The Wave Function of the Universe

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To my mother and father.

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Abstract

In Quantum Cosmology, universe states are treated as wave function solutions to a zero-energy Schroedinger equation that is hyperbolic in its second derivatives of spatial geometries and matter-fields. In order to select one wave function (that would in principle correspond to our Universe) out of infinitely many, requires an appropriate boundary condition. The Hartle-Hawking No Boundary and the Vilenkin Tunneling proposals are examples of such boundary conditions. We review their applications and shortcomings in the context of the Inflationary Scenario.

Another boundary condition is that of S.W. Hawing and D.N. Page (1990) in the context of *wormholes*. Wormholes are generally considered to play a major role in setting the cosmological constant to zero and to provide a mechanism for black hole evaporation. It is significant that we are able to show that even the class of bulk matter wormhole instantons found by Carlini and Mijić (1990) are predicted in the quantum theory. However, unresolved issues and newfound problems seem to threaten the wormhole theory.

Furthermore, since there are no *a priori* notions of time (and space) present in the quantum theory, it is important to show exactly how the notion of time is recovered over distances much larger than the Planck scale. A good notion of time is also essential for any quantum theory to predict the correct classical behaviour for the Universe today. The issue of time inevitably re-emerges throughout our work.

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Chapter 1

Introduction

Thus far, numerous efforts to unify the four fundamental interactions of Nature into a single "Theory Of Everything" are still inadequate in more ways than one, and we are offered only rare glimpses of what ultimately makes the world tick. Predictions from Superstrings, Supergravity and Higher dimensional Kaluza-Klein Theories are largely well beyond the reach of our particle accelerators, and will probably remain so for the forseeable future. Whether these efforts are true milestones or merely conspicuous cults corrupting a generation of new scientists still remains to be seen.

Of late there appear to be'indirect' tests of such theories, for instance, low energy stringy actions also predict the existence of cosmological and black hole solutions [41, 29, 87, 123].

Nevertheless, we know that General Relativity and Modern Quantum Mechanics are but stepping stones in our effort to construct a (generally covariant) theory of quantum gravity. Such a theory will have significant consequences wherever gravitational fluctuations are large and of the order of the Planck curvature m_p^2, and effects are nowhere else more noticeable than in the early stages of cosmic evolution. Since any change in the topology should effectively interact with most forms of matter-fields then present, the early Universe provides an ideal laboratory for testing predictions from such theories. An understanding of the quantum nature of gravitation is essential in order to explore the emergence of classical spacetime and the origin of the Universe as we know it.

Although the Universe is at present far from equilibrium, it had to be smooth to a very high degree in the very distant past. Yet it must have allowed sufficiently large density fluctuations required for galaxy formation to take place. These problems, amongst others have been addressed in the so-called Inflationary Scenario first suggested by Alan Guth (1981) [52].

However, what inflation does not resolve is the question of the initial conditions necessary for the field equations of General Relativity to predict a very large, isotropic and homogeneous Universe that contains very little trace of any cosmological constant. Hence the question of selecting our Universe from an infinite set of possible universes that could equally well have evolved from some initial state after the Big Bang, lies beyond the scope of General Relativity. In recent years, this issue has been tackled in a branch of quantum gravity known as called Quantum Cosmology.

1.1 Quantum Cosmology

The arena of Quantum Cosmology is an infinite dimensional Superspace of all possible three-geometries and matter-field configuration on a given constant three-surface. Dynamical laws are constrained by a second quantized, second order hyperbolic differential equation on Superspace, known as the Wheeler-De Witt equation. It has infinitely many solutions, and requires an associated boundary condition to pick out a unique wave function for the Universe. Apparently, a boundary condition alone does not suffice to select a unique steepest descent contour in the so-called Euclidean Path Integral Formulation [58] (see Chapter 4).

After the pioneering work of De Witt [22] many attempts have been made (Wheeler [142],Misner [110], Vilenkin [133], Hartle and Hawking [65]), to interpret various facets of the theory:

1) There are no *a priori* notions of time and space present in the theory.

2) The Wheeler -De Witt equation is second- order in its various functional derivatives. Hence there is no direct way of defining a good, i.e. positive definite *probability density.*

3) Problems such as the horizon, flatness and monopole problems are explained via an inflationary phase early in the cosmic evolution. One therefore requires appropriate *boundary conditions* for an inflationary phase to take place. In order to reach its current entropy, the universe had to be very smooth in the past. Yet one requires sufficiently large density fluctuations and gravitational waves consistent with the observed isotropic CMBR, to allow galaxy - formation. Again, appropriate boundary conditions are needed to predict this behaviour.

1.2 Mini-Superspace

The quantum state $\Psi(h_{ij}(X), \phi(X), S)$, of a closed universe contains a three-surface S on which the three-metric is h_{ij} and matter-field configuration $\phi(X)$. This wave functional satisfies the Wheeler -De Witt equation and momentum constraints, obtained by quantization of the Hamiltonian for the Einstein scalar action for gravity and matter-fields. It provides an amplitude from which predictions concerning the outcome of large scale observations are extracted.

The space of all three-metrics $h_{ij}(X)$ and matter-field configurations

$\phi(X)$) at a point X on a three-surface S, is called *Superspace.* It is an infinite-dimensional space with the so-called *Wheeler-De Witt metric* G_{ij} that has hyperbolic signature at every point X on the three-surface S. This signature is independent of the four-dimensional spacetime metric $g_{\mu\nu}$ signature.

Since the real universe appears to be homogeneous and isotropic on very large scales, we restrict ourselves to Friedmann-Robertson-Walker metrics only. All but a finite number of degrees of freedom of the metric and matter-fields in Superspace are "frozen": We therefore approximate the problem of defining a wave functional for the universe to a problem in quantum mechanics. We now deal with a finite-dimensional *Mini-Superspace* whose intrinsic quantities exclude an explicit time-parameter.

1.3 Boundary Conditions

The quantum theory of boundary conditions essentially involves selecting one solution of an infinite set of solutions to the Wheeler- De Witt equation. Numerous proposals have been encountered since De Witt (1967) [22]. The most studied proposals of recent years are the *No Boundary* proposal of S.W.Hawking and J.B.Hartle [74, 77, 65] and the *Tunneling* boundary condition of A.Vilenkin and A. Linde [133, 134, 135, 136, 137, 138, 139, 57, 109, 124].

The Hartle-Hawking proposal regards the three-surface B as the *only* boundary of a compact four-manifold $\mathcal{M}$, on which the spacetime metric $g_{\mu\nu}$ induces a three-metric $\tilde{h}_{ij}$ and a matter-field $\tilde{\phi}$ on B. The path integral over all such $g_{\mu\nu}$ and ϕ, and all $\mathcal{M}$ in principle leads to the No Boundary wave function, depending on the choice of contour.

The Tunneling boundary condition of Vilenkin and Linde attempts to draw a parallel between quantum creation of the universe from nothing and tunneling in ordinary quantum mechanics. The "outgoing mode" formulation of this proposal due to A.Vilenkin [139] states that one selects the solution to the WDW equation that is *everywhere bounded* and consists of *only outgoing modes* at singular boundaries of (Mini) Superspace. This proposal has been more successful in defining a unique solution to the WDW equation.

The first difference between the two proposals is that the Hartle-Hawking wave function is real, consisting of a sum of "expanding" and "contracting" solutions, while the Vilenkin proposal corresponds to only one of these two. The wave functions also predict different amount of inflation, depending on the initial value of ϕ most favoured by each proposal [77, 115].

1.4 Probability Measure

Like the Klein-Gordon equation, the WDW equation has an associated conserved probability current that allows negative probabilities. Authors Caves [11, 12], Hartle [69] and Page [116] have suggested a measure that is the square modulus $|\Psi|^2 dV$ over a volume element dV of Mini-Superspace. Since this definition is analogous to the probability measure of Quantum Mechanics, a further elucidation on the role of a clock (i.e. time in ordinary Quantum Mechanics) is required.

Naively, one looks for strong peaks in the wave function, and hence makes predictions. For example, classical behaviour is predicted if the wave function is strongly peaked about one or more classical configurations, while interference between distinct configurations should be negligible.

1.5 Predicting inflation

We briefly reconsider the Hartle-Hawking No Boundary and Vilenkin Tunneling proposals to illustrate the problem of defining a good probability measure, and in the same breath highlight a major difference in the two proposals when applied to a massive scalar field potential:

(a) Both proposals have a wave function that is peaked about the same set of inflationary solutions to the classical Einstein field equations. However, their respective conditional probability measure differs. In particular, for the HH wave function to be bounded it is peaked about some minimum value of the scalar field ϕ_{min}, and since this is small, the major contribution to the probability density comes from the region close to ϕ_{min}. It would therefore appear that the HH wave function predicts insufficient inflation. (Hawking and Page (1986) [79] gets around this by saying that the contribution from regions away from ϕ_{min} outweighs the contribution from the peak at ϕ_{min}, thus predicting sufficient inflation. For values of the scalar field potential comparable to the Planck mass m_p, it may also be necessary to include higher order corrections to the Einstein- Hilbert action [139]).

(b) On the other hand, the Vilenkin wave function has a probability density that is small for small $\phi \approx \phi_{min}$. This means that the largest contribution to its probability density comes from regions away from ϕ_{min}. This straightforward prediction of sufficient inflation seems more appealing.

1.6 Wave packets

A coherent state in Mini-Superspace corresponds to a wave packet sharply peaked along a single classical trajectory. Besides a Hamilton-Jacobi equation, we also need *the principle of constructive interference* for canonical

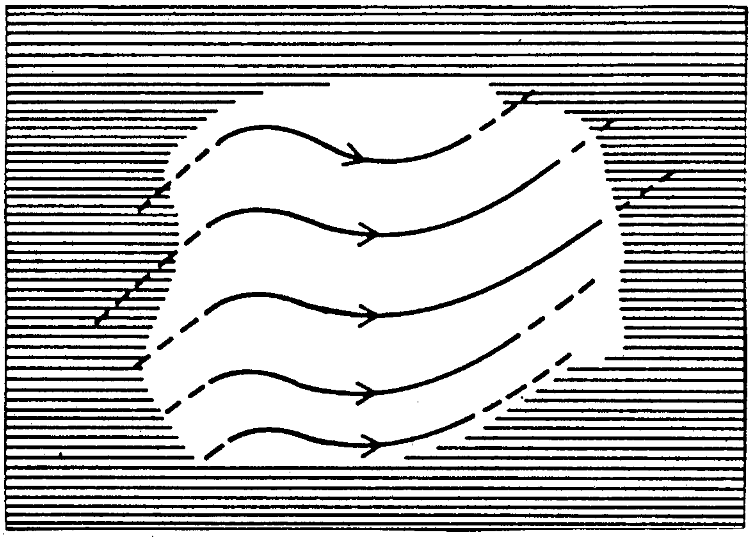

Figure 1.1: A schematic illustration of the behaviour of a typical wave function. In certain regions the wave function indicates that the notion of classical spacetime is an appropriate one, denoted by bold lines in the figure. Those regions of Superspace describable by quantum laws of physics are denoted by the shaded region in the figure.

variables to be correlated according to classical laws. Mini-Superspace does not have a natural time-label. However, one may define an affine parameter along the history of the classical path along which the wave function is peaked. *Essentially, classical spacetime is a concept appropriate to certain regions of configuration space* (fig. 1.1).

The absence of an external observer deprives one the usual quantum mechanical interpretation : for this one might have to resort to the "relative-state formalism" of Everett [67, 91]. To facilitate a good probability interpretation, Kazama and Nakayama [91] introduces a massless scalar field weakly coupled to matter fields and a large scale factor. It serves as a "desirable" clock, since the probability density is positive-definite.

1.7 Wormholes

Wormholes are topological fluctuations in a semi-classical or quantum theory of gravity. In the context of Hawking [72] and Wheeler [143], quantum fluctuations occur at the Planck length $10^{-33} cm$ where the spatial geometry has a foam-like structure, with "ripples", "bubbles" and "handles" appearing and disappearing. The feature of the wormhole was first introduced in three dimensions by Wheeler to resolve the problem of charge-singularity in the Maxwell field equations.

We know from a Geroch no-go theorem [33, 73] that a globally hyperbolic manifold cannot undergo topological fluctuations, since it is $R^1 \otimes S$, with S constant time three-surface; such fluctuations require singularities in Lorentzian spacetime.

Euclidean spacetime provides the perfect framework: A wormhole is a four- dimensional instanton, i.e. a solution to the classical Euclidean Einstein field equations with a finite action. One may picture it as a tube or small closed spatial geometry (known as a baby universe) that splits off and rejoins a unique large Lorentzian parent universe, or merely a link between two parent universes. In quantum gravity, it is a topological fluctuation in the ground state, and appears as a saddle point in the path integral in quantum gravity (although not always in the semi-classical limit).

More recently, Hawking and Page (1990) [85] have suggested wormholes to be solutions of the WDW equation for arbitrary matter content, or no matter content but just pure gravity. For such wave functions to be wormhole solutions, they need to satisfy the "Hawking-Page" boundary conditions:

(1) The wave functions are regular as the three-geometry collapses to zero,

(2) The wave functions are exponentially damped at large three- geometries.

The advantage of this definition is that wormholes might become the mechanism for black hole evaporation Hawking [71, 81]. It also supports the theses

(a) that wormholes are the reason why the cosmological constant is zero (Baum [3], Hawking [77], Coleman [13]),and

(b) the "big fix": Wormholes are considered to form a dilute gas (interactions between end-points of wormholes are negligible) linking otherwise disconnected large smooth universes. Each universe model is governed by dynamics with a set of coupling constants $\{a_i\}$. In the third quantized theory describing such dynamics, the probability distribution over different sets is sharply peaked at one fixed set of constants, thus randomly selecting our universe from an ensemble of possible universes.

1.8 Summary of Chapters' contents

We now take a cursory glance at what lies ahead. In Chapter Two we derive the Wheeler-De Witt equation in its most general form, using canonical quantization. The problem of finding the wave function of the Universe is then narrowed down to the arena of Mini-Superspace, in which most of the degrees of freedom have been "frozen" out. In particular, we write down the Wheeler-De Witt equation in two dimensions, and look at ways of recovering classical spacetime, using for instance the WKB approximation.

In order to make predictions in Quantum Cosmology, we need a good definition of the Probability Measure. We are able to arrive at the notion of

Conditional Probability only after considering an alternative to the Copenhagen Interpretation - the post-Everret Relative State Formalism of Quantum Mechanics. This we briefly outline in Chapter Three.

The Fourth Chapter deals with the problem of proposing Boundary Conditions to select a wave function for the Universe in two-dimensional Mini-Superspace. The Tunneling proposal of A.Vilenkin and the No Boundary proposal of J.B. Hartle and S.W. Hawking are discussed and compared in some detail.

Chapter Five describes the essential features of inflationary models, starting with the problem that inflation could mean that the Universe is infinitely old. Notwithstanding this, we outline the basic features of the power-law scalar field potential in Mini-Superspace and its prediction of an era with sufficient inflation. Following this, we investigate the existence of a unique measure for sufficient inflation, on the space of wave functions, to answer the question "How probable is Inflation ?" In this regard, we find that the original approach of Gibbons and Grishchuk [38] is applicable to a Mini-Superspace model containing an arbitrary power-law potential, i.e. slightly more general than the massive scalar field model used previously.

The Issue of Time has been debated since its inception by our ancestors, and in Chapter Six we relate to it in the context of quantum gravity. The so-called "Arrow of time" is discussed in some detail, and we use the Decoherence Functional as a criterion for the emergence of classical spacetime in a region of Superspace where the usual notion of the Hamiltonian acquires meaning. We postulate the need for a wave packet in such a region, and reflect on ways to introduce a judicious clock into the formalism. This we shall exploit to its full in our new, and original, treatment of bulk matter wormholes in the wave packet context in Chapter Nine.

Topological fluctuations in quantum gravity known as *wormholes* have been a subject of great scrutiny in recent years. We look at wormholes as they first made their appearance in Modern Literature - i.e. as Euclidean solutions to the Einstein field equations, known as *instantons.* We discuss the Hawking-Tolman and Giddings-Strominger wormholes for their generality : many authors have been able to find instantons either identical or very similar to these two. So in Chapter Seven we disclose the need for a more general class of wormholes, other than instantons, to explain for instance the evaporation of black holes and provide a mechanism for setting the cosmological constant to zero.

Chapter Eight explores the Hawking-Page proposal that wormholes are solutions of the Wheeler-De Witt equation in Superspace, satisfying asymptotic boundary conditions. We also illustrate how free massive scalar field wormhole-states are derived in a fairly straightforward fashion compared to the approximate results obtained by other authors. In addition, we formulate a new and approach, other than that of [85, 95, 96], to finding wormholes as solutions of the Wheeler-De Witt equation for a power-law potential in general.

With the advent of Wave Functions in Superspace, the Machian idea that only intrinsic quantities should appear in the formulation of a physical theory reaches near-perfection. We are able to construct wormhole states containing bulk matter satisfying the strong-energy condition $\gamma > 2/3$. There is no explicit time-parameter present in the theory. However, to obtain correlations between canonical variables, we construct a wave packet, and this in turn yields a material clock in the guise of a "bulk matter field ξ". Thus, we improve on previous literature by shedding new light on the possibility of having a larger class of quantum wormhole solutions. This is essentially what we achieve in Chapter Nine. We also postulate the existence of a relation between the Lorentzian perfect fluid index γ and its Euclidean counterpart γ_e. This we show explicitly in the Appendix.

In Chapter Ten we address some of the controversies that surround Wormhole Theory. The Coleman Mechanism for setting $\Lambda = 0$ is outlined, while the issue of the "big fix" of the coupling constants is summarized. Finally, we mention the main features of the Third Quantization of Gravity in the context of Parent and Baby Universes, and review the third quantized Uncertainty Principle. Unfortunately, in the elementary case of a massless scalar field, the third quantized Mini-Superspace theory makes a prediction that is almost certainly wrong.

Finally, Chapter Eleven outlines our conclusions and insights. We discuss the difficulties that surface in the process of extracting predictions from solutions to the Wheeler-De Witt equation. Open questions and obstacles that threaten the foundations of the Wormhole Theory are laid bare.

Chapter 2

The Wheeler-De Witt Equation

2.1 The Hamiltonian formulation

The general formalism of Quantum Cosmology starts with the Hamiltonian formulation of General Relativity. We look at a model with a homogeneous scalar field $\phi(X) = \phi$ that represents the matter fields and has Lagrangian

$$\mathcal{L} = \frac{m_p^2}{16\pi}(R - \frac{\Lambda}{2}) - \frac{1}{2}[g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi + V(\phi)]. \tag{2.1}$$

R is the Ricci scalar curvature, Λ the cosmological constant and the Planck length $= \sqrt{16\pi/m_p^2}$, where $m_p^2 = G^{-1}$, and G is Newton's constant. We have adopted units in which $\hbar = c = 1$. The metric $g_{\mu\nu}$ is that of a four-dimensional manifold $\mathcal{M}$ and has a standard form:

$$ds^2 = g_{\mu\nu}dX^\mu dX^\nu \tag{2.2}$$

Embedded in the four-manifold $\mathcal{M}$ is the three-surface S on which the three-metric is h_{ij}, (i,j = 1, 2, 3 ; μ, ν = 0,1,2,3). If we decompose the metric-element (2.2) we arrive at the Lorentzian (3 + 1)-form

$$ds^2 = -(N^2 - N_i N^i)dt^2 + 2N_i dX^i dt + h_{ij}dX^i dX^j$$

where N and N_i are the lapse and shift functions respectively:

They are arbitrary in that they describe the way in which the choice of coordinates on one three-surface is related to that on an adjacent three-surface. Once the "thin sandwich problem of Wheeler" has been solved, i.e. once the *horizontal stacking* for the shift N_i is resolved, we can address the problem of *vertical stacking* for the lapse N.

In the formulation of the Einstein-Hilbert action the matter term

$$I_{matter} = \int d^4X\sqrt{-g}\,\mathcal{L}(\phi)$$

is the integral over the ϕ-dependent part of the Lagrangian $\mathcal{L} = \mathcal{L}(grav) + \mathcal{L}(\phi)$ weighted by the determinant of the four-metric $g_{\mu\nu}$. The gravitational part of the action is just

$$I_{gravity} = \frac{m_p^2}{16\pi}\int_{\mathcal{M}} d^4X\sqrt{-g}\,(\,R - 2\Lambda\,) + \frac{m_p^2}{8\pi}\int_{\partial M} d^3X\sqrt{h}\,K\,.$$

The second part of the gravitational action is the integral over the trace K of the extrinsic curvature K_{ij} at the boundary ∂M of the four-manifold $\mathcal{M}$. It reads

$$K_{ij} = \frac{1}{2N}\left(\frac{-\partial h_{ij}}{\partial t} + 2D_{(i}N_{j)}\right)$$

with D_i the covariant derivative in the three-surface. The gravitational action in terms of the $(3+1)$ variables now becomes

$$I_{gravity} = \frac{m_p^2}{16\pi}\int_{\mathcal{M}} d^3X\,dt\,N\sqrt{h}\,[\,K_{ij}K^{ij} - K^2 + {}^{(3)}R - 2\Lambda\,]\,.$$

If we include the matter-part, the Hamiltonian form of the action takes the form

$$I_{total} = \int_{\mathcal{M}} d^3X\,dt\left[\,\dot{h}_{ij}\pi^{ij} + \dot{\phi}\pi_\phi - N\mathcal{H} - N^i\mathcal{H}_i\,\right] \qquad (2.3)$$

Here, π^{ij} and π_ϕ are the momenta conjugate to the three- metric h_{ij} and scalar field ϕ respectively. The Hamiltonian is a sum of constraints, with the lapse N and shift N^i playing the role of Lagrange multipliers. The momentum constraint is

$$\mathcal{H}_i = -\frac{m_p^2}{8\pi} D_j \pi_i^j + \mathcal{H}_i^{matter} = 0 , \tag{2.4}$$

where $\mathcal{H}_i^{matter}$ is the Hamiltonian for the matter-field contribution to momentum. There is the more important Hamiltonian constraint

$$\mathcal{H} = \frac{16\pi}{m_p^2} G_{ijkl} \pi^{ij} \pi^{kl} - \frac{m_p^2}{16\pi} \sqrt{h} (\, ^{(3)}R - 2\Lambda \,) + \mathcal{H}^{matter} = 0 \tag{2.5}$$

where G_{ijkl} is the De Witt metric on *Superspace*, the space of all three-metrics and matter field configurations $(h_{ij}(X), \phi(X))$ on a three-surface S. The signature of the De Witt metric is hyperbolic at every point X in the three-surface S, independent of the signature of spacetime. It is given by

$$G_{ijkl} = \frac{1}{2} \sqrt{h^{-1}} \, (\, h_{ik} h_{jl} + h_{il} h_{jk} - h_{ij} h_{kl} \,) \, . \tag{2.6}$$

Also, $\mathcal{H}^{matter}$ is the matter-field contribution to the Hamiltonian constraint (2.5), and is explicitly defined in the next section. From the Lagrangian in (2.3) we may express the momenta π_{ij} conjugate to h_{ij} as

$$\pi_{ij} = -\sqrt{h}(K_{ij} - h_{ij} K) \, .$$

In a similar fashion the energy of the matter-field can be expressed in terms of the momentum conjugate to the field π_ϕ and the field itself [65].

2.2 Canonical Quantization

The wave functional $\Psi(h_{ij}, \phi)$ on the infinite dimensional manifold called Superspace, describes the quantum state of our system of interacting three-metrics h_{ij} and matter-fields ϕ. The Dirac quantization procedure means that such a wave functional is annihilated by the operator versions of the classical constraints (2.4) and (2.5). We therefore introduce the conjugate momenta

$$\pi^{ij} \rightarrow -i\frac{\delta}{\delta h_{ij}} \tag{2.7}$$

$$\pi_\phi \rightarrow -i\frac{\delta}{\delta \phi}\,. \tag{2.8}$$

The momentum constraint is

$$\mathcal{H}_i\Psi = \left(\frac{m_p^2}{8\pi}iD_j\frac{\delta}{\delta h_{ij}} + \mathcal{H}_i^{matter}\right)\Psi = 0 \tag{2.9}$$

This implies that the wave functional is invariant under three- dimensional diffeomorphisms, i.e. configurations (h_{ij}, ϕ) that are related by coordinate transformations

$$X^i \rightarrow X^i - \xi^i$$

in the three-surface S (Halliwell [61]). To show this, we restrict attention to the case of no matter, and assume that the three-manifold is compact. Then we may write

$$\Psi[h_{ij} + D_{(i}\xi_{j)}] = \Psi[h_{ij}] + \int_{\mathcal{M}} d^3x D_{(i}\xi_{j)}\frac{\delta\Psi}{\delta h_{ij}}\,,$$

and integrating by parts in the last term, the boundary term vanishes since the three-manifold is assumed to be compact. Therefore the change in Ψ is

$$\delta\Psi = -\int_{\mathcal{M}} d^3x\xi_j D_i\left(\frac{\delta\Psi}{\delta h_{ij}}\right) = \frac{1}{2i}\int_{\mathcal{M}} d^3x\xi_i\mathcal{H}^i\Psi\,.$$

This shows that the wave functions satisfying 2.9 are unchanged.

The Hamiltonian constraint (2.5) becomes the so-called Wheeler-De Witt equation

$$\left(-\frac{16\pi}{m_p^2}G_{ijkl}\frac{\delta}{\delta h_{ij}}\frac{\delta}{\delta h_{kl}} + \frac{16\pi}{m_p^2}\gamma_{ij}\frac{\delta}{\delta h_{ij}} - \frac{m_p^2}{16\pi}\sqrt{h}[\,^{(3)}R - 2\Lambda\,] + \mathcal{H}^{matter}\right)\Psi = 0 \tag{2.10}$$

This equation describes the dynamical evolution of the wave functional in Superspace. It has infinitely many solutions and requires a boundary condition to pick out just one solution. Explicitly, the matter-field contribution to the Wheeler-De Witt equation reads

$$\mathcal{H}^{matter} = \sqrt{h}\left(\frac{1}{2h}\frac{\delta^2}{\delta\phi^2} + V(\phi)\right)$$

The coefficients γ_{ij} in the Wheeler-De Witt equation depend on the choice of operator-ordering in the quantization procedure, and becomes important only at or above the Planck curvature

$$R \geq \frac{m_p^2}{16\pi} .$$

There is a good reason for this. We have to bear in mind that the curvature scalar R is a function of the momenta π_{ij} conjugate to the three-metric h_{ij}, and therefore depends crucially on the operator- ordering in the Dirac quantization procedure. It also contributes to the action I_{total} of (2.3), and hence the Hamiltonian constraint (2.5).

Now for three-geometries that are much larger than the Planck size (i.e. $h^{1/3} \gg 16\pi/m_p^2$), the wave functional $\Psi(h_{ij}, \phi)$ is predominantly described by contributions from the intrinsic curvature scalar ${}^{(3)}R$ (and possibly Λ) and any matter-fields present. In that case, anyhow, the curvature scalar R is small ($\ll 16\pi/m_p^2$), so that predictions made from Ψ are largely insensitive to the choice of operator-ordering. But for three-geometries of the order of the Planck size or smaller, R may reach scales of $m_p^2/16\pi$ or bigger, and make a large contribution to the Hamiltonian. Consequently, solutions of the Wheeler-De Witt equation (2.10) will then depend crucially on the coefficients γ_{ij}.

2.3 The problem of operator-ordering

Predictions in quantum cosmology depend on how we resolve the operator-ordering problem. That would be equivalent to defining a differential operator on Superspace. Hawking and Page (1986) [79] proposed that the correct choice should be the Laplacian in a natural metric defined on Superspace. This Laplacian is then covariant in Superspace, and reads

$$-\frac{16\pi}{m_p^2}\sqrt{(-DetG)^{-1}}\frac{\delta}{\delta h_{ij}}G^{ijkl}\sqrt{-DetG}\frac{\delta}{\delta h_{kl}}+\frac{1}{2}\sqrt{h^{-1}}\frac{\delta^2}{\delta\phi^2}\sim\nabla^2_{Sup}\,.$$

This means that γ_{ij} is fixed for all values i,j = 1, 2, 3. But the omission of first derivative terms means that the natural metric on Superspace does not depend linearly on the lapse N; i.e. the Hamiltonian $\mathcal{H}$ is not a linear function of N and so the lapse does not serve as a Lagrange multiplier. They [79] do suggest that the nonlinear dependence on N would cancel out in theories like Supergravity which contain equal numbers of fermionic and bosonic degrees of freedom.

The advantage of this choice of operator-ordering is that the Wheeler-De Witt equation is invariant under arbitrary coordinate transformations on Superspace.

2.4 Mini-Superspace

The full formalism of quantum cosmology on the infinite-dimensional Superspace is too difficult to deal with in practice. We therefore only deal with "toy models" of the complete theory, in which all but a finite number of degrees of freedom of the metric h_{ij} and matter-fields ϕ are suspended: Such models are finite dimensional *Mini- Superspace models.* It is until now not yet clear if such models are indeed part of a systematic approximation to the full theory. In fact, setting most of the field modes and their momenta to zero violates the uncertainty principle (J.Halliwell [61]).

For instance, we construct a simple Mini-Superspace model with the metric and matter-field homogeneous and isotropic. That is, if we supposedly solve the problem of horizontal stacking for the shift, and gauge it to zero:

$$N^i = 0 \, ,$$

we proceed by taking a homogeneous lapse

$$N = N(t) \, .$$

Furthermore, we restrict the three-metric h_{ij} to be homogeneous, thus making it dependent on a finite number of functions w^r, r = 1, 2, 3, ...n - 1, all functions of the time-parameter t. The four-dimensional spacetime metric

$$ds^2 = -N^2(t)dt^2 + h_{ij}(t)dX^i dX^j \tag{2.11}$$

is now homogeneous and isotropic, and results in the dimensional reduction of the full natural metric in Superspace to $M_{rs}(w)$, where $r, s = 0, 1, 2...., n$ in Mini-Superspace. This reduced metric is now n-dimensional and has indefinite signature (-,+,+,+...). with the n^{th} component of w^r representing the matter-field ϕ. The Mini-Superspace metric element reads

$$dS^2 = M_{rs}(w) dw^r dw^s \, .$$

The Lagrangian for this model may now be abbreviated as

$$\mathcal{L} = \frac{1}{2N(t)} M_{rs}(w) \, \dot{w}^r \dot{w}^s - N(t) W(w) \tag{2.12}$$

with the Mini-Superspace potential $W(w)$ containing the curvature scalar ${}^{(3)}R$ intrinsic to the three-geometry h_{ij}, the cosmological constant Λ, and the matter-field potential $V(\phi)$:

$$W(w) = -\left(\frac{m_p^2}{16\pi}\right)^2 \sqrt{h}\left(\frac{1}{2}\, {}^{(3)}R - \Lambda\right) + \left(\frac{m_p^2}{16\pi}\right) \sqrt{h} V(\phi) \, ,$$

with the Planck mass $m_p = \sqrt{G^{-1}}$. For convenience we set $m_p^2 = 16\pi$, to make further calculations more readible. The canonical momenta are then defined as

$$\pi_r = \frac{\partial \mathcal{L}}{\partial \dot{w}^r} = M_{rs}\frac{\dot{w}^s}{N},$$

so that the canonical Hamiltonian reads

$$\pi_r \dot{w}^r - \mathcal{L} = N\,\mathcal{H}\,.$$

The Hamiltonian form of the action is

$$I = \int dt(\ \pi_r \dot{w}^r - N\,\mathcal{H}\ )$$

and indicates that the lapse N is a Lagrange multiplier, enforcing the Hamiltonian constraint

$$\mathcal{H}(\ w^r, \pi_r\ ) = \frac{1}{2}M^{rs}\pi_r\pi_s + W(w) = 0\,. \qquad (2.13)$$

The canonical quantization

$$\pi_r \longrightarrow -i\frac{\partial}{\partial w^r}$$

leads to the non-trivial operator ordering issue discussed in Section 2.3, since the metric M^{rs} depends on w. The most general Wheeler-De Witt equation in Mini-Superspace now reads

$$\left(-\frac{1}{2}M^{rs}\partial_r\partial_s + \gamma^r\partial_r + \xi\Re + W(w)\right)\ \Psi(w^r) = 0 \qquad (2.14)$$

and is covariant under w^r coordinate transformations. Here $\gamma^r\partial_r$ and ΞRe represent the vector and scalar part (in Superspace) of the operator-ordering ambiguities. ξ is an arbitrary constant and $\Re$ is the curvature of the metric M_{rs}. If we now accept the Hawking and Page [79] argument for choosing the Laplacian in the metric M_{rs} as a way of resolving the operator-ordering ambiguity, we impose

$$\gamma_r = 0\ ,$$

with r = 1, ..., n-1. This choice of operator ordering has been made previously, in particular by De Witt [22], and is true if one replaces $M^{rs}\partial_r\partial_s$ in equation 2.14 with ∇^2_{Sup}. The coefficient ξ may be taken to be zero as in [79], or it may be taken to be the *conformal coupling*

$$\xi = -\frac{(n-2)}{8(n-1)}$$

for $n \geq 2$, as in [55, 111]. This occurs if the metric part of the Hamiltonian constraint (2.13) is *conformally covariant*: the theory is invariant under rescaling of the lapse function $N \rightarrow \tilde{N} = \Omega^2 N$, the potential $W \rightarrow \Omega^{-2}W$ and the metric $M_{rs} \rightarrow \tilde{M}_{rs} = \Omega^2 M_{rs}$, where $\Omega = \Omega(w)$ is an arbitrary conformal factor.

