Classification of second-order conformally-superintegrable systems

Author:
Capel, Joshua

Publication Date:
2014

DOI:
https://doi.org/10.26190/unsworks/16817

Classification of Second-Order Conformally-Superintegrable Systems

Joshua Jordan Capel

A dissertation submitted in fulfilment
of the requirements for the degree of
Doctor of Philosophy

The School of Mathematics and Statistics
The University of New South Wales

Submitted for examination: 31 October 2013
Final Revision: 6 June 2014

PLEASE TYPE

THE UNIVERSITY OF NEW SOUTH WALES
Thesis/Dissertation Sheet

Surname or Family name: CAPEL

First name: JOSHUA

Other name/s: JORDAN

Abbreviation for degree as given in the University calendar:

School: SCHOOL OF MATHEMATICS AND STATISTICS

Faculty: FACULTY OF SCIENCE

Title:

Classification of Second-Order Conformally-Superintegrable Systems

Abstract 350 words maximum: (PLEASE TYPE)

Over the last half century the study of superintegrable systems has established itself as an interesting subject with connections to some of the earliest known dynamical systems in mathematical-physics. Systems with constants second-order in the momenta have been particularly well studied in recent years. This thesis provides a classification of non-degenerate (maximum parameter) three-dimensional second-order superintegrable systems over conformally-flat spaces. I show that, up to Stäckel equivalence, such systems can be put into correspondence with a 6 points in the extended complex plane with an action induced by the conformal-group in three dimensions. I use this correspondence, and the tools of classical-invariant theory, to determine the inequivalent orbits under this action and show there are only 10 conformal-classes. This answers an open problem by showing that no unknown systems exist on the sphere.

Additional interest in these systems comes from studying their algebra of constants. In the three-dimensional maximum-parameter case this algebra is generated by the iterated Poisson brackets of the 6 linearly independent second-order constants and is known to close at finite order. These 6 second-order constants are necessarily functionally dependent, and up to now the explicit relation for their dependence has only been known on a case-by-case basis. In this thesis I demonstrate a quartic identity which provides the functional relation for a general system.

Originality Statement

I hereby declare that this submission is my own work and to the best of my knowledge it contains no materials previously published or written by another person, or substantial proportions of material which have been accepted for the award of any other degree or diploma at UNSW or any other educational institution, except where due acknowledgement is made in the thesis. Any contribution made to the research by others, with whom I have worked at UNSW or elsewhere, is explicitly acknowledged in the thesis. I also declare that the intellectual content of this thesis is the product of my own work, except to the extent that assistance from others in the project's design and conception or in style, presentation and linguistic expression is acknowledged.

Signed Joshua Jordan Capel
Date 31 October 2013

Copyright Statement

I hereby grant the University of New South Wales or its agents the right to archive and to make available my thesis or dissertation in whole or part in the University libraries in all forms of media, now or here after known, subject to the provisions of the Copyright Act 1968. I retain all proprietary rights, such as patent rights. I also retain the right to use in future works (such as articles or books) all or part of this thesis or dissertation. I also authorise University Microfilms to use the 350 word abstract of my thesis in Dissertation Abstract International (this is applicable to doctoral theses only). I have either used no substantial portions of copyright material in my thesis or I have obtained permission to use copyright material; where permission has not been granted I have applied/will apply for a partial restriction of the digital copy of my thesis or dissertation.

Signed Joshua Jordan Capel
Date 31 October 2013

Authenticity Statement

I certify that the Library deposit digital copy is a direct equivalent of the final officially approved version of my thesis. No emendation of content has occurred and if there are any minor variations in formatting, they are the result of the conversion to digital format.

Signed Joshua Jordan Capel
Date 31 October 2013

Abstract

Over the last half century the study of superintegrable systems has established itself as an interesting subject with connections to some of the earliest known dynamical systems in mathematical-physics. Systems with constants second-order in the momenta have been particularly well studied in recent years. This thesis provides a classification of non-degenerate (maximum parameter) three-dimensional second-order superintegrable systems over conformally-flat spaces. I show that, up to Stäckel equivalence, such systems can be put into correspondence with a 6 points in the extended complex plane with an action induced by the conformal-group in three dimensions. I use this correspondence, and the tools of classical-invariant theory, to determine the inequivalent orbits under this action and show there are only 10 conformal-classes. This answers an open problem by showing that no unknown systems exist on the sphere.

Additional interest in these systems comes from studying their algebra of constants. In the three-dimensional maximum-parameter case this algebra is generated by the iterated Poisson brackets of the 6 linearly independent second-order constants and is known to close at finite order. These 6 second-order constants are necessarily functionally dependent, and up to now the explicit relation for their dependence has only been known on a case-by-case basis. In this thesis I demonstrate a quartic identity which provides the functional relation for a general system.

Acknowledgements

This thesis would not have been possible without the help of so many people in so many ways. I am grateful to my supervisor, Dr Jonathan Kress for the guidance, insight, encouragement and patience he gave to me during my candidature.

I'd also like to thank the academic, administrative and computing staff at the UNSW School of Mathematics and Statistics for their technical assistance, encouragement and support during my candidature.

To my friends and fellow PhD students, Amirah Rahman, Helen Macdonald, Michelle Dunbar, thank you for your support, encouragement and friendship. And finally thank you to my family for always supporting me, even when the road seemed long and unending.

Dedication

To Lee and Dinh, whose love and support helped me to persevere.

Contents

List of Figures

List of Tables

Chapter 1

Introduction

This thesis is concerned with maximum-parameter (non-degenerate) superintegrable systems over conformally-flat complex spaces with constants of the motion second-order in the momenta. Two-dimensional systems over conformally-flat spaces and three-dimensional systems over flat space (both complex) have both been completely classified [28, 16, 17]. The classifications use, in part, algebraic varieties (i.e. the zero set of *irreducible* polynomial ideals). These algebraic varieties come from examining the integrability conditions for such systems. The varieties are foliated by the nonlinear action of a Lie group (e.g. the Euclidean group or the conformal group) and the leaves of this foliation, or more precisely, the closure of the leaves with respect to the classical topology, are described by subvarieties. Working over complex spaces guarantees these subvarieties are connected (even if points from other algebraic sets are removed) and this fact simplifies the arguments needed to give a complete classification.

Before presenting the results obtained in this thesis a brief historical account of superintegrability will be given. A more complete introduction can be found in the upcoming review of Miller, Post and Winternitz [60].

1.1 Historical Background

Dynamical systems with exact-solutions have played an in important role in the development of mathematics and physics. In classical mechanics these system are characterised by there being sufficiently many constants of motion to allow the trajectories to be determined via integration (if the energy level set of an orbit is compact then this integration corresponds to the well known action-angle coordinates). An important subclass of these systems are given by systems that possess an overabundance of constants of motion, placing tighter algebraic restrictions on the trajectories obtained. In the last half-century the study and recognition of these types of systems has been growing steadily, and these systems have been given a new name: 'superintegrable' (older terms for such systems include: 'degenerate integrability' and 'non-Abelian integrability').

Despite the recency of the term 'superintegrable' such systems are actually amongst the oldest studied, two prime examples being the harmonic oscillator and the Kepler-Coulomb potentials. One of the ways the superintegrability of the Kepler-Coulomb potential manifests itself in the classical case is through Kepler's three laws of planetary motion. These relations were determined in the early 17th century from data collected at that time and are quite striking for their algebraic simplicity. These laws predate the discovery of the inverse square law and the Newton/Leibniz invention of calculus, both of which were needed to investigate this system thoroughly.

The superintegrability of the Kepler-Coulomb potential also manifest itself in its quantum mechanical analogue, the hydrogen atom. Here the spectrum of the energy operator (the Balmer series) is calculable from algebraic methods alone. The Balmer series, like Kepler's three laws, was determined from observational data, and, somewhat analogously to the classical case, predated the invention of Schrödinger's wave equation and the foundation of modern quantum mechanics. Both facts follow from their respective superintegrability and, at least in the quantum case, many superintegrable system known today share similar properties.

The systematic search for superintegrable systems can be traced back to the 1965-67 papers of Y. A. Smorodinskii, P. Winternitz *et al* [34, 61, 1] who were interested in two and three-dimensional quantum systems with 'higher symmetries', meaning symmetries that do not come from geometric symmetries.

The phrase 'superintegrable' comes from the work of Rauch-Wojciechowski in 1983, whose work on the Calogero-Moser system [62] which provided some of the first examples of superintegrable systems with constants of motion of order higher than two.

Interest in superintegrability was spurred at the end of the 20th century by the work of N. W. Evans in 1990 which gave a complete classification of three-dimensional real systems with second-order constants [30].

There has been shown to be close relationship between Exact and Quasi-Exact Solvability (QES) and superintegrable systems [20, 41, 59, 58], and it has been conjectured that all maximally-superintegrable systems are exactly solvable [50].

An important tool in the construction and classification of superintegrable systems is given by the coupling-constant-metamorphosis (CCM) [35] and its close cousin the Stäckel transform [5], both introduced in the mid 1980s. When focusing on second-order superintegrable systems the CCM and Stäckel transform coincide, but for general systems they are distinct [52, 23].

For the last half-century the most extensively known superintegrable cases are those with second-order constants separating in two or more coordinate systems. Second-order integrability (super or not) and separation of variables are intimately connected through theorems of Stäckel [38, 56, 5, 11, 13] showing that separability of a system implies the existence sufficiently many mutually commuting Killing tensors[1] to give integrability and, conversely giving conditions necessary to ensure that second-order integrability gives rise to separability.

This strong connection to separation of variables has lead to powerful classification results regarding systems with second-order constants. However, this close connection also limits these techniques from being generalised to systems with higher-order constants. Recent work of Kalnins *et al* in the classification of two and three-dimensional systems over complex Euclidean space has lead to classification results that explore the algebraic geometry present in these systems [16, 17, 15]. This thesis is a continuation of that work. This is important as recently superintegrable systems have been found which do not possess separation of variables [53].

The most recent interest in superintegrability has been spurred by the 2009 paper of Tremblay, Turbiner and Winternitz [31] containing what are now known as the TTW potentials. The TTW potentials were conjectured to be superintegrable for rational choice of its parameter k, and this conjecture was

[1] the leading part of the second-order constants are killing tensors for the pseudo-Riemannian manifold

validate shortly thereafter [54, 19, 22, 37]. These systems provides examples of superintegrable systems with constants of arbitrarily high degree.

By generalising the TTW systems from two dimensions to higher dimensions, superintegrable systems have been found over non-conformally flat spaces, possessing constants of arbitrarily high degree [18].

Another aspect, and probably the primary aspect, of superintegrable systems that has been of interest to the research community is the various links to special function theory that exist. Special functions can arise as the eigenfunction for the symmetry operators in quantum superintegrable systems, and in the interbasis expansion coefficients for multiseparable systems [21, 29]. They also arise in models of quadratic algebras of superintegrable systems [51]. Very recently the Askey-Wilson scheme for orthogonal polynomials has been shown to arise from the contractions of models of the quadratic algebras of second-order superintegrable systems [25].

A family of exactly-solvable two-dimensional Hamiltonians has been found with wave functions related to Laguerre and exceptional Jacobi polynomials [55]. And superintegrable systems with non-polynomial constants are also known [42].

Tool such as supersymmetry and ladder operators have been used in the construction of superintegrable systems [45, 46, 47]. And superintegrable systems having potentials linked to Painlevé transcendents have also been found [44].

Superintegrable systems have also appeared in applications, such as studying the Hartmann potential [39, 43], which was introduced in quantum chemistry to describe ring-shaped molecules like benzene [3].

1.2 Outline of this thesis

Chapter 2 begins by presenting the idea of a second-order superintegrable system over a conformally-flat space and discusses the classical structure theory as derived by Kalnins, Kress and Miller [12]. After this relevant background material is provided the chapter concludes by generalising a result that simplified the differential closure of the ideal of integrability conditions in Euclidean space [17]. The corresponding result is given for spaces of constant curvature.

In chapter 3 the local action of the rotation group $SO(3,\mathbb{C})$ is examined and a large set of rotation representations is defined. These representations are used to examine the space of higher-order constants, recreating for the most part the work of Kalnins *et al* [12] regarding the closure of the quadratic algebra. The use of rotation representations allows the calculations to be carried further than previous attempts and analyses the space of eighth-order constants. The results of this analysis closes an open question in the literature [36] by proving that the functional relation between the 6 second-order constants and the 5 parameters is a quartic relation. The form of this quartic relation is given explicitly for a general system.

In chapter 4 the notion of a conformally-superintegrable (Laplace-type) system is discussed. How these are related, by Stäckel transform and conformal-scaling, to the superintegrable systems in chapter 2 is also discussed. The action of the conformal group on these systems is then examined and the concepts necessary for the classification results in chapter 5 are presented. Specifically it is shown that the classification of such systems depends solely on a single 7-dimensional $SO(3,\mathbb{C})$ representation.

Chapter 5 gives the full classification of maximally-superintegrable, second-order maximum-parameter systems by determining 10 conformal-classes of Stäckel equivalent systems. These classes are determined by studying the orbits of the conformal group on a seven-dimensional manifold. The orbits are completely described by a set of algebraic varieties. The subvariety structure obtained by examining the containment of the associated polynomial ideals reveals a hierarchical structure consistent with the hierarchical structures already known [26].

In chapter 6 conclusions are drawn and the future directions that this research could take are discussed.

Appendix A contains a description of the connection between $SO(3,\mathbb{C})$ representations and $SL(2,\mathbb{C})$ representations. This information is particularly important for the techniques employed in chapter 3.

Appendix B defines a Hilbert basis for the space of covariants of a 6th degree binary form. This 6th degree binary form, and the covariants constructed from it, carry representations of the conformal group in three dimensions. This Hilbert basis is used to concisely describe the classification results in chapter 5.

And finally, for ease of reference, appendix C contains a list of the various notations and variable names used throughout this thesis.

Chapter 2

Second-Order Superintegrable Systems

After reviewing the necessary concepts needed to study classical mechanics, this chapter will then review the classical structure theory for three-dimensional second-order superintegrable systems, this working follows closely that of Ref. [12], which is one of the five papers by Kalnins, Kress and Miller that established the structure theory for second-order superintegrable systems in 2 and 3 dimensions [10, 11, 12, 13, 14].

In §2.1 Hamiltonian mechanics is briefly discussed and definitions relevant to the systems studied in this thesis are given.

In §2.2 the definition of a maximally superintegrability system is given.

In §2.3 the structure result regarding second-order superintegrable systems over three-dimensional conformally-flat space is reviewed, following the discussion in Ref. [12]. Importantly it is shown that the potential of a maximally superintegrable second-order system satisfies a set of linear PDEs which admits solution depending on 5 parameters (including the one trivial additive term).

In §2.4 the Stäckel transform is defined and it is shown how it can be used to map between second-order superintegrable systems over different conformally flat spaces. Only distinguishing systems up to Stäckel equivalence make the classification easier.

Finally, in §2.5 the algebraic ideal of integrability conditions for superintegrable potentials on the 3-spheres is shown to be generated by 6 quadratics, generalising a result that was only previously known for flat space.

2.1 Hamiltonian Mechanics

An important method in classical mechanics is the *Hamiltonian formalism.* This allows the dynamics of a classical system to be encoded by a single function over the position-momentum phase space. In this thesis only autonomous (time-independent) Hamiltonians are considered. More details on Hamiltonian mechanics can be found in, for example, V.I. Arnold's *"Classical Mechanics"* [2].

Start by considering an n-dimensional momentum-position phase space $(\mathbf{p}, \mathbf{q}) \in \mathbb{C}^{2n}$, where the components of $\mathbf{q}$ are the position coordinates $q_i(t)$, the components of $\mathbf{p}$ are the generalised-momentum coordinate $p_i(t)$ and t denotes the time. In this thesis only natural Hamiltonians of the form $H =$ "kinetic energy" + "potential energy" will be considered. More precisely, the following definition will be used for the Hamiltonian.

Definition 2.1.1. A classical system defined over a (pseudo)-Riemannian manifold with contravariant metric tensor $g^{ij}(\mathbf{q})$ and potential function $V(\mathbf{q})$ has a *Hamiltonian* of the form

$$\mathcal{H}(\mathbf{p}, \mathbf{q}) = \sum_{i,j=1}^{3} g^{ij}(\mathbf{q}) p_i p_j + V(\mathbf{q}). \tag{2.1.2}$$

The metric is non-degenerate (i.e. $\det(g^{ij}) \neq 0$) and symmetric ($g^{ij} = g^{ji}$). Furthermore, as the spaces considered in this thesis are conformally flat, there will a coordinate system such that the metric takes the form $g^{ij} = \lambda(\mathbf{q})\delta^i_j$.

The dynamics of such a system are given by Hamilton's equations, which are the differential equations

$$\frac{dq_i}{dt} = \frac{\partial \mathcal{H}}{\partial p_i}, \quad \frac{dp_i}{dt} = -\frac{\partial \mathcal{H}}{\partial q_i}, \qquad i = 1, \ldots, n. \tag{2.1.3}$$

Solutions of these give the trajectories of the system.

Definition 2.1.4. If $\mathcal{A}(\mathbf{p}, \mathbf{q}), \mathcal{B}(\mathbf{p}, \mathbf{q})$ are function on the phase space, their *Poisson commutator* or *Poisson bracket* is defined to be

$$\{\mathcal{A}, \mathcal{B}\}_{PB} = \sum_{i=1}^{n} \left(\frac{\partial \mathcal{A}}{\partial p_i} \frac{\partial \mathcal{B}}{\partial q_i} - \frac{\partial \mathcal{A}}{\partial q_i} \frac{\partial \mathcal{B}}{\partial p_i} \right). \tag{2.1.5}$$

The Poisson bracket is a bi-linear, skew-symmetric Lie bracket. That is, given three functions on the phase space $\mathcal{A}(\mathbf{p},\mathbf{q}),\mathcal{B}(\mathbf{p},\mathbf{q}),\mathcal{C}(\mathbf{p},\mathbf{q})$, the Poisson bracket can be shown to satisfy the following identities

$$\begin{aligned}\{\mathcal{A},\mathcal{B}\}_{PB} &= -\{\mathcal{B},\mathcal{A}\}_{PB}\,,\\ \{\mathcal{A},s\mathcal{B}+t\mathcal{C}\}_{PB} &= s\{\mathcal{A},\mathcal{B}\}_{PB}+t\{\mathcal{A},\mathcal{C}\}_{PB}\,,\\ \{\mathcal{A},\{\mathcal{B},\mathcal{C}\}_{PB}\}_{PB}+\{\mathcal{B},\{\mathcal{C},\mathcal{A}\}_{PB}\}_{PB}+\{\mathcal{C},\{\mathcal{A},\mathcal{B}\}_{PB}\}_{PB} &= 0,\\ \{\mathcal{A},\mathcal{B}\mathcal{C}\}_{PB} &= \mathcal{B}\{\mathcal{A},\mathcal{C}\}_{PB}+\{\mathcal{A},\mathcal{B}\}_{PB}\mathcal{C}.\end{aligned} \tag{2.1.6}$$

Using the Poisson bracket, Hamilton's equations (2.1.3) take on the symmetric form

$$\frac{dq_i}{dt}=\{\mathcal{H},q_i\}_{PB}\,,\quad \frac{dp_i}{dt}=\{\mathcal{H},p_i\}_{PB}\,,\qquad i=1,\ldots,n. \tag{2.1.7}$$

If the Poisson bracket of two functions vanishes they are said to be in involution. Any autonomous function $\mathcal{A}(\mathbf{p},\mathbf{q})$ in involution with the Hamiltonian $\mathcal{H}$ will be a constant along any trajectory thanks to the generic equation

$$\frac{d\mathcal{A}}{dt}=\{\mathcal{H},\mathcal{A}\}_{PB}+\frac{\partial A}{\partial t}. \tag{2.1.8}$$

Such an $\mathcal{A}$ will be referred to as a *constant of the motion*.

The Poisson bracket associates a vector field to each differentiable function on the phase space and in particular the Hamiltonian vector field $X_{\mathcal{H}}$ is given by

$$X_{\mathcal{H}}=\{\mathcal{H},\bullet\}\,. \tag{2.1.9}$$

The flow of the the vector field $X_{\mathcal{H}}$ follows the trajectories of the system.

If $\mathcal{A}$ is a constant of the motion then

$$X_{\mathcal{A}}(\mathcal{H})=\{\mathcal{A},\mathcal{H}\}=-X_{\mathcal{H}}(\mathcal{A})=0$$

and so the Hamiltonian $\mathcal{H}$ is conserved under the flow of the vector field $X_{\mathcal{A}}$. For this reason a constant of the motion $\mathcal{A}$ is often referred to as a *symmetry*.

2.2 Integrability and (Maximal) Superintegrability

An n-dimensional Hamiltonian $\mathcal{S}_1(\mathbf{p},\mathbf{q})=\mathcal{H}(\mathbf{p},\mathbf{q})$ is said to be integrable in the Liouville sense if their exist $n-1$ additional functionally independent constants $\mathcal{S}_i(\mathbf{p}, \mathbf{q})$, $i \in \{2, \dots, n\}$ all of which are in mutual involution. That is

$$\{\mathcal{S}_i, \mathcal{S}_j\}_{PB} = 0, \quad i, j \in \{1, \dots, n\}. \tag{2.2.1}$$

Maximal-*super*integrability is characterised by there being an additional $n - 1$ functionally independent functions $\mathcal{S}_k(\mathbf{p}, \mathbf{q})$, $k \in \{n+1, \dots, 2n-1\}$ in involution with (at least) the Hamiltonian.

In this thesis the maximal-superintegrability assumption will be relaxed to just requiring $2n - 1$ second-order constants of the motion (including the Hamiltonian). So the starting hypothesis is a system with $2n-1$ second-order functions $\mathcal{S}_i$ (with $\mathcal{S}_1 = \mathcal{H}$) such that

$$\{\mathcal{H}, \mathcal{S}_j\}_{PB} = 0, \quad i = 1, \dots, 2n - 1. \tag{2.2.2}$$

That such systems are also integrable is seen to hold a posteriori.

Note that $2n-1$ is the maximum possible number of functionally independent constants that can exist for an unconstrained system, and that locally such constants exist for any Hamiltonian system. However the main interest in this thesis is in constants that are polynomial in the momentum and defined up to the existence of lower-dimensional singularities such as poles and branch points.

2.3 Classical Structure Theory and the Bertrand-Darboux equations

Consider a conformally-flat pseudo-Riemannian manifold, for which coordinates $\mathbf{x} = (x_1, x_2, x_3)$ can be found such that the metric has the form

$$ds^2 = \lambda(\mathbf{x}) \left(dx_1^2 + dx_2^2 + dx_3^2 \right). \tag{2.3.1}$$

The conformal factor $\lambda(\mathbf{x})$ will often be expressed in the form

$$\lambda(\mathbf{x}) = e^{G(\mathbf{x})} \tag{2.3.2}$$

as this tends to give equations that are, at least visually, simpler.

As per (2.1.2) the natural Hamiltonian over this metric is of the form

$$H = \frac{p_{x_1}^2 + p_{x_2}^2 + p_{x_3}^2}{\lambda(\mathbf{x})} + V(\mathbf{x}). \tag{2.3.3}$$

It will be assumed that the Hamiltonian (2.3.3) is second-order maximally superintegrable, that is, admits 4 additional second-order constants of the form

$$L = \sum_{i,j=1}^{3} a^{ij}(\mathbf{x}) p_{x_i} p_{x_j} + W(\mathbf{x}), \quad a^{ij} = a^{ji}. \tag{2.3.4}$$

that Poisson commute with the Hamiltonian, that is

$$\{H, L\}_{PB} = 0. \tag{2.3.5}$$

Splitting the conditions given by (2.3.5) by degree in the momenta gives two independent conditions, one third-order and the other first-order. Explicitly these conditions are

$$\left\{ \frac{p_{x_1}^2 + p_{x_2}^2 + p_{x_3}^2}{\lambda}, \sum_{i,j=1}^{3} a^{ij} p_{x_i} p_{x_j} \right\}_{PB} = 0,$$
$$\left\{ \frac{p_{x_1}^2 + p_{x_2}^2 + p_{x_3}^2}{\lambda}, W \right\}_{PB} + \left\{ V, \sum_{i,j=1}^{3} a^{ij} p_{x_i} p_{x_j} \right\}_{PB} = 0. \tag{2.3.6}$$

The first condition in (2.3.6) implies that the purely second-order part of L is a second-order Killing tensor on the manifold.

Conditions (2.3.6) can be written out explicitly as

$$\sum_{i,j,k=1}^{3} \left(\frac{2}{\lambda} a^{jk}_{,i} p_{x_i} p_{x_j} p_{x_k} + \left(a^{jk} \frac{\lambda_{,j} p_{x_k}}{\lambda^2} + a^{jk} \frac{\lambda_{,k} p_{x_j}}{\lambda^2} \right) p_{x_i} p_{x_i} \right) \equiv 0,$$
$$\sum_{i=1}^{3} \frac{W_{,i}}{\lambda} p_{x_i} - 2 \sum_{i,j=1}^{3} a^{ij} V_{,i} p_{x_j} \equiv 0, \tag{2.3.7}$$

where a comma in the subscript followed by an index i indicates a partial derivative with respect to the variable x_i. Since the momentum is unrestricted the coefficients of the monomials in the p_x's must vanish independently, this gives four types of conditions (for distinct i, j, k)

$$\frac{4}{\lambda}(a^{jk}_{,i} + a^{ik}_{,j} + a^{ij}_{,k}) \equiv 0,$$
$$\frac{4}{\lambda}a^{ik}_{,k} + \frac{2}{\lambda}a^{kk}_{,i} + 2\sum_{r=1}^{3} a^{ri}\frac{\lambda_{,r}}{\lambda^2} \equiv 0,$$
$$\frac{2}{\lambda}a^{ii}_{,i} + 2\sum_{r=1}^{3} a^{ir}\frac{\lambda_{,r}}{\lambda^2} \equiv 0,$$
$$\frac{2}{\lambda}W_{,i} - 2\sum_{r=1}^{3} a^{ir}V_{,r} \equiv 0. \tag{2.3.8}$$

Setting $\lambda = \exp(G)$ the equations above can be rewritten as

$$a^{jk}_{,i} + a^{ik}_{,j} + a^{ij}_{,k} \equiv 0,$$
$$2a^{ij}_{,j} + a^{jj}_{,i} \equiv -\sum_{r=1}^{n} a^{ir}G_{,r} \equiv a^{ii}_{,i},$$
$$W_{,i} \equiv \lambda\sum_{r=1}^{n} a^{ir}V_{,r}. \tag{2.3.9}$$

The compatibility conditions $\partial_{x_i}W_{,j} = \partial_{x_j}W_{,i}$ lead to the Bertrand-Darboux equations

$$\begin{pmatrix} 0 & a^{12} & a^{11}-a^{22} & a^{31} & -a^{23} \\ a^{13} & 0 & -a^{23} & a^{21} & a^{11}-a^{33} \\ a^{23} & -a^{23} & -a^{13} & a^{22}-a^{33} & a^{12} \end{pmatrix} \begin{pmatrix} V_{,33}-V_{,11} \\ V_{,22}-V_{,11} \\ V_{,12} \\ V_{,23} \\ V_{,13} \end{pmatrix}$$
$$= \frac{1}{\lambda}\begin{pmatrix} (\lambda a^{12})_{,1} - (\lambda a^{11})_{,2} \\ (\lambda a^{13})_{,1} - (\lambda a^{11})_{,3} \\ (\lambda a^{13})_{,2} - (\lambda a^{12})_{,3} \end{pmatrix} V_{,1} + \frac{1}{\lambda}\begin{pmatrix} (\lambda a^{22})_{,1} - (\lambda a^{12})_{,2} \\ (\lambda a^{23})_{,1} - (\lambda a^{12})_{,3} \\ (\lambda a^{23})_{,2} - (\lambda a^{22})_{,3} \end{pmatrix} V_{,2} + \frac{1}{\lambda}\begin{pmatrix} (\lambda a^{23})_{,1} - (\lambda a^{13})_{,2} \\ (\lambda a^{33})_{,1} - (\lambda a^{13})_{,3} \\ (\lambda a^{33})_{,2} - (\lambda a^{23})_{,3} \end{pmatrix} V_{,3}. \tag{2.3.10}$$

Each second-order constant gives a set of conditions and as will be discussed in the next section, for sufficiently many independent constants these can be used to give a set of linear PDEs governing $V(\mathbf{x})$.

Maximum-Parameter (non-degenerate) Potentials

Most of this section is based on the 2005 paper of Kalnins *et al* [12], however that paper was found to contain a slight error and the correct statements about the relationship between functional-linear independence and maximum-parameter (non-degenerate) systems can be found in their subsequent 2006 paper [14].

The Bertrand-Darboux equations (2.3.10) for the four independent second-order constants (not including the Hamiltonian) can be written in the form $B\mathbf{v} = \mathbf{b}$ where B is a 12×5 matrix, $\mathbf{b}$ is a 12×1 vector and

$$\mathbf{v} = \begin{pmatrix} V_{,33} - V_{,11} \\ V_{,22} - V_{,11} \\ V_{,12} \\ V_{,23} \\ V_{,13} \end{pmatrix}.$$

If it is assumed that the potential contains sufficiently many parameters such that the value of $V_{,1}, V_{,2}$ and $V_{,3}$ can freely be specified at a generic point (that is, the potential depends non-degenerately on the three coordinates) then the second-order constants are functionally-linearly independent (i.e. the only functions $m_i(\mathbf{x})$ satisfying $\sum_{i=1}^4 m_i(\mathbf{x})S_i(\mathbf{x}) \equiv 0$ are the trivial ones). The functionally-linearly independence of the four constants means the matrix

$$A = \begin{pmatrix} a^{33}_{(1)} - a^{11}_{(1)}, & a^{33}_{(1)} - a^{11}_{(1)}, & a^{12}_{(1)}, & a^{12}_{(1)}, & a^{23}_{(1)} \\ a^{33}_{(2)} - a^{11}_{(2)}, & a^{33}_{(2)} - a^{11}_{(2)}, & a^{12}_{(2)}, & a^{12}_{(2)}, & a^{23}_{(2)} \\ a^{33}_{(3)} - a^{11}_{(3)}, & a^{33}_{(3)} - a^{11}_{(3)}, & a^{12}_{(3)}, & a^{12}_{(3)}, & a^{23}_{(3)} \\ a^{33}_{(4)} - a^{11}_{(4)}, & a^{33}_{(4)} - a^{11}_{(4)}, & a^{12}_{(4)}, & a^{12}_{(4)}, & a^{23}_{(4)} \end{pmatrix}$$

has rank 4 at a generic point, where $a^{ij}_{(k)}$ are the coefficients in the four constants.

From the fact that the matrix A above has rank 4 it can be shown that the 12×5 matrix B has rank 5. Thus there exists a 5×5 submatrix which is invertible in an open set around the generic point and hence the vector $\mathbf{v}$ can be solved for. This means the potential V satisfies a set of 5 linear, second-order PDEs. These PDEs can be written in the symmetric (but redundant) form,

$$\begin{aligned}
V_{,11} &= \mathcal{V}_{ee} + A_1^{11}V_{,1} + A_2^{11}V_{,2} + A_3^{11}V_{,3}, \\
V_{,22} &= \mathcal{V}_{ee} + A_1^{22}V_{,1} + A_2^{22}V_{,2} + A_3^{22}V_{,3}, \\
V_{,33} &= \mathcal{V}_{ee} + A_1^{33}V_{,1} + A_2^{33}V_{,2} + A_3^{33}V_{,3}, \\
V_{,12} &= \qquad A_1^{12}V_{,1} + A_2^{12}V_{,2} + A_3^{12}V_{,3}, \\
V_{,13} &= \qquad A_1^{13}V_{,1} + A_2^{13}V_{,2} + A_3^{13}V_{,3}, \\
V_{,23} &= \qquad A_1^{23}V_{,1} + A_2^{23}V_{,2} + A_3^{23}V_{,3}.
\end{aligned} \tag{2.3.11}$$

where

$$\mathcal{V}_{ee} = \frac{(V_{,11} + V_{,22} + V_{,33})}{3}.. \tag{2.3.12}$$

Here the coefficient functions A_k^{ij} are unknown, but cannot depend in any way on the parameters in V. The five parameters are the value of the potential V, the value of the 3 first order derivatives $V_{,1}$, $V_{,2}$, $V_{,3}$ and the combination of the second-order derivatives $\mathcal{V}_{ee}$ above[1].

The symmetric form of equations (2.3.11) comes at the price of redundancy. To avoid placing restriction on the first order parameters the coefficient functions must satisfy the 3 linear conditions

$$A_i^{11} + A_i^{22} + A_i^{33} = 0, \quad i \in \{1, 2, 3\}. \tag{2.3.13}$$

The second-order superintegrable systems in this thesis will be assumed to depend on all five parameters (maximum-parameters) and this assumption allows integrability conditions to be obtained relating the A_k^{ij}'s and their derivatives (2.3.11). The simplest integrability conditions are given by 5 linear conditions

$$\begin{aligned}
A_1^{22} - A_2^{12} &= A_1^{33} - A_3^{13}, \\
A_2^{11} - A_1^{12} &= A_2^{33} - A_3^{23}, \\
A_3^{11} - A_1^{13} &= A_3^{22} - A_2^{23}, \\
A_3^{12} &= A_2^{13}, \\
A_3^{12} &= A_1^{23}.
\end{aligned} \tag{2.3.14}$$

[1] In the papers by Kalnins *et al* the chosen second order parameter was $V_{,11}$, however the choice of $\mathcal{V}_{ee}$ has the advantage of not favoring any particular coordinate direction.

Taking the conditions (2.3.13) and (2.3.14) together puts 8 linearly independent conditions on 18 coefficient functions A_k^{ij} and hence there can be at most 10 independent ones. Continuing the theme of notational symmetry, these coefficient functions can be parameterised by in the following way,

$$\begin{array}{lll}
A_1^{11} = -4S^1 - R_2^{12} - R_3^{13}, & A_2^{11} = 2S^2 + R_1^{12}, & A_3^{11} = 2S^3 + R_1^{13}, \\
A_1^{22} = 2S^1 + R_2^{12}, & A_2^{22} = -4S^2 - R_1^{12} - R_3^{23}, & A_3^{22} = 2S^3 + R_2^{23}, \\
A_1^{33} = 2S^1 + R_3^{13}, & A_2^{33} = 2S^2 + R_3^{23}, & A_3^{33} = -4S^3 - R_1^{13} - R_2^{23}, \\
A_1^{23} = Q^{123}, & A_2^{23} = R_2^{23} - 3S^3, & A_3^{23} = R_3^{23} - 3S^2, \\
A_1^{13} = R_1^{13} - 3S^3, & A_2^{13} = Q^{123}, & A_3^{13} = R_3^{13} - 3S^1, \\
A_1^{12} = R_1^{12} - 3S^2, & A_2^{12} = R_2^{12} - 3S^1, & A_3^{12} = Q^{123},
\end{array} \qquad (2.3.15)$$

where 10 new variable names have been defined

$$(Q^{123}, R_1^{12}, R_2^{12}, R_1^{13}, R_3^{13}, R_2^{23}, R_3^{23}, S^1, S^2, S^3). \qquad (2.3.16)$$

These ten functions will be referred to as the set $\{\mathbf{Q}, \mathbf{R}, \mathbf{S}\}$ if needed. These variables were not chosen just because they are symmetric, but also because the sets $\{\mathbf{Q}, \mathbf{R}\}$ and $\{\mathbf{S}\}$ form irreducible rotation representations (as will be shown in chapter 3).

The $(5 \implies 6)$ Theorem

Another very useful result from the literature is the so called $(5 \implies 6)$ theorem which guarantees that the 5 functionally-independent second-order constants (including the Hamiltonian) can be extended to a set of six linearly-independent second-order constants. The importance of this theorem is, at any regular point in the system, and for any prescribed values of the six a^{ij}'s, there will be a second-order constant for which the a^{ij}'s take on the these values at the regular point. This fact can then be used when examining the consistency conditions from (2.3.7) and (2.3.11) and puts very strong restriction on the the derivatives of the $\{\mathbf{Q}, \mathbf{R}, \mathbf{S}\}$. In fact all the first derivative of the $\{\mathbf{Q}, \mathbf{R}, \mathbf{S}\}$ variables can be written as quadratics in the $\{\mathbf{Q}, \mathbf{R}, \mathbf{S}\}$ variables and the derivatives of $G(\mathbf{x}) = \ln(\lambda)$. Up to a permutation of indices, these 30 derivatives are described by the following six equations

$$\frac{\partial R_1^{12}}{\partial x_1} = -\frac{2}{3}R_2^{12}R_3^{23} + \frac{2}{3}R_3^{13}R_3^{23} + \frac{4}{3}Q^{123}R_2^{23} + \frac{5}{3}Q^{123}R_1^{13}$$
$$-R_1^{12}R_3^{13} - R_1^{12}\left(S^1 - \frac{1}{3}G_{,1}\right) + \left(R_3^{13} + 3R_2^{12}\right)\left(S^2 - \frac{1}{3}G_{,2}\right) + 2Q^{123}\left(S^3 - \frac{1}{3}G_{,3}\right), \tag{2.3.17}$$

$$\frac{\partial R_1^{12}}{\partial x_2} = \frac{3}{5}R_2^{12}R_3^{13} - \frac{1}{15}R_1^{13}R_2^{23} - \frac{11}{15}R_1^{12}R_3^{23} + \frac{8}{15}\left(R_1^{13}\right)^2 + \frac{1}{5}\left(R_1^{12}\right)^2 - \frac{4}{5}\left(R_2^{23}\right)^2 + \frac{8}{15}\left(R_3^{13}\right)^2$$
$$+ \frac{1}{5}\left(R_2^{12}\right)^2 - \frac{4}{5}\left(R_3^{23}\right)^2 + \frac{2}{15}\left(Q^{123}\right)^2$$
$$- \left(R_3^{13} + 3R_2^{12}\right)\left(S^1 - \frac{1}{3}G_{,1}\right) - R_1^{12}\left(S^2 - \frac{1}{3}G_{,2}\right) + R_1^{13}\left(S^3 - \frac{1}{3}G_{,3}\right), \tag{2.3.18}$$

$$\frac{\partial R_1^{12}}{\partial x_3} = -\frac{1}{3}Q^{123}R_2^{12} - \frac{1}{3}Q^{123}R_3^{13} + \frac{1}{3}R_2^{23}R_1^{12} + \frac{1}{3}R_3^{23}R_1^{13}$$
$$- 2Q^{123}\left(S^1 - \frac{1}{3}G_{,1}\right) - R_1^{13}\left(S^2 - \frac{1}{3}G_{,2}\right) - R_1^{12}\left(S^3 - \frac{1}{3}G_{,3}\right), \tag{2.3.19}$$

$$\frac{\partial Q^{123}}{\partial x_1} = \frac{2}{3}R_1^{13}R_1^{12} - \frac{1}{3}R_3^{23}R_1^{13} + Q^{123}R_3^{13} - \frac{1}{3}R_2^{23}R_1^{12} + Q^{123}R_2^{12}$$
$$-Q^{123}\left(S^1 - \frac{1}{3}G_{,1}\right) + \left(R_2^{23} - R_1^{13}\right)\left(S^2 - \frac{1}{3}G_{,2}\right) + \left(R_3^{23} - R_1^{12}\right)\left(S^3 - \frac{1}{3}G_{,3}\right). \tag{2.3.20}$$

$$\frac{\partial\left(S^1 - \frac{1}{3}G_{,1}\right)}{\partial x_1} = -\frac{17}{90}R_2^{12}R_3^{13} + \frac{1}{30}R_1^{13}R_2^{23} + \frac{1}{30}R_1^{12}R_3^{23}$$
$$- \frac{7}{45}\left(R_3^{13}\right)^2 + \frac{1}{15}\left(R_3^{23}\right)^2 - \frac{7}{45}\left(R_1^{12}\right)^2 - \frac{11}{90}\left(Q^{123}\right)^2 - \frac{7}{45}\left(R_1^{13}\right)^2 - \frac{7}{45}\left(R_2^{12}\right)^2$$
$$+ \frac{1}{15}\left(R_2^{23}\right)^2 + \frac{1}{2}\left(S^2 - \frac{1}{3}G_{,2}\right)^2 + \frac{1}{2}\left(S^3 - \frac{1}{3}G_{,3}\right)^2 - \frac{1}{2}\left(S^1 - \frac{1}{3}G_{,1}\right)^2, \tag{2.3.21}$$

$$\frac{\partial\left(S^1 - \frac{1}{3}G_{,1}\right)}{\partial x_2} = -\frac{1}{9}R_3^{13}R_3^{23} - \frac{2}{9}Q^{123}R_2^{23} + \frac{1}{9}R_1^{12}R_3^{13} + \frac{1}{9}R_2^{12}R_3^{23} - \frac{2}{9}Q^{123}R_1^{13}$$
$$- \left(S^1 - \frac{1}{3}G_{,1}\right)\left(S^2 - \frac{1}{3}G_{,2}\right). \tag{2.3.22}$$

Here, to simplify the presentation, the coefficient functions S^i and the partial derivative $G_{,i}$ have been combined together. This is especially noticeable in the left-hand sides of (2.3.21) and (2.3.22) where second-order derivatives of $G(\mathbf{x})$ have been "hidden" by combining them with first-order derivatives of S^i. An indication of why the combination $S^i - \frac{1}{3}G_i$ appears here will become more apparent when the Stäckel transform is discussed in the next section, and again in chapter 4 when the conformal equivalence of these potentials is discussed.

