= -i\Sigma_{\mu\nu} \mathcal{O}(0)\,.
\end{aligned} \tag{2.11}$$

The other states of the theory are the descendants, given by derivatives of primary fields which can be created by acting with the translation operator P^μ on the primary state. If the operator insertion is away from the origin, then the state produced is a superposition of states with different scaling dimensions, as can be seen easily from rewriting the operator in the following manner

$$\mathcal{O}(x) = e^{iP\cdot x}\mathcal{O}(0)e^{-iP\cdot x}\,. \tag{2.12}$$

Lastly, it is useful to note that one can extract bounds on the scaling dimensions from unitarity considerations [15]. Seeing that time-reversal on the cylinder corresponds to an inversion, it then follows that K_μ is the hermitian conjugate of P_μ. For states to have positive norm, we need the following matrix to be positive definite

$$\langle \Delta, l|K_{\mu'}K_{\nu'}P_\nu P_\mu|\Delta, l\rangle\,. \tag{2.13}$$

From this kind of considerations it is possible to derive the following bounds on scaling dimensions of scalars and operators in symmetric traceless representations of spin l

$$\Delta \geq \frac{d}{2} - 1\,, \qquad \Delta_l \geq l + d - 2\,. \tag{2.14}$$

2.1.2 Correlation Functions

We now know from the previous discussion that a primary scalar field must transform as

$$\phi(x) \to \tilde{\phi}(\tilde{x}) = \left|\frac{\partial \tilde{x}}{\partial x}\right|^{-\Delta/d} \phi(x)\,. \tag{2.15}$$

It then follows that correlation functions of primary scalar operators satisfy

$$\langle \mathcal{O}_1(\tilde{x}_1) \ldots \mathcal{O}_n(\tilde{x}_n)\rangle = \left|\frac{\partial \tilde{x}}{\partial x}\right|_{x_1}^{-\Delta_1/d} \cdots \left|\frac{\partial \tilde{x}}{\partial x}\right|_{x_n}^{-\Delta_n/d} \langle \mathcal{O}_1(x_1) \ldots \mathcal{O}_n(x_n)\rangle\,, \tag{2.16}$$

which imposes strong constraints on such correlators. For example, the only non-vanishing one-point function is for the identity operator, which has $\Delta = 0$. Meanwhile, for two-point functions it is sufficient to consider behaviour under Poincaré transformations and dilatations to show that they must obey

$$\langle \mathcal{O}_1(x_1)\mathcal{O}_2(x_2)\rangle = \frac{\delta_{\Delta_1,\Delta_2}}{(x_1 - x_2)^{2\Delta_1}}\,. \tag{2.17}$$

For three-point functions it is necessary to also consider the transformation under inversion, from which we can then show that they are of the following form [16]

$$\langle \mathcal{O}_1(x_1)\mathcal{O}_2(x_2)\mathcal{O}_3(x_3)\rangle = \frac{C_{123}}{|x_1 - x_2|^{2\alpha_3}|x_1 - x_3|^{2\alpha_2}|x_2 - x_3|^{2\alpha_1}}\,, \tag{2.18}$$

where we introduce $\alpha_i = \Sigma - \Delta_i$, with $\Sigma = \frac{1}{2}\sum_i \Delta_i$. With four external points it is possible to form two independent conformally invariant cross ratios

$$u = \frac{x_{12}^2 x_{34}^2}{x_{13}^2 x_{24}^2}\,, \qquad v = \frac{x_{14}^2 x_{23}^2}{x_{13}^2 x_{24}^2}\,. \tag{2.19}$$

This implies that four-point functions are fixed only up to a function of the cross ratios. Assuming all operators have scaling dimension Δ we then have

$$\langle \mathcal{O}_1(x_1) \ldots \mathcal{O}_4(x_4)\rangle = \frac{G(u,v)}{|x_1 - x_2|^{2\Delta}|x_3 - x_4|^{2\Delta}}\,. \tag{2.20}$$

2.1.3 Operator Product Expansion

The notion of operator product expansion appears already in quantum field theories, as we can substitute a product of two local operators close to each other by a sum of local operators. However, while that notion is only asymptotic in QFT, in conformal field theories we *do* have a convergent OPE, as can be seen from radial quantization. Consider for example two operator insertions which create a state in a sphere around them. This state can in turn be seen as an expansion over eigenstates of the dilatation operator. Using the operator-state correspondence we then conclude that the operator insertions can be substituted by an infinite sum of primary and descendant operator insertions at the center of the sphere. More explicitely, we have

$$\mathcal{O}_1(x)\mathcal{O}_2(0) = \sum_k C_{12k}|x|^{\Delta_k-\Delta_1-\Delta_2}\left(\mathcal{O}_k(0) + \text{descendants}\right) , \quad (2.21)$$

where we sum only over the primary operators, while the contribution of descendants is totally fixed by conformal symmetry. It can be shown that the structure constants C_{ijk} of the OPE decomposition correspond exactly to the numeric coefficients of three-point functions. Meanwhile, the convergence of the expansion depends on which correlation functions we consider. The fact that they diverge as two operators approach each other implies that the radius of convergence of the OPE is the distance from the origin of the sphere to the closest operator in the correlation function.

The fact that we have a convergent operator product expansion has strong implications on the study of conformal field theories. Given any higher-point function, we can perform OPEs until we are left only with three operator insertions, which means that all the information from those correlators is encoded in the spectrum of anomalous dimensions and the set of structure constants of the theory. It is therefore possible to compute any correlation function of local operators as long as those two sets of numbers are known. However, in practice it might not be feasible to perform all the necessary sums, so the study of higher-point functions is also important.

2.1.4 Embedding Formalism

Let us finish this review of conformal symmetry with a brief explanation of the embedding formalism. The tools used so far were sufficient for the study of scalar correlation functions, but if one wants to classify also the allowed tensor structures for correlators with spin [17–19], then the embedding formalism provides the machinery required for a much more efficient analysis. The crucial observation was made already in

(2.6), where we saw that the conformal group $SO(2,d)$ acts linearly on an embedding space $\mathbb{R}^{2,d}$. Following the notations introduced in (2.5), and introducing the coordinates X^M for the space $\mathbb{R}^{2,d}$, the action of conformal transformations on the embedding space is [20]

$$\tilde{X}^M = \Lambda^M_N X^N \,. \tag{2.22}$$

In order to relate this with the d-dimensional physics we have to get rid of two degrees of freedom. First, we restrict to the null cone in embedding space

$$X^2 = 0 \,, \tag{2.23}$$

and then we take a section of the cone parametrized by

$$X(x) = (X^+, X^-, X^\mu) = (1, x^2, x^\mu) \,, \tag{2.24}$$

where the metric has a non-diagonal component $\eta_{+-} = -1/2$ associated to the light-cone coordinates.

Given the transformation rule (2.15) for primary fields in $\mathbb{R}^d$, the natural extension in embedding space is a homogeneity condition on the fields

$$\mathcal{O}(\lambda X) = \lambda^{-\Delta} \mathcal{O}(X) \,. \tag{2.25}$$

Let us now look at operators in a symmetric traceless representation, for which this formalism becomes especially powerful. The fields on the embedding space will also be in a symmetric traceless representation, but once again we have to eliminate two of the degrees of freedom associated to each index. First we will require the fields to be transverse

$$X^M \mathcal{O}_{M\ldots}(X) = 0 \,, \tag{2.26}$$

and since we are working on the null cone $X^2 = 0$, any component proportional to X^M projects to zero, so this gauge redundancy

$$\mathcal{O}_{M\ldots}(X) \to \mathcal{O}_{M\ldots} + X_M f(X) \tag{2.27}$$

reduces the number of degrees of freedom to what we need. Finally, we obtain the physical operator with the following projection on the section

$$\mathcal{O}_{\mu_1\ldots\mu_n}(x) = \frac{\partial X^{M_1}}{\partial x^{\mu_1}} \ldots \frac{\partial X^{M_n}}{\partial x^{\mu_n}} \mathcal{O}_{M_1\ldots M_n}(X)\Big|_{X(x)} \,. \tag{2.28}$$

Notice that tracelessness of $\mathcal{O}_{\mu\ldots\nu}$ follows simply from tracelessness of $\mathcal{O}_{M\ldots N}$.

Finally, since it is much easier to deal with polynomials than with tensors, let us pick a reference polarization vector z for each symmetric traceless field, and encode the operators by symmetric polynomials [21]

$$\mathcal{O}(z) = \mathcal{O}_{\mu_1\ldots\mu_n} z^{\mu_1} \ldots z^{\mu_n}\big|_{z^2=0} \,. \tag{2.29}$$

To understand why we should restrict to the $z^2 = 0$ surface, consider two polynomials which are equal up to terms depending only on z^2. That means that the respective tensors are related up to $\delta_{\mu_i\mu_j}$ terms, but that creates no ambiguity as the original tensor is obtained by the action of a symmetric traceless projector, for which such terms vanish. The action of the projector is obtained rather easily by applying the following differential operator on the polynomial,

$$D_\mu = \left(\frac{d}{2} - 1 + z \cdot \frac{\partial}{\partial z}\right) \frac{\partial}{\partial z^\mu} - \frac{1}{2} z_\mu \frac{\partial^2}{\partial z^2} \,. \tag{2.30}$$

Going back to embedding space, the symmetric and traceless tensors are encoded by the following polynomials

$$\mathcal{O}(X, Z) = \mathcal{O}_{M_1 \dots M_n} Z^{M_1} \dots Z^{M_n} \big|_{Z^2=0, Z\cdot X=0} \,. \tag{2.31}$$

Just like in (2.29), the resctriction to $Z^2 = 0$ selects a representative for the class of polynomials which are equal after application of a symmetric traceless tensor. Meanwhile, the condition $Z \cdot X = 0$ selects a representative in the class of polynomials which are equal up to the gauge redundancy of (2.27), while the transversality condition (2.26) takes the form

$$X \cdot \frac{\partial}{\partial Z} \mathcal{O}(X, Z) = 0 \,. \tag{2.32}$$

After we find the relevant polynomials in embedding space, we just need to use the following prescription in order to go back to the physical space

$$\mathcal{O}(x, z) = \mathcal{O}(X(x), Z(x)) \,. \tag{2.33}$$

The embedding space polarization is related to the physical polarization in the following manner

$$Z(x) = (0, 2x \cdot z, z^\mu) \,. \tag{2.34}$$

At last, we can enumerate the tensor structures allowed by conformal symmetry in three-point functions. It can be shown that all allowed polynomials are [21]

$$\begin{aligned} V_{i,jk} &= \frac{(Z_i \cdot X_j)(X_k \cdot X_i) - (Z_i \cdot X_k)(X_j \cdot X_i)}{X_j \cdot X_k} \,, \\ H_{ij} &= (Z_i \cdot Z_j)(X_i \cdot X_j) - (X_i \cdot Z_j)(X_j \cdot Z_i) \,. \end{aligned} \tag{2.35}$$

We conclude that there are six building blocks, since $V_{i,jk} = -V_{i,kj}$ and $H_{ij} = H_{ji}$, with $i \neq j, k$. Three-point functions of operators in symmetric traceless representations are given by a sum over all possible polynomials one can form with those six structures

$$\langle \mathcal{O}_1 \mathcal{O}_2 \mathcal{O}_3 \rangle = \frac{\sum_{n_{ij}, m_i} C_{123}(\{n_{ij}\}, m_i) \prod_i V_i^{m_i} \prod_{i<j} H_{ij}^{n_{ij}}}{(X_1 \cdot X_2)^{\frac{\tau_1+\tau_2-\tau_3}{2}} (X_1 \cdot X_3)^{\frac{\tau_1+\tau_3-\tau_2}{2}} (X_2 \cdot X_3)^{\frac{\tau_2+\tau_3-\tau_1}{2}}}, \tag{2.36}$$

where $\tau_i = \Delta_i + S_i$ and S_i is the spin of the operator $\mathcal{O}_i$. The sum is over all n_{ij} and m_i that satisfy

$$m_i + \sum_{j \neq i} n_{ij} = S_i\,. \tag{2.37}$$

2.2 $\mathcal{N} = 4$ SYM

Maximally super-symmetric Yang-Mills theories were originally introduced by dimensionally reducing $\mathcal{N} = 1$ Yang-Mills in ten dimensions [22]. In this section we will consider the four dimensional theory with $\mathcal{N} = 4$ supersymmetry, which preserves all sixteen Poincaré supersymmetries and has in addition sixteen charges more due to conformal invariance . One of the reasons this theory has been so thoroughly studied is the fact that it also describes a stack of N D3-branes in string theory, which was later discovered to be equivalent to the propagation of type IIB closed strings in an $AdS_5 \times S^5$ background [5]. Furthermore, the theory seems to be invariant under an infinite tower of hidden symmetries, which make it integrable.

2.2.1 Field Content

The fields of the theory are the gauge bosons A_μ, six real scalars ϕ_{IJ}, four chiral fermions ψ^I_α and four anti-chiral fermions $\bar{\psi}_{\dot{\alpha} I}$. The fields are all massless and transform in the adjoint of the $SU(N)$ gauge group. For a gauge transformation U the transformation rules are

$$\begin{aligned} \phi_{IJ} &\to U \phi_{IJ} U^\dagger\,, & A_\mu &\to U A_\mu U^\dagger - i U \partial_\mu U^\dagger\,, \\ \psi^I_\alpha &\to U \psi^I_\alpha U^\dagger\,, & \psi_{\dot{\alpha} I} &\to U \psi_{\dot{\alpha} I} U^\dagger\,. \end{aligned} \tag{2.38}$$

The scalar fields also transform in a six-dimensional representation of the $SU(4)$ R-symmetry group, while the chiral and anti-chiral fermions transform in the fundamental and anti-fundamental representations respectively.

The action of the theory is

$$S = \frac{1}{g^2_{\mathrm{YM}}} \int \mathrm{d}^4 x (\mathcal{L}_k - V)\,, \tag{2.39}$$

where the kinetic terms for the gauge field, scalars and fermions are given by

$$\mathcal{L}_k = \mathrm{Tr}\left[-\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \frac{1}{4}(D_\mu\phi_{IJ})(D^\mu\phi^{IJ}) - i\bar{\psi}_{\dot{\alpha}I}\not{D}^{\dot{\alpha}\alpha}\psi^I_\alpha\right], \quad (2.40)$$

and the interaction terms are

$$V = \mathrm{Tr}\left[\psi^{\alpha I}[\psi^J_\alpha, \phi_{IJ}] + \bar{\psi}^{\dot{\alpha}}_I[\bar{\psi}_{\dot{\alpha}J}, \phi^{IJ}] - \frac{1}{4}[\phi_{IJ}, \phi_{KL}]^2\right]. \quad (2.41)$$

The action of the covariant derivative on a field χ is the usual

$$D_\mu\chi = \partial_\mu\chi - i[A_\mu, \chi]. \quad (2.42)$$

It is useful to relabel the scalar fields as three complex scalars

$$\begin{aligned} Z &= \phi_{12}, & X &= \phi_{13}, & Y &= \phi_{14}, \\ \bar{Z} &= \phi_{34}, & \bar{X} &= \phi_{42}, & \bar{Y} &= \phi_{23}. \end{aligned} \quad (2.43)$$

The supersymmetry of the theory relates all couplings to g_{YM}, so it is enough to study its behaviour under the renormalization group flow of g_{YM}. One can see that at one loop the number of scalars and fermions conspires such that one obtains a vanishing β-function. One can also see that conformal symmetry is preserved at the quantum level, by realizing that the Lagrangian belongs to a supermultiplet of protected operators. That means its dimension stays constant, which implies that the Yang-Mills coupling is invariant under rescaling. Said in another way, the β-function must vanish at all loop orders and we see that the powerful constraints imposed by supersymmetry make sure that the theory remains conformal at the quantum level [23].

2.2.2 Superconformal Algebra

As a conformal field theory, $\mathcal{N} = 4$ SYM is invariant under the $SO(2,4)$ conformal group in four dimensions. The Lorentz part of the algebra is $\mathfrak{so}(1,3) \cong \mathfrak{su}(2) \times \mathfrak{su}(2)$, so we can write the Lorentz boosts as two sets of traceless generators

$$L_\alpha^{\ \beta}, \qquad \dot{L}_{\dot{\alpha}}^{\ \dot{\beta}}, \quad (2.44)$$

corresponding to the two independent $\mathfrak{su}(2)$ algebras. The conformal algebra is comprised by the dilatation operator D, translations P_μ and special conformal generators K_μ, which we rewrite in the following way

$$P_{\alpha\dot{\alpha}} = \gamma^\mu_{\alpha\dot{\alpha}}P_\mu, \qquad K^{\dot{\alpha}\alpha} = \gamma_\mu^{\dot{\alpha}\alpha}K^\mu. \quad (2.45)$$

with γ_μ the off-diagonal blocks of the gamma matrices in four dimensions. The bosonic algebra is complemented by the global R-symmetry transformations $SO(6) \cong SU(4)$, which are generated by fifteen operators $R_I^{\ J}$, with $R_I^{\ I} = 0$.

Finally, $\mathcal{N} = 4$ SYM is a maximally superconformal gauge theory, so there are also thirty two fermionic generators. Just as P_μ and K_μ are raising and lowering operators in the conformal algebra, we can also separate the fermionic generators in two sets. By looking at the commutation relations, we realize the existence of sixteen generators which raise the dimension of an operator by 1/2, which we call supercharges

$$Q_{\alpha I}\,, \qquad \dot{Q}^I_{\dot{\alpha}}\,, \tag{2.46}$$

as well as sixteen superconformal charges wich lower the dimension of operators by 1/2

$$S^I_\alpha\,, \qquad \dot{S}_{\dot{\alpha} I}\,. \tag{2.47}$$

Just like in conformal fields theories we define conformal primaries as the lowest weight states. Here we say an operator $\mathcal{O}$ is a superconformal primary if it is annihilated by all superconformal charges

$$[S^I_\alpha, \mathcal{O}] = [\dot{S}_{\dot{\alpha} I}, \mathcal{O}] = 0\,. \tag{2.48}$$

Both the spinor indices α and $\dot{\alpha}$ as well as the R-symmetry indices are the same that label the Weyl fermions of the theory, with $\dot{Q}$ and S transforming in the fundamental of $SU(4)$, while Q and $\dot{S}$ transform in the anti-fundamental representation. It can be shown from the commutation relations of the superalgebra that a superconformal primary is also a conformal primary. Finally, by looking at the commutation relations for all the generators mentioned above, it is possible to recognize they form the $\mathfrak{psu}(2,2|4)$ superalgebra.

2.2.3 Representations

Primary operators and their descendants form irreducible representations of the algebra, which are infinite-dimensional due to its non-compact nature. In general we organize representations into modules which are closed under the action of the bosonic algebra, $\mathfrak{su}(2|2) \times \mathfrak{su}(4)$, and characterized by the charges of the bottom component under the Cartan generators. We can relate the modules of the representation by the action of $Q_{\alpha I}$ and $\dot{Q}^I_{\dot{\alpha}}$, which means that a typical representation has 2^{16} such modules.

However, whenever the lowest weight is also annihilated by some of the supercharges, the number of modules is smaller and we say we have a short representation. An important case we need to consider in detail is the half-BPS representation $\mathcal{V}$ with lowest weight Z, which is annihilated by half of the supercharges

$$Q_{\alpha 1}\,, \qquad Q_{\alpha 2}\,, \qquad \dot{Q}^3_{\dot{\alpha}}\,, \qquad \dot{Q}^4_{\dot{\alpha}}\,. \tag{2.49}$$

In the decomposition of the tensor product $\mathcal{V}^{\otimes L}$ there is a symmetric representation, whose lowest weight is given by the following single-trace operator

$$\mathrm{Tr}[Z^L]\,. \tag{2.50}$$

These operators are still annihilated by half the supercharges, and we call them chiral primaries. It can be shown from the commutation relations of the algebra that their scaling dimension is related to the Dynkin label of its $SU(4)$ representation $[0,J,0]$

$$\Delta = J\,. \tag{2.51}$$

Since the R-charge must be an integer, then the scaling dimension cannot vary continuously, which means that the representation is protected from quantum corrections. If we set $J = 2$, we obtain a very special supermultiplet as it contains both the Lagrangian of the theory as well as the stress-energy tensor and other conserved currents. The fact that the Lagrangian is the top component of this protected multiplet implies that the theory is conformal at the quantum level.

2.2.4 Planar Limit and AdS/CFT

If we look at the Feynman diagrams for correlation functions, it is natural to organize them by powers of $1/N$, which is equivalent to grouping them by their topology. It is then natural to take the planar limit of large N, where processes are dominated by planar diagrams. By sending the Yang-Mills coupling to a very small value, we obtain a new expansion parameter that is fixed in the planar limit, the 't Hooft coupling [2]

$$\lambda = g_{YM}^2 N\,. \tag{2.52}$$

$\mathcal{N} = 4$ SYM is the world-volume low energy theory of a stack of D3-branes, which is equivalent to the propagation of type IIB closed strings in an $AdS_5 \times S^5$ background. The parameters of the two sides of this duality relate in the following way

$$g_s = \frac{\lambda}{N}\,, \qquad \frac{R^2}{\alpha'} = \sqrt{\lambda}\,. \tag{2.53}$$

where R is the radius of both AdS and the five-sphere. In the planar limit the string coupling goes to zero, so we eliminate processes where strings split or join, which means we restrict to the first term in the genus expansion of string theory. It is also very important to note that this is a weak-strong duality, as the weakly coupled regime of $\mathcal{N} = 4$ SYM, with small λ, corresponds to strings with small tension, while at strong-coupling the string tension is very large and the string theory becomes classical. On one hand this kind of duality is difficult to check since one needs to perform calculations to all orders in perturbation theory on one side of the duality if one hopes to obtain a match. On the other hand, it becomes extremely powerful since it teaches us both about strongly coupled gauge theories and quantum gravity.

It is possible to formulate superstring theory in such a background by writing it as sigma-model on the following coset

$$\frac{PSU(2,2|4)}{SO(1,4)\times SO(5)}\,. \tag{2.54}$$

The bosonic part of $PSU(2,2|4)$ gives the isometries of $AdS_5 \times S^5$, while $SO(1,4) \times SO(5)$ corresponds to the set of transformations that leaves a point invariant. The algebra $\mathfrak{psu}(2,2|4)$ has a $\mathbb{Z}_4$ automorphism, which introduces a grading of the algebra currents

$$j = g^{-1}dg = j^{(0)} + j^{(1)} + j^{(2)} + j^{(3)}\,, \tag{2.55}$$

where g is a group element of $PSU(2,2|4)$. The superstring action is then written as [24]

$$S = \frac{\sqrt{\lambda}}{4\pi}\int \mathrm{STr}\left[\sqrt{-h}h^{\alpha\beta}j^{(2)}_{\alpha}j^{(2)}_{\beta} - \epsilon^{\alpha\beta}j^{(1)}_{\alpha}j^{(3)}_{\beta} + \epsilon^{\alpha\beta}\Lambda_{\alpha}j^{(2)}_{\beta}\right]\,, \tag{2.56}$$

where the last term in the action is a Lagrange multiplier that ensures supertracelessness of $j^{(2)}$.

2.2.5 From Local Operators to Spin-chains

In the context of this thesis we will consider only correlation functions of local observables. In order for these local observables to be gauge invariant, they must be constructed from traces which contract the gauge indices. The fields inside the trace can be any scalar, fermion or even field strength $F_{\mu\nu}$, and they each can have any number of covariant derivatives D_μ. A local observable is an operator composed of any number of such traces, but we can restrict our attention to single-trace operators since mixing with multi-trace operators in suppressed in the planar limit.

A useful way to study such class of operators is by thinking of them as spin-chains. The idea is that a trace with L fields can be mapped into a tensor product of L Hilbert spaces [10]

$$\mathcal{V}_1 \otimes \ldots \otimes \mathcal{V}_L\,, \tag{2.57}$$

where on each site $\mathcal{V}$ is the fundamental representation of $PSU(2,2|4)$. When considering the map between single-trace operators and spin-chains, it is important to remember that the cyclicity of the trace results in a shift symmetry of the chain. In the computation of the spectrum, we must renormalize the operators and then diagonalize the resulting mixing matrix, which corresponds to the spin-chain Hamiltonian. The eigenstates of the spin-chain then correspond to operators with definite scaling under dilatations, and their eigenvalues are the anomalous dimensions. Since the interaction terms in the Lagrangian are at most quartic, the one-loop dilatation operator is given by a spin-chain Hamiltonian with nearest neighbour interactions. However, as we go to higher loops we obtain a long-range spin-chain.

The ground state of the Hamiltonian corresponds to the protected operators

$$\mathcal{O}_{gs} = \mathrm{Tr}[Z^L]\,, \tag{2.58}$$

which break the $PSU(2,2|4)$ symmetry to $SU(2|2)_L \otimes SU(2|2)_R$. Two of the bosonic $\mathfrak{su}(2)$ factors are the Lorentz generators $L_\alpha^{\ \beta}$ and $\dot{L}_{\dot\alpha}^{\ \dot\beta}$. Meanwhile, for $Z = \phi_{12}$ it is easy to see that the only R-symmetry generators that annihilate the vacuum are the ones corresponding to transformations in the (12) or (34) R-symmetry planes. That means that the R-symmetry algebra breaks to $\mathfrak{su}(2) \times \mathfrak{su}(2) \times \mathfrak{u}(1)$ with generators

$$R_a^{\ b}\,, \qquad \dot{R}_{\dot a}^{\ \dot b}\,, \qquad J = \frac{1}{2}(R_1^{\ 1} + R_2^{\ 2} - R_3^{\ 3} - R_4^{\ 4})\,, \tag{2.59}$$

where we introduce new indices a and $\dot a$, which correspond to the original R-symmetry indices $I = 1,2$ and $I = 3,4$, respectively. In that language, the supercharges that annihilate the ground state (2.49) can be written as $Q_{\alpha a}$ and $\dot{Q}^{\dot a}_{\dot\alpha}$. Finally the residual algebra closes with the superconformal generators S^a_α and $\dot{S}_{\dot\alpha\dot a}$.

It turns out that not all fields correpond to elementary excitations of the spin-chain. To find out what are the elementary excitations we have to look at the charges

$$E = \Delta - J\,, \tag{2.60}$$

under the generator of $\mathfrak{u}(1)_E$, which is a central element common to both factors of the residual symmetry $\mathfrak{su}(2|2)_L \otimes \mathfrak{su}(2|2)_R$. We can see that the elementary excitations form an $(2|2)_L \otimes (2|2)_R$ representation with sixteen degrees of freedom $\Phi_{A\dot A}$, which have the following components

$$\begin{aligned} \Phi_{a\dot a} &= \phi_{a\dot a}\,, & \Phi_{a\dot\alpha} &= \psi_{\dot\alpha,a}\,, \\ \Phi_{\alpha\dot\alpha} &= D_{\alpha\dot\alpha}\,, & \Phi_{\alpha\dot a} &= \psi^{\dot a}_{\alpha}\,. \end{aligned} \tag{2.61}$$

Other fields that can appear in the single-trace operators show up in the spin-chain as composite excitations. For example, $\bar Z$ corresponds to a double excitation

$$\bar Z \sim \epsilon^{ab}\epsilon^{\dot a\dot b}\Phi_{a\dot a}\Phi_{b\dot b} \sim X\bar X + Y\bar Y\,. \tag{2.62}$$

The sites of the spin-chain transform in general in the fundamental representation of $\mathfrak{psu}(2,2|4)$. It is however possible to consider subsets of fields which are closed under the action of the dilatation operator, which we call closed sectors. There are several closed sectors we can consider at one-loop, from the $SU(2)$ Heisenberg spin-chain with excitation X, to the mixing between all the scalars in the $SO(6)$ sector, or the $SU(1|1)$ sector with a fermionic excitation. However, at higher loops only a few of the closed sectors survive, with the rank one cases being $SU(2)$ and $SL(2)$. The larger sector $SU(2|3)$ is also closed to all orders in perturbation theory, and corresponds to a spin-chain with three bosons and two fermions. Note that the mixing between fermions and bosons leads to a variation in the number of fields inside the trace, so we say that the spin-chain of $\mathcal{N}=4$ SYM is dynamical.

2.3 ABJM

Another example of an integrable CFT which has played a very important role in the study of AdS/CFT is ABJM theory. It originated from the discovery of an $\mathcal{N}=8$ superconformal theory in three dimensions describing the world-volume theory of two interacting M2-branes [25–27]. The naive Chern-Simons term of the action is not parity invariant, as one would expect from eleven-dimensional supergravity, but this problem is solved by taking the gauge group to be a product $U(N)\times\hat{U}(N)$ with two Chern-Simons terms of levels k and $-k$. ABJM generalizes the construction to more than two branes, but preserves only $\mathcal{N}=6$ supersymmetry, except for Chern-Simons level $k=1,2$ when there is an enhancement to $\mathcal{N}=8$ [28]. If the two factors in the gauge group are different, then we end up with ABJ theory [29], where the presence of integrability is still an open problem. From the open/closed string duality, the stack of branes in the near-horizon limit is equivalent to the propagation of closed strings in the orbifold background $AdS_4\times S^7/\mathbb{Z}_k$. Taking k large the circle effectively shrinks and we can instead describe the system with type IIA strings in $AdS_4\times\mathbb{CP}^3$.

2.3.1 Field Content

The gauge fields corresponding to the two Chern-Simons terms are A^μ and $\hat{A}^\mu$, which transform in the adjoint of the respective gauge groups. There are also four complex scalars Y^I and four Dirac fermions $\psi_{\alpha I}$ which transform in the bifundamental representation $(N,\bar{N})$ of the gauge group, while their complex conjugates transform in the anti-bifundamental representation $(\bar{N},N)$. A gauge transformation for such a gauge group is given by $(U,\hat{U})$, under which the fields transform as

$$\begin{aligned} Y^I &\to UY^I\hat{U}^\dagger\,, & A_\mu &\to UA_\mu U^\dagger - iU\partial_\mu U^\dagger\,, \\ \psi_{\alpha I} &\to U\psi_{\alpha I}\hat{U}^\dagger\,, & \hat{A}_\mu &\to \hat{U}\hat{A}_\mu\hat{U}^\dagger - i\hat{U}\partial_\mu\hat{U}^\dagger\,. \end{aligned} \tag{2.63}$$

The scalars Y^I transform in the fundamental representation of the $SU(4)$ R-symmetry, while the fermions transform in the anti-fundamental representation, and vice-versa for their complex conjugates.

The action of the theory is

$$S = \frac{k}{4\pi}\int \mathrm{d}^3x(\mathcal{L}_k - V_{\text{bos}} - V_{\text{ferm}})\,, \tag{2.64}$$

where $\mathcal{L}_k$ has the kinetic terms for the gauge fields, scalars and fermions

$$\begin{aligned} \mathcal{L}_k = \operatorname{Tr}\Bigg[&\epsilon_{\mu\nu\lambda}\left(A_\mu\partial_n uA_\lambda + \frac{2i}{3}A_\mu A_\nu A_\lambda\right) - (D_\mu Y^I)(D_\mu Y_I^\dagger) \\ &-\epsilon_{\mu\nu\lambda}\left(\hat{A}_\mu\partial_n u\hat{A}_\lambda + \frac{2i}{3}\hat{A}_\mu\hat{A}_\nu\hat{A}_\lambda\right) - i\psi^{I\dagger}\not{D}\psi_I\Bigg]\,, \end{aligned} \tag{2.65}$$

and the interaction terms are

$$\begin{aligned} V_{\text{ferm}} = \frac{i}{2}\operatorname{Tr}\Big[&Y_A^\dagger Y^A\psi^{\dagger B}\psi_B - Y^A Y_A^\dagger\psi_B\psi^{\dagger B} + \epsilon_{ABCD}Y^A\psi^{\dagger B}Y^C\psi^{\dagger D} \\ &-\epsilon^{ABCD}Y_A^\dagger\psi_B Y_C^\dagger\psi_D + 2Y^A Y_B^\dagger\psi_A\psi^{\dagger B} - 2Y_A^\dagger Y^B\psi^{\dagger A}\psi_B\Big]\,, \\ V_{\text{bos}} = -\frac{1}{12}\operatorname{Tr}\Big[&Y^I Y_I^\dagger Y^J Y_J^\dagger Y^K Y_K^\dagger + Y_I^\dagger Y^I Y_J^\dagger Y^J Y_K^\dagger Y^K \\ &+4Y^I Y_J^\dagger Y^K Y_I^\dagger Y^J Y_K^\dagger - 6Y^I Y_J^\dagger Y^J Y_I^\dagger Y^K Y_K^\dagger\Big]\,. \end{aligned} \tag{2.66}$$

The action of the covariant derivative on a bifundamental field χ is

$$D_\mu\chi = \partial_\mu\chi + iA_\mu\chi - i\chi\hat{A}_\mu\,. \tag{2.67}$$

As is well known, the Chern-Simons term is not gauge invariant, but changes by an integer multiple of $2\pi k$ under a gauge transformation. For the theory to be gauge invariant, the path integral must be gauge invariant which leads to a quantization condition for k

$$k \in \mathbb{Z}\,. \tag{2.68}$$

This has strong implications on the quantum behaviour of the theory. ABJM is classically conformal, but as the energy scale varies, the couplings could in principle vary as well. However, since k must be an integer, it cannot vary continuously with the energy, which means that the coupling is protected from renormalization. Since all other couplings are related to k due to supersymmetry, then the theory remains conformal at the quantum level.

2.3.2 Superconformal Algebra

As a three-dimensional conformal field theory, ABJM is invariant under the conformal group $SO(2,3) \cong Sp(4,\mathbb{R})$. The Lorentz algebra is $\mathfrak{so}(1,2) \cong \mathfrak{sl}(2,\mathbb{R})$, so we write it with the usual traceless generators $L_\alpha^{\ \beta}$. The rest of the algebra is composed by D, translations P_μ and special conformal generators K_μ, which we rewrite in a more convenient way

$$P_{\alpha\beta} = (C\Gamma^\mu)_{\alpha\beta} P_\mu \,, \qquad K^{\alpha\beta} = (C\Gamma^\mu)^{\alpha\beta} K_\mu \,, \tag{2.69}$$

with C the charge conjugation matrix and Γ_μ the Dirac matrices in three dimensions. The rest of the bosonic algebra is given by the generators of the global $SU(4)$ R-symmetry, which we write with $R_I^{\ J}$ as in $\mathcal{N}=4$ SYM.

A natural choice for the Cartan generators of $\mathfrak{su}(4)$ is

$$J = \frac{1}{2}(R_2^{\ 2} - R_3^{\ 3}) \,, \quad \dot{J} = \frac{1}{2}(R_1^{\ 1} - R_4^{\ 4}) \,, \quad J_3 = R_1^{\ 1} - R_2^{\ 2} - R_3^{\ 3} + R_4^{\ 4} \,. \tag{2.70}$$

This is convenient as later on the R-symmetry algebra will be broken into an $\mathfrak{su}(2)_G \times \mathfrak{su}(2)_{\dot{G}} \times \mathfrak{u}(1)$ subgroup, with J and $\dot{J}$ the Cartan charges of the $\mathfrak{su}(2)$ factors formed by $R_a^{\ b}$ and $R_{\dot{a}}^{\ \dot{b}}$, where a and $\dot{a}$ correspond to $I=2,3$ and $I=1,4$ respectively. Meanwhile, J_3 will turn out to be useful for distinguishing the two types of excitations in the spin-chain picture. The relation of these charges with the Dynkin labels $[p_1,q,p_2]$ is

$$J = \frac{p_1+q+p_2}{2} \,, \qquad \dot{J} = \frac{q}{2} \,, \qquad J_3 = p_1 - p_2 \,. \tag{2.71}$$

Finally, ABJM is also invariant under the action of twenty four supersymmetries. Once again, half of them are are raising operators, the supercharges $Q_{\alpha,IJ}$, while the superconformal generators $S^{\alpha,IJ}$ correspond to lowering operators. Since a superconformal primary $\mathcal{O}$ is a lowest weight of the representation, then it must satisfy

$$[S^{\alpha,IJ}, \mathcal{O}] = 0 \,. \tag{2.72}$$

The spinor indices are the same that label the Dirac fermions, but the fermionic generators have antisymmetric R-charge indices, so they transform in a six dimensional representation of the R-symmetry group. If we put all the commutation relations of these generators together, we obtain the superalgebra $\mathfrak{osp}(6|4,\mathbb{R})$.