2.5 Two-dimensional Mini-Superspace

We consider a simple model for which the four-geometries $g_{\mu\nu}$ are restricted to be spatially homogeneous and isotropic for a particular choice of lapse N, and hence characterized by a single scale factor a(t) (after a global rescaling of the metric by the factor $\sigma = \frac{2}{3\pi m_p^2}$):

$$ds^2 = -N^2(t)dt^2 + a^2(t)d\Omega_3^2 \qquad (2.15)$$

with $d\Omega_3^2$ the metric on a unit three-sphere for closed curvature (k = +1), a three-torus or flat space (k = 0), or a hyperbolic (open) space (k = -1). The spatial curvature scalar simplifies to

$$^{(3)}R = k(h^2 - h^j_i h^i_j) = \frac{6k}{a^2}.$$

The model now has a Friedmann-Robertson-Walker geometry. For the matter degrees of freedom, we select a spatially homogeneous scalar field $\phi = \phi(t)$. The Einstein-scalar action for this system is

$$I = \frac{1}{2}\int dtNa^3 \left[-\frac{\dot{a}^2}{N^2a^2} + \frac{\dot{\phi}^2}{N^2} + ka^{-2} - \Lambda - V(\phi) \right] \qquad (2.16)$$

(including the contribution from the boundary to remove $\ddot{a}$-terms). The field equations are derived in the usual fashion:

$$\ddot{\phi} = -3\dot{\phi}H - \frac{1}{2}\frac{\partial V}{\partial \phi} \tag{2.17}$$

$$\frac{\ddot{a}}{a} = -2\dot{\phi}^2 + \Lambda + V(\phi) \tag{2.18}$$

$$H^2 + ka^{-2} = \dot{\phi}^2 + \Lambda + V(\phi) \tag{2.19}$$

where $H = \dot{a}/a$ is the Hubble parameter, in the gauge N = 1. (In the gauge $N^2 = -1$ the field equations are Euclidean.) In the canonical quantization scheme described in the previous section, the conjugate momentum to the scale factor is π_a defined as

$$\pi_a^2 = -\frac{1}{a^p}\frac{\partial}{\partial a}\left(a^p \frac{\partial}{\partial a}\right)$$

where the operator-ordering ambiguity is reflected in the arbitrary constant p, and becomes important only for very small values of the scale factor when the spacetime curvature R exceeds the Planck curvature $\frac{m_P^2}{16\pi}$. The conjugate momentum to the scalar field is given by the operator π_ϕ and reads

$$\pi_\phi^2 = -\frac{\partial^2}{\partial \phi^2} \, .$$

The Wheeler-De Witt equation (2.14) takes the form

$$\left(\frac{\partial^2}{\partial a^2} + \frac{p}{a}\frac{\partial}{\partial a} - \frac{1}{a^2}\frac{\partial^2}{\partial \phi^2} + W(a,\phi)\right)\Psi(a,\phi) = 0 \tag{2.20}$$

where the superpotential

$$W(a,\phi) = -ka^2 + \Lambda a^4 + V(\phi)a^4 \, .$$

Notice that this equation is independent of the lapse N. The Mini-Superspace of this model is a two-dimensional manifold $0 < a < \infty$, $-\infty < \phi < \infty$ with metric $M_{rs}(w)$, appearing in the the action

$$I = \frac{1}{2}\int dt \left(\frac{1}{N}M_{rs}(w)\dot{w}^r\dot{w}^s - NW(w)\right) \, .$$

(including the contribution from the boundary to remove $\ddot{a}$-terms). The field equations are derived in the usual fashion:

$$\ddot{\phi} = -3\dot{\phi}H - \frac{1}{2}\frac{\partial V}{\partial \phi} \tag{2.17}$$

$$\frac{\ddot{a}}{a} = -2\dot{\phi}^2 + \Lambda + V(\phi) \tag{2.18}$$

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Notice that this equation is independent of the lapse N. The Mini-Superspace of this model is a two-dimensional manifold $0 < a < \infty$, $-\infty < \phi < \infty$ with metric $M_{rs}(w)$, appearing in the the action

$$I = \frac{1}{2}\int dt \left(\frac{1}{N} M_{rs}(w)\dot{w}^r \dot{w}^s - NW(w) \right) \, .$$

If we compare this with the Einstein-scalar action 2.16, we are able to write the Mini-Superspace metric element explicitly,

$$dS^2 = N^{-1}(-ada^2 + a^3 d\phi^2) , \tag{2.21}$$

where the lapse N is constant on every surface of homogeneity. It has a nonsingular boundary at $a = 0$ with ϕ finite. Singular boundaries occur when at least one of the two variables is infinite. The solutions to the Wheeler-De Witt equation are the wave functions $\Psi(a, \phi)$, functions of the two variables (a, ϕ), and independent of the time t.

How many Einstein field equations are there in Mini-Superspace? Well, if the reduced metric M_{rs} is n-dimensional, we anticipate $\frac{1}{2}n(n+1)$ field equations. The momentum constraints (2.9) constitute n of these equations, and are trivially satisfied (we were able to prove this in the case of compact three-manifolds with no matter content). So in principle we are left with $\frac{1}{2}n(n-1)$ remaining equations to solve. Symmetry considerations (such as the Copernican principle) may reduce this number even further.

For example, we consider only Friedman-Robertson-Walker geometries, so that there is only one gravitational variable, namely the scale factor ($w^0 = a$). We also represent any form of matter by a single, spatially homogeneous scalar field ($w^1 = \phi$). Therefore the whole of Mini-Superspace is further reduced to a two-dimensional system, with its metric element given by equation 2.21. So although there are maximally three ($\frac{1}{2} \cdot 2(2+1)$) field equations, two of these constitute momentum constraints (2.9). They are trivially satisfied, and simply imply that the wave function Ψ is independent of the choice of coordinates on the three-surface S [139]. We are therefore required to solve only one equation: the so-called Wheeler-De Witt equation (2.20).

Attempts to solve and interpret equation 2.20 for various scalar potentials $V(\phi)$ are encountered in much of the recent literature on inflation and related issues. More recently, Hawking and Page [85] have proposed that quantum wormholes are solutions to 2.20 provided they satisfy the appropriate boundary conditions.

2.6 Classical Spacetime

Before we venture on a discussion of the various issues regarding the interpretation of the wave functional, we briefly explain what prediction of classical spacetime in the context of quantum cosmology constitutes:

It was mentioned earlier that a suitable wave functional should predict that the canonical variables (Section 2.4) are strongly correlated according to classical laws. Any single or superposition of such wave functional(s) should be strongly peaked about one or more classical phase- space configurations.

Secondly, there should be negligible interference between distinct configuration-paths. In principle one should be able to construct a coherent state, so that on following its evolution through Superspace, we would find that it follows one particular trajectory. In the Mini- Superspace formalism of Section 2.4, the wave functional

$$\Psi(w^r) = C(w^r)e^{im_p^2 S_0(w^r)}\tilde{\Psi}(w^r) ,$$

is such a solution to the Wheeler-De Witt equation 2.10. $C(w^r)$ is a slowly varying functional in Mini-Superspace coordinates w^r. We expand w^r around a classical trajectory as $w^r = w^r_{cl} + \delta w^r$, so that $\tilde{\Psi}(w^r)$ is a functional of the fluctuations δw^r. It also satisfies the Schroedinger equation along the classical trajectories w^r_{cl} in Mini- Superspace, about which the wave function $\Psi(w^r)$ is peaked. Here $S_0(w^r)$ is a solution to the Hamilton-Jacobi equation

$$\frac{1}{2}(\ \nabla S_0\ )^2 + W(w^r) = 0 \tag{2.22}$$

for S_0 real. For instance, if we introduce the tangent vector to such classical solutions (see D.N.Page [117])

$$\frac{\partial}{\partial t} = \nabla S_0 \cdot \nabla\ ,$$

then it can then be shown [54] that $\tilde{\Psi}$ obeys the functional Schroedinger equation

$$i\frac{\partial \tilde{\Psi}}{\partial t} = \mathcal{H}_2 \tilde{\Psi}\ .$$

Generally, Schwinger-Tomonaga Hamiltonian $\mathcal{H}_2$ acts as a perturbation Hamiltonian to some fixed background Mini-Superspace. In the case of a gravitational background consisting of a Friedmann-Robertson- Walker metric 2.15, with purely inhomogeneous scalar firld perturbations for a potential $V(\phi) = m^2\phi^2$, and after expansion of $\delta\phi$ in the three-sphere harmonics Q_{nlm}, the Hamiltonian reads

$$\mathcal{H}_2 = \sum_{nlm} \frac{1}{2} a^{-3} \left[-\frac{\partial^2}{\partial f^2_{nlm}} + (\ m^2 a^6 + (n^2 - 1)a^4\ ) f^2_{nlm} \right]\ ,$$

after expansion in the three-sphere harmonics Q_{nlm}. So in the semi-classical limit, Quantum Cosmology reduces to quantum field theory on a fixed curved spacetime background. At least in this sense, we may speak of a semi-classical domain emerging from (Mini-)Superspace. The important advantage of such a region is that for any quantum theory of gravity to make predictions, their observation would be through correlation with the semi-classical domain [64]. Its meaning appears to be similar to the *quasi-classical domain* of Gell-Mann and Hartle [31]. In fact, in [70] it is shown on more general grounds that if $|C|^2$ (the density in Superspace) is conserved along the classical trajectories, then $\tilde{\Psi}$ satisfies a Schroedinger equation in the field representation for the *quantum* matter field δw_M in the classical background. It is also pointed out in [70] that this classical correlation should be implemented by some sort of coarse graining; as we point out in Section 6.3, one would probably need a wave packet construction.

Since $S_0(w^r)$ is a solution to the Hamilton-Jacobi equation, we can immediately write down a first integral to the classical field equations:

$$\pi_r = \frac{\partial S_0}{\partial w^r} , \tag{2.23}$$

and define a set of solutions to the field equations. The wave functional $\Psi(w^r)$ is an approximate solution to the Wheeler-De Witt equation and is therefore peaked about such a set of solutions to the field equations. The slowly varying function $C(w^r)$ in fact corresponds to the usual WKB prefactor.

2.7 The WKB approximation

Since the Wheeler-De Witt equation is a Klein-Gordon type equation, we instinctively associate with it the conserved current

$$\mathcal{J} = \frac{i}{2}(\Psi^* \nabla \Psi - \Psi \nabla \Psi^*) \tag{2.24}$$

-*Conserved* by virtue of the fact that

$$\nabla \cdot \mathcal{J} = 0 .$$

However, since (Mini)-Superspace has an *indefinite* metric signature, negative probabilities could occur if we define the probability measure in this fashion. This fact, first pointed out by De Witt [22], has prompted alternate approaches to arriving at the correct measure. For instance, the measure on sets of inflationary solutions was studied in [79, 82]. A measure on the set of solutions to the Wheeler-De Witt equation was also introduced by Gibbons and Grischuk [38]. An earlier attempt to make classical predictions in quantum cosmology was made by Gibbons *et al* [39]. We reflect more on these proposals in Chapter 3, entitled Probability Measure.

Since the Mini-Superspace wave functional may be expanded to the first order in the Planck mass m_p^2

$$\Psi(w) = Ce^{-iS} + \bar{C}e^{iS} + O(m_p^{-2}) \tag{2.25}$$

for complex prefactors C and $\bar{C}$, we insert these functionals into the Wheeler-De Witt equation (2.14), after recovering Planck units :

$$\left[-\frac{1}{2m_p^2} \nabla_{Sup} + m_p^2 W(w) \right] \Psi(w) = 0 \ ,$$

By equating powers of m_p^2 and splitting S into real and imaginary parts, $S = S_0(w) - iI$, and provided S_0 is a rapidly-varying function of w^r compared to I, then S_0 is a solution to the Lorentzian Hamilton-Jacobi equation 2.22 while simultaneously satisfying

$$\nabla I \cdot \nabla S_0 = 0.$$

It is clear from the Wheeler-De Witt equation that the wave functional is oscillatory in the region $W(w^r) \gg 0$; this loosely corresponds to regions of Mini-Superspace for which the four-dimensional spacetime is classical. In fact, this is precisely the type of wave functional the WKB approximation (2.25) yields. (In general, the result will depend also on the possibility of separating the Wheeler-De Witt equation, which will in turn depend on the existence of eventual Killing-symmetries in Superspace.) Furthemore, equation 2.23 defines the first integral to the classical field equations, and for a given (n - 1)- surface Σ at the beginning of classical evolution, effectively

Chapter 3

Probability Measure

3.1 Predictions in Quantum Cosmology

An issue that is very much under scrutiny of late is a more satisfactory and less heuristic scheme for the extraction of predictions from the wave functional in accordance with the Copenhagen interpretation, as outlined in Chapter 2. The best currently available approach (according to J.J. Halliwell [64]) employs not the wave functional, but the so-called *decoherence functional* as its central tool [62, 64, 31]. It has a number of features that may be perfectly suited to quantum cosmology :

It applies to closed systems; the Copenhagen interpretation applies to systems that interact with an external observer.

It assumes no *a priori* separation of quantum and classical domains as in the Copenhagen interpretation.

It does not rely on notions of measurement or observations by an external agency (fig. 3.1).

It focuses on histories rather than events at a single moment, a possible remedy to the problem of time in Quantum Gravity. This also translates the "Many Worlds" interpretation of Hugh Everett III (1957) [26] (the idea that the Universe splits up into many copies of itself whenever a measurement is

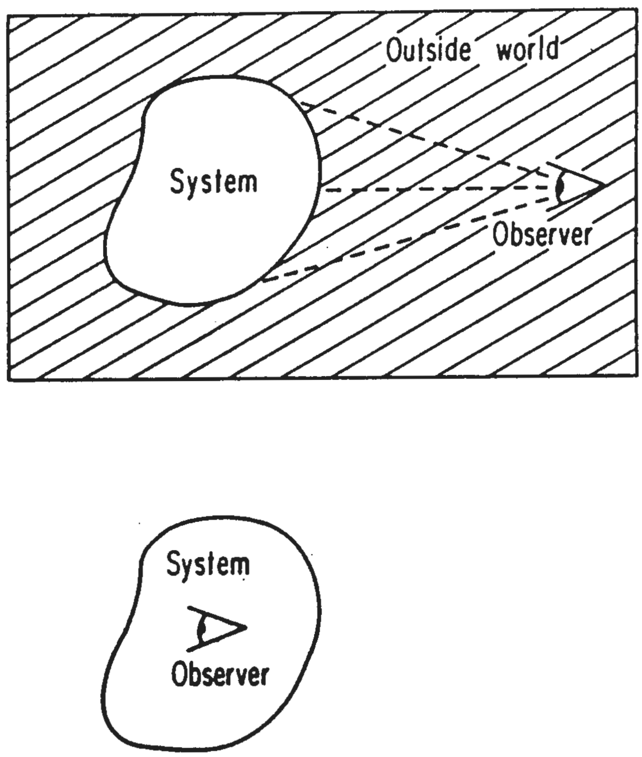

Figure 3.1: The difference between laboratory physics and cosmology: An external observer studying the system may control the external conditions, and use them as boundary conditions when determining what is going on inside the system. In cosmology, the observer is inside the system, and there is no outside world onto which the specification of boundary conditions can be passed.

performed) into a statement about the wave function of the entire system. As formulated by Hartle (1986) [67]:

"If the wave function for the closed system is strongly peaked about a particular region of configuration space, then we predict the correlations associated with that region; if it is very small, we predict the lack of the corresponding correlation; if it is neither strongly peaked, nor very small, we make no prediction."

Most attempts to interpret the wave functional have adopted this basic idea. One may henceforth think of the wave functional (and hence in this sense, Mini-Superspace models of quantum cosmology) as an approximation to the decoherent histories approach [64].

3.2 Conditional Probabilities

Such an interpretational scheme suggests that it is necessary to determine those quantities for which the theory gives probabilities close to one or zero. Hartle [67, 54] argues that we may arrive at the usual statistical interpretation of ordinary quantum mechanics through the *Quantum Mechanics of Individual Systems* (QMIS) :

Consider a closed, individual system Ψ, consisting of a large number of identical subsystems

$$\Psi(w_1, ...w_N) = \psi_1(w_1)\psi_2(w_2)...\psi_N(w_N) \ .$$

It is claimed [54] that in general we should not deal with probabilities for Ψ, but only for subsystems ψ_n. Only if Ψ is an exact eigenstate of some observable Q (with eigenvalue q) is there certainty of observing the vale q; if Ψ is an approximate eigenstate, then one should look for peaks.

Now since Ψ is the wave functional of an individual system, the probability of it being an eigenstate of some observable Q is either one or zero. But its identical subsystems $\{\psi(w_n)\}$ may be eigenstates of some relative frequency operator f_a^N, an operator defining the relative probability for the N^{th} to be peaked at a value $w = a$. Then it can be shown [47] that as $N \to \infty$, the subsystem probabilities are given by the square modulus of its wave functional, i.e.

$$f_a^\infty \Psi = |\psi(a)|^2 \Psi \ .$$

In the limit of large N, Ψ becomes an eigenstate of f_a^∞ with eigenvalue $|\psi(a)|^2$. In this way the Everett formulation of quantum mechanics is designed to deal with correlations internal to an isolated, individual system. In particular, it is designed to describe correlations in an isolated system consisting of an observer and an observed subsystem. Halliwell [54] deals with correlations in the wave functionals of quantum mechanics and quantum cosmology for such closed systems.

Predictions are extracted from the wave functionals using the interpretation of quantum cosmology proposed by Geroch [34], Hartle [66] and Wada [132]. One regards a strong peak as a definite prediction. A useful tool for identifying correlations between coordinates and momenta is the Wigner distribution function, important in the discussion of classical behaviour in quantum mechanical systems. It serves a good purpose in the study of scalar field fluctuations in inflationary universe models.

Furthermore, a weak form of the *Anthropic Principle* may be employed to make predictions:

We the observers look out into a Universe with conditions suitable for our own existence; hence we should restrict attention to those plausible histories of the Universe which exist long enough that make evolution of life possible. That is, we restrict our attention to a certain subset of the possible histories of the Universe, and make predictions within that subset. In this way we study only *Conditional Probabilities* (see also A.Vilenkin (1988) [88] for its application).

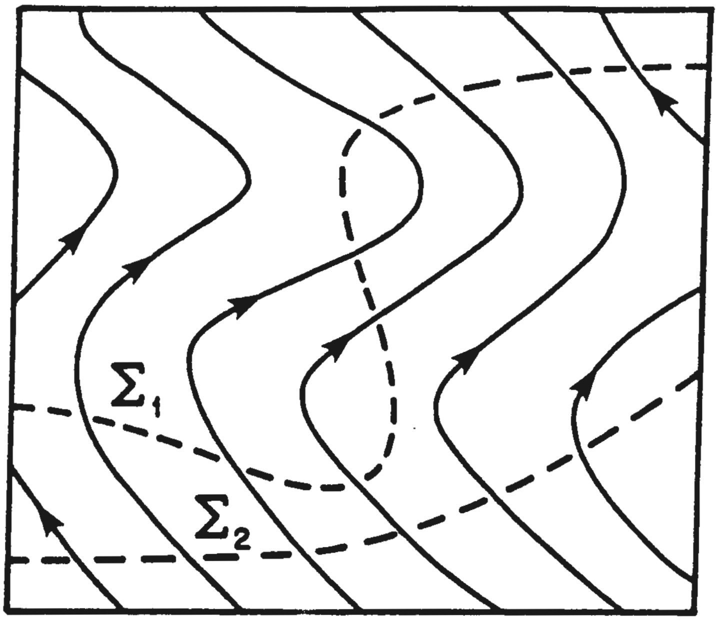

Figure 3.2: The integral curves of the current $\mathcal{J}$ (bold lines) and some possible choices for the hypersurfaces Σ_β (dashed lines). Σ_1 is a bad choice because the flow of $\mathcal{J}$ intersects Σ_1 more than once. Σ_2 is a good choice because the flow intersects it once and only once.

3.3 Conserved Measure

The regions in which the wave functional is rapidly oscillating in w^r we regard as the semiclassical domain. It was also stated in the previous chapter that certain contributions from a rapidly oscillating wave functional are peaked about classical configurations. Therefore we deduced (equation 2.23) a strong correlation between coordinates w^r and momenta π_r, which is a first integral that may be solved to yield an n-parameter set of classical equations. Given some (n - 1)- dimensional hypersurface Σ in Mini-Superspace as the beginning of classical evolution, we may solve the classical equations derived from equation 2.23 to arrive at a pencil B of the congruence of paths with tangent vector $\nabla_{\alpha}^{Sup} S_0$. The conserved current (2.26) allows the construction of a non-negative probability measure:

Suppose there is a family of hypersurfaces $\{\Sigma_\beta\}$, parameterized by β, that cut across the flow of $\mathcal{J}$. Then the conservation of current implies that for each β, a probability measure on the pencil B of the congruence of paths (i.e. through the intersection of the hypersurface $B \cap \Sigma_\beta$,for some β) is just the flux $\mathcal{J}$ across the surface:

$$dP = \mathcal{J} \cdot d\Sigma \ . \tag{3.1}$$

This measure is conserved along the pencil B of flow, due to conservation of the current $\mathcal{J}$. But (3.1) is not always positive; it vanishes for real Ψ, and becomes negative where the pencil B of flow intersects the same hypersurface Σ_β more than once (see fig. 3.2) due to the possibility of expanding and contracting universes. However, by suitable choice of the hypersurfaces $\{\Sigma_\beta\}$ in the semi-classical regime, one may construct a sensible Probability measure.

We have seen that the Wheeler-De Witt equation is independent of time. However, the parameter β labelling the family of hypersurfaces $\{\Sigma_\beta\}$ may be chosen to be the same as the affine parameter along the integral of curves $\nabla_{\alpha}^{Sup} S_0$ - i.e. the time t. Thus, four-dimensional spacetime may emerge over such regions of Mini- Superspace over which an appropriate family of hypersurfaces $\{\Sigma_t\}$ is defined.

The probability measure (3.1) on possible histories of the universe is commonly not normalizable over the entire hypersurface Σ. Given a pencil B of trajectories of the current flow through hypersurface Σ_2, one may calculate the *Conditional Probability* $P(\ 1 \mid 2\ )$ for that same pencil B to pass through another hypersurface Σ_1 :

$$P(\ 1 \mid 2\ ) = \frac{\int_{B\cap\Sigma_1} \mathcal{J} \cdot d\Sigma}{\int_{B\cap\Sigma_2} \mathcal{J} \cdot d\Sigma} \tag{3.2}$$

where Σ_1 is chosen in such a way that its intersection with B is a subset of universes which possess features that resemble our universe.

Instead, Hawking and Page [79, 82] uses a more traditional probability measure

$$dP = |\Psi(w^r)|^2\ dV \tag{3.3}$$

over a volume element dV of Mini-Superspace. This is indeed positive- definite, and some authors argue that it reduces to (3.1) in the limit in which the volume element dV is taken to be a hypersurface of codimension one slightly thickened (i.e. copies of the same hypersurface densely stacked) along the direction of the flow of J. However, this measure fall short in other respects (Kuchař (1992) [99]).

Chapter 4

Boundary Conditions

The standard *hot big bang* model leaves many features of the Universe unexplained. For instance, the observational fact that the Universe is spatially very flat at present means that it must have started out flat to within one part in 10^{60}. This is known as the "flatness problem".

The "horizon problem" arises out of the extreme uniformity of the Universe at very large scales, so that it consists of vast regions that could never have been in causal contact throughout their entire classical history.

In order for galaxies to form, fluctuations in the matter density need to have occurred in the very early Universe. How did these fluctuations originate? To resolve some of these problems, Alan Guth [52] proposed the so-called Inflationary Universe Scenario, in which the Universe experience a brief ($\sim\ 10^{-30}$seconds) period of inflation from an initial size of $\sim\ 10^{-28}$ centimetres to $\sim\ 1 metre$.

However, this Scenario cannot address the question of *initial conditions* necessary for the Einstein field equations of General Relativity to predict the correct classical behaviour of the Universe from an initial state of very high curvature and density. The second order quantized version of GR known as quantum cosmology, addresses this question in the context of *boundary conditions* in Superspace. Halliwell [63] is more conservative, and points out that quantum cosmology should be seen mainly as an "effective theory" until a more detailed and satisfactory theory of Quantum Gravity emerges.

Modulo contour ambiguities in the Euclidean Path Integral (see later), an appropriate boundary condition in Superspace selects one wave function of the Universe out of an infinite set of solutions to the Wheeler-De Witt equation. Initially, De Witt [22] suggested that mathematical consistency alone should lead to a unique wave function. Numerous proposals motivated by considerations of simplicity, naturalness, etc. have since been considered. We concentrate on recent "Tunneling" boundary condition of Vilenkin [133]-[139] and Linde [103, 104, 105] and the "No Boundary" proposal of Hartle and Hawking (Hawking [74, 77] and Hartle and Hawking [65]).

4.1 The Tunneling wave function

One particular proposal to determine the quantum state of the Universe is based on the picture of spontaneous nucleation into a de Sitter spacetime from nothing, after which it evolves along standard inflationary lines. The nucleation process is a nonsingular event, often referred to as "quantum tunneling from nothing". That, however, does not exclude the possibility of singular events such as black holes or a "big crunch" from occurring after nucleation.

In the semi-classical framework, evolution under the potential barrier corresponds to evolution in imaginary time, so that the tunneling process is an instanton. This regular Euclidean solution may be matched to a Lorentzian solution at the nucleation point.

The so-called Tunneling Boundary condition for the wave function Ψ as formulated by Vilenkin (1988) [139] is that
"At singular boundaries of Superspace, the wave function includes only outgoing modes (carrying flux out of Superspace)."

The definition of ingoing and outgoing modes is similar to that of positive- and negative-frequency modes, with the direction toward the boundary playing the role of "time" direction. We briefly summarize what is meant by a boundary in Superspace [110]. It consists of singular configurations which have points or regions with infinite three-curvature ${}^{(3)}R$, or where the scalar field is infinite, or its gradient $(\partial_i\phi)^2$ diverges, including configurations of infinite three- volume.

It is important to note that for a three-metric $h_{ij} = \Omega^2\tilde{h}_{ij}$, where $\tilde{h}_{ij}$ has a unit determinant, then the configurations with $\Omega \to 0$ but $\tilde{h}_{ij}$ and ϕ

nonsingular do not necessarily correspond to four-dimensional singularities. It is assumed that we can divide the boundary of Superspace into two parts:

1) The nonsingular boundary of Superspace, that includes three-geometries h_{ij} which can be attributed to the slicing of only *regular* four-geometries $g_{\mu\nu}$.

2) The singular boundary of Superspace, which includes the rest of the boundary.

We express the semi-classical wave function as

$$\Psi = \sum_n C_n e^{iS_n} \tag{4.1}$$

which is necessarily complex, and where the phase S_n satisfies the Hamilton-Jacobi equation in Superspace (see Chapter 2, Section 2.6)

$$\frac{1}{2}(\nabla S_n)^2 + W = 0 \ ,$$

and the current for the n^{th} term of (4.2)

$$\mathcal{J}_n = -| C_n |^2 \nabla S_n . \tag{4.2}$$

The tunneling boundary condition essentially means that any congruence of classical paths defined by S_n are allowed to end at the singular boundary of Superspace, but none are allowed to begin there. That is, the vectors $-\nabla S_n$ should point out of Superspace at the boundaries. In addition, a supplement to the boundary condition is that

$$| \Psi | < \infty \ . \tag{4.3}$$

For a Mini-Superspace model with a homogeneous and isotropic scalar field ϕ, and FRW metric (see eqn 2.15, Chapter 2) for a closed universe, the

Wheeler-De Witt equation 2.20 without a cosmological constant ($\Lambda = 0$) has a superpotential

$$W(a, \phi) = -a^4[\,1 - a^2 V(\phi)\,] \tag{4.4}$$

where the potential is assumed to be a slowly varying function of ϕ and far from the barrier $V(\phi) = 1/a^2$:

$$\left| \frac{dV(\phi)}{d\phi} \right| \ll max\{\,|\,V(\phi)\,|\,,\ 1/a^2\,\}\,. \tag{4.5}$$

(There is also the condition $V \ll 1$ for the classical approximation to remain valid.) This justifies omitting the ϕ-derivative term in the Wheeler-De Witt equation (2.20), which now reads

$$\left[\, a^2 \frac{\partial^2}{\partial a^2} + pa \frac{\partial}{\partial a} + W(a, \phi) \right]\, \Psi(a, \phi) = 0\,. \tag{4.6}$$

Since the factor-ordering p does not affect semi-classical probabilities we may choose $p = -1$ and introduce a new variable

$$\eta = -(2V)^{-2/3}(\,1 - a^2 V\,)\,,$$

so that $\Psi(\eta)$ satisfies

$$\left[\, \frac{\partial^2}{\partial \eta^2} + \eta \,\right]\, \Psi = 0\,.$$

With hindsight [139, 140] we choose Airy function solutions with appropriate asymptotic forms

$$Ai(\eta) \;\approx\; \frac{1}{2\sqrt{\pi}} z^{-1/4} e^{-2\eta^{3/2}/3}\,, \tag{4.7}$$

$$Ai(-\eta) \;\approx\; \frac{1}{\sqrt{\pi}} \eta^{-1/4} \sin\left(\frac{2}{3}\eta^{3/2} + \frac{\pi}{4} \right) \tag{4.8}$$

in the limit $\eta \rightarrow \infty$. Now the Tunneling wave function Ψ_T has to satisfy the requirement that only the outgoing wave should be present in the classically allowed region ($i\Psi^{-1}\partial\Psi/\partial a > 0$ for $V > a^{-2}$). Hence for $V(\phi) < 0$,

$$\Psi_T = \frac{Ai(\mid \eta \mid)}{Ai(\mid \eta_0 \mid)} \tag{4.9}$$

where $\eta_0 = \eta(a = 0)$. In the classically allowed region far from the barrier, η is large and positive while η_0 is large and negative, so that the asymptotic forms apply. We therefore write the approximation for $a^2V(\phi) > 1$

$$\Psi_T \approx e^{i\pi/4}(\, a^2V - 1\,)^{-1/4} \exp\left[\, -\frac{1 + i(\, a^2V - 1\,)^{3/2}}{3V}\,\right] \tag{4.10}$$

and in the classically forbidden region $a^2V(\phi) < 1$ for both positive and negative values of $V(\phi)$,

$$\Psi_T = (\, 1 - a^2V\,)^{-1/4} \exp\left[\, \frac{(\, 1 - a^2V\,)^{3/2} - 1}{3V}\,\right] . \tag{4.11}$$

It is possible to obtain the Hartle-Hawking wave function Ψ_H by the transformation

$$\Psi_H = \Psi_T(\, V \rightarrow e^{-i\pi}V\,,\ a \rightarrow e^{i\pi/2}a\,) .$$

For a general three-metric $h_{ij} \rightarrow e^{i\pi}h_{ij}$ and the corresponding transformation of the potential $V(\phi)$, the Superspace Wheeler-De Witt equation remains invariant [139]. Equation 4.11 is interpreted to describe an ensemble of classical universes after nucleation. We proceed to determine the probability distribution for the initial states of the Universe, characterized by the scalar field ϕ at the barrier $V(\phi) = a^2$ and the initial conditions $\dot{a} = 0, \dot{\phi} = 0$. The conserved current (2.24 in Chapter 2) has components

$$j^a = \frac{i}{2}a^p(\, \Psi^*\partial_a\Psi - \Psi\partial_a\Psi^*) \tag{4.12}$$

$$j^\phi = -\frac{i}{2}a^{p-2}(\, \Psi^*\partial_\phi\Psi - \Psi\partial_\phi\Psi^*\,) \tag{4.13}$$

and continuity equation

$$\partial_a j^a + \partial_\phi j^\phi = 0 .$$

Since the classical configurations represented by the wave function 4.11 include only expanding de Sitter spacetimes

$$a \approx V^{-1/2} cosh(V^{1/2} t) \ , \ \phi \approx const.$$

problems with negative probabilities do not arise. Hence the scale factor is a good "time variable", so that at every "instant" of scale factor a, the component j^a can be interpreted as the probability density for ϕ, provided it is properly normalized. We formally the probability density from equations 4.11-4.12, with $p = 1$:

$$P_T(a, \phi) = C_T \exp\left[-\frac{2}{3V(\phi)} \right] , \tag{4.14}$$

where

$$C_T^{-1} = \int_{[V(\phi)>0]} d\phi \exp\left[-\frac{2}{3V(\phi)} \right] . \tag{4.15}$$

has been defined so that $P_T(a, \phi)d\phi$ is the probability for the scalar field to be between ϕ and $\phi + d\phi$ at the "instant" when the scale factor has value a. Since the probability is obviously independent of a (since ϕ is approximately constant along pencils of classical trajectories), the conservation of probability is trivially satisfied:

$$\partial_a \int j^a d\phi = 0 \ .$$

Proper normalization requires that the integral 4.15 converges; this occurs if

1) $V(\phi) < 0$ as $\phi \to \pm\infty$, or

2) $V(\phi) \to 0$ faster than $2/3 \ln \mid \phi \mid$, or if

3) ϕ is a cyclic variable in a finite range $0 < \phi < \phi_0$ where the points $\phi = 0$ and ϕ_0 are identified.