There are still integrability conditions remaining relating the values of $\{\mathbf{Q}, \mathbf{R}, \mathbf{S}\}$ to the derivatives of $G(\mathbf{x})$, so the these values cannot be specified arbitrarily. These conditions are needed to ensure the coefficient functions are compatible with the metric. The lowest order integrability conditions are quadratic in $\{\mathbf{Q}, \mathbf{R}, \mathbf{S}\}$, two illustrative examples being

$$\begin{aligned}\frac{4}{3}R_2^{12}R_3^{13} - \frac{4}{3}R_3^{23}R_1^{12} + \frac{4}{3}R_1^{13\,2} + \frac{4}{3}R_3^{13\,2} - \frac{4}{3}R_3^{23\,2} - \frac{4}{3}R_2^{23\,2}\\ - 4(S^1 + \frac{1}{6}G_{,1})R_2^{12} + 4(S^2 + \frac{1}{6}G_{,2})R_1^{12}\\ + 2(S^3 + \frac{1}{6}G_{,3})R_1^{13} - 2(S^1 + \frac{1}{6}G_{,1})R_3^{13} + 2(S^2 + \frac{1}{6}G_{,2})R_3^{23} - 2(S^3 + \frac{1}{6}G_{,3})R_2^{23}\\ - 2G_{,1}S^1 + 2G_{,2}S^2 - 6S^{1^2} + 6S^{2^2} + \frac{1}{3}G_{,1}^2 - \frac{1}{3}G_{,2}^2 - G_{,11} + G_{,22} \equiv 0,\end{aligned} \tag{2.3.23}$$

$$\begin{aligned}+ \frac{4}{3}Q^{123}R_1^{13} + \frac{4}{3}Q^{123}R_2^{23} - \frac{2}{3}R_3^{13}R_1^{12} - \frac{2}{3}R_3^{23}R_2^{12} + \frac{2}{3}R_3^{23}R_3^{13}\\ + 2(S^1 + \frac{1}{6}G_{,1})R_1^{12} + 2(S^2 + \frac{1}{6}G_{,2})R_2^{12} + 2(S^3 + \frac{1}{6}G_{,3})Q^{123}\\ - S^2G_{,1} - G_{,2}S^1 - 6S^2S^1 + \frac{1}{3}G_{,2}G_{,1} - G_{,12} \equiv 0.\end{aligned} \tag{2.3.24}$$

Permuting the indices in (2.3.23) and (2.3.24) gives the full set of five quadratic integrability conditions.

In Ref. [17] these integrability condition were important in classifying all Euclidean superintegrable systems. A possible method for classifying systems over other conformally flat spaces would be to specify the metric for the space of interest and solve the (2.3.20)-(2.3.22) subject to the integrability condition (2.3.23)-(2.3.24). Chapter 4 resolves this issue by showing that the metric need never be specified, and the subsequent classification obtained in chapter 5 covers all systems over conformally flat spaces.

2.4 Conformal Equivalence of Superintegrable Systems

Coupling Constant Metamorphosis, Stäckel Transform

Two useful tools in the classification of second-order superintegrable systems are the coupling constant metamorphosis (CCM) [35] and the Stäckel transform [5]. The CCM works by exchanging the roles of the the energy of the system with a particular parameter choice. In the context of second-order superintegrable systems the term in the potential associated with the chosen parameter is used to conformally scaled the metric.

The Stäckel transform works by scaling the system by potentials (the Stäckel multiplier) that separate in the same coordinate system as the system begin transformed. If one is only interested in Stäckel transforms that take maximally-superintegrable second-order systems to maximally-superintegrable second-order system then this transformations coincides with the CCM. For a discussion of the similarities and the differences of the CCM and the Stäckel transform outside the realm of second-order maximally-superintegrable systems see the recent paper of S. Post [52].

The following theorem gives an explicit description of the Stäckel transform for second-order superintegrable systems.

Theorem 2.4.1. *If $H = H_0 + V + \alpha U = E$ is a Hamiltonian with constant of the motion $\mathcal{L}(\alpha) = L + \alpha W_U$ then the transformed Hamiltonian $\widetilde{H} = \dfrac{H_0 + V}{U}$ will have symmetry $\mathcal{L}(-\widetilde{H}) = L - \widetilde{H} W_U$.*

Proof. This elegant proof is due to Kalnins *et al* [23]. Firstly, given functions of the form $G(\mathbf{x}, \mathbf{p}), F(a, \mathbf{x}, \mathbf{p})$ where $a = \tau(\mathbf{x}, \mathbf{p})$ then

$$\{F, G\} = \{F(a, \mathbf{x}, \mathbf{p}), G(\mathbf{x}, \mathbf{p})\}|_{a=\tau(\mathbf{x},\mathbf{p})} + \partial_a F(a, \mathbf{x}, \mathbf{p})|_{a=\tau(\mathbf{x},\mathbf{p})} \{\tau(\mathbf{x}, \mathbf{p}), G(\mathbf{x}, \mathbf{p})\}.$$

Since $L(\alpha)$ is a symmetry

$$\{H_0 + V + \alpha U, L(\alpha)\} = 0,$$

and so

$$\{H_0 + V, L(\alpha)\} = -\alpha\{U, L(\alpha)\}.$$

Using these it can be shown that

$$\begin{aligned}\{\widetilde{H}, L(\alpha)\} &= \left\{\frac{H_0 + V}{U}, L(\alpha)\right\} \\ &= -\frac{H_0 + V}{U^2}\{U, L(\alpha)\} + \frac{\{H_0 + V, L(\alpha)\}}{U} \\ &= -\frac{H_0 + V}{U^2}\{U, L(\alpha)\} + \frac{-\alpha\{U, L(\alpha)\}}{U} \\ &= -\frac{\widetilde{H} + \alpha}{U}\{U, L(\alpha)\}.\end{aligned}$$

Therefore

$$\begin{aligned}\{\widetilde{H}, L(-\widetilde{H})\} &= \left(\partial_\alpha L(\alpha)\left\{\widetilde{H}, \widetilde{H}\right\} - \frac{\widetilde{H} + \alpha}{U}\{U, L(\alpha)\}\right)\Bigg|_{\alpha = -\widetilde{H}} \\ &= 0\end{aligned}$$

Thus $L(-\widetilde{H})$ is a constant for the Hamiltonian $\widetilde{H}$. □

Importantly for second-order superintegrable systems the new constants $\widetilde{\mathcal{L}} = L - \widetilde{H}W_U$ are still second-order, and so the Stäckel transform preserves second-order superintegrability. Thus the potential $\widetilde{V} = \frac{V}{U}$ is second-order maximally-superintegrable over the conformally-flat metric

$$\widetilde{\lambda} = U\lambda.$$

Substituting $\widetilde{V} = \frac{V}{U}$ into the PDE (2.3.11) a new set of $\{\widetilde{\mathbf{Q}}, \widetilde{\mathbf{R}}, \widetilde{\mathbf{S}}\}$ variables are obtained. Specifically, the action of the transform gives

$$\begin{aligned}\widetilde{R}^{ij}_j &= R^{ij}_j, \\ \widetilde{Q}^{123} &= Q^{123}, \\ \widetilde{S}^i &= S^i + \frac{1}{3}F_{,j},\end{aligned} \tag{2.4.2}$$

where $U = \exp(F)$. Noting that the new $\widehat{G}$ is given by $\widetilde{G} = G + F$, it should be clear that the combination

$$S^i - \frac{1}{3}G_{,i} \tag{2.4.3}$$

is invariant under the Stäckel transform, and this hints at why this term appears in (2.3.21) and (2.3.22).

Example: Self Equivalence of V_{II}

To see the Stäckel transform in action consider the Hamiltonian

$$H = p_{x_1}^2 + p_{x_2}^2 + p_{x_3}^2 + a(x_1^2 + x_2^2 + x_3^2) + b\frac{x_1 - ix_2}{(x_1 + ix_2)^3} + \frac{c}{(x_1 + ix_2)^2} + \frac{d}{x_3^2} + e. \tag{2.4.4}$$

This system is V_{II}, taken from reference [17]. As discussed the Stäckel equivalent Hamiltonian $\widetilde{H} = \frac{H}{(x_1+ix_2)^2}$ will be superintegrable over the space with flat metric

$$ds^2 = \frac{dx_1^2 + dx_2^2 + dx_3^2}{(x_1 + ix_2)^2}.$$

The coordinate change

$$\begin{aligned} x_1 &= \frac{-1 + z_1^2 + z_2^2 + z_3^2}{2(z_1 + iz_2)} \\ x_2 &= i\frac{1 + z_1^2 + z_2^2 + z_3^2}{2(z_1 + iz_2)} \\ x_3 &= \frac{z_3}{z_1 + iz_2} \end{aligned} \tag{2.4.5}$$

puts this Hamiltonian back into the standard Euclidean coordinates and it is clear from direct inspection that it has the same form as above

$$\widetilde{H} = p_{z_1}^2 + p_{z_2}^2 + p_{z_3}^2 - b(z_1^2 + z_2^2 + z_3^2) - a\frac{z_1 - iz_2}{(z_1 + iz_2)^3} + \frac{e}{(z_1 + iz_2)^2} + \frac{d}{z_3^2} + c. \tag{2.4.6}$$

So in this case the system V_{II} is Stäckel equivalent to itself.

2.5 The algebraic ideal of integrability conditions on the 3-sphere

This chapter will conclude with a new result regarding the integrability conditions (2.3.23) and (2.3.24). In Euclidean space Kalnins *et al* showed that maximum-parameter superintegrable systems over flat space can be put into correspondence with the orbits of the Euclidean group acting on a 6-dimensional[2] algebraic variety embedded in $\mathbb{C}^{10}$ [17]. Specifically the 10 coefficient functions (2.3.16) satisfy a set of six quadratic equations (equations (24) in Ref. [17]).

[2] Hilbert-dimension 6

Five of these can be given by the setting $G \equiv 0$ in (2.3.23) and (2.3.24) and index permutations thereof. The sixth quadratic is given by

$$2\left(R_1^{13}\right)^2 + 2\left(R_2^{23}\right)^2 + 2\left(R_3^{13}\right)^2 + 2\left(R_3^{23}\right)^2 + 2\left(R_1^{12}\right)^2 + 2\left(R_2^{12}\right)^2 + 3\left(Q^{123}\right)^2 + R_1^{13}R_2^{23} + R_1^{12}R_3^{23} + R_2^{12}R_3^{13} - 45\left(S^1\right)^2 - 45\left(S^2\right)^2 - 45\left(S^3\right)^2. \quad (2.5.1)$$

The square of this quadratic was found in the ideal generated by the 5 other quadratics (2.3.23) and (2.3.24), and their partial derivatives. Taking all six quadratics generated a radical ideal (i.e. if A^n was in the ideal for a positive integer n, then so was A) which completely covered the space of integrability conditions. Conceptually, this allowed Euclidean superintegrable systems to be put into correspondence with the 6-dimensional algebraic variety defined by this ideal.

In this section the result is extended to all constant curvature spaces. That is, for a constant curvature metric there are six conditions quadratic in the variables (2.3.16) such that the ideal generated from these 6 quadratics covers the full space of integrability conditions.

To examine a constant curvature metric the sectional curvature will need to be calculated. To calculated the sectional curvature, two sufficiently generic vectors on the manifold need to be given, and these are chosen to be

$$X = \frac{\partial}{\partial x_1} + r\frac{\partial}{\partial x_2}$$

and

$$Y = \frac{\partial}{\partial x_2} + q\frac{\partial}{\partial x_3}.$$

Varying q, r will alter the section spanned by X, Y.

Calculating the sectional curvature is straight forward procedure and in terms of the vectors above the sectional curvature is given by

$$
\begin{aligned}
K(X,Y) = -\frac{1}{4}\Big(& (2\lambda_{,33}\lambda + {\lambda_{,1}}^2 + 2\lambda_{,22}\lambda - 2{\lambda_{,2}}^2 - 2{\lambda_{,3}}^2) r^2 q^2 \\
& + \left({\lambda_{,3}}^2 - 2{\lambda_{,2}}^2 + 2\lambda_{,22}\lambda + 2\lambda_{,11}\lambda - 2{\lambda_{,1}}^2\right) \\
& + (-2{\lambda_{,1}}^2 - 2{\lambda_{,3}}^2 + {\lambda_{,2}}^2 + 2\lambda_{,33}\lambda + 2\lambda_{,11}\lambda) q^2 \\
& + (-4\lambda_{,13}\lambda + 6\lambda_{,1}\lambda_{,3}) rq \\
& + (4\lambda_{,12}\lambda - 6\lambda_{,1}\lambda_{,2}) rq^2 \\
& + (4\lambda_{,23}\lambda - 6\lambda_{,2}\lambda_{,3}) q \Big) \Big/ \Big(\lambda^3 \left(r^2 q^2 + q^2 + 1\right)\Big). \qquad (2.5.2)
\end{aligned}
$$

A constant curvature metric would have a sectional curvature independent of the section chosen (i.e. independent of the parameters q, r). The only way this can be true is if the numerator is a scalar multiple of $(r^2q^2 + q^2 + 1)$. Demanding that the coefficients of q, rq, rq^2, rq in the numerator vanish yields the conditions

$$
\begin{aligned}
6\lambda_{,1}\lambda_{,2} - 4\lambda_{,12}\lambda &= 0, \\
6\lambda_{,1}\lambda_{,3} - 4\lambda_{,13}\lambda &= 0, \\
6\lambda_{,2}\lambda_{,3} - 4\lambda_{,23}\lambda &= 0. \qquad (2.5.3)
\end{aligned}
$$

Demanding that the ratio of the remaining coefficients make the numerator a scalar multiple of the denominator yields

$$
\begin{aligned}
3((\lambda_{,2})^2 - (\lambda_{,1})^2) - 2(\lambda_{,22} - \lambda_{,11})\lambda &= 0, \\
3((\lambda_{,3})^2 - (\lambda_{,2})^2) - 2(\lambda_{,33} - \lambda_{,22})\lambda &= 0, \\
3((\lambda_{,1})^2 - (\lambda_{,3})^2) - 2(\lambda_{,11} - \lambda_{,33})\lambda &= 0. \qquad (2.5.4)
\end{aligned}
$$

Under the conditions above the sectional curvature can be shown to take the form

$$
K = \frac{1}{12}\frac{3\left((\lambda_{,1})^2 + (\lambda_{,2})^2 + (\lambda_{,3})^2\right) - 4\left(\lambda_{,11} + \lambda_{,22} + \lambda_{,33}\right)\lambda}{\lambda^3}. \qquad (2.5.5)
$$

Furthermore (2.5.3)-(2.5.4) can be used to show K is constant on the manifold, and hence no further conditions need be considered.

Using (2.5.3),(2.5.4) and (2.5.5) it can be shown that the metric satisfies the following PDEs

$$
\begin{aligned}
\lambda_{,11} &= -\frac{2\lambda^3 6K - 15(\lambda_{,1})^2 + 3(\lambda_{,2})^2 + 3(\lambda_{,3})^2}{12\lambda},\\
\lambda_{,22} &= -\frac{2\lambda^3 6K + 3(\lambda_{,1})^2 - 15(\lambda_{,2})^2 + 3(\lambda_{,3})^2}{12\lambda},\\
\lambda_{,33} &= -\frac{2\lambda^3 6K + 3(\lambda_{,1})^2 + 3(\lambda_{,2})^2 - 15(\lambda_{,3})^2}{12\lambda},\\
\lambda_{,12} &= \frac{3\lambda_{,1}\lambda_{,2}}{2\lambda},\\
\lambda_{,13} &= \frac{3\lambda_{,1}\lambda_{,3}}{2\lambda},\\
\lambda_{,23} &= \frac{3\lambda_{,2}\lambda_{,3}}{2\lambda}.
\end{aligned}
\tag{2.5.6}
$$

Making the substitution $\lambda = \exp(G)$ gives the equations

$$
\begin{aligned}
G_{11} &= -\lambda K + \frac{1}{4}(G_1)^2 - \frac{1}{4}(G_2)^2 - \frac{1}{4}(G_3)^2,\\
G_{22} &= -\lambda K - \frac{1}{4}(G_1)^2 + \frac{1}{4}(G_2)^2 - \frac{1}{4}(G_3)^2,\\
G_{33} &= -\lambda K - \frac{1}{4}(G_1)^2 - \frac{1}{4}(G_2)^2 + \frac{1}{4}(G_3)^2,\\
G_{12} &= \frac{1}{2}G_2G_1,\\
G_{13} &= \frac{1}{2}G_1G_3,\\
G_{23} &= \frac{1}{2}G_2G_3.
\end{aligned}
\tag{2.5.7}
$$

Additionally the sectional-curvature can be expressed in the form

$$
6\lambda K = -\frac{1}{2}\left((G_1)^2 + (G_2)^2 + (G_3)^2\right) - 2(G_{11} + G_{22} + G_{33}). \tag{2.5.8}
$$

Substituting (2.5.7) into (2.3.23)-(2.3.24) gives

$$
\begin{aligned}
&4R^{13}_1R^{12}_2 + 4R^{13}_3R^{23}_2 - 4R^{23}_2R^{12}_2 - 8Q^{123}R^{23}_3 - 8Q^{123}R^{12}_1 - 12R^{13}_1\left(S^1 + \frac{1}{6}G_1\right)\\
&-12Q^{123}\left(S^2 + \frac{1}{6}G_2\right) - 12R^{13}_3\left(S^3 + \frac{1}{6}G_3\right) + 36\left(S^3 + \frac{1}{6}G_3\right)\left(S^1 + \frac{1}{6}G_1\right) \equiv 0,
\end{aligned}
\tag{2.5.9}
$$

$$
\begin{aligned}
&- 8\left(R_2^{23}\right)^2 + 8\left(R_3^{13}\right)^2 - 8\left(R_3^{23}\right)^2 + 8\left(R_1^{13}\right)^2 - 8R_1^{12}R_3^{23} + 8R_2^{12}R_3^{13} \\
&\quad - 12\left(2R_2^{12} + R_3^{13}\right)\left(S^1 + \frac{1}{6}G_1\right) + 12\left(2R_1^{12} + R_3^{23}\right)\left(S^2 + \frac{1}{6}G_2\right) \\
&+ 12\left(R_1^{13} - R_2^{23}\right)\left(S^3 + \frac{1}{6}G_3\right) - 36\left(S^1 + \frac{1}{6}G_1\right)^2 + 36\left(S^2 + \frac{1}{6}G_2\right)^2 \equiv 0.
\end{aligned}
\tag{2.5.10}
$$

Just like in the Euclidean case the 5 quadratic given by the index permutations of (2.5.9) and (2.5.10) conditions do not form an ideal closed under differentiation. However, the ideal closes after two derivatives. Examining the quartics in the ideal, and being inspired by the form of (2.5.8), it can be proven that the square of

$$
\begin{aligned}
&2\left(R_1^{13}\right)^2 + 2\left(R_2^{23}\right)^2 + 2\left(R_3^{13}\right)^2 + 2\left(R_3^{23}\right)^2 + 2\left(R_1^{12}\right)^2 + 2\left(R_2^{12}\right)^2 \\
&\qquad + 3\left(Q^{123}\right)^2 + R_1^{13}R_2^{23} + R_1^{12}R_3^{23} + R_2^{12}R_3^{13} \\
&- 45\left(S^1 + \frac{1}{6}G_1\right)^2 - 45\left(S^2 + \frac{1}{6}G_2\right)^2 - 45\left(S^3 + \frac{1}{6}G_3\right)^2 - 27\lambda K
\end{aligned}
\tag{2.5.11}
$$

is in the ideal and hence can added to the ideal. Taking (2.5.11) alongside the 5 quadratic given by (2.5.9) and (2.5.10) gives, just like the Euclidean case, an ideal which is closed under differentiation.

Chapter 3

Rotationally Adapted Variables and the Algebra of Constants

The purpose of this chapter is two-fold, firstly it redefines our variables in a rotationally adapted form by writing them as $SO(3, \mathbb{C})$ representations. The use of representations is a very important technique in this thesis and the classification result in chapter 5 hinges on their use. Secondly, this chapter examines the space of higher-order constants and provides a rigorous proof for the existence of a quartic identity between the second-order constants. The explicit form of this quartic identity is then given directly from the defining equations and their integrability conditions.

In §3.1 the action of the Euclidean group is examined and then used in §3.2 to define a set of $SO(3, \mathbb{C})$ representations.

In §3.3 the rotation representations are used to study the dimension of the space of constants of order $2, 3, 4, 6$ and 8, calling on results from the Ref. [12] as needed. This examination proves that is a linear relationship between in the space of (up to) the quartic monomials and this identity provides the functional relationship between the 6 linearly-independent second-order constants, answering a open question from the literature [36].

Finally in §3.4 the algebra of constants generated from the iterated Poisson brackets of the second-order constants is briefly discussed. This algebra is known to close polynomially and has been called a *'quadratic algebra'*, although this terminology does not appear to be standard outside the field of superintegrable systems. Part of the general structure of the quadratic algebra is demonstrated by explicitly stating the form of the fourth-order identity between the second-order constants[1] proven to exist in §3.3. Until now the general form of the quartic identity was unknown, but here it is written out explicitly as a one dimensional rotation representation (see figure 3.1).

3.1 Local Action of the Euclidean Group

The Action of the Lie Group $SO(3,\mathbb{C})$ and Lie algebra $\mathfrak{so}(3,\mathbb{C})$

Consider a differentiable change of coordinates $\mathbf{x} = F(\mathbf{u})$ which fixes the origin and preserves the form of the Euclidean metric

$$dx_1^2 + dx_2^2 + dx_3^2 = du_1^2 + du_2^2 + du_3^2.$$

The action of this change of variables on the Hamiltonian is

$$\frac{1}{\lambda(\mathbf{x})}\left(p_{x_1}^2 + p_{x_2}^2 + p_{x_3}^2\right) + V(\mathbf{x}) = \frac{1}{\lambda(F(\mathbf{u}))}\left(p_{u_1}^2 + p_{u_2}^2 + p_{u_3}^2\right) + V(F(\mathbf{u})).$$

This group is parameterised by the Lie group of complex orthogonal matrices and the action of the Lie algebra can be used to create rotation representations from the most of the variables (e.g. x_i, p_{x_i}, A_k^{ij}, V, $V_{,i}$,... etc).

Consider, for example, the subgroup of rotations that fixes the x_1 axis. As a coordinate change this is given by

$$\begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos(t) & \sin(t) \\ 0 & -\sin(t) & \cos(t) \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \qquad (3.1.1)$$

for varying t. The form of the Hamiltonian (2.3.3) is unchanged by this coordinate transformation and, of course, the second-order constants remain second-order constants. So the set up in chapter 2 still holds and the potential must satisfy as set of PDEs of the form (2.3.11). This allows the action of the rotation group to be induced on the coefficient functions (2.3.16). For example, returning to the original names for the coefficient functions (i.e. A_k^{ij}),

[1] making the identity eighth-order in the momenta

the change of variables (3.1.1) turns $V_{,x_1x_2}$ into

$$\begin{aligned}\cos(t)V_{,u_1u_2} + \sin(t)V_{,u_1u_3} = A_1^{12}V_{,u_1} + A_2^{12}(\cos(t)V_{,u_2} + \sin(t)V_{,u_3})\\ + A_3^{12}(-\sin(t)V_{,u_2} + \cos(t)V_{,u_3}),\end{aligned} \quad (3.1.2)$$

and similarly $V_{,x_1x_3}$ becomes

$$\begin{aligned}\cos(t)V_{,u_1u_3} - \sin(t)V_{,u_1u_2} = A_1^{13}V_{,u_1} + A_2^{13}(\cos(t)V_{,u_2} + \sin(t)V_{,u_3})\\ + A_3^{13}(-\sin(t)V_{,u_2} + \cos(t)V_{,u_3}).\end{aligned} \quad (3.1.3)$$

These can be used to derive the equations

$$\begin{aligned}V_{,u_1u_2} &= (A_1^{12}\cos(t) - A_1^{13}\sin(t))V_{,u_1}\\ &\quad + (A_2^{12}\cos(t)^2 - (A_2^{13} + A_3^{12})\sin(t)\cos(t) + A_3^{13}\sin(t)^2)V_{,u_2}\\ &\quad + (A_3^{12}\cos(t)^2 + (A_2^{12} - A_3^{13})\cos(t)\sin(t) - A_2^{13}\sin(t)^2)V_{,u_3},\\ V_{,u_1u_3} &= (A_1^{13}\cos(t) + A_1^{12}\sin(t))V_{,u_1}\\ &\quad + (A_2^{13}\cos^2(t) + (A_2^{12} - A_3^{13})\cos(t)\sin(t) - A_3^{12}\sin^2(t))V_{,u_2}\\ &\quad + (A_3^{13}\cos^2(t) + (A_2^{13} + A_3^{12})\cos(t)\sin(t) + A_2^{12}\sin^2(t))V_{,u_3}.\end{aligned} \quad (3.1.4)$$

The new coefficients can now be read off from equations like those above, for example

$$\begin{aligned}\widetilde{A}_1^{12}(\mathbf{u};t) &= (A_1^{12}\cos(t) - A_1^{13}\sin(t)),\\ \widetilde{A}_1^{13}(\mathbf{u};t) &= (A_1^{13}\cos(t) + A_1^{12}\sin(t)).\end{aligned} \quad (3.1.5)$$

This expresses the new coefficient functions in terms of the old ones, and allows an action of the $SO(3,\mathbb{C})$ Lie group to be induced on the A_k^{ij}.

The action of the $\mathfrak{so}(3,\mathbb{C})$ Lie algebra is given by examining the derivative of this action at $t = 0$, denoting this operator by J_1 the Lie algebra action corresponding to (3.1.5) is

$$\begin{aligned}J_1(A_1^{12}) &= \left.\frac{d\widetilde{A}_1^{12}(\mathbf{u};t)}{dt}\right|_{t=0} = -A_1^{13},\\ J_1(A_1^{13}) &= \left.\frac{d\widetilde{A}_1^{13}(\mathbf{u};t)}{dt}\right|_{t=0} = -A_1^{12}.\end{aligned} \quad (3.1.6)$$

The Lie algebra action of rotations around the x_1, x_2 and x_3 axes will be denoted J_1, J_2 and J_3 respectively. The action of this Lie algebra on the coordinates (with the identification $\widetilde{\mathbf{x}} = \mathbf{u}$) is shown table 3.1, and the action on the coefficient functions $\{\mathbf{Q}, \mathbf{R}, \mathbf{S}\}$ is given in table 3.2.

$J_\alpha(x_i)$	J_1	J_2	J_3
x_1	0	$-x_3$	x_2
x_2	x_3	0	$-x_1$
x_3	$-x_2$	x_1	0

Table 3.1: The $\mathfrak{so}(3,\mathbb{C})$ action on the coordinates

$J_\alpha(\bullet)$	J_1	J_2	J_3
S^1	0	$-S^3$	S^2
S^2	S^3	0	$-S^1$
S^3	$-S^2$	S^1	0
R_1^{12}	R_1^{13}	$-2Q^{123}$	$3R_2^{12} + R_3^{13}$
R_2^{12}	$2Q^{123}$	$-R_2^{23}$	$-3R_1^{12} - R_3^{23}$
R_1^{13}	$-R_1^{12}$	$-3R_3^{13} - R_2^{12}$	$2Q^{123}$
R_3^{13}	$-2Q^{123}$	$3R_1^{13} + R_2^{23}$	R_3^{23}
R_2^{23}	$3R_3^{23} + R_1^{12}$	R_2^{12}	$-2Q^{123}$
R_3^{23}	$-3R_2^{23} - R_1^{13}$	$2Q^{123}$	$-R_3^{13}$
Q^{123}	$R_3^{13} - R_2^{12}$	$R_1^{12} - R_3^{23}$	$R_2^{23} - R_1^{13}$

Table 3.2: The $\mathfrak{so}(3,\mathbb{C})$ action on the coefficient functions

Rotationally Adapted Variables (and the standard normalisation)

The action of the $\mathfrak{so}(3,\mathbb{C})$ Lie algebra shown in table 3.2 makes it clear that the span of the $\{\mathbf{S}\}$ variables is an invariant subspace under this action. This is because the action of $SO(3,\mathbb{C})$ on the S^i's forms an irreducible representation.

It is clear from direct inspection that $\{\mathbf{Q},\mathbf{R}\}$ variables also form a representation, but it is not completely clear whether or not the $\{\mathbf{Q},\mathbf{R}\}$ variables can be decomposed into the direct sum of smaller representations. To answer this question systematically, the following standard Lie algebra operators are introduced

$$J_+ = iJ_1 - J_2, \quad J_0 = iJ_3, \quad J_- = iJ_1 + J_2, \tag{3.1.7}$$

which will be referred to as the raising, level-set and lowering operators respectively. These operators satisfy the commutation relations

$$[J_0, J_+] = J_+, \quad [J_+, J_-] = 2J_0, \quad [J_0, J_-] = -J_-. \tag{3.1.8}$$

The irreducible subspaces are now identified by their so-called highest weight vectors, that is, eigenvectors of J_0 vanishing under the raising operator J_+. On the 10-dimensional space of coefficient functions there are two such highest weight vectors. One with eigenvalue $+1$ and one with eigenvalue $+3$. These split the 10 variables into a 3-dimensional and a 7-dimensional representation. Specifically the eigenvectors under J_0 are given by

$$\begin{aligned} X_{\pm 1} &= iS_2 \pm S_1, \\ X_0 &= -S_3\sqrt{2} \end{aligned} \tag{3.1.9}$$

and

$$\begin{aligned} Y_{\pm 3} &= R_1^{12} + \frac{1}{4}R_3^{23} \pm i\left(R_2^{12} + \frac{1}{4}R_3^{13}\right), \\ Y_{\pm 2} &= \frac{1}{4}\sqrt{6}\left(i\left(R_1^{13} - R_2^{23}\right) \mp 2Q^{123}\right), \\ Y_{\pm 1} &= \frac{1}{4}\sqrt{15}\left(R_3^{23} \mp iR_3^{13}\right), \\ Y_0 &= -\frac{1}{2}i\sqrt{5}\left(R_1^{13} + R_2^{23}\right). \end{aligned} \tag{3.1.10}$$

Here the eigenvalue is indicated in the subscript of the X, Y variables. These eigenvectors have been scaled so as to satisfy the following relations for an n-dimensional representation. Given an eigenvector F_m with eigenvalue m in a representation whose highest weight is $l = (n-1)/2$, the action of the raising and lowering operators is given by

$$\begin{aligned} J_+F_m &= \sqrt{(l-m)(l+m+1)}F_{m+1}, \\ J_-F_m &= \sqrt{(l+m)(l-m+1)}F_{m-1}, \\ J_0F_m &= mF_m. \end{aligned} \tag{3.1.11}$$

This normalisation for an rotation representation is a quite common one and will be appear again briefly when describing the results in chapter 5.

Another normalisation, which was found to be convenient in describing the result of the following section, comes from considering rotation representations as binary forms. Further details about this interpretation of rotation representations can be found in appendix A. Because of the conciseness of the binary form description the rotation representations described in the rest of this chapter will be given as binary forms.

Consider an $(n+1)$-dimensional representation with eigenvectors F_m. A new set of variable a_k can be defined from the identification below

$$\begin{aligned}\mathcal{A}_n(r,s) &= \sum_{k=0}^{n}(-1)^k\sqrt{\binom{n}{k}}F_{-\frac{n}{2}+k}s^k r^{n-k}\\ &= \sum_{k=0}^{n}\binom{n}{k}a_k s^k r^{n-k}\\ &= a_0 s^n + n a_1 r s^{n-1} + \frac{n(n-1)}{2}a_2 r^2 s^{n-2} + \ldots + a_n r^n. \end{aligned} \tag{3.1.12}$$

The induced action of the rotation Lie algebra on the new variables a_k is given by

$$\begin{aligned} J_+(a_k) &= -(n-k)a_{k+1}.\\ J_0(a_k) &= \left(k - \frac{n}{2}\right)a_k,\\ J_-(a_k) &= -k a_{k-1}. \end{aligned} \tag{3.1.13}$$

The nice thing about this scaling is that the raising operator acts in an identical manner near the highest-weight vector, e.g. $J_+(a_{n-1}) = -a_n$ regardless of the dimension of the representation. Because of this a combination of a_{n-k} that vanishes under J_+ for an $(n+1)$-dimensional representation will still vanish if the coefficient b_{m-k} from an $(m+1)$-dimensional representation is substituted, provided m and n are larger than the smallest given k. So this removes a lot of the representation dependent factors that appear when considering the construction of highest-weight vectors.

Constructing rotation representation: tranvectants and partial derivatives

Given a set of rotation representations further representations can easily be constructed. One construction method is to use the transvectant operator. Given two representations as binary forms $A(r,s), B(r,s)$ their nth transvectant is defined to be

$$(A,B)^{[n]} = \frac{1}{(n!)^2}\sum_{k=0}^{n}(-1)^k\binom{n}{k}\frac{\partial^n A}{\partial^k r \partial^{n-k} s}\frac{\partial^n B}{\partial^{n-k} r \partial^k s}. \tag{3.1.14}$$

The binary form $(A, B)^{[n]}$ also carries a rotation representations. By recursively applying the transvectant to the newly constructed representations a large set of representations can be obtained. The representations obtained from the transvectant have eigenvectors which are higher-order polynomials of the eigenvectors of the input representations. Conversely, every representation constructed from higher order polynomials of a given base set of representations can be written out in terms of transvectants of those base representation (for a proof see Ref. [49] regarding the transvectant and the joint covariants of binary forms).

Partial derivatives with respect to the x_i's can also be used to map representations to representations. These representations can also be systematically constructed as follows[2]. Consider the operators

$$\begin{aligned}\partial_+ &= i\partial_2 + \partial_1,\\ \partial_0 &= \partial_3,\\ \partial_- &= i\partial_2 - \partial_1,\end{aligned} \tag{3.1.15}$$

which will be referred to as the raising derivative, the level-set derivative and the lowering derivative. Being derivatives these satisfy the obvious commutation relations

$$[\partial_-, \partial_+] = 0, \quad [\partial_+, \partial_0] = 0, \quad [\partial_-, \partial_0] = 0.$$

More interestingly are the commutators of the partial derivatives with the Lie algebra action, shown in table 3.3. The first thing to notice is that, given an eigenvector $\mathbf{v}_\lambda$ of the J_0 operator with eigenvalue λ the following hold,

$$\begin{aligned}J_0\partial_+(\mathbf{v}_\lambda) &= (\lambda + 1)\partial_+(\mathbf{v}_\lambda),\\ J_0\partial_0(\mathbf{v}_\lambda) &= \lambda\partial_0(\mathbf{v}_\lambda),\\ J_0\partial_-(\mathbf{v}_\lambda) &= (\lambda - 1)\partial_-(\mathbf{v}_\lambda).\end{aligned} \tag{3.1.16}$$

So $\partial_+(\mathbf{v}_\lambda), \partial_0(\mathbf{v}_\lambda), \partial_-(\mathbf{v}_\lambda)$ are also eigenvectors of J_0 with raised, unchanged and lowered eigenvalues respectively. Furthermore, if $\mathbf{v}_\lambda$ is a highest weight vector then the commutation relations imply

$$J_+\partial_+(\mathbf{v}_\lambda) = 0, \tag{3.1.17}$$

[2]If the partial derivatives are considered to be a three-dimensional representations (with associated binary form) then the construction that follows can also be described in the form of a transvectant.

$[J_\alpha, \partial_\beta]$	∂_+	∂_0	∂_-
J_+	0	$-\partial_+$	$-2\partial_0$
J_0	∂_+	0	$-\partial_-$
J_-	$-2\partial_0$	$-\partial_-$	0

Table 3.3: Table of J, ∂ commutators

i.e. $\partial_+(\mathbf{v}_\lambda)$ is a highest weight vector. Unfortunately the same statement does not apply to ∂_0 and ∂_-, but a similar statement is given by the following theorem.

