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# Running and Running of the Running of the Scalar Spectral Index in Warm Inflation

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**Abstract:** The next generation of cosmological observations are expected to improve the measurements of several quantities connected to the primordial inflation in the early Universe. These quantities include, for example, improved measurements for the spectral index of the scalar curvature of the primordial power spectrum and to also bring a better understanding on the scaling dependence of the primordial spectrum. This includes the running of the tilt and possibly, also, the running of the running. In this paper, we investigate the possibility of generating large runnings in the context of warm inflation. Useful analytical expressions for the runnings are derived in the context of warm inflation in the large dissipation regime. The results are compared to and discussed for some well-motivated primordial inflaton potentials that have recently been of interest in the literature.

**Keywords:** inflation; warm inflation; cosmic microwave background

## 1. Introduction

The final release of the *Planck* Cosmic Microwave Background (CMB) data [1] has severely constrained many primordial observables, such as the scalar spectral index, $n_s$, the tensor-to-scalar ratio, $r$, the primordial non-Gaussianities and the primordial isocurvature spectrum, leaving us with the simplest vanilla inflationary models as the most preferred ones. However, the presence of the running, $\alpha_s$, and the running of running, $\beta_s$, of the scalar spectral index that were inferred by the *Planck* data may hint otherwise. For the $\Lambda$CDM model, the *Planck* 2018 TT(TT,TE,EE)+lowE+lensing data constrains the scalar spectral index, its running and the running of its running as [1]

$$\begin{aligned} n_s &= 0.9587 \pm 0.0056 \ (0.9625 \pm 0.0048), \\ \alpha_s &= 0.013 \pm 0.012 \ (0.002 \pm 0.010), \\ \beta_s &= 0.022 \pm 0.012 \ (0.010 \pm 0.013), \end{aligned} \tag{1}$$

all at 68% CL. According to *Planck*, such values for the running and running of running of the scalar spectral index yield a better fit to the low-$\ell$ deficit of the TT spectrum. It is interesting to note that in these constraints there is a slight preference for a positive and rather large $\beta_s \sim 10^{-2}$, which is also larger than the running of the spectral index ($\beta_s > \alpha_s$). This is unexpected, when compared, for instance, to the values derived from standard inflationary models [2], where a hierarchy such as $n_s \gg \alpha_s \gg \beta_s$ is expected. It has been shown in Refs. [3,4] that if we consider the empirical relation $n_s - 1 \propto 1/N$, where $N$ is the number of $e-$foldings, then in these vanilla models one expects

$$|\alpha_s| \sim \frac{1}{N^2} \lesssim 10^{-4}, \qquad |\beta_s| \sim \frac{1}{N^3} \lesssim 10^{-5}. \tag{2}$$

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The *Planck*-preferred vanilla inflationary models in general also produce negative $\alpha_s$ and $\beta_s$. Therefore, such inflation models have been considered to be in tension with the current observations by *Planck* [4,5]. Going beyond these vanilla models does not seem to improve the scenario either. It was shown in Ref. [5] that single field inflation models and non-interacting two-field models are incapable of producing a $\beta_s$ larger than $\alpha_s$. Slow-roll violating models and those with non-trivial evolution of sound speed could be able to provide an exception to this trend, but with a considerable amount of fine-tuning [5].

The *Planck*-preferred vanilla inflaton potential models in general consider the traditional cold inflation (CI) scenario, where the dynamics of the inflaton field $\phi$ is assumed to be independent of its couplings with other fields during inflation. However, such interaction terms play a major role at the end of inflation, when the inflaton field needs to release its energy density in the form of radiation through decay processes to reheat the Universe and, hence, leading to a radiation dominated Universe as required by the big bang cosmology. On the other hand, in the warm inflation (WI) scenario [6], the couplings between the inflaton and other fields are strong enough such that their effects on the inflaton dynamics cannot be ignored. In WI (for reviews on WI, see, e.g., Refs. [7,8]), the inflaton field is able to keep dissipating its energy such that a non-negligible radiation bath throughout inflation can be produced, while preserving the flatness of the inflaton potential required for slow-rolling of the inflaton field. The presence of such a quasi-equilibrium thermal radiation bath during WI helps transiting smoothly from the inflationary accelerated phase to the radiation dominated phase after inflation ends, avoiding a phase of reheating in between. In WI, the inflaton dynamics are modified with respect to the one in CI due to the presence of an extra friction term, $\Upsilon\dot{\phi}$, that accounts for the energy transfer between the inflaton field and the radiation bath present during inflation. Due to such modified dynamics, the inflationary observables, such as $r$, $n_s$ and the non-Gaussianity parameter $f_{\rm NL}$, are also modified with respect to those obtained in CI. These changes have certain advantages. For example, some of the inflaton potentials excluded by data in the context of CI can be made in tune with the observations in the WI context. One such example is the quartic chaotic potential $\lambda\phi^4$ [9]. Moreover, it has recently been shown that, while CI fails to comply with the recently proposed Swampland Conjectures in String Theory [10–12], WI can easily accommodate those criteria [13–18]. Hence, WI provides a way to construct inflationary models that can be consistent as effective models that could descend from an ultraviolet complete quantum gravity, despite the swampland conjectures barring them from constructing de Sitter vacua in String Landscapes.