For sufficiently slow growth of the potential at large ϕ, the initial state leads to the "chaotic" inflation of Linde (1984) [103, 104, 105]. The largest nucleation probability is at the highest maximum of $V(\phi)$, corresponding to the initial condition required in the New Inflationary Scenario. Hence the Tunneling wave function naturally predicts inflation.

On the contrary, if the maximum of $V(\phi)$ is very close to zero, the initial density of the Universe is much lower than Planck density m_p^4, so that the whole history of the Universe is semi-classical.

4.2 The Hartle-Hawking wave function

4.2.1 The Path Integral Formulation

The alternative to the canonical quantization procedure discussed in the previous sections is the *path integral method*: The wave functional is an Euclidean functional integral over a certain class of four-metrics $[g_{\mu\nu}]$ and matter-fields $[\Phi]$, weighted by e^{-I_E}, where I_E is the Euclidean action of the gravity plus matter system. The wave functional

$$\Psi[\,\tilde{h}_{ij}, \tilde{\Phi}, B\,] = \sum_{\mathcal{M}} \int \mathcal{D}g_{\mu\nu}\mathcal{D}\Phi\; e^{-I_E} \tag{4.16}$$

is the sum taken over the class of manifolds $\mathcal{M}$ for which the three-surface B is part of their boundary, and over the class of four- metrics $[g_{\mu\nu}]$ and matter-fields $[\Phi]$ which induce the three-metric $\tilde{h}_{ij}$ and the matter-field configuration $\tilde{\Phi}$ on B. The path integral (4.16) is weighted by the Euclidean action (and not the Lorentzian action) in order to pick out Ψ as the ground state wave functional, and possibly to more easily deal with topology, avoiding the obstructions due to singularities in the Lorentzian theory. The measure $\mathcal{D}g_{\mu\nu}$ includes the product of the differentials $dg_{00}\cdots dg_{33}$ for each member of the class $[g_{\mu\nu}]$, and similarly for the measure $\mathcal{D}\Phi$.

A particular problem with this formulation is that the gravitational action is not bounded from below, so the path integral diverges if we integrate over real Euclidean metrics. Only by integrating along a complex contour in the space of complex four-metrics does the integral converge. Nor is there any unique contour, or for that matter, an exact and explicit prescription for the scale (or conformal) factor contour. So the wave functional depends crucially on which contour one chooses. Although there is no precise relationship, this problem is closely related to that of choosing boundary conditions on the wave functional. (More on these matters later.)

Provided that the action I_E, measure, and class of paths summed over respect an invariance generated by the Hamiltonian constraints, Halliwell and Hartle (1990) [62] have formally shown that the wave functionals generated by the path integral (2.15) indeed satisfy the Wheeler-De Witt equation (2.10) and the momentum constraints (2.9). This formulation of the wave functionals is essential for a discussion of the so-called Hartle-Hawking wave function of the Universe.

4.2.2 The No Boundary proposal

This proposal made by Hartle and Hawking [65] is essentially a topological statement about the class of histories summed over. The No Boundary proposal says that the three-surface B is the *only* surface of a *compact* four-manifold M, on which the four-metric $g_{\mu\nu}$ induces $\tilde{h}_{ij}$ on B, and the matter-field configuration ϕ matches $\tilde{\phi}$ on B. (See fig. 4.1.)

For a manifold M of the form $\mathcal{R} \otimes B$, with closed four-geometries that have vanishing shift $N^i = 0$ and constant lapse $\dot{N} = 0$, the path integral reduces to

$$\Psi_{NB}[\, \tilde{h}_{ij},\ \tilde{\phi},\ B\,] = \int dN \int \mathcal{D}h\mathcal{D}\phi\ e^{(-I_E[\,h\,,\,\phi\,,\,N\,])}\ .$$

In two-dimensional Mini-Superspace, such a path integral will have an Euclidean action I_E, for the homogeneous and isotropic scalar field ϕ and Friedmann-Robertson-Walker metric. (This is obtained by the substitution

$$t \longrightarrow -i\tau$$

in the Einstein scalar action 2.16 in Chapter 2.) If we represent the final surface B by $\tau = 1$ in terms of time-parameter τ, and label the initial point by $\tau = 0$, then for the four-geometry to close in a regular way as the scale

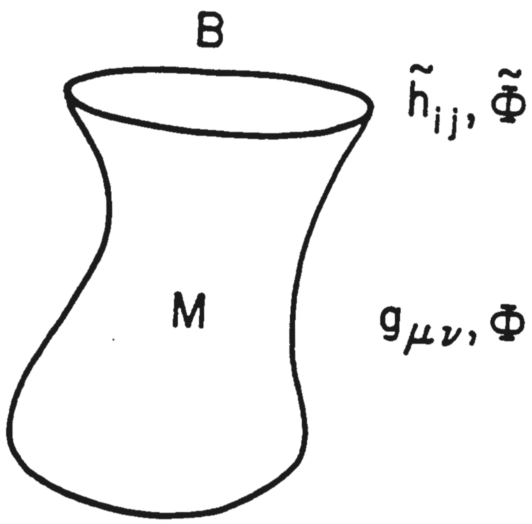

Figure 4.1: A pictorial representation of the class of histories summed over in the calculation of the No Boundary wave functional $\Psi[\tilde{h}_{ij}, \tilde{\phi}, B]$.

factor tends to zero we have to impose the initial condition

$$a(0) = 0 \ or \ \frac{1}{N}\frac{da}{d\tau}(0) = 1, \tag{4.17}$$

but not necessarily both [58]. Since the Euclidean action I_E leads to the Euclidean field equations (i.e. the Euclidean analogue of the Lorentzian field equations 2.17 - 2.19) it is easy to verify that this condition compels the scalar field to satisfy the initial condition

$$\frac{d\phi}{d\tau}(0) = 0 \tag{4.18}$$

Hence, we conclude that the No Boundary proposal applied to Mini-Superspace is equivalent to specifying *initial conditions* (4.17 and 4.18) for solutions to the field equations. Furthermore, the fact that the four-metrics induce $\tilde{h}_{ij}$ and ϕ matches $\tilde{\phi}$ on B, is translated into the final condition that

$$a(1) = \tilde{a} \ , \ and \ \phi(1) = \tilde{\phi} \tag{4.19}$$

in Mini-Superspace. However, these boundary conditions still fall short in giving unique solutions to the field equations (for an example, see [64]). But now the path integral has a number of saddle-points (i.e. where $\frac{\partial I_E}{\partial N} = 0$), each of which contribute to the integral by an amount $\sim e^{-I_E^k}$, with I_E^k the action of the solution corresponding to saddle-point k.

Nor do the boundary conditions (4.2 - 4.4) restrict the complex contour along which the lapse N is integrated. In fact, for every contour there exists a different path integral wave function solution Ψ_{NB}.

In an attempt to determine which saddle-point makes the dominant contribution to the path integral, Halliwell and Louka (1989) [58] found that there are a number of inequivalent contours along which the path integral converges, each dominated by different saddle-points, again leading to different forms of the wave function. Indeed, the No Boundary wave function of Hartle and Hawking is uniquely determined only after supplementing extra information to fix the contour. For instance, Hartle and Halliwell (1989) [56] suggested restricting the possible contours on the grounds of mathematical consistency and physical prediction.

Hartle and Hawking [65] gave heuristic arguments to support their thesis that a saddle-point will provide the dominant contribution only if the chosen contour in the path integral may be distorted into a *steepest-descent contour* along which the saddle-point is a global maximum. This allows them to derive a semiclassical form for the No Boundary wave function:

a) the wave function should be exponentially growing in the scale factor a in the classically forbidden region $a^2V(\phi) < 1$

$$\Psi_{NB} = (\ 1 - a^2V\ )^{-1/4} \exp\left[\frac{1 - (\ 1 - a^2V\ )^{3/2}}{3V} \right], \tag{4.20}$$

b) and contains equal contributions from contracting and re-expanding modes for the classically allowed region $a^2V > 1$:

$$\Psi_{NB} = 2(\ a^2V - 1\ )^{-1/4} \exp\left[\frac{1}{3V}\right] \cos\left[\frac{(\ a^2V - 1\ )^{3/2}}{3V} - \frac{\pi}{4}\right] . \tag{4.21}$$

This wave function represents an ensemble of both expanding universes and contracting universes. Since the total wave function is real, the current J is identically zero. Anyway, the probability distribution for expanding universes is readily given by equations 4.14 and 4.15 of Section 4.1, and reads

$$P_{NB}(a,\phi) = C_{NB} \exp\left[\frac{2}{3V(\phi)}\right] \tag{4.22}$$

where

$$C_{NB}^{-1} = \int_{[\ V(\phi)>0\ ]} d\phi \exp\left[\frac{2}{3V(\phi)}\right] . \tag{4.23}$$

Clearly the integral diverges when $V(\phi) = 0$ for certain values of ϕ. The probability distribution appears to be normalizable only if

1) $V(\phi)$ is strictly positive, and

2) ϕ has a finite range.

Since the maximum nucleation probability now corresponds to the true minimum of $V(\phi)$, it is not so clear how inflation will be predicted.

4.3 No-Boundary vs Tunneling proposal

We have seen that both the Tunneling and the No Boundary wave functions are peaked at about the same set of solutions to the field equations, whose first integral is equation 2.23 of Chapter 2. These solutions are inflationary for a slowly varying scalar field $\phi \approx \phi_0 = constant$ along pencils of trajectories. To determine which pencils lead to sufficient inflation in order to resolve outstanding problems in the standard model of the Hot Big Bang therefore depends directly on ϕ_0.

It is therefore interesting to see which of the two proposals is more reasonable in its predictions. It is clear from 4.14 and 4.23 that the respective probability distributions differ by a sign:

$$dP_{NB/T} \approx \exp\left[\pm \frac{2}{3V(\phi)} \right] d\phi \tag{4.24}$$

(+) for the No Boundary, (-) for the Tunneling proposal. Hawking and Page (1986) [79] argue that values of ϕ for which the initial density is too small should be excluded, and suggest that we should calculate *conditional probabilities* with the condition the density of the Universe is over a given range. We therefore assume that for a chaotic potential, the initial value of ϕ lies in a certain range

$$\phi_{min} < \phi < \phi_{max} \, .$$

As outlined by Vilenkin [141], sufficient inflation is achieved if $\phi_0 > \phi_{min}$, and is not achieved if $\phi_0 < \phi_{max}$. Given the range (ϕ_{min}, ϕ_{max}), suppose that $\phi = \phi_{suf}$ is the value of the scalar field within this range when sufficient inflation has occurred. Then the probability for sufficient inflation is the conditional probability (see equation 3.2 in Chapter 3)

$$\frac{\int_{\phi_{suf}}^{\phi_{max}} d\phi \; e^{(\pm 2/3V)}}{\int_{\phi_{min}}^{\phi_{max}} d\phi \; e^{(\pm 2/3V)}} \, .$$

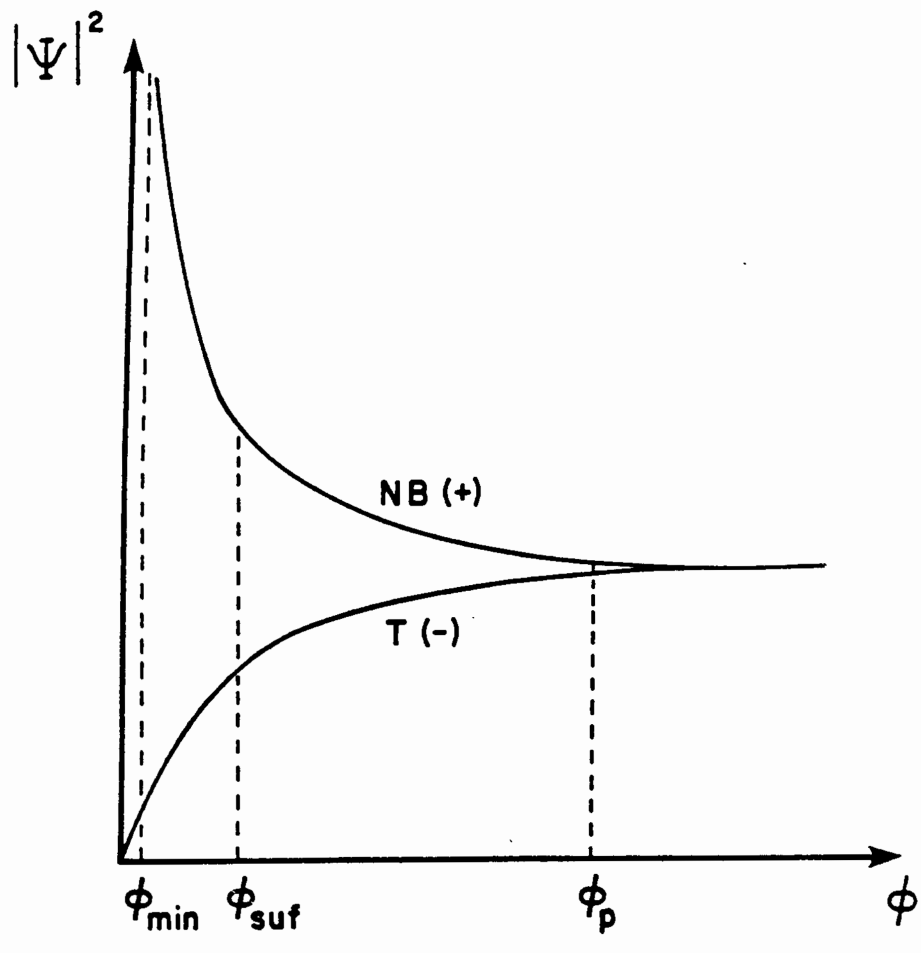

Figure 4.2: A plot of $|\Psi|^2$ against ϕ on a hypersurface of constant scale factor, for the Tunneling wave function and one component of the No Boundary wave function.

The Tunneling proposal (-) has by far its largest contribution from the region $\phi > \phi_{suf}$, so that the conditional probability for inflation is of the order unity (fig. 4.2).

The No Boundary proposal (+) receives its largest contribution from the region very close to ϕ_{min}, a very small cut-off. The conditional probability is therefore very close to zero, so that sufficient inflation is *not* predicted. On the other hand, Hawking and Page [79] assumes that $\phi_{max} \to \infty$, and argue that despite the peak close to ϕ_{min}, the contribution to the integral from this region is overwhelmingly outweighed by that from very large values of ϕ.

Chapter 5

A Measure of Inflation

5.1 An infinite period of inflation?

D.N. Page [117] claims that the No Boundary proposal applied to FRW Mini- Superspace suggests that the Universe may have had an infinite period of inflation.

If the scalar potential $V(\phi)$ rises monotonically well past unity for large $|\phi|$, so that

$$V(\phi) \gg 1 \tag{5.1}$$

$$\textit{and } 0 < \left| \frac{d \ln V(\phi)}{d\phi} \right| \ll 6 \tag{5.2}$$

for all larger values of $|\phi|$, then the wave function of the Universe is a solution of equation 2.20 with $p = 1$ and $\Lambda = 0$:

$$\Psi = J_0(z) \tag{5.3}$$

$$\textit{with } z = \frac{1}{3} a^3 \sqrt{V(\phi)}\,, \tag{5.4}$$

i.e. a zero order Bessel function J_0. For large z the wave function oscillates rapidly with the WKB form 2.25

$$\Psi = \Psi_+ + \Psi_- = Ce^{iS} + \bar{C}e^{-iS} ,$$

where the prefactor is just $C = (2\pi z)^{-1/2}$, and the phase

$$S = \frac{\pi}{4} - z$$

obeys the Hamilton-Jacobi equation 2.22, i.e.

$$(\nabla S)^2 + a^3 V(\phi) - ka = 0$$

for curvature $k = -1, 0, +1$, in the Mini-Superspace metric 2.21 with the lapse constant $N = 1$:

$$ds^2 = -a\, da^2 + a^3\, d\phi^2 . \tag{5.5}$$

The integral curves of ∇S represent the trajectories 2.23 of the semi-classical wave packets of which Ψ is a superposition, and along each of these trajectories there exists an affine time parameter t, satisfying

$$\frac{d}{dt} = \nabla S \cdot \nabla .$$

So the No Boundary wave function gives a superposition of a subset of all allowed semi-classical wave packets. For instance, the wave function (3) gives classical solutions for which $|\phi|$, V and z are all large have the inflationary form

$$\frac{\dot{a}}{a} \simeq \sqrt{V(\phi)} , \tag{5.6}$$

$$\dot{\phi} \simeq -\frac{1}{6\sqrt{V(\phi)}}\frac{dV(\phi)}{d\phi} , \tag{5.7}$$

where the derivatives are with respect to the conformal time t. The solutions are then labelled by one parameter ϕ_0, say at the first root $j_{0,1}$ of the Bessel function $J_0(z)$. A quantum regime exists for $z \leq j_{0,1}$, where there is no good classical notion of time. The classical regime occurs for $z \geq j_{0,1}$, since Ψ has an oscillatory WKB form there. The probability per unit time t contributed by Ψ_+ along a pencil of trajectories is proportional to the flux F of the conserved current $\mathcal{J}$ of Ψ_+, so that the flux per range of ϕ_0 is asymptotically constant for large ϕ_0, i.e.

$$\frac{dF}{d\phi_0} = \frac{3}{2\pi} + O(\phi_0^{-2}) \, .$$

This leads to a divergence of the total flux at $|\phi_0| = \infty$, hence the probability per unit time is dominated by contributions from arbitrarily large $|\phi_0|$, where the potential energies diverge.

Page [117] now tries to assess the amount of time the classical solution spends in the inflationary regime described by equations 5.6 and 5.7. It is clear from equation 7 that the time taken for ϕ to drop from ϕ_0 to some fixed value where inflation ends, will diverge as ϕ_0 becomes large and if the potential $V(\phi)$ does not increase faster than quadratic in ϕ. The power-law scalar field potential

$$V(\phi) = \frac{m^2}{2p}\phi^{2p} \tag{5.8}$$

is that of a free massive scalar field for $p = 1$, and chaotic if $p = 2$. For $0 < p \leq 2$, equation 7 yields

$$t - t_0 \simeq \frac{3(|\phi_0|^{2-p} - |\phi|^{2-p})}{m(2-p)\sqrt{p/2}} \xrightarrow{p\to 2} \frac{3}{m} \ln\left(\frac{\phi_0}{\phi}\right) , \tag{5.9}$$

which diverges as $\phi_0 \to \infty$.

Since the largest contribution of the trajectories come from large ϕ_0, almost all have an arbitrarily long period of inflation. For large values of the potential, the spatial curvature becomes negligible in the classical field equations 2.17 - 2.19, so we may put $k = 0$. They integrate to give for $\phi \gg p/3$,

$$a = a_0 \left(\frac{\phi_0}{\phi}\right)^{1/3} \exp\left(\frac{3}{2p}(\phi_0^2 - \phi^2) + \frac{p^2 - 3p}{54}(\phi^{-2} - \phi_0^{-2}) + O(\phi^{-4}, \phi_0^{-4})\right)$$

with

$$a_0^3 = \frac{3j_{0,1}}{m}\sqrt{2p}\,\phi_0^{-p} ,$$

and

$$t = \frac{3}{m\sqrt{p/2}}\left(\frac{-\phi^{2-p}}{2-p} + \frac{2-p}{18}\phi^{-p} - \frac{p^4 - 12p^3 + 24p^2}{648(p+2)}\phi^{-p-2} + O(\phi^{-p-4})\right) + const.$$

Inflation requires that

$$\dot{H}/H^2 \ll 1 ,$$

i.e. equation 2.18 gives

$$\dot{H}/H^2 = \frac{p^2}{3}\phi^{-2}[1 + O(\phi^{-2})]$$

for the Hubble parameter H, so that we restrict

$$|\phi| \geq p . \tag{5.10}$$

This leads to a duration of inflation of the order

$$\Delta t \simeq \frac{3(|\phi_0|^{2-p} - p^{2-p})}{m(2-p)\sqrt{p/2}} , \tag{5.11}$$

which diverges as $|\phi_0| \to \infty$, for $0 < p \leq 2$.

Since $|\phi|$ decreases monotonically from infinity during inflation, we may use it as a time coordinate. For arbitrarily large $|\phi_0|$, it would appear that the scale factor a diverges, but since this gives effectively a $k = 0$ model, we rescale the comoving coordinates to have a finite range and make the *rescaled* a finite: An explicit choice could be

$$a_{new} = \phi^{-1/3} \exp\left(-\frac{3}{2p}\phi^2 + O(\phi^{-2})\right) \tag{5.12}$$

during inflation. The spacetime four-metric now approximates to

$$-\frac{18}{m^2 p}\phi^{2-2p}d\phi^2 + \phi^{-2/3}e^{-3\phi^2/p}(dx^2 + dy^2 + dz^2)\,, \tag{5.13}$$

which gives a Universe that has an infinite classical history, yet the spacetime is still singular in the sense of being geodesically incomplete [73]. The proper time to go from $a = 0$ to some finite $a = a_1$ is just

$$\int_0^{a_1} \frac{H^{-1}da}{(P^2 + a^2)^{1/2}}\,,$$

which is finite for positve H and positive spatial momentum $P^2 = \sum_{x,y,z} a^4(dx'/d\tau)^2$. The null and spacelike geodesics are also incomplete: For a null geodesic with affine parameter λ,

$$\Delta\lambda = \int_0^{a_1} \frac{da}{PH}\,,$$

while the spacelike geodesic with proper length s has

$$\Delta s = \int_0^{a_1} \frac{H^{-1}da}{(P^2 - a^2)^{1/2}}\,,$$

which are both finite. Hence the age of the Universe may be infinite , even though its size is finite. This is a counter-example to the common notion that the Universe must have a finite age and that its classical evolution could not have started at curvatures above the Planck value. We believe it more likely that these results highlight the shortcomings associated with models constructed from quantum Mini-Superspace. This is due to the many strong assumptions made when "freezing" out extra degrees of freedom. It may also be because Einstein-Hilbert gravity and quantum cosmology is just an effective theory at large scales which is missing higher order corrections comming, for instance, from string theory.

5.2 Higher order corrections

Given the appropriate action 2.16 that result in the field equations 2.17 - 2.19 with vanishing cosmological constant and power-law potential 5.8, we now discuss in greater detail how the Hartle-Hawking No Boundary proposal enables us to impose initial conditions to these equations. In addition, the phase S of the WKB approximation is obtained by analytical continuation of the Euclidean action for compact metrics and regular matter-fields. Similar to the previous section, we are able to solve the *Lorentzian* field equations in terms of an affine time variable t, but now more accurately [25]. We see that such solutions exhibit a period of exponential inflation, as anticipated. The requirements for sufficient inflation are then outlined.

5.2.1 Lorentzian initial conditions

In the Euclidean regime the No Boundary proposal is equivalent to the initial conditions 4.17 and 4.18. The potential V of the scalar field ϕ acts as an effective cosmological constant when ϕ is large and roughly constant $\phi \simeq \phi_0$. Since inflation does not last forever (the present Universe is not expanding exponentially), the effective cosmological constant must eventually vanish as time passes. Thus, the full set of initial conditions are conditions 4.17 and 4.18 with the addenda

$$\phi(\tau = 0) = \phi_0 \,. \tag{5.14}$$

The corresponding value for the potential is $H_0^2 = m^2\phi_0^{2p}/2p$, in terms of the Hubble constant at $\tau = 0$. Since the Euclidean Path Integral (EPI) is taken over compact four-metrics, the scale factor $a(\tau)$ has to vanish for some value of τ we can choose to be zero [25]. Thus, for large ϕ_0, we have

$$a(\tau) = H_0^{-1} \sin(H_0\tau) \,,$$

consistent with the initial conditions 4.17, 4.18 and 5.14. Now we perform the analytic continuation to Lorentzian spacetime:

In the WKB ansatz 2.25 for the wave function, the phase S is chosen to satisfy the Hamilton-Jacobi equation 2.22 while the No Boundary proposal picks out a solution to this equation that corresponds to the analytic continuation of the Euclidean action

$$I_E = -\left(\frac{2p}{3m^2\phi^{2p}}\right)\left(1-(1-a^2m^2\phi^{2p}/2p)^{3/2}\right) . \tag{5.15}$$

This corresponds to the action of the smaller part of a four-sphere of radius $\sqrt{2p}/m\phi^p$, bounded by a three-sphere of radius a, and it generalizes what has been done in the case of the massive scalar field [75]. The solution to the Hamilton-Jacobi equation 2.22 is therefore the analytic continuation of I_E, and at large ϕ it is given by the phase

$$S \simeq -\frac{2p}{3m^2\phi^{2p}}(a^2m^2\phi^{2p}/2p-1)^{3/2} . \tag{5.16}$$

The application of this method yields, for $\tau = \pi/2H_0 + it$, see [100] :

$$a(t) = H_0^{-1}\cosh\left(H_0 t\right)$$

at very small times t, for large and constant $\phi(t) = \phi_0$. So the minimum value of a in the Lorentzian regime is equal to the maximum value of a in the Euclidean regime, while the initial conditions for a differ vastly :

$$a(t=0) = \frac{\sqrt{2p}}{m\phi_0^p} = a_0 \tag{5.17}$$

$$\dot{a}(t=0) = 0 . \tag{5.18}$$

The initial conditions for the scalar field are

$$\phi(t=0) = \phi_0 \tag{5.19}$$

$$\dot{\phi}(t=0) = 0 . \tag{5.20}$$

The phase S defines the first integrals 2.23 of the system,

$$\dot{a} = (a^2m^2\phi^{2p}/2p-1)^{1/2} \quad (5.21)$$

$$\dot{\phi} = -\frac{p\dot{a}}{\phi a}+\frac{4p^2}{3m^2\phi^{2p+1}}\left(\frac{\dot{a}}{a}\right)^3 , \quad (5.22)$$

besides the Friedmann constraint

$$a^2\dot{\phi}^2-\dot{a}^2+a^2m^2\phi^{2p}/2p-1=0\,. \quad (5.23)$$

The phase itself, up to the first order correction to that of Section 1, reads

$$S \simeq -\frac{m}{3\sqrt{2p}}a^3\phi^p\left(1-\frac{3p}{m^2a^2\phi^{2p}}\right)\,. \quad (5.24)$$

which is an approximation of equation 5.16 for large a, such that $a \sim e^{Ht}/2H$ (i.e. equation 5.24 holds in the range $[t_1,t]$. The Lorentzian Hartle-Hawking trajectories [106] then explicitly have

$$\dot{\phi} = -\frac{p}{3\sqrt{2p}}m\phi^{p-1}$$

and

$$\frac{\dot{a}}{a} = m\phi^p/\sqrt{2p}\,.$$

Integrate over the time interval [t_1 , t], we find

$$\phi(t) = \left(\phi_1^{2-p}-(2-p)\frac{p}{3\sqrt{2p}}m(t-t_1)\right)^{1/(2-p)}$$

$$a(t) = a_2\exp\left[\frac{m}{\sqrt{2p}}\int_{t_1}^{t}\left(\phi_1^{2-p}-(2-p)\frac{p}{3\sqrt{2p}}m(t'-t_1)\right)^{p/(2-p)}dt'\right] ,$$

for every $p \neq 2$, with $a_2 = H_0^{-1}\cosh(H_0t_1)$.

For the case of the chaotic potential $p=2$ [105], the integrals yield

$$\phi = \phi_1 e^{-\frac{2m}{3\sqrt{2p}}(t-t_1)}$$

and

$$a = a_2 \exp\left\{\frac{3\phi_1^2}{4}\left[1 - \exp\left(-\frac{4m}{3\sqrt{2p}}(t-t_1)\right)\right]\right\} .$$

For very early times when $t - t_1$ is very small, the inflationary formula for any p is approximately

$$a(t) \simeq a_2 \exp\left[m\phi_1^p(t-t_1)/\sqrt{2p}\right] \simeq \frac{1}{2}a_0 \exp\left(\frac{m}{\sqrt{2p}}\phi_1^p t\right) , \qquad (5.25)$$

where the scale factor at the end of inflation is $\simeq a_2 \exp(3\phi_1^2/2p)$ and assuming that $\phi_1 \simeq \phi_0$.

5.2.2 Minimal conditions for sufficient inflation

In order to solve the horizon and flatness problems [52, 105] the inflationary formula has to satisfy the condition

$$a(t) \geq 10^{28} a_0 \simeq a_0 \exp(65) .$$

If we put $t = \beta t_f$ where t_f denotes the time at the end of inflation, and for β in the interval (0,1], we find the constraint

$$\frac{m}{\sqrt{2p}}\phi_1^2 t_1 + \frac{3}{2}\phi_1^2 \ln(\phi_1/\phi_f) \geq \frac{(65 + \ln 2)}{\beta} \qquad (5.26)$$

for the chaotic potential, where

$$\frac{3}{2}\ln(\phi_1/\phi_f) = m(t_f - t_1)/\sqrt{2p} .$$

Generally, for $p > 2$, we have the constraint

$$\frac{m}{\sqrt{2p}}\phi_1^p t_1 + \frac{3}{p(p-2)}\left(\frac{\phi_1}{\phi_f}\right)^p \phi_f^2 \geq \frac{(65 + \ln 2)}{\beta} . \qquad (5.27)$$

In this case, we have the duration of the inflationary era

$$t_f - t_1 = 4\left(\frac{2p\phi_1}{m^2}\right)^{1/2}\left[1 - \left(\frac{\phi_f}{\phi_1}\right)^{1/2}\right] .$$

These are minimal requirements since a thorough inflationary model also has to solve other problems, e.g. the origin of the energy density fluctuations to account for galaxy-formation.

5.3 How probable is Inflation ?

5.3.1 Towards a representative model

So far we have seen how the No Boundary proposal leads to a wave function that possess, amongst others, the feature of sufficient inflation. However, this is not the only possible choice of boundary condition for a wave function defined in Superspace, so the question arises whether a sufficiently long period of inflation is a property of a "typical" wave function.