Theorem 3.1.18. *Given an $(n+1)$-dimensional representation of the form* (3.1.12)*, where $n \geq 2$, there are three representations made from the first-order partial derivatives and they have the highest-weight vectors*

$$b_{n+1} = \partial_+ (a_n), \tag{3.1.19}$$

$$c_n = \partial_0 (a_n) - \partial_+ (a_{n-1}), \tag{3.1.20}$$

$$d_{n-1} = \partial_- (a_n) - 2\partial_0 (a_{n-1}) + \partial_+ (a_{n-2}). \tag{3.1.21}$$

Proof. Applying the operator J_+ to b_{n+1} above, the relations (3.1.13) and the commutators in table 3.3 can be used to show

$$\begin{aligned} J_+ (b_{n+1}) &= J_+ \partial_+ (a_n) \\ &= \partial_+ J_+ (a_n) \\ &= 0, \end{aligned} \tag{3.1.22}$$

and

$$\begin{aligned} J_+ (c_n) &= J_+ \Big(\partial_0 (a_n) - \partial_+ (a_{n-1}) \Big) \\ &= \Big(\partial_0 J_+ (a_n) - \partial_+ (a_n) \Big) - \partial_+ J_+ (a_{n-1}) \\ &= -\partial_+ (a_n) + \partial_+ (a_n) \\ &= 0, \end{aligned} \tag{3.1.23}$$

and finally

$$\begin{aligned}
J_+\left(d_{n-1}\right) &= J_+\bigg(\partial_-\left(a_n\right) - 2\partial_0\left(a_{n-1}\right) + \partial_+\left(a_{n-2}\right)\bigg) \\
&= \bigg(\partial_- J_+\left(a_n\right) - 2\partial_0\left(a_n\right)\bigg) - 2\bigg(\partial_0 J_+\left(a_{n-1}\right) - \partial_+\left(a_{n-1}\right)\bigg) + \partial_+ J_+\left(a_{n-2}\right) \\
&= -2\partial_0\left(a_n\right) + 2\partial_0\left(a_n\right) + 2\partial_+\left(a_{n-1}\right) - 2\partial_+\left(a_{n-1}\right), \\
&= 0. \qquad (3.1.24)
\end{aligned}$$

This proves that each element vanishes under J_+. Since (3.1.19)-(3.1.21) are each clearly eigenvectors of the J_0 operator (with three distinct eigenvalues) these form highest-weight vectors for three distinct representations. A simple count of dimensions shows these are sufficient to cover the space of first-order derivatives. □

For the special case where a_n is a one-dimensional representation the three-dimensional representation given by (3.1.19) suffices to cover the space of first derivatives.

In the next section these constructions will be employed to study the dimension of the space of constants. For this some new notation will be defined. Given a binary form carrying a representation $\mathcal{R}(r,s)$ the three representations constructed from its first derivatives will be denoted

$$\partial_+^C(\mathcal{R}), \quad \partial_0^C(\mathcal{R}), \quad \partial_-^C(\mathcal{R}).$$

Letting π be the projection of a representation onto its highest weight vector, these will be defined so they satisfy

$$\begin{aligned}
\pi(\partial_+^C(\mathcal{R})) &= \frac{-i}{2}\partial_+(r_n), \\
\pi(\partial_0^C(\mathcal{R})) &= i\bigg(\partial_0(r_n) - \partial_+(r_{n-1})\bigg), \\
\pi(\partial_-^C(\mathcal{R})) &= \frac{-i}{2}\bigg(\partial_-(r_n) - 2\partial_0(r_{n-1}) + \partial_+(r_{n-2})\bigg). \qquad (3.1.25)
\end{aligned}$$

The scaling in the highest weight eigenvector of (3.1.25) was chosen partly to make the factors in the next section appear as simple as possible, and in some sense is natural given the choice of variables in that section.

3.2 More Rotationally Adapted Variables

In Ref. [12] the closure of the algebra of constants (the quadratic algebra) was proven by considering the maximal possible dimension of the space of constants up to order 6. This same approach could be used to rigorously establish the existence of the quartic-identity between the second-order constants, but until now this has not been done. This section begins by using rotation representations to re-derive the result of Ref. [12]. The primary purpose of this re-derivation is to show how representations can be used to achieve this goal. These techniques are then carried further to prove the existence of the aforementioned quartic identity. In the final section of this chapter the rotation representations are used to explicitly demonstrate the form of this quartic identity.

The motivation to use representation is two-fold: on the one-hand it makes some of the results more concise to state (for example the quartic identity in figure 3.1 would span several pages if written out directly) and on the other hand, by restricting attention to the highest weight vectors it also focuses the calculations onto a much smaller set of equations which, ultimately, allowed the computations to be done in a case which seemed infeasible before.

Further $\mathfrak{sl}(2,\mathbb{C}) \simeq \mathfrak{so}(3,\mathbb{C})$ representations

The coefficients functions $\{\mathbf{Q}, \mathbf{R}, \mathbf{S}\}$ have been set up as rotation representation in the previous section. The purpose of this section is to set up the rest of the variables as rotation representations.

To start with, observe that the form of the Hamiltonian

$$H = \frac{1}{\lambda}\left(p_{x_1}^2 + p_{x_2}^2 + p_{x_3}^2\right) + V(x_1, x_2, x_3), \tag{3.2.1}$$

is invariant under rotations. This makes the Hamiltonian a one-dimensional rotation representation. To make this more explicit consider the change of coordinates

$$\begin{aligned} w_0 &= -ix_1 + x_2, \\ w_1 &= -ix_3, \\ w_2 &= ix_1 + x_2. \end{aligned} \tag{3.2.2}$$

The action of the $\mathfrak{so}(3)$ Lie algebra on these coordinates is given by

$J_\alpha(w_\beta)$	w_0	w_1	w_2
J_+	0	$-w_0$	$-2w_1$
J_0	w_0	0	$-w_2$
J_-	$-2w_1$	$-w_2$	0

Obviously w_0 is a highest weight vector in a three-dimensional representation, and a written as binary form (cf. appendix A) this is

$$\begin{aligned}\mathcal{W}(r,s) &= w_2 r^2 + 2w_1 rs + w_0 s^2 \\ &= (-ix_1 + x_2)r^2 - 2ix_3 rs + (ix_1 + x_2)s^2.\end{aligned}$$

The canonical-momenta for (3.2.2) are

$$\begin{aligned}p_{w_0} &= \frac{1}{2}\left(ip_{x_1} + p_{x_2}\right), \\ p_{w_1} &= ip_{x_3}, \\ p_{w_2} &= \frac{1}{2}\left(-ip_{x_1} + p_{x_2}\right).\end{aligned} \tag{3.2.3}$$

and the action of the $\mathfrak{so}(3)$ Lie algebra is given by

$J_\alpha\left(p_{w_\beta}\right)$	p_{w_0}	p_{w_1}	p_{w_2}
J_+	p_{w_1}	$2p_{w_2}$	0
J_0	$-p_{w_0}$	0	p_{w_2}
J_-	0	$2p_{w_0}$	p_{w_1}

Clearly p_{w_2} is a highest weight vector in a three-dimensional representation. As a binary form this representation is expressible as

$$\mathcal{P}(r,s) = p_{w_0}s^2 - p_{w_1}rs + p_{w_2}r^2. \tag{3.2.4}$$

Using these representations the Hamiltonian can be re-written using the transvectant operator (3.1.14). Specifically

$$\begin{aligned}H &= \frac{2\left(\mathcal{P},\mathcal{P}\right)^{[2]}}{\lambda} + V \\ &= \frac{4p_{w_0}p_{w_2} - p_{w_1}^2}{\lambda} + V.\end{aligned} \tag{3.2.5}$$

The unrestricted first derivatives of the potential (i.e. the first order parameters) can be set up as a 3-dimensional rotation representation

$$\mathcal{V}(r,s) = \frac{\partial V}{\partial w_0}s^2 - \frac{\partial V}{\partial w_1}rs + \frac{\partial V}{\partial w_2}r^2. \tag{3.2.6}$$

The second-order parameter can be set up as the one-dimensional representation

$$\mathcal{V}_{ee} = \frac{4}{3}\left(\frac{\partial^2 V}{\partial w_0 \partial w_2} - \frac{1}{4}\frac{\partial^2 V}{\partial {w_1}^2}\right). \tag{3.2.7}$$

In terms of the original coordinates the representation $\mathcal{V}_{ee}$ is just the symmetric second-order parameter

$$\mathcal{V}_{ee} = \frac{1}{3}\left(V_{,11} + V_{,22} + V_{,33}\right).$$

This '*ee*' notation will be used elsewhere to denote symmetric combinations of the second order derivatives.

The seven-dimensional representation (3.1.10) re-expressed as a binary form is

$$\begin{aligned}\mathcal{Y}(r,s) = &\left(R_1^{12} + \frac{1}{4}R_3^{23} + iR_2^{12} + \frac{i}{4}R_3^{13}\right) r^6 \\ &+ \left(3Q^{123} + \frac{3i}{2}R_2^{23} - \frac{3i}{2}R_1^{13}\right) r^5 s \\ &+ \frac{15}{4}\left(R_3^{23} - iR_3^{13}\right) s^2 r^4 + 5i\left(R_1^{13} + R_2^{23}\right) r^3 s^3 + \frac{15}{4}\left(R_3^{23} + iR_3^{13}\right) r^2 s^4 \\ &+ \left(-3Q^{123} + \frac{3i}{2}R_2^{23} - \frac{3i}{2}R_1^{13}\right) r s^5 \\ &\left(R_1^{12} + \frac{1}{4}R_3^{23} - iR_2^{12} - \frac{i}{4}R_3^{13}\right) s^6,\end{aligned} \tag{3.2.8}$$

and the binary form version of the three-dimensional representation (3.1.9) is

$$\mathcal{X}(r,s) = (iS_2 + S_1)r^2 + 2S_3 rs + (iS_2 - S_1)s^2. \tag{3.2.9}$$

The derivatives of the conformal factor $\lambda = \exp(G)$ also will appear, and so there is a need to define the representations

$$\mathcal{Z}(r,s) = (i\partial_{x_2}G + \partial_{x_1}G)r^2 + 2\left(\partial_{x_3}G\right) rs + (i\partial_{x_2}G - \partial_{x_1}G)s^2 \tag{3.2.10}$$

and

$$\mathcal{Z}_{ee} = \frac{4}{3}\left(\frac{\partial^2 G}{\partial w_0 \partial w_2} - \frac{1}{4}\frac{\partial^2 G}{\partial {w_1}^2}\right). \tag{3.2.11}$$

Concise Forms for the Governing Equations

These representations and the derivative representations allow for the governing equations to be written in more compact form. For example equations the 5 PDEs (2.3.11) can be written as the single representation

$$\partial_+^C \partial_+^C(V) = -\frac{1}{15}\left(\mathcal{Y}, \partial_+^C(V)\right)^{[2]} + 3i\left(\mathcal{X}, \partial_+^C(V)\right)^{[0]}. \tag{3.2.12}$$

where ∂_+^C has been defined in (3.1.25).

Similarly the 30 derivatives of the $\{\mathbf{Q}, \mathbf{R}, \mathbf{S}\}$ variable given by (2.3.17)-(2.3.20) can expressed as the following 6 representations

$$\begin{aligned}
\partial_+^C(\mathcal{Y}) &= 2i\,(\mathcal{Y}, \mathcal{X}_C)^{[0]} - \frac{1}{675}\,(\mathcal{Y}, \mathcal{Y})^{[2]},\\
\partial_0^C(\mathcal{Y}) &= 0,\\
\partial_-^C(\mathcal{Y}) &= \frac{-i}{10}\,(\mathcal{Y}, \mathcal{X}_C)^{[0]} - \frac{2}{225}\,(\mathcal{Y}, \mathcal{Y})^{[4]},\\
\partial_+^C(\mathcal{X}_C) &= \frac{i}{2}\,(\mathcal{X}_C, \mathcal{X}_C)^{[0]} - \frac{2i}{2025}\,(\mathcal{Y}, \mathcal{Y})^{[4]},\\
\partial_0^C(\mathcal{X}_C) &= 0,\\
\partial_-^C(\mathcal{X}_C) &= \frac{-i}{4}\,(\mathcal{X}_C, \mathcal{X}_C)^{[0]} - \frac{11i}{90}\,(\mathcal{Y}, \mathcal{Y})^{[6]},
\end{aligned} \tag{3.2.13}$$

where $\mathcal{X}_C = \mathcal{X} - \frac{1}{3}\mathcal{Z}$.

3.3 The Algebra of Constants (Quadratic Algebra)

Since the Poisson-commutator of two constants of the motion is again a constant of the motion and since the Poisson-commutator of an order d_1 constant with a order d_2 constant is generically of order $d_1 + d_2 - 1$, the order of the constants obtained under repeatedly taking the Poisson-commutators (starting with the second-order constants) will grow without bound. In this way the second-order constants will typically generate an infinite dimensional Lie algebra of constants. The constant in this algebra cannot be independent but they need not be related by polynomial equations. However, in the case of a maximum-parameter second-order superintegrable system it has been shown that this algebra closes at the third-order. That is to say, all constants of order greater than 3 are polynomial in the second-order constant and their Poisson-brackets (i.e. polynomials in L_i and $\{L_j, L_k\}_{PB}$).

Second-order Constants in Rotationally Adapted Variables

Second-order constants, already discussed in §2.3, are now revisited in the rotationally adapted framework. This allows the techniques used in the rest of this chapter to be demonstrated before the equations get too complicated to write down in full. The technique used here is to rewrite the constants (2.3.4) in such a way that makes the rotationally adapted variables obvious (i.e. with the variables explicitly chosen to form irreducible rotation representations).

Beginning with the purely second-order part, there are only two representations second order in the momenta,

$$(\mathcal{P},\mathcal{P},)^{[0]} = \mathcal{P}^2 \text{ and } (\mathcal{P},\mathcal{P},)^{[2]}. \tag{3.3.1}$$

In order to make the one-dimensional representation corresponding to the second-order part of the second-order constant, it is necessary to define the two representations that balance the size of (3.3.1). This means a 5-dimensional representations

$$\begin{aligned}\mathcal{A}_4(r,s) &= \sum_{i=0}^{4}\binom{4}{i} a_i^{(4)} r^i s^{4-i}\\ &= a_0^{(4)}s^4 + 4a_1^{(4)}rs^3 + 6a_2^{(4)}r^2s^2 + 4a_3^{(4)}r^3s + a_4^{(4)}r^4,\end{aligned} \tag{3.3.2}$$

and a 1-dimensional

$$\mathcal{A}_0(r,s) = a_0^{(0)}. \tag{3.3.3}$$

Using these the leading part of the second order constant can be defined as

$$K_2 = \left(\mathcal{A}_4,\mathcal{P}^2\right)^{[4]} + \Big(\mathcal{A}_0,(\mathcal{P},\mathcal{P},)^{[2]}\Big)^{[0]}, \tag{3.3.4}$$

which allows for all 6 second-order monomials in the momenta.

The zeroth-order part in the second-order constant can also be expressed as a representation, however this one is not required to balance any momenta and so it can simply be defined as the 1-dimensional representation

$$\mathcal{B}_0(r,s) = b_0^{(0)}. \tag{3.3.5}$$

So the zeroth-order part by will be defined to be

$$K_0 = \mathcal{B}_0. \tag{3.3.6}$$

Using the above, the rotationally adapted form of (2.3.4) is given by

$$\begin{aligned} L &= K_2 + K_0 \\ &= \left(\mathcal{A}_4, \mathcal{P}^2\right)^{[4]} + \left(\mathcal{A}_0, (\mathcal{P}, \mathcal{P},)^{[2]}\right)^{[0]} + \mathcal{B}_0. \end{aligned} \tag{3.3.7}$$

The equivalence between the form (2.3.4) and (3.3.7) is given by

$$\begin{aligned} a_4^{(4)} &= a^{22} - a^{11} - 2ia^{12}, \\ a_3^{(4)} &= -a^{13} - ia^{23}, \\ a_2^{(4)} &= \frac{1}{3}(a^{22} + a^{11} - 2a^{33}), \\ a_1^{(4)} &= a^{13} - ia^{23}, \\ a_0^{(4)} &= a^{22} - a^{11} + 2ia^{12}, \end{aligned} \tag{3.3.8}$$

and

$$a_0^{(0)} = \frac{2}{3}(a^{11} + a^{22} + a^{33}), \tag{3.3.9}$$

and finally

$$b_0^{(0)} = W. \tag{3.3.10}$$

For the types of constants that are of interest in this thesis, the term $\mathcal{B}_0$ will be linear in the parameters (i.e. linear in the coefficients of $\mathcal{V}_{ee}$ and $\mathcal{V}(r,s)$). To construct such a $\mathcal{B}_0$ we define a pair of representations of the same size as the pair (3.2.6) and (3.2.7), namely

$$\begin{aligned} \mathcal{F}_{0,0}(r,s) &= f_0^{(0,0)}, \\ \mathcal{F}_{0,3}(r,s) &= \sum_{k=0}^{2} \binom{2}{k} f_k^{(0,3)} r^k s^{2-k}. \end{aligned} \tag{3.3.11}$$

Using these a suitable form for $\mathcal{B}_0$ is given by

$$\begin{aligned} \mathcal{B}_0 &= \left(\mathcal{F}_{(0,0)}, \mathcal{V}_{ee}\right)^{[0]} + \left(\mathcal{F}_{(0,3)}, \mathcal{V}\right)^{[2]} \\ &= f_0^{(0,0)} \mathcal{V}_{ee} + f_2^{(0,3)} V_{,w_0} + f_1^{(0,3)} V_{,w_1} + f_0^{(0,3)} V_{,w_2} \end{aligned} \tag{3.3.12}$$

which suffices to cover all four first-order monomials in the parameters. Note that the parameter V (i.e. the value of the potential at the regular point) does not need to appear in this set up because, without loss of generality, it can be assumed to be zero.

Taking the Poisson bracket of this second-order constant with the Hamiltonian gives the first order condition

$$\left\{K_0, \frac{4p_{w_0}p_{w_2} - p_{w_1}^2}{\lambda}\right\}_{PB} + \{K_2, V\}_{PB} \equiv 0.$$

Expanding this, there are 12 monomials in coefficients of $\mathcal{V}_{ee}, \mathcal{V}(r,s), \mathcal{P}(r,s)$. The coefficients of these 12 monomials must vanish identically, giving a set of 12 conditions. From these 12 conditions a set of 4 highest-weight vectors can be constructed. Highest weight vectors are, by construction, adapted to the current orientation of w_0, w_1, w_2. But since there is nothing special about the current orientation these 4 highest weight vectors must vanish independent of the choice of coordinates. The only way this is possible is if the entire representation vanishes identically and this implies the following conditions, written in the form of representations,

$$\begin{aligned}
\partial_+^C(\mathcal{F}_{0,0}) &= \frac{-5i}{2}(\mathcal{X}_2, \mathcal{F}_{0,0})^{[0]} - \frac{1}{2}\mathcal{F}_{0,3},\\
\partial_0^C(\mathcal{F}_{0,3}) &= \frac{5i}{4}(\mathcal{X}_2, \mathcal{F}_{0,3})^{[1]},\\
\lambda\mathcal{A}_4 - 2\partial_+^C(\mathcal{F}_{0,3}) &= \mathcal{F}_{0,0}\left(\frac{52}{2025}(\mathcal{Y},\mathcal{Y})^{[4]} - \frac{14i}{45}(\mathcal{X},\mathcal{Y})^{[2]} - \frac{2i}{27}(\mathcal{Z},\mathcal{Y})^{[2]} - 2(\mathcal{X},\mathcal{X})^{[0]}\right)\\
&\quad - \frac{2}{15}(\mathcal{F}_{0,3},\mathcal{Y})^{[2]} + i(\mathcal{F}_{0,3},\mathcal{X})^{[0]},\\
\lambda\mathcal{A}_0 - \frac{2}{3}\partial_-^C(\mathcal{F}_{0,3}) &= \mathcal{F}_{0,0}\left(\frac{29}{27}(\mathcal{Y},\mathcal{Y})^{[6]} - \frac{5}{6}(\mathcal{X},\mathcal{X})^{[2]} - \frac{5}{9}(\mathcal{X},\mathcal{Z})^{[2]} + \frac{5}{54}(\mathcal{Z},\mathcal{Z})^{[2]} - \frac{10}{3}\mathcal{Z}_{ee}\right)\\
&\quad + \frac{10i}{3}(\mathcal{F}_{0,3},\mathcal{X})^{[2]}.
\end{aligned} \tag{3.3.13}$$

These leave the six variables in the two representations $\partial_+^C(\mathcal{F}_{0,3})$, $\partial_-^C(\mathcal{F}_{0,3})$ unrestricted (or alternatively $\mathcal{A}_4, \mathcal{A}_0$ unrestricted). Likewise the values of $\mathcal{F}_{0,3}$ and $\mathcal{F}_{0,0}$ are also unrestricted, but the value of these four variables only affects the trivial additive constant, and so can be neglected.

From the third-order condition

$$\left\{K_2, \frac{4p_{w_0}p_{w_2} - p_{w_1}^2}{\lambda}\right\}_{PB} \equiv 0$$

two more highest weight vectors can be found, representing the conditions

$$\begin{aligned}
\partial_+^C(\mathcal{A}_4) &= 0,\\
4\partial_-^C(\mathcal{A}_4) + 10\partial_+^C(\mathcal{A}_0) &= \frac{5i}{3}(\mathcal{A}_4, \mathcal{Z}_2)^{[2]} + 5i(\mathcal{A}_0, \mathcal{Z}_2)^{[0]}.
\end{aligned} \tag{3.3.14}$$

Constructing the raising, lowering and level-set derivative representations of (3.3.13) gives a further 8 conditions

$$
\begin{aligned}
\partial_+^C\left(\partial_+^C\left(\mathcal{F}_{0,0}\right)\right) &= \ldots, \\
\partial_-^C\left(\partial_+^C\left(\mathcal{F}_{0,0}\right)\right) &= \ldots, \\
\partial_+^C\left(\partial_0^C\left(\mathcal{F}_{0,3}\right)\right) &= \ldots, \\
\partial_-^C\left(\partial_0^C\left(\mathcal{F}_{0,3}\right)\right) &= \ldots, \\
\lambda\partial_+^C\left(\mathcal{A}_4 - 2\partial_+^C\left(\mathcal{F}_{0,3}\right)\right) &= -\frac{i}{2}\lambda\mathcal{A}_4\mathcal{Z} + \ldots, \\
\lambda\partial_0^C\left(\mathcal{A}_4 - 2\partial_+^C\left(\mathcal{F}_{0,3}\right)\right) &= -\frac{i}{8}\lambda\left(\mathcal{A}_4, \mathcal{Z}\right)^{[1]} + \ldots, \\
\lambda\partial_-^C\left(\mathcal{A}_4 - 2\partial_+^C\left(\mathcal{F}_{0,3}\right)\right) &= -\frac{i}{12}\lambda\left(\mathcal{A}_4, \mathcal{Z}\right)^{[2]} + \ldots, \\
\lambda\partial_+^C\left(\mathcal{A}_0 - \frac{2}{3}\partial_-^C\left(\mathcal{F}_{0,3}\right)\right) &= -\frac{i}{2}\lambda\mathcal{A}_0\mathcal{Z} + \ldots,
\end{aligned} \tag{3.3.15}
$$

where the terms not written down are just the appropriate derivative constructions applied to the left-hand sides of (3.3.13). The missing terms contain (at worst) first-order derivative of the $\mathcal{F}_{i,j}$ representations, and so the left-hand sides of (3.3.15) can be used to determine how many of the first-order derivatives of the $\mathcal{A}_i$ and second-order derivatives of the $\mathcal{F}_{i,j}$ can be solved for. In total the 10 conditions from (3.3.14) and (3.3.15) cover the 6 representations constructed from second-order derivatives of $\mathcal{F}_{0,0}, \mathcal{F}_{0,3}$ and the 4 first-order derivatives of $\mathcal{A}_0, \mathcal{A}_4$. Thus the aforementioned variables (which will be denoted $\{\partial\mathcal{A}, \partial^2\mathcal{F}\}$) can be solved for explicitly in terms of lower order derivatives.

The conclusion, therefore, is that a second-order constants depends only on the value of $\partial_+^C\left(\mathcal{F}_{0,3}\right), \partial_-^C\left(\mathcal{F}_{0,3}\right)$. The representation $\mathcal{F}_{0,3}$ is a three-dimensional representation and therefore $\partial_+^C\left(\mathcal{F}_{0,3}\right), \partial_-^C\left(\mathcal{F}_{0,3}\right)$ are respectively of dimension 5 and 1. So the space of purely[3] second-order constants has dimension at most 6. This is consistent with the ($5 \implies 6$) theorem which states that the space of purely second-order constants is *exactly* 6.

Third-order Constants

This section examines the classical structure theory for the space of third order constants. To do so, note there are two representations in the 10-dimensional

[3]meaning the additive constant is begin ignored.

space spanned by the monomials cubic in the momenta. These are the 7-dimensional representation

$$\left(\mathcal{P}^2, \mathcal{P}\right)^{[0]} = \mathcal{P}^3 \tag{3.3.16}$$

and the three-dimensional representation

$$\left(\mathcal{P}^2, \mathcal{P}\right)^{[2]} = 4\mathcal{P}\left(\mathcal{P}, \mathcal{P}\right)^{[2]} = 2p_{w_2}(4p_{w_0}p_{w_2} - p_{w_1}^2)r^2 + \cdots . \tag{3.3.17}$$

Recycling the previous notation, a pair of representation balancing (3.3.16) and (3.3.17) are defined via

$$\begin{aligned}\mathcal{A}_6(r,s) &= \sum_{i=0}^{6} \binom{6}{i} a_i^{(6)} r^i s^{6-i} \\ &= a_0^{(6)} s^6 + 6a_1^{(6)} rs^5 + 15a_1^{(6)} r^2 s^4 + \ldots + a_6^{(6)} r^6,\end{aligned} \tag{3.3.18}$$

and

$$\begin{aligned}\mathcal{A}_2(r,s) &= \sum_{i=0}^{2} \binom{6}{i} a_i^{(2)} r^i s^{2-i} \\ &= a_0^{(2)} s^2 + 2a_1^{(2)} rs + a_2^{(2)} r^2.\end{aligned} \tag{3.3.19}$$

Using these the *purely* third-order part of the constant is then defined to be

$$K_3 = \left(\mathcal{A}_6, \mathcal{P}^3\right)^{[6]} + \left(\mathcal{A}_2, \left(\mathcal{P}^2, \mathcal{P}\right)^{[2]}\right)^{[2]}. \tag{3.3.20}$$

Similarly to the case of second-order constants, the linear part of the third-order constant will be assumed linear in the parameters. There is one representation linear in the momentum

$$\mathcal{P}$$

and two representations linear in the parameters

$$\mathcal{V}, \mathcal{V}_{ee}.$$

The representation $\mathcal{P}$ only needs a 3-dimensional representation (a degree 2 binary form) to balance it out under transvection. It is not too difficult to determine that there four possible ways such a representation could be constructed from transvectants with $\mathcal{V}, \mathcal{V}_{ee}$. Defining the representations,

$$\begin{aligned}
\mathcal{F}_{2,0}(r,s) &= f_0^{(2,0)},\\
\mathcal{F}_{2,1}(r,s) &= f_0^{(2,1)},\\
\mathcal{F}_{2,2}(r,s) &= f_0^{(2,2)}s^2 + 2f_1^{(2,2)}rs + f_2^{(2,2)}r^2,\\
\mathcal{F}_{2,3}(r,s) &= \sum_{i=0}^{6}\binom{4}{i} f_i^{(2,3)} r^i s^{4-i}\\
&= f_0^{(2,3)}s^4 + 4f_1^{(2,3)}rs^3 + 6f_1^{(2,3)}r^2s^2 + 4f_3^{(2,3)}r^3s + f_4^{(2,3)}r^4,
\end{aligned} \tag{3.3.21}$$

a suitable form for the linear part is be given by

$$K_1 = \left((\mathcal{F}_{2,0}, \mathcal{V}_{ee})^{[0]} + (\mathcal{F}_{2,1}, \mathcal{V})^{[0]} + (\mathcal{F}_{2,2}, \mathcal{V})^{[1]} + (\mathcal{F}_{2,3}, \mathcal{V})^{[2]}, \mathcal{P}\right)^{[2]}. \tag{3.3.22}$$

The third order constant can now be written in the form

$$\mathcal{K}_3 = K_3 + K_1. \tag{3.3.23}$$

The Poisson-bracket of the Hamiltonian and $\mathcal{K}_3$ is fourth-order in the momenta and consists of fourth, second and zeroth-order components that must vanish independently. Considering the equation which is zeroth-order in the momenta

$$\{V, K_1\}_{PB} \equiv 0$$

there are 3 highest-weight conditions can be found

$$f_4^{(2,3)} = 0, f_2^{(2,1)} = 0, f_0^{(2,0)} = 0.$$

As before, there is nothing special about this particular orientation and so these conditions will only be satisfied if

$$\mathcal{F}_{2,0} = 0, \mathcal{F}_{2,1} = 0, \mathcal{F}_{2,3} = 0. \tag{3.3.24}$$

Keeping in mind (3.3.24), the equations quadratic in the momenta are given by

$$\left\{V, \left(\mathcal{A}_6, \mathcal{P}^3\right)^{[6]} + \left(\mathcal{A}_2, \left(\mathcal{P}^2, \mathcal{P}\right)^{[2]}\right)^{[2]}\right\}_{PB} + \left\{\mathcal{H}_0, \left(\mathcal{F}_{21}, (\mathcal{V}, \mathcal{P})^{[1]}\right)^{[2]}\right\}_{PB} \equiv 0. \tag{3.3.25}$$

From these conditions four highest-weight conditions can be calculated and these imply, written as representations, the following conditions

$$\begin{aligned}
\mathcal{A}_6 &= \frac{-4}{9\lambda}\left(\mathcal{F}_{2,2},\mathcal{Y}\right)^{[1]}, \\
\mathcal{A}_2 &= \frac{i}{2\lambda}\left(\mathcal{F}_{2,2},\mathcal{X}+\frac{1}{6}\mathcal{Z}\right)^{[1]}, \\
\partial^C_+\left(\mathcal{F}_{2,2}\right) &= \frac{-1}{45}\left(\mathcal{F}_{2,2},\mathcal{Y}\right)^{[2]} - \frac{3i}{2}\left(\mathcal{F}_{2,2},\mathcal{X}\right)^{[0]}, \\
\partial^C_0\left(\mathcal{F}_{2,2}\right) &= \frac{-i5}{4}\left(\mathcal{F}_{2,2},\mathcal{X}-\frac{2}{15}\mathcal{Z}\right)^{[1]}.
\end{aligned} \tag{3.3.26}$$

This leaves only the values of $\mathcal{F}_{2,2}$ and $\partial^C_-(\mathcal{F}_{2,2})$ free.

Clearly all derivatives of $\mathcal{A}_6$ and $\mathcal{A}_2$ can be calculated using (3.3.26). Of the six possible ways to construct the representations for the second-order derivatives of $\mathcal{F}_{2,2}$, one is trivial,

$$\partial^C_-(\partial^C_0(\mathcal{F}_{2,2})) = 0$$

and two give the same representation up to scaling

$$\partial^C_+(\partial^C_0(\mathcal{F}_{2,2})) = 2\partial^C_0(\partial^C_+(\mathcal{F}_{2,2})).$$

The four remaining representations are independent and cover the space of second-order derivatives of $\mathcal{F}_{2,2}$ and hence all second-order derivatives of $\mathcal{F}_{2,2}$ can be solved for. Thus a third-order constant only depends on the values of the three-dimensional representation $\mathcal{F}_{2,2}$ and one-dimensional representation $\partial^C_-(\mathcal{F}_{2,2})$. This means the space of third-order constants has dimension ≤ 4. For most systems the dimension 4 is achieved, the exceptions are given by the following theorem (Corollary 5 in Ref. [12]).

Theorem 3.3.27. *Let V be a superintegrable maximum-parameter potential on a conformally flat space, not a Stäckel transform of the isotropic oscillator. Then the space of truly third-order constants of the motion is four-dimensional and is spanned by Poisson brackets of the second-order constants of the motion.*

Fourth-order constants

The space of fourth-order monomials in the momenta is 15-dimensional and splits into the three representations

$$\mathcal{P}^4, \quad \left(\mathcal{P}^2,\mathcal{P}^2\right)^{[2]}, \quad \left(\mathcal{P}^2,\mathcal{P}^2\right)^{[4]}. \tag{3.3.28}$$

These representations are of respective dimension 9, 5 and 1. To construct a generic fourth-order combination the following three representations, we define three representation of the same size as (3.3.28)

$$\begin{aligned}\mathcal{A}_8(r,s) &= \sum_{i=0}^{8}\binom{8}{i}a_i^{(8)}r^i s^{8-i}\\ &= a_0^{(8)}s^8 + 8a_1^{(8)}rs^7 + 28a_1^{(8)}r^2s^4 + \ldots + a_6^{(8)}r^6, \quad (3.3.29)\\ \mathcal{A}_4(r,s) &= \sum_{i=0}^{4}\binom{4}{i}a_i^{(4)}r^i s^{4-i}\\ &= a_0^{(4)}s^4 + 4a_1^{(4)}rs^3 + 6a_1^{(4)}r^2s^2 + a_3^{(4)}r^3s + a_4^{(4)}r^4, \quad (3.3.30)\\ \mathcal{A}_0(r,s) &= a_0^{(0)}. \quad (3.3.31)\end{aligned}$$

These are used to balance the sizes of (3.3.28) and, in some sense, "complete" them as a 1-dimensional representation. For this point forward representation like these will be referred to as *complementary* representation. For the second-order part, two representations complementary to those in (3.3.1) are defined to be

$$\begin{aligned}\mathcal{B}_4(r,s) &= \sum_{i=0}^{4}\binom{4}{i}b_i^{(4)}r^i s^{4-i}\\ &= b_0^{(4)}s^4 + 4b_1^{(4)}rs^3 + 6b_1^{(4)}r^2s^2 + \ldots + b_4^{(4)}r^4, \quad (3.3.32)\\ \mathcal{B}_0(r,s) &= b_0^{(0)}. \quad (3.3.33)\end{aligned}$$

And the finally the zeroth-order term is defined to be

$$\mathcal{C}_0(r,s) = c_0^{(0)}. \quad (3.3.34)$$

The representations $\mathcal{B}_4$, $\mathcal{B}_0$ and $\mathcal{C}_0$ must be respectively linear, linear and quadratic in the parameters. From the previous sections it should be clear that there are two first order representations in the parameters

$$\mathcal{V}, \quad \mathcal{V}_{ee}.$$

The space of monomials quadratic in the parameters splits into four representations

$$\mathcal{V}_{ee}^2, \quad \mathcal{V}_{ee}\mathcal{V}, \quad (\mathcal{V},\mathcal{V})^{[0]} = \mathcal{V}^2, \quad (\mathcal{V},\mathcal{V})^{[2]}. \quad (3.3.35)$$

Considering the most generic way in which representations can be transvected with $\mathcal{V}, V_{ee}$ to give representations of the form $\mathcal{B}_4$ and $\mathcal{B}_0$ leads to the following definitions

$$\begin{aligned}\mathcal{B}_4 &= \mathcal{F}_{4,0}\mathcal{V}_{ee} + (\mathcal{F}_{4,1},\mathcal{V})^{[0]} + (\mathcal{F}_{4,2},\mathcal{V})^{[1]} + (\mathcal{F}_{4,3},\mathcal{V})^{[2]}\,,\\ \mathcal{B}_0 &= \mathcal{F}_{0,0}\mathcal{V}_{ee} \qquad\qquad\qquad\qquad\qquad\qquad + (\mathcal{F}_{0,3},\mathcal{V})^{[2]}\,.\end{aligned} \tag{3.3.36}$$

The dimension of the representations $\mathcal{F}_{i,j}$ is implicit in the set up above[4] (e.g. $\mathcal{F}_{0,3}$ and $\mathcal{F}_{4,1}$ are of dimension 3).

Likewise the most general form of $\mathcal{C}_0$ is given by

$$\begin{aligned}\mathcal{C}_0 = \mathcal{G}_{0,0}\mathcal{V}_{ee}^2 &+ \mathcal{V}_{ee}\,(\mathcal{G}_{0,1},\mathcal{V})^{[0]} + \mathcal{V}_{ee}\,(\mathcal{G}_{0,2},\mathcal{V})^{[1]} + \mathcal{V}_{ee}\,(\mathcal{G}_{0,3},\mathcal{V})^{[2]}\\ &+ \left(\mathcal{G}_{0,4},(\mathcal{V},\mathcal{V})^{[2]}\right)^{[0]} + \left(\mathcal{G}_{0,5},\mathcal{V}^2\right)^{[0]} + \left(\mathcal{G}_{0,6},\mathcal{V}^2\right)^{[1]}\\ &+ \left(\mathcal{G}_{0,7},\mathcal{V}^2\right)^{[2]} + \left(\mathcal{G}_{0,8},\mathcal{V}^2\right)^{[3]} + \left(\mathcal{G}_{0,9},\mathcal{V}^2\right)^{[4]}\,,\end{aligned} \tag{3.3.37}$$

where the $G_{i,j}$ are chosen such that $\mathcal{C}_0$ will be a one-dimensional representation. Note in order for the final representation to be an invariant (i.e. one-dimensional) the representations $\mathcal{G}_{0,1}$, $\mathcal{G}_{0,2}$, $\mathcal{G}_{0,5}$, $\mathcal{G}_{0,6}$, $\mathcal{G}_{0,7}$ and $\mathcal{G}_{0,8}$ must all be identically zero, they are only shown here to indicate how the representations quadratic in the parameters will be constructed in later cases.

A generic fourth-order constant can now be written in the form

$$\mathcal{K}_4 = K_4 + K_2 + K_0, \tag{3.3.38}$$

where

$$\begin{aligned}K_4 &= \left(\mathcal{A}_8,\mathcal{P}^4\right)^{[8]} + \left(\mathcal{A}_4,\left(\mathcal{P}^2,\mathcal{P}^2\right)^{[2]}\right)^{[2]} + \left(\mathcal{A}_0,\left(\mathcal{P}^2,\mathcal{P}^2\right)^{[4]}\right)^{[0]}\,,\\ K_2 &= \left(\mathcal{B}_4,\mathcal{P}^2\right)^{[0]} + \left(\mathcal{B}_0,(\mathcal{P},\mathcal{P})^{[2]}\right)^{[0]}\,,\\ K_0 &= \mathcal{C}_0.\end{aligned} \tag{3.3.39}$$

The Poisson-Bracket $\{\mathcal{H},\mathcal{K}_4\}_{PB} \equiv 0$ splits into conditions based on the degree in the momenta. The 1st order conditions give

$$\{H_0,K_0\}_{PB} + \{V,K_2\}_{PB} \equiv 0.$$

[4]To determine the dimension it is helpful to remember that the rth transvectant of an nth-degree binary form and mth-degree binary form will be of degree $n+m-2r$.