The running and the running of the running of the scalar spectral index in the context of WI were first studied in Ref. [19], where several inflationary potentials were analyzed with two different forms of dissipative terms considered in WI, $\Upsilon_{\rm cubic} \propto T^3/\phi^2$ and $\Upsilon_{\rm linear} \propto T$ ($T$ being the temperature of the thermal bath). The WI models studied in Ref. [19] were, however, treated in the weak dissipative regime ($Q \ll 1$, where $Q$ is the ratio of the thermal friction term $\Upsilon\dot{\phi}$ and the Hubble expansion friction term $3H\dot{\phi}$ present in the inflaton equation of motion in WI). This was because the models studied in that reference could only lead to consistent observables, e.g., values for $r$ and $n_s$, in that specific dissipation regime of WI. In particular, it was shown in Ref. [19] that in all the models studied there, in the weak dissipative regime, there was still a large hierarchy between the values of $\alpha_s$ and $\beta_s$.

The aim of this paper is to study the running and the running of the running of the scalar spectral index in the context of WI, where WI is realized in a strong dissipative regime ($Q \gg 1$). This is motivated by the recent results in WI in the context of the swampland conjectures [13–16] and also on the solution of the so-called $\eta$-problem [20], which exactly favors WI being realized in the strong dissipative regime. For this purpose, we will focus mostly on a certain WI model, dubbed the Minimal Warm Inflation (MWI) [21], where WI has been shown to be possible to be realized in the strong dissipative regime and in which the dissipative term is proportional to the cubic power of the temperature of the thermal bath ($\Upsilon \propto T^3$). However, our results will be kept as general as possible, so that they can be extended to other models. In the original paper of MWI [21], a hybrid potential was used to produce a red-tilted scalar power spectrum. Later, MWI was studied with generalized exponential potentials in Ref. [22], and it was shown that this model is not only in accordance with the current observations (yielding the appropriate values for $n_s$ and $r$), but also are in tune with the swampland conjectures. We will analyze both the original MWI model, with the hybrid potential and the one with the generalized exponential potential to derive $\alpha_s$ and $\beta_s$ that are produced in these two scenarios. Note that although the running and the running of the running of the scalar spectral index in the context of WI were first studied in Ref. [19], they were estimated numerically. Here, however, we aim to produce explicit analytical expressions for these quantities in WI. To the best of our knowledge, this is the first time that such analytical analysis and derivation of $\alpha_s$ and $\beta_s$ in the context of WI are presented. Given the advent of new generations of cosmological observatories probing both the cosmic microwave background (CMB) measurements [23,24], the distribution of matter at low-redshift from optical, near-infrared and 21 cm intensity surveys [25], it is expected that theories of cosmic inflation will be further constrained with the more precise cosmological data. Thus, it is important to have such analytical expressions to help in our ability in finding possible models of interest and also in model building in WI.

The remainder of the paper is organized as follows. In Section 2, we briefly review WI and present the general expressions for $n_s$, $\alpha_s$ and $\beta_s$. In Section 3, we analyze the case of MWI with a hybrid potential. In Section 4, we turn our attention to the determination of $n_s$, $\alpha_s$ and $\beta_s$ in the recently studied generalized exponential potentials for the inflation in WI. Then, in Section 5, we discuss our findings and conclude.

## 2. Running and Running of the Running of Scalar Spectral Index in WI—General Expressions

The equation of motion of the inflaton field $\phi$ in WI is

$$\ddot{\phi} + 3H(1+Q)\dot{\phi} + V_{,\phi} = 0, \tag{3}$$

where $Q$ is the ratio of the two frictional terms, the friction due thermal bath and the friction due to Hubble expansion, present in the theory,

$$Q \equiv \frac{\Upsilon(T,\phi)}{3H}, \tag{4}$$

where the dissipation coefficient $\Upsilon \equiv \Upsilon(T,\phi)$ is in general a function of the temperature of the thermal radiation bath generated and can also depend on the inflaton amplitude (for examples of different forms of dissipation coefficients derived in quantum field theory and used in WI, see, e.g., Ref. [26]). In Equation (3), $V_{,\phi} \equiv dV/d\phi$, where $V$ is the potential of the inflaton field. As the inflaton dissipates part of its energy density, it can sustain a radiation bath, with energy density $\rho_r$ and whose evolution equation is given by