2.3.3 Representations

The representations of the superalgebra are infinte-dimensional and we organize them in modules which are closed under the bosonic subalgebra

$\mathfrak{so}(2,3) \times \mathfrak{su}(4)$. Each module is characterized by the Cartan charges of the bottom component: scaling dimension Δ, Lorentz spin S, and R-charges J, $\dot{J}$ and J_3 . The modules can be related to each other by the action of the supercharges $Q_{\alpha,IJ}$, which means that a typical representation will have 2^{12} such modules.

Let us now look at some of the most relevant short representations. We have the representation $\mathcal{V}$ with lowest weight Y^4 and $\bar{\mathcal{V}}$ whose lowest weight is the scalar $Y_1^\dagger$. They are both half-BPS representations as they are annihilated by half the supercharges

$$\begin{aligned} \mathcal{V}: &\qquad Q_{\alpha,12}\,, \qquad Q_{\alpha,13}\,, \qquad Q_{\alpha,23}\,, \\ \bar{\mathcal{V}}: &\qquad Q_{\alpha,12}\,, \qquad Q_{\alpha,13}\,, \qquad Q_{\alpha,14}\,. \end{aligned} \tag{2.73}$$

If we combine these representations together in the tensor product $(\mathcal{V} \otimes \bar{\mathcal{V}})^{\otimes L}$, there is an L-symmetric representation in its decomposition whose lowest weight corresponds to the following single-trace operator

$$\mathrm{Tr}[(Y^4 Y_1^\dagger)^L]\,. \tag{2.74}$$

These are 1/3-BPS operators since there are only four supercharges that annihilate both $\mathcal{V}$ and $\bar{\mathcal{V}}$ at the same time

$$Q_{\alpha,12}\,, \qquad Q_{\alpha,13}\,. \tag{2.75}$$

These operators are commonly known as chiral primaries, and their $\mathfrak{su}(4)$ Dynkin labels are $[\dot{J}, 0, \dot{J}]$, with $\dot{J}$ the R-charge under $\mathfrak{su}(2)_{\dot{G}}$. By looking at the superconformal algebra we can derive a relation between the scaling dimension and the R-charge of the chiral operators

$$\Delta = \dot{J} = L\,, \tag{2.76}$$

which protects these short multiplets from quantum corrections to their scaling dimensions.

2.3.4 Planar Limit and AdS/CFT

Also in ABJM we can organize Feynman diagrams by powers of $1/N$, so in the planar limit correlation functions are dominated by planar diagrams. Since k appears as an overall factor in front of the action, we can then see $1/k$ as a coupling constant. In the planar limit the expansion parameter is the 't Hooft coupling [2]

$$\lambda = \frac{N}{k}\,, \tag{2.77}$$

which remains fixed if we take k to also be very large. As menioned above, ABJM has an $AdS_4 \times \mathbb{CP}^3$ dual in type IIA string theory, so we can relate its parameters with the string tension and string coupling

$$g_s = \frac{\lambda^{5/4}}{N}\,, \qquad\qquad \frac{R^2}{\alpha'} = 4\pi\sqrt{2\lambda}\,, \tag{2.78}$$

with R the radius of the compact manifold $\mathbb{CP}^3$, which is twice the radius of AdS. Just like for $\mathcal{N}=4$ SYM, we have once again a weak-strong duality. As we take λ to be small, where ABJM is weakly coupled, the string background becomes highly curved and the string worldsheet is strongly coupled. When λ is large, we have strongly-coupled ABJM and semi-classical strings. Note that in the planar limit strings become free, so it is enough to take the first term in the genus expansion, with the topology of the sphere.

The superstring action in that background can be written as a sigma model on the supercoset

$$\frac{OSp(6|4)}{SO(1,3)\times U(3)}\,. \tag{2.79}$$

The algebra $\mathfrak{osp}(6|4,\mathbb{R})$ has a $\mathbb{Z}_4$ automorphism which introduces a natural grading for the algebra currents

$$j = g^{-1}dg = j^{(0)} + j^{(1)} + j^{(2)} + j^{(3)}\,, \tag{2.80}$$

where g is an element of the group $OSp(6|4,\mathbb{R})$. These currents are then used to write the action [30, 31]

$$S = -\sqrt{2\lambda}\int \mathrm{d}^2 z \mathrm{STr}\left[\sqrt{-h}h^{\alpha\beta}j^{(2)}_\alpha j^{(2)}_\beta + \epsilon^{\alpha\beta}j^{(1)}_\alpha j^{(3)}_\beta\right]\,. \tag{2.81}$$

Note that in ABJ we have two 't Hooft couplings λ and $\hat{\lambda}$ related to the two limits for the different ranks of the gauge group factors. The theory is unitary when $|N-\hat{N}|\le k$ which in the planar limit corresponds to $|\lambda-\hat{\lambda}|\le 1$.

2.3.5 Alternating Spin-chains

In our study of ABJM we will only consider correlation functions of local observables. In order to be gauge invariant, they must be made of a product of traces which contract the gauge indices. Given the structure of the gauge group, the fields inside the trace must alternate between the bifundamental and the anti-bifundamental representations.

As with $\mathcal{N}=4$ SYM, it is useful to think of single-trace operators as spin-chain states, which are also alternating due to the product nature of the gauge group and the representations of the matter fields. In order to solve the mixing problem for the spectrum, we must diagonalize the spin-chain Hamiltonian [11]. It turns out that in three dimensions only integrals with even number of loops have ultraviolet divergences, so the expansion of ABJM is in powers of λ^2. The two-loop spin-chain Hamiltonian has next-to-nearest neighbour interactions which include the six-boson interaction, and the range of interaction increases as we consider higher loops.

The ground state of the ABJM spin-chain is the protected operator introduced earlier

$$\mathcal{O}_{gs} = \mathrm{Tr}[(Y^4 Y_1^\dagger)^L]\,, \tag{2.82}$$

which breaks the symmetry to $\mathfrak{su}(2|2)\times\mathfrak{u}(1)$ [32]. One of the $\mathfrak{su}(2)$ factors corresponds to the Lorentz generators, while there is an $\mathfrak{su}(2)\times\mathfrak{u}(1)$ parametrized by $(R_a^{\ b}, R)$ which is preserved in the breaking of the R-symmetry to $\mathfrak{su}(2)_G \times \mathfrak{su}(2)_{\dot{G}} \times \mathfrak{u}(1)$

$$R_I^{\ J} \longrightarrow R_a^{\ b} + R_{\dot{a}}^{\ \dot{b}} + R_a^{\ \dot{b}} + R_{\dot{a}}^{\ b} + R\,. \tag{2.83}$$

The indices a and $\dot{a}$ correspond to the original R-symmetry indices $I = 2,3$ and $I = 1,4$ respectively. Under the breaking of the R-symmetry the fermionic generators decompose as

$$\begin{aligned} Q_{\alpha,IJ} &\longrightarrow Q_\alpha^{a\dot{a}} + Q_\alpha + \bar{Q}_\alpha\,, \\ S^{\alpha,IJ} &\longrightarrow S^\alpha_{a\dot{a}} + S^\alpha + \bar{S}^\alpha\,. \end{aligned} \tag{2.84}$$

In this language, the supercharges that annihilate the ground state (2.75) are written as $Q_\alpha^{a\dot{1}}$ and finally the algebra closes with the superconformal generators $S^\alpha_{a\dot{1}}$.

Other single-trace operators can be built by replacing some of the fields in the trace, but not all correspond to elementary excitations in the spin-chain picture. The elementary excitations can be found by looking at the charges under the bosonic generator $D - \dot{J}$ of $\mathfrak{u}(1)_E$, where $\dot{J}$ is the Cartan generator of $\mathfrak{su}(2)_{\dot{G}}$. They transform as $(2|2)_A \oplus (2|2)_B$, where the two fundamental representations of $\mathfrak{su}(2|2)$ have opposite charges J_3 under $\mathfrak{u}(1)$. We denote the A and B fundamental representations by Φ_A and $\dot{\Phi}_{\dot{A}}$ repectively, and they correspond to the fields

$$\begin{aligned} \Phi_A =& (Y^3, -Y^2|\psi_{\alpha 1})\,, \\ \dot{\Phi}_{\dot{A}} =& (Y_2^\dagger, Y_3^\dagger|\psi_\alpha^{\dagger 4})\,. \end{aligned} \tag{2.85}$$

If the single-trace operator has excitations other than these fundamental ones, then they correspond to composite excitations. For example, the scalars Y^1, $Y_4^\dagger$ and the derivatives D_μ are made up of the following fundamental excitations [33]

$$\begin{aligned} Y^1 Y_1^\dagger &\sim Y^2 Y_2^\dagger + Y^3 Y_3^\dagger\,, \\ Y^4 Y_4^\dagger &\sim Y^2 Y_2^\dagger - Y^3 Y_3^\dagger\,, \\ D_\mu Y^4 Y_1^\dagger &\sim \psi_{\alpha 1} (C\Gamma_\mu)^{\alpha\beta} \psi_\beta^{\dagger 4}\,. \end{aligned} \tag{2.86}$$

In a similar manner, one can see that $\psi_{\alpha 2}$, $\psi_\alpha^{\dagger 2}$, $\psi_{\alpha 3}$ and $\psi_\alpha^{\dagger 3}$ are double-excitations while $\psi_{\alpha 4}$ and $\psi_\alpha^{\dagger 1}$ are triple excitations.

Finally, the spin-chain sites transform in a representation of $\mathfrak{osp}(6|4,\mathbb{R})$, but it is often easier to study a set of fields which is closed under the action of the spin-chain Hamiltonian. Given the choice of vacuum we have made, the simplest closed sectors are the rank one groups $SU(2)$ and $SU(1|1)$. The first is composed of a single scalar excitation on either the even or the odd sites, while the second corresponds to a single fermionic excitation. A good playground for the study of integrability in ABJM is usually the $SU(2) \times SU(2)$ sector, as it has excitations in both even and odd sites of the spin-chain while being simple enough due to the fact that the excitations are decoupled. Just like in $\mathcal{N} = 4$ SYM, one could also consider the sector of scalar fields, but this is only closed at two loops.

3. Superstring Theory

String theory originated as a proposal for a theory of hadrons, but it was discarded after the appearance of QCD. Interestingly, the presence of a spin two excitation, one of the reasons for its failure as a theory of the strong force, was crucial to turn it into a candidate for a theory of quantum gravity. In string theory the basic object is a one-dimensional string that spans a two-dimensional worldsheet in spacetime. String interactions are given by joining and splitting of strings, and the fact that they do not single out a point in spacetime makes it free of short distance divergences. Another feature that makes it so compelling is that unlike quantum field theory, each order in perturbation theory is given by a single worldsheet topology.

Even if string theory does not turn out to be the correct theory of quantum gravity, it is undeniable that it has had a huge impact in theoretical physics. One such example is the AdS/CFT duality discovered by Maldacena [5], in which string theories in AdS_{d+1} spaces were found to be dual to conformal field theories in d dimensions. This provides an excellent framework for the study of strongly coupled gauge theories and several observables like amplitudes, correlation functions and Wilson loops can now be computed to all orders in perturbation theory, at least in the planar limit.

In the context of this thesis, superstring amplitudes are important because they are dual to correlation functions of conformal field theories. In this chapter we introduce superstring theory in flat space as well as the vertex operators necessary for the study of three-point functions in $\mathcal{N} = 4$ SYM at strong coupling. In the treatment of the superstring we will mostly follow the conventions of [34, 35].

3.1 Wolrdsheet CFTs

The coordinates of the two-dimensional worldsheet spanned by the string are the proper time σ_0 and the spacial coordinate $\sigma_1 \cong \sigma_1 + 2\pi$. Switching to Euclidean signature, we can form two new variables $w = \sigma_1 - i\sigma_0$ and $\bar{w} = \sigma_1 + i\sigma_0$, which parametrize the cylinder formed by the worldsheet. It is however useful to use a conformal map $z = e^{iw}$, so that the string worldsheet is parametrized by the complex plane, where we can use the tools of complex analysis.

3.1.1 Conformal symmetry in two dimensions

We will see later that the worldsheet theory is superconformal, so let us first introduce the basic tools for conformal field theories in two dimensions. In chapter 2 we mentioned that an infinitesimal conformal transformation has at most quadratic terms. However, in two dimensions that is not true, and any holomorphic or anti-homorphic transformation

$$z \to f(z)\,, \qquad\qquad \bar{z} \to g(\bar{z})\,, \tag{3.1}$$

is conformal. The stress-energy tensor must be traceless and if the theory is also supersymmetric, then there is a gamma-traceless supercurrent G_μ

$$T^\mu_\mu = 0\,, \qquad\qquad \Gamma^\mu_{\alpha\beta} G_{\mu\beta} = 0\,. \tag{3.2}$$

In two dimensions, we can then see that there are only two non-vanishing components for each of them. Furthermore, the conservation laws show that the two non-vanishing components must be either holomorphic or anti-holomorphic.

If an operator is a primary, then under a conformal transformation it obeys

$$\phi(z,\bar{z}) \to \left(\frac{\partial f}{\partial z}\right)^h \left(\frac{\bar{\partial}\bar{f}}{\bar{\partial}\bar{z}}\right)^{\bar{h}} \phi(z,\bar{z})\,, \tag{3.3}$$

where the two weights h and $\bar{h}$ can be related to the dimension and spin of the operator

$$\Delta = h + \bar{h}\,, \qquad\qquad S = h - \bar{h}\,. \tag{3.4}$$

From the transformation rule, we can show that a Fourier series of an holomorphic operator on the cylinder becomes a Laurent expansion on the plane, shifted by the weight of the operator

$$\phi(z) = \sum_n \phi_n z^{n-h}\,. \tag{3.5}$$

In radial quantization the action of the charges on the fields is given by the countour integral

$$[Q, \phi(z)] = \oint_z \frac{\mathrm{d}w}{2\pi i} \chi(w) T(w) \phi(z)\,. \tag{3.6}$$

We can then derive the leading singularities in the OPE of the stress-energy tensor with primary fields to be

$$T(z)\phi(w) = \frac{h\phi(w)}{(z-w)^2} + \frac{\partial\phi(w)}{z-w} + \dots\,. \tag{3.7}$$

One can also show that the stress-energy tensor and the supercurrent must have the following OPEs

$$\begin{aligned}
T(z)T(w) &= \frac{c}{2(z-w)^4} + \frac{2T(w)}{(z-w)^2} + \frac{\partial T(w)}{z-w} + \dots\,, \\
T(z)G(w) &= \frac{3G(w)}{2(z-w)^2} + \frac{\partial G(w)}{z-w} + \dots\,, \\
G(z)G(w) &= \frac{2c}{3(z-w)^3} + \frac{2T(w)}{z-w} + \dots\,,
\end{aligned} \tag{3.8}$$

where c is the central charge of the CFT. These equations show that the stress-energy tensor is not a primary operator, while the supercurrent is. The mode expansions of the currents are

$$\begin{aligned}
T(z) &= \sum_n L_n z^{n-2}\,, & G(z) &= \sum_r G_r z^{r-3/2}\,, \\
\tilde{T}(\bar{z}) &= \sum_n \tilde{L}_n \bar{z}^{n-2}\,, & \tilde{G}(\bar{z}) &= \sum_r \tilde{G}_r \bar{z}^{r-3/2}\,,
\end{aligned} \tag{3.9}$$

where $n \in \mathbb{Z}$ for the stress-energy tensor, while for the supercurrent r could be either integer of half-integer. By putting together the OPE relations (3.8) and the expansions (3.9), we can derive the commutation relations of the superconformal algebra in two dimensions

$$\begin{aligned}
[L_m, L_n] &= (m-n)L_{m+n} + \frac{c}{12}m(m^2-1)\delta_{m,-n}\,, \\
[L_m, G_r] &= (\frac{m}{2} - r)G_{m+r}\,, \\
[G_r, G_s] &= 2L_{r+s} + \frac{c}{12}(4r^2-1)\delta_{r,-s}\,.
\end{aligned} \tag{3.10}$$

This algebra is called super-Virasoro, and there is a similar copy for the anti-holomorphic currents.

3.1.2 The string matter sector

The string propagates in a spacetime described by the coordinates X^M. Just like for particles the action is the worldline length, here it is natural to introduce the Nambu-Goto action, which minimizes the worldsheet surface

$$S_{\rm NG} = \frac{1}{2\pi\alpha'}\int \mathrm{d}^2\sigma \sqrt{-\det\gamma}\,, \tag{3.11}$$

where the prefactor is interpreted as the string tension and γ is the induced metric on the worldsheet

$$\gamma_{ab} = \frac{\partial X^M}{\partial\sigma^a}\frac{\partial X^N}{\partial\sigma^b}\eta_{MN}\,. \tag{3.12}$$

The square root in the action is not very compelling, so we introduce the Polyakov action, whose linearity in the target-space coordinates comes with an independent field for the worldsheet metric h_{ab}

$$S_P = \frac{1}{4\pi\alpha'} \int \mathrm{d}^2\sigma \, \sqrt{-h} h^{ab} \partial_a X^M \partial_b X^M \eta_{MN} \, . \tag{3.13}$$

The equation of motion for h_{ab} shows that it is proportional to the induced metric, which means that the equations of motion for X^M are exactly the same as the ones obtained from the Nambu-Goto action.

The Polyakov action enjoys several symmetries. On one hand there are the global Poincaré transformations of the fields X^M. But there are also local symmetries: the action is invariant under worldsheet diffeomorphisms and local rescalings (also known as Weyl transformations)

$$\begin{aligned} \sigma^a &\to \sigma^a + \xi^a(\sigma) \, , \\ h_{ab} &\to e^{\phi(\sigma)} h_{ab} \, . \end{aligned} \tag{3.14}$$

The worldsheet metric h_{ab} has three degrees of freedom, but we can use the local symmetries to select a conformal gauge for the metric

$$h_{ab} = \eta_{ab} \, . \tag{3.15}$$

By computing the variation of the action to a perturbation of the worldsheet metric we obtain the stress-energy tensor. Imposing the equations of motion on h_{ab} is then equivalent to requiring the vanishing of the stress-energy tensor.

It is also important to note that even after fixing the worldsheet metric, there is still a residual gauge symmetry. By introducing the complex coordinates z and $\bar{z}$ from the previous discussion, we can see that the residual symmetry is composed of holomorphic and anti-holomorphic transformations, so it corresponds to the group of conformal transformations in two dimensions. Unlike the higher-dimensional cases studied in the previous chapter, this group is now infinite dimensional.

If we want our string theory to have fermionic excitations in its spectrum, then we must modify the Polyakov action. There are several ways to do so, but here we will take the RNS approach to the superstring. We shall then supplement the Polyakov action with two new ingredients: the anticommuting Dirac spinor Ψ^M, which transform as a spacetime vector, and a world-sheet gravitino χ. We can also use worldsheet supersymmetry to fix the gravitino degrees of freedom, and the gauge fixed action simplifies to

$$S_{\mathrm{m}} = \frac{1}{4\pi} \int \mathrm{d}^2 z \left(\frac{2}{\alpha'} \partial X^M \bar{\partial} X_M + \psi^M \bar{\partial} \psi_M + \tilde{\psi}^M \partial \tilde{\psi}_M \right) \, , \tag{3.16}$$

with ψ and $\tilde{\psi}$ the components of the Dirac spinor in the Majorana-Weyl basis. The equations of motion for the matter fields are

$$\partial\bar{\partial}X^M = 0\,, \qquad \bar{\partial}\psi^M = 0\,, \qquad \partial\tilde{\psi}^M = 0\,, \tag{3.17}$$

which imply that the they also split into two parts: ∂X^M and ψ^M are holomorphic while $\bar{\partial}X^M$ and $\tilde{\psi}^M$ are anti-holomorphic. The first two are usually called right-moving fields while the later are left-moving.

3.1.3 The string ghost sector

The previous discussion did not take into account that we also have to perform a path integral over the matter fields and the worldsheet metric. After gauge fixing we integrate over a slice that cuts once through each equivalence class. We must determine the measure of that integration, which can be done with the Faddeev-Popov method. The inverse of the Faddeev-Popov determinant is given by

$$\frac{1}{\Delta[h]} = \int \mathcal{D}\phi\mathcal{D}\xi\mathcal{D}\mu \exp\left(\int \mathrm{d}^2 z\sqrt{|h|}\mu^{ab}(\phi h_{ab} + \nabla_a\xi_b)\right)\,. \tag{3.18}$$

We can integrate out the Weyl parameter ϕ, which forces μ^{ab} to be traceless. By substituting the parameters μ^{ab} and ξ_a by anti-commuting ghost fields b and c, we then obtain the Fadeev-Popov determinant. We should follow the same logic and fix the worldsheet supersymmetries arising in the superstring, which leads us to introduce a pair of bosonic ghosts, β and γ. Finally, in the conformal gauge these determinants can be expressed through the ghost action

$$S_{\text{gh}} = \frac{1}{2\pi}\int \mathrm{d}^2 z\,(b\bar{\partial}c + \tilde{b}\partial\tilde{c} + \beta\bar{\partial}\gamma + \tilde{\beta}\partial\tilde{\gamma})\,. \tag{3.19}$$

Just like for the matter sector, the equations of motion are

$$\begin{aligned} &\bar{\partial}b = 0\,, \qquad &\partial\tilde{b} = 0\,, \qquad &\bar{\partial}c = 0\,, \qquad &\partial\tilde{c} = 0\,, \\ &\bar{\partial}\beta = 0\,, \qquad &\partial\tilde{\beta} = 0\,, \qquad &\bar{\partial}\gamma = 0\,, \qquad &\partial\tilde{\gamma} = 0\,, \end{aligned} \tag{3.20}$$

which imply that b, c, β and γ are holomorphic while $\tilde{b}$, $\tilde{c}$, $\tilde{\beta}$ and $\tilde{\gamma}$ are anti-holomorphic.

There are two further complications that we ignored in the previous discussion. The first concerns the parameters of the metric that cannot be removed using diffeomorphisms and Weyl invariance, called metric moduli. For the sphere there are no such moduli, but the torus has a complex modulus τ since we can at best bring the metric to the form

$$ds^2 = |d\sigma^0 + \tau d\sigma^1|^2\,. \tag{3.21}$$

The other issue is related to the conformal Killing group, when there are global symmetries that are not fixed by the choice of the metric. For example in the sphere there are six conformal Killing vectors, with infinitesimal transformations labeled by three complex parameters

$$\delta z = a_0 + a_1 z + a_2 z^2 \,, \tag{3.22}$$

which exponentiate to the $PSL(2,\mathbb{C})$ group. The Riemann-Roch theorem relates the number of metric moduli μ and the number of conformal killing vectors κ with the genus of the surface

$$\mu - \kappa = 6g - 6 \,. \tag{3.23}$$

By taking these two points into account when doing the gauge fixing of the path integral one can show that each metric moduli leads to a b ghost insertion and each conformal Killing vector removes an integration of a vertex operator while inserting a c ghost. Therefore, for the calculation on the sphere we need the following correlator of c ghosts

$$\langle c_1 c_2 c_3 \tilde{c}_1 \tilde{c}_2 \tilde{c}_3 \rangle = \left| \det \begin{pmatrix} 1 & 1 & 1 \\ z_1 & z_2 & z_3 \\ z_1^2 & z_2^2 & z_3^2 \end{pmatrix} \right|^2 = |z_{12}|^2 |z_{13}|^2 |z_{23}|^2 \,, \tag{3.24}$$

which can be understood as the Jacobian of the transformation that eliminates three of the integrations.

The ghost action also possesses a $U(1)$ symmetry generated by the currents

$$j_{b,c}(z) = -b(z)c(z) \,, \qquad j_{\beta,\gamma}(z) = -\beta(z)\gamma(z) \,. \tag{3.25}$$

The c and γ fields have charge $+1$ under these transformations, while the ghosts b and β have charges -1. These currents are not world-sheet primary fields, as they have cubic singularities in the OPE with the stress-energy tensor. The coefficients of those singularities have a physical meaning as they give the background charges

$$Q_{bc} = -3 \,, \qquad Q_{\beta\gamma} = 2 \,. \tag{3.26}$$

3.1.4 Currents and Primary Fields

If we assume that the path integral of a total derivative vanishes, the following equation

$$\int \mathcal{D}\Phi \frac{\partial}{\partial \phi_i(w)} \left(\phi_j(z) e^{-S} \right) = 0 \tag{3.27}$$

can be used to obtain differential equations for the propagators. For the matter fields we then obtain the following OPE singularities

$$\begin{aligned} X^M(z)X^N(w) &= -\frac{\alpha'}{2}\eta^{MN}\log|z-w|^2\,, \\ \psi^M(z)\psi^N(w) &= \frac{\eta^{MN}}{z-w}\,, \end{aligned} \tag{3.28}$$

while for the ghosts we get

$$\begin{aligned} b(z)c(w) &= \frac{1}{z-w}\,, \qquad & b(z)b(w) = c(z)c(w) = \mathcal{O}(z-w)\,, \\ \beta(z)\gamma(w) &= -\frac{1}{z-w}\,, \qquad & \beta(z)\beta(w) = \gamma(z)\gamma(w) = \mathcal{O}(1)\,. \end{aligned} \tag{3.29}$$

The stress-energy tensor can be obtained by varying the worldsheet metric, so it must be computed before setting the conformal gauge for the metric. We have seen before that the stress-energy tensor has a holomorphic component $T_{zz} = T(z)$ and an anti-holomorphic one $T_{\bar{z}\bar{z}} = \tilde{T}(\bar{z})$. The holomorphic component can be split into two parts

$$\begin{aligned} T_{\mathrm{m}}(z) &= -\frac{1}{\alpha'}\partial X^M \partial X_M - \frac{1}{2}\psi^M\partial\psi_M\,, \\ T_{\mathrm{gh}}(z) &= -2b\partial c + c\partial b - \frac{3}{2}\beta\partial\gamma - \frac{1}{2}\gamma\partial\beta\,, \end{aligned} \tag{3.30}$$

coming from the matter and ghost sectors of the action. Meanwhile, the supercurrent is given by

$$\begin{aligned} G_{\mathrm{m}}(z) &= i\left(\frac{2}{\alpha'}\right)^{1/2}\partial X_M\psi^M\,, \\ G_{\mathrm{gh}}(z) &= c\partial\beta + \frac{3}{2}\beta\partial c - 2b\gamma\,. \end{aligned} \tag{3.31}$$

By computing the OPE of the matter and ghost fields with the stress-energy tensor as in equation (3.7), we can understand what are their transformation rules under conformal transformations. It turns out that the only field of the action that is not a primary is X^M. The conformal weights of the primary fields are

$$\begin{aligned} h(\partial X^M) &= 1\,, \qquad & h(b) &= 2\,, \qquad & h(\beta) &= \frac{3}{2}\,, \\ h(\psi^M) &= \frac{1}{2}\,, \qquad & h(c) &= -1\,, \qquad & h(\gamma) &= -\frac{1}{2}\,, \end{aligned} \tag{3.32}$$

and analogously for the anti-holomorphic fields. It will also be useful to study the plane-wave operator $e^{ik\cdot X}$. We can show that it is a primary operator with weights

$$h = \bar{h} = \frac{\alpha'}{4}k^2\,, \tag{3.33}$$

which can be used to derive the OPE

$$e^{ik_1\cdot X(z)}e^{ik_2\cdot X(0)} = |z|^{\alpha' k_1\cdot k_2} e^{i(k_1+k_2)X(0)} + \dots\,. \tag{3.34}$$

From this equation we can show that the correlator of plane-waves imposes conservation of momenta

$$\langle \prod_j e^{ik_j\cdot X(z_j,\bar z_j)}\rangle = \delta(\sum_j k_j) \prod_{j<k} |z_{jk}|^{\alpha' k_i\cdot k_j}\,, \tag{3.35}$$

where we introduce the short-hand notation $z_{ij} = z_i - z_j$.

Finally, by doing the OPE of the stress-energy tensor with itself as in (3.8), we obtain an expression for the central charge of the superstring

$$c = \frac{3D}{2} - 26 + 11\,. \tag{3.36}$$

Conformal symmetry plays a crucial role in eliminating states with negative norm, so we would like the strings to remain conformal at the quantum level. However, the Weyl symmetry becomes anomalous in the quantum theory, since the trace of the stress-energy tensor has a non-vanishing vacuum expectation value

$$\langle T^a_a\rangle = -\frac{c}{12}R\,, \tag{3.37}$$

with c the central charge and R the worldsheet Ricci scalar. We then conclude that the superstring can only be consistent if the target-space is ten-dimensional [36].

3.1.5 RNS sectors

When we consider the theory on the cylinder, it is important to specify the periodicity conditions of the fields. The spacetime coordinates X^M must be periodic so the field expansion is

$$i\partial X^M(z) = (\alpha'/2)^{1/2} \sum_{n\in\mathbb{Z}} \frac{\alpha^M_n}{z^{n+1}}\,. \tag{3.38}$$

The ghosts must have the same periodicity as the corresponding matter field so we have

$$b(z) = \sum_{n\in\mathbb{Z}} \frac{b_n}{z^{m+2}}\,, \qquad c(z) = \sum_{n\in\mathbb{Z}} \frac{c_n}{z^{m-1}}\,. \tag{3.39}$$

Meanwhile, the fields ψ^M and $\tilde\psi^M$ live on the double cover of the complex plane, so they are either periodic or anti-periodic, which correpond to the Ramond and Neveu-Schwarz sectors respectively. Notice that these are periodicity conditions on the cylinder, so in the plane the expansions are

$$\psi^M(z) = \sum_r \frac{\psi_r^M}{z^{r+1/2}}\,, \quad \beta^M(z) = \sum_r \frac{\beta_r^M}{z^{r+3/2}}\,, \quad \gamma^M(z) = \sum_r \frac{\gamma_r^M}{z^{r-1/2}}\,, \tag{3.40}$$

where r is integer for the R sector and half-integer in the NS sector. Notice that the half-integer conformal weight of the fermions introduces a branch cut in the R sector while it removes the branch cut coming from anti-periodicity of the NS sector. In a closed string, where we consider both left and right movers, we can choose the periodicity conditions independently, so we obtain four different sectors: NS-NS, NS-R, R-NS and R-R.

In the quantum theory, we have the following commutators for the modes of the matter fields

$$\begin{aligned} [\alpha_m^M, \alpha_n^N] = [\tilde{\alpha}_m^M, \tilde{\alpha}_n^N] &= m\,\eta^{MN}\delta_{m,-n}\,, \\ \{\psi^M, \psi^N\} = \{\tilde{\psi}^M, \tilde{\psi}^N\} &= \eta^{MN}\delta_{m,-n}\,, \end{aligned} \tag{3.41}$$

while for the ghosts we have

$$\begin{aligned} [\gamma_r, \beta_s] = [\tilde{\gamma}_r, \tilde{\beta}_s] &= \delta_{r,-s}\,, \\ \{c_m, b_n\} = \{\tilde{c}_m, \tilde{b}_n\} &= \delta_{m,-n}\,. \end{aligned} \tag{3.42}$$

Finally, we can substitute the mode expansions in the expression for the stress-energy tensor (3.30), with normal ordering for the creation and annihilation operators. That procedure is unambiguous for all modes except L_0, for which we must introduce a normal ordering constant. In order to obtain the commutation relations (3.10), that normal order constant is given by

$$a_{NS} = -\frac{1}{2}\,, \qquad\qquad a_R = 0\,. \tag{3.43}$$

3.2 Spin fields and Bosonization

We will now look at the ground state of both the NS and R sectors and see that we need to introduce a new kind of operator, the spin field. The spin fields have non-trivial OPEs with the matter fields, so we will introduce the idea of bosonization where, at the cost of manifest Lorentz invariance, we substitute the interacting RNS CFT by one of free bosons.

3.2.1 The ground state

The matter ground state

By the state-operator correspondence, the state $|\mathbf{1}\rangle$ is equivalent to the insertion of the identity operator. If we consider the action of the matter field on this state, by regularity we have that

$$\begin{aligned}
\alpha_m^M|\mathbf{1}\rangle = \tilde{\alpha}_m^M|\mathbf{1}\rangle = 0 \qquad & \text{for } m \geq 0\,, \\
\psi_n^M|\mathbf{1}\rangle = \tilde{\psi}_n^M|\mathbf{1}\rangle = 0 \qquad & \text{for } n \geq 1/2\,.
\end{aligned} \tag{3.44}$$

Since all positive modes annihilate $|\mathbf{1}\rangle$, and there are no zero modes ψ_0^M in the NS sector, then it correspond to the NS ground state $|0\rangle_{\text{NS}}$. Since the NS ground state is a Lorentz singlet and excited states are created by the action of spacetime vectors, one sees that the NS Hilbert space is composed only of states of integer spin.

However, in the R sector there are zero modes ψ_0^M which do not increase the energy of the state, so the vacuum is degenerate. In fact the ψ_0^M form a Clifford algebra, so they create a spinor representation of $SO(1, d-1)$. We can then label the R sector ground states by

$$|A\rangle_{\text{R}} = |A_1 \dots A_5\rangle\,, \tag{3.45}$$

with $A_i = \pm 1/2$. We can see the ground state in the R sector as the action of a spin field operator on the NS vacuum [37, 38]

$$|A\rangle_{\text{R}} = \Theta^A|0\rangle_{\text{NS}}\,. \tag{3.46}$$

Since the excited states of the R sector are created by modes which are spacetime vectors, then the whole R sector Hilbert space is composed of states of half-integer spin. Note that in ten dimensions the Dirac representation splits into two sixteen dimensional Weyl representations, with opposite chiralities.

The ghost ground state

Let us now analyze the ground state in the fermionic ghost sector. Due to the different weights of the ghost fields, the action of the ghost modes on $|\mathbf{1}\rangle$ is now

$$\begin{aligned}
b_n|\mathbf{1}\rangle = 0 \qquad & \text{for } n \geq -1\,, \\
c_n|\mathbf{1}\rangle = 0 \qquad & \text{for } n \geq 2\,.
\end{aligned} \tag{3.47}$$

By looking at the commutation relations with L_0 we can see that any positive mode lowers the energy of the state, which means that $|\mathbf{1}\rangle$ cannot be the ground state. We can introduce a new state by acting with a c ghost

$$|\downarrow\rangle = c(0)|\mathbf{1}\rangle = c_1|\mathbf{1}\rangle\,. \tag{3.48}$$

Since c is a fermion, we see that $|\downarrow\rangle$ is annihilated by all positive ghost modes, so it must be a ground state of the ghost system. However, we are still free to act with the zero mode c_0 which does not increase the energy of the state, which implies that this ghost system has a degenerate ground state

$$\begin{aligned} c_0|\downarrow\rangle &= |\uparrow\rangle\,, & c_0|\uparrow\rangle &= 0\,,\\ b_0|\downarrow\rangle &= 0\,, & b_0|\uparrow\rangle &= |\downarrow\rangle\,. \end{aligned} \tag{3.49}$$

When we introduce BRST quantization in section 3.3.1, we will see that $|\uparrow\rangle$ is not a physical state, so the ground state of this ghost system is

$$|0\rangle_{\mathrm{bc}} = c(0)|\mathbf{1}\rangle\,. \tag{3.50}$$

The superghost ground state

Finally, let us look at the ground state of the superghost CFT. Since γ also has a negative weight, then we will run into the same problem where $|\mathbf{1}\rangle$ is not annihilated by some of the lowering operators

$$\begin{aligned} \beta_n|\mathbf{1}\rangle &= 0 \qquad n \geq -\frac{1}{2}\,,\\ \gamma_n|\mathbf{1}\rangle &= 0 \qquad n \geq \frac{3}{2}\,. \end{aligned} \tag{3.51}$$

However, since γ is bosonic, the solution presented above does not work in this case. We must rewrite the fields β and γ as a product of two fermions, where we bosonize one of them [39, 40]

$$\beta(z) = e^{-\phi(z)}\partial\xi(z)\,, \qquad \gamma(z) = e^{\phi(z)}\eta(z)\,. \tag{3.52}$$

The leading singularities in the OPE of these fermions must be such that we reproduce the ones of β and γ, which is achieved with

$$\begin{aligned} \phi(z)\phi(w) &= -\log(z-w) + \ldots\,, & \partial\xi(z)\partial\xi(w) &= \mathcal{O}(z-w) + \ldots\,,\\ \eta(z)\xi(w) &= \frac{1}{z-w} + \ldots\,, & \eta(z)\eta(w) &= \mathcal{O}(z-w) + \ldots\,. \end{aligned} \tag{3.53}$$

The stress-energy tensor for these new CFTs can be derived from a point-splitting technique we will introduce in section 3.2.3

$$\begin{aligned} T_\phi &= -\frac{1}{2}\partial\phi\partial\phi - \partial^2\phi\,,\\ T_{\eta\xi} &= -\eta\partial\xi\,, \end{aligned} \tag{3.54}$$

from which we derive the following weights for the primary operators

$$h(e^{l\phi}) = -\frac{l^2}{2} - l\,, \qquad h(\eta) = 1\,, \qquad h(\xi) = 0\,. \tag{3.55}$$

Using the fact that the zero mode of η annihilates $|\mathbf{1}\rangle$, it can be shown that the following state of weight 1/2

$$|0\rangle_{\mathrm{NS}} = e^{-\phi}|\mathbf{1}\rangle \tag{3.56}$$

is also annihilated by $\gamma_{1/2}$, which makes it the ground state in the NS sector.