From these conditions a set of eight highest-weight vectors can be determined. In terms of the representations these give the conditions

$$\partial_+^C(\mathcal{G}_{0,0}) = -\frac{1}{2}\mathcal{G}_{0,3} - 5i(\mathcal{G}_{0,0},\mathcal{X}_2)^{[0]}, \tag{3.3.40}$$
$$\partial_0^C(\mathcal{G}_{0,3}) = \frac{5i}{2}(\mathcal{X}_2,\mathcal{G}_{0,3})^{[1]}, \tag{3.3.41}$$

$$\begin{aligned}\lambda\mathcal{F}_{4,0} - 2\partial_+^C(\mathcal{G}_{0,3}) &= \mathcal{G}_{0,0}\left(\frac{104}{2025}(\mathcal{Y},\mathcal{Y})^{[4]} - \frac{28i}{45}(\mathcal{X},\mathcal{Y})^{[2]} - \frac{4i}{27}(\mathcal{Z},\mathcal{Y})^{[2]} - 4(\mathcal{X},\mathcal{X})^{[0]}\right)\\ &\quad - \frac{2}{15}(\mathcal{G}_{0,3},\mathcal{Y})^{[2]} + 6i(\mathcal{G}_{0,3},\mathcal{X})^{[0]} + 2\mathcal{G}_{0,9}, \quad (3.3.42)\\ \mathcal{F}_{0,0} - \frac{2}{3}\partial_+^C(\mathcal{G}_{0,3}) &= \mathcal{G}_{0,0}\left(\frac{58}{27}(\mathcal{Y},\mathcal{Y})^{[6]} - \frac{5}{3}(\mathcal{X},\mathcal{X})^{[2]} - \frac{10}{9}(\mathcal{X},\mathcal{Z})^{[2]} + \frac{5}{27}(\mathcal{Z},\mathcal{Z})^{[2]} - \frac{20}{3}\mathcal{Z}_{ee}\right)\\ &\quad + 5i(\mathcal{G}_{0,3},\mathcal{X})^{[2]} + 2\mathcal{G}_{4,0}, \quad (3.3.43)\\ \mathcal{F}_{4,3} - \frac{2}{15}\partial_+^C G_{0,9} &= \mathcal{G}_{0,3}\left(\frac{52}{30375}(\mathcal{Y},\mathcal{Y})^{[4]} - \frac{2}{15}(\mathcal{X},\mathcal{X})^{[0]} - \frac{14i}{675}(\mathcal{Y},\mathcal{X})^{[2]} - \frac{2i}{405}(\mathcal{Y},\mathcal{Z})^{[2]}\right)\\ &\quad - \frac{2}{675}(\mathcal{G}_{0,9},\mathcal{Y})^{[2]} - \frac{4}{15}(\mathcal{G}_{0,4},\mathcal{Y})^{[0]} + \frac{2i}{15}(\mathcal{G}_{0,9},\mathcal{X})^{[0]}, \quad (3.3.44)\end{aligned}$$

$$\begin{aligned}&\mathcal{F}_{0,3} - \frac{4}{3}\partial_-^C(\mathcal{G}_{0,9}) + \frac{2}{3}\partial_+^C\mathcal{G}_{0,4} =\\ &\quad\mathcal{G}_{0,3}\left(\frac{29}{27}(\mathcal{Y},\mathcal{Y})^{[6]} - \frac{5}{6}(\mathcal{X},\mathcal{X})^{[2]} - \frac{5}{9}(\mathcal{X},\mathcal{Z})^{[2]} + \frac{5}{54}(\mathcal{Z},\mathcal{Z})^{[2]} - \frac{10}{3}\mathcal{Z}_{ee}\right)\\ &\qquad + \frac{5i}{3}(\mathcal{G}_{0,9},\mathcal{X})^{[2]}, \quad (3.3.45)\end{aligned}$$

$$\begin{aligned}&\mathcal{F}_{4,1} + \frac{4}{5}\partial_-^C(\mathcal{G}_{0,9}) - 4\partial_+^C(\mathcal{G}_{0,4}) =\\ &\quad\frac{1}{6}\left(\mathcal{G}_{0,3}, -\frac{6}{5}(\mathcal{X},\mathcal{X})^{[0]} - \frac{14i}{75}(\mathcal{X},\mathcal{Y})^{[2]} - \frac{2i}{45}(\mathcal{Z},\mathcal{Y})^{[2]} + \frac{52}{3375}(\mathcal{Y},\mathcal{Y})^{[4]}\right)^{[2]}\\ &\qquad + 12i(\mathcal{G}_{0,4},\mathcal{X})^{[0]} - \frac{4i}{5}(\mathcal{G}_{0,9},\mathcal{X})^{[2]} - \frac{4}{25}(\mathcal{G}_{0,9},\mathcal{Y})^{[4]}, \quad (3.3.46)\end{aligned}$$

$$\begin{aligned}&\mathcal{F}_{4,2} - \frac{1}{3}\partial_0^C(\mathcal{G}_{0,9}) =\\ &\quad\frac{5}{144}\left(\mathcal{G}_{0,3}, -\frac{6}{5}(\mathcal{X},\mathcal{X})^{[0]} - \frac{14i}{75}(\mathcal{X},\mathcal{Y})^{[2]} - \frac{2i}{45}(\mathcal{Z},\mathcal{Y})^{[2]} + \frac{52}{3375}(\mathcal{Y},\mathcal{Y})^{[4]}\right)^{[1]}\\ &\qquad + \frac{i}{3}(\mathcal{G}_{0,9},\mathcal{X})^{[1]} - \frac{1}{120}(\mathcal{G}_{0,9},\mathcal{Y})^{[3]}. \quad (3.3.47)\end{aligned}$$

These are 8 representation cover 30 out of the 54 variables in the set $\{\mathcal{F}, \partial\mathcal{G}\}$, leaving 24 to be freely specified..

Examining components of $\{\mathcal{H}, \mathcal{K}_4\}_{PB} \equiv 0$, which are third-order in the momenta, gives the equation

$$\{H_0, K_2\}_{PB} + \{V, K_4\}_{PB} \equiv 0.$$

The subsequent equations can be used to solve for 8 of the representation in the set $\{\mathcal{A}, \partial\mathcal{F}\}$. Using ∂^C (3.1.25) the partial derivatives of (3.3.40)-(3.3.47) can be constructed. This construction gives an additional 20 representations whose highest derivatives are in the set $\{\mathcal{A}, \partial\mathcal{F}, \partial^2\mathcal{G}\}$. Together these 28 representations cover 126 of the 147 possible variables in $\{\mathcal{A}, \partial\mathcal{F}, \partial^2\mathcal{G}\}$, leaving 21 to be freely specified.

Finally the 5th order conditions

$$\{H_0, K_4\}_{PB} \equiv 0$$

gives rise to 3 representation, namely

$$\begin{aligned}
\partial_+^C \mathcal{A}_8 &= 0, \\
\partial_+^C \mathcal{A}_4 + \frac{1}{27}\partial_-^C \mathcal{A}_8 &= -\frac{i}{336}\left(\mathcal{Z}, \mathcal{A}_4\right)^{[2]} - \frac{i}{2}\left(\mathcal{Z}, \mathcal{A}_0\right)^{[0]}, \\
\partial_+^C \mathcal{A}_0 + \frac{36}{5}\partial_-^C \mathcal{A}_4 &= -\frac{21i}{10}\left(\mathcal{Z}, \mathcal{A}_4\right)^{[2]} - i\left(\mathcal{Z}, \mathcal{A}_0\right)^{[0]}.
\end{aligned} \tag{3.3.48}$$

These 3 representations cover 21 of 45 possible the variables in the set $\{\partial\mathcal{A}\}$.

Taking the partial derivatives of the 28 representations obtained so far for $\{\mathcal{A}, \partial\mathcal{F}, \partial^2\mathcal{G}\}$ gives an 50 representations on the set $\{\partial\mathcal{A}, \partial^2\mathcal{F}, \partial^3\mathcal{G}\}$ independent from three in (3.3.48). In total these 53 representations completely cover the 289 variables in the set $\{\partial\mathcal{A}, \partial^2\mathcal{F}, \partial^3\mathcal{G}\}$ and hence prove that all 289 variables can be solved for in terms of lower order derivatives.

So, ignoring the trivial additive term (i.e. the values of $\mathcal{G}$) the 24 unrestricted variables in the set $\{\mathcal{F}, \partial\mathcal{G}\}$ and the 21 unrestricted variables in the set $\{\mathcal{A}, \partial\mathcal{F}, \partial^2\mathcal{G}\}$ implies that the space of fourth-order constants is at most 45-dimensional. A set of 24 of these possible dimensions correspond to the six purely second-order constants multiplied with the four parameters and hence the space of *truly* fourth-order constants is at most 21-dimensional. This bound of 21 is achieved and the following theorem holds (taken from [12])

Theorem 3.3.49. *Defining the The 21 distinct standard monomials $L_{(i)}L_{(j)}$ form a basis for the space of purely fourth-order symmetries, where the 6 linearly independent constants have been denoted $L_{(i)}$, $i \in \{1, \ldots, 6\}$*

Sixth-order constants

The space of the even-order monomials up to degree 6 in the momenta splits into 10 representation (including the constant representation). Ordered by descending degree and size these are

$$\begin{aligned}
&K_6 : \mathcal{P}^6, \left(\mathcal{P}^3, \mathcal{P}^3\right)^{[2]}, \left(\mathcal{P}^3, \mathcal{P}^3\right)^{[4]}, \left(\mathcal{P}^3, \mathcal{P}^3\right)^{[6]}, \\
&K_4 : \mathcal{P}^4, \left(\mathcal{P}^2, \mathcal{P}^2\right)^{[2]}, \left(\mathcal{P}^2, \mathcal{P}^2\right)^{[4]}, \\
&K_2 : \mathcal{P}^2, \left(\mathcal{P}\ \ , \mathcal{P}\ \ \right)^{[2]}, \\
&K_0 : 1.
\end{aligned} \tag{3.3.50}$$

To each of these representations we will assign a complementary representation

$$\begin{aligned}
&K_6 : \mathcal{A}_{12}, \mathcal{A}_8, \mathcal{A}_4, \mathcal{A}_0, \\
&K_4 : \mathcal{B}_8, \mathcal{B}_4, \mathcal{B}_0, \\
&K_2 : \mathcal{C}_4, \mathcal{C}_0, \\
&K_0 : \mathcal{D}_0.
\end{aligned} \tag{3.3.51}$$

Where, like before, these are of equal dimension to (3.3.50).

Using this set up we parameterise a sixth-order constant via the form

$$\begin{aligned}
\mathcal{K}_6 = \left(\mathcal{A}_{12}, \mathcal{P}^6\right)^{[12]} + \Big(\mathcal{A}_8, \left(\mathcal{P}^3, \mathcal{P}^3\right)^{[2]}\Big)^{[8]} + \ldots \\
+ \left(\mathcal{B}_8, \mathcal{P}^4\right)^{[8]} + \Big(\mathcal{B}_4, \left(\mathcal{P}^2, \mathcal{P}^2\right)^{[4]}\Big)^{[4]} + \ldots \\
+ \left(\mathcal{C}_4, \mathcal{P}^2\right)^{[4]} + \Big(\mathcal{C}_0, (\mathcal{P}, \mathcal{P})^{[2]}\Big)^{[0]} + \mathcal{D}_0.
\end{aligned} \tag{3.3.52}$$

For the following discussion, the terms in $\mathcal{K}_6$ which are nth order in the momenta will be referred to as K_n.

The representations $\mathcal{B}_i, \mathcal{C}_i$ must be respectively linear and quadratic in the parameters and the explicit form for these can be deduced from (3.3.36) and (3.3.37). Borrowing the notation from the previous cases, the complementary that are defined in the construction of $\mathcal{B}_i$'s will be labeled $\mathcal{F}_{i,k}$ and the complementary representations in $\mathcal{C}_i$'s will be labeled $\mathcal{G}_{i,k}$.

The term $\mathcal{D}_0$ will be assumed cubic in the parameters and therefore be constructed by balancing the representations

$$\mathcal{V}_{ee}^3, \quad \mathcal{V}_{ee}^2\mathcal{V}, \quad \mathcal{V}_{ee}\left(\mathcal{V},\mathcal{V}\right)^{[2]}, \quad \mathcal{V}_{ee}\mathcal{V}^2, \quad \left(\mathcal{V}^2,\mathcal{V}\right)^{[2]}, \quad \mathcal{V}^3.$$

Explicitly $\mathcal{D}_0$ can be written in the form

$$\begin{aligned}\mathcal{D}_0 = {\mathcal{V}_{ee}}^3\mathcal{H}_{0,0} + {\mathcal{V}_{ee}}^2\left(\mathcal{H}_{0,3},\mathcal{V}\right)^{[2]} + \mathcal{V}_{ee}\ \left(\left(\mathcal{H}_{0,4},\left(\mathcal{V},\mathcal{V}\right)^{[2]}\right)^{[0]} + \left(\mathcal{H}_{0,9},\mathcal{V}^2\right)^{[4]}\right)\\ + \left(\mathcal{H}_{0,10},\left(\mathcal{V}^2,\mathcal{V}\right)^{[2]}\right)^{[0]} + \left(\mathcal{H}_{0,11},\left(\mathcal{V}^2,\mathcal{V}\right)^{[2]}\right)^{[1]} + \left(\mathcal{H}_{0,12},\left(\mathcal{V}^2,\mathcal{V}\right)^{[2]}\right)^{[2]}\\ + \left(\mathcal{H}_{0,13},\mathcal{V}^3\right)^{[0]} + \left(\mathcal{H}_{0,14},\mathcal{V}^3\right)^{[1]} + \left(\mathcal{H}_{0,15},\mathcal{V}^3\right)^{[2]}\\ + \left(\mathcal{H}_{0,16},\mathcal{V}^3\right)^{[3]} + \left(\mathcal{H}_{0,17},\mathcal{V}^3\right)^{[4]} + \left(\mathcal{H}_{0,18},\mathcal{V}^3\right)^{[5]} + \left(\mathcal{H}_{0,19},\mathcal{V}^3\right)^{[6]}\end{aligned} \tag{3.3.53}$$

for appropriately sized representations $\mathcal{H}_{i,j}$. Like before, seven trivial representation appear for the sake of notational completeness.

In what follows the notation $(n)_m$ will be used as a shorthand for 'n variables forming a set of m representations', if the subscript is dropped it won't change the meaning of the following sentences.

The condition

$$\{H,K_0\}_{PB} + \{V,K_2\}_{PB} \equiv 0$$

can be shown to put $(60)_{14}$ linearly independent restrictions (that is 60 condition which lie inside 14 representations of varying dimensions) on the set $\{\mathcal{G},\partial\mathcal{H}\}$ and consequently leave $(60)_{14}$ parameters free[5]. The condition

$$\{H,K_2\}_{PB} + \{V,K_4\}_{PB} \equiv 0$$

puts $(100)_{18}$ restrictions on the set $\{\mathcal{F},\partial\mathcal{G}\}$ and a further $(176)_{36}$ conditions can be constructed for the set $\{\partial\mathcal{G},\partial^2\mathcal{H}\}$, giving a total of $(276)_{54}$ restrictions, and leaving $(84)_{16}$ parameters free. The condition

$$\{H,K_4\}_{PB} + \{V,K_6\}_{PB} \equiv 0$$

gives $(84)_{12}$ conditions on the set $\{\mathcal{A},\partial\mathcal{F}\}$ and an additional $(624)_{110}$ condition can be constructed for the set $\{\partial\mathcal{F},\partial^2\mathcal{G},\partial^2\mathcal{H}\}$, giving a total of $(712)_{122}$ conditions and leaving $(56)_{10}$ parameters free. Finally the condition

$$\{H,K_6\}_{PB} \equiv 0$$

[5]There are $(120)_{28}$ parameters in total.

gives $(36)_4$ conditions on the set $\{\partial\mathcal{A}\}$ and an additional $(1308)_{196}$ can be constructed for the set $\{\partial\mathcal{A}, \partial^2\mathcal{F}, \partial^3\mathcal{G}, \partial^4\mathcal{H}\}$ and gives a total of $(1344)_{200}$ conditions leaving 0 parameters free. Hence the equations close at this level.

So the maximum number of sixth order constants is $(200)_{40}$ (ignoring the additive constant). There $(60)_{14}$ second-order constants with coefficients quadratic in the parameters and $(84)_{16}$ combinations of the second-order in the second-order constants and linear in the parameters. Thus the space of *purely* sixth-order constants at most $(56)_{10}$-dimensional. The following theorem states that this bound is achieved (see Ref. [12]).

Theorem 3.3.54. *The 56 distinct standard monomials $L_{(i)}L_{(j)}L_{(k)}$ form a basis for the space of purely sixth-order symmetries, where the* 6 *linearly independent constants have been denoted $L_{(i)}$, $i \in \{1, \ldots, 6\}$*

Eighth-order constants

We now venture into new territory and apply the technique to the case of eighth-order constants. Previous attempts (by the author) to do this without exploiting the rotational-adapted variables met with limited success. The analysis will make it clear that there is necessarily an identity at this level between the second-order constants.

To start, note the space spanned by the even-order monomials up to degree eight in the momenta splits into fifteen representation (including one constant representation). Ordered by descending degree and size these are

$$
\begin{aligned}
&K_8 : \mathcal{P}^8, \left(\mathcal{P}^4, \mathcal{P}^4\right)^{[2]}, \left(\mathcal{P}^4, \mathcal{P}^4\right)^{[4]}, \left(\mathcal{P}^4, \mathcal{P}^4\right)^{[6]}, \left(\mathcal{P}^4, \mathcal{P}^4\right)^{[8]}, \\
&K_6 : \mathcal{P}^6, \left(\mathcal{P}^3, \mathcal{P}^3\right)^{[2]}, \left(\mathcal{P}^3, \mathcal{P}^3\right)^{[4]}, \left(\mathcal{P}^3, \mathcal{P}^3\right)^{[6]}, \\
&K_4 : \mathcal{P}^4, \left(\mathcal{P}^2, \mathcal{P}^2\right)^{[2]}, \left(\mathcal{P}^2, \mathcal{P}^2\right)^{[4]}, \\
&K_2 : \mathcal{P}^2, \left(\mathcal{P}\ \ , \mathcal{P}\ \ \right)^{[2]}, \\
&K_0 : 1.
\end{aligned}
\tag{3.3.55}
$$

To each of these representations will be assigned a representation

$$\begin{aligned}
&K_8 : \mathcal{A}_{16},\ \ \mathcal{A}_{12},\ \ \mathcal{A}_8,\ \ \mathcal{A}_4,\ \ \mathcal{A}_0,\\
&K_6 : \mathcal{B}_{12},\ \ \mathcal{B}_8,\ \ \mathcal{B}_4,\ \ \mathcal{B}_0,\\
&K_4 : \mathcal{C}_8,\ \ \mathcal{C}_4,\ \ \mathcal{C}_0,\\
&K_2 : \mathcal{D}_4,\ \ \mathcal{D}_0,\\
&K_0 : \mathcal{E}_0,
\end{aligned} \tag{3.3.56}$$

which have the same dimensions as (3.3.55).

From these representations the eighth-order constant will be defined by

$$\begin{aligned}
\mathcal{K}_8 = \left(\mathcal{A}_{16}, \mathcal{P}^8\right)^{[16]} &+ \left(\mathcal{A}_{12}, \left(\mathcal{P}^4, \mathcal{P}^4\right)^{[2]}\right)^{[12]} + \dots\\
&+ \left(\mathcal{B}_{12}, \mathcal{P}^6\right)^{[12]} + \left(\mathcal{B}_8, \left(\mathcal{P}^3, \mathcal{P}^3\right)^{[2]}\right)^{[8]} + \dots\\
&+ \left(\mathcal{C}_8, \mathcal{P}^4\right)^{[8]} + \left(\mathcal{C}_4, \left(\mathcal{P}^2, \mathcal{P}^2\right)^{[4]}\right)^{[4]} + \dots\\
&+ \left(\mathcal{D}_4, \mathcal{P}^2\right)^{[4]} + \left(\mathcal{D}_0, \left(\mathcal{P}, \mathcal{P}\right)^{[2]}\right)^{[0]} + \mathcal{E}_0,
\end{aligned} \tag{3.3.57}$$

and like the previous cases, the terms nth-order in the momenta will be denoted by K_n.

Before making the assumptions about where the parameters appear, I will list all conditions that arise from the vanishing of the Poisson-bracket with the Hamiltonian. As no assumptions have been made these condition cover all the previous (even-order) cases. The condition $\{H_0, K_0\}_{PB} + \{V, K_2\}_{PB} = 0$ gives

$$\partial_+^C(\mathcal{E}_0) = \lambda \left(\frac{1}{12} \left(\mathcal{D}_4, \mathcal{V}\right)^{[2]} + \frac{1}{2} \left(\mathcal{D}_0, \mathcal{V}\right)^{[0]} \right). \tag{3.3.58}$$

The condition $\{H_0, K_2\}_{PB} + \{V, K_4\}_{PB} = 0$ gives

$$\begin{aligned}
\partial_+^C(\mathcal{D}_4) &= \lambda \left(\frac{1}{28} \left(\mathcal{C}_8, \mathcal{V}\right)^{[2]} + 6 \left(\mathcal{C}_4, \mathcal{V}\right)^{[0]} \right),\\
\partial_-^C(\mathcal{D}_4) + \frac{5}{2}\partial_+^C(\mathcal{D}_0) &= \lambda \left(\frac{7}{2} \left(\mathcal{D}_4, \mathcal{V}\right)^{[2]} + \frac{5}{3} \left(\mathcal{D}_0, \mathcal{V}\right)^{[0]} \right)\\
&\quad + \frac{5i}{24} \left(\mathcal{Z}, \mathcal{D}_4\right)^{[2]} + \frac{5i}{4} \left(\mathcal{Z}, \mathcal{D}_0\right)^{[0]}.
\end{aligned} \tag{3.3.59}$$

The condition $\{H_0, K_4\}_{PB} + \{V, K_6\}_{PB} = 0$ gives

$$\begin{aligned}
\partial_+^C(\mathcal{C}_8) &= \lambda\left(\frac{1}{44}(\mathcal{B}_{12},\mathcal{V})^{[2]} + \frac{45}{2}(\mathcal{B}_8,\mathcal{V})^{[0]}\right), \\
\partial_-^C(\mathcal{C}_8) + 27\partial_+^C(\mathcal{C}_4) &= \lambda\left(\frac{495}{112}(\mathcal{B}_8,\mathcal{V})^{[2]} + 81(\mathcal{B}_4,\mathcal{V})^{[0]}\right) \\
&\quad + \frac{9i}{112}(\mathcal{Z},\mathcal{C}_8)^{[2]} + \frac{27i}{2}(\mathcal{Z},\mathcal{C}_4)^{[0]}, \\
\partial_-^C(\mathcal{C}_4) + \frac{5}{36}\partial_+^C(\mathcal{C}_0) &= \lambda\left(\frac{9}{8}(\mathcal{B}_4,\mathcal{V})^{[2]} + \frac{1}{8}(\mathcal{B}_0,\mathcal{V})^{[0]}\right) \\
&\quad + \frac{7i}{24}(\mathcal{Z},\mathcal{C}_4)^{[2]} + \frac{5i}{36}(\mathcal{Z},\mathcal{C}_0)^{[0]}.
\end{aligned} \tag{3.3.60}$$

The condition $\{H_0, K_6\}_{PB} + \{V, K_8\}_{PB} = 0$ gives

$$\begin{aligned}
\partial_+^C(\mathcal{B}_{12}) &= \lambda\left(\frac{1}{60}(\mathcal{A}_{16},\mathcal{V})^{[2]} + 56(\mathcal{A}_{12},\mathcal{V})^{[0]}\right), \\
\partial_-^C(\mathcal{B}_{12}) + \frac{195}{2}\partial_+^C(\mathcal{B}_8) &= \lambda\left(\frac{70}{11}(\mathcal{A}_{12},\mathcal{V})^{[2]} + 780(\mathcal{A}_8,\mathcal{V})^{[0]}\right) \\
&\quad + \frac{13i}{264}(\mathcal{Z},\mathcal{B}_{12})^{[2]} + \frac{195i}{4}(\mathcal{Z},\mathcal{B}_8)^{[0]}, \\
\partial_-^C(\mathcal{B}_8) + \frac{9}{5}\partial_+^C(\mathcal{B}_4) &= \lambda\left(\frac{13}{14}(\mathcal{A}_8,\mathcal{V})^{[2]} + \frac{24}{5}(\mathcal{A}_4,\mathcal{V})^{[0]}\right) \\
&\quad + \frac{11i}{112}(\mathcal{Z},\mathcal{B}_8)^{[2]} + \frac{9i}{5}(\mathcal{Z},\mathcal{B}_4)^{[0]}, \\
\partial_-^C(\mathcal{B}_4) + \frac{1}{36}\partial_+^C(\mathcal{B}_0) &= \lambda\left(\frac{22}{27}(\mathcal{A}_4,\mathcal{V})^{[2]} + \frac{2}{63}(\mathcal{A}_0,\mathcal{V})^{[0]}\right) \\
&\quad + \frac{3i}{8}(\mathcal{Z},\mathcal{B}_4)^{[2]} + \frac{i}{24}(\mathcal{Z},\mathcal{B}_0)^{[0]}.
\end{aligned} \tag{3.3.61}$$

Finally the condition $\{H_0, K_8\}_{PB} = 0$ gives

$$\begin{aligned}
\partial_+^C(\mathcal{A}_{16}) &= 0, \\
\partial_-^C(\mathcal{A}_{16}) + 238\partial_+^C(\mathcal{A}_{12}) &= \frac{17i}{480}(\mathcal{Z},\mathcal{A}_{16})^{[2]} + 119i(\mathcal{Z},\mathcal{A}_{12})^{[0]}, \\
\partial_-^C(\mathcal{A}_{12}) + \frac{195}{28}\partial_+^C(\mathcal{A}_8) &= \frac{5i}{88}(\mathcal{Z},\mathcal{A}_{12})^{[2]} + \frac{195i}{28}(\mathcal{Z},\mathcal{A}_8)^{[0]}, \\
\partial_-^C(\mathcal{A}_8) + \frac{2}{5}\partial_+^C(\mathcal{A}_4) &= \frac{13i}{112}(\mathcal{Z},\mathcal{A}_8)^{[2]} + \frac{3i}{5}(\mathcal{Z},\mathcal{A}_4)^{[0]}, \\
\partial_-^C(\mathcal{A}_4) + \frac{1}{112}\partial_+^C(\mathcal{A}_0) &= \frac{11i}{24}(\mathcal{Z},\mathcal{A}_4)^{[2]} + \frac{i}{56}(\mathcal{Z},\mathcal{A}_0)^{[0]}.
\end{aligned} \tag{3.3.62}$$

Now assuming that the representations $\mathcal{B}_i, \mathcal{C}_i, \mathcal{D}_i$ are respectively linear, quadratic and cubic in the parameters, the necessary forms can be deduced from (3.3.36), (3.3.37) and (3.3.53). As might be obvious from the previous notational-recycling the complementary representations contained in $\mathcal{B}_i$ are labeled $\mathcal{F}_{i,j}$, the representations contained in $\mathcal{C}_i$ are labeled $\mathcal{G}_{i,j}$, and the ones comprising $\mathcal{D}_i$ are labeled $\mathcal{H}_{i,j}$.

The new term $\mathcal{E}_0$ comes from balancing the nine representations fourth-order in the parameters. Explicitly, the form is

$$\begin{aligned}\mathcal{E}_0 = {\mathcal{V}_{ee}}^4\mathcal{I}_{0,0} + {\mathcal{V}_{ee}}^3\left(\mathcal{I}_{0,3},\mathcal{V}\right)^{[2]} + {\mathcal{V}_{ee}}^2\left(\left(\mathcal{I}_{0,4},\left(\mathcal{V},\mathcal{V}\right)^{[2]}\right)^{[0]} + \left(\mathcal{I}_{0,9},\mathcal{V}^2\right)^{[4]}\right) \\ + \mathcal{V}_{ee}\left(\left(\mathcal{I}_{0,12},\left(\mathcal{V}^2,\mathcal{V}\right)^{[2]}\right)^{[2]} + \left(\mathcal{I}_{0,17},\mathcal{V}^3\right)^{[6]}\right) \\ + \left(\mathcal{I}_{0,20},\left(\mathcal{V}^2,\mathcal{V}^2\right)^{[4]}\right)^{[0]} + \left(\mathcal{I}_{0,25},\left(\mathcal{V}^2,\mathcal{V}^2\right)^{[2]}\right)^{[4]} + \left(\mathcal{I}_{0,31},\mathcal{V}^4\right)^{[6]}\end{aligned} \tag{3.3.63}$$

for appropriately sized representations $\mathcal{I}_{0,i}$. Since the analysis will stop at the eighth-order constants the trivial terms that would contribute when constructing higher-order constants haven't been written down (like the terms written down when considering the $\mathcal{D}_0$ in (3.3.53)).

Again the notation $(n)_m$ will be used as shorthand to refer to n variables forming a set of m representation, and dropping the subscript m doesn't change the meaning of the following sentences.

The condition

$$\{H, K_0\}_{PB} + \{V, K_2\}_{PB} \equiv 0$$

puts $(105)_{21}$ restrictions on the set $\{\mathcal{H}, \partial\mathcal{I}\}$ leaving $(120)_{24}$ parameters free. The condition

$$\{H, K_2\}_{PB} + \{V, K_4\}_{PB} \equiv 0$$

puts $(200)_{34}$ restrictions on the set $\{\mathcal{G}, \partial\mathcal{H}\}$ and a further $(311)_{57}$ conditions can be constructed for the set $\{\partial\mathcal{H}, \partial^2\mathcal{I}\}$, giving a total of $(511)_{91}$ restrictions, and leaving $(209)_{37}$ parameters free. The next condition

$$\{H, K_4\}_{PB} + \{V, K_6\}_{PB} \equiv 0,$$

puts $(220)_{28}$ restrictions on the set $\{\mathcal{F}, \partial\mathcal{G}\}$ and a further $(1198)_{192}$ conditions can be constructed for the set $\{\partial\mathcal{G}, \partial^2\mathcal{H}, \partial^2\mathcal{I}\}$. This gives a total of $(1408)_{220}$ conditions on the set and leaves $(224)_{34}$ parameters free. The penultimate Poisson-Brackets

$$\{H, K_6\}_{PB} + \{V, K_8\}_{PB} \equiv 0,$$

puts $(2736)_{338}$ restrictions on the set $\{\mathcal{A}, \partial\mathcal{F}\}$ and a further $(2736)_{388}$ conditions can be constructed for the set $\{\partial\mathcal{F}, \partial^2\mathcal{G}, \partial^2\mathcal{H}, \partial^3\mathcal{I}\}$. This gives a total of $(2880)_{404}$ conditions on the set and leaves $(126)_{38}$ parameters free. The final Poisson-Bracket

$$\{H, K_8\}_{PB} \equiv 0$$

puts $(55)_5$ restrictions on the set $\partial\mathcal{A}$ and a further $(4787)_{603}$ conditions can be constructed for the set $\{\partial\mathcal{A}, \partial^2\mathcal{F}, \partial^3\mathcal{G}, \partial^4\mathcal{H}, \partial^5\mathcal{I}\}$. This gives a total of $(4842)_{608}$ conditions and leaves no parameters free, hence the equations close at this level.

Taking into account the additive constants (i.e. the representations $\mathcal{I}_{i,j}$) the space of eight order constants is of dimension at most $(714)_{142}$. However the monomials of which are purely fourth-order in terms of the six second-order constants and the 4 parameters can be easily shown to have dimension

$$(35)_9 + (120)_{24} + (210)_{38} + (224)_{34} + (126)_{38} = (715)_{143}.$$

Here the term $(35)_9$ comes from the $\binom{4+3}{3} = 35$ monomials purely quadratic in the 4 parameter. At the other extreme the term $(126)_{38}$ comes from $\binom{6+3}{3}$ monomials purely quadratic in the 6 second-order constants. The rest of the terms come from the mixed monomials and can easily be enumerated using basic combinatorics.

So the space of monomials is larger (by one) than the space of possible constants of this type. Thus these monomials are not linearly independent and there necessarily exists a linear-combination of these monomials which is identically zero. That is, there exists a quartic identity between the second-order constants and the parameters. In the next section this identity will be given explicitly.

3.4 The Quartic Identity

For a maximum-parameter second-order system the algebra formed from the iterated Poisson-commutators of the second-order constant closes at the third order in the momenta, that is, given any constant derived from Poisson commutators in this manner and which is of order greater than 4, then it must be a polynomial in the second and third order constants [12, 8]. As only 5 of the six second-order constant can be independent it is necessary that they satisfy a functional relation, and for the known systems and this identity is a quartic (or at most quartic). The result of the previous section prove that a similar quartic identity would necessarily exist for any unknown system as well. By choosing a canonical form for the second-order constants, a general formula can be found.

The canonical basis

Thanks to the $(5 \implies 6)$ theorem it is known, that at any regular point in the systems $\mathbf{x}_{rp}$, and for any particular set of values for a_0^{ij} there will be a second-order constant which takes these values, i.e. $a^{ij}(\mathbf{x}_{rp}) = a_0^{ij}$. Since each constant of the motion allows an additive parameter these can also be assumed to have $W|_{\mathbf{x}_{rp}} = 0$.

The canonical basis is defined as follows. For any prescribed regular point in the system let L_{ij} be the linearly independent constants that satisfy the following,

$$
\begin{aligned}
L_{11}|_{\mathbf{x}_{rp}} &= p_{x_1}^2, \\
L_{22}|_{\mathbf{x}_{rp}} &= p_{x_2}^2, \\
L_{33}|_{\mathbf{x}_{rp}} &= p_{x_3}^2, \\
L_{12}|_{\mathbf{x}_{rp}} &= 2p_{x_1}p_{x_1}, \\
L_{13}|_{\mathbf{x}_{rp}} &= 2p_{x_1}p_{x_3}, \\
L_{23}|_{\mathbf{x}_{rp}} &= 2p_{x_2}p_{x_3}.
\end{aligned}
\tag{3.4.1}
$$

For example, taking the Euclidean superintegrable system with potential

$$V_{IV} = a(4x_1^2 + x_2^2 + x_3^2) + bx_1 + \frac{c}{x_2^2} + \frac{d}{x_3^2} + e \tag{3.4.2}$$

and defining the regular point to be with the regular point $\mathbf{x}_{rp} = (1, 1, 1)$ the canonical second-order constants are

$$
\begin{aligned}
L_{11} &= p_{x_1}^2 + 4a(x_1-1)(x_1+1) + (x_1-1)b,\\
L_{22} &= p_{x_2}^2 + a(x_2-1)(x_2+1) - c\frac{(x_2-1)(x_2+1)}{x_2^2},\\
L_{33} &= p_{x_3}^2 + a(x_3-1)(x_3+1) - d\frac{(x_3-1)(x_3+1)}{x_3^2},\\
L_{12} &= 2p_{x_2}(p_{x_2} - p_{x_2}x_1 + x_2p_{x_1}) + a(2x_2^2x_1 + 2x_2^2 - 4)\\
&\quad + b\frac{(x_2-1)(x_2+1)}{2} - c\frac{2(x_1-1)}{x_2^2},\\
L_{13} &= 2p_{x_3}(p_{x_3} - p_{x_3}x_1 + x_3p_{x_1}) + a(2x_3^2x_1 + 2x_3^2 - 4)\\
&\quad + b\frac{(x_3-1)(x_3+1)}{2} - d\frac{2(x_1-1)}{x_3^2},\\
L_{23} &= p_{x_2}^2 + p_{x_3}^2 - (x_3p_{x_2} - x_2p_{x_3})^2 + a(x_3^2 + x_2^2 - 2)\\
&\quad - c\frac{(x_3-1)(x_3+1)}{x_2^2} - d\frac{(x_2-1)(x_2+1)}{x_3^2}.
\end{aligned}
\tag{3.4.3}
$$

Returning to the general case, the canonical second-order constants can be split into two rotation representations. As binary forms these are the 5-dimensional representation

$$
\begin{aligned}
\mathcal{M}(r,s) = (L_{11} + iL_{12} - L_{22})r^4 &+ 2(iL_{23} + L_{13})r^3s\\
&+ 2(2L_{33} - L_{11} - L_{22})r^2s^2\\
&+ 2(iL_{23} - L_{13})rs^3 + (L_{11} - iL_{12} - L_{22})s^4,
\end{aligned}
\tag{3.4.4}
$$

and the one-dimensional representation

$$
\mathcal{L}_0 = L_{11} + L_{22} + L_{33}. \tag{3.4.5}
$$

By the result in the previous section, there is an eighth-order identity in the form of a quartic between the 6 canonical constants and the four parameters of the potential. To find this identity all that needs to be done is to set up a generic fourth-order combination of the canonical constants L_{ij} and the parameters. According to the previous section this eighth-order constant will be determined completely by knowing the value of (up to) 5th order partial derivatives. So, once the linear combination has been set up, finding the coefficients which cause this combination to vanish at a sufficiently high number of derivatives at the regular point will give the identity.

A first pass over the equations shows that the coefficients of terms which are either zeroth-order or linear in the second-order constants must be identically zero. Hence the quartic identity in the canonical constants will be of the form

$$Q = \sum \alpha^{(ij),(kl),(mn),(op)} L_{ij} L_{kl} L_{mn} L_{op} + \sum \beta^{(ij),(kl),(mn),(q)} L_{ij} L_{kl} L_{mn} \mathcal{V}_q + \sum \gamma^{(ij),(kl),(q),(r)} L_{ij} L_{kl} \mathcal{V}_q \mathcal{V}_r \quad (3.4.6)$$

where the q in $\mathcal{V}_q$ is from the set $\{1, 2, 3, ee\}$ (i.e. the four parameters) and the sums are over all possible combinations of variables.