Gibbons and Grishchuk [38] attempted to clarify this issue using a model of a two dimensional Mini-Superspace describing a free massive scalar field ϕ in a closed FRW universe with scale factor a. Various aspects of this model have been studied at both classical [42, 39] and quantum [77, 79, 51] level. We broaden the scope of their [38] arguments somewhat by applying it to a scalar field with the power-law potential 5.8 already encountered . Of course, we can instead of $m^2/2p$, simply read $\lambda/2p$, the self-interaction constant used in [25, 75, 85, 103, 104, 105, 106, 117], for $p \neq 1$.

The Wheeler-De Witt equation 2.20 has the form

$$\left(\frac{1}{a^q}\frac{\partial}{\partial a}a^q\frac{\partial}{\partial a} - \frac{1}{a^2}\frac{\partial^2}{\partial\phi^2} - a^2 + \frac{m^2}{2p}\phi^{2p}a^4\right)\Psi(a,\phi) = 0\ , \tag{5.28}$$

where q reflects the factor-ordering ambiguity. In the inflationary regime as outlined in Section 2, we define

$$H^2 = \frac{m^2}{2p}\phi_0^{2p} = const.$$

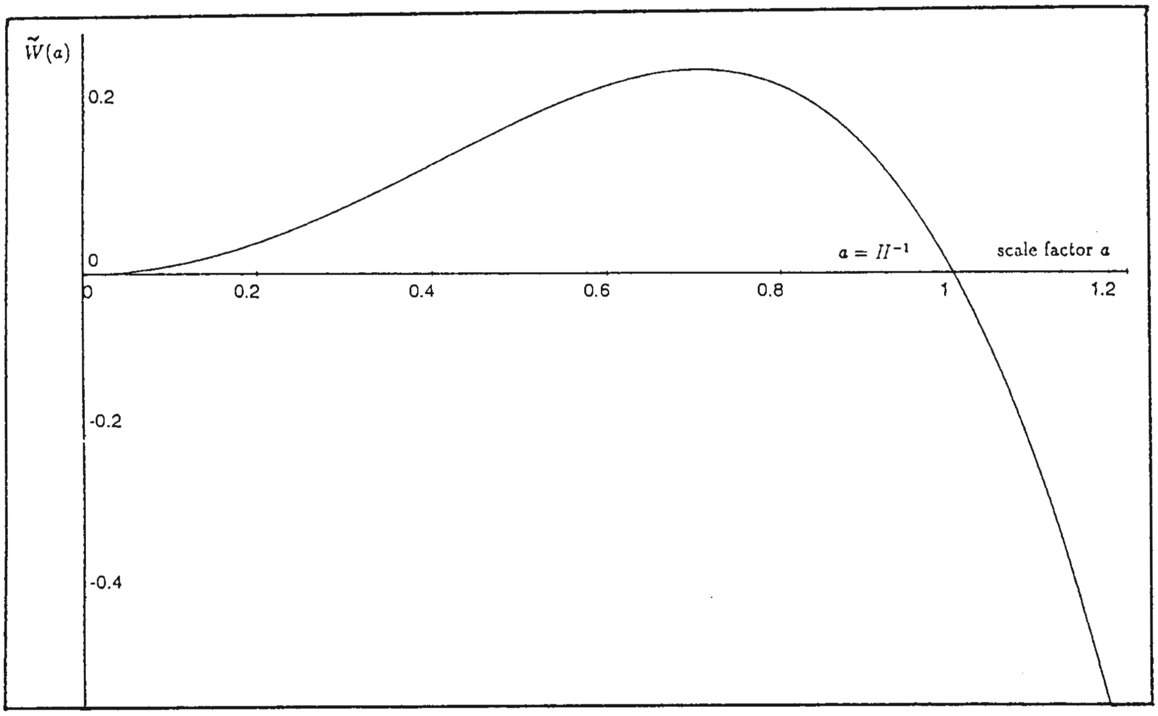

Figure 5.1: The potential $\tilde{W}(a) = a^2 - H^2a^4$ for $H^2 = 1.0$.

One neglects the term $-(1/a^2)(\partial^2/\partial\phi^2)$ in this regime, so that the model exhibits the features of a closed universe with a cosmological constant H^2. The case $q = -1$ [38] has equation

$$\left(a\frac{d}{da}\frac{1}{a}\frac{d}{da} - a^2 + H^2a^4\right)\Psi(a) = 0\ . \tag{5.29}$$

The solutions to this equation are Infeld, Macdonald and Hankel special functions with argument $\pm(H^2a^2-1)$ [38, 51]. Equation 5.29 has the form of the Schroedinger equation for a one-dimensional problem with superpotential $\tilde{W}(a) = a^2 - H^2a^4$. The turning point is at $a = H^{-1}$ (see fig. 5.1).

The ordinary semiclassical probability for the system to tunnel from one classically allowed region to another, has the value

$$D = \left|\frac{\Psi(a_2)}{\Psi(a_1)}\right|^2 ,$$

always less than unity for quantum tunneling. We define a similar quantity D in quantum cosmology, except that its physical interpretation is not so clear:

$$D = \left| \frac{\Psi(H^{-1})}{\Psi(0)} \right|^2 . \tag{5.30}$$

Provisionally, we define D as the probability coefficient describing the creation of the Universe from "nothing". It is then likely that the wave functions predicting $D < 1$ describe quantum tunneling. It is possible to show [51] that provided $H \gg 1$, the Hartle-Hawking wave function gives

$$D = \exp\left(\frac{3\pi}{GH^2}\right) \gg 1 .$$

To answer the question of how many such wave functions there are, we consider the space of all possible wave functions and introduce a suitable measure on this space.

Since the system has only two linearly independent states, we introduce an arbitrarily chosen, suitably normalized basis of states $|1>$ and $|2>$. A general state can be expanded as

$$|\Psi>= Z_1|1> + Z_2|2> ,$$

where Z_1 and Z_2 are complex constants. Then D can only depend on the ratio $\zeta = Z_1/Z_2 \equiv x \exp{(ib)}$, parameterizing the points on a two-sphere. In fact, it was shown [51] that in the approximation $H \ll 1$,

$$D \sim H^{-2/3} x^{-2} \exp\left(-\frac{6\pi}{GH^2}\right) .$$

The set of possible wave functions is in 1-1 correspondence with the points on the two-sphere. The effective physical (unitary) transformations acting on the space of quantum states is the rotation group $SO(3) = SU(2)/C_2$, where C_2 is the group consisting of +1 and -1. This acts on the two-sphere in the usual way provided b is the longitudinal angle and $x = cotan(\frac{1}{2}\theta)$, where θ is the co-latitude.

In terms of the ratio ζ, the line-element on the two-sphere is

$$4(1+\zeta\bar{\zeta})^{-2}d\zeta \wedge d\bar{\zeta},$$

and the volume element

$$dV = 4(1+\zeta\bar{\zeta})^{-2}d\zeta \wedge d\bar{\zeta}. \tag{5.31}$$

The quantum analogue of the principle of general covariance is that the measure is invariant on the space of quantum states, which will be the volume element of the two-sphere:

$$dV = \sin\theta d\theta db\ . \tag{5.32}$$

If we define a new variable [51] $y = \arctan x = (\pi - \theta)/2$, then $dV = 2\sin 2y dy db$, with $0 \leq y \leq \pi/2$ and $0 \leq b \leq 2\pi$, and the surface area corresponding with the wave functions $D > 1$:

$$y < y_0 = \exp\left(-\frac{3\pi}{GH^2}\right) \quad 0 \leq b \leq 2\pi$$

is very small compared to the surface area of the two-sphere. So the ratio of wave functions that predict $D > 1$ (among them the Hartle-Hawking wave function) at the point $\theta = \pi$ and those that predict $D < 1$ is just

$$y_0^2 \ll 1\ ,$$

i.e. very small indeed. Hence the probability of finding a wave function with $D > 1$ is minute.

5.3.2 Asymptotically flat curvature

In the limit that the curvature term $-a^2$ becomes negligible [38, 117] (see Section 1), the Wheeler-De Witt equation reduces to

$$\left(\frac{1}{a}\frac{\partial}{\partial a}a\frac{\partial}{\partial a} - \frac{1}{a^2}\frac{\partial^2}{\partial\phi^2} + \frac{m^2}{2p}\phi^{2p}a^4\right)\Psi(a,\phi) = 0. \tag{5.33}$$

In the WKB approximation 2.25, the phase S satisfies the Hamilton-Jacobi equation 2.22 and has the form [38]

$$S(a,\phi) = -a^3 F(\phi) \tag{5.34}$$

and F satisfies the equations

$$\dot{\phi} = -\frac{dF}{d\phi} \tag{5.35}$$

$$\frac{\dot{a}}{a} = 3F. \tag{5.36}$$

Different solutions to F correspond to different trajectories in the $\phi - \dot{\phi}$ plane of fig. 5.2 starting from the Big Bang, the repulsive knots K_1 and K_2. All trajectories are woven around the stable focus P that corresponds to the final stages of inflationary expansion with $k = 0$. The boundary of the circle correspond to infinity, $\dot{\phi}^2 + m^2\phi^2/2p = \infty$. The two attractive separatrices S_1 and S_2 correspond to the solutions [25]

$$F(\phi) = \pm\frac{m}{3\sqrt{2p}}\phi^p \tag{5.37}$$

and represent the Hartle-Hawking wave function

$$\Psi \approx \left[\exp\left(-i\frac{m}{3\sqrt{2p}}a^3\phi^p\right) + \exp\left(+i\frac{m}{\sqrt{2p}}a^3\phi^p\right)\right].$$

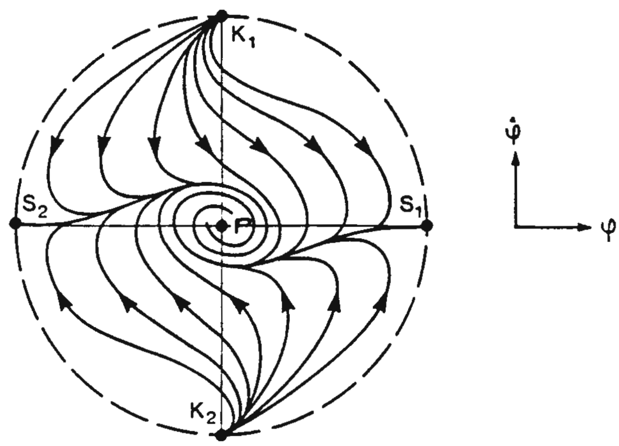

Figure 5.2: The compactified $\phi - \dot{\phi}$ phase plane.

Now for different solutions F_n denoted by the discrete index n, there are different wave functions Ψ_n that may sum to an arbitrary wave function of the form

$$\Psi = \sum_n \Psi_n.$$

It is possible to show that to every trajectory in the $\phi - \dot{\phi}$ plane one can assign a conserved quantity Q_n, corresponding to different F_n. Some of these solutions will be "unfavourable" as opposed to "favourable" with regards to having sufficient inflation. That is, for N different wave functions Ψ_n forming a linear superposition, with n' denoting those peaked around favourable trajectories, and n'' those not, such that their totals N', N'' add linearly $N' + N'' = N$, then the wave function

$$\Psi = \sum_{n'}^{N'} \Psi_{n'} + \sum_{n''}^{N''} \Psi_{n''}$$

can be characterized by the number P reflecting the amount of inflation,

$$P = \sum_{n'}^{N'} Q_{n'} \left(\sum_{n}^{N} Q_n \right)^{-1} .$$

So what is the mean value of P ? Well, as in the case of the cosmological constant model, the state $|\Psi>$ can be expressed as a sum of the basis states $|n>$,

$$|\Psi> = \sum_{n=1}^{N} Z_n |n>$$

where the coefficients $|Z_n|^2 = Q_n$, each n. We assume such bases $|n>$ are normalized. Then the space of physical states may be parameterized by $N-1$ complex ratios $\{\zeta_n\}$, with $\zeta_n = Z_n/Z_N$, $n = 1, 2, ..., N-1$.

They form coordinates to an $(n-1)$-dimensional complex manifold, known as "complex projective space" CP^{N-1}. There is an effective symmetry $SU(N)/C_N$, where C_N is the cyclic group generated by multiplication by $\exp(2\pi i/N)$. The CP^{N-1} space is homogeneous with respect to this group, so there is a unique invariant measure in terms of coordinates $\{\zeta_n\}$, given by the Riemannian volume measure with respect to the invariant metric on CP^{N-1} (known as the "Fubini- Study" metric). It is given by

$$dV = \left(1 + \sum_{i=1}^{N-1} |\zeta^i|^2 \right)^{-N} \Pi_{i=1}^{N-1} d\zeta^i \wedge d\bar{\zeta}^i . \qquad (5.38)$$

The "amount of inflation" P over the CP^{N-1} space endowed with this measure has an average value

$$\bar{P} = \int P(Q_1, Q_2, ... Q_n) dV = \frac{N'}{N} . \qquad (5.39)$$

A reasonable choice of states $|n>$ can be obtained by dividing the "quantum boundary" where the energy density $\dot{\phi}^2 + m^2\phi^{2p}/2p$ reaches its Planck value

m_p^4, which constitutes a Cauchy surface for all trajectories in $\phi - \dot{\phi}$ space, into N equal intervals. One of the trajectories in a given interval can play the role of a representative. It is then possible to show that $N'/N = 1 - \beta m/\sqrt{2p} m_p$, where $\beta = O(1)$. Thus, inflation indeed seems to be a property of a typical wave function provided $m \ll m_p$ and the power $p = O(1)$.

Chapter 6

Time in Quantum Cosmology

"The physical space I have in mind (which already includes time) is therefore nothing but the dependence of the phenomena on one another. A completed physics that knew this would have no need of separate concepts of space and time because these would already have been encompassed."

-Ernest Mach (1866)

"By an old sundial motto, the time thou killest will in time kill thee."

-Karel Kuchař (1992)

6.1 The problem of Time

A fundamental problem in quantum cosmology is the lack of a natural probabilistic interpretation of the wave function [91], as outlined in previous chapters. Closely related to this is the "problem of time" in any generally covariant theory. (The concept of general covariance applies to a theory like

General Relativity, for instance, in which gravitational phenomena are described by the spacetime metric alone; no one family of spacelike surfaces is preferred over any other [70].) We also learnt that the concept of probability is tenable only when one can specify with respect to which time variable it is conserved (see for instance Section 2.6, Chapter 2).

Due to the peculiar role that time plays in the usual framework of Hamiltonian quantum mechanics, the latter is insufficiently general for quantum cosmology. The observable Universe seems to have a fixed classical geometry that yields the notion of "preferred time" in quantum mechanics. Despite the presence of many foliating families of spacelike surfaces in the spacetimes of special relativity, different choices of such families to define a preferred time of quantum mechanics all give equivalent results. Similarly, General Relativity is generally covariant.

However, since we expect quantum fluctuations of spacetime in the very early Universe, there is no fixed background to define a notion of causality. So quantum mechanics constructed from two different choices of preferred spacelike surfaces may not be unitarily equivalent [99, 89]. The fact that spacetime is treated as a dynamical quantum variable may compel us to formulate a Hamiltonian quantum mechanics with time variable other than a family of spacelike surfaces in spacetime. This would be a generalization of familiar quantum mechanics provided the usual formulation with a preferred time variable emerges in the appropriate limit [70]. The generalization of quantum mechanics with the spacelike hypersurface as preferred time variable is just one such possibility.

6.2 The Arrow of Time

We briefly concentrate on the intriguing disparity observed in the time symmetry of the fundamental laws of physics and the time symmetries we encounter in the real Universe. To mention those peculiar to cosmology [32, 118, 146]:

(a) The thermodynamic arrow of time - approximately isolated systems almost all evolve towards equilibrium in the same direction of time.

(b) The arrow of time of the approximately uniform expansion of the Universe.

(c) The arrow of time supplied by the growth of inhomogeneity in the expanding Universe.[1]

Such time asymmetries could arise from time-symmetric dynamical laws solved with time-asymmetric boundary conditions [32]. For example, (a) is implied by an initial condition that would make conditions in the very early Universe far from equilibrium. Asymmetries (b) and (c) may follow from an initial Big Bang of sufficient spatial homogeneity and isotropy, given the attractive nature of gravity.

Since Quantum Cosmology is primarily a theory of the boundary condition(s) for our Universe, it is the perfect environment to address the origin of time asymmetries. Hawking [78], Page [114] and others [63, 100] investigate the emergence of the thermodynamic arrow of time from the No

[1]Other asymmetries are the Psychological arrow of time - we remember the past but not the future, the time-direction inherent in Retarded Electromagnetic Radiation, and the arrow of time supplied by the CP non- invariance of the weak interactions and the CPT invariance of field theory.

Boundary proposal. In the classical framework, Penrose [118] and others [82, 78, 114, 53, 80] have imposed time-asymmetric initial and final conditions on the Einstein field equations.

In the Copenhagen interpretation, the laws of quantum mechanics normally incorporate an arrow of time in the sense that for exhaustive sets of alternative histories $\{a_k\}$ at instants $t_1 < t_2 < ... < t_n$, the probability for a particular history in the exhaustive set of histories is given by

$$p(a_n, ..., a_1) = Tr\left[P^n_{a_n}(t_n)...P^1_{a_1}(t_1)\rho P^1_{a_1}(t_1)...P^n_{a_n}(t_n)\right] , \tag{6.1}$$

where $\{P^k_{a_k}(t_k)\}$ is the set of projection operators in the Heisenberg picture representing an exhaustive set of alternatives $\{a_k\}$ at time t_k, and the density matrix ρ describes the initial state of the system, and with usual time-ordering from the density matrix to the trace [70]. This formula therefore exhibits an asymmetry between 'future' and 'past', defining the arrow of time in ordinary quantum mechanics that in turn implies the familiar notion of causality. The *conditional probabilities* for future are

$$p(a_n, ..., a_{k+1}|a_k, ..., a_1) = \frac{p(a_n, ..., a_1)}{p(a_k, ..., a_1)} . \tag{6.2}$$

The *present* time t lies between the instants t_k and t_{k+1}. These probabilities can be expressed in terms of an *effective density matrix* $\rho_{eff}(t_k)$ at the instant t_k, and reads

$$Tr\left[P^n_{a_n}(t_n)...P^{k+1}_{a_{k+1}}(t_{k+1})\rho_{eff}(t_k)P^{k+1}_{a_{k+1}}(t_1)...\right] ,$$

where the effective density matrix is

$$\rho_{eff}(t_k) = \frac{P^k_{a_k}(t_k)...P^1_{a_1}(t_1)\rho P^1_{a_1}(t_1)...P^k_{a_k}(t_k)}{p(a_k, ...a_1)} . \tag{6.3}$$

Given the history $(a_1,...,a_k)$, then the effective state of the Universe at the time t_k is given by the density matrix $\rho_{eff}(t_k)$, as seen in the Copenhagen quantum mechanics of measured subsystems. Projection operators $\{P^k_{a_k}(t_k)\}$ describe alternative outcomes of measurements on subsystems.

For a closed system such as the Universe described by a spacetime with negligible gross fluctuations, the density matrix ρ can be seen as describing its initial condition. But consistent probabilities p are predicted only for those sets of histories for which there is negligible interference between individual members of the set as a consequence of the particular initial ρ. This is known as *decoherence* between sets of histories.

Hartle [70], Griffiths [49] and Aharonov et al [2] formulated a "neutral-time" quantum mechanics for cosmology, that is devoid of the effective density matrix $\rho_{eff}(t)$ that enables one to compute future probabilities from past histories. In fact in this new formulation, probabilities for the individual members of a set of alternative histories $\{a_k\}$ depend on Heisenberg operators (Hermitian and positive) ρ_i and ρ_f that represent initial and final conditions for the Universe respectively. That is, this formulation of quantum mechanics need not have a fundamental arrow of time. Here, the probabilities are defined as

$$p(a_n, ..., a_1) = NTr\left[\ \rho_f P^n_{a_n}(t_n)...P^1_{a_1}(t_1)\rho_i P^1_{a_1}(t_1)...P^n_{a_n}(t_n)\ \right] \tag{6.4}$$

where

$$N^{-1} = Tr[\,\rho_f\rho_i\,]\,.$$

In the case of $\rho_f \propto I$, the identity matrix, we arrive back at the Copenhagen formulation 6.3. This generalized quantum framework allows for the possibility of violation of causality, with advanced and retarded effects.

For instance, the imposition of time-symmetric (statistical) boundary conditions on a classical cosmology means that the entropy must behave time- symmetrically provided the coarse-graining is itself time-symmetric. The thermodynamic arrow of time will run backwards on one side of the moment of time symmetry as compared to the other side. This does not mean that individual histories (fine-graining) need necessarily be time-symmetric.

Penrose [118] estimated values of the initial low entropy for our Universe at the Big Bang, so we basically know its initial condition with respect to coarse-grainings defining the classical domain of familiar experience [32]. The problem of finding the final condition is somewhat more intricate:

a) If the Universe is closed and has a lifespan of the order of its present age since the Big Bang, then there are ample examples of models within our Universe with relaxation times comparable to (or even longer than) the timespan between the Big Bang and the final Big Crunch. This will enable us to detect (or infer) the existence of a time-symmetric final condition of our Universe from experiments on phenomena that remain out of equilibrium long enough for them to be affected by such a final condition. For example, radioactive material with very long half-lives, singularities contained within black holes, or black holes with life-time to decay by the Hawking radiation longer than the Hubble time.

b) However, if the lifespan of the Universe is much longer than its present age, such systems might be difficult to find. This would mean that we will never be able to detect the existence of a time-symmetric final condition.

We can expect that the wave function for the Universe gives an ensemble of classical solutions very much like that obtained from the WKB approximation, with different probabilities. For instance, closed geometries will be a probability distribution over possible lifespans of the Universe. Both the No Boundary and the Tunneling proposal predict very long lifespans for the

Universe (see [82, 139]).

6.2.1 Decoherence

We stated earlier that the quantum mechanics of a closed system such as the Universe as a whole predicts probabilities only for sets of alternative histories that *decohere.* So the minimal requirement on any theory of boundary conditions is that the universe exhibit a decoherent set of histories that corresponds to the semi-classical domain of everyday experience.

The coherence between individual histories in an exhaustive set of coarse grained histories $\{a\}$ is measured by the *decoherence functional*, a complex-valued functional on each pair of histories (a, a'),

$$D(a,a') = NTr\left[\rho_f C_{a'} \rho_i C_a^\dagger\right] . \tag{6.5}$$

Here we have abbreviated the strings of projective operators in equation 6.4 by C_a. Decoherence occurs when the real parts of the off-diagonal elements of the functional (those between two histories with any $a_k \neq a'_k$) vanish with sufficient accuracy. (More generally, it should occur when the off-diagonal elements of D are sufficiently small for any $a_k \neq a'_k$.) Under these conditions the probabilities p in equation 6.4 satisfy the usual sum rules of probability, and are in fact just the diagonal elements of D.

An extreme example of boundary conditions that are inconsistent with the existence of a semi-classical regime is when the final density matrix equals the initial density matrix

$$\rho_f = \rho_i \equiv \rho .$$

It is possible to show [32] that the probabilities of the different projections P remain constant in time, so that there is no dynamics nor any second law of thermodynamics. This is in contradiction with experience.

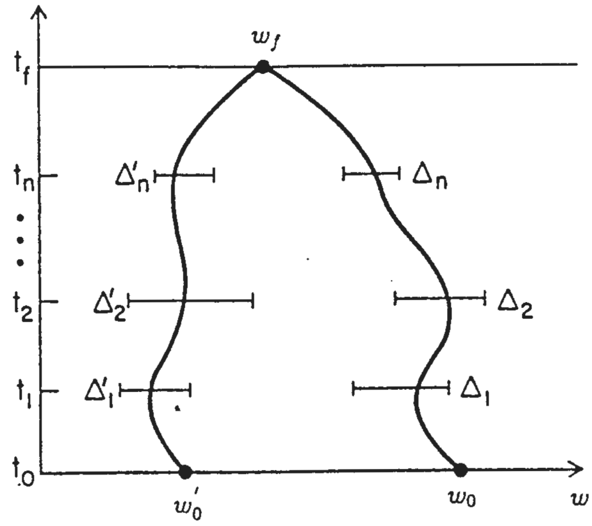

Figure 6.1: The sum-over-histories construction of the decoherence functional

Finally, besides predicting a semi-classical domain of familiar experience, the boundary conditions must also lead to probabilities that are strongly peaked at histories that are correlated by classical dynamics. I.e. we must still be able to derive the classical equations of motion.

So for the sum-over-histories quantum mechanics [70] the decoherence functional (fig. 6.1) is naturally defined on a set of coarse-grained histories $\{h_i\}$ as

$$D(h_i, h_j) = \int_{h_i,C} \delta g \delta\phi \int_{h_j,C} \delta g' \delta\phi' e^{i(S[g,\phi]-S[g',\phi'])/\hbar} \; . \tag{6.6}$$

Here S is the action for gravity and matter-fields. The integral is over four-metrics g and matter-field configurations ϕ that lie in the partition h_i. Similarly for the integral over g' and ϕ' over h_j. It is assumed that the initial and final conditions on the histories are incorporated in the sum over histories as conditions C on the fine-grained histories.

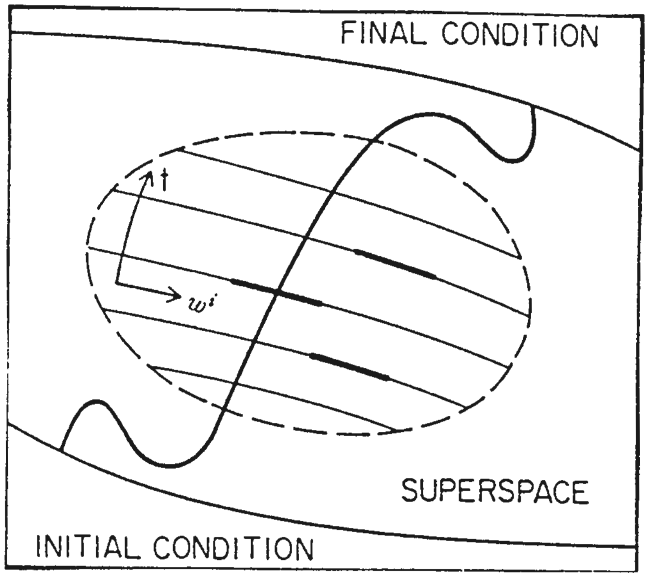

Figure 6.2: Recovery of Hamiltonian physics in the late Universe. Here is a schematic representation of the Superspace of all three-geometries and matter-field configurations. The region surrounded by the dotted line contains the large three-geometries of the late Universe.

In this formulation, there is no purely geometric quantity that uniquely labels a spacelike hypersurface. A Hamiltonian formulation may, however be approximated in a restricted *domain* of Superspace (see fig. 6.2), for special coarse-grainings and for particular initial conditions.

Suppose that the initial conditions were such that for coarse-grainings defined by sufficiently unrestricted regions of Superspace, in a regime of three-geometries much larger than the Planck scale, only a single spacetime geometry $\hat{g}$ contributed to the sum defining the decoherence functional. Then the remaining sum over ϕ in the functional integral defines a Quantum Field Theory (QFT) on the background spacetime $\hat{g}$, with the approximation

$$D(h_i, h_j) \simeq \int_{h_i, \mathcal{C}} \delta\phi \int_{h_j, \mathcal{C}} \delta\phi' e^{i(S[\hat{g},\phi]-S[\hat{g},\phi'])/\hbar} . \tag{6.7}$$

This is true if the action can be decomposed as $S = S(\hat{g}) + S_M(\hat{g}, \phi)$.

6.3 The need for a Wave Packet

Despite other desirable features (like inflation), the Hartle-Hawking (1983) "no boundary" proposal fails to address the issue of time and probability interpretation in a satisfactory manner. Any viable theory of quantum cosmology should be capable of naturally describing the emergence of classical spacetime. Kazama and Nakayama (1985) [91] argue that since the "no boundary" proposal does not exhibit a localized wave- packet structure, the argument of how classical behaviour can emerge out of the wave function is not convincing.

On the other hand, Vilenkin's prescription is of no use in models where there are no modes with outgoing flux only through the singular boundaries of Superspace, or where the flux turns around within Superspace and crosses only the nonsingular boundary [10]. If Vilenkin's condition is modified by choosing the phase S_n and pre-factor in such a way that the superposition $\Psi = \sum_n C_n e^{iS_n}$ is a wave packet, then (at least in the case of the conformally coupled scalar field considered in [10]) there are several possible wave functions for the Universe. It therefore seems that this proposal also needs to be improved.

The *Wave Packet proposal* for the wave function of the Universe [94, 10, 93] corresponds to a so called "final condition":

The quantum evolution must lead to the present classical Universe, i.e. the wave function of the Universe must approach a *Wave Packet* characterizing the presently observed cosmological data [10]. Also, the wave packet must go to zero as the scale factor grows to infinity (which means that the'returning' packets should be present'ab initio'). The wave packet then plays the role of a final condition from which we will *retrodict* the evolution of the Universe backwards in time.

Only if the wave function permits a probabilistic interpretation, and a wave packet can be constructed, can the gradient ∇S, of the classical action determine the classical trajectories (equation 2.23). This makes the *principle of constructive interference* indispensable. Consider superpositions of WKB solutions of the Wheeler-De Witt equation 2.20 which are of the form (compare with 2.25)

$$\Psi_n(a,\phi) = C_n(a,\phi)e^{-iS_n(a,\phi)} + \bar{C}_n(a,\phi)e^{iS_n(a,\phi)} ,$$

the pre-factors C_n and $\bar{C}_n$ being slowly varying amplitudes. These wave functions are extended all over configuration space. They interfere destructively everywhere except where the phase $S_n(a,\phi)$ has a saddle point with respect to the wave number n:

$$\left[\frac{\partial S_n}{\partial n}\right]_{n=\bar{n}} = 0 . \tag{6.8}$$

$S_{\bar{n}}(a,\phi)$ is a solution to the Hamilton -Jacobi equation 2.22 and yields classical trajectories in configuration space. Together with the principle of constructive interference, the general Hamilton-Jacobi equation corresponding to the Superspace Wheeler-De Witt equation 2.10 is equivalent to all 10 Einstein field equations [35]. The second derivatives $\partial^2 S_n/\partial n^2$ are a measure of the dispersion of the wave packet around the classical trajectories [93].

Since the Universe may be viewed as an isolated, individual system (in the sense described by Hartle [67, 54], see Chapter 3) there is a characteristic absence of an external observer. The so-called *relative state formalism of Everett* [26] was designed to deal with exactly this situation. If we regard the total system as composed of two subsystems, one the observing apparatus, the other the observed system, then the total wave function is just a superposition of eigenstates of the respective systems. All the possible results of measurement are contained in such a superposition.

It then becomes possible for a probability interpretation to emerge in a purely natural fashion; i.e. it is not something given *a priori* to the wave function. Kazama and Nakayama [58] then illustrates using a simple model due to von Neumann, that in the absence of an external observer, the emergence of classical spacetime requires the total wave function representing all the possible outcomes of measurement itself must be localized.

6.4 In search of a desirable time variable

In order to arrive at a good probability interpretation for wave packets in simple Mini-Superspace models, we need to specify a desirable time variable that will lead to conserved probability current. For instance, a bad choice would be the scale factor a in closed FRW models, since the wave function will be multi-valued with respect to a, and the semi-classical treatment will fail around the turning points. Matter-fields that are essential in driving the evolution of the scale factor a (such as scalar fields with chaotic potentials) do not qualify either:

To give a good probability interpretation for the wave packet, a good clock should not disturb or be disturbed by the system being observed in regions where the scale factor is large, i.e. it should decouple from the rest of the system. In addition, it should be monotonic with respect to the time t of the comoving frame. These features will guarantee that the square modulus of the wave packet is approximately conserved with respect to the desired time variable. An example of such a material clock is the homogeneous, isotropic and massless scalar field [91]. We shall see that this concept of a material clock is useful in the construction of wave packets for bulk matter wormholes (Chapter 9).

There are many other proposals about the problem of time in Quantum Gravity, the emergence of semiclassical spacetime and the question of the'arrow' of time ([90] and for a most recent and interesting approach [47]).