Meanwhile, in the R sector, we need to find a state that is annihilated by γ_1. Given the branch cut in the R sector expansions, we need to find an operator that also has a square root branch cut, which is achieved with

$$|A\rangle_{\mathrm{R}} = \Theta^A e^{-\phi/2}|\mathbf{1}\rangle\,. \tag{3.57}$$

The weight of the exponential factor is 3/8, which combines exactly with the weight of the spin field to form a vertex of conformal weight 1, which ensures locality of the OPE.

3.2.2 Bosonization

The worldsheet fermions ψ^M and the spin fields Θ^A form an interacting CFT, so it can be non-trivial to obtain some of the correlators of the theory. This problem is circumvented by the introduction of bosonization, where we map the RNS fields to operators built from five free bosons [41]. A disadvantage of this method is that the expressions at intermediate steps are not manifestly Lorentz covariant.

The new free bosons ϕ_i have an OPE analogous to that of the target-space scalars X^M

$$\phi_i(z)\phi_j(0) = \delta_{i,j}\log z\,. \tag{3.58}$$

From this we can show that for each scalar ϕ_i, we have the following OPE for their exponentials

$$e^{p\phi_i(z)}e^{q\phi_i(0)} = z^{pq}e^{(p+q)\phi_i(0)}\,. \tag{3.59}$$

This is quite useful as we recognize the fields of weight 1/2, $e^{-\phi_i}$ and e^{ϕ_i}, to be equivalent to a set of complex fermions. In order to make contact with our Majorana-Weyl fermions, we have to rewrite them in the Cartan-Weyl basis

$$\begin{aligned}\psi^{\pm e_0} &= \frac{1}{\sqrt{2}}(\pm\psi^0 + \psi^1)\,,\\ \psi^{\pm e_i} &= \frac{1}{\sqrt{2}}(\psi^{2i} \pm i\psi^{2i+1})\,, \qquad \text{for } i = 1,\ldots,4\,.\end{aligned} \tag{3.60}$$

The identification with the auxiliary bosons is done in the following way

$$\psi^{\pm e_j} = e^{\pm\phi_j}c_{\pm e_j}\,. \tag{3.61}$$

Notice that we had to introduce a new element in the expression, the Jordan-Wigner cocycle factors $c_{\pm e_j}$ [42, 43]. While the bosonized operators built from a single scalar anticommute, the same is not true for operators built from different scalars, as they are independent. Therefore, the cocycle factors make sure that all bosonized operators have the correct commutations relations. This can be done explicitely if the cocycle is given by

$$c_{\pm e_j} = (-1)^{N_1+\ldots+N_{j-1}}\,, \tag{3.62}$$

with N_j the fermion number operator.

So far we have not gained much since it was already simple to compute the OPE for the Majorana-fields. The real power of this technique is in the treatment of the spin fields describing the ground state of the R sector. The spin fields map the NS ground state to the ground state of the R sector, which means that they must introduce square root branch cuts. We can see from equation (3.59) that this is achieved by the operator

$$\Theta^A = \exp\left[\sum_i A_i\phi_i\right] c_A\,, \tag{3.63}$$

with $A_i = \pm 1/2$. Since the weight of the exponential is

$$h(e^{l\phi_i}) = l^2/2\,, \tag{3.64}$$

then we obtain the correct conformal weight for the worldsheet fermions and a conformal weight of 5/8 for the spin fields.

It is useful to combine the notation for the bosonization of the matter fields together with the bosonization of the superghosts. For example, for the Cartan-Weyl fermions we get

$$\psi^{\pm e_j} e^{-\phi} = e^{\lambda\cdot\phi} c_\lambda\,, \qquad \lambda = \pm e_j - e_6\,. \tag{3.65}$$

The correlator for n fields with such six-dimensional weights is

$$\langle e^{\lambda_1\cdot\phi(z_1)} c_{\lambda_1}\ldots e^{\lambda_n\cdot\phi(z_n)} c_{\lambda_n}\rangle = \delta(\sum_j \lambda_j + 2e_6)\prod_{j<k} z_{jk}^{\lambda_i\cdot\lambda_j} e^{i\pi\lambda_j\cdot M\cdot\lambda_k}\,, \tag{3.66}$$

where the scalar products now have a Lorentzian signature $(1,\ldots,1,-1)$ and we have implemented the cocycle relations with a lower-triangular sign matrix

$$M = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 \\ -1 & 1 & -1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 & 0 & 0 \\ -1 & -1 & -1 & -1 & 1 & 0 \end{pmatrix}\,. \tag{3.67}$$

In order to recover convariant expressions after using bosonization, it is important to know that the gamma matrices in the Cartan-Weyl basis are

$$\left(\Gamma^{\pm e_k}\right)^A{}_B = \sqrt{2}e^{i\pi(\pm e_k)\cdot M\cdot A}\delta(\pm e_k + A, B)\,. \tag{3.68}$$

The relation to gamma matrices in a covariant basis is exactly analogous to equation (3.60). For the charge conjugation matrix we have

$$C^{AB} = \pm\delta(A+B)e^{-i\pi A_{-1/2}\cdot M\cdot A_{-1/2}}\,, \tag{3.69}$$

where the sign corresponds to the chirality of A and $A_{\pm 1/2}$ denotes the extended weight $(A, \pm 1/2)$.

3.2.3 Point-splitting

When studying operators at higher massive levels, it is necessary to consider the product of several RNS fields at the some point. To find the bosonized version of these operators we use the point-splitting method. The idea is to evaluate the product at separated points, subtract the singular term and then take the limit of the separation going to zero [44]. Specifically, we have

$$\begin{aligned} \psi^{e_i}\psi^{e_i}(z) &= \psi^{-e_i}\psi^{-e_i}(z) = 0\,, \\ \psi^{e_i}\psi^{-e_i}(z) &= -\psi^{-e_i}\psi^{e_i}(z) = \partial\phi_i\,. \end{aligned} \tag{3.70}$$

If we include also a spin field, then we have

$$\begin{aligned} \psi^{\pm e_k}(\psi\Theta)^A = 4\sqrt{2}\delta(A_k \mp \frac{1}{2})e^{\pm i\pi e_k\cdot M\cdot A}\boldsymbol{\Theta}^{A\pm e_k} \\ + (2A_k \pm 6e_k)\cdot\partial\phi(\Gamma^{\pm e_k}\Theta)^A\,, \end{aligned} \tag{3.71}$$

where $\boldsymbol{\Theta}^A$ is defined exactly like the spin fields (3.63), but for non-spinorial weight A.

3.3 Superstring Spectrum

Now that we have introduced the worldsheet CFTs describing the superstring, we need to understand its spectrum. From light-cone quantization one can see that timelike components of the worldsheet fields do not contribute to the spectrum. Here we will introduce the BRST charge, whose cohomology selects the physical states in a covariant way. Then we will also have to consider the GSO projection in order to obtain a supersymmetric spectrum free of tachyons.

Finally, we will introduce the notion of superghost picture and the vertex operators from the massless and first massive level, as well as from the leading Regge trajectory.

3.3.1 BRST quantization

Looking at both the matter and ghost actions together, we can see that there is a new symmetry between matter and ghost fields. This BRST invariance is a manisfestation of the gauge freedom that was fixed, and it is given by the BRST current [45]

$$j_B = cT_{\mathrm{m}} + \gamma G_{\mathrm{m}} + \frac{1}{2}(cT_{\mathrm{gh}} + \gamma G_{\mathrm{gh}})\,, \tag{3.72}$$

which can be seen to transform as a tensor from the OPE with the stress energy tensor. The BRST charge is then given by

$$Q_B = \frac{1}{2\pi i}\oint \mathrm{d}z j_B - \mathrm{d}\bar{z}\tilde{j}_B\,. \tag{3.73}$$

The fact that the BRST operator generates gauge transformations implies that any physical state $|\chi\rangle$ must be annihilated by the BRST charge

$$Q_B|\chi\rangle = 0\,. \tag{3.74}$$

A very important feature of the BRST charge is that it is nilpotent

$$Q_B^2 = 0\,, \tag{3.75}$$

so any state $Q_B|\chi\rangle$ could in principle be a physical state as well. However, since the BRST charge is hermitian, one can show that a BRST exact term must always be orthogonal to all physical states

$$\langle\psi|Q_B|\chi\rangle = 0\,. \tag{3.76}$$

We conclude that any two physical states that differ by such a null state must be physically equivalent, so physical states are in the cohomology of the BRST charge.

In order to find physical states, it is useful to split Q_B into three parts

$$Q_B = Q_0 + Q_1 + Q_2\,, \tag{3.77}$$

where Q_k carries superghost number k. Since physical states have a definite ghost number, this implies that Q_0, Q_1 and Q_2 must annihilate them independently.

When studying the ground state of the ghost system, we found that it was degenerate, and at that point we simply chose the state annihilated by the zero mode b_0 to be the ground state. In order to motivate this choice, note that its commutator with the BRST charge is

$$\{Q_B, b_0\} = L_0\,. \tag{3.78}$$

We can then see that by requiring the zero mode of b to annihilate any physical state, we ensure that BRST invariance implies that string states are on-shell. Analogously, physical states in the superstring should also be annihilated by the zero mode of the superghost β.

Finally, let us derive some implications of BRST invariance on physical states. We have seen before that the ground state of the ghost system is generate by $c(0)$. Let us then denote a vertex operator as $c(0)V_h(0)|\mathbf{1}\rangle$, with V_h a field of weight h built from matter and superghost fields. The vertex operator can only be annihilated by Q_0 if the vertex is a conformal primary with weight $h = 1$. We have also seen before that the number of conformal Killing vectors of a given surface determines the number of c ghost insertions. So while $c(0)V_h(0)$ corresponds to a ghost number one representative of the vertex, we also need to find the ghost number zero representation. It turns out that the following integrated vertex

$$\int \mathrm{d}z V_h(z)\,, \tag{3.79}$$

is also annihilated by the BRST charge for conformal weight $h = 1$.

3.3.2 GSO projection

So far we have eliminated states with negative norm from the spectrum by requiring BRST invariance. However, while the simplest vertex in the R sector

$$\Theta^A(0)e^{-\phi(0)/2}e^{ik\cdot X(0)}c(0) \tag{3.80}$$

corresponds to a massless string state, in the NS sector the simplest possibility is

$$e^{-\phi(0)}e^{ik\cdot X(0)}c(0)\,, \tag{3.81}$$

which corresponds to a tachyonic string with imaginary mass.

In order to eliminate the tachyonic string, which indicates an instability of the theory, we need to perform the GSO projection [46]. Physical states must obey one further constraint

$$Q_{\mathrm{GSO}}|\chi\rangle = |\chi\rangle\,. \tag{3.82}$$

In the NS sector, the projection is performed with the following operator

$$Q_{\mathrm{GSO}} = -e^{i\pi F}\,, \tag{3.83}$$

with F the worldsheet fermion number. The projection then selects states with an odd number of worldsheet fermions, which clearly removes the tachyonic state from the spectrum.

Meanwhile, in the R sector the projector is

$$Q_{\mathrm{GSO}} = \pm e^{i\pi F}\Gamma_{11}\,, \tag{3.84}$$

with Γ_{11} the chirality matrix in ten dimensions. We can see that in the R sector the projection into even or odd fermion number depends on the chirality of the state, but the choice of overall sign is purely a convention. For closed strings we have to project both left and right movers, and depending on the choice of sign in (3.84), we can have two different theories. In Type IIA, we pick different signs, while in Type IIB we pick the same projection for left and right movers, so the theory becomes chiral.

A key observation is that the GSO projection gives the same number of bosonic and fermionic degrees of freedom at any mass level, making the spectrum supersymmetric. It is also important that the BRST charge commutes with the operator $e^{i\pi F}$, so that the previous discussion on BRST invariance still holds for a subsector with definite fermion number.

3.3.3 Superghost pictures

The background superghost charge imposes a condition on the total charge of the vertex operators. That means that we might have to consider vertex operators in different superghost pictures. From requiring BRST invariance of an NS vertex operator $V_{\mathrm{NS}}e^{-\phi}$, we obtain the following constraint on the OPE with the supercurrent

$$G(z)V_{\mathrm{NS}}(w) = \frac{1}{z-w}W_{\mathrm{NS}}(w) + \dots\,, \tag{3.85}$$

which means that the vertex operator must be the lowest component of a superconformal primary field. We can then relate the two components in the following way

$$W_{\mathrm{NS}} = G_{-1/2}\tilde{G}_{-1/2}V_{\mathrm{NS}}\,. \tag{3.86}$$

For a vertex operator in the R sector $V_{\mathrm{R}}e^{-\phi/2}$, closure under the BRST charge requires that it is a highest weight of the super-Virasoro algebra. To obtain the higher superghost picture we have to find the next element in the representation of the superconformal algebra

$$W_R = G_{-1}\tilde{G}_{-1}V_{\mathrm{R}}\,. \tag{3.87}$$

More systematically, if we want to find the vertex operator in a higher superghost picture we have to use the following formula

$$V^{(g+1)}(z) = -[Q_B, \xi(z)V^{(g)}(z)]\,. \tag{3.88}$$

At first it might look like we are creating a BRST exact state, but that is not true. The subtlety is that only $\partial\xi$ appears in the bosonization of the superghost fields. That means that the zero mode ξ_0 does not belong to the superghost algebra so the new vertex is not actually BRST exact. However, the nilpotency of the BRST charge still guarantees that the new vertex operator is BRST closed.

3.3.4 Vertex operators

Given that the Poincaré transformations of the target-space coordinates are global symmetries of the superstring action, we want to form vertex operators that are eigenstates of the corresponding charges. For example, the translations

$$P^M = \frac{2}{\alpha'} \oint \frac{\mathrm{d}z}{2\pi i} i\partial X^M \,, \tag{3.89}$$

are diagonalized by the exponential $e^{ik\cdot X}$ with eigenvalue k^M, which contributes to the vertex with weight $\alpha' k^2/4$. For a string state with mass m, we then see that it is annihilated by Q_0 if the remainder of the vertex has weight $1 - \alpha' m^2/4$. This will fix the type of terms that can appear in the vertex operator at a given level, while invariance under Q_1 and Q_2 will give constraints on the coefficients of each allowed term.

For simplicity, in the remainder of this section we will drop the ghost and plane-wave factors from the vertex operators, and we will discuss only the right movers contribution to the closed strings, as the results for left movers are analogous.

Massless States

After the GSO projection, the string excitations with lowest energies are massless states. Let us start by considering closed massless strings in the NS sector. The only possible vertex we can write in the (-1) superghost picture is

$$V_{\mathrm{NS}}^{(-1)} = \epsilon_M \psi^M e^{-\phi} \,. \tag{3.90}$$

BRST invariance demands transversality of the polarization

$$\epsilon_M k^M = 0 \,. \tag{3.91}$$

This condition makes the vertex BRST closed, but we must remember that the physical state is defined up to BRST exact terms. It turns out that the vertex becomes exact if the polarization is proportional to the momentum k^M, which leaves us with 8 degrees of freedom. We will also need this vertex in the (0) superghost picture , so we use equation (3.88) to obtain

$$V_{\mathrm{NS}}^{(0)} = \epsilon_M \sqrt{\frac{2}{\alpha'}} \left(i\partial X^M + \frac{\alpha'}{2} (k\cdot\psi)\psi^M \right) \,. \tag{3.92}$$

Meanwhile, in the R sector the only vertex operator we can write at the massless level is

$$V_{\mathrm{R}}^{(-\frac{1}{2})} = t_A \Theta^A e^{-\phi/2} \,. \tag{3.93}$$

In order for the operator to be BRST invariant the polarization must satisfy

$$(t\not{k}) = 0 \,, \tag{3.94}$$

which eliminates half of the degrees of freedom.

As expected, we obtained the same number of bosonic and fermionic degrees of freedom and the polarizations transform in representations of the little group $SO(8)$.

First Massive Level

The next string level is given by operators with $k^2 = -4/\alpha'$, so the NS vertex in superghost picture (-1) is a combination of the following terms [44]

$$\epsilon_{MN} i\partial X^M \psi^N + \alpha_{MNP}\psi^M\psi^N\psi^P + \sigma_M \partial\psi^M \,, \tag{3.95}$$

where we already eliminated the term $\partial\phi\,\psi^M$ by the addition of a total derivative to the vertex. By adding BRST exact terms we can also make ϵ_{MN} symmetric and eliminate the term $\sigma_M\partial\psi^M$. Finally, requiring closure under the BRST charge imposes further constraints on the polarizations

$$\epsilon_{MN}\eta^{MN} = 0\,, \qquad \epsilon_{MN}k^M = 0\,, \qquad \alpha_{MNP}k^M = 0\,. \tag{3.96}$$

The symmetric, transverse and traceless tensor ϵ_{MN} has 44 degrees of freedom, while the antisymmetric transverse tensor α_{MNP} has 84, and they both correspond to representations of $SO(9)$. Using the ghost picture changing prescrition (3.88) we obtain the vertices at zero ghost charge

$$\begin{aligned} V^{(0)}_{\mathrm{NS},1} &= \epsilon_{MN}\frac{2}{\alpha'}\left(i\partial X^M\left(i\partial X^N + \frac{\alpha'}{2}k\cdot\psi\psi^N\right) + \frac{\alpha'}{2}\partial\psi^M\psi^N\right)\,, \\ V^{(0)}_{\mathrm{NS},2} &= \alpha_{MNP}\sqrt{\frac{2}{\alpha'}}\left(3i\partial X^M + \frac{\alpha'}{2}k\cdot\psi\psi^M\right)\psi^N\psi^P\,. \end{aligned} \tag{3.97}$$

In the R sector, there are two vertices one can form at the first massive level, but there are several constraints imposed by BRST invariance. On one hand the vertex must be written as the following combination of the two terms

$$V^{(-1/2)}_{\mathrm{R}} = t_{M,A}\left(i\partial X^M\Theta^A - \frac{1}{8}\frac{\alpha'}{2}\psi^M(\not{k}\not{\psi}\Theta)^A\right)e^{-\phi/2}\,, \tag{3.98}$$

but it also imposes transversality on the polarization. Finally, by adding BRST exact terms we can add a gamma-traceless constraint on the polarization

$$t_{M,A}\,k^M = 0\,, \qquad (\not{t}_M)_A = 0\,, \tag{3.99}$$

which leaves us with 128 degrees of freedom.

Leading Regge Trajectory

In general, at higher massive levels the complexity of the vertex operators increases very fast due to the large number of $SO(9)$ representations allowed. However, closed string states in the leading Regge trajectory are quite simple due to the relation between their spin S and mass level n

$$S = n + 1\,. \tag{3.100}$$

These are states in the NS sector, so they must have at least one ψ field in the vertex. However, if we want to create a state totally symmetric in its $n+1$ indices, then the only possibility is

$$V^{(-1)} = \epsilon_{M_1 \dots M_{n+1}} i\partial X^{M_1} \dots i\partial X^{M_n} \psi^{M_{n+1}} e^{-\phi}\,, \tag{3.101}$$

with a totally symmetric, traceless and transverse polarization

$$\epsilon_{M_1 \dots M_{n+1}} \eta^{M_i M_j} = 0\,, \qquad\qquad \epsilon_{M_1 \dots M_{n+1}} k^{M_i} = 0\,. \tag{3.102}$$

3.3.5 Supercharges

We had already seen that the target-space was Poincaré invariant, but now we observed that there is also spacetime supersymmetry. The Poincaré algebra then gets enlarged to super-Poincaré, which includes the anticommutator

$$\{Q^A_{L,R}, Q^B_{L,R}\} = -2\,(\Gamma_{11}\Gamma_M C)^{AB}\, P^M\,, \tag{3.103}$$

with C the charge conjugation matrix and Γ_{11} the projector onto positive chirality. The left and right supercharges are given by

$$Q^A_L = \oint \frac{\mathrm{d}z}{2\pi i} \Theta^A e^{-\phi/2}\,, \qquad Q^A_R = \oint \frac{\mathrm{d}\bar{z}}{2\pi i} \bar{\Theta}^A e^{-\tilde{\phi}/2}\,. \tag{3.104}$$

It will be useful later to use the supercharges in a higher superghost picture, which we write as

$$Q^A_L = \frac{1}{\alpha'^{1/2}} \oint \frac{\mathrm{d}z}{2\pi i} i\partial X^M (\Gamma_M \Theta)^A e^{\phi/2}\,. \tag{3.105}$$

3.4 String Interactions

We will now briefly discuss the origin of the string coupling and the prescription for computing superstring amplitudes in flat space.

3.4.1 Genus expansion

In order to consider strings on a curved background, it is natural to modify the bosonic part of the Polyakov action by changing the flat Minkowski metric into a general curved one

$$S = \frac{1}{4\pi\alpha'} \int \mathrm{d}^2\sigma \sqrt{-h} h^{ab} G_{MN}(X) \partial_a X^M \partial_b X^N \,. \tag{3.106}$$

This might seem confusing as the graviton is an excitation of the superstring in flat space, so a curved background should be obtained from a coherent background of string states. These two pictures can be shown to be equivalent if we consider a metric infinitesimally close to the flat one, $G_{MN}(X) = \eta_{MN} + \chi_{MN}(X)$. By expanding the exponential, we see that the first term obtained corresponds to the Polyakov action, while higher orders correspond to the insertion of the graviton vertex operators in the Polyakov path integral.

Analogously, we can include the backgrounds for the other massless string states in the following way

$$S = \frac{1}{4\pi\alpha'} \int \mathrm{d}^2\sigma \sqrt{-h} \left(h^{ab} G_{MN} + \epsilon^{ab} B_{MN} \right) \partial_a X^M \partial_b X^N + \alpha' R \Phi(X) \,, \tag{3.107}$$

with R the worldsheet Ricci scalar, B_{MN} the antisymmetric tensor, and the dilaton given by Φ and the trace of G_{MN}. We can see that for $G_{MN} = \eta_{MN}$, $B_{MN} = 0$ and $\Phi = \Phi_0$ the theory is still Weyl invariant and the last term of equation (3.107) can be integrated, producing a topological term in the path integral

$$S = S_P + \Phi_0 (2 - 2g) \,, \tag{3.108}$$

which gives different weights to different topologies. We can identify e^{Φ_0} as the string coupling g_s so that each handle added to the worldsheet comes with an additional power of g_s. We conclude that adding a background dilaton corresponds to changing the string coupling and the full superstring theory can be seen as an expansion over topologies, or genus expansion.

3.4.2 Scattering Amplitudes

In general we cannot consider correlation functions involving closed string states at specific spacetime points, as they would depend on the choice of coordinates which is not a physical quantity. The objects of interest in string theory are then correlators of string states coming from infinity, which are gauge invariant observables and correspond to elements of the S-matrix. The scattering process is then a worldsheet with n legs that extend to infinity. Each leg has the topology of a cylinder, but we can use a conformal transformation and map each string state to the insertion of a local vertex operator.

Considering the first term in the genus expansion of string theory we have then a sphere with n local insertions. The sphere has six conformal Killing vectors which can be used to fix the positions on the worldsheet for three of the vertex operators. The amplitude for three closed string states is then given by

$$\langle c\tilde{c}V_1(0)\, c\tilde{c}V_2(1)\, c\tilde{c}V_3(\infty)\rangle\,. \tag{3.109}$$

The insertions of the three c ghost fields cancel the ghost background charge. Analogously, the choice of superghost pictures for the vertex operators must add up to -2 to cancel the background superghost charge.

Since the vertex operators in the NS sector have integer superghost charges, while in the R sector they are half-integer, we conclude that we must always have an even number of vertex operators in the R sector. In the case of three-point superstring amplitudes the correlators that we need to evaluate are then

$$\begin{aligned}&\langle c\tilde{c}V_{\rm NS}^{(-1)}(0)\, c\tilde{c}V_{\rm NS}^{(-1)}(1)\, c\tilde{c}V_{\rm NS}^{(0)}(\infty)\rangle\,,\\ &\langle c\tilde{c}V_{\rm NS}^{(-1)}(0)\, c\tilde{c}V_{\rm R}^{(-1/2)}(1)\, c\tilde{c}V_{\rm R}^{(-1/2)}(\infty)\rangle\,.\end{aligned} \tag{3.110}$$

4. Strong Coupling

In this chapter we will study correlation functions of $\mathcal{N} = 4$ SYM at strong coupling. AdS/CFT implies that the generating functional for correlators in the gauge theory is equivalent to the string action subject to some boundary conditions. Each operator in the gauge theory corresponds to the insertion of a string vertex operator at the boundary of AdS and so the correlators of the gauge theory become amplitudes in the string theory.

There are many types of correlators one can compute, but we shall focus in the case when point-like propagation in the bulk is still valid, so that interactions are localized at a point in the bulk. We will consider non-protected operators, so the supergravity approximation will not suffice and we will need to perform superstring computations.

The study of these operators is very important as they are usually associated with wrapping corrections that need to be resummed at strong coupling. There is currently a proposal for evaluating structure constants of $\mathcal{N} = 4$ SYM at any value of the coupling, but unfortunately it is not known yet how to efficiently resum its new kind of wrapping corrections. Once such a framework is proposed, the results obtained from our string amplitudes will provide invaluable tests of its validity at strong coupling.

4.1 Types of Correlators

When all operators in a correlation function are protected, the dual string states are all massless and it is sufficient to consider the low energy limit of the superstring. In this limit strings become point-like and we need only to compute Witten diagrams in type IIB supergravity. The operators of $\mathcal{N} = 4$ SYM couple to sources $\phi_0(\vec{x}_i)$ on the boundary of AdS. In order to compute the string action subject to such boundary conditions we need to find the boundary to bulk propagator $K_\Delta(\vec{z}, \vec{x}_i)$ [6], so that we can write the fields in the bulk

$$\phi(\vec{z}) = \int \mathrm{d}^d x_i K_\Delta(\vec{z}, \vec{x}_i)\phi_0(\vec{x}_i)\,. \tag{4.1}$$

In the case of a scalar field the propagator is simply the Green function for a Laplace operator

$$K_\Delta(\vec{z}, \vec{x}_i) = \frac{\Gamma(\Delta)}{\pi^2\Gamma(\Delta-2)}\left(\frac{z}{z^2 + (\vec{x}-\vec{x}_i)^2}\right)^\Delta\,. \tag{4.2}$$

For a three-point function one would have to consider the cubic Witten diagram, where all propagators meet at a point in the bulk and we integrate that point over AdS. In this case the integral can be evaluated to [47]

$$\begin{aligned}&\int \frac{\mathrm{d}^d z \mathrm{d}z_0}{z_0^{d+1}} K_{\Delta_1}(z,\vec{x}_1) K_{\Delta_2}(z,\vec{x}_2) K_{\Delta_3}(z,\vec{x}_3) \\ &\qquad = \frac{\sqrt{\Delta_1-1}\sqrt{\Delta_2-1}\sqrt{\Delta_3-1}}{2^{5/2}\pi\Gamma(\Delta_1)\Gamma(\Delta_2)\Gamma(\Delta_3)} \frac{\Gamma(\alpha_1)\Gamma(\alpha_2)\Gamma(\alpha_3)\Gamma(\Sigma-2)}{|x_{12}|^{2\alpha_3}|x_{13}|^{2\alpha_2}|x_{23}|^{2\alpha_1}}\,. \end{aligned} \tag{4.3}$$

In general, a correlation function of protected operators is given by a sum over Witten diagrams. This method is quite powerful, but it also requires the computation of many Witten diagrams which correspond to a single superstring amplitude.

In the realm of unprotected operators, there are still several cases one must consider separately, as they require different techniques. On one hand we have semi-classical operators whose anomalous dimensions grow as $\Delta \sim \sqrt{\lambda}$. These operators correspond to string solutions that are extended in AdS, and the saddle point is given by minimizing an effective action that includes a contribution from the vertex operators. It is simpler to study the case when two of the operators are of this type and the third is much lighter [48–50], since in this case one can integrate the insertion of the light operator over the classical solution formed by the other two. The case when all three operators are semi-classical is more complicated but has been tackled in [51–54].

Finally, there are operators we will call "short" which have anomalous dimensions that scale as $\Delta \sim \lambda^{1/4}$, and are dual to string states at lower massive levels [7]. In this case the size of the strings is much smaller than the radius of AdS, so one can use a flat-space approximation [55–58]. Effectively the propagation in AdS is analogous to that of a half-BPS operator with the same scaling dimension.

There are however some differences between the case of short and point-like strings. In supergravity a cubic coupling might have derivatives, but one can eliminate these terms by a field redefinition. In this way one obtains the so called extended chiral primaries, which mix with multi-trace operators [59]. In general the contribution of the multi-trace operators has no effect on the three-point function, but it could play a role in the extremal limit.

However, in the case of short strings we should view the scalar Witten diagram as the exponential contribution to the semiclassical vertex operators of [60, 61], rather than a Witten diagram in a supergravity computation. For example, for correlation function with spin operators, the spacetime dependence in this case is given by (2.36), which differs from the spacetime dependence obtained through a scalar cubic Witten diagram. We will see later in this chapter that the remainder of the spacetime dependence comes from the superstring amplitude, which is not at all as one would expect from a supergravity computation. This indicates that the vertex operators we have found correspond to primaries and not their extended version.

4.2 Flat-space approximation

Since the size of the operators we consider is small when compared to the radius of AdS, we can treat strings as point-like particles with action

$$S = \frac{1}{2}\int_{-1}^{+1} \mathrm{d}s \left(\frac{\dot{x}_\mu \dot{x}^\mu + \dot{z}^2}{z^2} e^{-1} + \Delta^2 e \right) , \tag{4.4}$$

where $e(s)$ is the einbein and x^μ and z are the Poincaré coordinates. The propagation in AdS is given by a cubic Witten diagram but for operators of large dimensions it suffices to take the saddle-point which fixes the position of the intersection point [62]

$$e^{-\Delta_1 \mathcal{L}_1(\vec{x}_1) - \Delta_2 \mathcal{L}_2(\vec{x}_2) - \Delta_3 \mathcal{L}_3(\vec{x}_3)} , \tag{4.5}$$

where $\mathcal{L}_i(\vec{x}_i)$ represents the geodesic distance from the boundary point $\vec{x}_i$ to the intersection point $\vec{z} = (z, \vec{x})$

$$\mathcal{L}_i(\vec{x}_i) = \log \frac{z}{z^2 + (\vec{x} - \vec{x}_i)^2} . \tag{4.6}$$

4.2.1 Saddle-point and canonical momenta

There is a unique solution to the saddle-point equations obtained by varying the position of the intersection point [63, 64]

$$\begin{aligned} x^\mu &= \frac{\alpha_1\alpha_2 x_{12}^2 x_3^\mu + \alpha_1\alpha_3 x_{13}^2 x_2^\mu + \alpha_2\alpha_3 x_{23}^2 x_1^\mu}{\alpha_1\alpha_2 x_{12}^2 + \alpha_1\alpha_3 x_{13}^2 + \alpha_2\alpha_3 x_{23}^2} , \\ z^2 &= \frac{\alpha_1\alpha_2\alpha_3 \Sigma x_{12}^2 x_{13}^2 x_{23}^2}{(\alpha_1\alpha_2 x_{12}^2 + \alpha_1\alpha_3 x_{13}^2 + \alpha_2\alpha_3 x_{23}^2)^2} , \end{aligned} \tag{4.7}$$

and if we plug this solution back into (4.5), we obtain the large Δ_i limit of the full Witten diagram. We can also compute the fluctuations of the action, which shows that the interaction is over a small region of AdS. It is then natural to evaluate the string amplitude at the intersection point using flat-space vertex operators, but one needs first to understand what are the momenta characterizing the vertices.

The saddle-point configuration is formed by three semi-circle geodesics meeting at the point (4.7), so we can evaluate the canonical momentum

$$\Pi^\mu = -i\frac{\dot{x}^\mu}{z^2}\,, \qquad\qquad \Pi^z = -i\frac{\dot{z}}{z^2}\,, \tag{4.8}$$

which satisfies $\Pi^2 = -\Delta^2$. It turns out the saddle-point equations correspond exactly to conservation of the canonical momenta at the intersection point, which means that they are the momenta that should enter the vertex operators [64]. For Π_1 we have

$$\begin{aligned}
\Pi_1^\mu &= \frac{2\sqrt{\alpha_1\alpha_2\alpha_3\Sigma}|x_{12}||x_{13}||x_{23}|}{\alpha_1\alpha_2 x_{12}^2 + \alpha_1\alpha_3 x_{13}^2 + \alpha_2\alpha_3 x_{23}^2}\left(\alpha_3\frac{x_{21}^\mu}{x_{12}^2} + \alpha_2\frac{x_{31}^\mu}{x_{13}^2}\right)\,,\\
\Pi_1^z &= \Delta_1 - \frac{2\alpha_2\alpha_3\Sigma x_{23}^2}{\alpha_1\alpha_2 x_{12}^2 + \alpha_1\alpha_3 x_{13}^2 + \alpha_2\alpha_3 x_{23}^2}\,,
\end{aligned} \tag{4.9}$$

with analogous expressions for Π_2 and Π_3. The ten-dimensional momenta are given by combining these expressions with the momentum on the five-sphere, $k^M = (\Pi^\mu, \Pi^z, \vec{J})$, so that we have

$$k^2 = -\Delta^2 + J^2 \approx -4n\sqrt{\lambda}\,, \tag{4.10}$$

with n the string mass level.