If equations governing the remaining coefficients of (3.4.6) are examined, which are not listed here for reason of space, it quickly becomes apparent that the equations would look more balanced if the $\alpha^{\mathbf{I}}$ coefficients were quadratic in the variables $\{\mathbf{Q}, \mathbf{R}, \mathbf{S}\}$ and if the coefficients $\beta^{\mathbf{I}}$ were linear in $\{\mathbf{Q}, \mathbf{R}, \mathbf{S}\}$. Making this assumption a set of non-trivial coefficients can be found which cause the linear combination to vanishes at the regular point up to fifth-order derivatives. Hence an explicit form for the quartic identity has been found[6]. This result was also verified by testing it at various regular points in the known systems.

The quartic identity that was obtained is very long, containing 1641 terms and covering 4 or more pages when written out directly. An example of the purely quartic terms that appear are given by

$$\frac{2(6S^{3,(0)} + G^{(0)}_{x_3} - 2R^{23,(0)}_2)Q^{123,(0)}}{9} L_{22} L_{12}^3 - \frac{(6S^{3,(0)} + G^{(0)}_{x_3} - 2R^{23,(0)}_2)^2}{18} L_{22}^2 L_{33}^2,$$

where the notation $A^{(0)} = A|_{\mathbf{x}_{rp}}$ has been used. Likewise an example of the quadratic terms is

$$(2L_{11}L_{33} - \frac{1}{2}L_{13}^2){V^{(0)}_{x_2}}^2 + (L_{12}L_{23} - 2L_{22}L_{13})V^{(0)}_{x_3}V^{(0)}_{x_1} + (L_{13}L_{23} - 2L_{33}L_{12})V^{(0)}_{x_2}V^{(0)}_{x_1}.$$

Despite the length of the identity found, it has the structure of a one-dimensional rotation representation, and this means it can be written out in terms of transvectants of simpler representations.

Using the notation $\mathcal{Y}, \mathcal{X}, \mathcal{Z}, \mathcal{V}, \mathcal{V}_{ee}, \mathcal{Z}, \mathcal{Z}_{ee}$ and λ to denote the value of those representations at the chosen regular point, and expressing the second-order

[6] The results of the previous section are enough to show the coefficients in an an eighth-order constant need only been known up to their 5th derivative to completely determine which eight-order constants is being referred to

constants as the representation (3.4.4) and (3.4.5), the quartic identity can be drastically simplified to a more manageable form and is given in figure 3.1.

Example: The quartic identity for V_{IV}

Taking the system V_{VI} given at (3.4.2), the representations related to the structural equations take the form, at the regular point $\mathbf{x} = (1, 1, 1)$,

$$\begin{aligned} \mathcal{Y}(r,s) =& \frac{3}{4}\left(r^6 + 3r^4s^2 + 8ir^3s^3 + 3r^2s^4 + s^6\right), \\ \mathcal{X}(r,s) =& \frac{i}{5}\left(r^2 - 2irs + s^2\right), \\ \lambda =& 1, \\ \mathcal{Z}(r,s) =& 0, \\ \mathcal{Z}_{ee} =& 0, \end{aligned} \tag{3.4.7}$$

and the representations coming from the parameter take on the form

$$\begin{aligned} \mathcal{V}(r,s) =& ((1+4i)a + \frac{i}{2}b - c)s^2 + 2i(a-d)rs + ((1-4i)a - \frac{i}{2}b - c)r^2, \\ \mathcal{V}_{ee} =& 4a + 2c + 2d. \end{aligned} \tag{3.4.8}$$

Redefining the basis of 2nd-order constants to be the same as the one given by Daskaloyannis [8] (which is specifically adapted to the sub-algebras within the quadratic algebra of system IV)

$$\begin{aligned} H &= L_{11} + L_{22} + L_{33} - 2a + c + d + ib, \\ A_1 &= L_{11} - 4a + ib, \\ A_2 &= L_{22} + a + c, \\ B_1 &= -iL_{33} - id + \frac{1}{2}L_{13} + ia + \frac{1}{4}b, \\ B_2 &= L_{33} - L_{23} + L_{22} + c + d, \\ F &= -\frac{1}{2}L_{12} - \frac{1}{4}b + iL_{22} - ia + ic, \end{aligned} \tag{3.4.9}$$

the quartic identity is given by

$$
\begin{aligned}
16A_2B_1\Big(A_1+A_2\Big)F-8\Big(A_1+A_2\Big)^2F^2-8A_2\Big(A_1A_2B_2+A_1^2B_2+A_2B_1^2\Big)\\
-8\Big(A_1B_2^2-4FB_1B_2\Big)a+4\Big(A_2B_1B_2+B_2(A_1+A_2)F\Big)b-8A_1\Big(A_1+A_2\Big)^2c\\
+8\Big(A_1A_2B_2+2(A_1+A_2)F^2-2A_2FB_1\Big)H-8H^2F^2-8A_1A_2^2d\\
-4\Big(FB_2b-4A_1c(A_1+A_2)\Big)H+32\Big(B_1^2c+F^2d\Big)a+8\Big(A_2Fd+B_1(A_1+A_2)c\Big)b\\
32A_1acd-8A_1H^2c-8B_1Hbc-\frac{1}{2}B_2^2b^2+2db^2c\equiv 0. \quad (3.4.10)
\end{aligned}
$$

$$
\begin{aligned}
&\left[\frac{1}{12}(\mathcal{Z},\mathcal{Z})^{[2]} - 5\left(\mathcal{X}+\frac{\mathcal{Z}}{6},\mathcal{X}+\frac{\mathcal{Z}}{6}\right)^{[2]} + \frac{14}{27}(\mathcal{Y},\mathcal{Y})^{[6]} - 2\mathcal{Z}_{ee}\right]\mathcal{L}_0{}^4 \\
&+\left(\frac{-16}{2025}(\mathcal{Y},\mathcal{Y})^{[4]} + \frac{4i}{15}\left(\mathcal{Y},\mathcal{X}+\frac{\mathcal{Z}}{6}\right)^{[2]} - 9\left(\mathcal{X}+\frac{\mathcal{Z}}{6}\right)^2 - \frac{9}{20}\mathcal{M}(\mathcal{Y},\mathcal{Y})^{[6]},\mathcal{M}\right)^{[4]}\mathcal{L}_0{}^3 \\
&+\left(\frac{2}{675}(\mathcal{Y},\mathcal{Y})^{[2]} + 6i\mathcal{Y}\left[\mathcal{X}+\frac{\mathcal{Z}}{6}\right],\mathcal{M}^2\right)^{[8]}\mathcal{L}_0{}^3 + \left(\mathcal{Y}^2,\mathcal{L}_0\mathcal{M}^3 + \frac{\mathcal{M}^2}{24}(\mathcal{M},\mathcal{M})^{[2]}\right)^{[12]} \\
&+\left(\frac{2}{14175}(\mathcal{Y},\mathcal{Y})^{[4]} + \frac{i}{315}\left(\mathcal{X}+\frac{\mathcal{Z}}{6},\mathcal{Y}\right)^{[2]} - \frac{3}{8}\left(\mathcal{X}+\frac{\mathcal{Z}}{6}\right)^2,(\mathcal{M},\mathcal{M})^{[2]}\right)^{[4]}\mathcal{L}_0{}^2 \\
&+\frac{9}{8}\left[\left(\mathcal{X}+\frac{\mathcal{Z}}{6},\mathcal{X}+\frac{\mathcal{Z}}{6}\right)^{[2]} - \frac{1}{12}(\mathcal{Z},\mathcal{Z})^{[2]} + 2\mathcal{Z}_{ee}\right](\mathcal{M},\mathcal{M})^{[4]}\mathcal{L}_0{}^2 \\
&+\left(\frac{4}{22275}(\mathcal{Y},\mathcal{Y})^{[2]} + \frac{i\mathcal{Y}}{4}\left(\mathcal{X}+\frac{\mathcal{Z}}{6}\right),\mathcal{M}(\mathcal{M},\mathcal{M})^{[2]}\right)^{[8]}\mathcal{L}_0 \\
&+\left(\frac{1}{22680}(\mathcal{Y},\mathcal{Y})^{[4]} - \frac{i}{3024}\left(\mathcal{X}+\frac{\mathcal{Z}}{6},\mathcal{Y}\right)^{[2]},\left((\mathcal{M},\mathcal{M})^{[2]},\mathcal{M}\right)^{[2]}\right)^{[4]}\mathcal{L}_0 \\
&-\frac{221}{15120}\mathcal{L}_0(\mathcal{Y},\mathcal{Y})^{[6]}\cdot\left((\mathcal{M},\mathcal{M})^{[2]},\mathcal{M}\right)^{[4]} + \frac{i}{768}\mathcal{L}_0\left(\left(\mathcal{X}+\frac{\mathcal{Z}}{6},\mathcal{Y}\right)^{[1]},\left((\mathcal{M},\mathcal{M})^{[2]},\mathcal{M}\right)^{[1]}\right)^{[6]} \\
&-\frac{1}{32}\left[\left(\mathcal{X}+\frac{\mathcal{Z}}{6},\mathcal{X}+\frac{\mathcal{Z}}{6}\right)^{[2]} + \frac{1}{12}(\mathcal{Z},\mathcal{Z})^{[2]} - 2\mathcal{Z}_{ee}\right]\cdot\left((\mathcal{M},\mathcal{M})^{[2]},\mathcal{M}\right)^{[4]}\mathcal{L}_0 \\
&+\frac{1}{106920}\left((\mathcal{Y},\mathcal{Y})^{[2]},\left[(\mathcal{M},\mathcal{M})^{[2]}\right]^2\right)^{[8]} + \frac{1}{4536}\left((\mathcal{M},\mathcal{M})^{[2]},\mathcal{M}\right)^{[4]}\left((\mathcal{Y},\mathcal{Y})^{[4]},\ \mathcal{M}\right)^{[4]} \\
&+\frac{1}{5544}(\mathcal{M},\mathcal{M})^{[4]}\left[\left(\left((\mathcal{M},\mathcal{M})^{[2]},\mathcal{Y}\right)^{[2]},\mathcal{Y}\right)^{[6]} - \frac{42}{5}\left(\left((\mathcal{M},\mathcal{Y})^{[4]},\mathcal{Y}\right)^{[2]},\mathcal{M}\right)^{[4]} + 180(\mathcal{M},\mathcal{M})^{[4]}(\mathcal{Y},\mathcal{Y})^{[6]}\right] \\
&+\left[\left(\mathcal{V},5\mathcal{X}+\frac{2}{3}\mathcal{Z}\right)^{[2]} + 2\mathcal{V}_{ee}\right]\mathcal{L}_0{}^3\lambda + \left(9\mathcal{V}\left[\mathcal{X}+\frac{\mathcal{Z}}{6}\right] - \frac{2i}{15}(\mathcal{Y},\mathcal{V})^{[2]},\mathcal{M}\right)^{[4]}\lambda \\
&-i\left(\mathcal{Y}\mathcal{V},3\mathcal{L}_0\mathcal{M}^2 + \frac{1}{8}\mathcal{M}(\mathcal{M},\mathcal{M})^{[2]}\right)^{[8]}\lambda + \left(\frac{3}{8}\mathcal{V}\left[\mathcal{X}+\frac{\mathcal{Z}}{6}\right] - \frac{i}{630}(\mathcal{Y},\mathcal{V})^{[2]},(\mathcal{M},\mathcal{M})^{[2]}\right)^{[4]}\mathcal{L}_0\lambda \\
&-\frac{9}{8}\left[\left(\mathcal{X}+\frac{\mathcal{Z}}{6},\mathcal{V}\right)^{[2]} + 2\mathcal{V}_{ee}\right](\mathcal{M},\mathcal{M})^{[4]}\mathcal{L}_0\lambda \\
&+\frac{i}{1536}\left((\mathcal{Y},\mathcal{V})^{[1]},\left((\mathcal{M},\mathcal{M})^{[2]},\mathcal{M}\right)^{[1]}\right)^{[6]}\lambda + \frac{i}{6048}\left((\mathcal{Y},\mathcal{V})^{[2]},\left((\mathcal{M},\mathcal{M})^{[2]},\mathcal{M}\right)^{[2]}\right)^{[4]}\lambda \\
&+\frac{1}{32}\left[\left(\mathcal{X}+\frac{\mathcal{Z}}{6},\mathcal{V}\right)^{[2]} - 2\mathcal{V}_{ee}\right]\left((\mathcal{M},\mathcal{M})^{[2]},\mathcal{M}\right)^{[4]}\lambda \\
&-\frac{3}{2}(\mathcal{V},\mathcal{V})^{[2]}\mathcal{L}_0{}^2\lambda^2 - \frac{9}{4}\left(\mathcal{V}^2,\mathcal{M}\right)^{[4]}\mathcal{L}_0\lambda^2 - \frac{3}{32}\left(\mathcal{V}^2,(\mathcal{M},\mathcal{M})^{[2]}\right)^{[4]}\lambda^2 + \frac{9}{16}(\mathcal{V},\mathcal{V})^{[2]}\cdot(\mathcal{M},\mathcal{M})^{[4]}\lambda^2 \equiv 0.
\end{aligned}
$$

Figure 3.1: The Quartic Identity

Chapter 4

Structure Theory of Second-Order Conformally-Superintegrable Systems

A classification of second-order superintegrable systems (i.e. the types discussed in chapter 2) is hindered by the need to specify the conformally-flat metric beforehand. A way around this obstruction is to instead consider conformal classes of potentials. That is, only distinguish systems which are not related via a Stäckel transform. This leads naturally to the study of the so-called *conformally*-superintegrable systems (also known as a Laplace-type system). Every superintegrable system over a conformally flat space is, by default, conformally superintegrable and every conformally superintegrability systems is Stäckel equivalent to a superintegrable system (see theorem 4.1.8).

The notion of conformal-superintegrability was introduced by Kalnins *et al* for the purpose of classifying superintegrable systems over conformally flat spaces [27]. It was shown that the second-order conformally-superintegrable systems over flat spaces can be put into correspondence with a 10-dimensional manifold. It is this correspondence that forms the starting point for the classification in chapter 5. This 10-dimensional manifold possess an action induced by the conformal group in three dimensions (which is a 10-dimension Lie group), and the foliation of the space under this actions provides the classification result.

In §4.1 the notion of a conformally-superintegrable system is defined and how these relate to the superintegrable systems is discussed. Specifically a conformally-superintegrable system is Stäckel equivalent to a superintegrable system, and a superintegrable system yields a conformally superintegrable one via a conformal scaling. This allows the classical structure theory for conformally-superintegrable systems to be determined from the discussion in chapter 2. Most importantly the potential for a conformally-superintegrable systems over a conformally flat-space statisfies a set of linear PDEs. However, unlike the superintegrable case the 5th parameter is no longer a trivial additive one.

In §4.2 the local action of the conformal group is studied by considering the action of a conformal change of coordinates. It is shown that this action decomposes the space of coefficient functions (denoted $\{\boldsymbol{\mathcal{Q}}, \boldsymbol{\mathcal{R}}, \boldsymbol{\mathcal{S}}\}$) into a 3-dimensional and a 7-dimensional component. These are almost the same as the $SO(3,\mathbb{C})$ representations $\mathcal{X}, \mathcal{Y}$ ((3.1.9) and (3.1.10) respectively) introduced in chapter 3. The local action of the conformal group is then shown to act transitively on variables $\{\boldsymbol{\mathcal{S}}\}$.

In §4.3 the non-local action of the conformal group is examined through translation of the regular point. This requires examining the partial derivatives, which take polynomials in $\{\boldsymbol{\mathcal{Q}}, \boldsymbol{\mathcal{R}}, \boldsymbol{\mathcal{S}}\}$ to higher degree polynomials in $\{\boldsymbol{\mathcal{Q}}, \boldsymbol{\mathcal{R}}, \boldsymbol{\mathcal{S}}\}$. The goal in chapter 5 is to, as was done for the Euclidean case [17], create polynomial ideals constructed from the $\{\boldsymbol{\mathcal{Q}}, \boldsymbol{\mathcal{R}}, \boldsymbol{\mathcal{S}}\}$ variables which are closed under translation of the regular point (i.e. under the action of ∂_x). Theorems 4.2.12 and 4.3.4 are both very relevant in this regard.

4.1 Classical Structure Theory for a Conformally-Superintegrable System

Consider a classical system with a Hamiltonian over a conformally-flat space. Without loss of generality this can be assumed to take the form

$$H(\mathbf{p}, \mathbf{x}) = \frac{{p_{x_1}}^2 + {p_{x_2}}^2 + {p_{x_3}}^2}{\lambda(\mathbf{x})} + V(\mathbf{x}). \tag{4.1.1}$$

This is the same form as (3.2.1). However instead of a second-order constant, consider a second-order *conformal*-constant. This will be a function of the form

$$L = \sum_{i,j=1}^{3} a^{ij}(\mathbf{x}) p_{x_i} p_{x_j} + W(\mathbf{x}), \qquad a^{ij} = a^{ji}, \tag{4.1.2}$$

such that the Poisson commutator of the Hamiltonian $\mathcal{H}$ and the conformal-constant gives

$$\{H, L\}_{PB} = \rho H, \tag{4.1.3}$$

where $\rho(\mathbf{p}, \mathbf{x})$ is polynomial in momentum coordinates. In this case ρ_L will first order in the momentum.

Notice that any Hamiltonian H will have infinitely many *trivial* conformal-constants of the form $F(\mathbf{x}, \mathbf{p})H$ for any differentiable function $F(\mathbf{x}, \mathbf{p})$. So two conformal-constants will only be considered different if their difference is not a multiple of the Hamiltonian (or equivalently, all second-order conformal-constants are assume to have a traceless second-order component). So while a superintegrable Hamiltonian would quite naturally be called a second-order constant for the system it describes, this identification makes a conformally-superintegrable Hamiltonian equivalent to zero.

Based on the discussion above, the obvious definition for a maximally conformally-superintegrable system is for the Hamiltonian H to possess $2n-2$ independent and inequivalent conformal-constants.

Lemma 4.1.4. *Every conformally-superintegrable system over a conformally-flat space can be conformally scaled to a conformally-superintegrable system over flat space.*

Proof. By hypothesis the systems possess a Hamiltonian of the form

$$H = \frac{H_0}{\lambda} + V \tag{4.1.5}$$

where $H_0 = p_{x_1}^2 + p_{x_2}^2 + p_{x_3}^2$. This Hamilton will possess conformal constants L which satisfy

$$\{H, L\}_{PB} = \rho_L H. \tag{4.1.6}$$

Scaling the Hamiltonian by the conformal factor λ gives the new Hamiltonian $\widetilde{H} = \lambda H = H_0 + \lambda V$. The Poisson-Bracket of this new Hamiltonian and the original conformal-constant satisfies

$$\begin{aligned}\left\{\widetilde{H}, L\right\}_{PB} &= \{\lambda H, L\}_{PB} \\ &= \lambda \{H, L\}_{PB} + H\{\lambda, L\}_{PB} \\ &= \lambda \rho_L H + H\{\lambda, L\}_{PB} \\ &= \left(\rho_L + \frac{\{\lambda, L\}_{PB}}{\lambda}\right)\lambda H \\ &= \left(\rho_L + \frac{\{\lambda, L\}_{PB}}{\lambda}\right)\widetilde{H}. \end{aligned} \tag{4.1.7}$$

The factor

$$\widetilde{\rho}_L = \rho_L + \frac{\{\lambda, L\}_{PB}}{\lambda}$$

is also polynomial in the momentum and hence L is also a conformal-constant of $\widetilde{H}$, which is clearly a Hamiltonian over flat space. □

So henceforth the conformally-superintegrable systems under consideration will be assumed to be over flat space, conformally scaling if necessary. There do exist superintegrable systems which are not Stäckel equivalent to conformally flat systems [18], but these are not the subject of this thesis. Note that a superintegrable system is a conformally-superintegrable system with conformal-constants for which $\rho_L = 0$, so an immediately corollary of lemma 4.1.4 is that all superintegrable systems over conformally-flat spaces are equivalent (by conformal scaling) to a conformally-superintegrable one.

Theorem 4.1.8. *If $\mathcal{H} = H_0 + \alpha U$ is a Hamiltonian with a conformal-constant $\mathcal{L}(\alpha) = L_0 + \alpha W_U$, then the new Hamiltonian $\widetilde{\mathcal{H}} = \dfrac{\mathcal{H}}{U}$ has constant $\mathcal{L}(-\widetilde{H})$.*

Proof. This proof is almost identical to the proof of theorem 2.4.1. Recall that, given functions of the form $G(\mathbf{x}, \mathbf{p}), F(a, \mathbf{x}, \mathbf{p})$ where $a = \tau(\mathbf{x}, \mathbf{p})$ then

$$\{F, G\}_{PB} = \left[\{F(a, \mathbf{x}, \mathbf{p}), G(\mathbf{x}, \mathbf{p})\}_{PB}\right]_{a=\tau(\mathbf{x},\mathbf{p})} + \left[\partial_a F(a, \mathbf{x}, \mathbf{p})\right]_{a=\tau(\mathbf{x},\mathbf{p})}\{\tau(\mathbf{x}, \mathbf{p}), G(\mathbf{x}, \mathbf{p})\}_{PB}.$$

Consider the conformal constant L which, by hypothesis, satisfies a relation of the form

$$\{H + \alpha U, L(\alpha)\}_{PB} = \rho(H + \alpha U),$$

and so

$$\{H, L(\alpha)\}_{PB} = -\alpha\{U, L(\alpha)\}_{PB} + \rho(H + \alpha U).$$

Using these it can be shown

$$\begin{aligned}
\{\widetilde{H}, L(\alpha)\}_{PB} &= \left\{\frac{H}{U}, L(\alpha)\right\}_{PB} \\
&= -\frac{H}{U^2}\{U, L(\alpha)\}_{PB} + \frac{\{H, L(\alpha)\}_{PB}}{U} \\
&= -\frac{H}{U^2}\{U, L(\alpha)\}_{PB} + \frac{-\alpha\{U, L(\alpha)\}_{PB} + \rho(H+\alpha U)}{U} \\
&= -\frac{\widetilde{H}+\alpha}{U}\{U, L(\alpha)\}_{PB} + \frac{\rho(H+\alpha U)}{U}.
\end{aligned}$$

So

$$\begin{aligned}
\{\widetilde{H}, L(-\widetilde{H})\}_{PB} &= \left[\partial_\alpha L(\alpha)\left\{\widetilde{H}, \widetilde{H}\right\}_{PB} - \frac{\widetilde{H}+\alpha}{U}\{U, L(\alpha)\}_{PB} + \frac{\rho(H+\alpha U)}{U}\right]_{\alpha=-\widetilde{H}} \\
&= \frac{\rho(H-\widetilde{H}U)}{U} \\
&= 0.
\end{aligned}$$

Thus $L(-\widetilde{H})$ is a constant of the motion for the transformed Hamiltonian. □

Theorem 2.4.1 is a special case of this theorem with $\rho = 0$. Although a subtle difference in the proof is the reason behind $\rho(H - \widetilde{H}U)$ vanishing.

In the superintegrable case an arbitrary constant can be added to the Hamiltonian without altering the superintegrability, however adding an arbitrary constant here would destroy the conformally-superintegrability of the Hamiltonian. Also note that if $\mathcal{L}(\alpha)$ is one of the trivial conformal-constant of the form $F(\mathbf{x})(H_0 + \alpha U)$ then $\mathcal{L}(-\widetilde{\mathcal{H}}) \equiv 0$.

Nondegenerate (maximum-parameter) Potentials

Because of the Stäckel equivalence between superintegrable and conformally-superintegrable systems, results regarding the structure theory of conformally-superintegrable potentials can easily be determined from the structure theory of the superintegrable systems. So if a superintegrable potential $\widehat{V}$ is scaled via the conformal-factor λ then the new potentials

$$V = \widehat{V}\lambda,$$

can be shown to satisfy a set of PDEs written in the symmetric (but once again redundant) form

$$
\begin{aligned}
V_{,11} &= \mathcal{V}_{ee} + \mathcal{A}_1^{11} V_{,1} + \mathcal{A}_2^{11} V_{,2} + \mathcal{A}_3^{11} V_{,3} + \mathcal{A}_0^{11} V, \\
V_{,22} &= \mathcal{V}_{ee} + \mathcal{A}_1^{22} V_{,1} + \mathcal{A}_2^{22} V_{,2} + \mathcal{A}_3^{22} V_{,3} + \mathcal{A}_0^{22} V, \\
V_{,33} &= \mathcal{V}_{ee} + \mathcal{A}_1^{33} V_{,1} + \mathcal{A}_2^{33} V_{,2} + \mathcal{A}_3^{33} V_{,3} + \mathcal{A}_0^{33} V, \\
V_{,12} &= \qquad \mathcal{A}_1^{12} V_{,1} + \mathcal{A}_2^{12} V_{,2} + \mathcal{A}_3^{12} V_{,3} + \mathcal{A}_0^{12} V, \\
V_{,13} &= \qquad \mathcal{A}_1^{13} V_{,1} + \mathcal{A}_2^{13} V_{,2} + \mathcal{A}_3^{13} V_{,3} + \mathcal{A}_0^{13} V, \\
V_{,23} &= \qquad \mathcal{A}_1^{23} V_{,1} + \mathcal{A}_2^{23} V_{,2} + \mathcal{A}_3^{23} V_{,3} + \mathcal{A}_0^{23} V.
\end{aligned} \tag{4.1.9}
$$

Like before, the second-order parameter is $\mathcal{V}_{ee} = \frac{(V_{,11}+V_{,22}+V_{,33})}{3}$. Note that, unlike the superintegrable case, the value of V is no longer just an additive parameter. This dependence on 5 parameter shouldn't be surprising as the additive constant of the superintegrable case C becomes a 5th term in the potential $C\lambda$ under the conformal scaling.

The redundancy in the PDEs above takes the same form as before and can be expressed as

$$
\mathcal{A}_i^{11} + \mathcal{A}_i^{22} + \mathcal{A}_i^{33} \equiv 0 \qquad i = 0, \dots, 3.
$$

Like before the integrability conditions can be use to show these 18 coefficient functions $\mathcal{A}_k^{ij}$ depend only on a subset of 10 coefficient function. These 10 coefficients, which will be referred to as $\{\boldsymbol{\mathcal{Q}}, \boldsymbol{\mathcal{R}}, \boldsymbol{\mathcal{S}}\}$, and for $\mathcal{A}_k^{ij}$, $k \neq 0$, these take essentially the same form as (2.3.16). If we wish to express these coefficients function in terms of the superintegrable system from which they were derived they can be expressed in terms of the old variables $\{\mathbf{Q}, \mathbf{R}, \mathbf{S}\}$ and the conformal factor, $\lambda = \exp(G)$, via the following formulas

$$
\begin{aligned}
\mathcal{R}_k^{ij} &= R_k^{ij}, \\
\mathcal{Q}^{123} &= Q^{123}, \\
\mathcal{S}^i &= S^i - \frac{1}{3} G_{,i}.
\end{aligned} \tag{4.1.10}
$$

The remaining 6 coefficients $\mathcal{A}_0^{ij}$ can be expressed as quadratics in the variables $\{\boldsymbol{\mathcal{Q}}, \boldsymbol{\mathcal{R}}, \boldsymbol{\mathcal{S}}\}$. These can be determined by index permutation in the following two illustrative examples

$$\begin{aligned}\mathcal{A}_0^{11} &= \frac{4}{9}(\mathcal{R}_1^{13})^2 - \frac{8}{9}(\mathcal{R}_2^{23})^2 + \frac{4}{9}(\mathcal{R}_1^{12})^2 + \frac{4}{9}(\mathcal{R}_3^{13})^2 - \frac{8}{9}(\mathcal{R}_3^{23})^2 + \frac{4}{9}(\mathcal{R}_2^{12})^2 \\ &\quad - 2\mathcal{R}_2^{12}\mathcal{S}^1 + 2(\mathcal{S}^3)^2 - 4(\mathcal{S}^1)^2 + \frac{8}{9}\mathcal{R}_2^{12}\mathcal{R}_3^{13} - 2\mathcal{S}^1\mathcal{R}_3^{13} + 2(\mathcal{S}^2)^2 + 2\mathcal{S}^3\mathcal{R}_1^{13} \\ &\quad + 2\mathcal{S}^2\mathcal{R}_1^{12} - \frac{4}{9}\mathcal{R}_3^{23}\mathcal{R}_1^{12} - \frac{4}{9}\mathcal{R}_1^{13}\mathcal{R}_2^{23}, \\ \mathcal{A}_0^{12} &= 2\mathcal{Q}^{123}\mathcal{S}^3 + 2\mathcal{R}_1^{12}\mathcal{S}^1 + 2\mathcal{S}^2\mathcal{R}_2^{12} - 6\mathcal{S}^2\mathcal{S}^1 + \frac{4}{3}\mathcal{Q}^{123}\mathcal{R}_1^{13} + \frac{4}{3}\mathcal{Q}^{123}\mathcal{R}_2^{23} \\ &\quad - \frac{2}{3}\mathcal{R}_3^{13}\mathcal{R}_1^{12} - \frac{2}{3}\mathcal{R}_3^{23}\mathcal{R}_2^{12} + \frac{2}{3}\mathcal{R}_3^{23}\mathcal{R}_3^{13}. \end{aligned} \tag{4.1.11}$$

Alternatively, using the concise notation introduced in chapter 3 these can be recovered by writing out the PDEs (4.1.9) as the following $SO(3,\mathbb{C})$ representation

$$\begin{aligned}\partial_+^C\partial_+^C(V) &= -\frac{1}{15}(\mathcal{Y},\partial_+^C(V))^{[2]} + 3i(\mathcal{X},\partial_+^C(V))^{[0]} \\ &\quad + V\left(\frac{2}{675}(\mathcal{Y},\mathcal{Y})^{[4]} + \frac{i}{15}(\mathcal{Y},\mathcal{X})^{[2]} + \frac{3}{2}(\mathcal{X},\mathcal{X})^{[0]}\right).\end{aligned} \tag{4.1.12}$$

If equations (2.3.17)-(2.3.20) are rewritten in the new $\{\boldsymbol{\mathcal{Q}},\boldsymbol{\mathcal{R}},\boldsymbol{\mathcal{S}}\}$ variables the following illustrative derivatives can be derived

$$\begin{aligned}\frac{\partial\mathcal{R}_1^{12}}{\partial x_1} &= -\frac{2}{3}\mathcal{R}_2^{12}\mathcal{R}_3^{23} + \frac{2}{3}\mathcal{R}_3^{13}\mathcal{R}_3^{23} + \frac{4}{3}\mathcal{Q}^{123}\mathcal{R}_2^{23} + \frac{5}{3}\mathcal{Q}^{123}\mathcal{R}_1^{13} \\ &\quad - \mathcal{R}_1^{12}\mathcal{R}_3^{13} - \mathcal{R}_1^{12}\mathcal{S}^1 + \left(\mathcal{R}_3^{13} + 3\mathcal{R}_2^{12}\right)\mathcal{S}^2 + 2\mathcal{Q}^{123}\mathcal{S}^3,\end{aligned} \tag{4.1.13}$$

$$\begin{aligned}\frac{\partial\mathcal{R}_1^{12}}{\partial x_2} &= \frac{3}{5}\mathcal{R}_2^{12}\mathcal{R}_3^{13} - \frac{1}{15}\mathcal{R}_1^{13}\mathcal{R}_2^{23} - \frac{11}{15}\mathcal{R}_1^{12}\mathcal{R}_3^{23} \\ &\quad + \frac{8}{15}\left(\mathcal{R}_1^{13}\right)^2 + \frac{1}{5}\left(\mathcal{R}_1^{12}\right)^2 - \frac{4}{5}\left(\mathcal{R}_2^{23}\right)^2 + \frac{8}{15}\left(\mathcal{R}_3^{13}\right)^2 + \frac{1}{5}\left(\mathcal{R}_2^{12}\right)^2 - \frac{4}{5}\left(\mathcal{R}_3^{23}\right)^2 \\ &\quad + \frac{2}{15}\left(\mathcal{Q}^{123}\right)^2 - \left(\mathcal{R}_3^{13} + 3\mathcal{R}_2^{12}\right)\mathcal{S}^1 - \mathcal{R}_1^{12}\mathcal{S}^2 + \mathcal{R}_1^{13}\mathcal{S}^3,\end{aligned} \tag{4.1.14}$$

$$\begin{aligned}\frac{\partial\mathcal{R}_1^{12}}{\partial x_3} &= -\frac{1}{3}\mathcal{Q}^{123}\mathcal{R}_2^{12} - \frac{1}{3}\mathcal{Q}^{123}\mathcal{R}_3^{13} + \frac{1}{3}\mathcal{R}_2^{23}\mathcal{R}_1^{12} + \frac{1}{3}\mathcal{R}_3^{23}\mathcal{R}_1^{13} \\ &\quad - 2\mathcal{Q}^{123}\mathcal{S}^1 - \mathcal{R}_1^{13}\mathcal{S}^2 - \mathcal{R}_1^{12}\mathcal{S}^3,\end{aligned} \tag{4.1.15}$$

$$\begin{aligned}\frac{\partial\mathcal{S}^1}{\partial x_1} &= -\frac{17}{90}\mathcal{R}_2^{12}\mathcal{R}_3^{13} + \frac{1}{30}\mathcal{R}_1^{13}\mathcal{R}_2^{23} + \frac{1}{30}\mathcal{R}_1^{12}\mathcal{R}_3^{23} \\ &\quad - \frac{7}{45}\left(\mathcal{R}_3^{13}\right)^2 + \frac{1}{15}\left(\mathcal{R}_3^{23}\right)^2 - \frac{7}{45}\left(\mathcal{R}_1^{12}\right)^2 - \frac{11}{90}\left(\mathcal{Q}^{123}\right)^2 - \frac{7}{45}\left(\mathcal{R}_1^{13}\right)^2 - \frac{7}{45}\left(\mathcal{R}_2^{12}\right)^2 \\ &\quad + \frac{1}{15}\left(\mathcal{R}_2^{23}\right)^2 + \frac{1}{2}\left(\mathcal{S}^2\right)^2 + \frac{1}{2}\left(\mathcal{S}^3\right)^2 - \frac{1}{2}\left(\mathcal{S}^1\right)^2,\end{aligned} \tag{4.1.16}$$

$$\frac{\partial \mathcal{S}^1}{\partial x_2} = -\frac{1}{9}\mathcal{R}_3^{13}\mathcal{R}_3^{23} - \frac{2}{9}\mathcal{Q}^{123}\mathcal{R}_2^{23} + \frac{1}{9}\mathcal{R}_1^{12}\mathcal{R}_3^{13} + \frac{1}{9}\mathcal{R}_2^{12}\mathcal{R}_3^{23} - \frac{2}{9}\mathcal{Q}^{123}\mathcal{R}_1^{13} - \mathcal{S}^1\mathcal{S}^2, \tag{4.1.17}$$

$$\begin{aligned}\frac{\partial \mathcal{Q}^{123}}{\partial x_1} &= \frac{2}{3}\mathcal{R}_1^{13}\mathcal{R}_1^{12} - \frac{1}{3}\mathcal{R}_3^{23}\mathcal{R}_1^{13} + \mathcal{Q}^{123}\mathcal{R}_3^{13} - \frac{1}{3}\mathcal{R}_2^{23}\mathcal{R}_1^{12} \\ &\quad + \mathcal{Q}^{123}\mathcal{R}_2^{12} - \mathcal{Q}^{123}\mathcal{S}^1 + \left(\mathcal{R}_2^{23} - \mathcal{R}_1^{13}\right)\mathcal{S}^2 + \left(\mathcal{R}_3^{23} - \mathcal{R}_1^{12}\right)\mathcal{S}^3. \end{aligned} \tag{4.1.18}$$

As before, all other derivatives can be found though index permutation. The full set of equations can also be derived directly from (3.2.13). Perhaps surprisingly, the integrability conditions for equations (4.1.13)-(4.1.18) are satisfied identically. This lack of restrictions means that there will be a solution to (4.1.9) for any given 10-tuple $(\boldsymbol{\mathcal{Q}}, \boldsymbol{\mathcal{R}}, \boldsymbol{\mathcal{S}}) \in \mathbb{C}^{10}$.

The 10-dimensional space of initial conditions is acted on by the conformal group in three-dimensions (which is a 10-dimensional group) and at a generic point the action is rank 10. So it should be expected that the bulk of this space corresponds to a single orbit under the conformal group (and hence a single conformal class). However there may (and do) exist points for which the action is less than rank 10, and so the task now is to find these points and determine which lower-dimensional orbits they belong to.

The following section answers part of this question on a local level, that is, determining what the orbit of a fixed regular point from a system is under a conformal change of coordinates and subsequent conformal rescaling.

4.2 The Local action of the Conformal Group

The local-action of the conformal group is given by conformal changes of variables, i.e. a change of variables $\mathbf{x} = F(\mathbf{u})$ that simply scales the metric by a conformal factor. So in our new variables the flat space metric is given by

$$ds^2 = dx_1^2 + dx_2^2 + dx_3^2 = \lambda\left(du_1^2 + du_2^2 + du_3^2\right).$$

Under this change of the coordinates the leading part of the conformally-superintegrable Hamiltonian is scaled by λ^{-1},

$$H = p_{x_1}^2 + p_{x_1}^2 + p_{x_1}^2 + V(\mathbf{x}) = \frac{p_{u_1}^2 + p_{u_1}^2 + p_{u_1}^2}{\lambda} + V(F(\mathbf{u})).$$

By lemma 4.1.4 this can be conformally scaled to flat-space conformally-superintegrable Hamiltonian in the standard flat space coordinates

$$\widetilde{H} = p_{u_1}^2 + p_{u_1}^2 + p_{u_1}^2 + V\lambda, \tag{4.2.1}$$

making $V\lambda$ a conformally-superintegrable potential on flat-space.