Chapter 7

Spacetime Wormholes

7.1 A survey on known solutions

Overview

We now turn our attention to one of *the* most interesting features in quantum gravity: the so- called *Wormhole.* These are gravitational instantons, i.e. exact solutions of the classical Euclidean Einstein field equations with *finite action.* Giddings and Strominger [43] and Hawking [81] were the first to introduce wormhole solutions in "canonical" Einstein gravity.

Semiclassical gravitational instantons joining two asymptotically flat manifolds in Mini-Superspace appear in [43, 1, 92, 102]; asymptotically flat space with a closed FRW universe [9, 101, 126], and a de Sitter space with a closed FRW or another de Sitter space [4, 17, 45, 57, 109, 124] have previously been found. Wormhole solutions have been discussed extensively in [7, 14, 95, 96, 48, 28, 18, 5, 6, 20, 21, 36] and [46, 108, 112, 37, 113, 131, 144, 60, 145]. Hawking and Page [85] and Campbell and Garay [88] have initiated investigations into the existence of quantum wormholes as solutions to the Wheeler-De Witt equation which satisfy appropriate asymptotic boundary conditions (Section 4).

The Yang-Mills instanton

Hosoya and Ogura [88] discovered a spherically symmetric classical wormhole solution of an SU(2) Yang Mills magnetic field coupled to gravity with a cosmological constant. Rey [124] studied a version with time dependent magnetic and electric fields. The wormhole solution is SO(4) symmetric, and describes a particle moving in a double well potential. The explicit analytic solutions are elliptic integrals, but a discrete set of wormholes exist for appropriate boundary conditions. The existence of a conserved energy density makes the spectrum of solutions similar to that of Giddings and Strominger [43] and Coleman and Lee [14]. See Section 3 for an outline of the Giddings-Strominger axionic wormhole.

The massive charged scalar field instanton

A minimally coupled charged scalar field was studied by Abbott and Wise [1], Coleman and Lee [14] and Lee [102]. Due to U(1) symmetry, the theory has a conserved current J^μ, that yields an associated conserved charge $Q = \int d\Sigma_\mu J^\mu$, integrated over a three-sphere containing the wormhole mouth.

If we restrict the model to be that of a massless Goldstone boson, an equivalence with the Giddings- Strominger [43] wormhole emerges. This is because the current is a vector density that is equivalent to a three-form in axionic theory: $J_\mu = \epsilon_\mu^{\alpha\beta\gamma} H_{\alpha\beta\gamma}$. The time-time component of the Einstein field equations essentially describes the motion of a particle in a repulsive a^{-4} potential, where a is. It comes from infinity and bounces off the barrier at the turning point (the minimum radius of the wormhole throat) and returns to infinity.

It appears that such wormholes may be able to simulate the formation and decay of blackholes: the size of the black hole collapsing under the collective

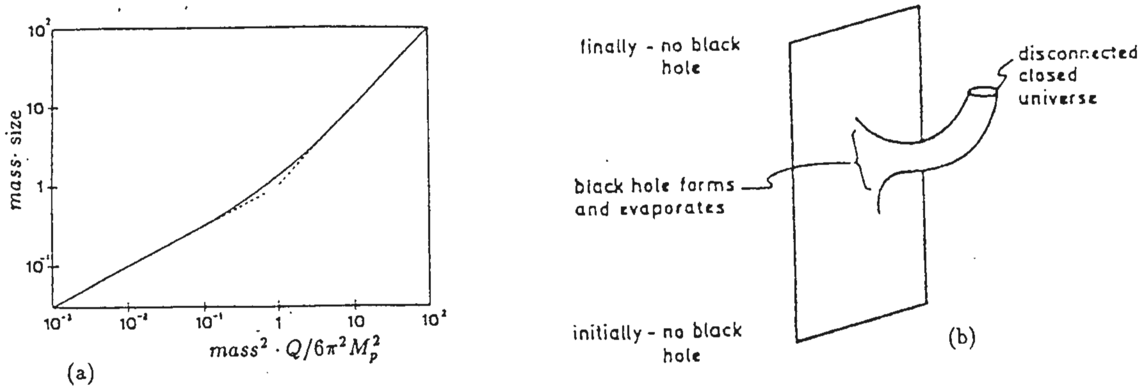

Figure 7.1: (a) Wormhole charge Q as a function of the wormhole size. The dotted curves show large-wormhole and small-wormhole limits. (b) An illustration of black hole evaporation.

mass m of Q mesons, is proportional to the charge Q (fig. 7.1). For large Q, the wormhole size grows to that of a black hole. The action corresponding to the insertion of a wormhole mouth into a region of constant background field f is found to be

$$I = -Q \ln \left[4f \left(\frac{G}{\pi} \right)^{1/2} \right] .$$

Double periodic wormhole solutions

Massive charged scalar field wormholes similar to the above were numerically analyzed by Midorikawa [109]. New boundary conditions to the same Einstein field equations yield single period instantons connecting two universes of the same size. The potential is restricted to have a local maximum at a finite value of the scalar field.

For a different potential (see fig. 7.2), a wormhole of double period connects two universes of different sizes. Such a double periodic solution implies

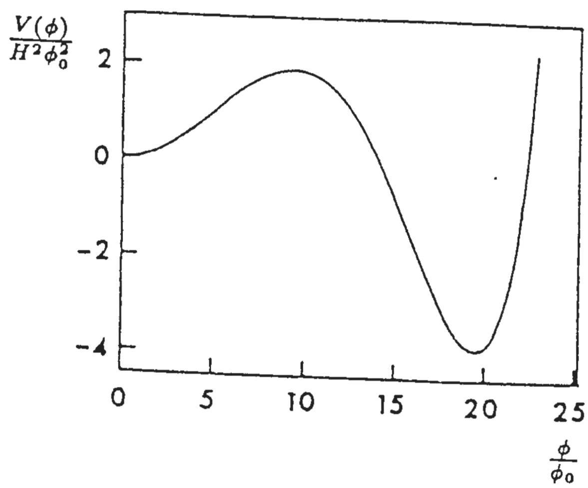

Figure 7.2: The potential V for double period instanton.

the creation of hot universe with a large cosmological constant from a cold universe with a small constant. The existence of universes with different Λ's may be useful for a large universe to evolve. Even if wormholes set small Λ to zero (the Coleman mechanism) in our Universe, the large Λ may stay finite.

Theories with axion *and* scalar fields

Lavrelashvili, Rubakov and Tinyakov [101] and Rubakov and Tinyakov [126] explored a theory containing a scalar field and an axionic field. They found a gravitational instanton whose analytic continuation is a closed expanding universe born at minimal radius and then undergoing inflation. There is a conserved axion charge present that lead to wormhole solutions for small radii. However, its contribution to the energy-momentum tensor decrease as a^{-4} (as in Giddings and Strominger [43]), so the universe quickly enters an inflationary phase as the scalar field undergoes damped oscillations.

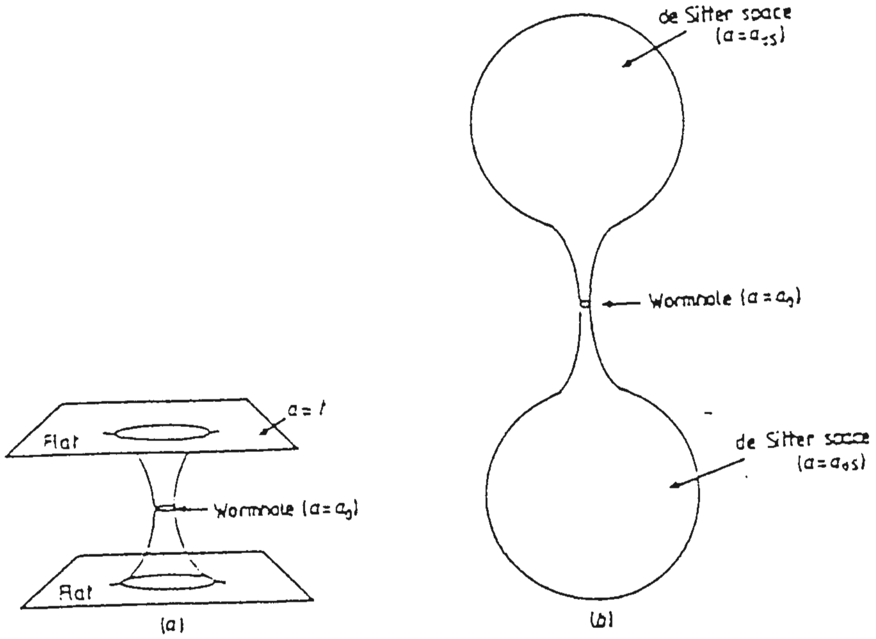

Figure 7.3: (a) A wormhole that connects two asymptotically flat Euclidean regions. Two dimensions are suppressed; each circle around the throat represents a three-sphere. (b) A wormhole connecting two de Sitter spacetimes.

Non-linear gravity coupled to axionic and scalar matter

Non-linear gravity wormhole instantons in the context of a theory containing additional scalar and axion fields were found by Coule and Maeda [17]. An antisymmetric tensor axionic field H is coupled to a scalar field with an arbitrary potential. Again the axion current H^μ is conserved, defining a quantized charge.

For an approximately flat, non-zero scalar potential there exists a wormhole with throat- radius a_0 which connects two asymptotically de Sitter spaces with radius a_{DS}, provided $a_0 \ll a_{DS}$. (see fig. 7.3). For zero scalar potential, the wormhole connects two flat regions $a = t$ as $\mid t \mid \rightarrow \infty$.

This theory is shown to be equivalent to a theory for a conformally coupled scalar field. Similar solutions exist for some generalized Einstein theories of gravity, e.g. a higher derivative gravity minimally coupled to an axion.

Wormhole solutions are also found for the case of a non-minimally coupled scalar in an effective theory derived from string theory.

7.2 The Hawking-Tolman wormhole

This is an asymptotically flat solution to a metric that does not satisfy the Einstein field equations. However, subsequent work [7,34] has shown that this wormhole is indeed a solution to the Einstein field equations. Gonzalez-Diaz considers pure gravity with a cut-off in the scale factor a. The same model has also been reproduced from a perfect fluid equation of state $p = \rho/3$ in [8, 9], and its quantum version occurs in Chapter 9 if $\gamma = 4/3$. We give a brief outline of the Hawking-Tolman wormhole:

It has a conformally flat metric

$$ds^2 = \left[1 + \frac{b^2}{|x - x_0|^2} \right]^2 dx^\mu dx_\mu \tag{7.1}$$

which is an asymptotic Euclidean metric that looks like it has a singularity at the point x_0. However, this is a mere coordinate singularity, with the regions $x^2 < b^2$ and $x^2 > b^2$ having similar geometry. The metric describes two asymptotically flat regions connected by a throat with radius $2b$ at the three-sphere (see fig. 7.4), also known as a *baby universe*. Typically, b will be of the order of the Planck length, so when the separation of the two ends is much greater than the Planck length, we may neglect their interaction.

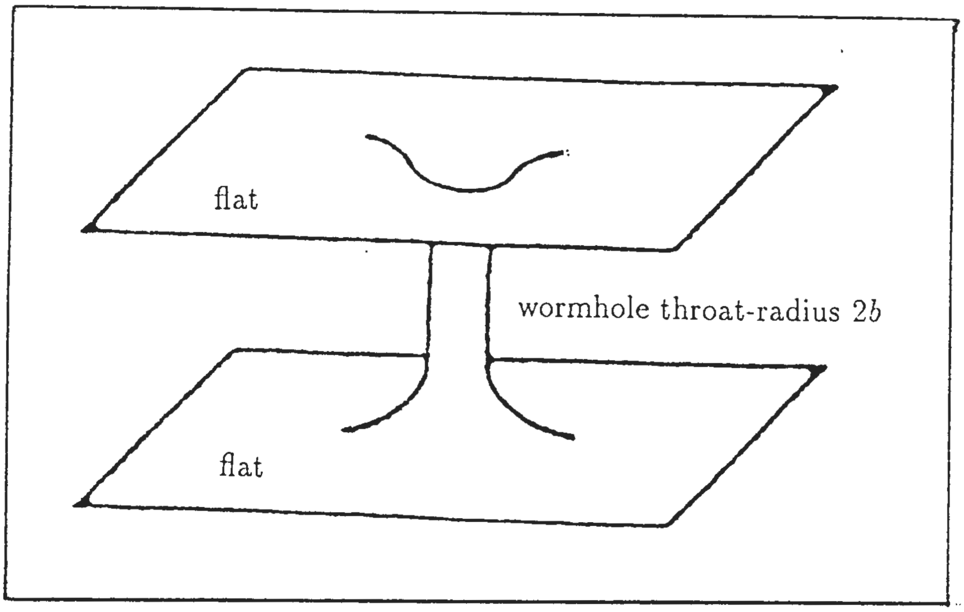

Figure 7.4: The Tolman-Hawking wormhole with throat-radius $2b$ connects two asymptotically flat Euclidean regions.

The metric is not a solution of the Einstein equations since $R_{\mu\nu} \neq 0$ although the Ricci scalar $R = 0$. The total gravitational action has its only contribution from the boundary term

$$S_b = -\frac{1}{8\pi G}\int d^3x\sqrt{h}(\ K - K_0\ ) = \frac{3\pi b^2}{G}$$

where K is the trace of the extrinsic curvature of the boundary, and K_0 that of the boundary embedded in flat space.

It was shown by Gonzalez-Diaz [45] that the above wormhole solution can be obtained in a pure gravity Mini-Superspace model with a positive cosmological constant, provided a cut-off in the scale factor is introduced. For the Euclidean metric

$$ds^2 = N_e^2 d\tau^2 + a^2(\tau)d\Omega_3^2$$

the action reads

$$I_e = -\frac{1}{16\pi G} \int d\tau N_e a \left[1 + \frac{\dot{a}^2}{N^2} - a^2 \Lambda \right] . \tag{7.2}$$

For a constant m, the transformation

$$a^2 \rightarrow a^2 - m^2$$

is equivalent to having a minimum radius m for the Euclidean three- sphere. The new time-coordinate is $dt = (1 - \frac{m^2}{a^2})^{1/2} d\tau$. For conformal time $d\eta = \frac{d\tau}{a}$, and defining $a' = \frac{da}{d\eta}$, the equations of motion have

$$\frac{1}{2} a'^2 + W(a,m) = 0 \tag{7.3}$$

$$a'' = -\frac{\partial W}{\partial a} \tag{7.4}$$

with

$$W(a,m) = \frac{1}{2} \left[m^2 (1 + m^2 \Lambda) - (1 + 2m^2 \Lambda) a^2 + \Lambda a^4 \right] . \tag{7.5}$$

This may be viewed as describing the motion of a particle of zero energy in the potential W. With $\Lambda = 0$ we have

$$a = (m^2 + \tau^2)^{\frac{1}{2}}$$

representing two asymptotically flat regions connected by a Tolman-Hawking wormhole of radius m. For $\Lambda > 0$, periodic wormhole solutions

$$a = \Lambda^{-1/2} \left[m^2 \Lambda + cos^2 (\Lambda^{\frac{1}{2}} \tau) \right]^{\frac{1}{2}} \tag{7.6}$$

occur in the region

$$m < a < (m^2 + \Lambda^{-1})^{\frac{1}{2}} .$$

In the Lorentzian framework ($\tau \rightarrow \pm i\tau$), this represents a Tolman universe with maximum radius m and a de Sitter universe with minimum radius $\sqrt{m^2 + \Lambda^{-1}}$. The instanton describes the tunneling between these two classical regions.

Numerical calculations have revealed that a conformal scalar field (see Halliwell and Laflamme [57]) in Mini-Superspace may have less physical significance due to a negative effective gravitational constant $\tilde{G} = (\ 1 - \phi^2\ )^{-1} G$, where ϕ is restricted to values greater than one. Starobinsky [129] has suggested that there may exist bounded regions where $G = const. > 0$ in a more detailed analysis that includes anisotropies.

7.3 The Giddings-Strominger axionic wormhole

An axionic field minimally coupled to gravity has Euclidean action

$$S = \frac{1}{16\pi G} \int d^4x \sqrt{g}\ (\ -R + H^2\ ) + (\textit{topological and boundary terms})\ . \tag{7.7}$$

The 3-form $H \doteq dB$ is the axion strength such that $dH = 0$. One may now derive the Einstein field equations

$$G_{\mu\nu} = 3 H_{\mu\alpha\beta} H_{\nu}^{\alpha\beta} - \frac{1}{2} g_{\mu\nu} H_{\alpha\beta\gamma} H^{\alpha\beta\gamma}\ , \tag{7.8}$$

$$\nabla_{[\mu} H_{\alpha\beta\gamma]} = 0\ . \tag{7.9}$$

Giddings and Strominger [43] make the spherically symmetric ansatz

$$ds^2 = dt^2 + a^2 d\Omega_3^2\ , \tag{7.10}$$

$$H_{ijk} = b^2 \epsilon_{ijk} \tag{7.11}$$

with Euclidean FRW-metric scale factor a, while ϵ_{ijk} is the volume element normalized to integrate to $2\pi^2$ on surfaces of constant a. All other components of H vanish. The axion current H^μ is now conserved, allowing one to define an axionic charge flow down the wormhole,

$$Q = \frac{1}{G} \int_{\Sigma(b)} H d\Omega_3 = \frac{2\pi^2 b^2}{G} \tag{7.12}$$

if the three-surface Σ, of radius b encloses the origin $t = 0$. The time-time component of the equations of motion (1.8) depends crucially on the charge. The equation is

$$\dot{a}^2 - 1 = -\frac{3b^4}{a^4}\,. \tag{7.13}$$

Its solution in parameterized form reads

$$a^2 = b^2 \cosh 2\eta \tag{7.14}$$

where

$$t \doteq b \int \sqrt{\cosh 2\eta}\; d\eta\,. \tag{7.15}$$

The Euclidean metric is invariant under the transformation $a \rightarrow -a$, so it represents two asymptotically flat regions as $\mid a \mid \rightarrow \infty$ that are connected by a throat with minimum radius b and three-sphere cross-sections. The extrinsic curvature K of the boundary at minimum throat-size b is zero. The wormhole instanton describes tunneling between an initial three-surface Σ_i of topology R^3, and a final surface Σ_f of topology $R^3 \oplus S^3$ (see fig. 7.5).

The instanton action reads

$$S = \frac{3\mid Q \mid}{8}\,, \tag{7.16}$$

so that nucleation of closed baby FRW universes are suppressed for large maximum radii $b = \sqrt{\frac{G|Q|}{2\pi^2}}$ large relative to the Planck size. The fields and their first derivatives on Σ_i and Σ_f are real when analytically continued back to the Lorentzian regime. This is obvious for R^3, but on S^3 this is ensured by

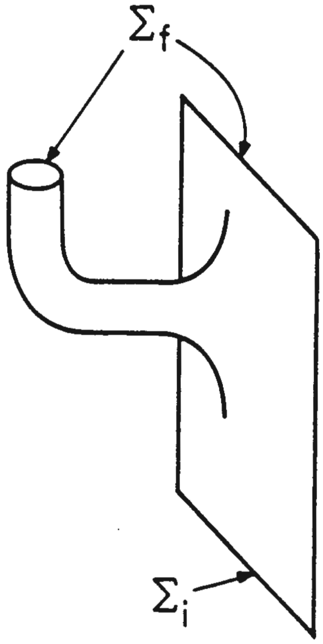

Figure 7.5: Tunneling from a topology R^3 initial geometry Σ_i to a topology $R^3 \oplus S^3$ final geometry Σ_f.

the ansatz that the time components of H vanish, while the time derivative of the metric vanishes because it is a minimal surface.

7.4 The Wheeler-De Witt equation

In an effort to find a more general class of wormholes, Hawking and Page [85] and Campbell and Garay [7] regarded wormholes as full quantum solutions of the 2^{nd} quantized Wheeler-De Witt equation. This is crucial to finally provide a mechanism for black hole evaporation suggested by Hawking [81], due to a lack of macroscopic wormhole instantons with arbitrary matter content. It should also facilitate the construction of a more fundamental theory of topological fluctuations in gravity.

The boundary conditions required for the wave function solution to represent an asymptotically flat wormhole will be reviewed in Chapter 8. We also derive exact wave functions for wormholes for the free massive scalar field.

In Chapter 9 we proceed in this new programmme by finding the quantum analogue of the FRW bulk matter instantons found by Carlini [8] and Carlini and Mijić [9].

7.4.1 Wormhole wave functions

Hawking-Page

In the ansatz $ds^2 = N_e^2 dt^2 + a^2 d\Omega_3^2$, Hawking and Page [85] solve the WDW equation for a minimally coupled massless scalar field ϕ by means of the separation

$$\Psi = c(a) e^{ik\phi}$$

where c(a) satisfies

$$\left[\frac{d^2}{da^2} + \frac{1}{a}\frac{d}{da} + \left(\frac{k^2}{a^2} - a^2 \right) \right] c(a) = 0 \; . \tag{7.17}$$

This has two independent solutions $J_{\pm \frac{ik}{2}}\left(\frac{ia^2}{2} \right)$. These are eigenstates of the operator $-i\frac{\partial}{\partial\phi}$ with eigenvalue k, and carry a conserved charge $Q = 2\pi^2 k$. This continuous set of solutions oscillates for $0 < a < k^{\frac{1}{2}}$, and correspond to classical Lorentzian FRW solutions with scalar flux Q, bouncing between a singularity and a sphere of maximum radius $k^{\frac{1}{2}}$.

For $a > k^{\frac{1}{2}}$, Ψ decreases like $e^{-a^2/2}$. There appears to be an irregularity as $a \to 0$, but by a coordinate transformation $x = a \sinh\phi$ and $y = a\cosh\phi$, one can derive a discrete spectrum

$$\Psi = \sum_n \Psi_n(x)\Psi_n(y)$$

with $\Psi_n(x) = H_n(x)e^{-x^2/2}$ (H_n are Hermite polynomials). Then each member of the spectrum is just a product of harmonic oscillator wave functions with the same energy, and therefore regular at the origin.

The Killing vector to the WDW equation, $\partial_\phi = y\partial_x + x\partial_y$ can be expressed in terms of harmonic creation and annihilation operators $a, a^\dagger$, as

$$\partial_\phi = a_x a_y - a_x^\dagger a_y^\dagger$$

so that the ∂_ϕ eigenstates $|k>$ is a sum of harmonic eigenstates $|n>$:

$$|k>= \sum_n c_n(k)|n> ,$$

with c_n satisfying the recursive relation

$$ikc_n = (n+1)c_{n+1} - nc_{n-1}$$

which can be solved iteratively in terms of hypergeometric functions. So the eigenstates $|k>$ are superpositions of regular harmonic oscillators that are regular everywhere and damped at infinity. A similar result is found for the case of a conformally invariant scalar field in [85, 94, 96].

Kantowski-Sachs

Campbell and Garay [7] study a spacetime that has the same metric

$$ds^2 = N_e^2 d\tau^2 + a^2 dr^2 + b^2 d\Omega_3^2$$

as that of the interior of a Schwarzschild black hole. A more general form for the operator-ordering is considered. Two kinds of wormhole solutions are studied, one with the asymptotic behaviour $R^3 \otimes S^1$ ($a \rightarrow a_0$, $b \rightarrow \tau$) and asymptotic ground state $\Psi \approx e^{-ab}$, and the other $R^2 \otimes S^2$ ($a \rightarrow \tau, b \rightarrow b_0$) and ground state $\Psi \approx e^{-a^2/4}$. Regular solutions are found by a Fourier transform of the explicit continuous ones, and reads

$$\Psi_{\lambda_0 \theta_0} = \exp\left(-ab\cosh\left(cos\theta_0 log\frac{a}{r_0} + \phi sin\theta_0 + \lambda_0\right)\right),$$

with constants θ_0,r_0 and excitations λ_0 of the wormhole state. For $R^3 \otimes S^1$ solutions, $\lambda_0 = 0$ gives a continuous set of degenerate ground states, while $\lambda_0 \neq 0$ gives excited states. However, in the case of $R^2 \otimes S^2$ solutions, $\lambda_0 = \theta_0 = 0$ is the only (ground) state and it corresponds to pure gravity.

Chapter 8

Wormholes in Superspace

8.1 Exact HP wormhole states

Hawking and Page (1990) [85] argue that wormholes are to be regarded as solutions of the quantum-mechanical Wheeler-De Witt Equation. The boundary conditions that these wave functions have to obey are that they be exponentially damped for large three-geometries, and regular when the three-geometries collapse to zero.

They found a continuous family of solutions with a massless scalar field, and of a conformal field, that correspond to instanton solutions found by Giddings and Strominger (1988) [42] . These wave functions are damped at infinity, but they oscillate infinitely near zero radius. The trick is to express such solutions as an infinite sum of a discrete family of solutions that are well-behaved both at infinity and zero radius.

Furthermore, well-behaved solutions were constructed only approximately for a massive scalar field. Explicit formulas for their asymptotic form were given.

As pointed out by C.Kiefer (1988) [93], the WKB approximation for the (Hawking-Page Wormhole) wave function breaks down near the turning point of the potential, i.e. as we approach the wormhole throat, thus making it difficult to construct wave packets following classical trajectories in such regions. Classical trajectories are shown to be represented by non- overlapping wave packets *only* for discrete values of the mass of the scalar field, and only in regions which are not too close to the turning point.

Kiefer [93] investigates the correspondence of Mini-Superspace quantum gravity with classical cosmology. He uses a Born-Oppenheimer type approximation to explicitly construct generalized coherent states in the case of a massive scalar field. Coherent states are known to be important to relate quantum theory to classical physics.

In this chapter (Section 3), we derive the exact solutions to the WDW equation for the massive scalar field. We also observe that they are regular everywhere, and are damped at infinity. This confirms Hawking and Page [85]. It shows that such solutions exist only for *discrete* values of the mass of the scalar field, consistent with Kiefer [93].[1]

[1] Also see Page and Kim (1992) [95].

8.2 Wormhole Representation

We consider the possibility of wormholes as solutions to the Wheeler-De Witt equation

$$H\Psi = 0 \tag{8.1}$$

obeying certain boundary conditions.

If we regard S as a cross-sectional three-surface of a wormhole that separates two asymptotically Euclidean regions, then the quantum states of a wormhole can be represented by the wave functions $\Psi_n(h_{ij}, \phi)$ where h_{ij} is the three-metric and ϕ the matter-fields on S. The wave functions obey equation 8.1 at all finite non-zero three-metrics h_{ij}.

If the wave functions $\Psi_n(h_{ij}, \phi)$ are to correspond to wormholes they should obey certain boundary conditions :

a) The boundary condition when h_{ij} is large should express that the four-metric is asymptotically Euclidean. Unlike the case of the No Boundary wave function which grows with the size of the three-surface, the wormhole wave function will be damped at a large three-surface.

b) The boundary condition when h_{ij} is small should indicate that the four-metric is non-singular. In Mini-Superspace models it means that the wave function should be regular, or go as a power of the scale factor a as a approaches zero.

Specifically, in the case of the Mini-Superspace model with the usual FRW four-metric (2.15)

$$ds^2 = [\, -N^2(t)dt^2 + a^2(t)d\Omega_3^2\,] \tag{8.2}$$

here $d\Omega_3^2$ is the metric of a three-sphere of unit radius, real N is the lapse of a Lorentzian metric for a Friedmann universe. If N is imaginary, the metric is that of an Euclidean wormhole (i.e. an instanton).

The No Boundary wave function (4.16) of Hartle and Hawking,

$$\Psi(h_{ij},\phi) = \int d[\,g_{\mu\nu}\,]d[\,\phi\,]e^{-I[\,g,\phi\,]} \tag{8.3}$$

is a path integral over all compact metrics and matter fields with the appropriate boundary values.

It increases as $e^{\frac{1}{2}a^2}$ where a is the radius of the three-surface S. The wormhole wave function decreases like $e^{-\frac{1}{2}a^2}$ for large a. The latter case indicates that such solutions are asymptotically Euclidean, and the ground state wormhole corresponds to a vacuum state.

In the path integral formulation, the wormhole ground state is therefore a path integral over all asymptotically Euclidean metrics and all asymptotically zero matter fields that have the given values on the surface S. Excited states of the wormhole are other solutions to the WDW equation that are damped at large radius and regular at $a = 0$. Regularity at the origin indicates that these solutions are nonsingular.

8.3 Quantum Wormholes

8.3.1 The minimally coupled massive scalar field

In the case of a closed Friedmann universe with scale factor a and metric (8.2) containing a homogeneous massive scalar field ϕ the WDW equation (8.1) reads

$$\left(a^{2-p}\frac{\partial}{\partial a}a^p\frac{\partial}{\partial a} - \frac{m_p^2}{16\pi}\frac{\partial^2}{\partial\phi^2} + \frac{m_p^2}{16\pi}m^2\phi^2a^6 - (\frac{m_p^2}{16\pi})^2ka^4\right)\Psi(a,\phi) = 0 \tag{8.4}$$

where the matter field potential is $m^2\phi^2$ and the curvature of the space-time closed ($k = +1$), flat ($k = 0$) or open ($k = -1$). We for the time being write

$\frac{m_p^2}{16\pi} = 1$ and recover it later on.

Kiefer [93] obtains approximate wave function solutions by means of an adiabatic approximation in the Born-Oppenheimer ansatz for $\ln a$, a technique used in Molecular Physics and also previously in Quantum Gravity. In Kim[96] and Kim and Page[95] we see that the wave functions can be expanded by a basis of eigenfunctions. However, the Symanzik scaling law allows for a suitable choice of coordinates (a, η) where

$$\eta = \frac{1}{2}\phi^2 a^3 \tag{8.5}$$

by which the wave function Ψ becomes separable:

$$\Psi_\lambda(a, \phi) = \psi_\lambda(\eta)\Phi_\lambda(\eta) \tag{8.6}$$

with separation constant λ, while the WDW equation separates into

$$\left(2\eta\frac{d^2}{d\eta^2} + \frac{d}{d\eta} + \lambda - 2m^2\eta\right)\Phi_\lambda(\eta) = 0 \tag{8.7}$$

$$\left(\frac{1}{a}\frac{d^2}{da^2} + \frac{p}{a^2}\frac{d}{da} + \lambda - ka\right)\psi_\lambda(a) = 0 \tag{8.8}$$

We solve for $\Phi_\lambda(\eta)$ by writing

$$\Phi_\lambda(\eta) = e^{-m\eta}y_\lambda(\eta) \tag{8.9}$$

so that $y_\lambda(x)$, where $x = 2m\eta$, satisfies

$$\left(x\frac{d^2}{dx^2} + (\frac{1}{2} - x)\frac{d}{dx} + \frac{\lambda - m}{4m}\right)y_\lambda(x) = 0 \tag{8.10}$$

The general solution is a combination of Kummer functions

$$\begin{aligned}\Phi(\eta) &= e^{-m\eta}U[\,(m-\lambda)/4m, 1/2; 2m\eta\,] \\ &+ \Phi_0\, e^{-m\eta}M[\,(m+\lambda)/4m, 1/2; -2m\eta\,]\end{aligned}$$

The "final" boundary condition for wormholes Kiefer (1990) [94] is that the wave function decays to zero exponentially as the scale factor, hence η, goes to infinity. This compels the constraint $\Phi_0 = 0$ in the complete solution. Since the second term is exponentially increasing it is more appropriate to the Hartle-Hawking No Boundary proposal.

This is just another form of the Hermite equation, so that for appropriate normalization $N_n = (2^n N!)^{-\frac{1}{2}}$, the solutions $y(\eta)$ are normalized Hermite polynomials

$$y_n(\eta) = N_n H_n[\, 2\sqrt{m\eta}\,] \tag{8.11}$$

provided $\lambda = (2n+1)m$ for $n = 0, 1, 2, \ldots$ Therefore

$$\Phi_n(\eta) = e^{-m\eta} N_n H_n[\, 2\sqrt{m\eta}\,]\,. \tag{8.12}$$

This is an exact eigenfunction that is equivalent [modulo prefactor $(ma^3)^{1/4}$] to the adiabatic solutions obtained in the Born- Oppenheimer ansatz [93] and the Symanzik scaling law (see Kim [96]). In addition, differential equation 8.8 in the scale factor a is the zero-energy Schroedinger equation for the wave function $\psi_\lambda(a)$ with potential $ka^4 - \lambda a^3$. In the WKB approximation [93] with factor-ordering p $= +1$,

$$\psi_n(a) \sim [\lambda a^3 - ka^4]^{-1/4}\,\cdot$$

$$\cos\left[\left|(\frac{a}{2} - \frac{\lambda}{4})[\lambda a - a^2]^{1/2} - \frac{\lambda^2}{8}[\arcsin(1 - \frac{2a}{\lambda}) + \frac{\pi}{2}]\right| - \frac{\pi}{4}\right]. \tag{8.13}$$

However, these solutions break down near the turning point $a_n = \lambda_n = (2n+1)m$ for closed universe models with maximum radius a_n. That is, the wave functions do not appear to be regular there. So do they represent wormholes?