4.2.2 Flattening of the superconformal algebra

For each operator on the boundary we will have to find the correct flat-space vertex operator, and in order to better understand this map we should look at the superconformal algebra. In chapter 2 we introduced the supercharges of $\mathcal{N} = 4$ SYM: $Q_{\alpha I}$, $\dot{Q}^I_{\dot\alpha}$, $\dot{S}_{\dot\alpha I}$ and S^I_α. It is now more convenient to write the spinor indeces in a $SU(2,2) \cong SO(2,4)$ covariant way

$$Q_{aI} = (Q_{\alpha I}, \dot{S}_{\dot\alpha I})\,, \qquad\qquad S^{aI} = (\epsilon^{\alpha\beta}S^I_\beta, \epsilon^{\dot\alpha\dot\beta}\dot{Q}^I_{\dot\alpha})\,. \tag{4.11}$$

Q_{aI} is in the fundamental of both $SU(2,2)$ and $SU(4)$, while S^{aI} is in the anti-fundamental representaion. We can also see these indices as chiral spinor indices of $SO(2,4)$ and $SO(6)$ respectively. Finally we can combine them further in the following way

$$\begin{aligned}
Q_L^A &= Q_{aI} + \gamma^{-1}_{ab}\gamma^{10}_{IJ}S^{bJ}\,,\\
Q_R^A &= -i(Q_{aI} - \gamma^{-1}_{ab}\gamma^{10}_{IJ}S^{bJ})\,,
\end{aligned} \tag{4.12}$$

where γ^{-1} and γ^{10} are gamma matrices of $SO(2,4)$ and $SO(6)$ respectively. It is now more convenient to write the R-symmetry generators with antisymmetric $SO(6)$ indices R_{IJ} and also write the conformal algebra in a covariant way with the generators J_{mn} of (2.5). By choosing a basis where only the generators J_{-1m} and $R_{J,10}$ have non-vanishing expecation values we can identify the ten-dimensional momentum operator

$$P^M = (J_{-1m}, R_{J,10})\,. \tag{4.13}$$

We also rescale the generators with a small parameter ϵ

$$\hat{Q}^A_{L,R} = \sqrt{\epsilon} Q^A_{L,R}\,, \qquad \hat{P}^M = \epsilon P^M\,, \tag{4.14}$$

so that they stay finite in the limit of large scaling dimension Δ. In this limit, the superconformal algebra flattens to super-Poincaré, with the commutation relations

$$\{\hat{Q}^A_{L,R}, \hat{Q}^A_{L,R}\} = -(\Gamma_M C)^{AB} \hat{P}^M\,, \qquad \{\hat{Q}^A_L, \hat{Q}^A_R\} = 0\,. \tag{4.15}$$

In this way we can identify the boundary operators with flat-space string states.

4.2.3 Superconformal primaries

Let us now consider the superconformal primary, which is annihilated by all superconformal generators

$$S^I_\alpha \mathcal{O}(0) = S_{\dot{\alpha}} \mathcal{O}(0) = 0\,. \tag{4.16}$$

We need to understand the equivalent of these equations in terms of flat-space quantities. We can divide the ten-dimensional spinor index A with positive chirality into two parts $SO(1,3) \times SO(6)$, so that A decomposes into $(\alpha, \tilde{a})$ and $(\dot{\alpha}, \dot{\tilde{a}})$, where dotted and undotted indices denote negative and positive chirality respectively. The superprimary condition then becomes [64]

$$\begin{aligned} Q_L^{\alpha\tilde{a}} \mathcal{O} &= i\, Q_R^{\alpha\tilde{a}} O\,, \\ Q_L^{\dot{\alpha}\dot{\tilde{a}}} \mathcal{O} &= -i\, Q_R^{\dot{\alpha}\dot{\tilde{a}}} O\,, \end{aligned} \tag{4.17}$$

where α and $\dot{\alpha}$ label the boundary spacetime directions.

At the intersection point in the bulk there is an analogous condition, but the spinor indices label instead the spacetime directions perpendicular to the momentum of the state at the intersection point, which we label as $(0', \ldots, 3')$. This is equivalent to the following relation on the vertex operator V

$$Q^A_L V = i(i\Gamma^{0'}\Gamma^{1'}\Gamma^{2'}\Gamma^{3'})^A{}_B Q^B_R V\,, \tag{4.18}$$

where the factor $i\Gamma^{0'}\Gamma^{1'}\Gamma^{2'}\Gamma^{3'}$ gives ± 1 according to the chirality of the spinor in the relevant spacetime directions. We will find it easier to implement first a relation on a twisted vertex operator

$$Q_L^A V^{\mathrm{T}} = i\, Q_R^A V^{\mathrm{T}}\,, \tag{4.19}$$

and later find the implications of untwisting on the polarizations of the vertices.

In this work we will only consider bosonic excitations of the string, so we can have string states in the NS-NS and R-R sectors. Since the action of the supercharge exchanges an NS string by a state in the R sector, we can then see that the equations determining the vertex operator for a superconformal primary imply that it must be a linear combination of states from both sectors.

It is also important to note that when applying equation (4.19) we will have to make sure to match the superghost picture of the vertex operators on both sides of the equation. If we pick the NS-NS states to be in the $(-1,-1)$ picture, and R-R states to have $(-1/2,-1/2)$ superghost numbers, then the supercharge acting on the R-R state must be in the $(-1/2)$ picture, while the supercharge acting on NS-NS states much be in the $(1/2)$ superghost picture.

4.3 The vertex operators

In this section we will specify which operators of $\mathcal{N}=4$ SYM we will consider in the correlation functions at strong coupling, and we will show what are the vertex operators that describe the dual string states.

The dual of $\mathcal{N}=4$ SYM is type IIB string theory, which has the same chirality for left and right-moving states in the R sector. In the rest of this work we choose the convention of positive chirality.

4.3.1 Protected operators

The chiral primaries are given by

$$\mathcal{O}_{\mathrm{CP}} = C^{I_1K_1,\ldots,I_JK_J}\,\mathrm{Tr}[\phi_{I_1K_1}\ldots\phi_{I_JK_J}]\,, \tag{4.20}$$

with $C^{I_1K_1,\ldots,I_JK_J}$ a symmetric and traceless tensor on any pair of indices

$$C^{I_1K_1,\ldots,I_JK_J}\epsilon_{I_pK_pI_qK_q} = 0\,. \tag{4.21}$$

These states transform in the $[0,J,0]$ representation of $SU(4)$ and have protected dimension $\Delta = J$, as they are annihilated by half the supercharges.

Protected states are dual to massless strings, so they will map to the following vertices in the NS-NS and R-R sectors

$$\begin{aligned} W_1 &= g_c \varepsilon_{M\tilde{M}} \psi^M e^{-\phi} \tilde{\psi}^{\tilde{M}} e^{-\tilde{\phi}} e^{ik\cdot X} \,, \\ W_2 &= g_c t_{AB} \tilde{\Theta}^{\tilde{A}} e^{-\tilde{\phi}/2} \Theta^B e^{-\phi/2} e^{ik\cdot X} \,. \end{aligned} \tag{4.22}$$

We have seen before that the chiral primaries of $\mathcal{N} = 4$ SYM are the lowest weight states of short representations with 2^8 elements. In the previous chapter we have also seen that the open massless string has $8 + 8$ degrees of freedom, which implies that the closed string has 2^8 states, so we conclude that the counting of degrees of freedom matches. The solution to the twisted condition (4.19) is then

$$W^{\mathrm{T}} = \frac{1}{4}(W_1^{\mathrm{T}} + W_2^{\mathrm{T}}) \,, \tag{4.23}$$

with the polarizations given by

$$\begin{aligned} \epsilon^{\mathrm{T}}_{M\tilde{M}} &= \eta_{M\tilde{M}} - \frac{k_M q_{\tilde{M}} + k_{\tilde{M}} q_M}{k\cdot q} \,, \\ t^{\mathrm{T}}_{AB} &= \frac{1}{\sqrt{2}} \left(\frac{\alpha'}{2}\right)^{1/2} (C^{\dagger} \not{k})_{AB} \,. \end{aligned} \tag{4.24}$$

The vector q is an arbitrary light-like vector whose scalar product with the momentum k is non-vanishing and the twisted vertex corresponds to a linear combination of the dilaton and axion with some Kaluza-Klein momentum.

Another family of protected operators we will consider later are the relatives of the Lagrangian with some R-charge J and dimension $\Delta_{\mathcal{L}} = 4 + J$

$$\mathcal{L}_J = \mathrm{Tr}[F_{\mu\nu} F^{\mu\nu} Z^J] + \dots \,, \tag{4.25}$$

where the ellipsis denotes two types of terms we do not write explicitely: terms where the field strength $F_{\mu\nu}$ of $\mathcal{N} = 4$ SYM occupies different positions inside the trace, and also other terms with scalars and fermions.

At strong coupling this operator corresponds to the ten-dimensional dilaton with some Kaluza-Klein momentum, so the vertex is simply the NS-NS part of the twisted chiral above, with corrected normalization

$$V_{\mathcal{L}} = \frac{g_c}{\sqrt{8}} \left(\eta_{M\tilde{M}} - \frac{k_M q_{\tilde{M}} + k_{\tilde{M}} q_M}{k\cdot q}\right) \psi^M e^{-\phi} \tilde{\psi}^{\tilde{M}} e^{-\tilde{\phi}} e^{ik\cdot X} \,. \tag{4.26}$$

4.3.2 Konishi-like operators

We would like to consider the Konishi operator, which is dual to a string in the first massive level, with $\Delta_K \approx 2\lambda^{1/4}$ [55, 56]. However, in some cases we will need to add some R-charge to the operator in order to have non-vanishing three-point functions. These operators are described by Beisert in [65] and they correspond to a generalization of BMN operators outside the BMN regime $J \sim \sqrt{\lambda}$. The operators are given by

$$\mathcal{O}_J = \sum_{l}^{J} \sum_{K,L=1}^{4} \cos\frac{\pi(2l+3)}{J+3} \operatorname{Tr} \phi_{KL} Z^l \phi^{KL} Z^{J-l} . \tag{4.27}$$

Like the chiral primaries, these operators transform in the $[0, J, 0]$ representation of $SU(4)$, but their bare dimension is $\Delta = J + 2$. For $J = 0$ this operator becomes the Konishi and in the limit of large R-charge J it becomes the BMN operator of [66]. The Konishi-like operators we are going to consider will have R-charge $J \ll \lambda^{1/4}$ so that the anomalous dimension at strong coupling is still the main contribution to Δ_K.

These operators are also superprimaries, so the dual vertices are a combination of the following R-R and NS-NS vertex operators

$$\begin{aligned}
V_3 &= g_c(i\bar{\partial}X^M \tilde{\Theta}^A - \frac{1}{8}\frac{\alpha'}{2}\tilde{\psi}^M (\not{k}\tilde{\not{\psi}}\tilde{\Theta})^A)e^{-\tilde{\phi}/2} \\
&\qquad\qquad \times t_{MA,NB}(i\partial X^N \Theta^B - \frac{1}{8}\frac{\alpha'}{2}\psi^N (\not{k}\not{\psi}\Theta)^B)e^{-\phi/2} e^{ik\cdot X} , \\
V_1 &= g_c \varepsilon_{MN,\tilde{M}\tilde{N}} i\partial X^M \psi^N e^{-\phi} i\bar{\partial} X^{\tilde{M}} \tilde{\psi}^{\tilde{N}} e^{-\tilde{\phi}} e^{ik\cdot X} , \\
V_2 &= g_c \alpha_{MNP,\tilde{M}\tilde{N}\tilde{P}} \psi^M \psi^N \psi^P e^{-\phi} \tilde{\psi}^{\tilde{M}} \tilde{\psi}^{\tilde{N}} \tilde{\psi}^{\tilde{P}} e^{-\tilde{\phi}} e^{ik\cdot X} .
\end{aligned} \tag{4.28}$$

The open string has $128 + 128$ states, so there are degrees 2^{16} of freedom in the closed string at the first massive level, which corresponds exactly to the dimension of a typical representation of $\mathcal{N} = 4$ SYM. The solution to (4.19) is in this case

$$V^{\mathrm{T}} = \frac{1}{16}(V_1^{\mathrm{T}} + V_2^{\mathrm{T}} + V_3^{\mathrm{T}}) , \tag{4.29}$$

with the following twisted polarizations

$$\begin{aligned}
t^{\mathrm{T}}_{MA,NB} &= \frac{1}{\sqrt{2}}\left(\frac{\alpha'}{2}\right)^{1/2} (C^\dagger \not{k}(\hat{\eta}_{MN} - \frac{1}{9}\Gamma^R\Gamma^S \hat{\eta}_{RM}\hat{\eta}_{SN}))_{AB} , \\
\varepsilon^{\mathrm{T}}_{MN,\tilde{M}\tilde{N}} &= \frac{1}{2}(\hat{\eta}_{M\tilde{M}}\hat{\eta}_{N\tilde{N}} + \hat{\eta}_{M\tilde{N}}\hat{\eta}_{N\tilde{M}}) - \frac{1}{9}\hat{\eta}_{MN}\hat{\eta}_{\tilde{M}\tilde{N}} , \\
\alpha^{\mathrm{T}}_{MNP,\tilde{M}\tilde{N}\tilde{P}} &= \frac{1}{36}(\hat{\eta}_{M\tilde{M}}\hat{\eta}_{N\tilde{N}}\hat{\eta}_{P\tilde{P}} \mp 5 \text{ permutations}) .
\end{aligned} \tag{4.30}$$

In order for the polarizations to be transverse to the momentum, we defined the matrix $\hat{\eta}$ as

$$\hat{\eta}^{MN} = \eta^{MN} - \frac{k^M k^N}{k^2} . \tag{4.31}$$

4.3.3 Leading twist operators

The twist of an operator is given by the difference between the scaling dimension and the spin. We will now consider the twist two operators of $\mathcal{N}=4$ SYM that are R-charge singlets. There are three such operators, whose degeneracy at tree level is lifted at one loop, as can be seen from the action of the dilatation operator [67]. In general the eigenstates of the dilatation operator will be linear combinations of the three operators, but we are interested only in the one with the lowest eigenvalue, so that its string dual sits in the leading Regge trajectory. That operator is given by the following linear combination [68]

$$|\phi D^S \phi\rangle + \frac{2(S+1)}{S}|\bar{\psi} D^{S-1}\psi\rangle - \frac{2(S+1)(S+1)}{S(S-1)}|F D^{S-2} F\rangle \,, \tag{4.32}$$

where we defined the states

$$\begin{aligned}
|\bar{\psi} D^{S-1}\psi\rangle &= \sum_{k=0}^{S}(-1)^k \binom{S}{k}\binom{S}{k+1} \mathrm{Tr}[D_{\mu_1\ldots\mu_k}\bar{\psi}_{\dot{\alpha}I} D_{\mu_{k+1}\ldots\mu_{S-1}}\psi^I_\alpha] \,,\\
|F D^{S-2} F\rangle &= \sum_{k=0}^{S}(-1)^k \binom{S}{k}\binom{S}{k+2} \mathrm{Tr}[D_{\mu_1\ldots\mu_k}F_{\mu_{k+1}\nu} D_{\mu_{k+2}\ldots\mu_{S-1}}F_{\mu_S}{}^{\nu}] \,,\\
|\phi D^S \phi\rangle &= \sum_{k=0}^{S}(-1)^k \binom{S}{k}^2 \mathrm{Tr}[D_{\mu_1\ldots\mu_k}\phi_{IJ} D_{\mu_{k+1}\ldots\mu_S}\phi^{IJ}] \,.
\end{aligned} \tag{4.33}$$

At strong coupling this operator is dual to a string state with spin S at the n-th mass level. To make contact with (2.36) we introduce a polarization vector Z_M and turn it into a polynomial. The normalized vertex is then

$$V_S = g_c \frac{2^n}{\Gamma(S/2)\alpha'^n}(Z\cdot(i\partial X))^n (Z\cdot\psi) e^{-\phi} (Z\cdot(i\bar{\partial}X))^n (Z\cdot\tilde{\psi}) e^{-\tilde{\phi}} e^{ik\cdot X} \,, \tag{4.34}$$

which is explicitely symmetric in its S indices. BRST invariance of the state constrains the polarization (3.102), and in terms of the polarization vector the constraints become

$$Z^2 = 0 \,, \qquad Z\cdot k = 0 \,. \tag{4.35}$$

4.3.4 Untwisting

Finally, we have introduced the vertex operators at the massless and first massive level that satisfy the twisted condition (4.19), but now we need to understand how to untwist the vertex operators. It can be shown that the untwisting factor $i\Gamma^{0'}\Gamma^{1'}\Gamma^{2'}\Gamma^{3'}$ leads to the following transformation of gamma matrices

$$(i\Gamma^{0'}\Gamma^{1'}\Gamma^{2'}\Gamma^{3'})\Gamma^M(i\Gamma^{0'}\Gamma^{1'}\Gamma^{2'}\Gamma^{3'}) = (-1)^{\sigma(M)}\Gamma^M\,, \tag{4.36}$$

with

$$\sigma(M) = \begin{cases} 1 & M = 0', \ldots, 3' \\ 0 & M = 4, \ldots, 9 \end{cases} \tag{4.37}$$

With this in mind we can analyze the equation (4.18) and see that the solution is exactly the same as for the twisted case, but with the following modifications to the massless polarizations

$$\begin{aligned} \epsilon_{M\tilde{M}} &= (-1)^{\sigma(\tilde{M})}\epsilon^{\mathrm{T}}_{M\tilde{M}}\,, \\ t_{AB} &= (C^\dagger i\Gamma^{0'}\Gamma^{1'}\Gamma^{2'}\Gamma^{3'}\not{k})_{AB}\,. \end{aligned} \tag{4.38}$$

Analogously, at the first massive level we have

$$\begin{aligned} t_{MA,NB} &= (-1)^{\sigma(M)}\big(C^\dagger i\Gamma^{0'}\Gamma^{1'}\Gamma^{2'}\Gamma^{3'}\not{k}(\hat{\eta}_{MN} - \frac{1}{9}\Gamma^R\Gamma^S\hat{\eta}_{RM}\hat{\eta}_{SN})\big)_{AB}\,, \\ \epsilon_{MN,\tilde{M}\tilde{N}} &= (-1)^{\sigma(\tilde{M})}(-1)^{\sigma(\tilde{N})}\epsilon^{\mathrm{T}}_{MN,\tilde{M}\tilde{N}}\,, \\ \alpha_{MNP,\tilde{M}\tilde{N}\tilde{P}} &= (-1)^{\sigma(\tilde{M})}(-1)^{\sigma(\tilde{N})}(-1)^{\sigma(\tilde{P})}\alpha^{\mathrm{T}}_{MNP,\tilde{M}\tilde{N}\tilde{P}}\,. \end{aligned} \tag{4.39}$$

Since we have to untwist three operators, we need to introduce three different untwisting factors σ_i as the directions perpendicular to the momentum are different for each vertex operator. It is then useful to introduce a twisting factor that switches the sign of all components corresponding to AdS

$$\tilde{\sigma}(M) = \begin{cases} 1 & M = 0, \ldots, 4 \\ 0 & M = 5, \ldots, 9 \end{cases} \tag{4.40}$$

Its action on the metric is related to the action of the original untwisting factor in the following way

$$(-1)^{\sigma(M)}\eta^{MN} = (-1)^{\tilde{\sigma}(M)}\eta^{MN} - 2\frac{\tilde{k}^M\tilde{k}^N}{\Delta^2}\,, \tag{4.41}$$

where $\tilde{k}$ denotes the AdS projection of the momentum, and $\tilde{k}^2 = -\Delta^2$. It is also useful to rewrite the factor $i\Gamma^{0'}\Gamma^{1'}\Gamma^{2'}\Gamma^{3'}$ so that we don't have dependence on specific indices for each polarization

$$i\Gamma^{0'}\Gamma^{1'}\Gamma^{2'}\Gamma^{3'} = \frac{1}{\Delta_i}\Gamma^0\Gamma^1\Gamma^2\Gamma^3\Gamma^4\tilde{\not{k}}_i\,. \tag{4.42}$$

Just like in (4.36), we can relate the new product of gamma matrices with the new untwisting factor

$$(\Gamma^0\Gamma^1\Gamma^2\Gamma^3\Gamma^4)\Gamma^M(\Gamma^0\Gamma^1\Gamma^2\Gamma^3\Gamma^4) = (-1)^{\sigma(M)}\Gamma^M\,. \tag{4.43}$$

Finally, it is useful to note that for the massless vertex operators we had to choose an arbitraty light-like vector q whose scalar product with the momentum was non-vanishing. For practical purposes it is useful to choose $q^M = (-1)^{\sigma(M)}k^M$.

4.4 Structure Constants

Now that we have introduced all the vertex operators, we can move on to considering the superstring amplitudes that will correspond to the correlation functions of interest. We have seen that the propagation to the intersection point is just a large dimension approximation to the cubic Witten diagram, so the structure constant is given by

$$C_{123} = \frac{\sqrt{\Delta_1 - 1}\sqrt{\Delta_2 - 1}\sqrt{\Delta_3 - 1}}{2^{5/2}\pi}\frac{\Gamma(\alpha_1)\Gamma(\alpha_2)\Gamma(\alpha_3)\Gamma(\Sigma - 2)}{\Gamma(\Delta_1)\Gamma(\Delta_2)\Gamma(\Delta_3)}\mathcal{G}_{123}\,, \tag{4.44}$$

with the coupling given by

$$\mathcal{G}_{123} = \frac{8\pi}{g_c^2\alpha'}\langle V_{k_1}V_{k_2}V_{k_3}\rangle\langle\psi_{J_1}\psi_{J_2}\psi_{J_3}\rangle\,, \tag{4.45}$$

where $\langle V_{k_1}V_{k_2}V_{k_3}\rangle$ is the superstring amplitude and $\langle\psi_{J_1}\psi_{J_2}\psi_{J_3}\rangle$ the sphere overlap integral [69, 70]

$$\langle\psi_{J_1}\psi_{J_2}\psi_{J_3}\rangle = \frac{\prod_{i=1}^3\sqrt{(J_i+1)(J_i+2)}J_i!}{\sqrt{2}\pi^{3/2}\alpha_1!\alpha_2!\alpha_3!(\Sigma+2)!}\langle C^{J_1}C^{J_2}C^{J_3}\rangle\,. \tag{4.46}$$

The factor $\langle C^{J_1}C^{J_2}C^{J_3}\rangle$ is the unique $SO(6)$ invariant that can be formed with the polarizations of the spherical harmonics. Using the relation of g_c to the string coupling, $g_c = 4\pi^{5/2}g_s\alpha'^2$, we obtain the following relation with the gauge theory parameters

$$g_c = \frac{\pi^{3/2}}{N}\,. \tag{4.47}$$

Note that if Δ is large it is acceptable to use overlaps even for small J. To understand this we can think of particles in a box, where the interactions are local but the particles are unlocalized. In this case one can still use the overlap integrals over the whole box. In our problem, the condition for local interaction is satisfied when scaling dimensions are large, so we can take the R-charge to zero and obtain structure constants for the Konishi operator.

In order to obtain the relevant correlation functions we have computed several superstring amplitudes with massive states, which are in itself new results. However, in the remainder of this section we will focus on their implications for correlation function of $\mathcal{N} = 4$ SYM, and refer the reader to the appendices of Paper I and II for more details on superstring amplitudes.

4.4.1 Ward identities and protected correlators

Before we show the results obtained, let us present some checks of our methods.

While the twisted vertices satisfying (4.19) were originally meant purely as an auxiliary object, they were also useful in checking both the expressions found for the vertices as well as the Mathematica implementation of the contractions.

The twisted vertex operators are annihilated by the following sixteen flat-space supercharges

$$\tilde{Q}^A = Q_L^A - iQ_R^A \,. \tag{4.48}$$

Since the twisted vertex operators have non-zero momenta, we can write them as

$$V^{\mathrm{T}} = \{\tilde{Q}^A, \tilde{V}\} \,, \tag{4.49}$$

and use this to show that superstring amplitudes with three twisted vertices must vanish due to a supersymmetric Ward identity

$$\langle V_1^{\mathrm{T}} V_2^{\mathrm{T}} V_3^{\mathrm{T}} \rangle + \langle \tilde{V}_1 [\tilde{Q}^A, V_2^{\mathrm{T}}] V_3^{\mathrm{T}} \rangle + \langle \tilde{V}_1 V_2^{\mathrm{T}} [\tilde{Q}^A, V_3^{\mathrm{T}}] \rangle = 0 \,. \tag{4.50}$$

We can also derive a similar Ward identity even for the case when one of the vertex operators is untwisted. If we choose supercharges that have positive chirality in the four-dimensional space orthogonal to the momentum of the untwisted vertex

$$\tilde{Q}^+ = Q_L^{\alpha a} - iQ_R^{\alpha a} \,, \tag{4.51}$$

then

$$\langle V_1^{\mathrm{T}} V_2^{\mathrm{T}} V_3 \rangle + \langle \tilde{V}_1 [\tilde{Q}^+, V_2^{\mathrm{T}}] V_3 \rangle + \langle \tilde{V}_1 V_2^{\mathrm{T}} [\tilde{Q}^+, V_3] \rangle = 0 \,. \tag{4.52}$$

We cannot derive any such identity for the case of two untwisted vertices, as the charges that annihilate one of the them do not annihilate the other. This happens because the rotation from k_2 to k_3 mixes components with negative chirality in the four-dimensional subspace.

We can also check our results against correlators that were previously known in the literature. One such example is the case of three chiral operators, for which the flat-space amplitude takes a remarkably simple form

$$\langle WWW \rangle = g_c^2 \alpha' \frac{\alpha_1 \alpha_2 \alpha_3 \Sigma^5}{J_1^2 J_2^2 J_3^2}\,, \tag{4.53}$$

which combined with the large R-charge limit of (4.44) gives the following structure constant

$$C_{CCC} = \frac{\sqrt{J_1 J_2 J_3}}{N} \langle C^{J_1} C^{J_2} C^{J_3} \rangle\,. \tag{4.54}$$

This result coincides with the one derived in [69] with supergravity techniques. Let us emphasize however that in our method the scaling dimensions must be large, so the match for protected operators occurs only in the limit of large R-charges.

4.4.2 Structure constants with unprotected operators

We are now ready to summarize the main results obtained for correlators with primary operators dual to string states in the first mass level. For the case of three Konishi operators we are free to send all R-charges to zero, so the superstring amplitude simplifies to

$$\langle VVV \rangle = g_c^3 \frac{3^8}{2^9}\,. \tag{4.55}$$

The simplicity of the prime factors in this expression is remarkable as it comes after the manipulation of roughly a million terms in Mathematica. Plugging this result into (4.44) we obtain the structure contant for three Konishi operators at strong coupling

$$C_{KKK} = \frac{2^6 \pi^{1/2} \lambda^{1/4}}{N} \left(\frac{3}{4}\right)^{2\lambda^{1/4}+5/2}\,. \tag{4.56}$$

The structure constant is exponentially supressed, but that is a universal property of semiclassical three-point functions. The intuition for this stems from the fact that each prong in the three-point function is longer than the two-point semicircle geodesics. The action (4.5) then indicates that for large dimensions this leads to an exponential suppression.

Another important result is for two chiral primaries and one Konishi operator. In order to have R-charge conservation we need the chiral primaries to have opposite R-charges and their dimension is then $\Delta = J$. The most interesting regime in this case is near extremality $J \approx \lambda^{1/4}$, for which we get the following string coupling

$$\mathcal{G}_{123} = \frac{8\pi\sqrt{\lambda}}{N}\,, \tag{4.57}$$

which leads to the structure constant

$$C_{CCK} = \frac{\lambda^{3/8}}{N(2J - \Delta_K)}\,. \tag{4.58}$$

This case is especially interesting as we can see that the coupling does not vanish, which means that the structure constant develops a pole. We will study this in more detail in section 4.5.

Finally, we can look at the correlation function of two Konishi operators and one chiral primary of dimension J. In order to have R-charge conservation we will need to give some R-charge to one of the operators and make it a Konishi relative as in (4.27). Since we take $J \ll \lambda^{1/4}$, the R-charge will not make an appearance in the string coupling, leading to the following structure constant

$$C_{CKK} = \frac{\pi^{1/2}\lambda^{1/4}}{N} 2^{-J}\,. \tag{4.59}$$

In this case two of the operators are much heavier than the third one, so the intersection point (4.7) lies on the geodesic between them, and the exponential suppression scales with the dimension of the light operator.

4.4.3 From superstring to CFT building blocks

Let us now move on to the case of operators with spin, dual to states in the leading Regge trajectory. In [71] Schlotterer obtained the open superstring amplitude for three states of spin s_i in the leading Regge trajectory at mass levels n_i, obtaining

$$\begin{aligned}\sum_{i,j,k\in\mathcal{I}} &\frac{(\alpha'/2)^{-i-j-k}(is_3 + js_2 + ks_1 - ij - ik - jk)}{i!j!k!(s_1 - i - j)!(s_2 - i - k)!(s_3 - j - k)!}(Z_1\cdot k_2)^{s_1-i-j} \\ &\times (Z_2\cdot k_3)^{s_2-i-k}(Z_3\cdot k_1)^{s_3-j-k}(Z_1\cdot Z_2)^i(Z_1\cdot Z_3)^j(Z_2\cdot Z_3)^k\,,\end{aligned} \tag{4.60}$$

where the summation range is

$$\mathcal{I} = \{i, j, k \in \mathbb{N}_0 : s_1 - i - j \geq 0,\ s_2 - i - k \geq 0,\ s_3 - j - k \geq 0\}\,. \tag{4.61}$$

In our case this would be the contribution for only the left and right movers independently, but we can still see that the closed superstring amplitude will be a sum over all terms of the form

$$(Z_1\cdot k_2)^{S_1-i-j}(Z_2\cdot k_3)^{S_2-i-k}(Z_3\cdot k_1)^{S_3-j-k}(Z_1\cdot Z_2)^i(Z_1\cdot Z_3)^j(Z_2\cdot Z_3)^k\,. \tag{4.62}$$

We will now see that we can relate these superstring building blocks with the tensor structures allowed by conformal symmetry in (2.36). We will restrict here to the case of massive vertices, where we can send the R-charge to zero and use the expression (4.9) for the momenta k_i. The key observation is that the polarization at the intersection point Z^M lives on a four-dimensional plane perpendicular to the momentum, which we can map to the four-dimensional boundary space, where the polarization vector becomes z^μ. The transformation between these is

$$Z_i^M = z_i^\mu (P_i)_\mu^M\,, \tag{4.63}$$

where the matrix P_μ^M performs a change of basis. We can then map the superstring building blocks to boundary objects through

$$Z_i \cdot k_j = z_i^\mu (k_j^M (P_i)_\mu^M) = - \frac{2\sqrt{\alpha_1\alpha_2\alpha_3\Sigma}x_{jk}^2}{|x_{12}||x_{13}||x_{23}|\Delta_i} V_i \bigg|_{Z(z,x),X(x)}\,, \tag{4.64}$$

and

$$Z_i \cdot Z_j = z_i^\mu (P_i P_j^{\mathrm{T}})_{\mu\nu} z_j^\nu = \frac{1}{x_{ij}^2}\left(H_{ij} + \frac{2\alpha_i\alpha_j}{\Delta_i\Delta_j}V_iV_j\right)\bigg|_{Z(z,x),X(x)}\,. \tag{4.65}$$

Putting all terms of (4.62) together, we get the following expression

$$\prod_i V_i^{S_i - \sum_{j\neq i} n_{ij}} \prod_{i<j} |x_{ij}|^{S_k - S_i - S_j}\left(H_{ij} + \frac{2\alpha_i\alpha_j}{\Delta_i\Delta_j}V_iV_j\right)^{n_{ij}}\,. \tag{4.66}$$

A very important feature of this equation is that the spacetime dependence combines with the factor $|x_{12}|^{-2\alpha_3}|x_{13}|^{-2\alpha_2}|x_{23}|^{-2\alpha_1}$ from the geodesic propagation to the intersection point, reproducing exactly the spacetime dependence expected in correlation functions of operators with spin (2.36). This shows that the correct prescription for the geodesic propagation in AdS is with scalar cubic Witten diagram, with the flat-space vertex operators taking into account the different nature of the operators.

4.5 Extremality and Mixing

A three-point function is extremal when the dimension of one of the operators equals the sum of the dimensions of the other two. In general a single-trace operator mixes with multi-trace operators, but in the planar limit one can usually ignore such corrections. However, this is not true for extremal correlation functions, where mixing with double-trace operators becomes important.

When studying the three-point functions of chiral operators, the integration of the Witten diagram still produces a pole, but the coupling vanishes in the extremal limit. It can be seen from analytic continuation that the correlators remain finite, while in [72] the authors worked exactly at extremality and showed that the results are due to boundary terms.

Meanwhile, in our computation of the structure constant between Konishi and two chiral operators, we saw that the coupling remains finite while the correlator develops a pole. We can relate this to mixing with double-trace operators, but in order to do so we will first consider a toy model which leads to insight in the problem.

4.5.1 Toy Model

Consider a theory with a complex scalar field ϕ in d dimensions. Its scaling dimension is $\delta = \frac{d}{2} - 1$ and we will assume it is normalized such that its two-point function is

$$\langle \phi^*(x)\phi(y) \rangle = \frac{1}{|x-y|^{2\delta}} = \frac{1}{|x-y|^{d-2}}\,. \tag{4.67}$$

We can also add an interaction term to the Lagrangian

$$\mathcal{L}_{\text{int}} = \lambda \phi^* \phi\, \mathcal{O}(x)\,, \tag{4.68}$$

where $\mathcal{O}$ is an operator of dimension two and the coupling λ is dimensionless in any dimension. At one-loop, the three-point correlator is given by

$$\langle \phi^*(x_1)\phi(x_2)\mathcal{O}(x_3) \rangle = \lambda \int \mathrm{d}^d z \frac{1}{|z-x_1|^{2\delta}|z-x_2|^{2\delta}|z-x_3|^4}\,. \tag{4.69}$$

For $0 \leq d-4 \ll 1$ this becomes a conformal integral and so we obtain the three-point function expected from conformal symmetry

$$\langle \phi^*(x_1)\phi(x_2)\mathcal{O}(x_3) \rangle = \frac{\mathcal{C}}{|x_{12}|^{d-4}|x_{23}|^2|x_{13}|^2} \approx \frac{\mathcal{C}}{|x_{23}|^2|x_{13}|^2}\,. \tag{4.70}$$

Note that in this regime the correlation function becomes extremal, as the dimension of the complex scalar approaches half the dimension of $\mathcal{O}$.

Let us now assume that the complex scalars are much closer to each other than to the operator $\mathcal{O}$, so that $|x_{12}| = \epsilon \ll |x_{13}|$. In this case the integral is approximately

$$\langle \phi^*(x_1)\phi(x_2)\mathcal{O}(x_3) \rangle \approx \lambda \int_\epsilon d^d z \frac{1}{|z|^{4\delta}|z-x_{31}|^4}\,, \tag{4.71}$$

where we have introduced a cutoff ϵ around x_1. We will also assume there is a cutoff $\tilde{\epsilon}$ around $z = x_{31}$ which we can later take to zero. The dominant contribution to the integral is near $z = 0$ and $z = x_{31}$, hence we find that

$$\langle\phi^*(x_1)\phi(x_2)\mathcal{O}(x_3)\rangle \approx \frac{2\pi^2\lambda}{(d-4)|x_{13}|^d}\left(\left(\frac{|x_{13}|}{\epsilon}\right)^{d-4} - \left(\frac{\tilde{\epsilon}}{|x_{13}|}\right)^{d-4}\right). \tag{4.72}$$

Now we can consider two different limits of this expression. On one hand we can take $\tilde{\epsilon}$ to zero in a non-singular way for $d > 4$. The three-point function becomes

$$\langle\phi^*(x_1)\phi(x_2)\mathcal{O}(x_3)\rangle \approx \frac{2\pi^2\lambda}{(d-4)\epsilon^{d-4}|x_{13}|^4}, \tag{4.73}$$

and we recognize it to be of the form (4.70), thus fixing $\mathcal{C}$

$$\mathcal{C} = \frac{2\pi^2\lambda}{d-4}. \tag{4.74}$$

On the other hand, we could have considered a different limit, where $\tilde{\epsilon}$ is of the same order of ϵ and we take first the extremal limit $d \to 4$. In this case the integral becomes

$$\langle\phi^*(x_1)\phi(x_2)\mathcal{O}(x_3)\rangle \approx \frac{4\pi^2\lambda}{|x_{13}|^d}\log\frac{\epsilon}{|x_{13}|}. \tag{4.75}$$

This is the sort of term one computes for the anomalous dimension mixing between $:\phi^*\phi:$ and $\mathcal{O}$.