Inversion in the Sphere (Kelvin Inversion)

To study the effect of the local conformal group it is sufficient to examine a single conformal-change of variables, inversion in the sphere. Since a translation change of variables act trivially on form of the PDEs (4.1.9) it induces a trivial action on the coefficient functions $\{\boldsymbol{\mathcal{Q}}, \boldsymbol{\mathcal{R}}, \boldsymbol{\mathcal{S}}\}$. Thus the sphere of inversion can be assumed to be centred at the origin. The inversion with respect to the sphere of radius δ is then given by the change of variables

$$x_i = \delta^2 \frac{u_i}{{u_1}^2 + {u_2}^2 + {u_3}^2}. \tag{4.2.2}$$

Under this change of variables the Hamiltonian becomes

$$H = \frac{\left({u_1}^2 + {u_2}^2 + {u_3}^2\right)^2}{\delta^4} \left(p_{u_1}^2 + p_{u_2}^2 + p_{u_3}^2\right) + V(u_1, u_2, u_3)$$

which, via a conformal scaling, can be turned into the conformally-superintegrable system

$$\widetilde{H} = \frac{\delta^4 H}{\left({u_1}^2 + {u_2}^2 + {u_3}^2\right)^2} = \left(p_{u_1}^2 + p_{u_2}^2 + p_{u_3}^2\right) + \frac{\delta^2 V(u_1, u_2, u_3)}{\left({u_1}^2 + {u_2}^2 + {u_3}^2\right)^2}$$

Applying the change of variables to the PDEs (4.1.9) and making the substitution

$$V = \delta^{-4}(u_1^2 + u_1^2 + u_1^2)^2 \widetilde{V}$$

the action on $\{\boldsymbol{\mathcal{Q}}, \boldsymbol{\mathcal{R}}, \boldsymbol{\mathcal{S}}\}$ is given by the three illustrative equations

$$\begin{aligned}\widetilde{\mathcal{S}}^1 = &-\frac{u_1^2 - u_2^2 - u_3^2}{\left(u_1^2 + u_2^2 + u_3^2\right)^2}\delta^2 \mathcal{S}^1 - \frac{2u_1 u_2}{\left(u_1^2 + u_2^2 + u_3^2\right)^2}\delta^2 \mathcal{S}^2 \\ &- \frac{2u_3 u_1}{\left(u_1^2 + u_2^2 + u_3^2\right)^2}\delta^2 \mathcal{S}^3 + \frac{2u_1}{\left(u_1^2 + u_2^2 + u_3^2\right)},\end{aligned} \tag{4.2.3}$$

$$\begin{aligned}
\widetilde{\mathcal{R}}_1^{12} = & \frac{2u_2u_1\left(6u_3^2u_2^2 - 2u_2^2u_1^2 + u_2^4 + u_1^4 - 10u_3^2u_1^2 + 5u_3^4\right)}{\left(u_1^2+u_2^2+u_3^2\right)^4}\delta^2\mathcal{R}_3^{13} \\
& + \frac{2u_2u_1\left(2u_3^2u_2^2 - u_3^4 - 2u_3^2u_1^2 - 10u_2^2u_1^2 + 3u_1^4 + 3u_2^4\right)}{\left(u_1^2+u_2^2+u_3^2\right)^4}\delta^2\mathcal{R}_2^{12} \\
& + \frac{8u_3u_1^2u_2\left(2u_3^2 - 2u_2^2 + u_1^2\right)}{\left(u_1^2+u_2^2+u_3^2\right)^4}\delta\mathcal{R}_2^{23} \\
& + \frac{\left(-u_2^6 - 15u_1^4u_2^2 + 15u_1^2u_2^4 + u_1^6 + u_3^6 - u_2^4u_3^2 + u_2^2u_3^4 - u_1^2u_3^4 + 6u_1^2u_3^2u_2^2 - u_1^4u_3^2\right)}{\left(u_1^2+u_2^2+u_3^2\right)^4}\delta^2\mathcal{R}_1^{12} \\
& - \frac{2u_3u_2\left(u_3^4 - 10u_3^2u_1^2 + 2u_3^2u_2^2 - 6u_2^2u_1^2 + 5u_1^4 + u_2^4\right)}{\left(u_1^2+u_2^2+u_3^2\right)^4}\delta^2\mathcal{R}_1^{13} \\
& - \frac{4u_1^2\left(6u_3^2u_2^2 - u_3^2u_1^2 - u_2^4 - u_3^4 + u_2^2u_1^2\right)}{\left(u_1^2+u_2^2+u_3^2\right)^4}\delta^2\mathcal{R}_3^{23} \\
& + \frac{4u_3u_1\left(3u_2^4 + 2u_3^2u_2^2 - 8u_2^2u_1^2 + u_1^4 - u_3^4\right)}{\left(u_1^2+u_2^2+u_3^2\right)^4}\delta^2\mathcal{Q}^{123}, \qquad (4.2.4)
\end{aligned}$$

$$\begin{aligned}
\widetilde{\mathcal{Q}}^{123} = & \, 2u_3u_1\frac{\left(5u_2^4 - u_3^4 - 10u_1^2u_2^2 + u_1^4\right)}{\left(u_1^2+u_2^2+u_3^2\right)^4}\delta^2\mathcal{R}_1^{12} + 2u_3u_2\frac{\left(5u_1^4 - u_3^4 - 10u_2^2u_1^2 + u_2^4\right)}{\left(u_1^2+u_2^2+u_3^2\right)^4}\delta^2\mathcal{R}_2^{12} \\
& + 2u_1u_2\frac{\left(5u_3^4 - u_1^4 - 10u_2^2u_3^2 + u_2^4\right)}{\left(u_1^2+u_2^2+u_3^2\right)^4}\delta^2\mathcal{R}_2^{23} + 2u_1u_3\frac{\left(5u_2^4 - u_1^4 - 10u_3^2u_2^2 + u_3^4\right)}{\left(u_1^2+u_2^2+u_3^2\right)^4}\delta^2\mathcal{R}_3^{23} \\
& + 2u_2u_3\frac{\left(5u_1^4 - u_2^4 - 10u_3^2u_1^2 + u_3^4\right)}{\left(u_1^2+u_2^2+u_3^2\right)^4}\delta^2\mathcal{R}_3^{13} + 2u_2u_1\frac{\left(5u_3^4 - u_2^4 - 10u_1^2u_3^2 + u_1^4\right)}{\left(u_1^2+u_2^2+u_3^2\right)^4}\delta^2\mathcal{R}_1^{13} \\
& - \frac{\left(u_1^6 + u_2^6 + u_3^6 - 5u_2^2u_3^4 - 5u_1^2u_2^4 + 30u_1^2u_3^2u_2^2 - 5u_2^4u_3^2 - 5u_1^2u_3^4 - 5u_1^4u_2^2 - 5u_1^4u_3^2\right)}{\left(u_1^2+u_2^2+u_3^2\right)^4}\delta^2\mathcal{Q}^{123},
\end{aligned}$$

(4.2.5)

and as before the full set can be obtained through index permutation. For the variables $\{\boldsymbol{\mathcal{Q}}, \boldsymbol{\mathcal{R}}\}$ a simpler description of this action will be given by (4.2.15) when the rotations representation are reconsidered as binary forms (see appendix A).

Inversions in the sphere (for spheres of varying sizes and locations) and translations generate the full conformal group. For example, consider the change of variables

$$
\begin{aligned}
x_1 &= \frac{t^2 z_1}{z_1^2 + z_2^2 + (z_3+t)^2} \\
x_2 &= \frac{t^2 z_2}{z_1^2 + z_2^2 + (z_3+t)^2} \\
x_3 &= \frac{t^2 (z_3+t)}{z_1^2 + z_2^2 + (z_3+t)^2} - t,
\end{aligned}
$$

given by translating the coordinate system by $x_3 \mapsto x_3 + t$, inverting in a sphere of radius t^2 centred at the origin and then translating again by $x_3 \mapsto x_3 - t$. In the limit $t \to \infty$ this becomes the reflection

$$
\begin{aligned}
x_1 &= z_1, \\
x_2 &= z_2, \\
x_3 &= -z_3.
\end{aligned}
$$

Clearly all reflections can be generated this way, and by extension all rotations.

Continuous transformations

Although the local action of the conformal group is completely described by spherical inversions, it will be worth discussing continuous changes of variables as well. That is, the infinitesimal changes of variables around the identity.

The conformal group, as a Lie group, is 10-dimensional and can be identified with four types of actions: rotation, translations, dilations and Möbius transformations (i.e. translations conjugated with an inversion in the unit sphere). When considering the $\{\boldsymbol{\mathcal{Q}}, \boldsymbol{\mathcal{R}}, \boldsymbol{\mathcal{S}}\}$ variables the local action of translations is trivial (since the PDEs (4.1.9) don't change under this change of variables) and the action of rotations is the same as the superintegrable case (given on page 27).

Dilations: Consider the change of variables

$$u_i = x_i \exp(t).$$

Under this change of variables the Hamiltonian becomes

$$H = e^{2t}\left(p_{u_1}^2 + p_{u_2}^2 + p_{u_3}^2\right) + V(\mathbf{u}; t).$$

Conformally scaling this by $\exp(-2t)$ gives the conformally-superintegrable potential $\widetilde{V} = Ve^{-2t}$. The examining the derivatives around $t = 0$ gives the Lie algebra action

$$D(x_i) = x_i, \quad D(\mathcal{A}) = \mathcal{A},$$

where $\mathcal{A}$ is a stand-in for any of the $\{\boldsymbol{\mathcal{Q}}, \boldsymbol{\mathcal{R}}, \boldsymbol{\mathcal{S}}\}$ variables.

Möbius Transformations: The effect of a Möbius transformation is more complicated. Given a choice of direction a Möbius transformation can be constructed by an inversion in the unit sphere, a translation in the chosen direction and inversion in the unit sphere again. Choosing the direction to be x_1 gives the change of variables

$$\begin{aligned} u_1 &= \frac{x_1 - tr^2}{1 - 2tx_1 + t^2 r^2}, \\ u_2 &= \frac{x_2}{1 - 2tx_1 + t^2 r^2}, \\ u_3 &= \frac{x_3}{1 - 2tx_1 + t^2 r^2}, \end{aligned} \tag{4.2.6}$$

where $r^2 = x_1^2 + x_2^2 + x_3^2$. In these variables the Hamiltonian is given by

$$H = \left(1 + 2u_1 t + t^2(u_1^2 + u_2^2 + u_3^2)\right)^2 \left(p_{u_1}^2 + p_{u_2}^2 + p_{u_3}^2\right) + V(\mathbf{u}; t).$$

and so, scaling this back to the standard flat metric, the action on potential is given by

$$\widetilde{V} = \frac{V}{\left(1 + 2u_1 t + t^2(u_1^2 + u_2^2 + u_3^2)\right)^2}.$$

The corresponding Lie algebra action on the coordinates is given by

$$\begin{aligned} C_1(x_1) &= {x_1}^2 - {x_2}^2 - {x_3}^2, \\ C_1(x_2) &= 2x_2 x_1, \\ C_1(x_3) &= 2x_3 x_1. \end{aligned} \tag{4.2.7}$$

The action on the coefficient functions is

$$\begin{aligned} C_1(\mathcal{S}^1) &= 2 - 2x_3\mathcal{S}^3 - 2x_2\mathcal{S}^2 - 2x_1\mathcal{S}^1, \\ C_2(\mathcal{S}^1) &= 2x_1\mathcal{S}^2 - 2x_2\mathcal{S}^1, \\ C_3(\mathcal{S}^1) &= 2x_1\mathcal{S}^3 - 2x_3\mathcal{S}^1, \end{aligned} \tag{4.2.8}$$

and

$$\begin{aligned}
C_1(\mathcal{R}_1^{12}) &= -2x_1\mathcal{R}_1^{12} - 2x_2\left(\mathcal{R}_3^{13} + 3\mathcal{R}_2^{12}\right) - 4x_3\mathcal{Q}^{123},\\
C_1(\mathcal{R}_2^{12}) &= -2x_1\mathcal{R}_2^{12} + 2x_2\left(\mathcal{R}_3^{23} + 3\mathcal{R}_1^{12}\right) - 2x_3\mathcal{R}_2^{23},\\
C_1(\mathcal{R}_3^{13}) &= -2x_1\mathcal{R}_3^{13} - 2x_2\mathcal{R}_3^{23} + 2x_3\left(\mathcal{R}_1^{13} + 3\mathcal{R}_1^{13}\right),\\
C_1(\mathcal{R}_3^{23}) &= -2x_1\mathcal{R}_3^{23} + 2x_2\mathcal{R}_3^{13} + 4x_3\mathcal{Q}^{123}, \qquad (4.2.9)
\end{aligned}$$

and finally

$$C_1(\mathcal{Q}^{123}) = -2x_1\mathcal{Q}^{123} + 2x_2\left(\mathcal{R}_1^{13} - \mathcal{R}_2^{23}\right) + 2x_3\left(\mathcal{R}_1^{12} - \mathcal{R}_3^{23}\right). \qquad (4.2.10)$$

What is immediately apparent from (4.2.8) (and this is also apparent in (4.2.3)) is that the action on $\{\boldsymbol{\mathcal{S}}\}$ has a different form from the action on $\{\boldsymbol{\mathcal{Q}}, \boldsymbol{\mathcal{R}}\}$. In light of (4.1.10) this difference can be attributed to the conformal rescaling. Perhaps less obvious is the fact that, when restricted to the variables $\mathcal{A} \in \{\boldsymbol{\mathcal{Q}}, \boldsymbol{\mathcal{R}}\}$, the Lie algebra action can be written out in form

$$C_i(\mathcal{A}) = -2x_iD(\mathcal{A}) - 2J_1(x_i)J_1(\mathcal{A}) - 2J_2(x_i)J_2(\mathcal{A}) - 2J_3(x_i)J_3(\mathcal{A}). \qquad (4.2.11)$$

From this the following theorem can be proved.

Theorem 4.2.12. *Given a polynomial ideal I in the polynomial ring $\mathbb{C}[\boldsymbol{\mathcal{Q}}, \boldsymbol{\mathcal{R}}]$ which is closed under the action of the dilation Lie algebra action D and the rotations Lie algebra action J_1, J_2, J_3. The ideal I is then closed under the action of C_1, C_1, C_2.*

Proof. From (4.2.11) it is clear that, over the variable $\{\boldsymbol{\mathcal{Q}}, \boldsymbol{\mathcal{R}}\}$ the Lie algebra action C_i a linear combination of the dilation D and the rotations J_1, J_2, J_3. Since D and J are derivations (i.e. they satisfy a Leibniz rule) so are the C_i's. Thus (4.2.11) also holds when $\mathcal{A}$ is a polynomial in the variables $\{\boldsymbol{\mathcal{Q}}, \boldsymbol{\mathcal{R}}\}$, proving the theorem. □

This will be important in chapter 5 as it means any algebraic ideal closed under dilations and rotations is immediately closed under Möbius transformations.

Action of the Möbius Transformation on the $\mathfrak{sl}(2,\mathbb{C})$ representations

In appendix A the rotation representations are described as binary forms, and the action of $SO(3,C)$ is given by the action of $SL(2,\mathbb{C})$ on these binary forms. If attention is restricted to the 7-dimensional representation $\mathcal{Y}$ then it can be shown that the local action of the conformal group extends to the action $GL(2,\mathcal{C}) \simeq \mathbb{C} \oplus SL(2,\mathcal{C})$.

To see this, consider the action of the Möbius transformations (4.2.2). The induced action on the 7-dimensional rotations representation is given by (4.2.9)-(4.2.5) and looks rather complicated. However if this representation is considered to be the 6th-order binary form

$$\begin{aligned}\mathcal{Y}(r,s) &= \left(\mathcal{R}_1^{12} + \frac{1}{4}\mathcal{R}_3^{23} + i\mathcal{R}_2^{12} + \frac{i}{4}\mathcal{R}_3^{13}\right) r^6 + \ldots \\ &= a_6(r-\eta_1 s)(r-\eta_2 s)(r-\eta_3 s)(r-\eta_4 s)(r-\eta_5 s)(r-\eta_6 s) \quad (4.2.13)\end{aligned}$$

(given in full at (3.2.8)) then (4.2.9)-(4.2.5) can succinctly be described by mapping the roots of (4.2.13) via

$$\widehat{\eta}_i = \frac{(-u_3)\eta_i + (u_1 - iu_2)}{(u_1+iu_2)\eta_i + (u_3)} \quad (4.2.14)$$

and by scaling the leading coefficient via

$$\widehat{a}_6 = \delta^2 \frac{\prod_{i=1}^{6}((u_1+iu_2)\eta_i + (u_3))}{(u_1^2+u_2^2+u_3^2)^4} a_6.$$

Alternatively this can be described by the linear change of variables

$$\begin{pmatrix} r \\ s \end{pmatrix} = \frac{\delta^{1/3}}{(u_1^2+u_2^2+u_3^2)^{2/3}} \begin{pmatrix} -u_3 & u_1 - iu_2 \\ u_1+iu_2 & u_3 \end{pmatrix} \begin{pmatrix} \widehat{r} \\ \widehat{s} \end{pmatrix} \quad (4.2.15)$$

Similarly the refection $u_3 = -x_3$ can be modelled by the action on the roots via

$$\widehat{\eta}_i = -\eta_i \quad (4.2.16)$$

and by a trivial scaling of the leading coefficient

$$\widehat{a}_6 = a_6.$$

Alternatively this can be described by the linear change of variables of the binary form $\mathcal{Y}(r,s)$ under

$$\begin{pmatrix} r \\ s \end{pmatrix} = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \widehat{r} \\ \widehat{s} \end{pmatrix}. \tag{4.2.17}$$

The same analysis does not hold for the three-dimensional representation. In fact it is this difference which allows the following theorem to be proved.

Theorem 4.2.18. *Given a regular point in a second-order conformally-superintegrable system there is a local conformal transformation that takes the value of the 10 coefficient functions at the regular point from their initial values* $(\boldsymbol{\mathcal{Q}}_0, \boldsymbol{\mathcal{R}}_0, \boldsymbol{\mathcal{S}}_0)$ *to the values* $(\boldsymbol{\mathcal{Q}}_0, \boldsymbol{\mathcal{R}}_0, \mathbf{0})$*. That is the values of the* S_i *can be mapped to zero by a conformal group motion without changing the value of the* $\mathcal{Q}^{123}, \mathcal{R}^{ij}_j$*.*

Proof. Assume $S^3 \neq 0$ at the regular point. Performing an inversion in the sphere via (4.2.2) such that the new regular point satisfies $u_1 = u_2 = 0, u_3 \neq 0$ and then performing a subsequent reflection via (4.2.16) (this reflection is an important step as (4.2.14) has reflected the roots of the 7-dimensional representation in the imaginary axis) then the action on $\mathcal{S}^i$ can be shown to be

$$\widehat{\mathcal{S}}^1 = \delta^2 \frac{\mathcal{S}^1}{u_3^2}, \;\; \widehat{\mathcal{S}}^2 = \delta^2 \frac{\mathcal{S}^2}{u_3^2}, \;\; \widehat{\mathcal{S}}^3 = \delta^2 \frac{\mathcal{S}^3}{u_3^2} - u_3. \tag{4.2.19}$$

Choosing the dilation factor to be $\delta^2 = u_3^2$ means the seven dimensional $\mathcal{Y}(r,s)$ has retained its original value under this conformal motion. Meanwhile the $\mathcal{S}^i$'s have become

$$\widehat{\mathcal{S}}^1 = \mathcal{S}^1, \;\; \widehat{\mathcal{S}}^2 = \mathcal{S}^2, \;\; \widehat{\mathcal{S}}^3 = \mathcal{S}^3 - 2u_3. \tag{4.2.20}$$

Making the choice $u_3 = \frac{\mathcal{S}^3}{2}$, which is non-zero by hypothesis, gives $\widehat{\mathcal{S}}_3 = 0$. So S^3 has been mapped to zero under this action and the rest of the variables have retained their original value. By rotating either $\mathcal{S}^1$ or $\mathcal{S}^2$ into the place of $\mathcal{S}^3$, the same technique as above can used to set these to zero as well. Inverting all the rotations used then yields the result. □

An immediate corollary to this result is that the conformal group acts transitively on the value of the $\mathcal{S}^i$. Interpreted in the language of binary forms the local equivalence of a conformally-superintegrable systems is given by the following theorem.

Theorem 4.2.21. *Given a regular point in a maximum-parameter conformally-superintegrable system, there is a local conformal transformation that will map it to another such regular point if and only if the roots of their respective binary forms $\mathcal{Y}(r,s)$ are equivalent up to a general linear transformation.*

Proof. Every conformal change of variable can be constructed from inversion in the sphere (for varying centres) and dilations. The corresponding action on $\mathcal{Y}(r,s)$ is therefore repeated application of matrices of the form (4.2.15). These are easily shown to cover the entire space of $GL(2,\mathbb{C})$ matrices. Since this is the full set of local conformal actions, the result follows. □

4.3 Non-local Action of the Conformal Group

Thus far the examination of the action of the conformal group has been a purely local one, meaning that only a fixed point within a systems has been considered. To understand what effect a translation from one regular point to another regular point has on the values of the coefficient functions $\{\boldsymbol{\mathcal{Q}}, \boldsymbol{\mathcal{R}}, \boldsymbol{\mathcal{S}}\}$ will require an examination of the partial derivatives (4.1.18)-(4.1.17).

It is already clear from the local picture that any invariants will only depend on the value of the 7-dimensional representation $\mathcal{Y}$. And in the following chapter a very specific type of invariant is sought. These are polynomials ideals constructed from the variables in the $\mathcal{Y}$ representations whose zero-sets are closed under the action of the full-conformal group.

The search for such invariants can take place in two stages. The first stage is to consider polynomial ideals closed under dilations and rotations and, by extension all local-transformations. Such ideals are clearly generated by sets of homogeneous rotation representations and so these will be the focus of the following discussion.

The second-stage is to narrow these ideals down to ideals which are also closed under action the of the derivatives. Using a Taylor expansion it is easy to see that the zero-set of such an ideal will be invariant under translations off the regular point.

Consider the derivation

$$\widehat{\partial}_i(\mathcal{A}) = \partial_i(\mathcal{A}) + S^i\mathcal{D}(\mathcal{A}) - J_1(S^i)J_1(\mathcal{A}) - J_2(S^i)J_2(\mathcal{A}) - J_3(S^i)J_3(\mathcal{A}), \tag{4.3.1}$$

where ∂_i is just the partial derivative, and $\mathcal{D}$ and J_i are the Lie algebra operators. Applied to the variables $\{\boldsymbol{\mathcal{Q}}, \boldsymbol{\mathcal{R}}\}$ these are

$$
\begin{aligned}
\widehat{\partial}_1 R_1^{12} &= -\frac{2}{3}R_2^{12}R_3^{23} + \frac{2}{3}R_3^{13}R_3^{23} + \frac{4}{3}Q^{123}R_2^{23} + \frac{5}{3}Q^{123}R_1^{13} - R_1^{12}R_3^{13}, \\
\widehat{\partial}_2 R_1^{12} &= \frac{3}{5}R_2^{12}R_3^{13} - \frac{1}{15}R_1^{13}R_2^{23} - \frac{11}{15}R_1^{12}R_3^{23} + \frac{8}{15}\left(R_1^{13}\right)^2 + \frac{1}{5}\left(R_1^{12}\right)^2 \\
&\quad - \frac{4}{5}\left(R_2^{23}\right)^2 + \frac{8}{15}\left(R_3^{13}\right)^2 + \frac{1}{5}\left(R_2^{12}\right)^2 - \frac{4}{5}\left(R_3^{23}\right)^2 + \frac{2}{15}\left(Q^{123}\right)^2, \\
\widehat{\partial}_3 R_1^{12} &= -\frac{1}{3}Q^{123}R_2^{12} - \frac{1}{3}Q^{123}R_3^{13} + \frac{1}{3}R_2^{23}R_1^{12} + \frac{1}{3}R_3^{23}R_1^{13}, \\
\widehat{\partial}_1 Q^{123} &= \frac{2}{3}R_1^{13}R_1^{12} - \frac{1}{3}R_3^{23}R_1^{13} + Q^{123}R_3^{13} - \frac{1}{3}R_2^{23}R_1^{12} + Q^{123}R_2^{12}. \qquad (4.3.2)
\end{aligned}
$$

Even more compactly the full set of equations for these derivations can be written as

$$
\begin{aligned}
\widehat{\partial}_+^C(\mathcal{Y}) &= -\frac{1}{675}\left(\mathcal{Y}, \mathcal{Y}\right)^{[2]}, \\
\widehat{\partial}_0^C(\mathcal{Y}) &= 0, \\
\widehat{\partial}_-^C(\mathcal{Y}) &= -\frac{2}{225}\left(\mathcal{Y}, \mathcal{Y}\right)^{[4]}. \qquad (4.3.3)
\end{aligned}
$$

These only depend on the 7-dimensional representation, reinforcing the point that $\mathcal{Y}(r, s)$ is the only identifying characteristic of the conformally-superintegrable systems. More imporantantly, since (4.3.1) is a derivation the following theorem can be proved (which, in form, is almost identical to theorem 4.2.12).

Theorem 4.3.4. *Given a polynomial ideal I in the polynomial ring $\mathbb{C}[\boldsymbol{\mathcal{Q}}, \boldsymbol{\mathcal{R}}]$ which is closed under the action of the dilation operator D, the rotation operators J_1, J_2, J_3 and derivations $\widehat{\partial}_1, \widehat{\partial}_2, \widehat{\partial}_3$, then I is closed under the action of the partial derivatives $\partial_1, \partial_2, \partial_3$.*

Proof. From (4.3.1) it is clear that $\widehat{\partial}_i$ a linear combination of the dilation operator D and the rotation operators J_i and the partial derivatives ∂_i. Since D, J_i and ∂_i are all derivations (i.e. they satisfy a Leibniz rule) so is $\widehat{\partial}_i$. Thus (4.3.1) also holds when $\mathcal{A}$ is a polynomial in the variables $\{\boldsymbol{\mathcal{Q}}, \boldsymbol{\mathcal{R}}\}$. Thus proving the theorem. □

Chapter 5

Classification of Conformally-Superintegrable Systems

As was shown in the previous chapter, only knowledge of the 7-dimensional rotation representation $\mathcal{Y}(r,s)$, given at (4.2.13), will be needed to classify the systems into conformal classes. Furthermore the local action of the conformal group on this representation can be modelled as the action of $GL(2,\mathbb{C})$ on a binary form. In projective coordinates this binary form is just a univariate-polynomial and the action of $GL(2,\mathbb{C})$ is through fractional linear transformations (sometimes referred to as 'Möbius transformations' or the 'inversive group'). This makes the local classification of $\mathcal{Y}(r,s)$ a problem in classical invariant theory (see the book by P. Olver on this topic for more details [49]). Specifically, thinking of the polynomial $p(z) = \mathcal{Y}(z,1)$, the orbits are uniquely identified by knowing the multiplicities of roots and the cross-ratios between the roots.

In this chapter no distinction is made between the linear factors $(\alpha_i r - \beta_i s)$ of the binary form $\mathcal{Y}(r,s)$ and the roots $\eta_i = \frac{\beta_i}{\alpha_i} \in \mathbb{C}^*$ of the polynomial $p(z) = \mathcal{Y}(z,1)$. So the references made about the "roots" of the binary form should be taken to mean the binary form in projective coordinates.

An important fact to keep in mind while reading this chapter is that the action of $GL(2,\mathbb{C})$ acts transitively on triplets of roots. Given four distinct roots, three can be can be moved into predetermined canonical locations on the

Riemann sphere, with the location of the fourth root being uniquely determine by the cross ratio between the four roots.

Classifying the 7-dimensional representation using the roots of the associated binary form (meaning the linear factors) is conceptually simple, but due to Galois theory it is well known that for a general sextic these roots cannot be be expressed in terms of radicals of the coefficients of the binary form. So giving explicit expression for the roots will not be attempted. A more convenient method is to express the classifying information in terms of the representations that vanish given a particular root configuration. That is, rotationally closed sets of polynomials which vanish for all possible configurations of a given root structure.

A simple example is given by the discriminant of the binary sextic $\mathcal{Y}(r, s)$. The discriminant can be thought of as a one-dimensional representation (of order 10) and is a necessary and sufficient condition for the binary form to have a double root (i.e. a repeated factor). The conditions discussed in this chapter are generalisations of this observation.

In §5.1 two particular classifications are discussed, motivating the techniques that will be used in the rest of the chapter.

In §5.2 the method used to calculate the polynomial ideals corresponding to different multiplicity structures is discussed, however the ideals themselves won't be explicitly stated until the they are needed in the following section.

In §5.3 the complete classification of the conformally-superintegrable systems is given and a set of 10 conformal classes are derived. This proves there are no additional unknown classes. The results in this section are described using the Hilbert basis in appendix B.

Finally §5.4 conclude this chapter with a brief discussion of how these systems can be placed into a hierarchy, with all systems naturally occurring as the limit of one master system.

5.1 Examples of Differentially Closed Algebraic Ideals

Before discussing the general case it is worth going over two illustrative examples.

Example 5.1.1 (The zero sextic)**.** Consider the polynomial ideal

$$I_{[0]} = \langle Y_{+3}, \ldots, Y_{-3} \rangle,$$

where $\langle \mathbf{A} \rangle$ denotes the the ring of polynomials generated by $\mathbf{A}$. The zero set of $I_{[0]}$ corresponds to the vanishing of the 7-dimensional representation (3.1.10). Since $I_{[0]}$ is generated by elements of homogeneous degree the ideal is clearly closed under dilations. Since the generators form a complete $SO(3, \mathbb{C})$ representation the ideal is closed under rotations as well. Hence, by (4.2.11), the ideal will be closed Möbius transformations and thus closed under all local conformal transformations.

It is clear from (4.3.3) that $I_{[0]}$ is also closed under partial derivatives. So if all the coefficient functions Y_i are zero at any point in a conformally-superintegrable system then their derivatives are zero as well (to any order). This shows the algebraic-set given by $I_{[0]}$ is closed with respect to all conformal motions (i.e. the local ones induced by a conformal changes of variables and non-local ones induced by translation of the regular point).

So the second-order conformally-superintegrable systems can be split into two classes, those with coefficients whose values lie in the algebraic set defined by $I_{[0]}$ and those that don't. Since the action of the conformal group was proven transitive on the 3-dimensional representation (3.1.9), any two 10-tuples with $\mathbf{Q}, \mathbf{R} \equiv 0$ are related by a conformal motion (in this case a local one is always sufficient). Hence, up to Stäckel equivalence, there is only one such system in the [0]-type class.

Example 5.1.2 (The single factor sextic)**.** A more complicated example is given by considering the Hessian of the the binary form $\mathcal{Y}(r, s)$. The vanishing of the Hessian corresponds to a binary form having only one factor of multiplicity six, and thus gives an interesting geometric condition to consider. The Hessian of the binary form $\mathcal{Y}(r, s)$ is the covariant

$$\begin{aligned} H[\mathcal{Y}] &= 2\,(\mathcal{Y}, \mathcal{Y})^{[2]} \\ &= 150 \left(\frac{6}{\sqrt{15}} Y_{+3} Y_{+1} - Y_{+2}^2 \right) r^8 + \ldots + 150 \left(\frac{6}{\sqrt{15}} Y_{-3} Y_{-1} - Y_{-2}^2 \right) s^8 \\ &= \frac{1}{2} \left(B_{+4}^{(4)} r^8 - \sqrt{8} B_{+3}^{(4)} r^7 s + \ldots + B_{-4}^{(4)} s^8 \right). \end{aligned} \tag{5.1.3}$$

This corresponds to the covariant $\frac{1}{2} B_4$ given in the Hilbert basis (B.1.1). For the discussion below the coefficients of B_4 have been named $B_i^{(4)}$ and normalised by (3.1.11). As is hopefully clear, the subscript of $B_i^{(4)}$ indicates the eigenvalue and the superscript is just a reference to the name of the covariant in the Hilbert basis.

The action of the raising derivative (coming from (4.3.1)) on the highest weight coefficient of the Hessian (5.1.3) gives

$$\begin{aligned}\widehat{\partial}_+\left(B_{+4}^{(4)}\right) &= \left(i\widehat{\partial}_2+\widehat{\partial}_1\right)\left(B_{+4}^{(4)}\right)\\ &= \left(-6X_{+1}-\frac{28}{9\sqrt{15}}iY_{+1}\right)B_{+4}^{(4)}+\frac{14}{9\sqrt{3}}iY_{+2}B_{+3}^{(4)}-\frac{2\sqrt{7}}{9}iY_{+3}B_{+2}^{(4)}.\end{aligned} \tag{5.1.4}$$

Likewise the action of level-set derivative is

$$\begin{aligned}\widehat{\partial}_0\left(B_{+4}^{(4)}\right) &= \widehat{\partial}_3\left(B_{+4}^{(4)}\right)\\ &= \left(\sqrt{2}X_0+\frac{56}{9\sqrt{5}}iY_0\right)B_{+4}^{(4)}+\left(\sqrt{2}X_{+1}-\frac{77\sqrt{2}}{9\sqrt{15}}iY_{+1}\right)B_{+3}^{(4)}\\ &\quad+\frac{10\sqrt{14}}{9\sqrt{3}}iY_{+2}B_{+2}^{(4)}-\frac{\sqrt{14}}{3}iB_{+1}^{(4)}Y_{+3}.\end{aligned} \tag{5.1.5}$$

These show that the derivatives $\widehat{\partial}_+\left(B_{+4}^{(4)}\right)$ and $\widehat{\partial}_0\left(B_{+4}^{(4)}\right)$ are contained in the ideal generated from the coefficients of the Hessian (that is, the ideal $\langle B_{+4}^{(4)},\ldots,B_{-4}^{(4)}\rangle$). However the lowering derivative gives

$$\begin{aligned}\widehat{\partial}_-\left(B_{+4}^{(4)}\right) &= \left(i\widehat{\partial}_2-\widehat{\partial}_1\right)\left(B_{+4}^{(4)}\right)\\ &= \left(2X_{-1}+\frac{16\sqrt{3}}{\sqrt{5}}iY_{-1}\right)B_{+4}^{(4)}-\left(2X_0+\frac{8\sqrt{2}}{3\sqrt{5}}iY_0\right)B_{+3}^{(4)}\\ &\quad-\frac{194}{3\sqrt{105}}iY_{+1}B_{+2}^{(4)}+\frac{14\sqrt{7}}{3\sqrt{3}}iY_{+2}B_{+1}^{(4)}-\frac{2\sqrt{14}}{\sqrt{5}}iY_{+3}B_0^{(4)}\\ &\quad+\frac{176}{105\sqrt{15}}i\left(Y_{+1}^2-\frac{\sqrt{10}}{\sqrt{3}}Y_0Y_{+2}+\frac{\sqrt{5}}{\sqrt{3}}Y_{-1}Y_{+3}\right)Y_{+1},\end{aligned} \tag{5.1.6}$$

and the final term, $\left(Y_{+1}^2-\frac{\sqrt{10}}{\sqrt{3}}Y_0Y_{+2}+\frac{\sqrt{5}}{\sqrt{3}}Y_{-1}Y_{+3}\right)Y_{+1}$, is not in the ideal generated by coefficients of the Hessian. So this is not a differentially closed ideal.

If all the cubic terms from all of the partial derivatives are added to the set of generators then a differentially closed ideal is obtained. However these extra cubic conditions vanish for a binary form of the form $\mathcal{Y}(r,s)=a(r-cs)^6$

regardless of the value of a and c. The vanishing of the Hessian already implied that $\mathcal{Y}(r,s)$ would have a single multiplicity six factor, and so these extra cubics do not actually place any further restrictions beyond those already implied by the vanishing of the Hessian.

One way to understand this is to notice that the perfect square

$$\left(Y_{+1}^2 - \frac{\sqrt{10}}{\sqrt{3}}Y_0Y_{+2} + \frac{\sqrt{5}}{\sqrt{3}}Y_{-1}Y_{+3}\right)^2$$

lies in the ideal formed by the coefficients of the Hessian. This condition corresponds to the square of the covariant B_2 lying in the ideal (see appendix B). Taking ideal generated from the coefficient of B_4 and B_2 together gives a differentially closed ideal.

Additionally, it can be shown that B_0^2 lies in the ideal formed from the coefficients of B_4, and also vanishes for a binary form with a single multiplicity six factor. Taking the coefficients from the pair of covariants B_4 and B_0 generates another differentially closed ideal. Naturally taking the coefficients from all three covariants B_4, B_2 and B_0 also generates a differentially closed ideal. This last one, the ideal generated by B_4, B_2 and B_0, is important as it is a radical ideal, meaning if there is an element in the ideal of the form A^n for a positive integer n, then A is also contained in the ideal.

So four separate differentially closed ideals ideals have been found, $\langle B_4, \partial B_4\rangle$, $\langle B_4, B_2\rangle$, $\langle B_4, B_0\rangle$ and $\langle B_4, B_2, B_0\rangle$, but clearly they all correspond to the condition $B_4 \equiv 0$. Of course it should be clear that all these ideals have the same radical, given by $\langle B_4, B_2, B_0\rangle$, and this demonstrates that it would be best to work with radical ideals to remove ambiguity.

Insisting that the ideals used in this proof be radical ideals could pose a problem, although there do exist algorithms for calculating the radical of an ideal, in practice this is typically computationally expensive. Thankfully this issue can mostly be avoided by the techniques used in the following section, which immediately lead to radical ideals.

Definition 5.1.7 (The radical of an ideal)**.** Given an ideal I the *radical* of the ideal will ideal containing all elements A such that A^n in the ideal I for a positive integer n. The radical of the ideal I will be denoted $\sqrt{I}$.

Given any ideal I it can be closed differentially by adding the derivatives to set the generators of the ideal until the ideal stops growing (a fact checkable using Gröbner basis techniques). This is process gives an ascending chain of ideals and the Noetherian property of polynomial rings over $\mathbb{C}$ ensures that a maximal ideal exists in this chain (i.e. the ideal will always close under a finite number of derivatives).

Definition 5.1.8 (The differential closure of an ideal)**.** The differential closure of an ideal I will the ideal generated by I and it's partial derivatives under (4.3.1). The differential closure of the ideal I be denoted $\overline{I}$.

Lemma 5.1.9. *If an ideal I is closed under differentiation then so is the radical of the ideal $\sqrt{I}$. That is to say*

$$\overline{\sqrt{\overline{J}}} = \sqrt{\overline{J}}$$

for any ideal J.

Proof. If A is in the radical of the ideal I then $A^n \in I$ for some positive integer n. Taking the nth derivative of A^n with respect to x_i gives

$$\partial_i^n(A^n) = n!(\partial_i A)^n + O(A), \tag{5.1.10}$$

where the terms hidden in $O(A)$ are at least first order in A. Since $\partial_i^n(A^n) \in I$, rearranging (5.1.10) shows that $(\partial_i A)^n$ can be written as a combination of elements from $\sqrt{I}$. Since I is radical this means $\partial_i A \in \sqrt{I}$ as well. Hence $\sqrt{I}$ is differentially closed. □

5.2 Ideals obtained from Coincident Root-Loci

As was already mentioned, for a non-zero binary form $\mathcal{Y}(r, s)$, the vanishing of the Hessian (5.1.3) is equivalent to there being a single multiplicity 6 root of the polynomial $p(z) = \mathcal{Y}(z, 1)$. This is a simple algebraic condition and it would be worth trying to exploit this link between polynomial ideals and the algebraic structures of the roots.