The answer is yes, and we prove this by first showing that the gravitational contribution to the wave function is regular everywhere:

a) First of all, as we approach the origin for small values of the scale factor a, the factor-ordering p becomes important since the Ricci scalar curvature R grows bigger than the Planck curvature $m_p^2/16\pi$. We may then approximate equation 8.8 by neglecting the curvature term. It is then convenient to redefine the wave function

$$\psi_\lambda(a) \approx a^{(1-p)/2}\,\theta_\lambda(a)$$

so that

$$\left[a^2\frac{d^2}{da^2} + a\frac{d}{da} + \lambda a^3 - \frac{1}{4}(\,1-p\,)^2 \right]\,\theta_\lambda(a) = 0\,. \qquad (8.14)$$

We may then express the solutions in terms of a sum of Bessel and modified Bessel functions, in the process substituting $\lambda_n = (\,2n+1\,)m$:

$$\begin{aligned}\psi_n(a) \;&\approx\; A_1 a^{\frac{(1-p)}{2}}\, J_{\pm\frac{(1-p)}{3}}\left[\frac{2}{3}\sqrt{(2n+1)\,ma^3}\right] \\ &\quad + A_2 a^{\frac{(1-p)}{2}}\, Y_{\pm\frac{(1-p)}{3}}\left[\frac{2}{3}\sqrt{(2n+1)\,ma^3}\right]\,.\end{aligned}$$

We now see that for fixed n and p, and for the *plus sign* in this solution, the wave function has limiting form

$$A_3\frac{a^{(1-p)}}{2^{(1-p)/3}\Gamma[\,1+(1-p)/3\,]} - A_4\frac{2^{(1-p)/3}\Gamma[\,(1-p)/3\,]}{\pi}$$

(A_1, A_2, $A_3\, and\, A_4$ are constants in p, n and mass m) which is regular as the scale factor $a \to 0$ for any value of the factor-ordering $p < 1$. The specific case of $p = +1$ is trivial provided $A_2 = 0$, since the modified Bessel function Y_0 scales like $3\ln a/\pi$ in this limit.

b) Secondly, as the scale factor a increases away from the origin, the factor-ordering ambiguity becomes less significant. The WKB approximation wave packet (8.13) is generic only for closed universe models with turning points a_n much larger that the Planck radius l_p. Apart from the multiple integral formulation (Kim [96]), equation 8.8 lacks an explicit closed-form solution for arbitrary factor-ordering. We are able to construct an exact spectrum of states that is regular everywhere, by simply choosing the factor-ordering $p = 0$ with positive curvature $k = 1$. Now the equation for $\psi_n(a)$ reads

$$\left(\frac{d^2}{da^2} + (2n+1)ma - a^2\right)\psi_n(a) = 0 \tag{8.15}$$

If we now put $z = a - (n + \frac{1}{2})m$, the wave functions $\psi_n(a)$ are found to be (confluent hypergeometric) Kummer functions

$$\begin{aligned} &{}_1F_1\left(\frac{1}{4}[\,1-(n+1/2)^2m^2\,]\,,\,\frac{1}{2}\,;\,z^2\right)e^{-z^2/2} \\ +A_5\,&{}_1F_1\left(\frac{1}{4}[\,3-(n+1/2)^2m^2\,]\,,\,\frac{3}{2}\,;\,z^2\right)e^{-z^2/2}z\,, \end{aligned}$$

which are infinite series in $\frac{z^2}{2}$, but may be expressed as Hermite polynomials, provided that the mass is discrete. That is,

$$m^2 = 4r \tag{8.16}$$

for odd integers r (on recovering Planck-units $m^2 = \frac{m_P^2}{4\pi}r$). We arrive at a spectrum of harmonic oscillator wave functions

$$\psi_n(a) = e^{-\frac{1}{2}[a-(n+\frac{1}{2})m]^2}N_{\frac{1}{2}[(n+\frac{1}{2})^2m^2-1]}H_{\frac{1}{2}[(n+\frac{1}{2})^2m^2-1]}[\,a-(n+\frac{1}{2})m\,]. \tag{8.17}$$

So for each non-negative integer n, there exists a regular and exponentially damped wormhole state

$$\begin{aligned} \Psi_n(a,\phi) &= e^{-\frac{1}{2}[a-(n+\frac{1}{2})m]^2}N_{[(n+\frac{1}{2})^2m^2-1]}H_{[(n+\frac{1}{2})^2m^2-1]}[\,a-(n+\frac{1}{2})m\,] \\ &\quad\cdot e^{-\frac{1}{2}m\phi^2a^3}N_nH_n[\,\sqrt{2m\phi^2a^3}\,] \end{aligned}$$

The wormhole wave function is a discrete spectrum of such states :

$$\Psi(a,\phi) = \sum_{n=0}^{\infty} C_n \Psi_n(a,\phi) \tag{8.18}$$

for constant coefficients C_n. Another such spectrum exists if the factor-ordering $p = 2$, when we simply replace C_n by $\frac{C_n}{a}$. But this could mean that Ψ and its derivatives blow up as $a \to 0$.

8.4 The power-law potential $\frac{\kappa}{2q}\phi^{2q}$.

If we consider two-dimensional Mini-Superspace containing a homogeneous scalar field ϕ with a power-law potential $\frac{\kappa}{2q}\phi^{2q}$, the Wheeler-De Witt equation 2.20 takes the form

$$\left(a^{2-p}\frac{\partial}{\partial a}a^p\frac{\partial}{\partial a} - \frac{\partial^2}{\partial\phi^2} + \frac{\kappa}{2q}\phi^{2q}a^6 - ka^4 \right)\Psi(a,\phi) = 0\ . \tag{8.19}$$

Once again, we do not expand Ψ by a basis of eigenfunctions ([95, 96]), but in principle there exists a transformation between the class of solutions found by Kim [96] and our derivation. In general we define a new variable by means of the Symanzik scaling law as

$$\eta = \phi a^{\frac{3}{1+q}}$$

for q positive (see [95, 96]). Equation 8.19 then transforms as

$$\left[\frac{\partial^2}{\partial\eta^2} - \frac{\kappa}{2q}(\eta)^{2q} - a^{-\frac{6}{1+q}}\left(a^{2-p}\frac{\partial}{\partial a}a^p\frac{\partial}{\partial a} - ka^4 \right) \right]\Psi(a,\eta) = 0\ . \tag{8.20}$$

For separation constant λ, we may split this into

$$\left[\frac{\partial^2}{\partial\eta^2} + \lambda - \frac{\kappa}{2q}\eta^{2q} \right]\Phi(\eta) = 0\ , \tag{8.21}$$

and

$$\left[\frac{\partial^2}{\partial a^2} + \frac{p}{a}\frac{\partial}{\partial a} + a^2\left(\lambda a^{2(\frac{1-2q}{1+q})} - k \right) \right]\psi(a) = 0\ . \tag{8.22}$$

In principle these equations can be solved simultaneously. We were able to solve these two equations exactly for $q = +1$ and factor-ordering $p = 0$ in the previous section. But in general, (i.e. for $q = 2, 3, 4, ...$) this is rather intricate. The multiple integral formulation of Kim[96] solves equation 8.19 exactly by writing it as a system of infinitely many linear differential equations.

8.5 Conclusion

Lorentzian WKB wave packets were constructed for large n in Kiefer [93] by applying an appropriate boundary condition to approximate wave functions solutions of equation 8.4. However, these wave packets are badly behaved at the turning-points. Hawking and Page [85] used an asymptotic formulation to show that the wave functions $\Psi_n(a, \phi)$ are indeed well-behaved everywhere, while Kim [96] used a multiple integral formulation to derive the general solutions.

By means of a relatively simple coordinate transformation similar to Kim [96] and Page and Kim [95] we are able to obtain an exact spectrum of wormholes. This occurs under condition that the mass is a discrete multiple of the Planck mass, consistent with Kiefer's [93] approximate result. It therefore appears that the quantization of mass is a necessary requirement to construct quantum states for microscopic closed universes in the case of a free massive scalar field.

We also observe the significant role that the factor-ordering ambiguity plays for small radii. Since the Hawking-Page boundary condition requires either regularity in the limit of zero radius, or that the wave function go as a power of a, depending on the factor-ordering p, then at least in this sense our results are validated.

Chapter 9

Quantum Carlini-Mijić Wormholes

9.1 Closed bulk matter universes

Carlini (1992) [9] explored the fact that spacetime wormholes may be understood as analytic continuation of closed expanding universes. For every classical solution in standard cosmology with closed spatial geometry (k = +1) and a real scalar field ϕ that obeys the strong energy condition $\rho+3p > 0$, there is a wormhole instanton.

This was achieved by means of the Ellis and Madsen (1990) [107] procedure for solving the Einstein field equations, after which both the lapse N and the scalar field ϕ are Wick rotated to the Euclidean sector. This is perfectly consistent with the reality of the path integral at one loop, although the asymmetric rotation for the lapse in the gravitational and matter part of the action (Carlini and Mijić (1990) [9]) seems rather *ad hoc.* They find an infinite class of new instantons which also includes the Hawking and Giddings-Strominger wormholes as specific cases.

In order for wormholes to solve the problem of the cosmological constant *and* provide the mechanism for black hole evaporation, Hawking and Page [85] have proposed that wormholes are solutions to the Wheeler De Witt equation (see Chapter 7 and 8). For this reason it is essential that the class of bulk matter wormholes found by Carlini and Mijić [9] are predicted by quantum cosmology. We show that this is in fact the case under the condition that the wave function for wormholes satisfy the "final condition" for *wave packets* (see Kiefer [94, 93]).

Since the Wheeler-De Witt equation is independent of the lapse N, we are able to find wormhole solutions without having to invoke the *asymmetric* analytic continuation described in Carlini [8] and Carlini-Mijić [9]. It also becomes clear that the matter- field representation of the perfect fluid bulk-matter source outlined in Madsen and Ellis [107], in terms of a scalar field ϕ does not immediately yield a desirable time variable for a good probability interpretation. Instead we have to introduce a new "bulk matter field" ξ to serve as a material clock. We are able to construct wave packets that are strongly peaked along pencils of configuration space paths corresponding to the closed bulk matter universes of Carlini-Mijić [9].

The Lorentzian metric with the so-called Carlini-Mijić lapse $N \cdot a^{(4-3\gamma)/2}$ reads

$$ds^2 = -N^2 a^{4-3\gamma} d\tau^2 + a^2(\tau) d\Omega_3^2 \,. \tag{9.1}$$

We have put $\frac{2G}{3\pi} = 1$. The lapse constant N^2 is fixed with respect to the time τ, normally gauge equivalent to unity in the Lorentzian framework; in their analytic continuation scheme, CM [9] defines a Euclidean lapse constant $N_e^2 = -\,N^2$. The line-element $d\Omega_3^2$ is defined on a three-sphere (k = +1).

Our interest lies in classical closed models, for which we will derive the corresponding WDW equation. Consider a bulk matter source with perfect fluid equation of state

$$p = (\gamma - 1)\, \rho \tag{9.2}$$

with pressure p and energy density ρ. The Ellis-Madsen [107] procedure for solving the Einstein field-equations for the scale factor and scalar field via such a source requires solutions to

$$H^2 = \rho - ka^{-2}\ , \tag{9.3}$$

where H is the Hubble parameter and ρ the energy density for a perfect fluid source

$$\rho = \rho_m a_m^{3\gamma}\, a^{-3\gamma}\ . \tag{9.4}$$

The strong-energy condition requires $\gamma > 2/3$. For closed models $k = +1$. By the Ellis-Madsen [107] procedure, we may define a scalar field ϕ such that the energy density is the sum of kinetic and potential energy :

$$\rho = \frac{a^{3\gamma-4}}{2N^2}\, (\frac{d\phi}{d\tau})^2 + V(\phi)\ . \tag{9.5}$$

The conservation of energy requires

$$\rho^{-1} \frac{a^{3\gamma-4}}{N^2}\, (\frac{d\phi}{d\tau})^2 = \gamma = constant. \tag{9.6}$$

This leads to the scalar field evolution

$$\frac{d\phi}{d\tau} = \pm N \sqrt{\gamma \rho_m a_m^{3\gamma}}\ a^{2-3\gamma}\ , \tag{9.7}$$

where a_m represents the maximum radius for a particular closed universe. In the gauge $N = 1$ we can now solve the Friedmann equation for the scale factor from equations 9.3 and 9.4:

$$a(\tau) = [\ a_m^{3\gamma-2} - (1 - \frac{3\gamma}{2})^2\, \tau^2\ ]^{\frac{1}{3\gamma-2}}. \tag{9.8}$$

This in turn allows us to evaluate the scalar field ϕ in terms of the time variable τ (we set the integration constant $\phi_0 = 0$)

$$| \phi(\tau) | = \frac{\sqrt{2\gamma}}{3\gamma - 2} \tanh^{-1} \left[\frac{(3\gamma - 2)}{2a_m^{\frac{3\gamma}{2}-1}} \tau \right] . \tag{9.9}$$

From the energy density 9.4 we proceed to define a *bulk matter potential*

$$V(a) = V_m a^{-3\gamma} \tag{9.10}$$

where the constant

$$V_m = (1 - \frac{\gamma}{2}) a_m^{3\gamma - 2} . \tag{9.11}$$

These results reflect the fact that the perfect fluid representation in terms of an equation of state 9.2 with constant γ and equation 9.6 allows us to *impose* the kind of behaviour we want the model universe to exhibit. In principle, the general form of the scalar field potential that will lead to our choice of solutions to the field equations, can also be determined. This is precisely the point that Madsen and Ellis [107] demonstrates. To fix γ is equivalent to selecting one feature of the complete quantum theory such as, for instance, a massive scalar field where γ varies between 0 and 2. Since we are already aware of the nature of the classical solutions, we say that the wormhole is "on shell". We therefore anticipate that solutions to the corresponding Wheeler-De Witt equation exist only in the dilute-wormhole approximation.[1]

[1] A. Carlini has indeed pointed out to me that it could be dangerous to adopt this procedure for the potential at the quantum level. Instead, it should be interesting to construct the action 9.12, without a prior relation between the scalar field ϕ and the potential $V(a)$. I.e. we abandon any identification with the classical CM wormholes. This means that we work in some sort of 'mean field' approximation, with the behaviour of ρ and ϕ separated by some kind of adiabatic mechanism.

9.2 The Wheeler-De Witt Equation

We focus on the quantum behaviour for the potential V(a) in equation 9.10. The Lorentzian action that includes a cosmological constant Λ, here reads

$$S = \frac{1}{2}\int N d\tau \left[-\frac{a^{\frac{3\gamma}{2}-1}}{N^2}(\frac{da}{d\tau})^2 + \frac{a^{\frac{3\gamma}{2}+1}}{N^2}(\frac{d\phi}{d\tau})^2 + a^{5-\frac{3\gamma}{2}}(\ ka^{-2} - \Lambda - V(a)\ ) \right] \tag{9.12}$$

and the conjugate momenta (π_a, π_ϕ) are defined as

$$\pi_a = \frac{\partial \mathcal{L}}{\partial \dot{a}} = -\frac{a^{\frac{3\gamma}{2}-1}}{N}\ \dot{a} \tag{9.13}$$

$$\pi_\phi = \frac{\partial \mathcal{L}}{\partial \dot{\phi}} = \frac{a^{\frac{3\gamma}{2}+1}}{N}\ \dot{\phi} \tag{9.14}$$

for lapse constant N, cosmological constant Λ and Lagrangian $\mathcal{L}$. We can now write down the Hamiltonian

$$\mathcal{H} = \pi_a \dot{a} + \pi_\phi \dot{\phi} - \mathcal{L} \tag{9.15}$$

$$= \frac{N}{2} a^{1-\frac{3\gamma}{2}}\ [\ -\pi_a^2 + a^{-2}\pi_\phi^2 + (\ \Lambda + V(a)\ )a^4 - ka^2\ ] \tag{9.16}$$

The Hamiltonian constraint $\mathcal{H} = 0$ is quantized, leading to a zero- energy Schroedinger equation satisfied by a wave functional $\Psi(a,\phi)$ in Mini-Superspace :

$$\mathcal{H}\ \Psi(a,\phi) = 0\ , \tag{9.17}$$

with quantized conjugate momenta

$$\pi_a^2 \longrightarrow -a^{-p}\frac{\partial}{\partial a}\, a^p \frac{\partial}{\partial a} \tag{9.18}$$

$$\pi_\phi^2 \longrightarrow -\frac{\partial^2}{\partial \phi^2} \tag{9.19}$$

with factor-ordering p. The Wheeler-De Witt equation thus reads, for closed curvature ($k = +1$) and potential $V(a) = V_m a^{-3\gamma}$ with factor-ordering $p = +1$

$$[\frac{\partial^2}{\partial a^2} + \frac{1}{a}\frac{\partial}{\partial a} - \frac{1}{a^2}\frac{\partial^2}{\partial \phi^2} + \Lambda a^4 + V_m a^{4-3\gamma} - a^2\,]\Psi(a,\phi) = 0 \tag{9.20}$$

The separation of variables $\Psi(a,\phi) = \psi(a)\,\Phi(\phi)$ leads to separate equations for the matter-field

$$\left[\frac{d^2}{d\phi^2} + s^2\right]\,\Phi(\phi) = 0 \tag{9.21}$$

and for the scale factor

$$\left[\frac{d^2}{da^2} + \frac{1}{a}\frac{d}{da} + \Lambda a^4 + V_m a^{4-3\gamma} - a^2 + \frac{s^2}{a^2}\right]\,\psi(a) = 0 \tag{9.22}$$

where s is the separation constant. Unfortunately the equation for the scale factor 9.22 is difficult to solve even without the cosmological constant term. The WKB approximation can be found in principle, but the integrals are rather complicated to evaluate. We therefore resort to different means: simply introduce better coordinates. With a good choice of coordinates we are able to perform the WKB approximation far from the turning points, and consequently construct a wave packet solution.

9.3 Quantum bulk matter states

With a transformation of coordinates

$$x^2 = a^{3\gamma-2} \quad (9.23)$$
$$\xi^2 = \xi_m^2 \, \phi^2 a^{3(2-\gamma)} , \quad (9.24)$$

with constant factor ξ_m as yet undetermined, the WDW equation becomes :

$$\left[x^2\frac{\partial^2}{\partial x^2} + x\frac{\partial}{\partial x} + \frac{4x^{2(\frac{2-\gamma}{\gamma-2/3})}}{(3\gamma-2)^2}\left(-\xi_m^2\frac{\partial^2}{\partial \xi^2} + V_m + \Lambda x^{\frac{6\gamma}{3\gamma-2}} - x^2\right)\right]\Psi = 0 . \quad (9.25)$$

Suppose that the bare cosmological constant Λ is zero. Now introduce the separation ansatz

$$\Psi(x,\xi) = X(\, x\,)\,\Xi(\,\xi\,), \quad (9.26)$$

then the functions X(x) and $\Xi(\xi)$ respectively satisfy

$$\left[\frac{d^2}{d\xi^2} + \epsilon^2\right]\Xi(\xi) = 0 \quad (9.27)$$

with separation constant ϵ, and

$$\left[x^2\frac{d^2}{dx^2} + x\frac{d}{dx} + \left(\frac{2}{3\gamma-2}\right)^2 x^{2(\frac{2-\gamma}{\gamma-2/3})}[\omega^2 - x^2]\right] X(\,x\,) = 0 . \quad (9.28)$$

Here $\omega^2 = \epsilon^2 + V_m$, the separation constant ϵ having absorbed the factor ξ_m^{-1} temporarily. Also abbreviate the exponent in equation 9.28

$$2(\frac{2-\gamma}{\gamma-2/3}) = 2(\,n+1\,) \quad (9.29)$$

for n real and positive and $2/3 < \gamma_n < 2$. The upper limit (2) comes from the requirement that the sound wave velocity of the bulk matter should not be greater than the speed of light. (In the perspective outlined in the previous footnote, no lower limit besides $\gamma > 0$ related to the CM classical wormholes should be introduced now.) The Wheeler-De Witt equation has solutions

$$\Psi_\epsilon(x,\ \xi) = \exp(-i\epsilon\xi)\, X_\epsilon(x)\ . \tag{9.30}$$

The WKB approximation in X(x) in the Lorentzian region with the associated phase S(x) and Wheeler-De Witt potential W(x) is

$$X(x) = \frac{1}{\sqrt{xW^{1/2}(x)}} \exp[\ \pm iS(x)\ ] \tag{9.31}$$

where

$$S(x) = \int^x \sqrt{W(x')}\, dx' \tag{9.32}$$

and

$$W(x) = (\frac{n+2}{2})^2\, x^{2n}[\,\omega^2 - x^2\,]\ , \tag{9.33}$$

with

$$n = \frac{2-\gamma}{\gamma - 2/3} - 1\ . \tag{9.34}$$

This results in the phase

$$\begin{aligned} S(x,\omega) &= -\frac{n+2}{2} \int_x^\omega x^n \sqrt{\omega^2 - x^2}\, dx \\ &= -\frac{1}{2}[\,\omega x^{n+1} \cosh^{-1}\frac{\omega}{x} - x^{n+1}\sqrt{\omega^2 - x^2} \\ &\quad + (n+1)\int_x^\omega \omega x^n \cosh^{-1}\frac{\omega}{x}\, dx\,]\ . \end{aligned}$$

With the restriction $x \ll \omega$, we can approximate the integral

$$\frac{(n+1)\omega}{2} \int_x^\omega x^n \cosh^{-1}\frac{\omega}{x}\, dx \approx \frac{\omega}{2} x^{n+1} \cosh^{-1}\frac{\omega}{x}\ ,$$

so that for $\omega = \bar{\omega}$ and $x \ll \bar{\omega}$ the phase approximates to

$$S(x,\bar{w}) = -\bar{w}x^{n+1}\cosh^{-1}\frac{\bar{w}}{x} + \frac{x^{n+1}}{2}[\,\sqrt{\bar{w}^2 - x^2} - \frac{\bar{w}}{n+1}\,]\,. \qquad (9.35)$$

An interesting correlation with classical theory emerges by constructing a wave-packet solution by a superposition of WKB states. We also introduce Gaussian amplitudes of width b and centre $(\bar{\epsilon}) = [(\bar{w})^2 - V_m]^{1/2}$,

$$C[\epsilon,\bar{\epsilon}] = (\pi b^2)^{-1/4}\,\exp[\,-1/2b^2(\epsilon-\bar{\epsilon})^2\,]\,. \qquad (9.36)$$

We now integrate over all real values of the separation constant ϵ:

$$\Psi_{\pm}(x,\xi) = \int_{-\infty}^{\infty} d\epsilon \frac{C[\epsilon,\bar{\epsilon}]}{\sqrt{xW^{1/2}}}\,\exp[\,-i\epsilon\xi \pm iS(x,\bar{w})\,]\,. \qquad (9.37)$$

On evaluating this integral, we find the wave packet

$$\Psi_{\pm}(x,\xi) = c(\bar{\epsilon})\cdot\exp\left[-(b^2/2)[\,\xi \mp (\frac{\bar{w}}{\bar{\epsilon}})x^{n+1}\cosh^{-1}\frac{\bar{w}}{x}\,]^2\right]$$
$$\cdot\exp\left[-i\bar{\epsilon}[\,\xi \mp (\frac{\bar{w}}{\bar{\epsilon}})x^{n+1}\cosh^{-1}\frac{\bar{w}}{x} \pm \frac{x^{n+1}}{2\bar{\epsilon}}(\sqrt{\bar{w}^2-x^2} - \frac{\bar{w}}{n+1})\,]\right],$$

where the constants are related as:

$$(\frac{\bar{w}}{\bar{\epsilon}}) = \sqrt{1+\frac{V_m}{\bar{\epsilon}^2}}\,, \qquad (9.38)$$
$$c(\bar{\epsilon}) = \left(\frac{2b}{x}\right)^{\frac{1}{2}}\left(\frac{\pi}{W_{\bar{\epsilon}}(x)}\right)^{\frac{1}{4}}\exp[-\bar{\epsilon}\pi]\,. \qquad (9.39)$$

The wave function is therefore localized, with the gradient of the total phase yielding pencils of classical trajectories in configuration space. The probability current is conserved through surfaces of constant "time" ξ (see [91]), so that the probability density is normalizable and proportional to

$$\exp\left[-b^2[\,\xi \mp \frac{\bar{w}}{\bar{\epsilon}}\,x^{n+1}\cosh^{-1}\frac{\bar{w}}{x}\,]^2\right]. \qquad (9.40)$$

It is now clear that the wave function is peaked about the configuration -space paths

$$\mid \xi \mid = \frac{\bar{w}}{\bar{\epsilon}}\, x^{n+1} \cosh^{-1} \frac{\bar{w}}{x}\,. \tag{9.41}$$

At this point we re-introduce coordinates (a, ϕ),

$$x^2 = a^{3\gamma-2}$$

and

$$\xi^2 = \xi_m^2 \phi^2 a^{6-3\gamma}.$$

We recover the constant

$$\bar{\epsilon} \longrightarrow \bar{\epsilon} \cdot \xi_m^{-1}\,.$$

Substitute for $\bar{w}$ in the configuration-space paths, and we instantly identify this configuration as the expression arrived at in the classical theory by eliminating the classical time-coordinate τ from the solutions for $a(\tau)$ (9.8) and $\phi(\tau)$ (9.9):

$$a^{-(3\gamma-2)} = a_m^{-(3\gamma-2)}\ \cosh^2 (\frac{3\gamma-2}{\sqrt{2\gamma}}\ \mid \phi \mid)\,, \tag{9.42}$$

provided the constants take on values

$$\bar{w}^2 = a_m^{3\gamma-2} \tag{9.43}$$

$$\bar{\epsilon}^2 = \frac{(3\gamma-2)^2}{2\gamma}\, a_m^{3\gamma-2} \tag{9.44}$$

$$\xi_m = \frac{3\gamma-2}{\sqrt{\gamma}} \tag{9.45}$$

with

$$n+1 = \frac{2-\gamma}{\gamma - 2/3}\,. \tag{9.46}$$

By integrating over the continuous family of wormhole states $\{\Psi_w(x(a), \xi)\}$, with Gaussian amplitudes 9.36, we obtain a *wave packet* that is peaked about configuration-space paths 9.42, where the latter satisfies the classical equations of motion (2.17 - 2.19).

9.4 Wormhole states

The effect of the coordinate transformation on the super-potential is that the "kinetic energy" term $-s^2$ is now, in effect, stored in the potential term $V(a)$. The coefficient V_m is modified, $V_m \rightarrow V_m + \epsilon^2$. That is, the superpotential $E_s \rightarrow E_w$ (see figs. 9.1 and 9.2), where

$$E_w = a^4 - (V_m + \epsilon^2)\, a^{6-3\gamma}\,. \tag{9.47}$$

This is similar to the Ellis-Madsen procedure for representing bulk matter in terms of a scalar field ϕ: For constant γ, the kinetic energy T can be expressed in terms of the potential V, $T = \frac{\gamma}{\gamma-2}V$.[2]

Unfortunately, the scalar field ϕ does not facilitate solving the quantum mechanical WDW equation; it is a bad choice of coordinate. We therefore introduced the "bulk matter field"

$$\xi = \frac{3\gamma-2}{\sqrt{\gamma}}\, a^{3(1-\gamma/2)}\, \phi\,, \tag{9.48}$$

and obtained the superpotential E_w and an accurate wave packet solution.

Furthermore, the family of states $\{\Psi_w\}$ behave similar to the massless minimally coupled scalar field states described in Hawking [84] and Hawking and Page [85] (see Chapter 7, Section 4.1 for an outline of their results). In our case, the wave functions Ψ_w fall off exponentially for $a > a_m$, (where $a_m = w^{2/(3\gamma-2)}$), and correspond to asymptotically Euclidean four geometries. As in the above-mentioned example (for which $\gamma = 2$), these geometries cannot pinch off to non singular compact metrics (like those of the No

[2]In the perspective that A. Carlini proposes, this connection does not exist.

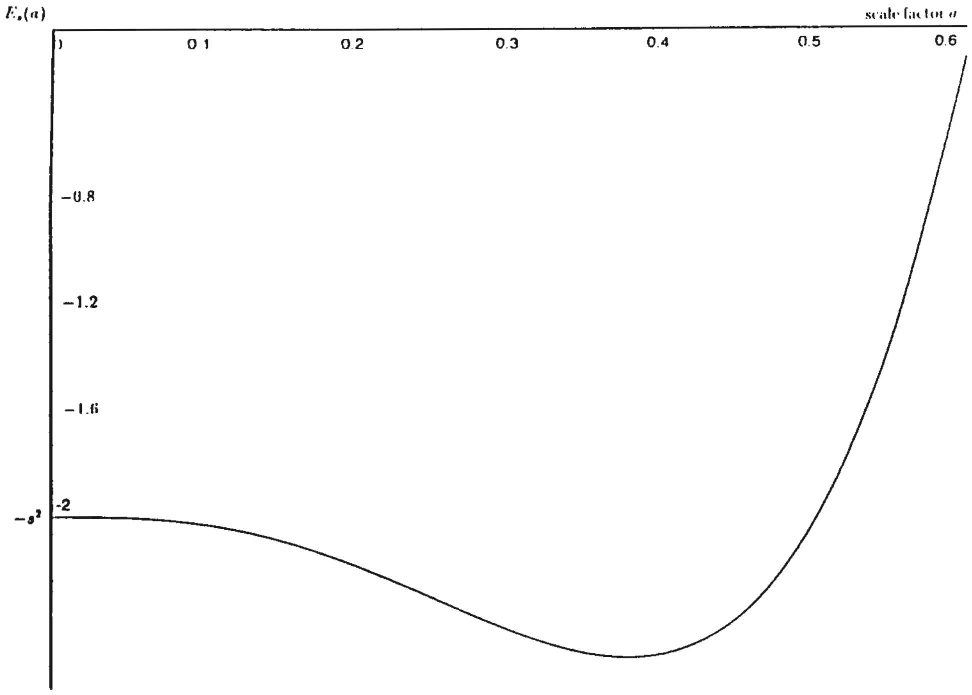

Figure 9.1: The potential $E_s(a) = -s^2 + a^4 - V_m a^{6-3\gamma}$. Here $s^2 = 2.0$, $V_m = 0.5$, and $\gamma = 5/3$.

Boundary proposal), due to a conserved flux $2\pi^2 iw$ of bulk matter particles passing through it. Such geometries correspond to that of wormholes, with a minimum throat-radius of order $w^{2/(3\gamma-2)}$.

The solutions $\{\Psi_w\}$ oscillate for $a < w^{2/(3\gamma-2)}$, and in this region correspond to classical Lorentzian Friedmann universes with bulk matter flux $2\pi^2 w$. These solutions (like their massless scalar counterparts) expand from $a = 0$ to a maximum radius of $w^{2/(3\gamma-2)}$, and then recollapse to $a = 0$.

We used the principle of constructive interference to arrive at an "on shell" wave packet solution (9.39) that is indeed regular near the origin $a = 0$. It is easy to show that the wave packet decays like $e^{-a^2/2}$ as $a \to \infty$, thus satisfying the Hawking-Page boudary conditions for wormholes to occur.