In comparing the two limits we see that the former approach uses a UV cutoff for one operator and dimensional regularization for the other in order to regulate the integral, while the latter approach imposes a UV cutoff at the positions of both operators. Using the UV dimreg combination, the mixing of the operators is given by

$$\begin{aligned}\mathcal{O}^{ren} &= \mathcal{O} - \lim_{d\to 4}\frac{2\pi^2\lambda}{d-4}\mu^{4-d} :\phi^*\phi:, \\ :\phi^*\phi:^{ren} &= :\phi^*\phi: + \lim_{d\to 4} 2\pi^2\lambda\mu^{d-4}\log(\epsilon\mu)\,\mathcal{O},\end{aligned} \tag{4.76}$$

where the powers of μ are necessary to match dimensions. The anomalous dimension matrix then has the form

$$\begin{pmatrix} 0 & \delta_{\mathcal{O},:\phi^*\phi:} \\ \delta_{:\phi^*\phi:,\mathcal{O}} & 0 \end{pmatrix}, \tag{4.77}$$

where

$$\delta_{:\phi^*\phi:,\mathcal{O}} = \mu\frac{\partial}{\partial\mu}2\pi^2\lambda\log(\epsilon\mu) = 2\pi^2\lambda = \delta_{\mathcal{O},:\phi^*\phi:}. \tag{4.78}$$

So we conclude that the splitting between the operators is $4\pi^2\lambda$.

4.5.2 Mixing in $\mathcal{N}=4$ SYM

We can easily adapt the previous discussion to the case of $\mathcal{N}=4$ SYM. The analog of the coefficient $\mathcal{C}$ is the structure constant between the Konishi and two chiral operators from equation (4.58)

$$\mathcal{C}_{123} \approx \frac{\lambda^{3/8}}{N} \frac{1}{2J-\Delta_K}\,, \tag{4.79}$$

while the role of the coupling is played by $1/N$, and extremality is approached at $J \sim \lambda^{1/4}$.

The structure constant for physical operators must be finite, so the pole indicates that there is mixing between the Konishi and the double-trace operators $\mathcal{O}_{J\bar{J}}(x)$. In order to find the mixing we need to regulate the double-trace operator using a point-splitting technique

$$\mathcal{O}^{\epsilon}_{J\bar{J}}(x) = \,:\mathcal{O}_J(x+\epsilon)\mathcal{O}_{\bar{J}}(x):\ . \tag{4.80}$$

The two-point function between the double-trace operator and Konishi can be related to the three-point function, so one obtains

$$\langle \mathcal{O}^{\epsilon}_{J\bar{J}}(x)\mathcal{O}_K(y)\rangle \approx \frac{\lambda^{3/8}}{N} \frac{1}{(2J-\Delta_K)|\epsilon|^{2J-\Delta_K}} \frac{1}{|x-y|^{2\Delta_K}}\,. \tag{4.81}$$

We can therefore extract the non-planar contribution to the two-point function from a planar three-point function. As one would expect, there is no mixing between Konishi and the double-trace operator in the limit of infinite N.

It is important to remember that Konishi is an R-charge singlet, so it can only mix with the R-charge singlet appearing in the tensor product of the two J-symmetric traceless representations. The two-point function (4.81) corresponds only to one of the components of the double-trace singlet, which we can write as

$$\mathcal{O}^{\epsilon}_{s,J} = \sum_{\vec{I}} C_{\vec{I}} : \mathcal{O}_{\vec{I}}\, \mathcal{O}_{-\vec{I}} :\,, \tag{4.82}$$

where we sum over over all states $\vec{I}$ in the representation $[0,J,0]$ of $SU(4)$. Since the state is an R-charge singlet it must be annihilated by all roots of the algebra, which implies that all coefficients $C_{\vec{I}}$ must be equal up to a phase. Since the dimension of the representation is

$$M = \frac{(J+1)(J+2)^2(J+3)}{12} \approx \frac{J^4}{12}\,, \tag{4.83}$$

then the two-point function of the Konishi operator with the double-trace singlet is

$$\langle \mathcal{O}^{\epsilon}_{s,J}(x)\mathcal{O}_K(y)\rangle \approx \frac{\lambda^{3/8}\sqrt{M}}{N} \frac{1}{(2J-\Delta)|\epsilon|^{2J-\Delta}} \frac{1}{|x-y|^{2\Delta}}\,. \tag{4.84}$$

Besides the pole, there is a logarithmic divergence in the UV cutoff, so we cancel both of these with the following renormalized operators

$$\mathcal{O}^{\epsilon,ren}_{s,J} = \mathcal{O}^{\epsilon}_{s,J} + \lim_{\Delta\to 2J} \frac{\lambda^{3/8}\sqrt{M}\log(\epsilon\mu)}{N}\mu^{2J-\Delta}\,\mathcal{O}_K\,,$$
$$\mathcal{O}^{ren}_{K} = \mathcal{O}_K - \lim_{\Delta\to 2J} \frac{\lambda^{3/8}\sqrt{M}}{N(2J-\Delta)}\mu^{\Delta-2J}\,\mathcal{O}^{\epsilon}_{s,J}\,, \tag{4.85}$$

with μ the renormalization scale. To leading order in $1/N$, the three-point function of two chiral operators and $\mathcal{O}^{ren}_K$ now becomes finite

$$\langle \mathcal{O}_J(x_1)\mathcal{O}_{\bar{J}}(x_2)\mathcal{O}^{ren}_K(x_3)\rangle \approx \frac{\lambda^{3/8}\sqrt{M}}{N|x_{13}|^{2J}|x_{23}|^{2J}}\log\frac{|x_{12}|}{\mu|x_{13}||x_{23}|}\,. \tag{4.86}$$

Analogously, the correlation function of the chiral operators with $\mathcal{O}^{ren}_{s,J}$ is also finite at leading order in $1/N$. The three-point functions are not of the form imposed by conformal symmetry anymore since the operators do not have definite scaling dimensions. To find the correct primary operators we need to diagonalize the anomalous dimension matrix

$$\Gamma = \begin{pmatrix} 0 & \delta_{sJ,K} \\ \delta_{K,sJ} & 0 \end{pmatrix}\,, \tag{4.87}$$

where the off-diagonal elements are given by

$$\delta_{sJ,K} = \delta_{K,sJ} = \frac{\lambda^{3/8}\sqrt{M}}{N}\,. \tag{4.88}$$

At the cross-over point $J = \Delta/2 \approx \lambda^{1/4}$, the mixed operators have dimensions

$$\Delta_\pm = 2J \pm \frac{\sqrt{M}\lambda^{3/8}}{N} \tag{4.89}$$

and thus the splitting is given by[1]

$$\Delta_+ - \Delta_- \approx \frac{\lambda^{7/8}}{\sqrt{3}\,N}\,. \tag{4.90}$$

Note that the splitting found is much larger that the leading correction to the double-trace dimension coming from supergravity [73].

The operators with definite scaling dimension at the cross-over point are

$$\mathcal{O}_\pm = \frac{1}{\sqrt{2}}(\mathcal{O}^{ren}_K \pm \mathcal{O}^{ren}_{s,J})\,, \tag{4.91}$$

[1] The published paper missed a factor of 2, which was meanwhile fixed in the preprint `arXiv:1410.4746`.

and their three-point functions with two chiral operators now have the expected spacetime dependence

$$\langle \mathcal{O}_J(x_1)\mathcal{O}_{\bar{J}}(x_2)\mathcal{O}_{\pm}(x_3)\rangle = \frac{1}{\sqrt{2}}\frac{1}{|x_{13}|^{\Delta_\pm}|x_{23}|^{\Delta_\pm}|x_{12}|^{2J-\Delta_\pm}}\,. \tag{4.92}$$

After Paper II was published, Korchemsky extended the results presented above by considering the splitting of operators away from the cross-over point [74]. His considerations are for generic CFTs and so he finds universal behaviours for the anomalous dimensions and OPE coefficients in the vicinity of the crossing.

5. Weak Coupling

Integrability of $\mathcal{N} = 4$ SYM paves the way for a finite coupling description of many of its observables. A paradigmatic example is the planar anomalous dimension of single trace operators. As the integrability framework is developed, it is crucial to compare its proposals with perturbative computations done by other means. For example, the perturbative calculations in [75–78] were crucial for understanding wrapping corrections in the context of the spectrum of anomalous dimensions.

More recently there has been great progress in the understanding of other observables at finite coupling. A striking example is the all-loop Hexagon proposal [14] for the three-point functions of $\mathcal{N} = 4$ SYM. The Hexagon form factors are also the building blocks for higher-point functions [79], so it is extremely important to ensure that we have a good understanding of those objects.

The Hexagon has been thoroughly checked up to three loops [80–82], but at four loops there is a contribution from two virtual particles which leads to a double pole singularity. In [83] the authors have provided a resolution to this issue by introducing a renormalization prescription, and matched the perturbative computations of [84, 85].

It is important to check if the regularization of the double pole singularity continues to hold without any change at the five loop level, so one must perform a perturbative computation of the OPE coefficients at five loops. While doing so we find the structure constant to be given in terms of massless propagator integrals, which are currently unknown. Nevertheless, we were able to devise a strategy that determines most of these integrals without any explicit integration.

5.1 Structure constant from four-point function

We will be considering the four-point function of the half-BPS operator in the $[0, 2, 0]$ representation of $SU(4)$. This is a very special operator, as it belongs to the same supermultiplet as the Lagrangian and the stress-energy tensor. Just like in (2.29) we encoded operators with spin as polynomials, it is also useful here to introduce the antisymmetric harmonic variables Y_{IJ} and write the chiral operator in the following way

$$\mathcal{O}(x, Y) = Y_{IJ} Y_{KL} \operatorname{Tr}[\phi^{IJ} \phi^{KL}](x) \,. \tag{5.1}$$

The operator is manifestly symmetric, and tracelessness is obtained by requiring that the harmonic variables are null

$$Y_{IJ}Y^{IJ} = 0\,. \tag{5.2}$$

The four-point function can be decomposed into a sum over conformal blocks, where each block corresponds to the contribution of a primary operator and its descendants to the OPE of two chiral primaries

$$\langle \mathcal{O}(x_1, Y_1) \ldots \mathcal{O}(x_4, Y_4)\rangle = \sum_k c_k^2 \frac{G_{\Delta_k, S_k}(u,v)}{x_{12}^4 x_{34}^4} \frac{\mathcal{F}_{m_k, n_k}(\sigma, \tau)}{y_{12}^{-4} y_{34}^{-4}}\,, \tag{5.3}$$

where we introduce the R-charge cross ratios

$$\sigma = \frac{y_{12}^2 y_{34}^2}{y_{13}^2 y_{24}^2}\,, \qquad \tau = \frac{y_{14}^2 y_{23}^2}{y_{13}^2 y_{24}^2}\,, \tag{5.4}$$

and the short-hand notation $y_{ij}^2 = Y_{i,IJ} Y_j^{IJ}$. The labels of the blocks $\mathcal{F}_{m,n}$ correspond to the exchange of operators in the $[n-m, 2m, n-m]$ representation of $SU(4)$.

Since we consider a four-point function of scalar operators, there can only be operators of even spin in the OPE. There is a closed expression for the conformal blocks in four dimensions, and in the OPE limit of small u the conformal block for an operator of dimension Δ and spin S simplifies to [86]

$$G_{\Delta,S}(u,v) \approx u^{\frac{\Delta - S}{2}} \left(\frac{v-1}{2}\right)^S {}_2F_1\left(\frac{\Delta+S}{2}, \frac{\Delta+S}{2}, \Delta+S, 1-v\right)\,. \tag{5.5}$$

For small u the leading contribution comes from operators with the lowest twist $\Delta - S$. The identity operator is the only operator with twist zero, and next we have several operators with twist two. There are six representations in the tensor product of two $[0,2,0]$ representations of $SU(4)$

$$20' \otimes 20' = 1 \oplus 15 \oplus 20' \oplus 84 \oplus 105 \oplus 175\,, \tag{5.6}$$

so for each twist we will have to consider six distinct channels in the OPE decomposition. In general the singlet channel is more difficult to study as it has a contribution from three twist two operators (4.32). Meanwhile, in the channel of the $20'$ there is a single twist two primary for each spin. For $S=2$ it corresponds to the primary $\mathrm{Tr}[ZD^2Z]$, which is in the same supermultiplet of the Konishi operator. The relation between the structure constants can be found from supersymmetry to be

$$c_{\mathcal{K}}^2 = 3 c_{20'}^2\,. \tag{5.7}$$

Since the spinless operator in the $20'$ channel is the chiral operator itself, which is protected, then the spin two operator is the first one appearing beyond tree level. Expanding both the anomalous dimension γ and the structure constant c_k in the coupling we obtain

$$u^{\gamma(\lambda)/2} c_{\mathcal{K}}^2(\lambda) = \sum_{n=0}^{5} \sum_{k=0}^{n} c_{n,k} \log(u)^k \lambda^n \,. \tag{5.8}$$

5.2 Perturbative Correlators

We have seen how we can obtain the structure constant of Konishi with two half-BPS operators from the OPE limit of a four-point function of chiral operators. The correlation function can be expanded as

$$C_4 = \langle \mathcal{O}(x_1, Y_1) \mathcal{O}(x_2, Y_2) \mathcal{O}(x_3, Y_3) \mathcal{O}(x_4, Y_4) \rangle = \sum_{l=0}^{\infty} g^{2l} C_4^{(l)} \,, \tag{5.9}$$

with the tree level component given by

$$\begin{aligned} C_4^{(0)} =& \frac{N^2-1}{(4\pi^2)^4} \left(\frac{y_{12}^2 y_{23}^2 y_{34}^2 y_{41}^2}{x_{12}^2 x_{23}^2 x_{34}^2 x_{41}^2} + \frac{y_{12}^2 y_{24}^2 y_{43}^2 y_{31}^2}{x_{12}^2 x_{24}^2 x_{43}^2 x_{31}^2} + \frac{y_{13}^2 y_{32}^2 y_{24}^2 y_{41}^2}{x_{13}^2 x_{32}^2 x_{24}^2 x_{41}^2} \right) \\ &+ \frac{(N^2-1)^2}{4(4\pi^2)^4} \left(\frac{y_{12}^4 y_{34}^4}{x_{12}^4 x_{34}^4} + \frac{y_{13}^4 y_{24}^4}{x_{13}^4 x_{24}^4} + \frac{y_{14}^4 y_{23}^4}{x_{14}^4 x_{23}^4} \right) . \end{aligned} \tag{5.10}$$

And we can see that the first line gives the connected contribution while the second corresponds to disconnected diagrams.

We can see already from the tree level result that it splits into six structures, corresponding to the six irreducible representations in the tensor product of two $[0, 2, 0]$ representations of $SU(4)$. In general, at higher loops the correlator will still be a linear combination of these six structures, each accompanied by a funtion of the cross ratios u and v. However, since the correlator is invariant under exchange of the four external points, we can show that only two of them are independent. Furthermore, in a superconformal theory such as $\mathcal{N} = 4$ SYM, it can be shown that supersymmetry imposes further constraints, and the quantum corrections to the four-point function take in fact a factorized form [87]

$$C_4^{(l)} = \frac{2(N^2-1)}{(4\pi^2)^4} R(x_i, Y_i) F^{(l)}(x_i) \,, \tag{5.11}$$

where the factor $R(x_i, Y_i)$ contains all the information on the harmonic variables

$$
\begin{aligned}
R =& \frac{y_{12}^2 y_{23}^2 y_{34}^2 y_{41}^2}{x_{12}^2 x_{23}^2 x_{34}^2 x_{41}^2}(x_{13}^2 x_{24}^2 - x_{12}^2 x_{34}^2 - x_{14}^2 x_{23}^2) + \frac{y_{13}^4 y_{24}^4}{x_{13}^2 x_{24}^2} \\
&+ \frac{y_{12}^2 y_{24}^2 y_{43}^2 y_{31}^2}{x_{12}^2 x_{24}^2 x_{43}^2 x_{31}^2}(x_{14}^2 x_{23}^2 - x_{12}^2 x_{34}^2 - x_{13}^2 x_{24}^2) + \frac{y_{14}^4 y_{23}^4}{x_{14}^2 x_{23}^2} \\
&+ \frac{y_{13}^2 y_{32}^2 y_{24}^2 y_{41}^2}{x_{13}^2 x_{32}^2 x_{24}^2 x_{41}^2}(x_{12}^2 x_{34}^2 - x_{13}^2 x_{24}^2 - x_{14}^2 x_{23}^2) + \frac{y_{12}^4 y_{34}^4}{x_{12}^2 x_{34}^2} \,. \qquad (5.12)
\end{aligned}
$$

The functions $F^{(l)}$ are given by l-loop Feynman diagrams with four external operators, but we can rewrite them as tree-level $(4+l)$-point correlators by using the method of Lagrangian insertion [88]. This equivalence stems from the fact that the derivative of a correlation function with respect to the coupling brings down insertions of the corresponding marginal operator

$$\lambda \frac{\partial}{\partial \lambda} \langle \mathcal{O}(x_1) \dots \mathcal{O}(x_4) \rangle = \int \mathrm{d}^4 x_5 \langle \mathcal{O}(x_1) \dots \mathcal{O}(x_4) \mathcal{L}(x_5) \rangle \,. \qquad (5.13)$$

Since $F^{(l)}$ are conformally covariant on the external points with weight $+1$, we write them as

$$F^{(l)} = \frac{x_{12}^2 x_{13}^2 x_{14}^2 x_{23}^2 x_{24}^2 x_{34}^2}{l!(-4\pi^2)^l} \int \mathrm{d}^4 x_5 \dots \mathrm{d}^4 x_{4+l} f^{(l)}(x_1, \dots, x_{4+l}) \,, \qquad (5.14)$$

so that the functions $f^{(l)}$ are conformally covariant on all $4+l$ points with weight $+4$. The explicit symmetry is $S_4 \times S_l$ corresponding to permutations of both external and internal points. So far these are very generic considerations, but in $\mathcal{N}=4$ SYM something special happens, as supersymmetry requires the functions $f^{(l)}$ to be completely symmetric under exchange of any two of the $4+l$ points, thus enhancing the symmetry to S_{4+l} [89]. It is useful to consider the following ansatz

$$f^{(l)}(x_1, \dots, x_{4+l}) = \frac{P^{(l)}(x_1, \dots, x_{4+l})}{\prod_{1 \le i < j \le 4+l} x_{ij}^2} \,, \qquad (5.15)$$

so that the numerator $P^{(l)}(x_1, \dots, x_{4+l})$ is a homogeneous polynomial in x_{ij}^2 of degree $l-1$ on each point. The reason we make this ansatz is because one can show that in the OPE limit of $x_i \to x_j$ the $(4+l)$-point function of chiral and Lagrangian operators cannot have a stronger divergence than x_{ij}^{-2}.

We then see that at one and two loops the polynomial is fixed from these considerations alone, as the only totally symmetric polynomials with the correct weights are

$$P^{(1)} = 1 \,, \qquad P^{(2)} = \frac{1}{48} x_{12}^2 x_{34}^2 x_{56}^2 + S_6 \text{ permutations} \,, \qquad (5.16)$$

where the coefficients are such that each term appears only once.

A crucial observation is that only planar conformal integrals contribute to the planar limit of the four-point function. At three loops there is only one planar conformal integral, but at four loops there are 3 while at five loops there are 7. In order to find the correct linear combination of polynomials at each loop order, one needs to study the Minkowski light-cone limit and require that it has the singular behaviour expected from the analysis of the logarithm of the correlation function [90, 91]. This limit is defined by taking the external points to be light-like separated from each other. In the end such techniques are sufficient for the determination of the planar integrand up to very high loop order. The three- and four-loop polynomials are

$$
\begin{aligned}
P^{(3)} &= \frac{1}{20} x_{12}^4 x_{34}^2 x_{45}^2 x_{56}^2 x_{67}^2 x_{73} + S_7 \text{ permutations}\,, \\
P^{(4)} &= \frac{1}{24} x_{12}^2 x_{13}^2 x_{16}^2 x_{23}^2 x_{25}^2 x_{34}^2 x_{45}^2 x_{46}^2 x_{56}^2 x_{78}^6 \\
&\quad + \frac{1}{8} x_{12}^2 x_{13}^2 x_{16}^2 x_{24}^2 x_{27}^2 x_{34}^2 x_{38}^2 x_{45}^2 x_{56}^4 x_{78}^4 \\
&\quad - \frac{1}{16} x_{12}^2 x_{15}^2 x_{18}^2 x_{23}^2 x_{26}^2 x_{34}^2 x_{37}^2 x_{45}^2 x_{48}^2 x_{56}^2 x_{67}^2 x_{78}^2 \\
&\quad + S_8 \text{ permutations}\,,
\end{aligned}
\tag{5.17}
$$

while at five loops we have

$$
\begin{aligned}
P^{(5)} = &-\frac{1}{2} x_{13}^2 x_{16}^2 x_{18}^2 x_{19}^2 x_{24}^4 x_{26}^2 x_{29}^2 x_{37}^2 x_{38}^2 x_{39}^2 x_{47}^2 x_{48}^2 x_{56}^2 x_{57}^2 x_{58}^2 x_{59}^2 x_{67}^2 \\
&+ \frac{1}{4} x_{13}^2 x_{16}^2 x_{18}^2 x_{19}^2 x_{24}^4 x_{26}^2 x_{29}^2 x_{37}^4 x_{39}^2 x_{48}^4 x_{56}^2 x_{57}^2 x_{58}^2 x_{59}^2 x_{67}^2 \\
&+ \frac{1}{4} x_{13}^4 x_{17}^2 x_{19}^2 x_{24}^2 x_{26}^2 x_{27}^2 x_{29}^2 x_{36}^2 x_{39}^2 x_{48}^6 x_{56}^2 x_{57}^2 x_{58}^2 x_{59}^2 x_{67}^2 \\
&+ \frac{1}{6} x_{13}^2 x_{16}^2 x_{19}^4 x_{24}^4 x_{28}^2 x_{29}^2 x_{37}^4 x_{38}^2 x_{46}^2 x_{47}^2 x_{56}^2 x_{57}^2 x_{58}^2 x_{59}^2 x_{68}^2 \\
&- \frac{1}{8} x_{13}^4 x_{16}^2 x_{18}^2 x_{24}^4 x_{28}^2 x_{29}^2 x_{37}^2 x_{39}^2 x_{46}^2 x_{47}^2 x_{56}^2 x_{57}^2 x_{58}^2 x_{59}^2 x_{69}^2 x_{78}^2 \\
&+ \frac{1}{28} x_{13}^2 x_{17}^2 x_{18}^2 x_{19}^2 x_{24}^8 x_{36}^2 x_{38}^2 x_{39}^2 x_{56}^2 x_{57}^2 x_{58}^2 x_{59}^2 x_{67}^2 x_{69}^2 x_{78}^2 \\
&+ \frac{1}{12} x_{13}^2 x_{16}^2 x_{17}^2 x_{19}^2 x_{26}^2 x_{27}^2 x_{28}^2 x_{29}^2 x_{35}^2 x_{38}^2 x_{39}^2 x_{45}^2 x_{46}^2 x_{47}^2 x_{49}^2 x_{57}^2 x_{58}^2 x_{68}^2 \\
&+ S_9 \text{ permutations}\,.
\end{aligned}
\tag{5.18}
$$

5.3 Four-point function in the OPE limit

We have seen above that we can obtain the perturbative four-point function without ever considering Feynman diagrams, which is a very impressive result. It would seem at this point that the problem is almost solved, since we know the integrand and only need to perform the integration. Up to three-loop order this task is doable, but beyond three loops there are no techniques for the evaluation of all the necessary conformal integrals.

5.3.1 Asymptotic expansions

However, it is important to note that ultimately our goal is finding the structure constant, so we only need to perform the integrations in the coincidence limit of small x_{12}, which can be done with the method of asymptotic expansions [92]. For simplicity, we use the conformal symmetry to set the point x_1 to zero and x_4 to infinity. Usually one would also use the degrees of freedom left to fix x_3, but we will not do so as the method of asymptotic expansions introduces spurious powers of $\log(x_3)$. The fact that these spurious terms must vanish introduces many constraints on the master integrals, so it is useful to leave x_3 generic. In the coincidence limit we have $|x_2| \ll |x_3|$ and the cross ratios become

$$u = \frac{x_2^2}{x_3^2} \approx 0\,, \qquad v = 1 - \frac{2x_2 \cdot x_3}{x_3^2} + u \approx 1\,. \tag{5.19}$$

The polynomial (5.18) leads to 200 inequivalent conformal integrals Φ_i. Due to the permutations of external points we need to consider six distinct versions for each of these

$$\begin{aligned} &\Phi_i(u,v)\,, && \Phi_i(v,u)\,, && \Phi_i(1/u,v/u)\,, \\ &\Phi_i(u/v,1/v)\,, && \Phi_i(1/v,u/v)\,, && \Phi_i(v/u,1/u)\,. \end{aligned} \tag{5.20}$$

In the limit of $u \to 0$ and $v \to 1$ we then have an expansion of each conformal integral in different regions of its parameters.

In what follows we will denote the integration points x_{4+i} by k_i. The key idea in the method of asymptotic expansions is to separate each integration into two regions, where the integration point k_i is either of the order of x_2 or of x_3. For example, if k_i is close to x_3 while k_j is very small, then we can expand the propagators

$$\begin{aligned} \frac{1}{(x_2-k_i)^2} &= \sum_{n=0}^{\infty} \frac{(2x_2\cdot k_i - x_2^2)^n}{(k_i^2)^{1+n}}\,, \\ \frac{1}{(x_3-k_j)^2} &= \sum_{n=0}^{\infty} \frac{(2x_3\cdot k_j - k_j^2)^n}{(x_3^2)^{1+n}}\,, \\ \frac{1}{(k_i-k_j)^2} &= \sum_{n=0}^{\infty} \frac{(2k_i\cdot k_j - k_j^2)^n}{(k_i^2)^{1+n}}\,. \end{aligned} \tag{5.21}$$

In general the expressions in (5.21) are only valid for $k_i^2 > x_2^2$ and $k_j^2 < x_3^2$, but we can extend the region of integration to the whole space. In doing so we introduce divergences which can be regulated via dimensional regularization, where we set the dimension to be $4-2\epsilon$. For a five-loop integral we will then have 2^5 different regions we need to consider, and schematically the asymptotic expansion gives

$$\Phi^{(5)}(u,v) = \sum_{\{\alpha\}} P^{(|\alpha|)}(x_2) P^{(|\bar{\alpha}|)}(x_3)\,, \tag{5.22}$$

where α denotes a subset of the internal points $\{k_i\}$, $\bar{\alpha}$ is its complement and $P^{(l)}(x_i)$ represents an l-loop massless propagator integral with external points at the origin and at x_i. This is a very important simplification as we transformed an integral with three external points into products of integrals with two external points. We can see that only two of the propagator type integrals obtained with asymptotic expansions are five-loop integrals. They correspond to the cases where the integration variables are either all close to x_2 or all close to x_3.

5.3.2 Tensor reduction

In general, the conformal integrals have numerators such as $(k_i - x_3)^2$ and the asymptotic expansions include regions where k_i is of the order of x_2. The integral in k_i will therefore contain numerators where the integration variable is contracted with external vectors, which we need to rewrite as a combination of scalar integrals. The algorithm for tensor reduction of an integral with denominators D_a is [82]

$$\int \mathrm{d}^d k_1 \ldots \mathrm{d}^d k_n \frac{N^{\mu_1 \ldots \mu_s}}{\prod_a D_a} = \sum_{j=0}^{[s/2]} P_j^{\mu_1 \ldots \mu_s}(x_3) I_j(x_3)\,, \tag{5.23}$$

where $I_j(x_3)$ are scalar integrals and the tensor structure is now encoded outside the integral with the factors

$$P_j^{\mu_1 \ldots \mu_s}(x) = \frac{x^{2j}}{4^j j!} (\Box_x)^j \left(x^{\mu_1} \ldots x^{\mu_s}\right)\,. \tag{5.24}$$

The scalar integrals can be defined through a matrix $M_{ij}(x_3)$ and a set of scalar integrals $J_j(x_3)$

$$I_i(x_3) = M_{ij}^{-1}(x_3) J_j(x_3)\,, \tag{5.25}$$

where

$$
\begin{aligned}
M_{ij}(x) &= \frac{k^{2i}}{4^i i!}(\Box_k)^i \left(\frac{x^{2j}}{4^j j!}(\Box_x)^j (k\cdot x)^s\right)\Bigg|_{k=x}\,, \\
J_j(x) &= \int \frac{\mathrm{d}^d k_1 \dots \mathrm{d}^d k_n}{\prod_a D_a} \frac{x^{2j}}{4^j j!}(\Box_x)^j \left(x^{\mu_1}\dots x^{\mu_s} N_{\mu_1\dots\mu_s}\right)\,.
\end{aligned}
\tag{5.26}
$$

In the end, the l-loop massless p-integrals we need to compute are of the form

$$
P(\{a_i,b_i,c_{ij}\})(x) = \int \prod_{i=1}^{l} \mathrm{d}^d k_i \prod_{i=1}^{l} \frac{1}{k_i^{2a_i}} \prod_{i=1}^{l} \frac{1}{(x-k_i)^{2b_i}} \prod_{1\le i<j\le l} \frac{1}{k_{ij}^{2c_{ij}}}\,,
\tag{5.27}
$$

where all exponents are integer. It is possible to write the dependence on the scale x explicitly so that we obtain a purely numeric integral

$$
P(\{a_i,b_i,c_{ij}\}) = P(\{a_i,b_i,c_{ij}\})(x) \times x^{2\sum_i (a_i+b_i+\sum_{j>i} c_{ij}) - d\,l}(x)\,.
\tag{5.28}
$$

If we use (5.28) we can see that many regions of the asymptotic expansions do not contribute as they lead to higher powers of u. In some cases one also finds scaleless integrals where an integration variable appears at most in one denominator. These integrals are vanishing so we can throw them away.

5.4 Bootstrap of master integrals

At the end of the day we have transformed conformal integrals into combinations of massless propagator integrals. The optimal way to evaluate them is to first reduce them to master integrals through Integration By Parts identities [93]. This is a very important step as it reduces the integrals that need to be evaluated to a smaller set of simpler integrals. The IBP identities follow from the fact that the integral of a total derivative is vanishing

$$
\int \prod_{i=1}^{5} \mathrm{d}^d k_i \frac{\partial}{\partial k_i} \left(n_j \prod_{i=1}^{5} \frac{1}{k_i^{2a_i}} \prod_{i=1}^{5} \frac{1}{(x-k_i)^{2b_i}} \prod_{1\le i<j\le 5} \frac{1}{k_{ij}^{2c_{ij}}} \right) = 0\,,
\tag{5.29}
$$

where n_j is a polynomial that depends on internal and/or external points. The IBP identities can be rewritten as

$$
\sum \alpha_k(\{a_i,b_i,c_{ij}\}) P(\{a_i+\delta^{(1)}_{k,i}, b_i+\delta^{(2)}_{k,i}, c_{ij}+\delta^{(3)}_{k,ij}\}) = 0\,,
\tag{5.30}
$$

where $\delta^{(l)}_{k,i} \in \{-1,0,1\}$ and the cofficients α_k depend on the exponents of the denominators.

These identities can be implemented very efficiently with the help of the computer programs FIRE5 [94] and LiteRed [95]. The five-loop integrals we consider are exactly at the boundary of feasibility, since in many cases the reductions required the use of a cluster node with 256 GB of RAM.

For lower loop level, all master integrals are known up to the required order of ϵ. However, at five loops there is barely any data on the ϵ expansion of the master integrals. In the remainder of this section we show how one can obtain information on the five-loop master integrals without any explicit integration.

5.4.1 Canceling divergences

One of the key ideas is to realize that both the method of asymptotic expansions and the reduction to master integrals introduce spurious divergences. Any massless propagator integral diverges at most as ϵ^{-5}, but the coefficients of its reduction to master integrals may also include poles in ϵ. If the highest divergence introduced by the coefficients is of order k, then the expansion in ϵ of the integral gets shifted to

$$P(\{a_i, b_i, c_{ij}\}) = \sum_{n=-5-k}^{\infty} P(\{a_i, b_i, c_{ij}\})^{(n)} \epsilon^n \,. \qquad (5.31)$$

The integral appears to have a divergence of order higher than five, but in fact all those higher divergences must vanish, which leads to the following constraints on the coefficients

$$P(\{a_i, b_i, c_{ij}\})^{(n)} = 0 \qquad \text{for } n < -5 \,. \qquad (5.32)$$

This idea of cancellation of spurious divergences can be taken even further. All the conformal integrals that appear in the four-point function happen to be convergent, but after performing the asymptotic expansions there is an apparent divergence up to ϵ^{-5}. Besides this, the conformal integrals depend only on the cross ratios, but the expansions also introduce a spurious dependence on the point x_3. In general the expansion of a conformal integral looks like

$$\sum_{n=0}^{5} \sum_{k=0}^{n} \sum_{l=0}^{n-k} c_{nkl} \epsilon^{-5+n} \log(x_3^2)^k \log(u)^l \,, \qquad (5.33)$$

where the coefficients c_{nkl} depend on the expansions of the master integrals. The fact that only powers of $\log(u)$ are allowed to appear in the conformal integral lead to another set of equations

$$c_{nkl} = 0 \qquad \text{for } n \neq 5 \text{ and } k \neq 0 \,. \qquad (5.34)$$

It is worth noting that these systems of equations are highly redundant and the existence of a solution strongly indicates that both the asymptotic expansions and the reduction to master integrals were performed correctly.

5.4.2 Magic Identities

Since a subintegral of a conformal integral is also conformal, then we know that we can express the subintegral through a function of cross ratios and some spacetime factor that takes into account the conformal weights of its external points. We can permute the internal points of the subintegral without changing the function of cross ratios, but that permutation will have a non-trivial effect both on the subintegral and the full integral. In that way we can obtain very different representations of the same conformal integral, and the fact that they are equivalent leads to the so called magic identities [96].

In order to understand better how to implement these identities, let us consider an example of a five-loop conformal integral

$$I = \int \frac{\mathrm{d}^4x_5\mathrm{d}^4x_6\mathrm{d}^4x_7\mathrm{d}^4x_8\mathrm{d}^4x_9\ x_{15}^6}{x_{16}^2x_{17}^2x_{18}^2x_{19}^2x_{25}^2x_{28}^2x_{35}^2x_{36}^2x_{37}^2x_{45}^2x_{46}^2x_{56}^2x_{57}^2x_{58}^2x_{59}^2x_{79}^2x_{89}^2}\,. \tag{5.35}$$

We can now look at the subintegral formed by the integration variables x_7, x_8 and x_9. The external points of the subintegral are x_1, x_2, x_3 and x_5, and so we have

$$I_{\mathrm{sub}} = \int \frac{\mathrm{d}^4x_7\mathrm{d}^4x_8\mathrm{d}^4x_9}{x_{17}^2x_{18}^2x_{19}^2x_{28}^2x_{37}^2x_{57}^2x_{58}^2x_{59}^2x_{79}^2x_{89}^2} = \frac{1}{x_{15}^6x_{23}^2}\Phi(\tilde{u},\tilde{v})\,, \tag{5.36}$$

where I expressed the subintegral as a function $\Phi(\tilde{u},\tilde{v})$ of its cross ratios

$$\tilde{u} = \frac{x_{12}^2x_{35}^2}{x_{13}^2x_{25}^2}\,, \qquad \tilde{v} = \frac{x_{15}^2x_{23}^2}{x_{13}^2x_{25}^2}\,. \tag{5.37}$$

If we exchange x_1 with x_2 and x_3 with x_5, then the function of the cross ratios $\Phi(\tilde{u},\tilde{v})$ remains the same but we have

$$I'_{\mathrm{sub}} = \int \frac{\mathrm{d}^4x_7\mathrm{d}^4x_8\mathrm{d}^4x_9}{x_{18}^2x_{27}^2x_{28}^2x_{29}^2x_{37}^2x_{38}^2x_{39}^2x_{57}^2x_{79}^2x_{89}^2} = \frac{1}{x_{15}^2x_{23}^6}\Phi(\tilde{u},\tilde{v})\,. \tag{5.38}$$

We can then go back to the full integral I and substitute I_{sub} by I'_{sub}. Taking into account the different spacetime factors we obtain

$$I' = \int \frac{\mathrm{d}^4x_5\mathrm{d}^4x_6\mathrm{d}^4x_7\mathrm{d}^4x_8\mathrm{d}^4x_9\ x_{15}^2x_{23}^4}{x_{16}^2x_{18}^2x_{25}^2x_{27}^2x_{28}^2x_{29}^2x_{35}^2x_{36}^2x_{37}^2x_{38}^2x_{39}^2x_{45}^2x_{46}^2x_{56}^2x_{57}^2x_{79}^2x_{89}^2}\,. \tag{5.39}$$

The equivalence between the conformal integrals I and I' is highly non-trivial as the integrals have different numerators and a different number of propagators between internal points.