Considering only the multiplicities of the factors of $\mathcal{Y}(r, s)$ there are 12 different possible configurations. These correspond to the 11 partitions of 6 and the case $\mathcal{Y}(r, s) \equiv 0$. These will denoted by

$$[111111], [21111], [2211], [222], [3111], [321], [33], [411], [42], [51], [6], [0],$$

where, for example, [21111] denotes five distinct factors and one factor of multiplicity two, and where [0] indicates the trivial zero-form.

While there exist a number of results in the literature describing ways to determine the ring of covariants vanishing under certain root multiplicity structure (e.g. ref. [6]) it is simpler for this particular problem to construct these rings using elimination ideals using the computer algebra system Singular [9].

Starting with a binary form with a single root in its most general form

$$a_0 s^6 + a_1 r s^5 + \cdots + a_6 r^6 = (\alpha r + \beta s)^6.$$

Equality between the coefficients of r, s above gives rise to the ideal

$$K = \langle a_0 - \beta^6, a_1 - 6\beta^5\alpha, a_2 - 15\beta^4\alpha^2, \cdots, a_6 - \alpha^6 \rangle. \tag{5.2.1}$$

The elimination ideal is now calculated by determining the intersection of the ideal K with the ideal $\langle a_0, a_1, \cdots, a_6 \rangle$. This eliminates α, β from the generators and yields an ideal generated by 15 homogeneous, second-order polynomials in the a_i's. In this case the generators are fairly obvious and can be computed by hand, however the other cases required the assistance of a computer. Explicitly, the generators are of the form

$$400a_0a_6 - a_3^2, \quad 225a_0a_6 - a_2a_4, \ \cdots, \quad 6a_4a_6 - 15a_5^2.$$

In terms of representations these conditions correspond to the coefficients of the covariants B_4, B_2 and B_0 discussed in example 5.1.2 above. Importantly this technique generates the full ideal (a radical ideal) of conditions for a type-[6] binary form with minimal effort.

These elimination ideals were used to calculate the ring of conditions for all of the root multiplicity structures above with one exception[1], the case of a single multiplicity two root (i.e. [21111]), for which it is well-known that the ring is generated from the vanishing of the discriminant.

These ideals are a good starting point for our classification, with some of the conformal-classes being classifiable simply based on the multiplicities of the factors of the associated binary sextic. For the remaining systems the cross-ratios of the roots must somehow be examined.

[1]Theoretically this computation will finish in finite time, but in practice the computation failed.

5.3 The Full Classification

There is now sufficient set up for the full classification of conformally superintegrable systems. For each Stäckel class found below a representative system is given. Nine of the ten possible Stäckel classes have representatives in flat space and a complete list of these can be found in Ref. [17]. The tenth class does not have a flat space equivalent, but does have a representative over a non-zero constant curvature space. This corresponds to the generic spherical potential and is given by the 4-dimensional Smorodinskii-Winternitz potential restricted to the 3-sphere.

Case [0]: The simplest ideal is given by the coefficients of the binary form $\mathcal{Y} = A_3$ (see the Hilbert basis (B.1.1)). This corresponds to the factor structure [0] and so this ideal will be denoted $I_{[0]}$. As was already discussed in example 5.1.1 above, this ideal is easily seen to be closed under differentiation (i.e. $\overline{I_0} = I_0$). Thus any system which satisfies the condition $I_{[0]} = 0$ at one point does so everywhere. The zero locus of $I_{[0]}$ is a single 7-tuple and hence there is only one conformal class represented by this ideal. A particular representative of a system in this class is given by the isotropic oscillator on flat space,

$$V_O = a(x^2 + y^2 + z^2) + bx + cy + dz + e. \tag{5.3.1}$$

The classifying binary form is, of course,

$$\mathcal{Y}(r, s) = 0$$

Case [6]: The ideal $I_{[6]}$ containing the conditions for the [6]-type root structure is given by the coefficients of the covariants

$$B_4^{[6]} = B_4,$$
$$B_2^{[6]} = B_2,$$
$$B_0^{[6]} = B_0.$$

The ideal $I_{[6]}$ is generated by 15 second-order polynomials and it can easily be shown that the cubic polynomials that arise from the derivatives are contained in $I_{[6]}$, i.e. $\overline{I_{[6]}} = I_{[6]}$. So just like the previous case this shows the vanishing of

$I_{[6]}$ locally implies it also vanishes globally. This implies the the root structure [6] is persistent feature when found at a regular point in a system (i.e. it remains a feature in an open set around that point). Since $I_{[0]} \neq 0$ is also a persistent feature the classifying binary form cannot degenerate into the [0]-class. The local action of the conformal group, through $GL(2, \mathbb{C})$, is transitive on three or fewer roots, therefore any two systems in this class can be put into correspondence using a (purely-local) conformal motion and hence this corresponds to a single conformal class.

A particular representative of the [6]-class systems is the (Euclidean superintegrable) system

$$V_A = a\left((x_1 - ix_2)^3 + 6({x_1}^2 + {x_2}^2 + {x_3}^2)\right) + b\left((x_1 - ix_2)^2 + 2(x_1 + ix_2)\right) + c(x_1 - ix_2) + dx_3 + e. \quad (5.3.2)$$

The classifying binary form for V_A is given by

$$\mathcal{Y}(r, s) = ix^6,$$

which has a leading coefficient that clearly vanishes nowhere.

Case [51]: The ideal $I_{[51]}$ of conditions for the [51] root structure is given by a subset of the $I_{[6]}$ generators

$$B_2^{[51]} = B_2,$$
$$B_0^{[51]} = B_0.$$

As before it is simple to show $\overline{I_{[51]}} = I_{[51]}$ and hence represents another persistent structure. Like before, the conformal group acts transitively on the set of [51] binary forms and so only one system can exist in this class. A suitable choice is the (Euclidean superintegrable) system

$$V_{VII} = a(x_1 + ix_2) + b\left(3(x_1 + ix_2)^2 + x_3\right) + c\left(16(x_1 + ix_2)^3 + (x_1 - ix_2) + 12x_3(x_1 + ix_2)\right) + d\left(5(x_1 + ix_2)^4 + (x_1^2 + x_2^2 + x_3^2) + 6\left(x_1 + ix_2\right)^2 x_3\right) + e \quad (5.3.3)$$

which has classifying binary form

$$\mathcal{Y}(r, s) = 24i\left((x_1 + ix_2)r - \frac{3}{2}s\right) r^5.$$

Note that this binary form can not degenerate into a [6]-type structure at any finite point.

Case [42]: The ideal of conditions for the [42] root structure is given by 5 representations, namely

$$\begin{aligned}B_0^{[42]} &= B_0,\\ C_1^{[42]} &= C_1,\\ D_8^{[42]} &= 27B_4^2 - 50B_2A_3^2,\\ D_6^{[42]} &= 20C_3A_3 + B_4B_2,\\ D_0^{[42]} &= D_0.\end{aligned}$$

Unlike the cases examined so far, this ideal is not closed under differentiation. By adding in the first derivatives the ideal closes and using Gröbner bases it can be shown that

$$I_{[6]}{}^3 \subset \overline{I_{[42]}} \subset I_{[6]}. \tag{5.3.4}$$

From this, and the fact that $I_{[6]}$ is radical, it can be concluded that the radical of the differential closure of $I_{[42]}$ is $I_{[6]}$ (i.e. $\sqrt{\overline{I_{[42]}}} = I_{[6]}$).

So as should have been expected given the non-closure of $I_{[42]}$, forcing the conditions for the [42] root structure to hold identically will only yield forms which are generically the [6] root structure (The type-[0] structure is also a possible degeneration, however this is just a further degeneration of [6]). This proves that no potential can have a [42] structure everywhere. However a binary form with a [42] structure gives valid values for $\mathcal{Y}(r, s)$ and hence there must exist a system with a [42] root structure at a non-generic point.

Case [33]: The ideal $I_{[33]}$ of conditions for the [33] root structure is generated by the coefficients of the 3 covariants

$$\begin{aligned}C_6^{[33]} &= C_6,\\ C_4^{[33]} &= C_4,\\ C_3^{[33]} &= 33B_0A_3 - 5C_3.\end{aligned}$$

This ideal is closed under differentiation. Since the only degenerations of the [33] root structure are [6] or [0] it is safe to conclude that a system with the [33] root structure at a regular point has the [33] root structure at every regular point. The transitivity of the conformal group on three or fewer roots means that this can only correspond to one conformal class of systems. A particular representative is given by the (Euclidean superintegrable) system

$$V_{OO} = a(4x_1^2 + 4x_2^2 + x_3^2) + bx_1 + cx_2 + \frac{d}{x_3^2} + e$$

which has classifying binary form

$$\mathcal{Y}(r,s) = \frac{6i}{x_3} r^3 s^3.$$

Case [411]: The ideal $I_{[411]}$ of conditions for the [411] root structure is generated by the coefficients of the 3 covariants

$$B_0^{[411]} = B_0,$$
$$C_1^{[411]} = C_1,$$
$$D_0^{[411]} = D_0.$$

The ideal $I_{[411]}$ is closed under differentiation. So if a potential has a [411] root structure at any point, it will do so in an open set around that point. The ideal $I_{[42]}$ contains the ideal $I_{[411]}$ (meaning the algebraic set satisfying $I_{[411]}$ contains the algebraic set satisfying $I_{[42]}$) and hence, even without explicitly checking, it's clear that the transient [42] structure will break up into the [411] structure under conformal motions. The action of the local action of the conformal group is transitive on the 3 roots and hence every system in this class is conformally related.

A particular representative is the (Euclidean superintegrable) system

$$V_V = a(4x_3^2 + x_1^2 + x_2^2) + bx_3 + \frac{c}{(x_1 + ix_2)^2} + d\frac{x_1 - ix_2}{(x_1 + ix_2)^3} + e$$

which has classifying binary form

$$\mathcal{Y}(r,s) = \left(\frac{9i}{x_1 + ix_2} r^2 - \frac{3i(-x_1 + ix_2)}{(x_1 + ix_2)^2} rs \right) s^4.$$

This potential has the [411] structure at most points and takes the [42] root structure on the hypersurface

$$x_1 - ix_2 = 0.$$

The [4111] class of superintegrable systems contains a second-flat space representative given by scaling the potential V_V above by the Stäckel multiplier $U = (x_1 + ix_2)^2$ and re-expressing the metric in the standard Euclidean coordinates. Specifically V_V is Stäckel equivalent to

$$V_{III} = a\left({x_1}^2 + {x_2}^2 + {x_3}^2\right) + b\frac{1}{(x_1 + ix_2)^2} + c\frac{x3}{(x_1 + ix_2)^3} + d\frac{{x_1}^2 + {x_2}^2 - 3{x_3}^2}{(x_1 + ix_2)^4} + e. \quad (5.3.5)$$

The potential V_{III} has classifying binary form

$$\mathcal{Y}(r,s) = \left(-\frac{9i}{(x_1 + ix_2)}r^2 - \frac{18ix_3}{(x_1 + ix_2)^2}rs + \frac{3i(x_1^2 + x_2^2 - 2x_3^2)}{(x_1 + ix_2)^3}s^2\right)s^4$$

which takes the [42] root structure on the hypersurface

$$x_1^2 + x_2^2 + x_3^2 = 0.$$

Case [321]: The ideal $I_{[321]}$ of conditions for the [321] root structure is given by the 5 representations

$$\begin{aligned}
D_0^{[321]} &= 11B_0^2 - 25D_0,\\
E_{+1}^{[321]} &= 3C_1B_0 - 5E_1,\\
F_{+8}^{[321]} &= 75(2610D_2 + 827B_2B_0){A_3}^2 - 100(125C_3B_2 + 144C_1B_4)A_3\\
&\quad + (3125{C_4}^2 + 5184{B_4}^2B_0),\\
F_{+6}^{[321]} &= 300(61{B_0}^2 - 115D_0){A_3}^2 - 20(5C_1B_2 + 22C_3B_0)A_3\\
&\quad - B_4(7B_2B_0 - 270D_2),\\
F_0^{[321]} &= 41{B_0}^3 - 75D_0B_0 - 125F_0.
\end{aligned}$$

This ideal is not closed under differentiation but it closes after 2 derivatives. By transitivity the location of the three roots is irrelevant and an educated guess would be that the algebraic-set corresponding to the differential closure of $I_{[321]}$ will be points satisfying either the $I_{[33]}$ ideal or the $I_{[51]}$ ideal. That is, it should be expected that

$$\sqrt{I_{[321]}} \stackrel{?}{=} I_{[51]} \cap I_{[33]}.$$

This intersection can be calculated by eliminating t from the convex combination of ideals $(1-t)I_{[33]} + tI_{[51]}$ (see, for example Hassett, chapter 4 [33]). Denoting this ideal by $I_{[51]\wedge[33]}$, the elimination above shows that it is generated by the two covariants

$$\begin{aligned}
C_4^{[51]\wedge[33]} &= C_4,\\
C_3^{[51]\wedge[33]} &= 33B_0A_3 - 5C_3.
\end{aligned}$$

Straightforward Gröbner basis calculations now show

$$\left(I_{[51]} \cap I_{[33]}\right)^3 \subset \overline{I_{[321]}} \subset I_{[51]} \cap I_{[33]},$$

and hence

$$\sqrt{\overline{I_{[321]}}} = I_{[51]} \cap I_{[33]}$$

as predicted. So any systems whose coefficient functions cause the ideals $I_{[321]}$ to vanish identically must lie in either the [33] or [51] classes (or their degenerations) and thus have already been classified.

Case [222]: The ideal of conditions for the [222] root structure is given by 5 representations, namely

$$\begin{aligned}
D_8^{[222]} &= 50B_2A_3^2 - 27B_4^2,\\
D_6^{[222]} &= 160B_0A_3^2 - B_4B_2 - 20C_3A_3,\\
D_4^{[222]} &= -3B_4B_0 + 25C_1A_3,\\
D_2^{[222]} &= B_2B_0 + 90D_2,\\
D_0^{[222]} &= 43B_0^2 - 75D_0.
\end{aligned}$$

This ideal is not closed under differentiation, but closes after one derivative. Straightforward calculations show

$$I_{[6]}^4 \subset \overline{I_{[222]}} \subset I_{[6]}.$$

Hence $\sqrt{\overline{I_{[222]}}} = I_{[6]}$ and any systems with coefficient functions that cause the ideal $I_{[222]}$ to vanish identically are in the class of [6] type systems and therefore have already been classified.

Case [3111]: The ideal $I_{[3111]}$ of conditions for the [3111] root structure is given by the 3 representations

$$\begin{aligned}
D_0^{[3111]} &= 11B_0^2 - 25D_0,\\
E_1^{[3111]} &= 3C_1B_0 - 5E_1,\\
F_0^{[3111]} &= 8B_0^3 - 125F_0.
\end{aligned}$$

This is a differentially closed ideal so we can conclude that a [3111] root structure is stable. Unlike previous cases the action of the conformal group is not automatically transitive, so the values of the roots cannot be assigned arbitrarily. To distinguish between different possible [3111] root structures only the cross-ratio of the roots needs to be be considered, this is clear because once the conformal group has been used to move three of the roots to a canonical location the cross-ratio determines the location of the fourth uniquely.

Denoting the 4 roots by $\mathbf{r} = (\eta_1, \eta_2, \eta_3, \eta_4)$ (where η_4 will be the triply repeated root) a cross-ratio[2] can be defined by

$$\lambda = \frac{(\eta_1 - \eta_2)(\eta_3 - \eta_4)}{(\eta_2 - \eta_3)(\eta_4 - \eta_1)}. \tag{5.3.6}$$

Assuming Y_{+3} is non-zero (or performing a small rotation such that it is non-zero), define

$$\mathbf{a} = \frac{1}{Y_{+3}} \left(Y_{-1}, Y_0, Y_{+1}, Y_{+2}\right),$$

which can be expressed as functions of the η_i using Vieta's formula. From this the Jacobian $\frac{\partial \lambda}{\partial \mathbf{x}}$ can be calculated via the formula

$$\begin{aligned} \frac{\partial \lambda}{\partial \mathbf{x}} &= \frac{\partial \lambda}{\partial \mathbf{r}} \frac{\partial \mathbf{r}}{\partial \mathbf{a}} \frac{\partial \mathbf{a}}{\partial \mathbf{x}} \\ &= \frac{\partial \lambda}{\partial \mathbf{x}} \left(\frac{\partial \mathbf{a}}{\partial \mathbf{r}}\right)^{-1} \frac{\partial \mathbf{a}}{\partial \mathbf{x}}. \end{aligned} \tag{5.3.7}$$

Without loss of generality the roots can be assumed take the values

$$\mathbf{r}_0 = \left(-1, 0, 1, \frac{1-\lambda}{1+\lambda}\right)$$

at the regular point. Substituting this into (5.3.7) gives,

$$\begin{aligned} \left.\frac{\partial \lambda}{\partial x_1}\right|_{\mathbf{r}_0} &= \frac{-8i(1-\lambda+\lambda^2)(\lambda-1)\lambda}{27(1+\lambda)^3}, \\ \left.\frac{\partial \lambda}{\partial x_2}\right|_{\mathbf{r}_0} &= \frac{-16(1-\lambda+\lambda^2)(\lambda-1)\lambda}{27(1+\lambda)^3}, \\ \left.\frac{\partial \lambda}{\partial x_3}\right|_{\mathbf{r}_0} &= 0. \end{aligned} \tag{5.3.8}$$

These imply that the action of the translations is rank 1 for almost all values of λ. The five possible exceptions correspond to points where $\lambda = 0, 1, \infty$ or $\exp(\pm i\pi/3)$, where the action is, to a first order approximation, rank 0.

The case $\lambda = 0, 1, \infty$ correspond to the degenerate root structure [411] and so have already been examined. The point $\lambda = \exp(\pm i\pi/3)$ will be considered below.

[2] What follows is actually the multi-ratio, which is equivalent to the standard cross-ratio by a permutation of the indices.

Subcase [3111]+Cross Ratio$= \exp\left(\pm\frac{i\pi}{3}\right)$**:** The analysis above shows that the first-order changes in the cross-ratio at value $\lambda = \exp(\pm i\pi/3)$ are zero with respect to first order changes in x_1, x_2, x_3. A higher changes in x_1, x_2, x_3 move λ away from this value and so to examine whether or not this really is a persistent feature the corresponding ideal will need to be generated. This will be referred to as the $[3111] + CR$ root structure.

Under the action of $GL(2, \mathbb{C})$, a canonical form of the binary form with the $[3111] + CR$ root structure is given by

$$\mathcal{Y}(r, s) = r^3(r^3 - s^3).$$

Performing a general linear transformation of the form

$$x \mapsto c_1x + c_2y, \qquad y \mapsto c_3x + c_4y,$$

and setting up an ideal in the same manner as (5.2.1), an elimination ideal can be calculated with respect to c_1, c_2, c_3, c_4. The ideal found in generated by the coefficients of the covariants

$$\begin{aligned} D_4^{[3111]+CR} &= 3600C_1A_3 + 288B_4B_0 - 125B_2^2, \\ D_2^{[3111]+CR} &= B_2B_0 - 10D_2, \\ D_0^{[3111]+CR} &= 11B_0^2 - 25D_0. \end{aligned} \tag{5.3.9}$$

This ideal is closed under differentiation and hence this case is actually a persistent one. Up to local equivalence there is only one binary form with the $[3111] + CR$ structure, and hence this represents a single conformal class. A particular representative of this class is the (Euclidean superintegrable) system

$$\begin{aligned} V_{VI} = a\left(x_3^2 - 2(x_1 - ix_2)^3 + 4(x_1^2 + x_2^2)\right) + b\left(2x_1 + 2ix_2 - 3(x_1 - ix_2)^2\right) \\ + c\,(x_1 - ix_2) + \frac{d}{x_3^2} + e \end{aligned} \tag{5.3.10}$$

which has classifying binary form

$$\mathcal{Y}(r, s) = 3i\left(r^3 + \frac{2}{x_3}s^3\right)r^3.$$

Subcase [3111]+Cross Ratio$\neq \exp\left(\pm\frac{i\pi}{3}\right)$**:** Since the action of a translation on the cross-ratio λ (c.f (5.3.7)) is rank 1 everywhere on the connected set

$\mathbb{C}^* \setminus \left\{0, 1, \infty, \exp\left(\pm\frac{i\pi}{3}\right)\right\}$ every point will lie in a single orbit under this action. Hence this there is a single conformal class of systems with this structure.

A particular representative is given by the (Euclidean superintegrable) potential

$$V_{II} \;=\; a\left({x_1}^2+{x_2}^2+{x_3}^2\right) + b\frac{(x_1-ix_2)}{(x_1+ix_2)^3} + c\frac{1}{(x_1+ix_2)^2} + d\frac{1}{{x_3}^2} + e \tag{5.3.11}$$

which has classifying binary form

$$\mathcal{Y}(r,s) = \left(\frac{6i}{x_3}r^3 + \frac{9i}{x_1+ix_2}r^2 s - \frac{3i(-x_1+ix_2)}{(x_1+ix_2)^2}s^3\right)s^3.$$

Using this binary form to calculate the covariants (5.3.9) yield

$$D_2^{[3111]+CR}(r,s) \propto \frac{s^4}{x_3^2(x_1+ix_2)^2}$$

verifying that it cannot also contain the $[3111]+CR$ root structure.

Case [2211]: The ideal $I_{[2211]}$ of conditions for the [2211] root structure is generated by a single covariant

$$G_6^{[2211]} = 50(10F_3^{(2)} + 2D_3B_0 + 55F_3^{(1)})A_3 - 4(43B_0^2 - 75D_0)C_6 + 75E_2B_4.$$

This ideal is not closed under differentiation, but closes after 3 derivatives. Straightforward calculations show that

$$\left(I_{[411]} \cap I_{[33]}\right)^4 \subset \overline{I_{[2211]}} \subset I_{[411]} \cap I_{[33]}.$$

Hence $\sqrt{\overline{I_{[2211]}}} = I_{[411]} \cap I_{[33]}$ and any systems with coefficient functions that identically satisfy polynomials in the $I_{[2211]}$ ideal are in the class of [411] systems or the class of [33] systems and thus have been classified.

Case [21111]: As is well known, the ideal of conditions for the [21111] root structure is generated by one condition, the discriminant. In terms of the Hilbert basis, the ideal $I_{[21111]}$ is generated by the single covariant

$$J_0^{[21111]} = 5393B_0^5 - 20125D_0B_0^3 + 18750D_0^2B_0 - 31875F_0B_0^2 + 56250F_0D_0 + 28125J_0.$$

The ideal $I_{[21111]}$ is not closed under differentiation, but closes after five derivatives. Straightforward calculations shown

$$\left(I_{[3111]}\right)^4 \subset \overline{I_{[21111]}} \subset I_{[3111]},$$

meaning $\sqrt{\overline{I_{[21111]}}} = I_{[3111]}$. Hence any systems with coefficient functions that identically satisfies the $I_{[21111]}$ ideal are in the [3111] class and have already been classified above.

Case [111111]: All systems with a persistent root of multiplicity 2 or greater have been been classified above. All that remains is to classify systems which correspond to binary forms with distinct linear factors. If all factors are distinct then there are only three independent absolute invariants. Any three independent cross-ratios will do and for the following discussion these will be chosen to be

$$\begin{aligned}\lambda_4 &= \frac{(\eta_1-\eta_2)(\eta_3-\eta_4)}{(\eta_2-\eta_3)(\eta_4-\eta_1)},\\ \lambda_5 &= \frac{(\eta_1-\eta_2)(\eta_3-\eta_5)}{(\eta_2-\eta_3)(\eta_5-\eta_1)},\\ \lambda_6 &= \frac{(\eta_1-\eta_2)(\eta_3-\eta_6)}{(\eta_2-\eta_3)(\eta_6-\eta_1)}.\end{aligned} \tag{5.3.12}$$

However, trying to repeat the sort of analysis that was done in the [3111] case is hampered by the complexity of the equations that arise.

An alternative approach is to define three absolute invariants strictly in terms of the coefficients of the binary form. This can be achieved by balancing out the covariant weight (see appendix A) and a suitable set is given by

$$\mathbf{I} = \left(\frac{D_0}{B_0^2}, \frac{F_0}{B_0^3}, \frac{J_0}{B_0^5}\right). \tag{5.3.13}$$

It is safe to assume B_0 is non-zero as doing otherwise this leads to the case [411]. Examining the action of translations on the absolute invariants in $\mathbf{I}$ should give equivalent results to examining the action of the cross-ratios provided the map between them is invertible. Since $\mathbf{I}$ can be expressed as a function of the cross-ratios[3], the Jacobian

$$\det\left(\frac{\partial\left(\frac{D_0}{B_0^2}, \frac{F_0}{B_0^3}, \frac{J_0}{B_0^5}\right)}{\partial(\lambda_4,\lambda_5,\lambda_6)}\right) \tag{5.3.14}$$

[3]as can all absolute invariants

can be calculated. The Jacobian (5.3.14) factors nicely and can be seen to vanish if and only if there is either a double root or if the condition

$$\lambda_4 - \lambda_5\lambda_6 = 0 \tag{5.3.15}$$

is satisfied (up to permutation of roots). Condition (5.3.15) is a well known object in the literature, going by the name of the $M_6 = -1$ multi-ratio condition and is an interesting object of study in its own right [40]. Written in terms of the roots (5.3.15) is equivalent to

$$\frac{(\eta_1 - \eta_2)(\eta_5 - \eta_3)(\eta_4 - \eta_6)}{(\eta_2 - \eta_5)(\eta_3 - \eta_4)(\eta_6 - \eta_1)} = -1. \tag{5.3.16}$$

Assuming the condition (5.3.15) is satisfied, the roots can be assumed to take the values

$$\mathbf{r}_0 = \left(-1, 0, 1, \frac{1 - \lambda_5\lambda_6}{1 + \lambda_5\lambda_6}, \frac{1 - \lambda_5}{1 + \lambda_5}, \frac{1 - \lambda_6}{1 + \lambda_6}\right).$$

The action of a translation on the value of $\lambda_4 - \lambda_5\lambda_6$ is (to first order) given by

$$\begin{aligned}
\left.\frac{\partial(\lambda_4 - \lambda_5\lambda_6)}{\partial x_1}\right|_{\mathbf{r}_0} &= 0, \\
\left.\frac{\partial(\lambda_4 - \lambda_5\lambda_6)}{\partial x_2}\right|_{\mathbf{r}_0} &= \frac{8(\lambda_5\lambda_6 - 1)(\lambda_5^2\lambda_6^2 - \lambda_5^2\lambda_6 - \lambda_5\lambda_6^2 + \lambda_5^2 + \lambda_6^2 - \lambda_5 - \lambda_6 + 1)^2}{27(1 + \lambda_5\lambda_6)(\lambda_6 - 1)(\lambda_5 - 1)(\lambda_5 + 1)(\lambda_6 + 1)}, \\
\left.\frac{\partial(\lambda_4 - \lambda_5\lambda_6)}{\partial x_3}\right|_{\mathbf{r}_0} &= \frac{-8i(\lambda_5^2\lambda_6^2 - \lambda_5^2\lambda_6 - \lambda_5\lambda_6^2 + \lambda_5^2 + \lambda_6^2 - \lambda_5 - \lambda_6 + 1)^2}{27(\lambda_6 - 1)(\lambda_5 - 1)(\lambda_5 + 1)(\lambda_6 + 1)}.
\end{aligned} \tag{5.3.17}$$

and so, remembering that all roots are distinct, this action will be rank zero only if the condition

$$\lambda_5^2\lambda_6^2 - \lambda_5^2\lambda_6 - \lambda_5\lambda_6^2 + \lambda_5^2 + \lambda_6^2 - \lambda_5 - \lambda_6 + 1 = 0 \tag{5.3.18}$$

is also satisfied. Likewise, calculating the action of a derivative on (5.3.18) shows that the action is rank zero (to first order) and so this is a promising candidate for a persistent feature.

Subcase [111111]; $M_6 = -1 + CR$: The ideal I_{M_6+CR} of covariants vanishing under conditions (5.3.15) and (5.3.18), can be calculated without too much effort. This ideal I_{M_6+CR} is generated by the coefficients of the two covariants

$$\begin{aligned} F_4^{M_6+CR} &= 360(49C_1B_0 - 48E_1)A_3 - 193B_2^2B_0 - 1896C_3C_1 \\ &\quad + 288D_0B_4 + 3276D_2B_2, \\ F_0^{M_6+CR} &= 97B_0^3 - 275D_0B_0 + 375F_0. \end{aligned} \tag{5.3.19}$$

Calculations using Gröbner bases show that the ideal I_{M_6+CR} is closed under differentiation and hence represents a persistent feature.

The geometry of the algebraic set corresponding to the I_{M_6+CR} will determine whether or not this corresponds to a single conformal class, or whether is it has several difference components that are inequivalent under the action of the conformal group. One could imagine a situation where the ideal breaks into two unconnected components or where one component is separated into two by removing points from the algebraic set where the rank of the action drops. Thankfully such a situation would show up algebraically due to the following theorem (taken from Corollary 4.16 in Ref. [48]).

Theorem 5.3.20. *Let $X \subset \mathbb{P}^n$ be an r-dimensional projective variety and let $Y \subsetneq X$ be a closed algebraic set. Then $X \setminus Y$ is connected in the classical topology.*

The locations where the rank of the action drops is defined by polynomial conditions and hence is an algebraic subset of the algebraic set defined by (5.3.18). A test using Maple indicates that condition (5.3.18) is absolutely irreducible over $\mathbb{C}$ (meaning its algebraic set is actually an algebraic variety). Theorem 5.3.20 implies that the set of points for which the action has maximal rank will be connected. So the action of the translations will be transitive on set of point satisfying (5.3.15) and (5.3.18) where the action is rank 1. Points where the rank of the action drops can be shown to correspond to roots of multiplicity 2 or higher and hence need not be considered.

Hence the ideal I_{M_6+CR} found corresponds to single conformal class. A particular representative of the systems lying in this class is given by the (Euclidean superintegrable) system

$$V_{IV} = a(4x_1^2 + x_2^2 + x_3^2) + bx_1 + \frac{c}{x_2^2} + \frac{d}{x_3^2} + e \tag{5.3.21}$$

which has classifying binary form

$$\mathcal{Y}(r,s) = \frac{3}{4x_2}(r^2+s^2)^3 + \frac{6i}{x_3}r^3s^3.$$

Subcase [111111]; Rank 3 Jacobian: Assuming now that the multi-ratio condition (5.3.15) is not satisfied, then the Jacobian (5.3.14) will be rank 3 and the absolute invariants **I** can be used to examine the rank of the action on the cross-ratios.

Calculating the the determinant of the Jacobian between the three absolute invariants (5.3.13) and the coordinates gives,

$$\det\left(\frac{\partial \mathbf{I}}{\partial \mathbf{x}}\right) = \det\left(\frac{\partial\left(\frac{D_0}{B_0^2},\frac{F_0}{B_0^3},\frac{J_0}{B_0^5}\right)}{\partial(x_1,x_2,x_3)}\right)$$
$$= \frac{\begin{pmatrix} 2521B_0^5 - 9625D_0B_0^3 + 6250D_0^2B_0 \\ -7500F_0B_0^2 + 65625F_0D_0 - 84375J_0 \end{pmatrix}}{2^5 3^{10} 5^6 B_0^{11}} O_0. \quad (5.3.22)$$

This shows that action is rank 3 away from

$$J_0^{Jac} = 2521B_0^5 - 9625D_0B_0^3 + 6250D_0^2B_0 - 7500F_0B_0^2 + 65625F_0D_0 - 84375J_0$$
$$= 0 \quad (5.3.23)$$

and

$$O_0 = 0. \quad (5.3.24)$$

A careful examination reveals that O_0 is a symmetric version of the $M_6 = -1$ condition, specifically

$$O_0 \propto \prod_{\sigma\in\Sigma}\begin{pmatrix} (r_{\sigma(1)} - r_{\sigma(2)})(r_{\sigma(3)} - r_{\sigma(4)})(r_{\sigma(5)} - r_{\sigma(6)}) \\ + \quad (r_{\sigma(6)} - r_{\sigma(2)})(r_{\sigma(2)} - r_{\sigma(3)})(r_{\sigma(4)} - r_{\sigma(5)}) \end{pmatrix} \quad (5.3.25)$$

where Σ is the subset of 15 elements of the permutation group that give the 15 different versions of the $M_6 = -1$ condition. By the discussion in the previous section systems satisfying this have already been considered. So henceforth O_0 will be assumed non-zero.

Returning to the main argument, the action of the translations will be rank 3 on the absolute invariants **I** away from $J_0^{Jac} = 0$ and $O_0 = 0$. The set $\mathbb{C}^3 \setminus \left\{\mathbf{I} \in \mathbb{C}^3 : J_0^{Jac} = 0, O_0 = 0\right\}$ is clearly connected (invoking theorem 5.3.20

if needed) and hence there can only be one orbit under this action. A particular representative of the system in this orbit is given by the conformally-superintegrable potential

$$V_S = \frac{a}{(1+x_1^2+x_2^2+x_3^2)^2} + \frac{b}{x_1^2} + \frac{c}{x_2^2} + \frac{d}{x_3^2} + \frac{e}{(-1+x_1^2+x_2^2+x_3^2)^2}.$$

The potential V_S above is Stäckel equivalent to the potential

$$V_{S'} = \frac{\alpha}{s_1^2} + \frac{\beta}{s_2^2} + \frac{\gamma}{s_3^2} + \frac{\delta}{s_4^2} + \epsilon$$

which is superintegrable over the 3-sphere $s_1^2 + s_2^2 + s_3^2 + s_4^2 = 1$.

Subcase [111111]; Rank 2 Jacobian; $J_0^{Jac} \equiv 0$: Taking the condition J_0^{Jac} on its own generates a closed ideal, which will be denoted I_{Jac}. The differential closure of I_{Jac} is easily seen by considering the following relations

$$\begin{aligned} \frac{\partial J_0^{Jac}}{\partial x} &= 5\,(X_{-1} - X_{+1})\, J_0^{Jac}, \\ \frac{\partial J_0^{Jac}}{\partial y} &= 5i\,(X_{-1} + X_{+1})\, J_0^{Jac}, \\ \frac{\partial J_0^{Jac}}{\partial z} &= 5\sqrt{2}\,(X_0)\, J_0^{Jac}. \end{aligned} \tag{5.3.26}$$

The Hilbert dimension of the the ideal I_{Jac} is 6 and, a check for absolutely irreducibility using Maple returns a positive result. This means (like I_{M_6+CR}) the ideal I_{Jac} gives an algebraic variety and hence is connected set. The local action of the conformal group is rank 4 and the action of a non-local transformation on the absolute invariants is rank 2 (which is clearly distinct from the local action). Hence the generic action on this space will be rank 6 and thus can only be one orbit under the action of the conformal group.

So I_{Jac} represents a single conformal class and a particular representative of the systems in this class is given by

$$V_I = a({x_1}^2 + {x_2}^2 + {x_3}^2) + \frac{b}{{x_1}^2} + \frac{c}{{x_2}^2} + \frac{d}{{x_3}^2} + e \tag{5.3.27}$$

which has the classifying binary form

$$\mathcal{Y}(r,s) = \frac{6i}{x_3} r^3 s^3 + \frac{3}{4x_2}(s^6 + 3r^2s^4 + 3r^4s^2 + r^6) - \frac{3i}{4x_1}(s^6 - 3r^2s^4 + 3r^4s^2 - r^6).$$

Subcase [111111]; Rank 1 Jacobian: Any further restrictions would necessarily show up when the Jacobian is rank 1. Examining the 2×2 subminors of the Jacobian under the restriction $J_0^{Jac} = 0$ gives an additional 14th order covariant that must vanish identically, namely

$$N_1^{Rank1} = (125F_0 + 49B_0^3 - 125D_0B_0)H_1 - 20(-25D_0 + 14B_0^2)J_1 + 150L_1B_0 \tag{5.3.28}$$

The ideal I_{Rank1} generated by the coefficients of the covariants N_1^{Rank1}, J_0^{Jac} closes after 3 derivatives. Straightforward calculations show

$$\left(I_{M_6+CR} \cap I_{[3111]}\right)^3 \subset \overline{I_{Rank1}} \subset I_{M_6+CR} \cap I_{[3111]}.$$

Hence $\sqrt{\overline{I_{Rank1}}} = I_{M_6+CR} \cap I_{[3111]}$ and all corresponding systems have already been classified.

This completes the classification. There are a total of 10 conformal classes and a given maximal-parameter, second-order conformally-superintegrable systems can be identified by determining which of the ideals above vanished. Table 5.1 shows the pattern of vanishing ideals for each of the representative systems.

A classification of maximal-parameter, second-order superintegrable systems over flat spaces and the sphere can now be determined by which of the 10 given conformally-superintegrable systems givens an appropriate metric. The conclusion is there are 10 Euclidean Systems (as was well known) labeled

$$I, II, III, IV, V, VI, VII, O, OO, A$$

and 6 systems on the Sphere, labeled

$$S, I, II, IV, VI, OO.$$

5.4 Limiting Diagrams of the Maximum-Parameter Systems

The ideals that were found when performing this classification gives a natural way to think about limiting from one class to another by considering the partial ordering put on these ideals via ideal containment. For example, the

	$I_{[0]}$	$I_{[6]}$	$I_{[51]}$	$I_{[411]}$	$I_{[33]}$	$I_{[3111]}$	$I_{[3111]+CR}$	I_{M_6+CR}	J_0^{Jac}
S									
I									0
II						0			0
III/V				0		0		0	0
IV								0	0
VI						0	0	0	0
VII			0	0		0	0	0	0
O	0	0	0	0	0	0	0	0	0
OO					0	0	0	0	0
A		0	0	0	0	0	0	0	0

Table 5.1: Vanishing irreducible ideals for the ten maximum-parameter systems

ideal $I_{[3111]+CR}$ contains the ideal $I_{[3111]}$ as a subideal, this means that the algebraic variety defined by $I_{[3111]}$ contains the algebraic variety defined by $I_{[3111]+CR}$. A particular $[3111]+CR$ system corresponds to a point in the variety defined by $I_{[3111]+CR}$. A motion (either local conformal or translation of the regular point) can move a point in the variety and take it arbitrarily close to a subvariety.

The structure of ideals and subideals is displayed in figure 5.1. The arrows in figure 5.1 point from subideal to superideal, and since ideal containment is transitive only a minimal set of arrows has been drawn. In terms of subvarieties (or thinking about limiting from one variety to another) the arrows should be reversed.