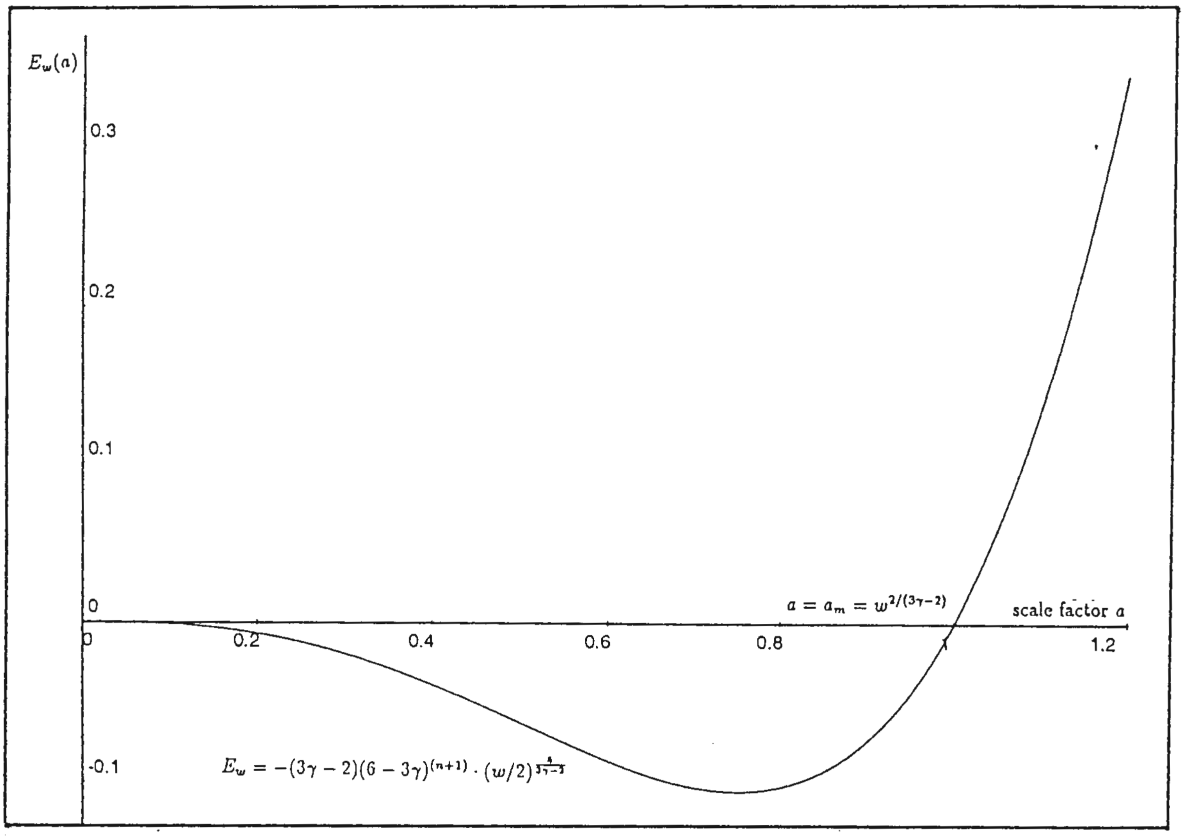

Figure 9.2: The potential $E_w(a) = a^4 - w^2 a^{6-3\gamma}$, for $w^2 = 1.0$, $\gamma = 5/3$ and the turning point $a_m = 1.0$.

For a non-zero bare cosmological constant Λ the superpotential (fig. 9.3) reads

$$E_w^{\Lambda} = -\Lambda a^6 + a^4 - w^2 a^{6-3\gamma} , \tag{9.49}$$

which leads to a second Lorentzian region for $a > a_s$, a second turning point. A quantum FRW universe tunnels through the potential barrier at $a_m < a < a_s$ to a large size de-Sitter spacetime. The Coleman mechanism for setting $\Lambda \to 0$ (see Coleman (1988) [13]), means that the second turning point $a_s \to \infty$.

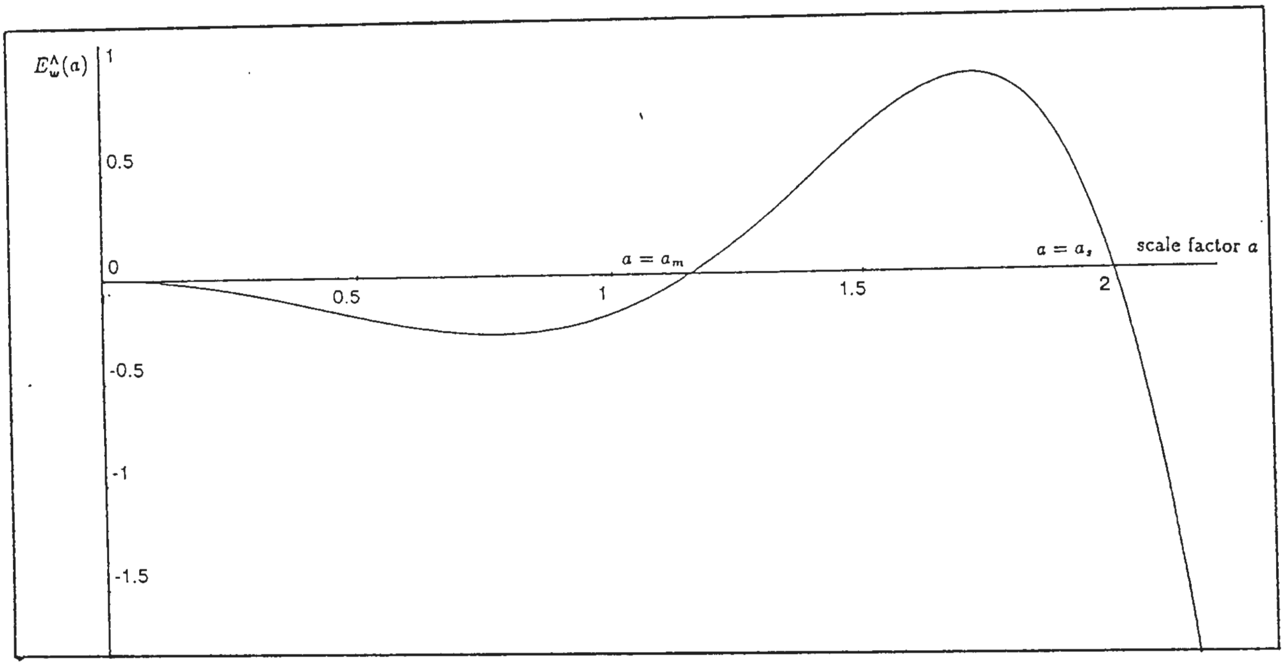

Figure 9.3: The potential $E_w^\Lambda(a) = a^4 - w^2 a^{6-3\gamma} - \Lambda a^6$. We have chosen $w^2 = 1.0$, with $\gamma = 4/3$ and $\Lambda = 3/16$.

9.4.1 Conditional Probability

We now compare our results with that of Kiefer [94]. The probability to tunnel from a Friedmann closed universe to the forbidden region $a_m < a \ll a_s$ is given by the tunneling amplitude

$$P(a \ll a_s) = \exp[-2S(a \ll a_s)] \tag{9.50}$$

$$S(a \ll a_s) = \int^{a_m} \sqrt{|E_w|}\, da \tag{9.51}$$

$$\approx \frac{\gamma - 2/3}{2-\gamma}\, a_m^2 \tag{9.52}$$

$$I.e.:\ P(a \ll a_s) \approx \exp[-2\frac{\gamma - 2/3}{2-\gamma}\, a_m^2\,]\,. \tag{9.53}$$

The probability for a universe from a large size de-Sitter spacetime to emerge from the forbidden region $a_m \ll a < a_s$ from the right is outlined in

Rubakov (1988) [125] for a conformal scalar field. A similar argument applies in our case. The wave packet Ψ is turned into an operator obeying the WDW equation in a third quantized version of our theory. It acts on states $|i>_F$ and $|j>_{dS}$ for Friedmann and de Sitter universes. If there are no universes present initially (i.e. $a_m = 0$) we choose the vacuum state $|0>_F$. Then the probability to obtain a large size de Sitter spacetime is

$$\sim \exp\left(\frac{2}{\Lambda}\right) \tag{9.54}$$

peaked at $\Lambda = 0$. So for a spectrum of baby-universe states $|i>_F$ with "bulk matter field" $\xi(\gamma)$, the conditional probability to tunnel from Friedmann universe via wormhole into de Sitter spacetime is

$$\frac{P(\ |j>_{dS}\ )}{P(\ |i>_F\ )} = \frac{P(\ a \gg a_s\ )}{P(\ a \ll a_s\ )} = \exp\left(\ \frac{2}{\Lambda} + 2\frac{\gamma - 2/3}{2-\gamma}\ a_m^2\ \right)\ . \tag{9.55}$$

This result is similar to Kiefer [94] modulo a coefficient in constant γ.

9.4.2 Conclusion

Carlini and Mijić [9] demonstrated how specific values of γ may represent wormhole instantons. We now see that for $2/3 < \gamma < 2$ there exist coherent states to the WDW equation representing Lorentzian closed universes with bulk matter sources. The condition that $\gamma > 2/3$ ensures that, as $a \to \infty$, $E_w \to +\infty$ (equation 9.47), and thus we recover an asymptotically Euclidean region. In addition, the coordinate transformation $x^2 = a^{3\gamma-2}$, $\xi^2 = (\frac{3\gamma-2}{\gamma})^2\phi^2 a^{6-3\gamma}$ is useful in that the parameter $\xi(a,\phi)$ serves as a judicious clock :

It easily decouples from the scale factor in the separation of variables, and more importantly, it is a suitable "time - variable" in the construction of a probability current density. The latter is conserved in "time ξ" and makes proper interpretation of the probability density effective.

On a more general note, it is clear that our choice of Gaussians are quite specific, since our WKB wave function is *either* expanding or contracting with respect to "time" ξ. We could equally well have chosen *symmetric* Guassians if we took the sum of equal amounts. Similarly, we could have introduced *antisymmetric* Guassians if we wanted correspondence with the Hartle-Hawking boundary proposal (see Kiefer [94, 93]).

Chapter 10

Issues in Wormhole theory

Initial excitement around the study of topological features known as wormholes and baby universes resided in the hope that they play a crucial role not just in setting the cosmological constant to zero (Hawking [76], Baum [3]) but also fixing the low energy interaction couplings of nature (Coleman [16]).

Integrating out wormhole fluctuations in the Euclidean Path Integral (EPI) gives an effective theory for gravity and matter-fields where the coupling constants become dynamical variables, sampled from a probability distribution. A saddle point analysis of the action functional in the EPI around large, smooth geometries shows that this distribution should be exponentially peaked at $\Lambda = 0$. This seemed to solve a crucial problem of both standard cosmology and particle physics. In his seminal work, Coleman [16] suggested a similar mechanism to fix the other coupling constants of nature, such as the gravitational constant.

In this chapter we briefly discuss the main features of the cosmological constant theory and the so-called big fix. There are, however, a lot of difficulties that threaten the wormhole theory. For instance, we saw in Chapter

4 that a well-defined formulation of Quantum Gravity in terms of the EPI is still lacking. The Euclidean action for gravity is unbounded from below, so we must choose a contour of integration for which the EPI will converge.

The proposal by Halliwell and Louka [58] for the use of "steepest- descent" methods in the space of complex four-geometries have been considered. Gibbons et al [40] initially proposed the rotation of the conformal degrees of freedom of the metric. This has not yet been implemented for more general and complicated cases, although it has been tried by Hartle and Schleich [68] in linear gravity. There is also the embarrassment that the choice of contour for the EPI may turn the peak at $\Lambda = 0$ into a broad distribution. A one loop estimate was performed by Polchinski [121].

The use of smooth geometries, and the distinction between large universes and wormholes in the derivation of Coleman's theory is still not fully justified. The issue of suppressing the amplitudes of "giant" wormholes, and the question of regulating the infrared divergence of the probability measure, still need attention. In addition, the meaning of the probability $\Psi(\Lambda)$ constructed from the EPI is not yet clear.

We also explore the idea of a "multi-universe" quantum field theory on Superspace, where 3^{rd} quantized operators create 2^{nd} quantized states in the field theory of a single universe. The field equation in the 3^{rd} quantized theory is non-linear, and represents a dynamical equation for the 2^{nd} quantized couplings. These couplings satisfy a 3^{rd} quantized *Uncertainty Principle.*

Euclidean 3^{rd} quantization theories agree with the main predictions of the Coleman mechanism. The Lorentzian version, on the other hand, predict that a peak at $\Lambda = 0$ should not occur. Different versions of a 3^{rd} quantization theory differ in their predictions, and are still to be implemented in a more realistic cosmological context.

Scepticism over wormhole theory seems to have grown, since progress to overcome these difficulties has been rather slow.

10.1 A theory of the cosmological constant

In a misguided effort to model a static Universe, Einstein [23] was obliged to introduce a free parameter Λ into the equation of motion

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = 8\pi G(T_{\mu\nu} + \Lambda g_{\mu\nu}) \; . \tag{10.1}$$

Even after Hubble's discovery that the Universe is expanding, the need for Λ persisted due to possibly non-zero vacuum mass density $< \rho >$ contribution to equation 1. We therefore write, for an effective cosmological constant Λ_{eff},

$$\Lambda_{eff} = \Lambda + < \rho > \; .$$

In a homogeneous and isotropic Universe like our own, with the expansion rate $H_0 \approx 75 \; km.sec^{-1} \; Mpc^{-1}$ smoothing out any gross effects of the spatial curvature, and a near-critical value of the total mass density $\leq 3H_0^2/8\pi G$, there is an upper bound to the effective cosmological constant:

$$\mid \Lambda_{eff} \mid \; \leq \; \frac{H_0^2}{8\pi G} \; \simeq \; 10^{-47} GeV^4. \tag{10.2}$$

This contradicts the predictions of Quantum Field Theory (QFT), e.g. for a free massive scalar field. The zero point energy summed over all modes, with a wave number cut-off m_p is of the order of $< \rho > \simeq m_p^4/16\pi^2 \; \simeq \; 10^{74} GeV^4$. This means that the bare Λ should be fine tuned to at least 121 significant places for the Universe to be large and flat with $\Lambda_{eff} \simeq 0$.

Hawking [76] studied a saddle point approximation dominated by large four-spheres in the EPI for gravity, in which $\Lambda > 0$ is treated as a dynamical variable. He then showed that probability of a given configuration is exponentially peaked at $\Lambda = 0$:

$$P(\Lambda) \sim \exp\left(\frac{3\pi}{G\Lambda}\right).$$

Baum [3] found a slightly different way out of the problem by considering a minimally coupled scalar field to make Λ dynamical without invoking topological fluctuations of gravity. He found the same peak at $\Lambda = 0$.

10.1.1 The Coleman mechanism

It was Coleman [40] who first gave a detailed mechanism for setting $\Lambda \to 0$, by giving a semiclassical analysis, based on a few debatable hypotheses, about the effects that wormholes have on Λ and other coupling constants.

The first assumption is that the EPI for Quantum Gravity is given by the Hartle-Hawking wave function, which is determined by a contour integration over all compact topologies approximated by large four-spheres, and eventually connected by microscopic wormholes. In the "dilute approximation" for wormholes, end-point interaction between wormholes are neglected, and they only interact with low energy physics. It also neglects the possibility that wormholes can divide into two or more, and have sizes far above the Planck scale $\sim m_p^{-1}$.

We present an outline of an argument due to Hawking in [84], as a summary of Coleman's original approach [16]. It considers an effective interaction $\theta_i(x_o)$, between a wormhole state i and low energy quantum fields ϕ, at a point x_0 on an asymptotically Euclidean region of spacetime. The other end of the wormhole i will join onto the same, or a different asymptotic region, at a point y_0. The effect between points x_0 and y_0 is equivalent to the insertion of the factor

$$\frac{1}{2}\int\int\sqrt{g(x_0)}\theta_i(x_0)\sqrt{g(y_0)}\theta_i(y_0)dx_0dy_0\,.$$

In the dilute wormhole approximation described earlier, the effect of n wormholes joining onto the asymptotic regions is given by a factor

$$\frac{1}{n!}\left(\frac{1}{2}\int\int\sqrt{g(x)}\theta_i(x)\sqrt{g(y)}\theta_i(y)dxdy\right)^n\,.$$

Here $n!$ compensates for overcounting identical wormholes. For an arbitrary wormhole configuration we have to sum over n, obtaining a factor of

$$\exp\left(\frac{1}{2}\int\int\sqrt{g(x)}\theta_i(x)\sqrt{g(y)}\theta_i(y)dxdy\right)\,.$$

This exponential is now regarded as a bi-local addition to the action. The bi-local action is

$$\frac{1}{2}\sum_i\int\int\theta_i(x)\theta_i(y)\,.$$

The bi-local action can be transformed into a sum of local terms, as performed by Klebanov, Susskind and Banks [97]. At this stage, position independent parameters, α are introduced in the identity

$$\exp\left(\frac{1}{2}\int d^4x\sqrt{g(x)}\theta(x)\int d^4y\sqrt{g(y)}\theta(y)\right)$$
$$=\frac{1}{\sqrt{\pi}}\int d\alpha e^{-\frac{1}{2}\alpha^2}\exp\left(-\frac{\alpha}{\sqrt{2}}\int d^4x\sqrt{g(x)}\theta(x)\right)\,.$$

The path integral now becomes

$$Z=\int d\alpha_iP(\alpha_i)Z(\alpha_i) \tag{10.3}$$

where

$$P(\alpha_i)=\exp\left(-\frac{1}{2}\alpha_i^2\right) \tag{10.4}$$

and

$$Z(\alpha_i) = \int d[\,\phi\,] \exp\left(- \int d^4x \sqrt{g} (\, L + \sum_j \alpha_j \theta_j \,) \right) . \qquad (10.5)$$

This is the formula for an ensemble of worlds with a statistical distribution of coupling constants, α_i. An observer in one of the members of the ensemble would have no way to deduce the existence of others. The quantum state of the universe is divided into non-interacting "superselection" sectors. Each sector is labelled by the coupling constants α_i, and an effective Lagrangian is the ordinary Lagrangian L plus an α-dependent term, $\alpha\theta$.

The integration variables are independent of position, so the effects of wormholes are to equalize the couplings in all the regions of spacetime. There is a spread of possible couplings, and different sectors are weighted by the probability distribution $P(\alpha_i)$. If one measures the strength of one of the effective interactions, the probability distribution collapses to the corresponding value of the coupling constants α_i. Any further measurement of that effective interaction will give the same strength.

The probability distribution $P(\alpha_i)$ for the couplings α_i is multiplied by the factor $Z(\alpha_i)$ given by equation 5, a path integral over all low energy fields ϕ, with effective interactions $\alpha_i\theta_i$. The path integral does not converge since the action is not bounded from below. We estimate $Z(\alpha_i)$ by looking for the saddle point with the greatest contribution to the path integral. Such a saddle point will be the that of a 4-sphere, with the lowest action

$$\Gamma = -\frac{3}{8G^2\Lambda} .$$

For a single sphere (see Hawking [76]) we may write $Z = \exp(-\Gamma)$, but for an arbitrary amount of spheres connected by wormholes (see fig. 10.1) there is the distribution

$$Z = \exp(\, \exp(\, -\Gamma\,)\,) . \qquad (10.6)$$

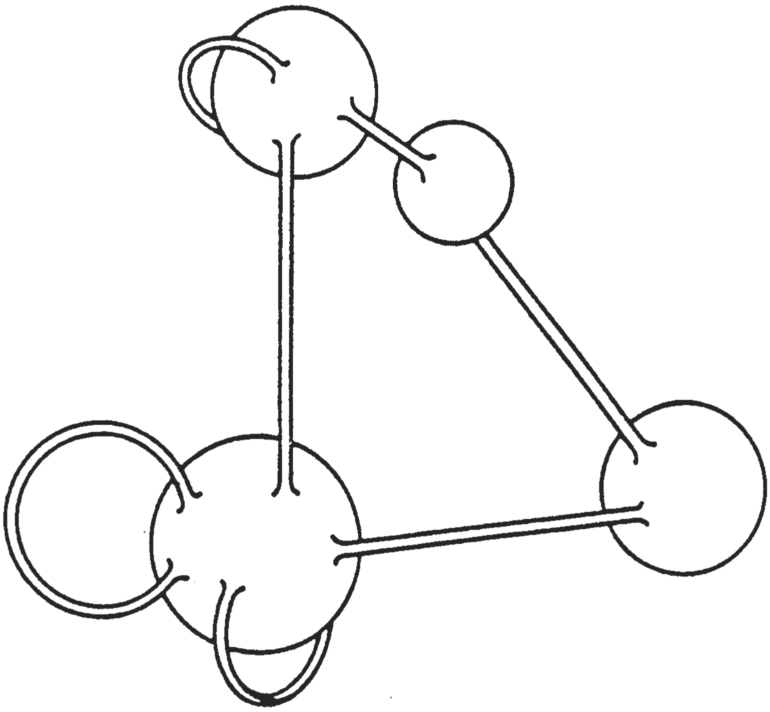

Figure 10.1: The large spheres represent parent universes, and the thin tubes baby universes. In the dilute approximation, these baby universes interact only via coupling to the parent universes.

Both the single and the double exponentials blow up rapidly as Λ approaches zero from above. This means that the probability distribution is peaked at those α_i for which $\Lambda = 0$.

In conclusion, our Universe is in contact with other large cool universes, through microscopic wormholes that set $\Lambda \rightarrow 0$. Even as our Universe undergoes inflation as a small hot Universe, the other large four-spheres still see $\Lambda = 0$ (fig. 10.2).

The approach of Klebanov et al [97] improves on that of Coleman [16] since it depends very little on the scale of the wormhole since it avoids the

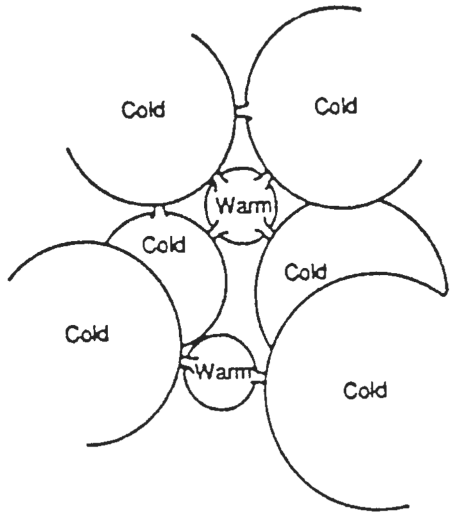

Figure 10.2: Small hot and large cool universes

controversy surrounding the behaviour of metrics and manifolds at the Planck scale. Nor does [97] assume any semi-classical approximations, since it is based on a bilocal effective interaction.

10.1.2 The "big fix"

The fundamental idea that wormholes might fix most, if not all of the constants of nature present in an effective Lagrangian theory was first suggested by Coleman [16]. A better mechanism was proposed by Preskill [122] and Grinstein and Wise [50]. Since the dominant term in the action is $-\frac{3}{8G^2\Lambda}$, the probability distribution would be peaked at either $G = 0$ or $\Lambda = 0$. Since we observe $G(\alpha_i)$ to be non-zero, it has to have some minimum value, about which the probability distribution would be concentrated. We hope that this minimum would occur at a single value of the couplings α_i. There is as yet little agreement about the effective values of other couplings such as masses, Θ_{QCD} etc.

A flaw in this argument has been pointed out by Hawking [84]. The probability measure $P(\alpha_i)Z(\alpha_i)$ diverges strongly on the surface $G^2\Lambda = 0$. This means that the total measure of α_i- space is infinite. The only way to avoid such a divergence is for $\mu(\alpha_i) = PZ$ to be finite and positive, to predict a large concentration at an isolated point in α_i-space. To do this one needs an appropriate cut-off for the probability measure. But there is no unique way of doing this, and different such cut-offs give different results. The ambiguity in the choice of the cut-off is known as the *regulator problem* for the measure.

Coleman [16] introduces such a cut-off in α_i-space at Λ, so that the probability measure is finite yet highly peaked there. Preskill [122] proposes the volume cut-off at $G^2\Lambda^2$. Another alternative is $-\frac{1}{\Gamma}$, leading to $\Lambda = 0$ and a $P(\alpha_i)$ distribution of the other couplings.

The fact of the matter is that the probability measure diverges since the Einstein Hilbert action is not bounded below. An ad hoc way to make the path integral converge is to integrate the conformal factor over a complex contour. However, it is not yet clear if this will always work.

10.1.3 The contour problem

The idea of integrating along a complex contour was explored in Gibbons et al [92], but it fails when the metric is coupled to non-conformally invariant matter.

J.B. Hartle's original idea that the EPI should be calculated along the "steepest descent path" in the space of complex four-geometries was applied by Halliwell and Louko [58, 59, 60, 30] to a de Sitter Mini-Superspace model. Unfortunately there are many contours that make the path integral converge, and the question arises whether some "correct boundary condition" will determine the contour uniquely.

A peculiar consequence of a complex contour is that some saddle points in the path integral may have neither Euclidean nor Lorentzian signature. It should be interesting to apply Hartle's idea to more realistic cases which also include matter-fields and maybe metrics that are anisotropic.

Also, higher derivative gravitational corrections to the stationary point for large four-spheres in the effective action that include terms up to Λ^2, showed the surprising result that the peak at $\Lambda = 0$ disappears. Instead the (normalized) $Z(\alpha_i)$ becomes a uniform smooth distribution in Λ (see Elizalde and Gaztanaga [24]). This is somewhat disappointing.

10.1.4 The giant wormhole disaster

The dilute wormhole approximation excludes wormholes larger than the Planck size $\sim m_p^{-1}$. Yet "giant" wormholes of sizes $\gg m_p^{-1}$ might be of great use as a mechanism to explain the "evaporation" of black holes as suggested by Hawking [83]. The problem is that low energy QFT may be violated if macroscopic wormholes are free to join onto arbitrary regions of spacetime. This is the so-called *giant wormhole disaster.*

Fischler and Susskind [27] showed that the main assumptions in the Coleman mechanism for Λ are mutually inconsistent and give rise to wormholes of every size. Essentially, we assume that the path integral over small-wormhole fluctuations (i.e. wormholes of scale b, say, at Planck value m_p^{-1} or less) has been calculated resulting in an effective theory with probability distribution $Z(\alpha)$ (equation 3) for a single Universe . This distribution may be expanded as the sum

$$Z(\alpha) = \sum_N C_N \alpha^N ,$$

where the N^{th} term can be interpreted as representing N macroscopic wormholes inserted in the large (parent) Universe, with an average

$$< N >= \frac{1}{Z(\alpha)} \sum N C_N \alpha^N = \frac{\alpha}{Z} \frac{\partial Z}{\partial \alpha} .$$

For small $\Lambda_{eff}(b)$, the mean density of wormholes is the average $< N >$ divided by the volume, approximately

$$-\alpha \frac{\partial \Lambda_{eff}(b)}{\partial \alpha} .$$

On dimensional grounds $\Lambda_{eff} \sim (m_p^2 b^6)^{-1}$, so that the maximum wormhole density is $\sim$ the close packing density b^{-4} provided $\alpha \sim m_p^2 b^2$.

Preskill [122] suggested that interactions between microscopic instantons should "crowd out" large ones (see fig. 10.3a). This seems to violate the principle that short distance physics is effectively decoupled from long distance physics.

By dividing a large four-volume into k-cells that may (or may not) contain an instanton of size $2^{k-1}b$, for some fixed unit b, Polchinski [120] argued for "the return of the giant wormholes" : The EPI over all topologies on a k-cell is then the sum over all the instantons of sizes $2^{k-1}b$, $2^{k-2}b$, $2^{k-3}b$, ...(see fig. 10.3b).

The presence (or absence) of arbitrary instantons in such k-cells shifts the effective cosmological constant by an amount

$$b^{-4} \sum_k 2^{-4(k-1)} \ln(1 - \bar{n}_k) ,$$

where $\bar{n}_k$ is the fraction of k-cells occupied by instantons of size $2^{k-1}b$. This is well-defined under condition that $0 \leq \bar{n}_k \leq 1$, and in particular $\bar{n}_k$ can be

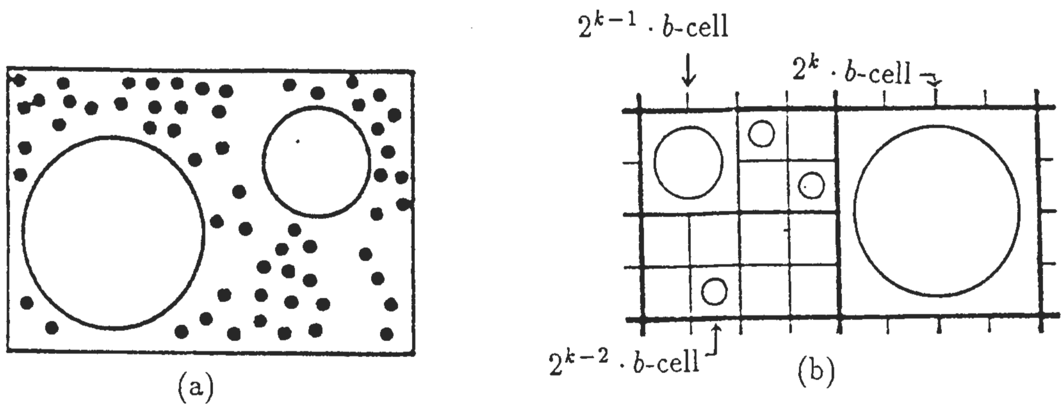

Figure 10.3: (a) Large wormholes "crowd out" small ones from spacetime. (b) A large instanton in a k-cell forbids any smaller instanton in that cell.

of order 1 for arbitrary large k, allowing the existence of giant wormholes. Also, it now becomes clear that the probability for the existence of a $2^{k-1}b$-size instanton at a given point depends on the probability that no larger instanton is found at that point. This demonstrates that violation of the decoupling principle is really just an illusion.

An "escape from the menace of the giant wormholes" was partly performed by Coleman and Lee [14] for a peculiar type of wormhole carrying a conserved global U(1) charge Q_k and of size $2^{k-1}b$, occuring only at stationary points of the EPI. Also assuming that wormhole induced terms in the effective Lagrangian are charge changing, the shift in the effective cosmological constant arising at the second order in α is

$$b^{-4}\sum_k B_k|\alpha_k|^2 2^{-4(k-1)}e^{-2S_k} ,$$

for dimensionless constants B_k and wormhole action $2S_k$. The fraction of the

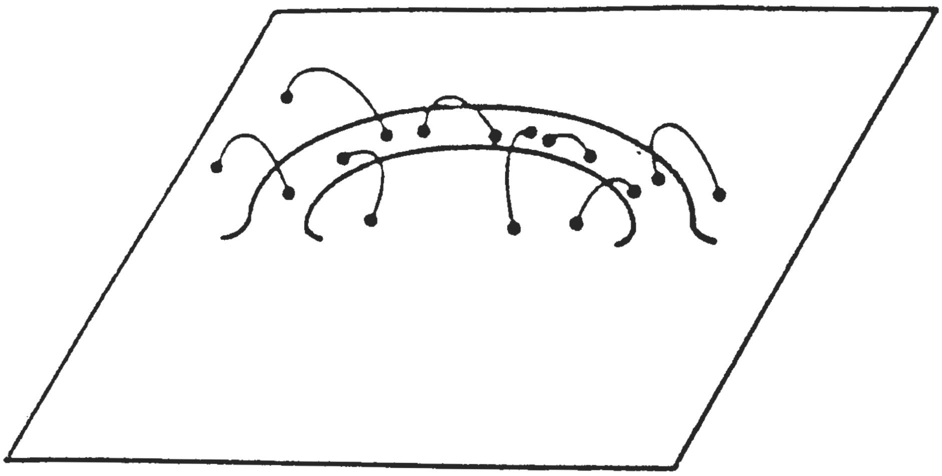

Figure 10.4: A large wormhole "bleeded" by small wormholes attached to it.

four-volume occupied by wormhole ends of type k is then

$$B_k|\alpha_k|^2 e^{-2S_k} .$$

It finally assumes that microscopic wormholes "bleed" the giants by inducing charge nonconservation interactions (see fig. 10.4). As charge flows into the throat of a large wormhole, it can be diverted into small wormholes, until there is too little charge left to support the large one, therefore destabilizing it. The giant wormhole becomes unstable when the mean square charge carried by the microscopic wormholes is greater than its own charge Q_K^2. That is, when the stability condition

$$\sum_{k<K} B_k|\alpha_k|^2 2^{-4(k-K)} e^{-2S_k} Q_k^2 \leq Q_K^2$$

is violated.