Usually this kind of magic identity is used to simplify a calculation. For example, if the integral I appeared in a physical problem then it would be wise to use I' instead as it is a much simpler integral. Nevertheless, we keep all equivalent versions of the conformal integrals as their asymptotic expansions lead to different expressions and in that way we can obtain even more constraints on the master integrals. Each conformal integral will lead to a set of equations like (5.34), so ultimately we have

$$I = \sum_{l=0}^{5} c_{50l} \log(u)^l \,, \qquad I' = \sum_{l=0}^{5} c'_{50l} \log(u)^l \,. \tag{5.40}$$

Since the integrals are equivalent, we can add the following constraint

$$c_{50l} = c'_{50l} \,. \tag{5.41}$$

5.4.3 Generalized ladder diagrams

At this point it is obvious that the more conformal integrals we analyze and the more p-integrals we reduce, the more information we gain on the ϵ expansion of master integrals. We will then consider a class of integrals introduced by Drummond [97] which generalizes ladder diagrams. At five loops there are sixteen such integrals, labeled by four letters a_i which can be either 1 or 2. The integrals are given by

$$L(a_1,\ldots,a_4) = \int \mathrm{d}^d x_{b_5} \frac{1}{x^2_{1b_1} x^2_{3b_5} x^2_{2b_5} x^2_{4b_5}} \prod_{i=1}^{4} \frac{\mathrm{d}^d x_{b_i}}{x^2_{b_i b_{i+1}} x^2_{b_i a_i} x^2_{4b_i}} \,. \tag{5.42}$$

We can see that half of the diagrams is mapped to the other half by some global symmetry, so we then have seven additional conformal integrals that we can evaluate.

For each conformal integral we consider the expansion of small u and $1-v$ in six regions of the parameter space as in equation (5.20). Just like before we use magic identities and the knowledge of allowed singularities in order to obtain more constraints on the master massless propagator integrals, with the advantage that in this case we can use the prescription of Drummond to evaluate the conformal integrals directly.

5.5 Results

While the original motivation of the project was the extraction of the structure constant of Konishi with half-BPS operators, along the way we found an extremely powerful method of obtaining five-loop master integrals without any explicit integration. We cannot in any way present here all the results obtained, but we will rather comment on some universal features of the integrals we have found.

5.5.1 Master Integrals

The simplicity of the methods used to determine master integrals is truly remarkable, as it amounts to solving a large system of linear equations, at least as long as all lower loop integrals are already known to the required order in ϵ. At the end of the day we obtained 1580 coefficients in the expansions of 179 master integrals, reaching terms of transcendentality weight 7 for as many as 159 of the integrals.

There are some universal transcendental features of the integrals which are worth commenting upon. However, a discussion on transcendentality is only sensible once the normalization of the integrals is specified. We have worked with the convention where the five-loop bubble diagram is given exactly by

$$\int \prod_{i=1}^{5} \frac{\mathrm{d}^d k_i}{k_i^2 (k_i - x)^2} = \frac{1}{\epsilon^5}\,, \tag{5.43}$$

for $x^2 = 1$.

For the lower loop master integrals, it had been observed that the transcendentality weight of the singular term ϵ^{-n} in an l-loop integral never exceeds $2l - 1 - n$. We have confirmed that this property of the singular terms still holds for the five-loop master integrals.

It was also found at lower loop level that the transcendentality weight in the coefficient of ϵ^{p-n} cannot exceed $k - n$, where k is the transcendentality weight at some higher power ϵ^p. Once again we confirmed that all the five-loop master integrals also have this property.

Let us emphasize that five-loop massless propagator integrals can be applied in many other problems. One such example is the evaluation of conformal integrals [98], where the integral is directly reduced master integrals. One then obtains a set of differential equations governing the master integrals, and their boundary value is given by the kind of propagator integrals we have considered. Something very analogous happens also in the computation of the photon-quark form factor of large N_c QCD [99], where the p-integrals enter as boundary values in the system of differential equations for the relevant master integrals.

Finally, the massless propagator integrals also appear in the study of integrable deformations of $\mathcal{N} = 4$ SYM and ABJM. In [100] the chiral limit of those theories has been considered, and the Feynman diagrams relevant for the spectrum are given by multi-loop massless propagator integrals. In that paper the authors proposed some relations between five-loop integrals from integrability considerations, which we have now confirmed to be true.

5.5.2 Structure constant at five loops

Regarding the structure constant of the Konishi operator with two half-BPS operators, the bootstrap of master integrals does not fix the result completely, but it provides nonetheless a much stronger simplification than we had expected. We have determined the following five-loop terms in the expansion of (5.8)

$$
\begin{aligned}
c_{5,1} &= 95072 + 14976\zeta(3) + 864\zeta(3)^2 + 34560\zeta(5) + 40320\zeta(7)\,, \\
c_{5,2} &= -30912 - 5184\zeta(3) - 8640\zeta(5)\,, \\
c_{5,3} &= 5328 + 864\zeta(3)\,, \\
c_{5,4} &= -504\,, \\
c_{5,5} &= \frac{108}{5}\,,
\end{aligned}
\tag{5.44}
$$

which matches the expected combination of the anomalous dimension and lower orders of the structure constant. The expression for $c_{5,0}$, which is proportional to the structure constant at five loops is

$$
\begin{aligned}
c_{5,0} = &-18\,P(\{0,0,0,1,1,0,0,1,0,1,0,1,1,1,1,1,1,0,0,0\})^{(2)} \\
&-216\,P(\{0,0,0,1,1,0,0,1,0,1,0,1,1,1,1,1,2,0,0,0\})^{(1)} \\
&+3\,P(\{0,0,0,1,1,0,1,1,0,0,0,1,1,1,1,1,1,0,0,0\})^{(2)} \\
&+8\,P(\{0,0,1,1,1,0,1,0,1,1,1,1,0,0,0,1,1,1,1,0\})^{(2)} \\
&-4\,P(\{0,0,1,1,1,1,2,0,1,2,1,1,-1,1,1,1,-1,1,-1,1\})^{(0)} \\
&-\frac{604341}{5}\zeta(2,6) + \frac{472878}{25}\zeta(3,5) - 1239119\zeta(9) - \frac{3015022\zeta(7)}{5} \\
&+\frac{1211838\zeta(3)\zeta(5)}{5} - \frac{29868\pi^4\zeta(5)}{25} - \frac{85998\zeta(5)}{5} + \frac{267951\zeta(3)^3}{5} \\
&+\frac{110202\zeta(3)^2}{5} - \frac{2824\pi^6\zeta(3)}{15} - \frac{34331\pi^4\zeta(3)}{75} - \frac{550372\zeta(3)}{15} \\
&+\frac{1590287\pi^8}{15000} + \frac{397\pi^6}{63} + \frac{517\pi^4}{25} - \frac{1771112}{15}\,,
\end{aligned}
\tag{5.45}
$$

with $\zeta(s_1,\ldots,s_n)$ multiple zeta functions and $P(\{n_1,\ldots,n_{20}\})^{(p)}$ the order in ϵ^p of the propagator integrals defined in (5.27).

The most complicated of these unknowns is a propagator integral that we have not been able to reduce to master integrals yet. We expect nevertheless that the reduction will be possible on a cluster node with 512

GB of RAM. The other four terms correspond to coefficients in the expansion of master integrals, which can be evaluated with HyperInt [101]. This package automatizes the computation of integrals that are linearly reducible. The main idea is to consider their Feynman parametrization and then find an order of integration of the Feynman parameters such that at each step the result is written in terms of hyperlogarithms multiplied by rational functions with linear denominators.

6. Integrability Review

Integrability first appeared in the context of condensed matter physics, with Bethe's solution of the Heisenberg model [102]. His ansatz consists on a superposition of plane waves, and terms with different orderings of the momenta come with S-matrix phase factors. The key point of the ansatz is that scattering factorizes and everything can be expressed as a function of the two-body S-matrix, which must obey the Yang-Baxter equation for consistency. The fact that the S-matrix factorizes can be related to the presence of an infinite number of symmetries, which is also a signature feature of integrable theories.

Integrability was first applied in the context of quantum field theories by Lipatov [103] and then introduced in the context of AdS/CFT by Minahan and Zarembo when studying the one-loop anomalous dimensions of scalar single-trace operators in $\mathcal{N} = 4$ SYM [10]. Meanwhile, there has been a lot of work to generalize their proposal, which led eventually to an extension to the full theory and also to all orders in perturbation theory. Later on, the same authors proved ABJM to be integrable at two loops [11], and since then other lower dimensional examples have been discovered, as well as possible deformations of these theories.

In the following sections we will review the construction of the Bethe ansatz in the simple $SU(N)$ spin-chain and then explain how these techniques can be used to solve for the spectrum of $\mathcal{N} = 4$ SYM and ABJM. We will finish with a review of Hexagon form factors, which are the building blocks of higher point functions. For an extended review of integrability one should check [104].

6.1 $SU(N)$ spin-chain

In this section we will consider single-trace operators which map to spin-chains where each site transforms in the fundamental representation of $SU(N)$. This is a very simple toy model but it already illustrates quite well many of the crucial features in the Bethe ansatz.

6.1.1 The Hamiltonian

At each site we have N possible spins which we label with ϕ^i, and we will consider a Hamiltonian with nearest neighbour interactions

$$H = \sum_n H_{n,n+1} \,. \tag{6.1}$$

The terms $H_{n,n+1}$ take values in the tensor product of two fundamental representations, which decomposes into the symmetric and antisymmetric representations

$$N \otimes N = \frac{N(N-1)}{2} \oplus \frac{N(N+1)}{2} \,. \tag{6.2}$$

We then conclude that a Hamiltonian with $SU(N)$ symmetry must be a linear combination of projectors into these two representations

$$H_{n,n+1} = c_A \Pi^A_{n,n+1} + c_S \Pi^S_{n,n+1} \,, \tag{6.3}$$

with the projectors given in terms of the permutation operator $\mathbb{P}$

$$\Pi^A = \frac{1}{2}(\mathbb{I} - \mathbb{P}) \,, \qquad \Pi^S = \frac{1}{2}(\mathbb{I} + \mathbb{P}) \,. \tag{6.4}$$

The cyclicity of the trace induces a shift symmetry of the spin-chain, which implies that the constants c_A and c_S cannot depend on n. The symmetry fixes the spin-chain Hamiltonian up to the ratio of these two constants, but we can shift the ground-state energy to any value we like by adding a term proportional to the identity $\mathbb{I}$. At the end, we are free to write the Hamiltonian as

$$H = \sum_n (\mathbb{I}_{n,n+1} - \mathbb{P}_{n,n+1}) \,. \tag{6.5}$$

With the specific setup we have chosen here, the spin-chain Hamiltonian is automatically integrable, but this is not what happens in general. Even for an $SU(N)$ spin-chain we could have considered sites transforming in another representation, leading to more terms in the tensor product decomposition. In that case integrability would appear only for very specific combinations of the projectors.

For a spin-chain with L sites, the spectrum is found by diagonalizing an $N^L \times N^L$ matrix. A way to do this is to find all representations in the tensor product of L fundamentals, and then find the eigenvalues for the lowest weight of each representation

$$\underbrace{N \otimes \ldots \otimes N}_{L} = \frac{(N+L-1)!}{L!(N-1)!} \oplus \ldots \tag{6.6}$$

However, this task becomes very difficult as we take L to be large, so we need to come up with a better strategy.

6.1.2 Bethe Ansatz

Since the Hamiltonian is integrable we will be able to diagonalize it by introducing the Bethe ansatz. We pick the ground state to be made up of L spins ϕ^1, as it is annihilated by the spin-chain Hamiltonian

$$H|\phi^1 \ldots \phi^1\rangle = 0\,, \tag{6.7}$$

like any other state in the totally symmetric representation. The excited states of the spin-chain are given by substituting any number of vacuum sites by ϕ^i, with $i \neq 1$.

The natural ansatz for the single-excitation state is given by

$$|\phi_p^a\rangle = \sum_n e^{inp} |\phi_n^a\rangle\,, \tag{6.8}$$

where p is the momentum of the excitation and $|\phi_n^a\rangle$ stands for a state where the n-th vacuum site is substituted by an excitation ϕ^a. A straightforward calculation shows that such a state is an eigenvector of the Hamiltonian with energy

$$E(p) = 4\sin^2\frac{p}{2}\,. \tag{6.9}$$

An excitation with zero momentum does not increase the energy of the ground state, but this is what one would expect as such states are in the symmetric representation whose lowest weight is the ground state.

Let us now consider a two-particle state, with momenta p and q. The Bethe ansatz is given by a superposition of plane-wave factors

$$|\phi_p^a\phi_q^b\rangle = \sum_{n<m} e^{i(pn+qm)}|\phi_n^a\phi_m^b\rangle + \hat{S}_{cd}^{ab} e^{i(qn+pm)}|\phi_n^c\phi_m^d\rangle\,, \tag{6.10}$$

where in general the S-matrix $\hat{S}_{cd}^{ab}$ can transform the flavours of the excitations. Notice however that our choice of vacuum breaks the $SU(N)$ symmetry of the Hamiltonian, and the scattering must be invariant under the residual symmetry $SU(N-1)$. The S-matrix acts on the tensor product of two fundamental representations, which like before decomposes into symmetric and antisymmetric representations

$$(N-1)\otimes(N-1) = \frac{(N-1)(N-2)}{2} \oplus \frac{N(N-1)}{2}\,, \tag{6.11}$$

so the S-matrix is fixed up to two functions of the momenta, $S_s(p,q)$ and $S_a(p,q)$. One can find the energy of such a state by looking at the region where the excitations are well separated, so that the Hamiltonian acts on each independently. The energy of the two-particle state must then equal the sum of energies of the single-particle states

$$E(p,q) = 4\sin^2\frac{p}{2} + 4\sin^2\frac{q}{2}\,. \tag{6.12}$$

Now that we have determined the energy, we just need to find the right S-matrix elements for which the Hamiltonian acts in a consistent way when excitations are close to each other. We start by acting on the lowest weight state in the symmetric representation $|\phi_p^2\phi_q^2\rangle$, for which we obtain the usual S-matrix of the Heiseberg model

$$S_s = -\frac{1+e^{i(p+q)}-2e^{iq}}{1+e^{i(p+q)}-2e^{ip}}\,. \tag{6.13}$$

Similarly, in order to find the antisymmetric component it suffices to act with the Hamiltonian on the state $|\phi_p^2\phi_q^3\rangle - |\phi_p^3\phi_q^2\rangle$, which leads to the simple scattering phase

$$S_a = 1\,. \tag{6.14}$$

Finally, we can introduce the rapidity variable $u = \frac{1}{2}\cot\frac{p}{2}$, so that the expressions become more compact

$$E(u) = \frac{1}{u^2+\frac{1}{4}}\,, \qquad S_s(u_1,u_2) = \frac{u_1-u_2-i}{u_1-u_2+i}\,. \tag{6.15}$$

6.1.3 Factorized Scattering and the Yang-Baxter Equation

The correct ansatz for the multi-particle state is given by a sum over the $n!$ different orderings of the excitations

$$|\phi_{p_1}^{a_1}\dots\phi_{p_n}^{a_n}\rangle = \sum_\sigma \sum_{m_1<\dots<m_n} \hat{S}(\{p\}_\sigma)^{a_1\dots a_n}_{b_1\dots b_n}\prod_j e^{ip_{\sigma_j}m_j}|\phi^{b_1}_{p_{\sigma_1}}\dots\phi^{b_n}_{p_{\sigma_n}}\rangle\,. \tag{6.16}$$

Some orderings in the multi-particle state require a single permutation of two excitations, and in that case the scattering is given by the two-body S-matrix found above. However, in general there are also new phase factors that need to be computed for each new multi-particle state. For example, for three particles one of the orderings is $\{p_3,p_2,p_1\}$, which in principle leads to a three-to-three scattering matrix. Here integrability plays a crucial role, as it requires the S-matrix to factorize into two-body scattering processes. To ensure there is no ambiguity in the way we do the decomposition, the S-matrix must satisfy the Yang-Baxter equation

$$\hat{S}_{12}\hat{S}_{13}\hat{S}_{23} = \hat{S}_{23}\hat{S}_{13}\hat{S}_{12}\,. \tag{6.17}$$

This equation is trivially satisfied by a scalar S-matrix, as in the case of the Heisenberg model, but it does impose strong constraints in a more complicated setup. We can for example check that the S-matrix we obtained in our $SU(N)$ model does satisfy the Yang-Baxter equation.

In [105,106] it was argued that this property of the S-matrix is in fact related to the presence of higher conserved charges, which guarantee that scattering can only rearrange the momenta of the particles. While the origin of these higher conserved charges is obscured by the approach we have taken to the Bethe ansatz, they can be explicitely constructed from the algebraic approach reviewed in [107], but which we will not present here.

6.1.4 Nested Bethe Equations

We now understand how to write any excited state in the $SU(N)$ spin-chain, but we still do not have a quantization condition for the rapidities. That comes about when we require L to be finite, in which case particles are in a circle and one needs to impose boundary conditions on the Bethe wavefunction (6.16)

$$\psi(m_1, m_2, \ldots, m_n) = \psi(m_2, \ldots, m_n, m_1 + L)\,. \tag{6.18}$$

These boundary conditions lead to the Bethe equations

$$e^{ip_j L} \prod_{k \neq j} \hat{S}(p_j, p_k) = 1\,. \tag{6.19}$$

The problem would be solved if the S-matrix was a number, but that is not the case so we need to diagonalize it by introducing several levels of excitations, in a procedure called the nested Bethe ansatz [108]. The level I vacuum $|0\rangle^{\mathrm{I}}$ is the spin-chain ground state, and the excitations at the first level are given by ϕ^2, whose scattering is given by

$$\hat{S}|\phi^2_{p_1}\phi^2_{p_2}\rangle = S^{\mathrm{I,I}}(p_1, p_2)|\phi^2_{p_2}\phi^2_{p_1}\rangle = S_s(p_1, p_2)|\phi^2_{p_2}\phi^2_{p_1}\rangle\,. \tag{6.20}$$

At the next level, the ground state $|0\rangle^{\mathrm{II}}$ is now given by the set of excitations in level I

$$|0\rangle^{\mathrm{II}} = |\phi^2_{p_1} \ldots \phi^2_{p_n}\rangle\,. \tag{6.21}$$

The single excitation at level II is given by substituting a level II vacuum site by ϕ^3, which carries a new auxiliary parameter q

$$|\chi_q\rangle^{\mathrm{II}} = \sum_{i}^{n} f(q, p_k) \prod_{j}^{k-1} S^{\mathrm{II,I}}(q, p_j)|\phi^2_{p_1} \ldots \phi^3_q \ldots \phi^2_{p_n}\rangle\,. \tag{6.22}$$

The factors $S^{\mathrm{II,I}}$ account for the scattering phase we have to pay to move the new excitation past the level I excitations, and f is related with the creation of an excitation at level II. In order to fix these two functions, we just require that the level II state is compatible with scattering of level I excitations

$$\hat{S}|\chi_q\rangle^{\mathrm{II}}_{\{p_1,p_2\}} = S^{\mathrm{I,I}}(p_1, p_2)|\chi_q\rangle^{\mathrm{II}}_{\{p_2,p_1\}}\,. \tag{6.23}$$

which fixes the scattering between level II vacuum sites and excitations to be

$$S^{\mathrm{II},\mathrm{I}}(u,v) = \frac{u - v + \frac{i}{2}}{u - v - \frac{i}{2}} \,. \tag{6.24}$$

The two-particle state at level II is now given by the usual superposition of states

$$\begin{aligned} |\chi_{q_1}\chi_{q_2}\rangle =& f(q_1,p_1)f(q_2,p_2)S^{\mathrm{II},\mathrm{I}}(q_2,p_1)|\phi^3_{q_1}\phi^3_{q_2}\rangle \\ &+ f(q_2,p_1)f(q_1,p_2)S^{\mathrm{II},\mathrm{I}}(q_1,p_1)S^{\mathrm{II},\mathrm{II}}(q_1,q_2)|\phi^3_{q_2}\phi^3_{q_1}\rangle \,, \end{aligned} \tag{6.25}$$

and a compatibility condition like (6.23) requires that the scattering between level II excitations is given by

$$S^{\mathrm{II},\mathrm{II}}(u,v) = \frac{u - v - i}{u - v + i} \,. \tag{6.26}$$

By continuing this process until there are no excitations left, we create $N-1$ sets of Bethe roots. At the end we can impose periodicity at each level, which leads to the Bethe equations

$$\left(\frac{u_i^{(k)} + \frac{i}{2}V_k}{u_i^{(k)} - \frac{i}{2}V_k}\right)^L \prod_{l=1}^{N-1}\prod_{j=1}^{N_k} \frac{u_i^{(k)} - u_j^{(l)} - \frac{i}{2}M_{kl}}{u_i^{(k)} - u_j^{(l)} + \frac{i}{2}M_{kl}} = -1 \,, \tag{6.27}$$

with N_k roots $u_i^{(k)}$ at level k. For a spin-chain with sites in the fundamental of $SU(N)$ the matrix M and the vector V obtained are

$$M = \begin{pmatrix} 2 & -1 & 0 & \cdots \\ -1 & 2 & -1 & \cdots \\ 0 & -1 & 2 & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix} , \qquad V = \begin{pmatrix} 1 \\ 0 \\ \vdots \end{pmatrix} , \tag{6.28}$$

which correspond exactly to the Cartan matrix and Dynkin labels of the spin-chain representation. The equations (6.27) hold for any integrable nearest neighbour spin-chain with sites trasnforming in a representation V of any Lie algebra with Cartan matrix M [109–111]. Finally, the cyclicity of the trace imposes a constraint on the momentum-carrying Bethe roots of the spin-chain

$$\prod_i \frac{u_i^{(1)} + \frac{i}{2}}{u_i^{(1)} - \frac{i}{2}} = 1 \,, \tag{6.29}$$

which means that the total momentum of the spin-chain must vanish.

Let us end this section with a remark on the number of auxiliary roots that is necessary to consider. The creation of excitations at each new level can be seen as the action of a raising operator of the algebra. These operators have well defined $SU(N)$ charges, and given the charges of the spin-chain states we want to study, we can derive exactly how many roots we need to add at each level.

6.2 Spectrum of $\mathcal{N} = 4$ SYM and ABJM

The nested Bethe ansatz presented in the previous section is a very powerful method for computing the eigenvalues of integrable spin-chains. This is especially obvious in the limit of very large L, which is not difficult to take with the integrable framework.

Let us now apply the Bethe ansatz technique to some of the integrable models that appear in the context of AdS/CFT. The mixing matrices for two-point fuctions of single-trace operators in $\mathcal{N} = 4$ SYM and ABJM correspond to the Hamiltonians of spin-chains where the sites transform in representations of $\mathfrak{psu}(2,2|4)$ and $\mathfrak{osp}(6|4,\mathbb{R})$ respectively.

Since the integrable construction is very similar for both theories, we will mostly focus on the four-dimensional case, and we will finish the section with some remarks on what needs to be changed in the case of ABJM.

6.2.1 One-loop Integrability in $\mathcal{N} = 4$ SYM

At one-loop in $\mathcal{N} = 4$ SYM we have a nearest-neighbour Hamiltonian similar to the one of the previous section. A crucial difference now is that the tensor product of two fundamental representations $\mathcal{V}$ of $\mathfrak{psu}(2,2|4)$ decomposes into an infinite number of representations $\mathcal{V}_j$. However, despite this apparent difficulty, Beisert was able to show by using an oscillator representation that the full one-loop Hamiltonian is of the form [67]

$$H = \sum_{n=1}^{L} \sum_{j=0}^{\infty} \left(\sum_{k=1}^{j} \frac{1}{k} \right) \Pi^{(j)}_{n,n+1} , \tag{6.30}$$

with $\Pi^{(j)}$ the projector onto the representation $\mathcal{V}_j$. If we restrict to the $SO(6)$ scalar sector, it can be shown that there are only three representations in the tensor product. The corresponding projectors can be written in terms of the identity $\mathbb{I}$, the permutation $\mathbb{P}$ and the trace operator $\mathbb{K}$ appearing in the work of Minahan and Zarembo. Like we did in the previous section, we can rescale the identity term by shifting the ground state energy. The only meaningful parameter in the $SO(6)$ Hamiltonian is then the ratio of coefficients for $\mathbb{P}$ and $\mathbb{K}$, whose value turns out to make the spin-chain integrable. Similarly, Beisert and Staudacher [112] showed that the full one-loop Hamiltonian in (6.30) also corresponds to an integrable model.

The one-loop anomalous dimensions of $\mathcal{N} = 4$ SYM can then be obtained by applying the Bethe ansatz techniques reviewed in the previous section. The Bethe equations are given by (6.27), by specifying the Cartan matrix of the super Lie algebra $\mathfrak{su}(2,2|4)$ and its fundamental representation. Note that for super algebras the Dynkin diagram is not unique, so there are several ways to write the Bethe equations depending on the choice we make. Despite the apparent ambiguity, it can be shown that all those choices are equivalent and lead to the same spectrum [112].

6.2.2 Perturbative Asymptotic Bethe Ansatz

So far we have used the Bethe ansatz to tackle only the problem of one-loop anomalous dimensions. A way to extend it to higher loops is by finding the dilatation operator at the corresponding loop order. For example, in $\mathcal{N} = 4$ SYM we know the three-loop Hamiltonian for the maximal compact $SU(2|3)$ sector [113, 114], which includes long-range interactions. However, as long as we work with asymptotically long spin-chains, then the states will always have regions where all excitations are well separated and the plane-wave ansatz still holds. There are then two effects that change the anomalous dimension at higher loops: the dispersion relation for the single excitations is corrected, and the S-matrix in the two-particle state is also modified.

In order to find the correction to the dispersion relation one needs to apply the higher loop Hamiltonian on the regions of the Bethe states where excitations are far from each other. Meanwhile we can obtain the scattering matrix by introducing the Perturbative Bethe Ansatz [115], where we modify the wave function close to the collision point. For example, in the $SU(2)$ sector of the spin-chain the two-particle state is

$$|\phi_p \phi_q\rangle = \sum_{n<m} (\phi(n,m) + S(p,q)\phi(m,n))|\phi_n \phi_m\rangle \,, \tag{6.31}$$

with

$$\begin{aligned} \phi(n,m) &= e^{ipn+iqm}(1 + \lambda A \delta_{m,n+1}) \,, \\ S(p,q) &= S^{(0)}(p,q) + \lambda S^{(1)}(p,q) \,. \end{aligned} \tag{6.32}$$

By looking at the action of the Hamiltonian when $m = n+1$ and $m = n+2$, and discarding terms of order $\mathcal{O}(\lambda^2)$, one can fix both A and the one-loop scattering matrix.

At higher loops the relation between the momentum p of the excitations and the Bethe rapidity u changes [116,117]. By looking at the BMN scaling it is possible to see that at higher loops the relation is modified to

$$e^{ip} = \frac{x^+}{x^-}\,, \tag{6.33}$$

where we introduce the Zhukowsky variables $x^\pm$ parametrized by the rapidity u as

$$x^\pm + \frac{1}{x^\pm} = \frac{u \pm \frac{i}{2}}{g}\,, \qquad g = \frac{\sqrt{\lambda}}{4\pi}\,. \tag{6.34}$$

Note that this map introduces a branch cut in the u complex plane, and the physical region is given by $|x^\pm(u)| > 1$. Surprisingly, despite the complexity of the dilatation operator at higher loops, the S-matrix remains quite compact. This is very important as the S-matrix is the only non-trivial element of the Bethe equations, which quantize the Bethe rapidities and determine the anomalous dimensions. The simple form of the S-matrix hints at the presence of a structure to all orders in perturbation theory, which we will uncover in the rest of this section.

6.2.3 Centrally extended $\mathfrak{su}(2|2)$ algebra

The choice we made for the ground state of equation (2.58) breaks the symmetry of the spin-chain to $SU(2|2)^2$. The building block for the multi-particle Bethe ansatz is then the $SU(2|2)$ S-matrix, and we will now use symmetry considerations to constrain it as much as possible. In general the full S-matrix is defined for on-shell states with vanishing total momentum P. However, integrability imposes factorized scattering, so the two-body S-matrix must then be invariant under the full off-shell symmetry group of the problem [118]. From the string point of view, this means that we must relax the level-matching condition, and the worldsheet momentum does not necessarily vanish. We can then see that the symmetry group $SU(2|2)$ is enlarged by two central elements [108], with charges

$$\mathfrak{P} = ige^{2i\xi}(e^{iP}-1)\,, \qquad \mathfrak{K} = -ige^{-2i\xi}(e^{-iP}-1)\,. \tag{6.35}$$

Naturally, as we go on-shell these central charges vanish and we go back to the original group of symmetries.

In what follows we will study the centrally extended algebra $\mathfrak{psu}(2|2) \ltimes \mathbb{R}^3$ and its fundamental representation. The commutation relations of the bosonic generators with a generic operator J are

$$\begin{aligned}
[L_\alpha{}^\beta, J_\gamma] &= \delta_\gamma^\beta J_\alpha - \frac{1}{2}\delta_\alpha^\beta J_\gamma\,, \qquad [L_\alpha{}^\beta, J^\gamma] = -\delta_\alpha^\gamma J^\beta + \frac{1}{2}\delta_\alpha^\beta J^\gamma\,,\\
[R_a{}^b, J_c] &= \delta_c^b J_a - \frac{1}{2}\delta_a^b J_c\,, \qquad [R_a{}^b, J^c] = -\delta_a^c J^b + \frac{1}{2}\delta_a^b J^c\,.
\end{aligned} \tag{6.36}$$

Meanwhile, the commutation relations of the fermionic generators in the extended algebra are modified to

$$\{Q^a_\alpha, Q^b_\beta\} = \epsilon_{\alpha\beta}\epsilon^{ab}\mathfrak{P}\,, \qquad \{S^\alpha_a, S^\beta_b\} = \epsilon^{\alpha\beta}\epsilon_{ab}\mathfrak{K}\,,$$
$$\{Q^a_\alpha, S^\beta_b\} = \delta^a_b L_\alpha^{\ \beta} + \delta^\beta_\alpha R_b^{\ a} + \frac{1}{2}\delta^a_b\delta^\beta_\alpha\mathfrak{C}\,, \tag{6.37}$$

with $\mathfrak{C}$ the central charge generating the $\mathfrak{u}(1)_E$, and $\mathfrak{P}$ and $\mathfrak{K}$ the new central elements.

The fundamental representation $(2|2)$ is composed by two bosons ϕ_a and two fermions ψ_α. The action of the bosonic generators follows from (6.36), while the most general action of the fermionic generators is

$$\mathfrak{Q}^\alpha_a|\phi_b\rangle = a\epsilon_{ab}\epsilon^{\alpha\beta}|\psi_\beta\rangle\,, \qquad \mathfrak{S}^a_\alpha|\phi_b\rangle = c\delta^a_b|\psi_\alpha\rangle\,,$$
$$\mathfrak{Q}^\alpha_a|\psi_\beta\rangle = b\delta^\alpha_\beta|\phi_a\rangle\,, \qquad \mathfrak{S}^a_\alpha|\psi_\beta\rangle = d\epsilon^{ab}\epsilon_{\alpha\beta}|\phi_b\rangle\,. \tag{6.38}$$

Closure of the algebra then implies that the action of the central charges on a generic state $|\chi\rangle$ is

$$\mathfrak{C}|\chi\rangle = (ad+bc)|\chi\rangle\,, \qquad \mathfrak{P}|\chi\rangle = ab|\chi\rangle\,, \qquad \mathfrak{K}|\chi\rangle = cd|\chi\rangle\,, \tag{6.39}$$

and it also constrains the parameters to obey $ad - bc = 1$. Notice that without central extension the parameters b and c would vanish, which would constrain the eigenvalue of $\mathfrak{C}$ to be constant. Since that central charge is the generator of $\mathfrak{u}(1)_E$, which gives the energy of the spin-chain state, then we see that the central charges play a fundamental role in this construction as they allow to form a representation with a continuous parameter. The fact that the representation obeys a dispersion relation is a special feature of these integrable models and it happens because the spin-chain Hamiltonian belongs to the group of symmetries of the spin-chain.

In order for the representation to be unitary, the parameters must be related by complex conjugation: $a = d^*$ and $b = c^*$. If we also want to match the central charges with the ones from the worldsheet analysis (6.35), it is convenient to parametrize the fundamental representation with the Zhukowsky variables from (6.33), and two parameters η and $\zeta = e^{2i\xi}$ [119]

$$a = \sqrt{g}\eta\,, \qquad c = -\sqrt{g}\frac{\eta}{\zeta x^+}\,,$$
$$b = \sqrt{g}\frac{i\zeta}{\eta}\left(\frac{x^+}{x^-} - 1\right)\,, \qquad d = \sqrt{g}\frac{x^+}{i\eta}\left(1 - \frac{x^-}{x^+}\right)\,, \tag{6.40}$$

where unitarity constrains η to be of the form

$$\eta = \sqrt{ix^- - ix^+}e^{i(\xi+\varphi)}\,. \tag{6.41}$$

From these expressions we can show that the dispersion relation is

$$C = E_p = \sqrt{1 + 16g^2 \sin^2(p/2)}\,. \tag{6.42}$$

By setting $\varphi = 0$ one obtains the representation in the string frame, more natural from the worldsheet theory point of view. However, we can also set $\varphi = -\xi$ thus obtaining the so called spin-chain frame. In that case we can understand the representation also from the perpective of a dynamical spin-chain where we need to introduce $Z^{\pm}$ markers that correspond to length changing effects.

6.2.4 Multi-particle states

We have obtained a representation of $\mathfrak{psu}(2|2) \ltimes \mathbb{R}^3$ which is characterized by the momentum p and the phase ζ. The one-particle state can then be identified with the module $V(p,\zeta)$, and we build the multi-particle state by tensoring a number of such modules [120]

$$V(p_1,\zeta_1) \otimes \ldots \otimes V(p_n,\zeta_n)\,. \tag{6.43}$$

While for the one-particle state we can just set the phase ζ to 1, that is not true for a state with several excitations. We know that the action of the central charges on the full multi-particle state is still given by (6.35), so we can only have a dependence on the total momentum P of the state. We can then show that happens only if the phases of the different modules satisfy the following relation

$$\zeta_i = \zeta_1 \prod_{l=1}^{i-1} e^{ip_l}\,. \tag{6.44}$$

The action of the generators on two-particle states is then given by

$$\begin{aligned} \mathfrak{B}(p_1,p_2,\zeta_1) &= \mathfrak{B}(p_1,\zeta_1) \otimes \mathbb{I} + \mathbb{I} \otimes \mathfrak{B}(p_2,\zeta_1 e^{ip_1})\,, \\ \mathfrak{F}(p_1,p_2,\zeta_1) &= \mathfrak{F}(p_1,\zeta_1) \otimes \mathbb{I} + \Sigma \otimes \mathfrak{F}(p_2,\zeta_1 e^{ip_1})\,, \end{aligned} \tag{6.45}$$

where $\mathfrak{B}$ and $\mathfrak{F}$ correspond to bosonic and fermionic generators repectively. We introduce the matrix

$$\Sigma = \mathrm{diag}(1,1,-1,-1)\,, \tag{6.46}$$

which ensures we get a minus sign when a fermionic generator goes past a fermion on the first site.