There are three pieces of information contained in the boxes in figure 5.1, the first is the name of the chosen representative of the system in the classification above, the second is the factor structure for the binary form $\mathcal{Y}(r,s)$ associated to the system, the third is a reference to the bracket notation of Bôcher where a partition of 5 indicates the generic separable coordinates in which the system separates [4], and the last piece of information is Hilbert dimensional of the ideal with regards to the seven variables from the 7-dimensional representation $\mathcal{Y}(r,s)$ (that is, ignoring the values of the three-dimensional representation $\mathcal{X}(r,s)$). The generic system, labeled S, has Hilbert dimension 7 as this corresponds to a generic point in $\mathbb{C}^7$. Most of the (minimal) degenerations shown by the arrows drop the Hilbert dimensional by one, the only exception to this is the degeneration from [6] to [0] where the two degrees of freedom coming from the position of the single root and the value of the leading coefficients are lost.

The systems denoted by O, OO and A only separate in non-generic separable coordinates and so do not have a Bôcher bracket associated to them. However it should be clear that, for the 7 classes that do, the partial-ordering of the Bôcher brackets (as partitions of five) is the same as the partial-ordering given by the ideal containment relations.

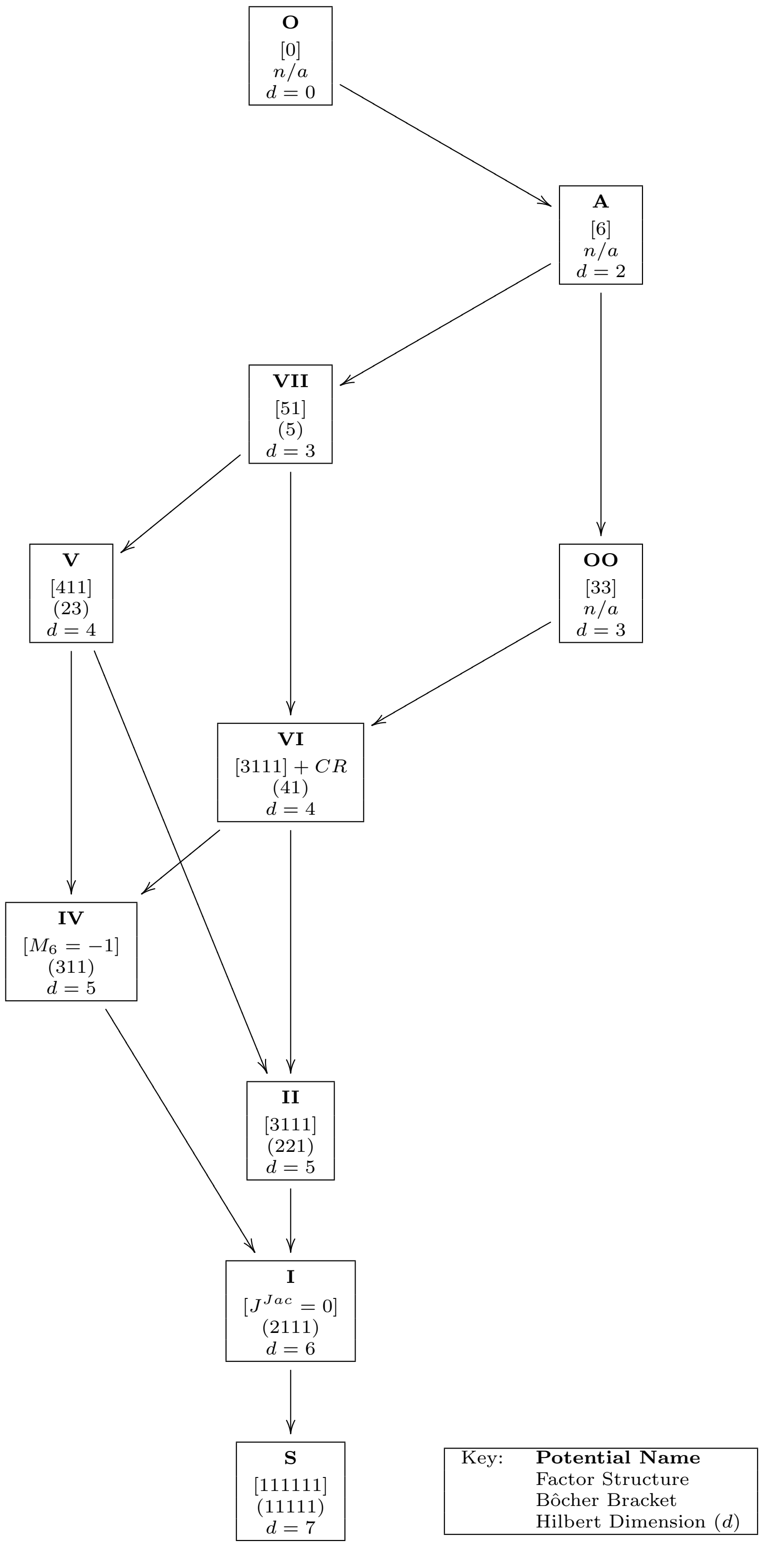

Figure 5.1: Subideal Containment Diagram

Chapter 6

Conclusions and Future Directions

The central result in this thesis is the classification of three-dimensional second-order systems with maximum-parameter (non-degenerate) potentials over conformally flat complex spaces. The classification made use of the algebraic-geometry that arose when considering the action of the conformal-group on the variety of integrability conditions. Importantly the techniques used did not depend on separation of variables (which is a purely second-order phenomenon) and hence should be applicable to a wide range of systems.

There are four natural directions to investigate next: systems in higher dimensions, systems depending on strictly fewer than the maximum of parameters (degenerate), systems with higher-order constants, and the quantum analogues of these classical systems.

The quantization procedure for maximal-parameter three-dimension second-order superintegrable systems has already been worked out in Ref. [14] and so there is no-foreseeable difficulty in carrying the classification over to the quantum case.

In the case of classifying a classical n-dimensional maximum-parameter second-order systems there is already a large class of systems known to corresponding to separation of variables in ellipsoidal coordinates and their degenerations [26]. If the 3-dimensional case is a good model for the n-dimensional cases (as they appear to be) then the techniques discussed here can be used to more efficiently examine the completeness of the list of known potentials and find the systems which are missing.

Only maximum-parameter (non-degenerate) second-order systems were discussed in this thesis, and in three-dimensional spaces the complete classification of systems depending on strictly fewer parameters is still an open question. However using the integrability conditions to to derive algebraic varieties already seem like a promising path towards a complete classification [15] and it would be interesting to try and take the techniques developed here and apply them to the degenerate potentials.

Since the classification result in this thesis was primarily based on the integrability conditions for second-order superintegrable systems (i.e. without appealing to separation of variables) the techniques used here should be applicable to systems with higher-order constants. The systems studied in this thesis are somewhat special as their classification only depends on the invariants of a single $GL(2,\mathcal{C})$ representation, however the techniques should also carry over to joint invariants between multiple representations.

The techniques used in this thesis are not without their drawbacks. They would most likely become intractable if the binary form model of the seven-dimensional representation hadn't been used as they required significant computing power. A deeper understanding of the algebraic geometry underlying the varieties which does not rely on such a specific model is a natural avenue that will be pursued. Each of the generalisations discussed above allow for ample new examples to be investigated and improvements to these techniques can be explored.

Finally a very important aspect of superintegrable systems, which wasn't discussed in this thesis, is the connection to special function theory. A particularly interesting example of this connection is given by the relationship between the contractions of the quantum quadratic algebras of 2-dimensional systems and the Askey-Wilson scheme for hypergeometric orthogonal polynomials [25].

Analogously to the 2d case all second-order maximum-parameter systems can be obtained from contraction of the generic spherical case (as can be seen in the limiting diagram in figure 5.1). A model of the quadratic algebra of the generic spherical potentials is given by two-variable Wilson polynomials [24] and it would be expected that suitable models for all the quadratic algebras of all 3-dimensional superintegrable systems could be given by taking appropriate limits of this master model (a task which, to the best of the Author's knowledge, hasn't been completed yet). The classification of the classical systems given in chapter 5 should provide information relevant to this limiting process.

The rotation representations in chapter 3 were effective at determining the dimension of the space of constants. It seems reasonable to expect that this technique could be extended to prove the existence of the quadratic algebra for higher-dimensional systems.

The subalgebras of the quadratic algebras is also a topic of interest to the research community [7] and the techniques in chapter 3 should be adaptable to describe such subalgebras in terms of invariants and joint-invariants. At the very least it seems feasible that the descriptions could be made more concise.

Appendix A

Representations of $\mathfrak{so}(3,\mathbb{C}) \simeq \mathfrak{sl}(2,\mathbb{C})$

In this thesis the irreducible representations of $\mathfrak{so}(3,\mathbb{C})$ are use to describe the results obtained, identify possible avenues of investigation and to simplify the computations involved. The main idea that is needed here is that all $\mathfrak{so}(3,\mathbb{C})$ representations can be modelled as odd dimensional $\mathfrak{sl}(2,\mathbb{C})$ representations. This identification allows the representations to be concisely described as a binary form. This description in terms of binary forms makes the tools of classical invariant theory relevant.

A.1 The isomorphism between $\mathfrak{so}(3,\mathbb{C}) \simeq \mathfrak{sl}(2,\mathbb{C})$

Before beginning a discussion of $\mathfrak{sl}(2,\mathbb{C})$ representations, recall some basic facts about the $\mathfrak{so}(3,\mathbb{C})$ Lie algebra. Examining the action of the rotation group on the coordinate the following matrix-infinitesimals are determined

$$J_1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix},$$
$$J_2 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix},$$
$$J_3 = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}. \qquad \text{(A.1.1)}$$

These matrices satisfy the commutations relations

$$[J_1, J_2] = J_3, \qquad [J_2, J_3] = J_1, \qquad [J_3, J_1] = J_2.$$

This Lie algebra is isomorphic to the $\mathfrak{sl}(2, \mathbb{C})$ Lie algebra whose generators are given by the Pauli matrices

$$A_1 = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix},$$
$$A_2 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix},$$
$$A_3 = \begin{pmatrix} -i & 0 \\ 0 & i \end{pmatrix}. \qquad \text{(A.1.2)}$$

Like the J_i above these satisfy the commutation relations

$$[A_1, A_2] = 2A_3, \qquad [A_2, A_3] = 2A_1, \qquad [A_3, A_1] = 2A_2,$$

so the identification $\frac{1}{2}A_i \simeq J_i$ provides the Lie algebra isomorphism (over $\mathbf{C}$).

Exponentiating these matrices, using the (absolutely convergent) sum

$$\exp(A) = I + A + \frac{1}{2}A^2 + \frac{1}{3!}A^3 + \cdots$$

gives the recognisable rotation matrices

$$\exp(tJ_1) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos(t) & -\sin(t) \\ 0 & \sin(t) & \cos(t) \end{pmatrix},$$

$$\exp(tJ_2) = \begin{pmatrix} \cos(t) & 0 & \sin(t) \\ 0 & 1 & 0 \\ -\sin(t) & 0 & \cos(t) \end{pmatrix},$$

$$\exp(tJ_3) = \begin{pmatrix} \cos(t) & -\sin(t) & 0 \\ \sin(t) & \cos(t) & 0 \\ 0 & 0 & 1 \end{pmatrix}. \tag{A.1.3}$$

Similarly, taking exponentials of the A_i gives the matrices

$$\exp\left(\frac{t}{2}A_1\right) = \begin{pmatrix} \cos(\frac{t}{2}) & i\sin(\frac{t}{2}) \\ i\sin(\frac{t}{2}) & \cos(\frac{t}{2}) \end{pmatrix},$$

$$\exp\left(\frac{t}{2}A_2\right) = \begin{pmatrix} \cos(\frac{t}{2}) & \sin(\frac{t}{2}) \\ -\sin(\frac{t}{2}) & \cos(\frac{t}{2}) \end{pmatrix},$$

$$\exp\left(\frac{t}{2}A_3\right) = \begin{pmatrix} \cos(\frac{t}{2}) - i\sin(\frac{t}{2}) & 0 \\ 0 & \cos(\frac{t}{2}) + i\sin(\frac{t}{2}) \end{pmatrix}. \tag{A.1.4}$$

These allow a homomorphism to be defined from the $SL(2,\mathbb{C})$ Lie group to the $SO(3,\mathbf{C})$ Lie group. Thus any $SO(3,\mathbb{C})$ representation can be pulled back to a $SL(2,\mathbb{C})$ representation. Importantly, the irreducible representations remain irreducible. In the following section the irreducible representations of $SL(2,\mathbb{C})$ will be discussed, an only those that are in correspondence with $SO(3,\mathbb{C})$ representations will be kept.

Irreducible Representations of $SL(2,\mathbb{C})$

The standard way to examine the irreducible, finite-dimensional representations of $SL(2,\mathbb{C})$ (or any semi-simple Lie algebra) is to split these into weight spaces and identify the representation by its highest weight vector. Specifically, for the Lie algebra $\mathfrak{sl}(2,\mathbb{C})$ the raising, lowering and level-set operators are defined via

$$\begin{aligned}
e &= \frac{1}{2}(iA_1 - A_2), \\
f &= \frac{1}{2}(iA_1 + A_2), \\
h &= \frac{i}{2}A_3.
\end{aligned} \tag{A.1.5}$$

These satisfy the commutation relations

$$[h, e] = e, \qquad [h, f] = -f, \qquad [e, f] = 2h. \tag{A.1.6}$$

From these it can easy be shown that an eigenvector $\mathbf{v}_\lambda$ of h, that is $h\mathbf{v}_\lambda = \lambda\mathbf{v}_\lambda$, then

$$\begin{aligned}
he\mathbf{v}_\lambda &= (\lambda + 1)e\mathbf{v}_\lambda, \\
hf\mathbf{v}_\lambda &= (\lambda - 1)f\mathbf{v}_\lambda.
\end{aligned} \tag{A.1.7}$$

So $e\mathbf{v}_\lambda$ is an eigenvector with eigenvalue $\lambda + 1$ and $f\mathbf{v}_\lambda$ is an eigenvector with eigenvalue $\lambda - 1$.

Rather than go through a full exposition on how to determining the properties of an irreducible $SL(2, \mathbb{C})$ representation we will skip straight to the model used to construct them.

There are two well-known two models that could be used, the first is to take a 2×2 complex matrix

$$M = \begin{pmatrix} a & c \\ b & d \end{pmatrix}$$

and act on the complex polynomial

$$p(z) = a_0 + \binom{n}{1}a_1 z + \binom{n}{2}a_2 z^2 + \ldots + a_n z^n$$

via

$$\widehat{p}(\widehat{z}) = (b\widehat{z} + d)^n p\left(\frac{a\widehat{z} + c}{b\widehat{z} + d}\right). \tag{A.1.8}$$

This identification induces an action of the coefficients of $p(z)$. This description gives a nice way to think about parameterising the representations by considering the roots of the polynomial.

The second, equivalent, model is is to work instead in homogeneous coordinates. Computationally, this description is easier to work with. Consider the binary form

$$Q(r,s) = a_0 s^n + \binom{n}{1} a_1 r s^{n-1} + \binom{n}{2} a_2 r^2 s^{n-2} + \ldots + a_n r^n. \tag{A.1.9}$$

The action of M on the coordinates will be defined to be

$$\begin{aligned} \widehat{r} &= ar + br, \\ \widehat{s} &= cs + ds. \end{aligned} \tag{A.1.10}$$

The identification

$$\widehat{Q}(\widehat{r},\widehat{s}) = Q(\widehat{r},\widehat{s}) \tag{A.1.11}$$

induces the same action as (A.1.8). The explicit link between these two model is given by the equation

$$s^n p\left(\frac{r}{s}\right) = Q(r,s).$$

In terms of the matrices e, h, f the action on r, s will be given by the coordinate changes

$$\begin{aligned} \mathbf{r} &= \exp(te)\widehat{\mathbf{r}}, \\ \mathbf{r} &= \exp(tf)\widehat{\mathbf{r}}, \\ \mathbf{r} &= \exp(th)\widehat{\mathbf{r}}. \end{aligned} \tag{A.1.12}$$

By defining the action of the Lie algebra via $\left.\frac{\partial \widehat{\mathbf{r}}}{\partial t}\right|_{t=0}$ and noting

$$\exp(tg)^{-1} = \exp(-tg)$$

yields the relations

$$\begin{aligned} e(r) &= s, & h(r) &= -\frac{r}{2}, & f(r) &= 0, \\ e(s) &= 0, & h(s) &= \frac{s}{2}, & f(s) &= s, \end{aligned} \tag{A.1.13}$$

which can be verified to satisfy the commutation relations (A.1.6). This action can be expressed using the operators

$$\begin{aligned} J_+ &= s\partial_r, \\ J_0 &= \frac{1}{2}\left(-r\partial_r + s\partial_s\right), \\ J_- &= r\partial_s, \end{aligned} \tag{A.1.14}$$

which will be convenient when applying this action to higher order combinations. This algebra has a Casimir given by

$$C = \frac{1}{2}\left(r\partial_r + s\partial_s\right)$$

which is related to the other operators via

$$J_1^2 + J_2^2 + J_3^2 = C(C+1).$$

Returning to the binary form (A.1.9), the action of the matrix M on the coefficients can be defined via the relation

$$\widehat{Q}(\widehat{\mathbf{a}}, \widehat{\mathbf{r}}) = Q\left(\mathbf{a}, M\widehat{\mathbf{r}}\right).$$

The corresponding Lie algebra action is given by

$$\begin{aligned} J_+(a_k) &= -(n-k)a_k \\ J_0(a_k) &= \left(k - \frac{n}{2}\right) a_k \\ J_-(a_k) &= -ka_k \end{aligned} \tag{A.1.15}$$

And finally, returning to the polynomial point of view, the action of (A.1.8) can be given by mapping the roots η_i of $p(z)$ to

$$\eta_i \to \frac{d\eta_i - c}{-b\eta_i + a}$$

and scaling the leading coefficient by

$$a_6 \to \prod_{i=1}^{n}(a - b\eta_i)a_6.$$

These binary forms define all the irreducible $SL(2,\mathbb{C})$ representations, however since $SL(2,\mathbb{C})$ is a double cover of $SO(3,\mathbb{C})$ one final point must be clarified. When $t = 2\pi$ (i.e. a full rotation) the matrices (A.1.3) are all the 3×3 identity matrix. However at $t = 2\pi$ the matrices (A.1.4) are the 2×2 anti-identity matrix $-I$. The action of $-I$ on the nth degree binary form (A.1.9) is to scaled it by $(-1)^n$, for this to make sense n must be an even integer. This reveals the well known fact that $SO(3,\mathbb{C})$ only has odd-dimensional representations.

A.2 The covariants of $Y(r, s)$

In chapter 4 the $SO(3,\mathbb{C})$ representation $\mathcal{Y}(r,s)$ is shown to also be a representation of the local action of the conformal group. This representation is isomorphic to one for $GL(2,\mathbb{C})$ for the action described above. It is natural therefore to talk about the covariants of $\mathcal{Y}(r,s)$, that is, combinations of r, s and the coefficients of $\mathcal{Y}(r,s)$ such that the action of a matrix in $GL(2,\mathbb{C})$ of is just some multiple of the determinant. A Hilbert basis for these covariants in given in appendix B.

The only difference between the $SL(2,\mathbb{C})$ representations above and the $GL(2,\mathbb{C})$ representations (i.e. the covariants) is given by the determinant. To each covariant a weight can be assigned, for example, the covariant B_0 (defined in appendix B) has covariant weight 6. So the action of a matrix with determinant Δ is given by

$$B_0 \mapsto \Delta^6 B_0.$$

Similarly D_0, F_0, H_0, J_0 and O_0 have covariant weights $12, 18, 24, 30$ and 90 repsectively. This information is important in chapter 5 as it allow the absolute invariants[1] $\mathbf{I} = \left(\frac{D_0}{B_0^2}, \frac{F_0}{B_0^3}, \frac{J_0}{B_0^5}\right)$ to be defined.

[1]Absolutely invariant under the local action of the conformal group (i.e. $GL(2,\mathbb{C})$)

Appendix B

A Hilbert Basis for the Binary Sextic

The polynomial ideals that arise in this thesis form sets of $\mathfrak{so}(3,\mathbb{C}) \simeq \mathfrak{sl}(2,\mathbb{C})$ representations created from the symmetric tensor products of a single irreducible 7-dimensional representation. From the discussion in appendix A it is clear the ideals can be described in terms of the covariants of a 6th order binary form. It has been known almost a century and a half that every covariant of a binary form can be be written as a (non-unique) polynomial in a finite set of basis covariants [32]. Abstractly, such a basis is guaranteed to exist by Hilbert's basis theorem (which applies to any Noetherian ring) but more concretely, a basis can be constructed using Gordan's method (see P. Olver, chapter 7 [49]).

The only tool we really need for constructing the Hilbert basis is the transvectant operator[1]

$$(A, B)^{[q]} = \frac{1}{(q!)^2} \sum_{k=0}^{q} (-1)^k \binom{q}{k} \frac{\partial^q A}{\partial r^k \partial s^{q-k}} \frac{\partial^q B}{\partial r^{q-k} \partial s^k}. \tag{B.0.1}$$

Defined this way the transvectant can be applied to any two functions of r,s. Any covariant can now be written as a linear combination of iterated transvectants. A simple example is given by the determinant of the Hessian

[1]the algorithm actual uses a closely related operator, the partial transvectant. However partial transvectants can be rexpressed in terms of full transvectants.

matrix (also known as the Hessian) which can be expressed as

$$H(f) = \det \begin{vmatrix} f_{,rr} & f_{,rs} \\ f_{,sr} & f_{,ss} \end{vmatrix} = f_{,rr}f_{,ss} - f_{,rs}^2 = 2\,(f,f)^{[2]}\,.$$

B.1 The Hilbert Basis

The ring of covariants of a degree-six polynomial has a Hilbert basis consisting of twenty-six elements [57], but such a basis is not (to the best of the author's knowledge) canonically defined. As such a particular choice will need to be given explicitly. Letting $\mathcal{Y}(r,s)$ to be 6th degree binary form defined at (3.2.8) the Hilbert basis used in this thesis is defined in the following recursive manner

$$\begin{aligned}
A_3 &= \mathcal{Y}, & & \\
B_4 &= 4(A_3,A_3)^{[2]}, & B_2 &= 576(A_3,A_3)^{[4]}, \\
B_0 &= 518400(A_3,A_3)^{[6]}, & & \\
C_6 &= \frac{1}{2}(A_3,B_4)^{[1]}, & C_4 &= \frac{1}{2}(A_3,B_2)^{[1]}, \\
C_3 &= \frac{2}{3}(A_3,B_2)^{[2]} + 5A_3B_0, & C_1 &= 96(A_3,B_2)^{[4]}, \\
D_5 &= \frac{2}{3}(A_3,C_3)^{[1]}, & D_3 &= (A_3,C_1)^{[1]}, \\
D_2 &= 4(A_3,C_1)^{[2]}, & D_0 &= 34560(A_3,C_3)^{[6]}, \\
E_4 &= (A_3,D_2)^{[1]}, & E_2 &= 36(A_3,D_2)^{[3]}, \\
E_1 &= 576(A_3,D_2)^{[4]}, & & \\
F_3^{(1)} &= (A_3,E_1)^{[1]}, & F_3^{(2)} &= -\frac{5}{2}(A_3,E_1)^{[1]} + 2(A_3,E_2)^{[2]}, \\
F_0 &= 4(C_1,C_1)^{[2]}, & & \\
G_2 &= \frac{1}{2}(B_2,E_1)^{[1]}, & G_1 &= 4(C_1,D_2)^{[2]}, \\
H_1 &= -48(A_3,G_2)^{[4]}, & & \\
I_2 &= 4(A_3,H_1)^{[2]}, & & \\
J_1 &= 576(A_3,I_2)^{[4]}, & J_0 &= 4(C_1,G_1)^{[2]}, \\
L_1 &= \frac{2}{3}(B_2,J_1)^{[2]} + \frac{1}{3}B_0J_1, & & \\
O_0 &= 4(L_1,C_1)^{[2]}. & & \qquad \text{(B.1.1)}
\end{aligned}$$

The notation chosen here (consisting of a capital letter with a numerical subscript) indicate of the type of representation each covariant represents. The position of the letter in the English alphabet is a reflection of the polynomial degree of the coefficients and the subscript indicates the weight of the highest-weight vector. This convention is also used when describing the ideals in chapter 5 with the addition of descriptive superscripts to distinguish them from the elements of the Hilbert basis.

Appendix C

Notation

This is a brief review of notation used in the thesis. Tables C.1 and C.2 can be used as a quick guide to most of the symbols used in this thesis.

Symbol	**Meaning**	**Definition**
$A_{,i}$	The partial derivative of A with respect to x_i	
λ	The conformal factor of the conformally flat metric	(2.3.1)
K	The sectional curvature	(2.5.2)
K_n	The nth order part of a constant (contextually dependent upon which constant is under examination)	§3.3
$\{\mathbf{Q}, \mathbf{R}, \mathbf{S}\}$	A linearly independent and symmetric choice of coefficient functions for the superintegrable systems	(2.3.16)
$\{\boldsymbol{\mathcal{Q}}, \boldsymbol{\mathcal{R}}, \boldsymbol{\mathcal{S}}\}$	A linearly independent and symmetric choice of coefficient functions for the conformally-superintegrable systems	(4.1.10)
J_+, J_-, J_0	The raising, lowering and level set operators	(3.1.7)
$\mathcal{Y}(r,s)$	The 7-dimensional representations constructed from the $\{Q, R\}$ variables	(3.2.8)
$\mathcal{X}(r,s)$	The 3-dimensional representations constructed from the $\{S\}$ variables	(3.2.9)

Table C.1: Symbols used in this thesis with their places of definition.

Symbol	Meaning	Definition
η_i	The roots of polynomial $p(z) = \mathcal{Y}(z, 1)$	
$\mathcal{Z}(r, s)$	The 3-dimensional representations constructed from the first-order derivatives of the conformal-factor $\lambda = \exp(G)$	(3.2.10)
$\mathcal{Z}_{ee}$	The 1-dimensional representations constructed from the second-order derivatives of the conformal-factor $\lambda = \exp(G)$	(3.2.11)
$\mathcal{P}(r, s)$	The 3-dimensional representations constructed from the momenta	(3.2.4)
$\mathcal{V}(r, s)$	The 3-dimensional representations constructed from the potential's parameters	(3.2.6)
$\mathcal{V}_{ee}$	The 1-dimensional representations constructed from the potential's parameters	(3.2.7)
$(m)_n$	Shorthand for n representations covering a space of m variables	§3.3
$\{A, B\}_{PB}$	The Poisson-Bracket/Poisson Commutator	(2.1.5)
$(A, B)^{[n]}$	The transvectant	(3.1.14)
$\partial_+, \partial_-, \partial_0$	The raising,lowering and level-set derivatives	(3.1.15)
$\partial_+^C, \partial_-^C, \partial_0^C$	The raising,lowering and level-set constructor derivatives	(3.1.25)
$\langle A, B, \ldots \rangle$	The polynomial ideal generated by $A, B, \ldots$	5.1.1
$\sqrt{I}$	The radical ideal of the ideal I	5.1.7
$\overline{I}$	The differential closure of the ideal I	5.1.8

Table C.2: More symbols used in this thesis with their places of definition.

Bibliography

[1] K. Valiev A.A. Makarov, Y. A. Smorodinsky and P. Winternitz. A systematic search for nonrelativistic systems with dynamical symmetries. *Il Nuovo Cimento A Series 10*, 52(4):1061–1084, 1967.

[2] V. I. Arnold. *Mathematical methods of classical mechanics*, volume 60. Springer, 1989.

[3] G. G. Blado. Supersymmetry and the hartmann potential of theoretical chemistry. *Theoretica chimica acta*, 94(1):53–66, 1996.

[4] M. Bôcher. *Über die Reihenentwickelungen der Potentialtheorie*. Teubner, 1894.

[5] C. P. Boyer, E. G. Kalnins, and W. Miller Jr. Stäckel-equivalent integrable hamiltonian systems. *SIAM Journal on Mathematical Analysis*, 17(4):778–797, 1986.

[6] J. V. Chipalkatti. Invariant equations defining coincident root loci. *Arch. Math. (Basel)*, 83(5):422–428, 2004.

[7] C. Daskaloyannis. Quadratic poisson algebras of two-dimensional classical superintegrable systems and quadratic associative algebras of quantum superintegrable systems. *Journal of Mathematical Physics*, 42(3):1100–1119, 2001.

[8] C. Daskaloyannis and Y. Tanoudis. Quadratic algebras for three-dimensional superintegrable systems. *Physics of Atomic Nuclei*, 73(2):214–221, 2010.

[9] G.-M.; Pfister G.; Schönemann H. Decker, W.; Greuel. Singular 3-1-6 — A computer algebra system for polynomial computations. 2012. http://www.singular.uni-kl.de.

[10] J. M. Kress E. G. Kalnins and W. Miller Jr. Second-order superintegrable systems in conformally flat spaces. i. two-dimensional classical structure theory. *Journal of mathematical physics*, 46:053509, 2005.

[11] J. M. Kress E. G. Kalnins and W. Miller Jr. Second order superintegrable systems in conformally flat spaces. ii. the classical two-dimensional stäckel transform. *Journal of mathematical physics*, 46:053510, 2005.

[12] J. M. Kress E. G. Kalnins and W. Miller Jr. Second order superintegrable systems in conformally flat spaces. iii. three-dimensional classical structure theory. *Journal of mathematical physics*, 46:103507, 2005.

[13] J. M. Kress E. G. Kalnins and W. Miller Jr. Second order superintegrable systems in conformally flat spaces. iv. the classical 3d stäckel transform and 3d classification theory. *Journal of mathematical physics*, 47:043514, 2006.

[14] J. M. Kress E. G. Kalnins and W. Miller Jr. Second-order superintegrable systems in conformally flat spaces. v. two-and three-dimensional quantum systems. *Journal of mathematical physics*, 47:093501, 2006.

[15] J. M. Kress E. G. Kalnins and W. Miller Jr. Fine structure for 3d second-order superintegrable systems: three-parameter potentials. *Journal of Physics A: Mathematical and Theoretical*, 40(22):5875, 2007.

[16] J. M. Kress E. G. Kalnins and W. Miller Jr. Nondegenerate 2d complex euclidean superintegrable systems and algebraic varieties. *Journal of Physics A: Mathematical and Theoretical*, 40(13):3399, 2007.

[17] J. M. Kress E. G. Kalnins and W. Miller Jr. Nondegenerate three-dimensional complex euclidean superintegrable systems and algebraic varieties. *Journal of Mathematical Physics*, 48:113518, 2007.

[18] J. M. Kress E. G. Kalnins and W. Miller Jr. Superintegrability in a non-conformally-flat space. *Journal of Physics A: Mathematical and Theoretical*, 46(2):022002, 2013.

[19] J. M. Kress E. G. Kalnins and Jr W. Miller. Tools for verifying classical and quantum superintegrability. *SIGMA*, 6(066):23, 2010.

[20] Jr. E. G. Kalnins, W. Miller and M. V. Tratnik. Families of orthogonal and biorthogonal polynomials on the n-sphere. *SIAM Journal on Mathematical Analysis*, 22(1):272–294, 1991.

[21] W. Miller Jr. E. G. Kalnins and G. S. Pogosyan. Superintegrability on the two-dimensional hyperboloid. 1997.

[22] W. Miller Jr. E. G. Kalnins and G. S. Pogosyan. Superintegrability and higher-order constants for classical and quantum systems. *Physics of Atomic Nuclei*, 74(6):914–918, 2011.

[23] W. Miller Jr E. G. Kalnins and S. Post. Coupling constant metamorphosis and nth-order symmetries in classical and quantum mechanics. *Journal of Physics A: Mathematical and Theoretical*, 43(3):035202, 2010.

[24] W. Miller Jr. E. G. Kalnins and S. Post. Two-variable wilson polynomials and the generic superintegrable system on the 3-sphere. *SIGMA. Symmetry, Integrability and Geometry: Methods and Applications*, 7:051, 2011.

[25] W. Miller Jr E. G. Kalnins and S. Post. Contractions of 2d 2nd order quantum superintegrable systems and the askey scheme for hypergeometric orthogonal polynomials. *arXiv preprint arXiv:1212.4766*, 2012.

[26] W. Miller Jr E. G. Kalnins, J. M. Kress, , and G. S. Pogosyan. Nondegenerate superintegrable systems in n-dimensional euclidean spaces. *Physics of Atomic Nuclei*, 70(3):545–553, 2007.

[27] W. Miller Jr E. G. Kalnins, J. M. Kress and S. Post. Laplace-type equations as conformal superintegrable systems. *Advances in Applied Mathematics*, 46(1):396–416, 2011.

[28] W. Miller Jr E. G. Kalnins, J. M. Kress and P. Winternitz. Superintegrable systems in darboux spaces. *Journal of Mathematical Physics*, 44:5811, 2003.

[29] Y. M. Hakobyan E. G. Kalnins, W. Miller Jr and G. S. Pogosyan. Superintegrability on the two-dimensional hyperboloid. ii. *Journal of Mathematical Physics*, 40:2291, 1999.

[30] N. W. Evans. Superintegrability in classical mechanics. *Physical Review A*, 41(10):5666, 1990.

[31] A. V. Turbiner F. Tremblay and P. Winternitz. An infinite family of solvable and integrable quantum systems on a plane. *Journal of Physics A: Mathematical and Theoretical*, 42(24):242001, 2009.

[32] P. Gordan. Beweis, dass jede covariante und invariante einer binären form eine ganze function mit numerischen coefficienten einer endlichen anzahl solcher formen ist. *Journal für die reine und angewandte Mathematik*, 69:323–354, 1868.

[33] B. Hassett. *Introduction to algebraic geometry.* Cambridge University Press, 2007.

[34] Y. A. Smorodinsky M. Uhlíř J. Friš, V. Mandrosov and P. Winternitz. On higher symmetries in quantum mechanics. *Physics Letters*, 16(3):354–356, 1965.

[35] B. Dorizzi J. Hietarinta, B. Grammaticos and A. Ramani. Coupling-constant metamorphosis and duality between integrable hamiltonian systems. *Physical review letters*, 53(18):1707–1710, 1984.

[36] W. Miller Jr. Second order superintegrable systems in three dimensions. *SIGMA. Symmetry, Integrability and Geometry: Methods and Applications*, 1:015, 2005.

[37] E. G. Kalnins, J. M. Kress, and W. Miller Jr. A recurrence relation approach to higher order quantum superintegrability. *SIGMA*, 7(031):24, 2011.

[38] E.G. Kalnins and Jr Willard Miller. Killing tensors and variable separation for hamilton-jacobi and helmholtz equations. *SIAM Journal on Mathematical Analysis*, 11(6):1011–1026, 1980.

[39] M. Kibler and P. Winternitz. Dynamical invariance algebra of the hartmann potential. *Journal of Physics A: Mathematical and General*, 20(13):4097, 1987.

[40] A. D. King and W. K. Schief. Tetrahedra, octahedra and cubo-octahedra: integrable geometry of multi-ratios. *Journal of Physics A: Mathematical and General*, 36(3):785, 2003.

[41] P. Letourneau and L. Vinet. Superintegrable systems: polynomial algebras and quasi-exactly solvable hamiltonians. *Annals of Physics*, 243(1):144–168, 1995.

[42] P. A. Damianou M. A. Agrotis and C. Sophocleous. The toda lattice is super-integrable. *Physica A: Statistical Mechanics and its Applications*, 365(1):235–243, 2006.

[43] G. H. Lamot M. Kibler and P. Winternitz. Classical trajectories for two ring-shaped potentials. *International journal of quantum chemistry*, 43(5):625–645, 1992.

[44] I. Marquette. Superintegrability with third order integrals of motion, cubic algebras, and supersymmetric quantum mechanics. ii. painlevé transcendent potentials. *Journal of Mathematical Physics*, 50:095202, 2009.

[45] I. Marquette. Supersymmetry as a method of obtaining new superintegrable systems with higher order integrals of motion. *Journal of Mathematical Physics*, 50:122102, 2009.

[46] I. Marquette. Construction of classical superintegrable systems with higher order integrals of motion from ladder operators. *Journal of Mathematical Physics*, 51:072903, 2010.

[47] I. Marquette. Classical ladder operators, polynomial poisson algebras, and classification of superintegrable systems. *Journal of Mathematical Physics*, 53:012901, 2012.

[48] D. Mumford. *Algebraic geometry*. Springer Berlin Heidelberg New York, 1976.

[49] P. J. Olver. *Classical invariant theory*, volume 44 of *London Mathematical Society Student Texts*. Cambridge University Press, Cambridge, 1999.

[50] A. V. Turbiner P. Tempesta and P. Winternitz. Exact solvability of superintegrable systems. *Journal of Mathematical Physics*, 42:4248, 2001.

[51] S. Post. *Models of second-order superintegrable systems*. PhD thesis, UNIVERSITY OF MINNESOTA, 2009.

[52] S. Post. Coupling constant metamorphosis, the stäckel transform and superintegrability. In *AIP Conference Proceedings*, volume 1323, page 265, 2010.

[53] S. Post and P. Winternitz. A nonseparable quantum superintegrable system in 2d real euclidean space. *Journal of physics. A, Mathematical and theoretical*, 44(16), 2011.

[54] C. Quesne. Superintegrability of the tremblay–turbiner–winternitz quantum hamiltonians on a plane for odd k. *Journal of Physics A: Mathematical and Theoretical*, 43(8):082001, 2010.

[55] S. Tsujimoto S. Post and L. Vinet. Families of superintegrable hamiltonians constructed from exceptional polynomials. *Journal of Physics A: Mathematical and Theoretical*, 45(40):405202, 2012.

[56] P. Stäckel. *Über die integration der Hamilton-Jacobischen differentialgleichung mittelst separation der variabeln...* Halle., 1891.

[57] J. J. Sylvester and F. Franklin. Tables of the generating functions and groundforms for the binary quantics of the first ten orders. *American Journal of Mathematics*, 2(3):223–251, 1879.

[58] A. V. Turbiner. Quasi-exactly-solvable problems and sl (2) algebra. *Communications in mathematical physics*, 118(3):467–474, 1988.

[59] A. V. Turbiner and A. G. Ushveridze. Spectral singularities and quasi-exactly solvable quantal problem. *Physics Letters A*, 126(3):181–183, 1987.

[60] S. Post W. Miller Jr and P. Winternitz. Classical and quantum superintegrability with applications. *arXiv preprint arXiv:1309.2694*, 2013.

[61] P. Winternitz, M. Uhlíř Y. A. Smorodinsky, and J. Friš. Symmetry groups in classical and quantum mechanics. *Yadern. Fiz.*, 4, 1966.

[62] S. Wojciechowski. Superintegrability of the calogero-moser system. *Physics Letters A*, 95(6):279–281, 1983.