10.2 A multi-universe 3^{rd}-quantized theory

The theory of Λ thus far makes no clear distinction between the nature of a single universe theory and effective interactions with other universes. This has been pointed out by Coule and Solomons [19] where the Wheeler-De Witt equation for a de Sitter spacetime is modified by the presence of bulk-matter wormholes. Generally, the Hartle-Hawking path integral used in the Coleman mechanism for Λ do not take such modifications into account. And what about interactions among wormholes themselves ?

A more fundamental framework in which small closed "baby universes" can interact with each other or with a macroscopic "parent universe" is achieved through third quantization. It is essentially a "multi- universe" system treated as a QFT on Superspace. Third quantized field operators act on a third quantized state with no universes, the so-called *void*, and create (and subsequently annihilate) quantized states in the field theory of a single universe. These operators obey the Wheeler- De Witt equation. Interactions then generalize this equation to a non-linear equation for spacetime couplings.

It is a gauge theory, therefore third quantized gauge symmetries are important in the construction of the action [130, 128]. Since Superspace is infinite dimensional, it is ill-defined because of non- renormalizability. An advantage over the second quantized theory is that topology-changing interactions are naturally described by a sum- over-smooth-four-geometries with fixed boundaries. This amounts to the addition of non-linear terms in its fundamental equation.

Strominger [130] postulates that a multi-universe system described by a Schroedinger state $\Psi[\Phi(X^i), X^0]$ of the third quantized Hilbert space obeys

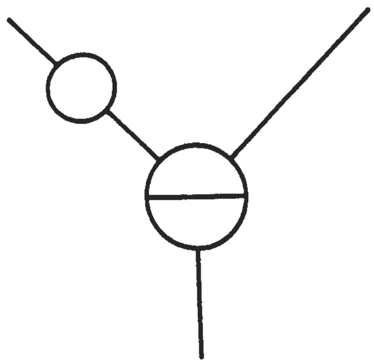

Figure 10.5: Iterating the basic joining-splitting interaction leads to arbitrarily complicated many-universe processes.

the third quantized Schroedinger equation

$$\mathcal{H}|\Psi> = i\frac{\partial}{\partial X^0}|\Psi> \tag{10.7}$$

where X^0 is a second quantized field operator that serve as a third quantized "time" coordinate, and $\mathcal{H}$ is the Hamiltonian of the third quantized action

$$S_E = \frac{1}{2}\int d^D X^\mu \sqrt{g}[\,(\nabla\Phi)^2 + m^2\Phi^2 + \frac{\lambda}{3}\Phi^3\,]\,,$$

where Φ is the second quantized wave function of the universe. Here the arbitrary weighting λ reflects the strength of multi- universe interactions like those in fig. 10.5.

X^μ is the D dimensional field configuration in the universe. We may define orthonormal eigenspaces $|n>$ for the universe number operator $\mathcal{N}$

$$\mathcal{N}|n> = n|n> \tag{10.8}$$

and decompose the state $|\Psi>$ at some moment X^0:

$$|\Psi> = \sum_n \Psi_n(X^0)|n>\ . \tag{10.9}$$

The probability amplitude for n universes at an instant X^0 is then $\Psi_n(X^0)$.

We now give an outline of the "single universe" approximation, that is to some extent valid for an observer in our Universe. Consider two separate classes of universes, the small ($\sim$ Planck scale) baby universes and large ($\sim$ Hubble scale) parent universes. The second quantized actions are for simplicity written in $D = 1$, as

$$S_{P,B} = \int d\tau \left(\frac{\dot{X}^2}{N} - N m_{P,B}^2 \right) . \tag{10.10}$$

The topology changing interactions are assumed to be (a) nucleation (or annihilation) of a baby by a parent universe, or (b) bifurcation of a baby universe (see fig. 10.6). The third quantized action reads

$$S_E[\Phi] =$$
$$\frac{1}{2g^2} \int dX \left(-(\nabla\Phi_P)^2 + m_P^2\Phi_P^2 - (\nabla\Phi_B)^2 + m_B^2\Phi_B^2 + \kappa\Phi_P^2\Phi_B + \frac{\lambda}{3}\Phi_B^3 \right) \tag{10.11}$$

with $\Phi_{P,B}$ acting as annihilation and creation operators for "babies" and "parents",and g^2 is a scaled out third quantization coupling. For very large m_P, pair production of parent universes is suppressed, and since the couplings preserve parent universe number modulo 2 we may restrict ourselves to the case of a single parent universe propagating in a plasma of baby universes. See fig. 10.7.

Parent-baby interaction may be described by the "hybrid" action

$$S_I = \int d\tau N \sum_i \mathcal{L}_i(\tau)\Phi_B^i \tag{10.12}$$

for local second quantized operators $\mathcal{L}_i$ on the parent universe, and third quantized baby field operator Φ_B^i. Replace the discrete index i by the continuous index k and introduce the Fourier transform $\tilde{\Phi}_B(k)$, then the action

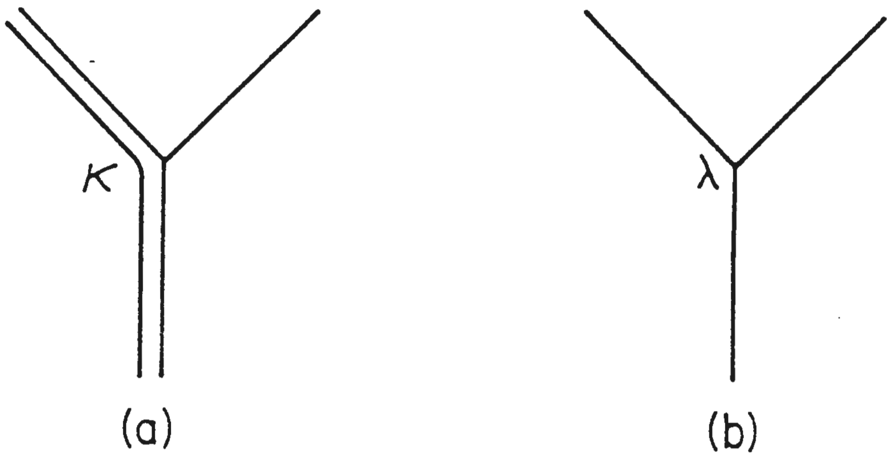

Figure 10.6: A double line represents a parent universe, and a single line a baby universe. (a) Nucleation (or annihilation) of a baby by a parent universe. (b) Bifurcation of a baby universe.

is equivalent to

$$S_I = \kappa \int_0^1 d\tau \Phi_B[\, X(\tau)\,],$$

so that the third quantized functional integral for the parent propagator in the bath becomes a second quantized path integral

$$G(X_i, X_f) = -ig^2 \int_{X_i}^{X_f} \mathcal{D}X(\tau) \int_0^\infty dN e^{iS_P + iS_I[\alpha]} \tag{10.13}$$

where

$$S_P + S_I[\alpha] = \int d\tau \left(\frac{\dot{X}^2}{N} - Nm^2 - N\kappa\alpha(X) \right) .$$

This looks like an ordinary second quantized action for a one dimensional universe. The effect of baby universes is summarized by the addition of an ordinary potential $\alpha(X)$ into the field theory [42, 15]. In the semi-classical limit of the third quantized theory, $g^2 \to 0$ in equation 10.13, the field op-

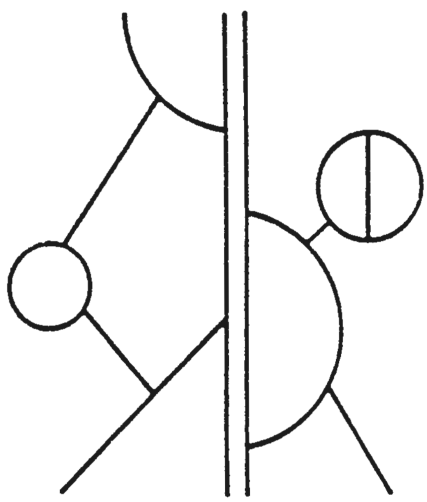

Figure 10.7: A parent universe propagating in a plasma of baby universes.

erators all commute and we can diagonalize Φ_B in terms of real time third quantized baby universe eigenstates $|\alpha(X)>$:

$$\Phi_B(X)|\alpha(X)> = \alpha(X)|\alpha(X)>$$

where the eigenvalues $\alpha(X)$ are constrained to obey the baby universe field equation

$$(\nabla^2 + m_B^2)\alpha(X) + \frac{\lambda}{2}\alpha^2(X) = 0 \tag{10.14}$$

in the absence of parent universe sources.

10.2.1 The Third Quantized Uncertainty Principle

Generally a baby universe is in a linear superposition of (orthogonal) eigenstates

$$|\alpha, \alpha'> = \beta|\alpha> + \beta'|\alpha'> \tag{10.15}$$

where $|\beta|^2 + |\beta'|^2 = 1$. For a desirable clock in the parent universe, we may calculate the correlation function of n-field operators at times $\tau_1 ... \tau_n$:

$$
\begin{aligned}
< X(\tau_1)...X(\tau_n) >_{\alpha,\alpha'} &= < \alpha, \alpha'| \int \mathcal{D}X(\tau) e^{iS_P + iS_I} X(\tau_1)...X(\tau_n)|\alpha, \alpha' > \\
&= |\beta|^2 \int \mathcal{D}X(\tau) e^{iS_P + iS_I[\alpha]} X(\tau_1)...X(\tau_n) \\
&\quad + |\beta'|^2 \int \mathcal{D}X(\tau) e^{iS_P + iS_I[\alpha']} X(\tau_1)...X(\tau_n) .
\end{aligned}
$$

This is a sum of ordinary correlation functions in universes with different couplings α and α'. Second quantized operators corresponding to observables in a single universe do not affect the baby universe state, so they cannot connect the states $|\alpha>$ and $|\alpha'>$. Two observers measuring different eigenvalues can never communicate.

We employ the Copenhagen interpretation to rephrase this result. Initially the coupling constants are not defined, but depend on a probability distribution. Performing some measurements which indicate that the coupling constants are α (α'), will collapse the wave function into the orthogonal eigenstates with respective probabilities $|\beta|^2$ and $|\beta'|^2$. All future measurements are then consistent with some definite coupling α (α'). When $g^2 \to 0$, after fixing the values of the parent universe potential $\alpha(X)$ and its first derivative $\frac{\partial \alpha(X)}{\partial X}$ at a given X, then 10.14 uniquely determines $\alpha(X)$ for each other value of X.

However, for $g^2 \neq 0$, the baby state is subject to quantum fluctuations and the results of measurements are expressed as conditional probabilities.

Measurements at X_1 and X_2 give results $\alpha_{1,2}$ for the parent universe potential $\alpha(X)$. So the conditional probability amplitude that the potential at an intermediate point X_3 will have value α_3 is given by

$$A(\alpha_3) = C \int \mathcal{D}\Phi(X) e^{iS_B[\Phi]} .$$

Here S_B is the third quantized baby universe action, and the path integral is over all paths obeying $\alpha_{1,2,3} = \Phi(X_{1,2,3})$, respectively, and is normalized to one. The *Uncertainty Principle* now reads as follow :

If X runs over an infinite range, $A(\alpha_3)$ is zero for all α_3. Even if X has finite range, there will be difficulties in measuring the first derivative of α at X_3: the "momentum" spread is very large immediately after a precise measurement of "position". As this would be the case in practice, we explore it in greater detail.

Suppose that the potential has not been measured exactly at X_1 and X_2, but has been determined to within a Gaussian of width λ around the values α_1 and α_2. Then the conditional probability amplitude for measuring the first derivative of the potential at X_3 to take the value

$$\frac{\partial \alpha}{\partial X}(X_3) = \alpha'$$

is given by the Fourier transform of $A(\alpha_3)$. For X_3 very near to X_2, it reads

$$A(\alpha') = \sqrt{2\pi\lambda} e^{-\alpha'^2/2\lambda} .$$

Now as the difference between the two field values X_1 and X_3 and the uncertainty λ of the measurement of the potential at X_1 go to zero, the spread in α' goes to infinity. This is equivalent to the statement that the momentum spread of a quantum mechanical particle is very large shortly after a precise measurement of its position. This inability to obtain precise measurements of coupling constants is known as the *uncertainty principle* for spacetime couplings.

10.2.2 Third Quantized Coherent States

Suppose that FRW Mini-Superspace containing a massless scalar field ϕ, is third-quantized as recommended by Giddings and Strominger [44]. This avoids difficulties with negative probabilities encountered in second quantized Mini-Superspace. Coherent states can then be constructed in such a model, and the Heisenberg uncertainty relation investigated. This suggestion was taken up by H.J. Pohle [119], and they exposed a peculiarity in the sense that quantum effects dominate in regions that are essentially classical in nature. This strange prediction may forecast problems for the third-quantization of gravity in general. Here we present a brief outline of their paper, Pohle [119].

A Friedmann-Robertson-Walker Mini-Superspace with its usual metric and containing a massless scalar field, was quantized in the sense of third quantization by Pohle [119]. The analogue of the Klein-Gordon equation for Mini-Superspace is the Wheeler-De Witt equation

$$\left(a^2\frac{\partial^2}{\partial a^2}+a\frac{\partial}{\partial a}-\frac{\partial^2}{\partial\phi^2}-a^4\right)\psi(a,\phi)=0\ , \tag{10.16}$$

The Hamiltonian operator of the system is

$$\hat{H}=\frac{1}{2}\int d\phi\left(-\frac{\delta^2}{\delta\psi(\phi)^2}+\frac{\partial^2\psi}{\partial\phi^2}+a^4\psi^2\right)\ , \tag{10.17}$$

where the functional derivative $\frac{\delta^2}{\delta\psi(\phi)^2}$ is performed with respect to the operator $\hat{\psi}(\phi,a)$ taken to be the "time-independent" c-number field $\psi(\phi)$. The amplitude to find an instantaneous field configuration $\psi(\phi)$ on a spacelike hypersurface in Mini-Superspace is given by the wave functional $\Psi[\psi,a]$ of the Schroedinger equation

$$\hat{H}\Psi=ia\frac{\partial}{\partial a}\Psi\ , \tag{10.18}$$

with time variable a. The third-quantized Lagrangian now reads

$$L^{(3)} = \frac{1}{2}\left[\left(a\frac{\partial\psi}{\partial a}\right)^2 - \left(\frac{\partial\psi}{\partial\phi}\right)^2 + a^3\psi^2\right] . \tag{10.19}$$

The inner product is defined in the usual way. Since we are interested in classical behaviour for the Universe, we look for coherent state solutions of the Schroedinger equation. An exact solution to this equation is given by the ansatz

$$\Psi[\phi, a] =$$

$$C\exp\left(-\frac{1}{4}\int d\phi d\phi'[D + iI] \times [\psi(\phi) - \eta(\phi, a)][\psi(\phi') - \eta(\phi', a)]\right)$$
$$\cdot\exp\left(i\int d\phi P(\phi, a)[\psi(\phi) - \eta(\phi, a)]\right) ,$$

where C is the normalization. All the functions indicated are real, and can in principle be determined from the Schroedinger equation. Due to translation invariance with respect to ϕ, the functions D and I have the following properties :

$$D(\phi, \phi', a) = D(\phi - \phi', a) \tag{10.20}$$
$$I(\phi, \phi', a) = I(\phi - \phi', a) . \tag{10.21}$$

By taking Fourier transforms we find

$$D(\phi, \phi', a) = \frac{1}{2\pi}\int dk \exp[-ik(\phi - \phi')]D(k, a) ,$$

and similarly for the function I. Substitute the ansatz into equation 18 and define

$$2A(k, a) = D(k, a) + iI(k, a) .$$

Then the function $A(k, a)$ satisfies the equation

$$ia\frac{\partial A}{\partial a} + A^2 + a^4 = k^2 . \tag{10.22}$$

Re-define

$$A = -\frac{a}{u}\frac{\partial u}{\partial a} \tag{10.23}$$

and it is now clear that we have been able to produce the Fourier transformed second quantized Wheeler-De Witt equation 16:

$$\left(a^2\frac{\partial^2}{\partial a^2} + a\frac{\partial}{\partial a} + k^2 - a^4\right) u(a,k) = 0 \; . \tag{10.24}$$

The solutions are combinations of independent modified Bessel functions $K_{ik/2}(\frac{a^2}{2})$ (also called Macdonald functions) and $L_{ik/2}(\frac{a^2}{2})$. The former is zero at infinity and trigonometric at zero a, while the latter goes like $\frac{1}{a}e^{a^2}$ as a goes to infinity. The general solution of equation 22 for $A(k,a)$ we write as

$$A(k,a) = -i\frac{a}{(r_1u_1 + r_2u_2)}\frac{\partial(r_1u_1 + r_2u_2)}{\partial a}$$

with r_1,r_2 complex functions of k. It is easy to show that the real part of A,

$$2D = \frac{w}{|\, r_1u_1 + r_2u_2\,|^2} \tag{10.25}$$

for constant w. The left-hand side of Heisenberg's uncertainty relation can readily be calculated, and is found to be

$$(\triangle\hat{\psi})^2(\triangle\hat{\pi})^2 = \frac{1}{4}\left[1 + \left(\frac{I}{D}\right)^2\right] .$$

The imaginary part of equation 22 leads to the result

$$I = \frac{a}{D}\frac{\partial D}{\partial a}$$

and therefore

$$\frac{I}{D} = -a\frac{\partial}{\partial a}D^{-1} \; .$$

Given the solution to D in equation 25 we arrive at the result

$$(\triangle\hat{\psi})^2(\triangle\hat{\pi})^2 \rightarrow \infty \;\; for\; a \rightarrow \infty\, . \tag{10.26}$$

This means that quantum effects dominate the Universe at large radius. Clearly we are contradicting the fact that spacetime is essentially classical at present. This seem to raise questions about the true meaning of third quantized gravity on Mini-Superspace and beyond.

Chapter 11

Conclusion

We now review the contents of our work, emphasizing areas where there are prospects for progress, and taking heed of the shortcomings.

Throughout, there has been a positive attempt to clarify the role that time plays in quantum cosmology, yet a generally covariant theory of quantum gravity should indicate a marked *absence* of time. Only in the 'classical limit' (in this context, General Relativity and Quantum Mechanics) should the notion of classical spacetime as we know it, enter the arena.

To start with, Chapter Two explores the general formulation of quantum cosmology, by using the Hamiltonian formalism. The Dirac quantization procedure results in an operator-ordering ambiguity that has received considerable attention, but still remains unresolved. This we consider to be the first indication that any predictions of quantum cosmology are to be taken with a large grain of salt. The attempt to construct Mini- Superspace models in which all except finitely many degrees of freedom are frozen violates the uncertainty principle [61]. It also skips the regularization problems.

Futhermore, we observe the Universe to be homogeneous and isotropic.

Any effective theory of quantum gravity should therefore *predict* the nature of the spacetime metric; to impose an FRW-metric and a homogeneous, isotropic matter-field onto two-dimensional Mini-Superspace merely begs the question.

Nevertheless, we are able to make some interesting remarks concerning the emergence of classical behaviour from the"quantum fuzz" that permeate throughout most of Superspace (Section 2.6). The WKB approximation leads to a conserved current that may have negative probability density. This raises the question of the role of time and the issue of predictions in quantum cosmology. First and foremost, the usual Copenhagen interpretation of Quantum Mechanics falls short due to the absence of an external observer [67, 26]. Instead, the (post)-Everett idea of splitting our single Universe (essentially an isolated,"closed" system) into many identical subsystems, allows us to retrodict its history using *Conditional Probabilities* [88, 34, 66, 132, 54], in Chapter 3.

The issue of choosing initial conditions to the classical Einstein field equations is translated into proposing an appropriate boundary condition in Superspace. The heuristic aim of such a proposal is to select a single wave function for the Universe that predicts sufficient inflation to resolve among others, the flatness-, horizon- and monopole problems of standard cosmology, and provide the seeds for galaxy-formation. The No Boundary proposal of Hartle and Hawking [74, 77, 65] seems to fall short in this regard, while the Tunneling proposal of Vilenkin and Linde [133, 134, 135, 136, 137, 138, 139] and [103, 104, 105] make predictions that appear more reasonable. However, the Tunneling wave function is not well- defined, since there is no guarantee that flux is carried out of Superspace at singular boundaries.

In practice, the WKB approximation results in an ensemble of possible wave functions to the Universe, that predict pencils of classical trajectories in configuration space, instead of a single classical path. More-over, if we require an initial inflationary phase along its classical evolution, it would appear [117] that the Universe has an infinite classical history, despite the spacetime being singular in the sense that it is geodesically incomplete. This misleading result emphasizes the need for more realistic models that contain a reliable time variable, and well- defined probability densities.

It would appear that the Hartle-Hawking wave function is not an example of a"typical" wave function of the Universe [51]. On the other hand, for an appropriate basis $\{|n>\}$ defined on the space of all wave functions for a Mini-Superspace model with a power-law potential, it seems plausible that sufficient inflation is a property of a typical wave function (Chapter 5).

What we definitely learn from quantum cosmology is that the usual framework of Hamiltonian quantum mechanics needs to be generalized - gravitational fluctuations of spacetime deny us any definite notion of causality, since spacetime itself becomes a dynamical variable [99, 89]. The"neutral-time " formalism [2, 49, 70] is devoid of a fundamental arrow of time, hence the probabilities for the individual members of a set of alternative histories depend on the initial and the final conditions of the Universe. The semi-classical domain of everyday experience emerge only when such boundary conditions lead to decoherence of alternative sets of histories in an appropriate fashion.

For quantum evolution to lead to the present classical Universe, the wave function must approach a wave packet that describe observed cosmological data [91, 10]. To obtain a good probability interpretation for the wave packet, we need to specify a judicious clock.

This is ideally manifested in the description of bulk-matter wormholes in

Mini-Superspace (Chapter 9). The wave function exhibits wave packet-like structure with a"bulk matter-field" clock ξ. Since they satisfy the strong-energy condition $\gamma > 2/3$, and the Wheeler-De Witt equation, these solutions are a substantial contribution to existing wormhole theory. In the Appendix we attempt to construct a relation between the *Lorentzian* perfect fluid index γ for bulk matter universes, and its analytic continuation γ_e for an Euclidean *exterior*.

Previously, wormhole instantons [4, 17, 85, 42, 9, 45, 57, 88, 101, 102, 126, 109, 124] were obtained by analytic continuation of the Lorentzian Einstein field equations to the Euclidean regime. Wormholes in Superspace were introduced to provide mechanisms for black hole evaporation and setting the cosmological constant to zero [85, 88]. We are able to improve on the Born-Oppenheimer approximation [93], as well as the asymptotic expansions of Hawking and Page [85] by producing exact quantum wormhole states for the free massive scalar field, provided the mass m^2 takes on discrete integer values.

The initial enthusiasm over wormholes and the Coleman mechanism [13] for setting Λ to zero, has largely subsided, and is slowly being replaced by cool realism:

There is no well-defined theory of quantum gravity to date.

The Euclidean action for gravity is not bounded from below.

Different choices for the contour of integration to make the EPI converge lead to different results. The idea of integrating along a complex contour [92] fails in the case of non-conformally invariant matter. A complex contour results in saddle points in the EPI with neither Euclidean nor Lorentzian signature.

The Coleman peak at $\Lambda = 0$ disappears and becomes a disappointingly smooth distribution in Λ in higher derivative corrections to the stationary point for large four-spheres in the effective action.

In the giant wormhole disaster, low energy QFT may be violated if macroscopic wormholes are free to join onto arbitrary regions of spacetime.

A multi-universe third quantized theory of interacting baby- and parent universes was formulated as a QFT on Superspace. The third quantized Uncertainty Principle states that there is *some uncertainty* in the prediction for the relation among coupling constants of the Universe. Unfortunately, the fact that a simple FRW-universe model containing a homogeneous, isotropic matter-field in third quantized theory instead reveals *maximum* uncertainty [119], merely adds to the existing doubts of the Mini-Superspace formalism.

Appendix A

Relating γ and γ_e

The Ellis -Madsen procedure [8, 9, 107] defined the index γ for bulk matter in a perfect fluid model as a function of the kinetic energy T and the potential energy V of a scalar field ϕ:

$$\gamma = \frac{2T}{T+V} \,. \tag{A.1}$$

In order to describe wormholes as analytic continuations of classical FRW closed universes, we rotate the scalar field such that

$$\phi \rightarrow i\phi \tag{A.2}$$

so that the *Lorentzian* index γ has to be redefined as an *Euclidean* index γ_e. Since the kinetic energy is a square function of the time derivative in ϕ, we see that $T_e \rightarrow -T$, while dynamical consideration tells us that the potential energy $V_e \rightarrow V$. Equation A.1 now reads

$$\gamma \rightarrow \gamma_e = \alpha \frac{-2T}{-T+V} \,, \tag{A.3}$$

where α is a constant of proportionality. Simple manipulation of equations A.1 and A.3 lead to

$$\begin{aligned} \frac{1}{\gamma} &= \frac{1}{2} + \frac{V}{2T} \\ \frac{\alpha}{\gamma_e} &= \frac{1}{2} - \frac{V}{2T} \,, \end{aligned}$$

which add up to

$$\frac{1}{\gamma} + \frac{\alpha}{\gamma_e} = 1 \,. \tag{A.4}$$

We have found a relation between the indices γ for the Lorentzian and Euclidean regimes respectively, barring the unknown α. It is also possible to formulate a tentative argument for evaluating this constant, which contains two crucial stages:

a) We demonstrate that the potential energy V_e of the scalar field ϕ approximates to a power-law potential

$$\frac{\kappa}{2q}\phi^{2q}$$

for small ϕ, for a perfect fluid model with index γ_e. This will allow us to separate the Wheeler-De Witt equation as in Section 8.4, provided that the power q is a function of γ_e.

b) Secondly, for ϕ sufficiently small, and for high order modes, the power-law potential model of Section 8.4 approximates to the closed bulk matter universe model described by equations 9.27 and 9.28 of Section 9.3. This implies that the power q may in turn be expressed as a function of the Lorentzian index γ.

The upshot of all this is firstly, that (a) results in

$$q = \frac{3\gamma_e}{3\gamma_e - 2} \tag{A.5}$$

and secondly, (b) leads to

$$q = \frac{\gamma}{2 - \gamma} \,. \tag{A.6}$$

Equating these two expressions for q results in the relation

$$3\gamma_e + \gamma = 3\gamma_e\gamma \,, \tag{A.7}$$

and finally,

$$\frac{1}{\gamma}+\frac{1}{3\gamma_e}=1\ . \tag{A.8}$$

This crucial relation matches the closed FRW universe bulk matter index to its analytic Euclidean continuation. It is easy to see that

$$\alpha=\frac{1}{3}\ . \tag{A.9}$$

A.1 (a) The scalar field potential

We give an outline to the derivation of the sinh potential in the framework used by CM[8, 9]. If this new potential is indeed the correct one, we are able to derive the asymptotic wave functions for $|\phi| \ll 1$ by means of the HP[85] procedure. These expansions are functions of a and ϕ, with coefficients that contain terms in $3\gamma_e - 2$, so that $\gamma_e = 2/3$ is a critical point.

With the Lorentzian metric 9.1

$$ds^2=-a^{4-3\gamma}d\tau^2+a^2(\tau)d\Omega_3^2 \tag{A.10}$$

the scale factor is the same as that of CM[8, 9], namely 9.8

$$a(\tau)=[a_m^{3\gamma-2}-(1-3\gamma/2)^2\tau^2]^{1/(3\gamma-2)}\ . \tag{A.11}$$

Here a_m corresponds to the maximum size of the FRW Universe. The scale factor ϕ satisfies the relation

$$\frac{a^{3\gamma-4}}{2}\,(\frac{d\phi}{d\tau})^2+V(\phi)=\rho\ , \tag{A.12}$$

where $V(\phi)$ is the scalar field potential that we are about to determine explicitely, and the ρ is the energy density of the perfect fluid bulk matter source. We recall from 9.4 that it reads

$$\rho=\rho_m a_m^{3\gamma}\,a^{-3\gamma}\ , \tag{A.13}$$

for an arbitrary constant ρ_m. The conservation of energy requires

$$\rho^{-1} a^{3\gamma-4} \left(\frac{d\phi}{d\tau}\right)^2 = \gamma = constant. \tag{A.14}$$

Substitute equation A.13 into equation A.14, to obtain the relation

$$\frac{d\phi}{d\tau} = \pm\sqrt{\rho_m \gamma a_m^{3\gamma}}\, a^{2-3\gamma} \,. \tag{A.15}$$

Now use equation A.11 to write

$$\phi(\tau) = \pm\sqrt{\rho_m \gamma a_m^{3\gamma}} \int_\tau dt' \frac{1}{[a_m^{3\gamma-2} - (1-3\gamma/2)^2 t'^2]} \,. \tag{A.16}$$

For convenience, we set the integration constant to zero. Depending on the sign $\pm$, we obtain two solutions to the integral. In CM[9], the $+$ sign gives the *arctanh* solution. This solution we call *interior* in the sense that the integral is defined for a closed Lorentzian universe. The model starts off from zero radius, evolves along classical lines to a maximum radius a_m and recollapses.

On the other hand, the $-$ sign leads to an *arccoth* solution:

$$\phi(\tau) = \frac{\sqrt{2\gamma_e}}{3\gamma_e - 2} \coth^{-1}\left[\frac{3\gamma_e - 2}{2a_m^{(3\gamma_e-2)/2}}\tau\right] . \tag{A.17}$$

In this case we have continued the evolution analytically *pass* its classical maximum, into a forbidden Euclidean regime which we call the *exterior* solution. We have also introduced the Euclidean index γ_e into our formalism. This will correspond to the index described for the instantons found by CM[9], for which $a \to \infty$ as $\tau \to \infty$. We now use equation A.12 to derive the scalar field potential, and find

$$V(\phi) = (-1)^{\frac{3\gamma_e}{3\gamma_e-2}} \frac{1-\gamma_e/2}{a_m^2} \sinh^{\frac{6\gamma_e}{3\gamma_e-2}}\left[\frac{3\gamma_e-2}{\sqrt{2\gamma_e}}\phi\right] . \tag{A.18}$$

There appears to be some ambiguity in the sign of this potential. If we take the scalar field to be imaginary, $\phi \to i\phi$, and if we then correctly write sin instead of sinh for the potential (A.18), we are able to remove this ambiguity.

Provided that $|\phi| \ll 1$, the potential(A.18) approximates to

$$V(\phi) = \frac{\kappa}{2q}\phi^{2q} \, , \tag{A.19}$$

of HP[85], where we have abbreviated

$$q = \frac{3\gamma_e}{3\gamma_e - 2} \, , \tag{A.20}$$

and

$$\kappa = \frac{1 - \gamma_e/2}{a_m^2} \frac{6\gamma_e}{3\gamma_e - 2} \left[\frac{3\gamma_e - 2}{\sqrt{2\gamma_e}} \right]^{\frac{6\gamma_e}{3\gamma_e - 2}} \, . \tag{A.21}$$

The important result is of course the exterior relation A.20. It is then possible to use the expansions of refs. [95, 96] to derive the appropriate wormhole wave functions. Alternatively, we may use our own formalism of Section 8.4 for a qualitative discussion.

A.2 (b) The Lorentzian interior

For closed Lorentzian bulk matter universe models we derived the ordinary differential equations 9.27 and 9.28, which we rewrite as

$$\left[\frac{d^2}{d\xi^2} + \epsilon^2 \right] \, \Xi(\xi) = 0 \tag{A.22}$$

with separation constant ϵ, and

$$\left[a^2 \frac{d^2}{da^2} + a \frac{d}{da} + (\epsilon^2 + V_m) a^{6-3\gamma} - a^4 \right] \, X(\, a \,) = 0 \, . \tag{A.23}$$

with the wave function

$$\Psi(a, \xi) = X(\, a \,) \, \Xi(\, \xi \,) \, . \tag{A.24}$$

We briefly compare these equations with that of the power-law potential of Section 8.4. For very small ϕ, and high order modes such that

$$\lambda \simeq \epsilon^2 \gg \frac{\kappa}{2q} = V_m \ , \tag{A.25}$$

we may say that the equations for the power-law potential approximates to equations A.22 and A.23, under condition that we compare the indices of the scale factor in the energy terms:

$$6 - 3\gamma = 4 + \frac{2 - 4q}{1 + q}$$

and solving for q, we arrive at relation A.6. This completes our argument for fixing $\alpha = \frac{1}{3}$.

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