So far we have associated phases ζ_i to the representation at each site and used these phases to create a well define multi-particle state. There is however a different construction with an Hopf algebra interpretation, where instead we consider a non-trivial tensor product [121]. We define the coproduct by modifying the graded tensor product in the following way

$$\begin{aligned} \Delta(\mathfrak{B}) &= \mathfrak{B}\hat{\otimes}\mathbb{I} + \mathbb{I}\hat{\otimes}\mathfrak{B}\,, \\ \Delta(Q^a_\alpha) &= Q^a_\alpha\hat{\otimes}\mathbb{I} + e^{iP/2}\hat{\otimes}Q^a_\alpha\,, \\ \Delta(S^\alpha_a) &= S^\alpha_a\hat{\otimes}\mathbb{I} + e^{-iP/2}\hat{\otimes}S^\alpha_a\,, \end{aligned} \tag{6.47}$$

where once again we denote the bosonic generators $R_a^{\ b}$ or $L_\alpha^{\ \beta}$ by $\mathfrak{B}$. Meanwhile, the coproduct obtained for the central charges is

$$\begin{aligned} \Delta(\mathfrak{P}) &= \mathfrak{P}\hat{\otimes}\mathbb{I} + e^{iP}\hat{\otimes}\mathfrak{P}\,, \\ \Delta(\mathfrak{K}) &= \mathfrak{K}\hat{\otimes}\mathbb{I} + e^{-iP}\hat{\otimes}\mathfrak{K}\,, \end{aligned} \tag{6.48}$$

which implies that Δ corresponds to an algebra homomorphism. Hopf algebras also have an antipode map S which in this case acts on the generators as

$$S(\mathfrak{B}) = -\mathfrak{B}\,, \quad S(Q^a_\alpha) = -e^{-iP/2}Q^a_\alpha\,, \quad S(S^\alpha_a) = -e^{iP/2}S^\alpha_a\,. \tag{6.49}$$

The reason we introduce this machinery is because the antipode map does play a crucial role in constraining the S-matrix by introducing crossing relations [121].

6.2.5 The $SU(2|2)$ S-matrix

We are now ready to derive the $SU(2|2)$ S-matrix. The fact that it is invariant under an $SU(2|2)$ symmetry implies that it commutes with all the generators $\mathfrak{J}$ of $\mathfrak{su}(2|2)$

$$\hat{S}_{12}(p_1,p_2)\mathfrak{J}(p_1,p_2,\zeta_1) = \mathfrak{J}(p_2,p_1,\zeta_1)\hat{S}_{12}(p_1,p_2)\,. \tag{6.50}$$

One can show that invariance under the bosonic algebra requires the S-matrix to be of the form [108, 122]

$$\begin{aligned} \hat{S}_{12}|\phi^1_a\phi^2_b\rangle &= A_{12}|\phi^2_{\{a}\phi^1_{b\}}\rangle + B_{12}|\phi^2_{[a}\phi^1_{b]}\rangle + \frac{1}{2}C_{12}\epsilon_{ab}\epsilon^{\alpha\beta}|\psi^2_\alpha\psi^1_\beta\rangle\,, \\ \hat{S}_{12}|\psi^1_\alpha\psi^2_\beta\rangle &= D_{12}|\psi^2_{\{\alpha}\psi^1_{\beta\}}\rangle + E_{12}|\psi^2_{[\alpha}\psi^1_{\beta]}\rangle + \frac{1}{2}F_{12}\epsilon_{\alpha\beta}\epsilon^{ab}|\phi^2_a\phi^1_b\rangle\,, \\ \hat{S}_{12}|\phi^1_a\psi^2_\beta\rangle &= G_{12}|\psi^2_\beta\phi^1_a\rangle + H_{12}|\phi^2_a\psi^1_\beta\rangle\,, \\ \hat{S}_{12}|\psi^1_\alpha\phi^2_b\rangle &= K_{12}|\psi^2_\alpha\phi^1_b\rangle + L_{12}|\phi^2_b\psi^1_\alpha\rangle\,. \end{aligned} \tag{6.51}$$

Meanwhile, we can use the fermionic symmetries to show that all coefficients can be obtained up to a single scalar function of the momenta.

We can fix the matrix part by setting A_{12} to $\frac{x_1^- - x_2^+}{x_1^+ - x_2^-}$, so that the full S-matrix of $\mathcal{N}=4$ SYM is

$$\mathbb{S}_{\mathfrak{psu}(2,2|4)} = S_0\, \hat{S}_{\mathfrak{psu}(2|2)} \otimes \hat{S}_{\mathfrak{psu}(2|2)}\,, \tag{6.52}$$

with an overall scalar factor defined as

$$S_0(p_1,p_2) = \frac{x_1^+ - x_2^-}{x_1^- - x_2^+}\,\frac{1-\frac{1}{x_1^- x_2^+}}{1-\frac{1}{x_1^+ x_2^-}}\,\frac{1}{\sigma^2(p_1,p_2)}\,, \tag{6.53}$$

and σ the dressing phase. The fact that the algebra is centrally extended plays a crucial role here, as the central charges ensure that the tensor product of two short representations decomposes into a single long representation, which is very unusual. One can also check that the S-matrix obtained from symmetry considerations automatically obeys the Yang-Baxter equation.

6.2.6 Crossing Equations and the Dressing Phase

In relativistic theories, the S-matrix depends on two Mandelstam variables, s and t, and crossing symmetry implies that the S-matrix should be invariant under exchange of those variables. It turns out that in two dimensions, unitarity and crossing symmetry are enough to fix the dressing phase.

In a non-relativistic model, the S-matrix is still unitary, but we do not have the same notion of crossing. However, it was understood by Janik that the antipode of the Hopf algebra introduced in (6.49) can be seen as a particle/anti-particle transformation. At the level of the representation, the antipode introduces a charge conjugation matrix $\mathcal{C}$. This leads to another representation for the generators of the algebra

$$\tilde{\mathfrak{J}}((x^\pm)^{2\gamma}) = \mathcal{C}\,\mathfrak{J}(x^\pm)\,\mathcal{C}^{-1}\,, \tag{6.54}$$

with exactly the same central charges. The crossing transformation of the Zhukowsky variables is

$$(x^\pm)^{2\gamma} = \frac{1}{x^\pm}\,, \tag{6.55}$$

which corresponds to flipping both energy and momentum of the excitation. Finally, this property of the generators together with the invariance of the S-matrix (6.50) leads to a constraint on the dressing phase [121]

$$\sigma(u_1^{2\gamma},u_2)\sigma(u_1,u_2) = \frac{x_1^- - x_2^-}{x_1^- - x_2^+}\,\frac{1-1/x_1^+ x_2^-}{1-1/x_1^+ x_2^+}\,. \tag{6.56}$$

The solution to the crossing equation was conjectured by Beisert, Eden and Staudacher [123], following the string analysis of [124] and trancendentality considerations. Later on, that solution was derived by considering the analytical properties of the S-matrix [125]. We know for example that the position of simple poles should match the structure of bound states, and higher-order poles are related to exchange of multi-particle states. A useful integral representation was obtained by Dorey, Hofman and Maldacena in [126]. While at weak coupling the dressing phase goes to one, at strong coupling we recover the dressing factor found by Arutyunov, Frolov and Staudacher [127]

$$\sigma_{AFS}(u_1,u_2) = \frac{1-1/u_1^+u_2^-}{1-1/u_1^-u_2^+}\left(\frac{u_1^-u_2^- - 1}{u_1^-u_2^+ - 1}\frac{u_1^+u_2^+ - 1}{u_1^+u_2^- - 1}\right)^{i(u_1-u_2)} . \quad (6.57)$$

6.2.7 Some remarks

At this stage we have the all-loop S-matrix, which enables us to write the all-loop Bethe equations for asymptotic spin-chains. Using the nested Bethe ansatz we obtain a set of Bethe equations which quantize the rapidities at the different levels, and the energy obtained corresponds to the anomalous dimension of a primary single-trace operator. The descendants are obtained by changing the state while keeping the energy fixed, which can be done by adding roots at infinity.

For short operators the spectrum also gets corrected by wrapping corrections, which correspond to planar Feynman diagrams wrapping the operators one or more times. The key idea is to understand that the finite-size problem can be related to a thermodynamic mirror model [128–130]. Using the Thermodynamic Bethe Ansatz [131–133] one can then obtain the anomalous dimension of short operators at intermediate values of the coupling [134, 135].

6.2.8 Integrability in ABJM

The integrability setup for ABJM follows very closely the previous discussion of $\mathcal{N}=4$ SYM. Despite the alternating nature of its spin-chain, the residual symmetry is generated by the same $\mathfrak{su}(2|2)$ algebra, supplemented by an $\mathfrak{u}(1)$ that distinguishes the two types of excitations. The S-matrix of ABJM must also be invariant under the off-shell symmetry algebra, which becomes the centrally extended algebra we studied before. A major difference is that now we have a different parametrization of the Bethe rapidity with the Zhukowsky variables

$$x^\pm + \frac{1}{x^\pm} = \frac{u \pm \frac{i}{2}}{h(\lambda)} . \quad (6.58)$$

In ABJM the function $h(\lambda)$ has a very non-trivial dependence on the coupling constant λ, with $h(\lambda) \sim \lambda$ at weak coupling and $h(\lambda)^2 \sim \lambda/2$ at strong coupling. The dispersion relation of ABJM is also very similar to the one of $\mathcal{N} = 4$ SYM

$$E_p = \frac{1}{2}C = \sqrt{\frac{1}{4} + 4h^2(\lambda)\sin^2(p/2)}\,, \tag{6.59}$$

where the different factor of $1/2$ can be traced back to the different R-charge of the vacuum sites.

Another difference is that in ABJM we can have four different types of scatterings, depending on the $\mathfrak{u}(1)$ charges of the incoming particles. For each scattering, the matrix part is exactly the same as in $\mathcal{N} = 4$ SYM, but now we have two new scalar factors

$$\begin{aligned} \mathbb{S}^{AA} &= \mathbb{S}^{BB} = S_0^{AA}\hat{S}_{\mathfrak{psu}(2|2)}\,, \\ \mathbb{S}^{AB} &= \mathbb{S}^{BA} = S_0^{AB}\hat{S}_{\mathfrak{psu}(2|2)}\,. \end{aligned} \tag{6.60}$$

Since A and B particles are related by complex conjugation, then the crossing equations relate the two dressing phases of the S-matrix. We can express the dressing factors of ABJM through the BES phase of $\mathcal{N} = 4$ SYM [136]

$$\begin{aligned} S_0^{AA}(p_1,p_2) &= \frac{1-\frac{1}{x_1^- x_2^+}}{1-\frac{1}{x_1^+ x_2^-}}\frac{1}{\sigma(p_1,p_2)}\,, \\ S_0^{AB}(u,v) &= \frac{x_1^+ - x_2^-}{x_1^- - x_2^+}\frac{1}{\sigma(p_1,p_2)}\,. \end{aligned} \tag{6.61}$$

An interesting property of this S-matrix is that it is reflectionless [137], which has been checked both at weak and strong coupling.

When we take the size of the operators to be small, we must also consider wrapping corrections in ABJM, which have been resummed through the TBA approach in [138]. In [139] the authors combined results from localization and integrability to obtain an exact formula for $h(\lambda)$.

6.3 Structure Constants from Hexagon Form Factors

Now that we have reviewed the solution to the spectral problem, the natural next step is to find an integrable framework for the computation of structure constants. We will focus on $\mathcal{N} = 4$ SYM where most of the progress has been made.

At weak coupling a series of papers [140–142] introduced the tailoring technique for tree level three-point functions in the $SU(2)$ sector, while

[143, 144] developed a similar approach for the other rank one sectors of the theory. At the same time, some progress was made at strong coupling in [54] for semi-classical operators. In what follows we will introduce the all-loop Hexagon proposal for N=4 SYM [14], which we will apply and amend accordingly in the next chapter in our own research on ABJM structure constants.

6.3.1 Intuition for Hexagons from Strings

From the perspective of the string worldsheet, the three-point function corresponds to the amplitude process for three closed strings. For the spectral problem, it was useful to consider first an infinite volume limit where we map the cylinder to the plane and then glue back by inserting a complete basis of states. Those states correspond to virtual particles wrapping the cylinder. Analogously, it will be useful to consider an asymptotic limit for the pair of pants, but in this case we can perform that expansion in several ways. If we first cut only one of the seams, we obtain the string field theory vertex that has been considered by [145]. By cutting another edge we obtain an octagon [146], and finally, in the full asymptotic limit where all three states are asymptotically large, we can see the string worldsheet as the product of two hexagons.

The hexagon form factors depend on three sets of Bethe rapidities coming from three different operators. While usual form factors probe incoming and outgoing excitations, the Hexagon form factors probe particles from three distinct infinities, which hints at a conical excess of π associated to each hexagon. In fact, this picture is corroborated by a string analysis, where one can find the existence of two points in the worldsheet with such a conical excess [51, 54].

6.3.2 The non-extremal setup

In order to avoid mixing with double-trace operators, we need to consider non-extremal correlation functions. It is then clear that operators at different points in spacetime must have different polarizations. Moreover, if the polarization depends on the spacetime position in the right way, then it is possible to preserve some supersymmetry [147]. Using conformal symmetry we can move the operators to a line, with one at the origin, another at $x = 1$, and the last at infinity. The single-trace operator at the origin is $\mathrm{Tr}[Z^L]$, but as we supertranslate the others along the line we obtain the following configuration

$$\mathrm{Tr}[Z^{L_1}](0)\,, \quad \mathrm{Tr}[(Z + \bar{Z} + Y - \bar{Y})^{L_2}](1)\,, \quad \mathrm{Tr}[\bar{Z}^{L_3}](\infty)\,, \qquad (6.62)$$

which preserves a residual $SU(2|2)$ symmetry. Each $SU(2)$ factor of the bosonic subalgebra corresponds to the $SO(3)$ group of rotations pre-

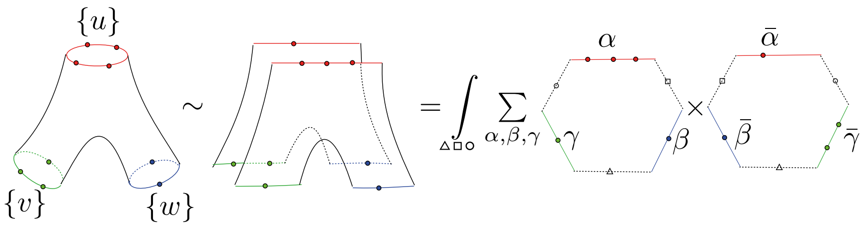

Figure 6.1. Each coloured circle in the pair of pants represents a spin-chain state. As we cut the worldsheet we obtain two hexagons, where we must sum over the partitions α, β and γ of the three sets of rapidities $\{u\}$, $\{w\}$ and $\{v\}$. The symbols on the dashed lines represent virtual particles that one has to integrate in order to go back to finite size.

served by the setup. An interesting feature of this construction is that some of the spin-chain excitations are longitudinal, which means that even a single excitation on one of the operators will lead to a non-vanishing three-point funtion due to absorption of excitations by the rotated vacua.

6.3.3 Structure constants as partitions over Hexagons

In the asymptotic limit we cut each operator in two parts, so the world-sheet is divided into two planes which probe a subset of the excitations from each of the three operators. We must therefore split each set of Bethe rapidities into two parts

$$\{u\} = \alpha \cup \bar{\alpha}\,, \qquad \{v\} = \beta \cup \bar{\beta}\,, \qquad \{w\} = \gamma \cup \bar{\gamma}\,, \tag{6.63}$$

and sum over all possible ways to distribute the excitations, as represented in Figure 6.1. We can also think of the hexagon as a projector $\langle\mathfrak{h}|$ acting on the tensor product of three spin-chain states

$$\langle\mathfrak{h}|(|\alpha\rangle|\beta\rangle|\gamma\rangle) = \langle\mathfrak{h}|\alpha|\beta|\gamma\rangle\,. \tag{6.64}$$

The structure constant should then be given by [14]

$$C_{123} = \sum_{\alpha,\beta,\gamma} \omega_{l_{13}}(\alpha,\bar{\alpha})\omega_{l_{12}}(\beta,\bar{\beta})\omega_{l_{23}}(\gamma,\bar{\gamma})\langle\mathfrak{h}|\alpha|\beta|\gamma\rangle\langle\mathfrak{h}|\bar{\gamma}|\bar{\beta}|\bar{\alpha}\rangle\,, \tag{6.65}$$

where l_{ij} are the bridge lengths between two given operators. They are given by the number of Wick contractions between the operators at tree level

$$l_{ij} = \frac{1}{2}(L_i + L_j - L_k)\,, \tag{6.66}$$

and must all be large in the asymptotic limit. Each partition of the Bethe rapidities comes with a splitting factor

$$\omega_l(\alpha,\bar{\alpha}) = \prod_i e^{ip(\bar{\alpha}_i)l} \prod_{j>i} S(\bar{\alpha}_i,\alpha_j)\,, \tag{6.67}$$

which has a clear physical interpretation. When we move the excitations $\bar{\alpha}_i$ to the second hexagon, we get a phase e^{ipl} for propagating over the bridge of length l. Along the way, these excitations also scatter with a number of particles that stay in the first hexagon, which produces a phase shift given by the product of S-matrices. There is no direct proof that the structure constants can be written in this form, but one can look at simple Wick contractions at tree level and check that one recovers the sum over partitions.

The computation of structure constants is then reduced to the evaluation of form factors with excitations in all three edges $\langle \mathfrak{h}|\alpha|\beta|\gamma\rangle$. In relativistic theories, form factors with both incoming and outgoing momenta can be related by crossing to form factors with only outgoing momenta. Using the crossing for Zhukowsky variables from (6.55) and the crossing relation for the dressing phase (6.56), we can relate any hexagon form factor to the one where all excitations are on the same edge

$$\langle \mathfrak{h}|\Phi_{A_1\dot{A}_1}\dots\Phi_{A_n\dot{A}_n}\rangle = \mathfrak{h}_{A_1\dot{A}_1,\dots,A_n\dot{A}_n}\,. \tag{6.68}$$

It is also important to understand how the flavour of the excitations $\Phi_{A\dot{A}}$ changes with crossing. They transform under the residual $SU(2|2)_L \otimes SU(2|2)_R$, and it can be shown that the left and right parts of the excitations transform as particle and anti-particle under the diagonal $SU(2|2)$ preserved by the Hexagon. It is then natural that left and right indices get exchanged under a crossing transformation

$$\Phi_{A\dot{B}} \underset{2\gamma}{\longrightarrow} \Phi_{B\dot{A}}\,. \tag{6.69}$$

Finally, note that in the hexagon we can cross excitations in two distinct ways: either we cross them clockwise and change their flavours, or we cross them anti-clockwise twice, in which case the flavour remains unchanged. It is quite non-trivial that both crossings give exactly the same result, despite the fact that the computation looks very different at intermediate stages.

6.3.4 Fixing the one- and two-particle form factors

Just like we did for the S-matrix, it is very useful to consider the implications of the residual symmetry on the Hexagon form factors. For a generic multi-particle state there is no hope that symmetry will fix it totally, but there are very strong contraints imposed on the one- and two-particle form factors.

We want the Hexagon form factors to be invariant under the symmetry group, which means that they must be annihilated by the action of the generators $\mathfrak{J}$ of the algebra defined in (6.37) and (6.36)

$$\langle \mathfrak{h}|\mathfrak{J}|\psi\rangle = 0\,. \tag{6.70}$$

It follows from bosonic symmetry that the non-vanishing elements of the one-particle form factor are

$$\langle\mathfrak{h}|\Phi_{a\dot{b}}\rangle = \epsilon_{a\dot{b}}\,, \qquad \langle\mathfrak{h}|\Phi_{\alpha\dot{\beta}}\rangle = \mathcal{N}(p)\epsilon_{\alpha\dot{\beta}}\,. \tag{6.71}$$

The relative coefficient $\mathcal{N}(p)$ is fixed by imposing fermionic symmetries, but it will still depend on the frame we are working with, either $\mathcal{N}(p) = 1$ for the spin-chain frame or $\mathcal{N}(p) = i$ for the string frame. We can see immediately that half the excitations produce a non-vanishing form factor, and they correspond exactly to the longitudinal excitations Y, $\bar{Y}$, D and $\bar{D}$.

The residual symmetry is also sufficient to fix the full two-particle form factor up to a single undetermined function of the momenta. It turns out that it is defined through the S-matrix of (6.51) and the one-particle form factors of equation (6.71)

$$\mathfrak{h}_{A\dot{A},B\dot{B}} = h_{12}(-1)^{\dot{f}_1 f_2}\hat{S}_{AB}^{CD}\;\mathfrak{h}_{D\dot{A}}\mathfrak{h}_{C\dot{B}}\,, \tag{6.72}$$

where $f_i, \dot{f}_i$ denote the fermion numbers of the left and right parts of the i-th excitation, while h_{12} is the undetermined overall factor. In order to obtain the two-particle hexagon we have to scatter the left parts of the excitations, and then use the one-particle hexagon to contract them with the right parts. As seen in the previous section, the two-particle representation is $(V(p_1,\zeta_1)\otimes\dot{V}(p_1,\dot{\zeta}_1))\otimes(V(p_2,\zeta_1 e^{ip_1})\otimes\dot{V}(p_2,\dot{\zeta}_1 e^{ip_1}))$, so invariance under the residual symmetry also imposes a condition on ζ_1 and $\dot{\zeta}_1$

$$\zeta_1\dot{\zeta}_1 = e^{-iP}\,. \tag{6.73}$$

In the spin-chain frame, this equation is interpreted as a condition on the eigenvalue of the Z marker under the action of the hexagon.

The two-particle Hexagon should also obey the form factor axioms, which include the Watson equation [148, 149]

$$\langle\mathfrak{h}|\mathbb{S}_{12}|\Phi_1\Phi_2\rangle = \langle\mathfrak{h}|\Phi_2\Phi_1\rangle\,. \tag{6.74}$$

The matrix part of this relation is trivially satisfied by (6.72), but it also introduces a contraint on h_{12}

$$\frac{h_{12}}{h_{21}} = S_{12}^0\,, \tag{6.75}$$

with S^0 the overall factor of the $SU(2|2)$ S-matrix written in (6.53).

6.3.5 The Multi-particle Form Factor

We cannot determine any other form factor fully from symmetry considerations, but the most natural guess is

$$\mathfrak{h}_{A_1\dot{A}_1,\ldots,A_n\dot{A}_n} = (-1)^{\sum_{i<j}\dot{f}_i f_j} \prod_{i<j} h_{ij} \langle \chi_{\dot{A}_n} \ldots \chi_{\dot{A}_1} | \hat{S} | \chi_{A_1} \ldots \chi_{A_n} \rangle \,. \tag{6.76}$$

with $\hat{S}$ is the matrix part of the factorized S-matrix. This ansatz automatically satisfies the Watson equations for the multi-particle form factors.

The scalar factors h_{ij} dressing each of the S-matrices are not the usual ones from the S-matrix solution, as we cannot require them to satisfy unitarity. In order to fix these scalar factors, we need to consider another of the form factor axioms, the so called decoupling condition [150]

$$\operatorname*{res}_{u_1=u_2} \langle \mathfrak{h} | \Phi(u_1^{2\gamma}) \Phi(u_2) \Phi(u_3) \ldots \rangle = \langle \mathfrak{h} | \Phi(u_3) \ldots \rangle \,, \tag{6.77}$$

which says the form factor must develop a kinematical pole whenever a particle-antiparticle pair decouples from the rest of the state. For a form factor of local operators there would also be another term with a product of S-matrices, but only the identity is compatible with the non-local nature of the Hexagon form factors. In the end, this implies that the scalar factor h_{12} must obey exactly the same crossing equation (6.56) as the dressing phase of $\mathcal{N}=4$ SYM. There are many possible solutions to this equation, but the correct one turns out to be [14]

$$h_{12} = \frac{x^- - y^-}{x^- - y^+} \frac{1 - \frac{1}{x^- y^+}}{1 - \frac{1}{x^+ y^+}} \frac{1}{\sigma_{12}} \,, \tag{6.78}$$

which has passed many non-trivial higher-loop tests.

6.3.6 General remarks

All we have considered so far was the asymptotic contribution to the three-point function. There are also new wrapping corrections that need to be considered when the bridge lengths are small, since the asymptotic description applies only for large l_{ij}. We will however not give any further detail as the wrapping corrections play no role in the work developed in this thesis.

Meanwhile, the Hexagon form factors have also been used for the study of higher point-functions. In [151] the authors have been able to compute four-point functions of half-BPS operators by performing an OPE expansion, where each structure constant is given as a product of two Hexagon form factors. By taking a certain limit on the sizes of the operators, one can make sure that the asymptotic result gives the leading contribution even at finite coupling. In [151] the authors also obtained a modification of the splitting factors for operators with excitations transforming under $\mathfrak{su}(2|2)^2$.

Finally, it has been understood recently that the Hexagon form factors are also the building block of higher-point functions. A precise prescription for the computation of four-point functions at any value of the coupling was given in [79]. The method is quite distinct from an OPE expansion, as it relies on a very different hexagonalization of the string wordsheet.

7. Hexagon Form Factors in ABJM

With the success of the Hexagon form factor approach to correlation functions of $\mathcal{N} = 4$ SYM, it is natural to attempt a generalization to other integrable theories. The solution for the spectrum of anomalous dimensions in ABJM is very similar to the one in $\mathcal{N} = 4$ SYM, so it is natural to study Hexagon form factors in ABJM.

In this chapter we will assume that like in $\mathcal{N} = 4$ SYM the structure constants are given as a partition of the Bethe roots over two hexagon form factors. There are three reasons why this assumption is reasonable. First, in $\mathcal{N} = 4$ SYM that structure is not proved, but rather observed from weak coupling tests. In the same way, we have evaluated structure constants at tree level in ABJM for operators in simple closed sectors, and we observed the same structure as in equation (6.67). Second, there are also hints of the existence of the hexagons at strong coupling. In [51, 54] it was found that the classical worldsheet in AdS has two points with conical excess of π. However, the analysis was done in an AdS_3 subspace of AdS_5, and the same result should therefore be valid for AdS_4. Finally, ABJM must also have an asymptotic limit for three-point functions. When the operators are all large, the three-point function becomes a superstring amplitude with the form of a pair of pants. The asymptotic limit should correspond to cutting the string worldsheet and in that way one should obtain two separated hexagons, like in $\mathcal{N} = 4$ SYM. These statements do not in any way prove the assumption, but the existence of such arguments both at weak and strong coupling indicates that it should be correct.

7.1 Non-extremal setup

If the three-point functions are indeed given as a partition of the Bethe roots over two hexagons, then there are several points that need to be understood. For example, one needs to find the form of the splitting factor in the case of operators with auxiliary Bethe roots. However, the most important problem is to understand if one can bootstrap the Hexagon form factors.

In $\mathcal{N} = 4$ SYM symmetry fixed the one- and two-particle form factors, which led to a natural ansatz for the multi-particle case. Here we will follow the same logic and see how far can symmetry take us.

7.1.1 Supertranslation

The vacua of the three-point functions are three chiral operators, which in principle can have different polarizations. In fact, in order to obtain a non-vanishing and non-extremal three-point function, all the polarizations need to be different. If the operators are at generic spacetime positions then there is no supersymmetry generator that annihilates all of them. However, following the ideas of [147], we can preserve some symmetry by carefully picking a polarization that depends on the spacetime position.

We can use conformal transformations to put all three operators in a line. The chiral primary $\mathrm{Tr}[(Y^4 Y_1^\dagger)^L](0)$ preserves an $SU(2|2)\times U(1)$ symmetry, so we need to perform a supertranslation and see what is the residual symmetry. The action of the supertranslation is given by

$$e^{\mathcal{T}a}(Y^4 Y_1^\dagger)(0)e^{-\mathcal{T}a}\,, \tag{7.1}$$

and it is built from the translation operator and the R-charge generators written in the $\mathfrak{su}(2)_G\times\mathfrak{su}(2)_{\dot G}\times\mathfrak{u}(1)$ form of (2.83)

$$\mathcal{T}=A^{\alpha\beta}P_{\alpha\beta}+B^{\dot a}{}_{\dot b}R_{\dot a}{}^{\dot b}+C^{\dot a}{}_{b}R_{\dot a}{}^{b}+D^{a}{}_{\dot b}R_{a}{}^{\dot b}\,, \tag{7.2}$$

where $A^{\alpha\beta}$, $B^{\dot a}{}_{\dot b}$, $C^{\dot a}{}_{b}$ and $D^{a}{}_{\dot b}$ are constants. This is the most generic supertranslation we can consider, as both $R_a{}^b$ and R are part of the symmetry algebra that annihilates the vacuum, which means they do not change the polarization.

The supersymmetry generators preserved by $\mathrm{Tr}[(Y^4Y_1^\dagger)^L]$ are the $Q_\alpha^{a\dot 1}$ and $S^\alpha_{a\dot 1}$ from (2.84), so we need to find the right linear combination that commutes with the supertranslation. The translation operator commutes with the supercharges, but the commutator with the superconformal charges is proportional to $Q_\gamma^{b\dot 2}$. We then conclude that we must include $R_{\dot 1}{}^{\dot 2}$ in the supertranslation as it is the only R-symmetry generator which can cancel the contribution from the translation. In the same way, we understand that $R_{\dot 2}{}^{\dot 1}$ is absent from $\mathcal{T}$ as its commutator with the fermionic generators produces terms that cannot be cancelled in any way.

The only fermionic generators that have a chance of being preserved by the supertranslation are then given by the following linear combination

$$\mathfrak{Q}^a_\alpha=\frac{1}{\sqrt2}Q_\alpha^{a\dot 1}+\frac{1}{\sqrt2}\epsilon^{ab}E_{\alpha\beta}S^\beta_{b\dot 1}\,, \tag{7.3}$$

where $E_{\alpha\beta}$ is a constant matrix. One can see that their commutator with

$$\mathcal{T}^{(4)}=A^{\alpha\beta}P_{\alpha\beta}+R_{\dot 1}{}^{\dot 2} \tag{7.4}$$

vanishes only if the matrix A and E are related in the following way

$$A = -\frac{1}{2}E^{-1} \,. \tag{7.5}$$

If one wants to preserve all four supersymmetry generators $\mathfrak{Q}^a_\alpha$, then $\mathcal{T}^{(4)}$ is the best one can do and we have to send all $C^{\dot{a}}_{\ b}$, $D^a_{\ \dot{b}}$ and $B^{\dot{1}}_{\ \dot{1}}$ to zero. For simplicity we will take E to be a matrix of determinant -1 in the remainder of the chapter.

We can consider a more generic supertranslation, but in that case we preserve at most two supersymmetries given by

$$\mathfrak{F}_\alpha = \rho_a \mathfrak{Q}^a_\alpha \,, \tag{7.6}$$

with ρ_a a generic vector. They commute with a supertranslation of the form

$$\mathcal{T}^{(2)} = A^{\alpha\beta}P_{\alpha\beta} + R_{\dot{1}}^{\ \dot{2}} + c^{\dot{a}}\rho_b R_{\dot{a}}^{\ b} + \epsilon^{ac}\rho_c d_{\dot{b}} R_a^{\ \dot{b}} \,, \tag{7.7}$$

where $c^{\dot{a}}$ and $d_{\dot{b}}$ are arbitrary constants.

7.1.2 Residual symmetry

We have now established what kind of supertranslation we should perform in order to preserve some of the supersymmetries, and we will now see what is the residual symmetry in each case. Since we are translating the operators along a line, it is clear that rotations around that line are a symmetry of the setup. The generator of this $SO(2)$ symmetry is given by

$$\mathfrak{B} = \frac{1}{2}B_{\alpha\beta}\epsilon^{\beta\gamma}L_\gamma^{\ \alpha} \,, \tag{7.8}$$

and we can show that it commutes with the supertranslation.

Let us start with the configuration that preserves four supersymmetries. It is useful to relabel the fermionic generators slightly

$$\mathfrak{Q}_a = \epsilon_{ab}\mathfrak{Q}^b_2 \,, \qquad \mathfrak{S}^a = \mathfrak{Q}^a_1 \,, \tag{7.9}$$

so that the commutation relations induced from (6.36) and (6.37) become

$$\begin{aligned}
&\{\mathfrak{Q}_a, \mathfrak{Q}_b\} = 0 \,, && \{\mathfrak{S}^a, \mathfrak{S}^b\} = 0 \,, \\
&[R_a^{\ b}, \mathfrak{Q}_c] = \delta^b_c \mathfrak{Q}_a - \frac{1}{2}\delta^b_a \mathfrak{Q}_c \,, && [\mathfrak{B}, \mathfrak{Q}_a] = -\frac{1}{2}\mathfrak{Q}_a \,, \\
&[R_a^{\ b}, \mathfrak{S}^c] = -\delta^c_a \mathfrak{S}^b + \frac{1}{2}\delta^b_a \mathfrak{S}^c \,, && [\mathfrak{B}, \mathfrak{S}^a] = \frac{1}{2}\mathfrak{S}^a \,, \\
&\{\mathfrak{Q}_a, \mathfrak{S}^b\} = \delta^b_a \left(\mathfrak{B} + \frac{1}{2}(\mathfrak{P} - \mathfrak{K})\right) + R^b_a \,.
\end{aligned} \tag{7.10}$$

This corresponds to an $\mathfrak{su}(1|2)$ algebra and since R commutes with all elements of this algebra and with the supertranslation, then the residual symmetry is $\mathfrak{su}(1|2) \times \mathfrak{u}(1)$. Note that the $\mathfrak{su}(1|2)$ algebra cannot be centrally extended, so the central charges are simply a redefinition of the $\mathfrak{u}(1)$ generator

$$\tilde{\mathfrak{B}} = \mathfrak{B} + \frac{1}{2}(\mathfrak{P} - \mathfrak{K}) \,. \tag{7.11}$$

We will however keep the $\mathfrak{su}(2|2)$ central charges explicit as it is useful to decouple their action from that of $\mathfrak{B}$ on the hexagon form factors.

If instead we consider the setup that preserves only two supersymmetries, then the only R-charge generator that commutes with the supertranslation is given by

$$\mathfrak{J} = R_a{}^b \epsilon^{ac} \rho_c \rho_b \,. \tag{7.12}$$

If we relabel the supersymmetry generators as $\mathfrak{Q} = \mathfrak{F}_1$ and $\mathfrak{S} = \mathfrak{F}_2$ then the commutation relations are

$$\begin{aligned}
&[\mathfrak{J}, \mathfrak{Q}] = 0 \,, && [\mathfrak{J}, \mathfrak{S}] = 0 \,, \\
&[\mathfrak{B}, \mathfrak{Q}] = -\frac{1}{2}\mathfrak{Q} \,, && [\mathfrak{B}, \mathfrak{S}] = \frac{1}{2}\mathfrak{S} \,, \\
&\{\mathfrak{Q}, \mathfrak{S}\} = \mathfrak{J} \,.
\end{aligned} \tag{7.13}$$

The residual symmetry is in this case only $\mathfrak{u}(1|1)$ as the generator R does not commute with the supertranslation $\mathcal{T}^{(2)}$.

7.1.3 Representations of the residual algebra

The elementary spin-chain excitations in ABJM transform as $(2|2)_A \oplus (2|2)_B$ under $\mathfrak{su}(2|2) \times \mathfrak{u}(1)$, so we have to understand how they transform under the residual symmetries.

Unfortunately, the $(2|2)$ representation does not decompose into $(1|1)^2$ under any of the residual algebras. This makes the generalization to ABJM much harder so we will henceforth consider only the supertranslation from equation (7.4), which preserves more symmetry.

The $(2|2)$ excitations ϕ_a and ψ_α transform as the typical four-dimensional representation $(1|2|1)$ of $\mathfrak{su}(1|2)$. There is no bosonic symmetry relating the two fermions anymore, so we relabel them in the following way

$$|\chi\rangle = |\psi_+\rangle \,, \qquad |\xi\rangle = |\psi_-\rangle \,. \tag{7.14}$$

The action of the generators on this representation can be induced from the action of $\mathfrak{su}(2|2)$ on the fundamental representation and we obtain

$$\begin{aligned}
\mathfrak{Q}_a|\phi_b\rangle &= \frac{\epsilon_{ab}}{\sqrt{2}}\,(a+c)\,|\xi\rangle\,, & \mathfrak{B}|\chi\rangle &= \frac{1}{2}|\chi\rangle\,,\\
\mathfrak{S}^a|\phi_b\rangle &= \frac{\delta^a_b}{\sqrt{2}}\,(a-c)\,|\chi\rangle\,, & \mathfrak{B}|\xi\rangle &= -\frac{1}{2}|\xi\rangle\,,\\
\mathfrak{Q}_a|\chi\rangle &= \frac{1}{\sqrt{2}}\,(b+d)\,|\phi_a\rangle\,, & \mathfrak{Q}_a|\xi\rangle &= 0\,,\\
\mathfrak{S}^a|\xi\rangle &= \frac{\epsilon^{ab}}{\sqrt{2}}\,(b-d)\,|\phi_b\rangle\,, & \mathfrak{S}^a|\chi\rangle &= 0\,,
\end{aligned} \tag{7.15}$$

where we used the parameters a, b, c and d defined in (6.40) for the fundamental representation of $\mathfrak{su}(2|2)$.

7.1.4 Rotated vacua

Finally, we can now apply the supertranslation obtained and see how it rotates the operators. For simplicity, we pick a direction for the translation which corresponds to the following choice for the matrix E

$$E = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}. \tag{7.16}$$

Looking at the form of the gamma matrices in three dimensions we can see that the relation between $P_{\alpha\beta}$ and P_μ is

$$P_{\alpha\beta} = \begin{pmatrix} P_0 + P_1 & P_2 \\ P_2 & P_0 - P_1 \end{pmatrix}, \tag{7.17}$$

which means that our choice corresponds to a translation along the x_2 direction. Using equation (7.1) we can then show that the rotated vacua are

$$\mathrm{Tr}[((Y^4 - aY^1)(Y_1^\dagger + aY_4^\dagger))^L](0,0,a)\,. \tag{7.18}$$

For operators at the origin, at infinity and at $a = 1$, we have

$$\begin{aligned}
&\mathrm{Tr}[(Y^4 Y_1^\dagger)^{L_1}](0,0,0)\,,\\
&\mathrm{Tr}[((Y^4 - Y^1)(Y_1^\dagger + Y_4^\dagger))^{L_2}](0,0,1)\,,\\
&\mathrm{Tr}[(-Y^1 Y_4^\dagger)^{L_3}](0,0,\infty)\,.
\end{aligned} \tag{7.19}$$

We have therefore obtained a setup where Wick contractions between any pair of operators are non-vanishing, which is ideal for the study of non-extremal three-point functions of single-trace operators.

It is worth remarking a few differences from the setup of $\mathcal{N}=4$ SYM. In our case a single elementary excitation cannot contract with any of the other vacua, so we do not have longitudinal excitations. This in fact indicates that form factors with an odd number of excitations will be vanishing.

Another difference is that the contraction of colour indices is different in ABJM, as the fields transform in bifundamental and anti-bifundamental representations of the gauge group. The number of Wick contraction between two operators at tree level is given by

$$l_{ij} = L_i + L_j - L_k \,. \tag{7.20}$$

Since each operator has an even number of fields, then the l_{ij} are either all even or all odd. It turns out that when they are all odd the three-point function must be vanishing due to a cancelation in the contraction of colour indices. The reason why this happens is that we need to sum all planar contributions, so we must consider all cyclic permutations of the fields inside the traces. There are $L_1L_2L_3$ contributions where the first field is in the bifundamental representation, and they all contribute equally. But there are also $L_1L_2L_3$ contributions where the first field in the trace is in the anti-bifundamental representation. However, due to the minus signs in (7.19), these two contributions cancel each other for l_{ij} odd, while they add up to $2L_1L_2L_3$ when they are all even.

The argument made here also holds for excited states if one uses the additional fact that the total momentum of Bethe states must be vanishing.

7.2 Hexagon form factors

Usually, the form factor of an operator $\mathcal{O}$ is defined as the expectation value between an incoming and outgoing on-shell state

$$\langle \theta_1, \ldots, \theta_n | \mathcal{O} | \theta_1, \ldots, \theta_m \rangle \,. \tag{7.21}$$

By using crossing we can consider form factors only with incoming particles. The fact that we have the form factor between two states corresponds to the fact that there is a past and future infinity.

Something analogous happens with three-point functions. From the string point of view we have a worldsheet with three infinities, which correspond to the states from the three operator insertions at the boundary. When we cut the worldsheet into two hexagons, each of them probes the three infinites, but we can use crossing to map excitations from one edge to another. The fact that each hexagon is associated with a point of conical excess in the worldsheet is related with the presence of three infinities and implies that two crossing transformations do not lead to the original configuration.

We can also think of the hexagon form factor as the action of a vertex $\langle\mathfrak{h}|$ on the tensor product of three spin-chain states [152–154]

$$\langle\mathfrak{h}|\psi_1|\psi_2|\psi_3\rangle\,. \tag{7.22}$$

By using crossing, it is enough to consider hexagon form factors where all excitations are in the same edge

$$\langle\mathfrak{h}|\Phi_{A_1}\dots\Phi_{A_n}\rangle\,. \tag{7.23}$$

Once this object is known for any number n of particles, then it is in principle possible to compute any three-point function of ABJM.

7.2.1 Constraints on the multi-particle form factors

In what follows we will use the residual symmetry to constrain the hexagon form factors. The fact that the vertex is invariant under an $SU(1|2)\times U(1)$ symmetry implies that it is annihilated by all generators of the algebra. There are some general remarks that can be done without specifying the number of excitations. In ABJM we have A and B particles, which have opposite charges under the abelian $U(1)$. Since that symmetry is preserved by the hexagon, we have

$$\langle\mathfrak{h}|R|\Phi_{A_1}\dots\Phi_{A_n}\rangle = 0\,, \tag{7.24}$$

which implies that the hexagon is only non-vanishing if the number of A and B particles is the same. A consequence of this is that the only non-vanishing hexagon form factors are the ones with an even number of excitations.

The fact that $\mathfrak{P}-\mathfrak{K}$ annihilates the hexagon also has a generic consequence. First we must remember that the multi-particle state is defined through the modules $V(p_1,\zeta_1)\otimes\dots V(p_n,\zeta_n)$, where the phases ζ_i obey equation (6.44). The symmetry then imposes a constraint on ζ_1

$$\zeta_1^2 = e^{-iP}\,, \tag{7.25}$$

where P is the total momentum of the state.

Finally, since the fermions have opposite charges under the bosonic generator $\mathfrak{B}$, we conclude that the hexagon form factors must have an equal number of χ and ξ excitations in order to be non-vanishing.

7.2.2 The two-particle Hexagon

The most fundamental hexagon of ABJM is then the two-particle form factor. We know that it must contain an A and B particle, which we write as Φ_A and $\Phi_{\dot{B}}$ respectively. There are in principle two structures allowed, $\langle\mathfrak{h}|\Phi_A\dot{\Phi}_B\rangle$ and $\langle\mathfrak{h}|\dot{\Phi}_A\Phi_B\rangle$. However, the three-point function with A-type excitations at the origin and B-type at infinity should be equivalent to the case where we have B-type excitations at the origin and A-type at infinity, which implies that those two structures should be equal. If we impose bosonic symmetries we can fix the hexagon up to three scalar functions

$$\begin{aligned}
\langle\mathfrak{h}|\phi_a^1\dot{\phi}_b^2\rangle &= \mathcal{A}_{12}\epsilon_{ab}\,,\\
\langle\mathfrak{h}|\chi^1\dot{\xi}^2\rangle &= \mathcal{B}_{12}\,,\\
\langle\mathfrak{h}|\xi^1\dot{\chi}^2\rangle &= \mathcal{C}_{12}\,.
\end{aligned}\tag{7.26}$$

By further requiring the fermionic charges of $\mathfrak{su}(1|2)$ to annihilate the two particle hexagon form factor, we understand that it depends only on a single scalar function $\mathcal{A}_{12}$

$$\begin{aligned}
\mathcal{B}_{12} &= \mathcal{A}_{12}\frac{b(p_1)+d(p_1)e^{iP/2}}{a(p_2)e^{iP/2}+c(p_2)e^{ip_2}}\,,\\
\mathcal{C}_{12} &= \mathcal{A}_{12}\frac{b(p_2)e^{-ip_1}+e^{-iP/2}d(p_2)}{a(p_1)e^{-iP/2}+c(p_1)}\,.
\end{aligned}\tag{7.27}$$

Unfortunately, and unlike the one-particle form factor in $\mathcal{N}=4$ SYM, the dependence on the momenta of the excitations is quite non-trivial, and we have not found any way to simplify (7.27).

7.2.3 The four-particle Hexagon

The hope for a bootstrap of the hexagon form factors in ABJM is that the two-particle form factor provides a inner-product between excitations which is then used in the multi-particle hexagon.

We will now impose $\mathfrak{su}(1|2)\times\mathfrak{u}(1)$ symmetry on the four-particle form factor and see how far it takes us. There are in general six structures that one can consider

$$\begin{aligned}
&\langle\mathfrak{h}|\Phi_A\Phi_B\dot{\Phi}_C\dot{\Phi}_D\rangle\,, \quad \langle\mathfrak{h}|\Phi_A\dot{\Phi}_B\Phi_C\dot{\Phi}_D\rangle\,, \quad \langle\mathfrak{h}|\Phi_A\dot{\Phi}_B\dot{\Phi}_C\Phi_D\rangle\,,\\
&\langle\mathfrak{h}|\dot{\Phi}_A\Phi_B\Phi_C\dot{\Phi}_D\rangle\,, \quad \langle\mathfrak{h}|\dot{\Phi}_A\Phi_B\dot{\Phi}_C\Phi_D\rangle\,, \quad \langle\mathfrak{h}|\dot{\Phi}_A\dot{\Phi}_B\Phi_C\Phi_D\rangle\,,
\end{aligned}\tag{7.28}$$

and they are all annihilated independently by the residual algebra. We will focus on the structure $\langle\mathfrak{h}|\Phi_A\Phi_B\dot{\Phi}_C\dot{\Phi}_D\rangle$ which from bosonic symmetries must be of the form

$$\begin{aligned}
\langle\mathfrak{h}|\phi_a\phi_b\dot{\phi}_c\dot{\phi}_d\rangle &= \mathcal{A}^1\epsilon_{ac}\epsilon_{bd}+\mathcal{A}^2\epsilon_{ad}\epsilon_{bc}\,,\\
\langle\mathfrak{h}|\phi_a\phi_b\dot{\chi}\dot{\xi}\rangle &= \mathcal{B}^{12}\epsilon_{ab}\,,\\
\langle\mathfrak{h}|\phi_a\phi_b\dot{\xi}\dot{\chi}\rangle &= \mathcal{C}^{12}\epsilon_{ab}\,,\\
\langle\mathfrak{h}|\chi\chi\dot{\xi}\dot{\xi}\rangle &= \mathcal{D}^{12}\,.
\end{aligned}\tag{7.29}$$

where $\mathcal{B}^{ij}$ corresponds to the case when the bosonic excitations are in the positions i and j, and analogously for $\mathcal{C}^{ij}$ and $\mathcal{D}^{ij}$. Once we impose fermionic symmetries, we are left with only three undetermined functions of the four momenta.

Unfortunately, we were not able to express the four-particle hexagon in terms of two-particle hexagons, and it is unclear if that task is difficult or impossible. On one hand, one can see that all non-vanishing components of the four-particle form factor $\langle\mathfrak{h}|\Phi_A\Phi_B\Phi_C\Phi_D\rangle$ coincide with the non-vanishing components of at least one of the following three structures

$$\begin{aligned}&\langle\mathfrak{h}|\Phi_1^A\Phi_2^B\rangle\langle\mathfrak{h}|\Phi_3^C\Phi_4^D\rangle\,,\\&\langle\mathfrak{h}|\Phi_1^A\Phi_3^C\rangle\langle\mathfrak{h}|\Phi_2^B\Phi_4^D\rangle\,,\\&\langle\mathfrak{h}|\Phi_1^A\Phi_4^D\rangle\langle\mathfrak{h}|\Phi_2^B\Phi_3^C\rangle\,.\end{aligned}\tag{7.30}$$

On the other hand, one can also argue that it might not be possible to express the four-particle hexagon in terms of the two-particle form factors. To understand the argument let us look at $\mathcal{N}=4$ SYM for a moment. In that case the prescription for the multi-particle form factor involved scattering the right parts of the excitations. In terms of the modules, after scattering we have

$$(V(p_1,\zeta_1)\otimes\dot{V}(p_1,\dot{\zeta}_1'))\otimes\ldots\otimes(V(p_n,\zeta_n)\otimes\dot{V}(p_n,\dot{\zeta}_n'))\,.\tag{7.31}$$

It turns out that the phases obey

$$\zeta_n\dot{\zeta}_n'=e^{-ip_n}\,,\tag{7.32}$$

which is exactly the condition for a well defined two-particle hexagon. In some sense this justifies the multi-particle conjecture as we see that after scattering the action of the projector $\langle\mathfrak{h}_1|\otimes\ldots\otimes\langle\mathfrak{h}_n|$ is well defined. From this point of view, in ABJM we would need to find a map

$$\rho:\ V(p_1,\zeta_1)\otimes\ldots\otimes V(p_n,\zeta_n)\to V(p_{\sigma_1},\zeta_1')\otimes\ldots\otimes V(p_{\sigma_n},\zeta_n')\,,\tag{7.33}$$

after which the action of the projector $\langle\mathfrak{h}_{12}|\otimes\ldots\otimes\langle\mathfrak{h}_{n-1,n}|$ is also well defined. However, it is possible to show that the map ρ must involve structures other than the scattering matrix $\hat{S}_{\mathfrak{psu}(2|2)}$. While this does not show that it is impossible to build the multi-particle form factor from the two-particle one, it does indicate that the map ρ must involve some sort of novel structure.

7.2.4 Watson equations

The hexagon form factors must also obey the form factor axioms, which include the Watson equation [148, 149]

$$\langle\mathfrak{h}|\mathbb{S}_{i,i+1}|\Phi_1\ldots\Phi_n\rangle=\langle\mathfrak{h}|\Phi_1\ldots\Phi_{i+1}\Phi_i\ldots\Phi_n\rangle\,.\tag{7.34}$$

For example, in $\mathcal{N}=4$ SYM this constraints were crucial in the determination of the scalar factor h_{12} from (6.78).

In ABJM, the Watson equation for the two-particle form factor is

$$\langle\mathfrak{h}|\dot\Phi^2_A\Phi^1_B\rangle-(\mathbb{S}_{21})^{CD}_{AB}\langle\mathfrak{h}|\Phi^1_C\dot\Phi^2_D\rangle=0\,, \tag{7.35}$$

with $\mathbb{S}$ the ABJM S-matrix. The solution to this equation is given by

$$\mathcal{A}_{21}=S_0^{AB}B_{21}\mathcal{A}_{12}+\frac{1}{2}S_0^{AB}C_{21}(\mathcal{B}_{12}-\mathcal{C}_{12})\,, \tag{7.36}$$

with B_{21} and C_{21} elements of the $\mathfrak{su}(2|2)$ S-matrix from (6.51), $\mathcal{B}_{12}$ and $\mathcal{C}_{12}$ given in (7.27) in terms of $\mathcal{A}_{12}$, and S_0^{AB} the scalar factor of the ABJM S-matrix in (6.60). The Watson equation then imposes a constraint on the scalar factor $\mathcal{A}_{12}$ of the two-particle hexagon

$$\frac{\mathcal{A}_{12}}{\mathcal{A}_{21}}=\frac{1}{\sigma(u,v)}f(u,v)\,, \tag{7.37}$$

with $f(u,v)$ a complicated fuction of the two rapidities. This means that $\mathcal{A}_{12}$ should depend on some sort of square root of the BES phase, which due to its non-trivial analytical structure might in fact require a new dressing phase.

Regarding the four-particle Hexagon, there are now two kinds of Watson equations we can impose on $\langle\mathfrak{h}|\Phi_A\Phi_B\dot\Phi_{\dot A}\dot\Phi_{\dot B}\rangle$. One of them relates it to an hexagon with A and B particles in different positions

$$(\mathbb{S}_{23})^{\dot CC}_{B\dot A}\langle\mathfrak{h}|\Phi^1_A\dot\Phi^3_{\dot C}\Phi^2_C\dot\Phi^4_{\dot B}\rangle=\langle\mathfrak{h}|\Phi^1_A\Phi^2_B\dot\Phi^3_{\dot A}\dot\Phi^4_{\dot B}\rangle\,. \tag{7.38}$$

One can show that all six structures in (7.28) are related to each other by this kind of Watson equation which exchanges A and B particles. The other two equations are

$$\begin{aligned}(\mathbb{S}_{12})^{CD}_{AB}\langle\mathfrak{h}|\Phi^2_C\Phi^1_D\dot\Phi^3_{\dot A}\dot\Phi^4_{\dot B}\rangle&=\langle\mathfrak{h}|\Phi^1_A\Phi^2_B\dot\Phi^3_{\dot A}\dot\Phi^4_{\dot B}\rangle\,,\\ (\mathbb{S}_{34})^{\dot C\dot D}_{\dot A\dot B}\langle\mathfrak{h}|\Phi^1_A\Phi^2_B\dot\Phi^4_{\dot C}\dot\Phi^3_{\dot D}\rangle&=\langle\mathfrak{h}|\Phi^1_A\Phi^2_B\dot\Phi^3_{\dot A}\dot\Phi^4_{\dot B}\rangle\,,\end{aligned} \tag{7.39}$$

and they provide constraints on the three undetermined functions of the four-particle form factor.

7.3 Tree level checks

All the discussion so far was based on symmetry, so it is important to validate the results obtained with perturbative data. Unfortunately, with a few exceptions [155–157] there is very little information on three-point functions of ABJM, so we will resort to simple tree level considerations.

7.3.1 Double excitations

Let us consider the case when only the operator at the origin is unprotected. We saw previously that the elementary excitations cannot be contracted with any of the rotated vacua. Meanwhile, we have found well-defined and non-vanishing hexagon form factors even when there are excitations only in one edge. This is an apparent contradiction which is resolved by considering the effect of double excitations.

Before we do so, it is important to find the relation between spin-chain excitations and the fields in the single-trace operators. From the transformations under bosonic symmetries we can show that we have

$$\begin{aligned} (\phi_1, \phi_2) &= (Y^3, -Y^2)\,, & (\chi, \xi) &= (\psi_{1+}, \psi_{1-})\,, \\ (\dot{\phi}_1, \dot{\phi}_2) &= (Y_2^\dagger, Y_3^\dagger)\,, & (\dot{\chi}, \dot{\xi}) &= (\psi_+^{4\dagger}, \psi_-^{4\dagger})\,. \end{aligned} \tag{7.40}$$

The solution for the hexagon with two scalar excitations given in (7.26) is non-vanishing for $Y^3 Y_3^\dagger$ and $Y^2 Y_2^\dagger$. It was shown in (2.86) that these mix with the double excitations $Y^4 Y_4^\dagger$ and $Y^1 Y_1^\dagger$, which in turn have non-vanishing Wick contractions with the rotated vacua.

We can also play the same game with the fermionic excitations. From the two-particle hexagon we see that the only non-vanishing contributions are from $\psi_{1+}\psi_-^{4\dagger}$ and $\psi_{1-}\psi_+^{4\dagger}$. Following (2.86) we conclude that these fields mix with the derivative D^2, whose propagator is proportional to x_2. This is exactly the direction along which we supertranslated the operators, so we see that it also leads to non-vanishing Wick contractions with the other vacua.

7.3.2 Crossing

In $\mathcal{N} = 4$ SYM the flavour of the excitations changes as we cross them to other edges of the hexagon. In order to understand how the flavour of excitations changes under crossing in ABJM we need to look at tree-level examples and match them with the non-vanishing entries of the two-particle hexagon form factor.

First, we need to derive the action of the supertranslation on the elementary excitations of the spin-chain, so that we identify correctly the supertranslated fields that correspond to the $(2|2)$ fundamental representation. The flavour of the scalar excitations is unchanged

$$\begin{aligned} e^{a\mathcal{T}} Y^2(0) e^{-a\mathcal{T}} &= Y^2(a)\,, & e^{a\mathcal{T}} Y^3(0) e^{-a\mathcal{T}} &= Y^3(a)\,, \\ e^{a\mathcal{T}} Y_2^\dagger(0) e^{-a\mathcal{T}} &= Y_2^\dagger(a) & e^{a\mathcal{T}} Y_3^\dagger(0) e^{-a\mathcal{T}} &= Y_3^\dagger(a)\,, \end{aligned} \tag{7.41}$$

while for the fermions we have a non-trivial transformation

$$\begin{aligned} e^{a\mathcal{T}} \psi_{1\alpha}(0) e^{-a\mathcal{T}} &= (\psi_{1\alpha} + a\psi_{4\alpha})(a)\,, \\ e^{a\mathcal{T}} \psi_\alpha^{4\dagger}(0) e^{-a\mathcal{T}} &= (\psi_\alpha^{4\dagger} - a\psi_\alpha^{1\dagger})(a)\,. \end{aligned} \tag{7.42}$$

If we insert Y^2 or Y^3 excitations at the origin, then the correlation function is non-vanishing for an operator with $Y_2^\dagger$ or $Y_3^\dagger$ at infinity. In terms of the (2|2) excitations, they correspond to the hexagon form factors

$$\langle \mathfrak{h}|\phi_1|\ |\dot{\phi}_2\rangle\,, \qquad \langle \mathfrak{h}|\phi_2|\ |\dot{\phi}_1\rangle\,. \tag{7.43}$$

We conclude that the flavour of the excitations cannot change upon crossing, otherwise we would obtain vanishing form factors.

Let us now consider the case when the operator at the origin has the fermionic excitations $\psi_{1\pm}$ while at infinity we have $\psi_{\mp}^{1\dagger}$. This corresponds to a non-vanishing correlator since the operators are translated along x_2 and the fermionic propagator is

$$\langle \psi_{I\alpha}(0)\psi_\beta^{J\dagger}(x)\rangle \propto \delta_I^J (C\Gamma^\mu)_{\alpha\beta} x_\mu\,. \tag{7.44}$$

In terms of hexagon form factors these configurations correspond to

$$\begin{aligned}\langle \mathfrak{h}|\psi_+|\ |\dot{\psi}_-\rangle &= \langle \mathfrak{h}|\chi|\ |\dot{\xi}\rangle\,,\\ \langle \mathfrak{h}|\psi_-|\ |\dot{\psi}_+\rangle &= \langle \mathfrak{h}|\xi|\ |\dot{\chi}\rangle\,.\end{aligned} \tag{7.45}$$

Once again, the hexagon form factors are only non-vanishing if crossing does not alter the flavour of the excitations.

7.3.3 Tree level data

We will now consider a few explicit examples of tree-level three-point functions. They are useful for checking the ratios of $\mathcal{A}$, $\mathcal{B}$ and $\mathcal{C}$ at weak coupling, but they can also be used as a guide for the multi-particle case. Ideally, one would consider a single unprotected operator, so that crossing would not be necessary. Unfortunately, this also corresponds to a more difficult setup, where we need to excite several levels of the nested Bethe ansatz in order to obtain a non-vanishing result. We will instead consider the case of two unprotected operators, and for simplicity we take them to be in the rank one sectors $\mathfrak{su}(2)$ and $\mathfrak{su}(1|1)$.

In the scalar case, we have Y^3 excitations on the operator at the origin, and and equal number of $Y_3^\dagger$ on the operator at infinity. At weak coupling, the S-matrix in the $\mathfrak{su}(2)$ sector is

$$S_{su(2)}(u,v) = S_0^{AA}(u,v)A(u,v) \xrightarrow[g\to 0]{} \frac{u-v-i}{u-v+i}\,. \tag{7.46}$$

At tree level the S-matrix then coincides with that of the $\mathfrak{su}(2)$ sector of $\mathcal{N}=4$ SYM. Since the one-loop Bethe states are the same, we can then recycle the results for correlation functions of $\mathcal{N}=4$ SYM at tree level. It is important to consider only X and $\bar{X}$ excitations of $\mathcal{N}=4$

SYM, so that we do not have contractions with the rotated vacuum. The matrix part of the hexagon becomes the domain wall partition function of a six-vertex model [158], and so we obtain

$$h^{su(2)}(\{u\},\{v\}) = \frac{\det\left[\frac{i}{(u_i-v_j)(u_i-v_j-i)}\right]\prod_{i,j}(v_i-u_j+i)}{\prod_{i<j}(u_i-u_j+i)(v_j-v_i-i)}\,. \tag{7.47}$$

These results also apply to ABJM, and so we obtain a tree level prediction for the undetermined function of the two-particle form factor

$$\langle\mathfrak{h}|\phi_1(u^{4\gamma})\dot{\phi}_2(v)\rangle = \mathcal{A}(u^{4\gamma},v) = \frac{-i}{u-v}\,. \tag{7.48}$$

Analogously, we can consider the $\mathfrak{su}(1|1)$ sector, where we have fermionic excitations ψ_{1+} in the operator at the origin and $\psi_-^{1\dagger}$ in the operator at infinity. The S-matrix in this sector is

$$S_{su(1|1)}(u,v) = S_0^{AA}(u,v)D(u,v) \underset{g\to 0}{\longrightarrow} -1\,, \tag{7.49}$$

which once again correponds exactly to the tree level scattering matrix in the $\mathfrak{su}(1|1)$ of $\mathcal{N}=4$ SYM. At weak coupling this setup is equivalent to the one considered by Caetano and Fleury [159] where they showed that the matrix part of the hexagon is a domain wall partition function on another six-vertex model

$$h^{su(1|1)}(\{u\},\{v\}) = i^n\frac{\prod_{i<j}(u_i-u_j)(v_i-v_j)}{\prod_{i,j}(v_i-u_j)}\,. \tag{7.50}$$

This corresponds to the hexagon of ABJM and we can obtain a prediction for the fermionic part of the two-particle form factors at weak coupling

$$\begin{aligned}\langle\mathfrak{h}|\psi_+(u^{4\gamma})\dot{\psi}_-(v)\rangle &= \mathcal{B}(u^{4\gamma},v) = \frac{-i}{u-v}\,,\\ \langle\mathfrak{h}|\psi_-(u^{4\gamma})\dot{\psi}_+(v)\rangle &= \mathcal{C}(u^{4\gamma},v) = \frac{-i}{u-v}\,.\end{aligned} \tag{7.51}$$

The ratios with $\mathcal{A}(u^{4\gamma},v)$ at weak coupling are then

$$\frac{\mathcal{B}(u^{4\gamma},v)}{\mathcal{A}(u^{4\gamma},v)} = 1\,, \qquad \frac{\mathcal{C}(u^{4\gamma},v)}{\mathcal{A}(u^{4\gamma},v)} = 1 \tag{7.52}$$

which correspond exactly to the weak coupling expansion of the ratios obtained in (7.27). Unfortunately, since crossing does not commute with perturbation theory, we are not able to obtain a prediction for $\mathcal{A}(u,v)$.

8. Epilogue

After the success of integrability in solving the spectrum of $\mathcal{N} = 4$ SYM in the planar limit, there has also been in recent years a great effort in the study of its correlation functions. Integrability has played a very important role in the determination of structure constants both at weak and strong coupling and recently an all-loop framework for the computation of these objects in $\mathcal{N} = 4$ SYM has been proposed. Despite its success, there is still much to be understood. One would like to resum wrapping corrections and effectively obtain three-point functions at intermediate values of the coupling. Besides that, it is important to extend these techniques to other theories, just like it happened for the spectrum.

The focus of this thesis was the study of correlation functions in integrable theories. We have studied both the weak and strong coupling limits of $\mathcal{N} = 4$ SYM and have obtained important data that should be matched with the Hexagon form factor approach. In the context of ABJM we have also studied the role of Hexagon form factors in the computation of its structure constants. In what follows we describe the main achievements of this thesis and also some of the possible directions for future work.

8.1 Structure constants at weak and strong coupling

In this thesis we have obtained several new results for structure constants at strong coupling. We have focused on correlators of short operators, whose dimensions scale as $\Delta \sim \lambda^{1/4}$ and for which one can use a flat-space approximation. By understanding the map between the gauge theory operators and the string vertex operators, we have been able to express three-point functions of $\mathcal{N} = 4$ SYM as superstring amplitudes in flat space. The main results of Paper I are the structure constants of the Konishi operator with half-BPS operators, and the realization that they are exponentiallly suppressed at strong coupling.

In Paper II we have studied correlation functions for short operators with spin. We were able to perform superstring amplitudes with states at higher mass levels by considering operators dual to string states in the leading Regge trajectory. We have also obtained a precise map between the building blocks of superstring amplitudes and the tensor structures allowed by conformal symmetry.

In Paper II we considered also the extremal limit for the three-point function of Konishi with two half-BPS operators. The fact that the string coupling does not vanish leads to a pole in the structure constant which indicates that at the extremal point Konishi does not have a definite scaling dimension and mixes with the double-trace singlet operator. We have understood how to correctly renormalize the operators and found the $1/N$ corrections to their scaling dimensions.

In this thesis we have also studied correlation functions at weak coupling. By considering a four-point function of protected operators one can extract the structure constant of the lowest twist operators at five loops. We used the technique of asymptotic expansions, where conformal integrals become massless propagator integrals in the OPE limit. The master integrals obtained were unknown, but we have obtained their ϵ expansions without any explicit integration, by considering the allowed divergences and magic identities of conformal integrals.

Finally, we took the first steps in a generalization of the Hexagon programme to ABJM. We studied the possible setups for the evaluation of non-extremal correlation functions and found the residual symmetry to be $\mathfrak{su}(1|2) \times \mathfrak{u}(1)$. We used that symmetry to fix the two-particle hexagon, and found that the four-particle form factors are determined only up to three scalar functions. We have also argued that it is not sufficient to use the S-matrix in order to write the multi-particle form factors in terms of two-particle hexagons. Finally, the Watson equations indicate the appearance of a square root of the BES dressing factor which needs to be investigated further.

8.2 Future work

There are several directions of work that should be pursued in the research of structure constants at strong coupling. One could try to obtain $1/N$ corrections by evaluating the next term in the genus expansion of superstring amplitudes, where the flat-space approximation should still be valid. It is also important to find the α' corrections of the correlators we studied. A change in the coupling can be seen as a change in the background geometry, and that effect can be obtained with the insertion of closed string states in the superstring amplitude. It is however unclear what those states should be and if they are compatible with the flat-space approximation. Perhaps even more important would be to find the vertex operators for strings propagating on an AdS background. The most suitable formalism for this endeavour is pure spinor but there are several difficulties one must overcome. On the technical side, one has to go beyond the existing expansions for the vertex operators [160], while a more conceptual problem is to understand exactly what is the correct prescription for the zero-mode integration [161].

At weak coupling there are also many points that should be investigated further. In order to claim a good understanding of the wrapping corrections in three-point functions, one needs to find the Hexagon form factors at five loops and match with perturbative results. It would also be interesting to use the technique of asymptotic expansions in order to study non-planar corrections to structure constants. There are indications that the Hexagon form factors can be used to find $1/N$ corrections to correlation functions, so it is important to obtain perturbative data relevant to that problem. It is known that for a four-point function of operators in the $20'$ the non-planar corrections appears at four loops [90], but one can also consider more generic four-point functions for which the non-planar corrections show up at lower loop order [162].

Finally, it is important to understand if we can extend the Hexagon form factor programme to other integrable theories. It would be natural to look at it in the context of AdS_3/CFT_2. In that case the symmetry takes the factorized form $\mathfrak{psu}(1|1)^2$ [163], so one could hope that the hexagon form factor bootstrap would follow closely the example of $\mathcal{N} = 4$ SYM. In the context of ABJM, there are many points that need to be considered further. It is essential to understand if the multi-particle form factors can be expressed in terms of the two-particle hexagon. Besides that, it is also important to find if the quantum corrections to the three-point function of protected operators can be explained with wrapping corrections of the hexagon form factors. Finally, since some supersymmetry generators annihilate all three vacua, there is hope that one can evaluate the three-point function with a localization technique [164].

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Svensk sammanfattning

Kvantfältteori är ett teoretisk ramverk som har varit extremt framgångsrikt i sin beskrivning av partikelfysik. Men trots alla dess applikationer är det väldigt svårt att gå bortom störningsteori och studera starkt kopplade system. Förutom detta finns också problemet att vi inte förstår allmän relativitetsteori då krökningen av rumtiden blir stark, vilket kräver en teori för kvantgravitation.

Det visar sig att strängteori är ett matematiskt ramverk som kan hjälpa oss lösa båda dessa frågor. Istället för att betrakta punkt-lika partiklar, är det fundementala objektet i strängteori ett endimensionellt objekt, en sträng, som sveper ut en två-dimensionell världsyta i rumtiden, och olika partiklar är olika excitationer av denna sträng. I en strängteori kan man också betrakta objekt med utbredning i fler dimensioner, så kallade D-bran, vilka är vad öppna strängar slutar på.

Någonting väldigt speciellt händer när vi studerar vissa konfigurationer av D-bran vid låga energier. Åena sidan beskrivs dynamiken av öppna strängar av en d-dimensionell konformal fält-teori. Men vi kan också se branen som en deformering av bakgrundsgeometrin, vilket i när-horisont regionen blir $AdS_{d+1} \times \mathcal{M}$ där $\mathcal{M}$ är någon kompakt mångfald. Från detta kan vi dra slutsatsen att det finns en dualitet mellan den d-dimensionella konforma fältteorin och en supersträngteori på AdS_{d+1} bakgrund. Denna dualitet kallas AdS/CFT, och är ett bra exempel på holografi.

Två av de mest studerade exemplena av AdS/CFT är givna av typ IIB strängar i $AdS_5 \times S^5$ och typ IIA strängar i $AdS_4 \times \mathbb{CP}^3$, vars duala konforma fältteorier ges av $\mathcal{N} = 4$ SYM i fyra dimensioner och ABJM i tre dimensioner. Dessa teorier är supersymmetriska, men i den så kallade "planar limit", vilket är en specifik gräns av teorins parametrar, tror vi att de blir integrerbara, eftersom de där har ett oändligt torn av gömda symmetrier. En av de huvudsakliga implikationerna av integrabilitet är att S-matrisen kan faktoriseras till två-till-två spridningsprocesser, vilket är en enorm förenkling. Notera att dessa är svaga/starka dualiteter, där den klassiska regimen av den ena teorin avbildas på den kvantmekaniska regimen av den andra.

I denna avhandling har vi studerat korrelationsfunktioner av dessa teorier. Vid stark koppling studerar vi korrelationsfunktioner av så kallade korta operatorer, vilkas dimensioner skalar som $\Delta \sim \lambda^{1/4}$, och för vilka vi kan approximera rumtiden som platt. I artikel I förstod vi hur man avbildar gaugeteori-operatorer till sträng vertex operatorer, och vi hittade strukturkonstanterna för Konishi-operatorn med skyddade halv-BPS operatorer.

I artikel II studerade vi tre-punktsfunktioner av operatorer med spin, och hittade en exakt funktion mellan byggklossarna av supersträngsamplituder och de tensorstrukturer som är tillåtna av konform symmetri. Vi har också studerat den extrema gränsen av tre-punktsfunktionerna av Konishi med två halv-BPS operatorer. Genom att renormalisera operatorerna kunde vi göra tre-punktsfunktionerna finita och vi hittade hur Konishi blandas med en dubbel-trace operator.

I denna avhandlingen har vi också studerat OPE gränsen av fyrpunktsfunktioner vid svag koppling. För att bestämma strukturkonstanterna vid fem loopar använde vi tekniken med asymptotisk serie. De resulterande integralerna var okända, men vi lyckades finna deras ϵ-utveckling utan någon explicit integrering, från enkla överväganden av tillåtna typer av divergenser och magiska identiteter från konforma integraler.

Slutligen studerade vi en generalisering av Hexagon-projektet för att hitta formfaktorer, för ABJM. Vi fann en icke-extrem setup med $\mathfrak{su}(1|2)\times\mathfrak{u}(1)$ överbliven symmetri. Två-partikel hexagonen är fixerad upp till en enda skalär funktion, vilket tyder på existensen av en ny typ av "dressing phase". Slutligen använde vi den överblivna symmetrin också på fyra-partikel hexagonen, men fann att den inte är totalt fixerad av symmetrin och att den inte kan uttryckas som en enkel produkt av spridningsmatriser.

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