$$\dot{\rho}_r + 4H\rho_r = 3HQ\dot{\phi}^2. \tag{5}$$

We define the slow-roll parameters $\epsilon_V$ and $\eta_V$ in the usual way,

$$\epsilon_V = \frac{M_{\rm Pl}^2}{2}\left(\frac{V_{,\phi}}{V}\right)^2, \tag{6}$$

$$\eta_V = M_{\rm Pl}^2 \frac{V_{,\phi\phi}}{V}, \tag{7}$$

where $M_{\rm Pl} \equiv 1/\sqrt{8\pi G} \simeq 2.4 \times 10^{18}$ GeV is the reduced Planck mass. In WI, another set of slow-roll parameters are also defined,

$$\epsilon_{\rm WI} = \frac{\epsilon_V}{1+Q}, \tag{8}$$

$$\eta_{\rm WI} = \frac{\eta_V}{1+Q}, \tag{9}$$

due to the fact that WI actually ends when $\epsilon_{\rm WI} \sim 1$, or, similarly, when $\epsilon_V \sim 1+Q$, while in CI the usual condition for the end of the accelerated inflationary regime is simply $\epsilon_V \sim 1$.

Besides $\epsilon_V$ and $\eta_V$, it is also useful to define the higher order slow-roll coefficients $\xi_V^2$ and $\omega_V^3$, as [27]

$$\xi_V^2 \equiv M_{\rm Pl}^4 \frac{V_{,\phi} V_{,\phi\phi\phi}}{V^2}, \tag{10}$$

and

$$\omega_V^3 = M_{\rm Pl}^6 \frac{V_{,\phi}^2 V_{,\phi\phi\phi\phi}}{V^3}, \tag{11}$$

which will appear in the equations for the running and for the running of the running to be derived later on.

The primordial scalar curvature power spectrum of a typical WI model can be written as [19,28]

$$\Delta_{\mathcal{R}}(k/k_*) = \left(\frac{H_*^2}{2\pi\dot{\phi}_*}\right)^2 \mathcal{F}(k/k_*), \tag{12}$$

where the subindex $*$ means that the quantities are evaluated at the Hubble radius crossing of the pivot scale $k_*$. Here, the function $\mathcal{F}(k/k_*)$ is given as

$$\mathcal{F}(k/k_*) \equiv \left(1 + 2n_* + \frac{2\sqrt{3}\pi Q_*}{\sqrt{3+4\pi Q_*}} \frac{T_*}{H_*}\right) G(Q_*), \tag{13}$$

where $n_*$ is the thermal distribution of the inflaton field due to the presence of the radiation bath and $G(Q_*)$ accounts for the effect of the coupling of the inflaton and radiation fluctuations [29–31].

The primordial scalar power spectrum $\Delta_{\mathcal{R}}(k/k_*)$ can be expanded in terms of the small scale-dependence as

$$\Delta_{\mathcal{R}}(k/k_*) \simeq A_s \left(\frac{k}{k_*}\right)^{n_s - 1 + \frac{\alpha_s}{2}\ln(k/k_*) + \frac{\beta_s}{6}\ln^2(k/k_*)}, \tag{14}$$

where $A_s$ is the scalar amplitude, $n_s$ is the scalar tilt, $\alpha_s$ is the running, and $\beta_s$ is the running of the running (or second running).

From Equation (14), the scalar spectral index $n_s$ can be calculated at the horizon crossing ($k = k_*$) as

$$n_s - 1 = \frac{d\ln\Delta_{\mathcal{R}}(k/k_*)}{d\ln(k/k_*)}\Big|_{k\to k_*}. \tag{15}$$

Likewise, the expressions for $\alpha_s$ and $\beta_s$ are determined, respectively, by

$$\alpha_s = \frac{dn_s(k/k_*)}{d\ln(k/k_*)}\Big|_{k\to k_*}, \tag{16}$$

and

$$\beta_s = \frac{d^2 n_s(k/k_*)}{d\ln(k/k_*)^2}\Big|_{k\to k_*} = \frac{d\alpha_s(k/k_*)}{d\ln(k/k_*)}\Big|_{k\to k_*}. \tag{17}$$

Using Hubble radius crossing $k_* = aH$, we can write $d\ln k = (d\ln k/dN)dN$, with $N = \ln a$ being the number of $e$-folds and $\epsilon_H = -\dot{H}/H^2$, then

$$\frac{d\ln k}{dN} = 1 - \epsilon_H \approx 1 - \epsilon_V/(1+Q). \tag{18}$$

The definitions given by Equations (15)–(17) can now be applied directly to the scalar curvature in WI, Equation (12). Once an appropriate functional form for the dissipation coefficient is given, explicit expressions for $n_s$, $\alpha_s$ and $\beta_s$ can be derived. Here, we work with the well-motivated functional form for the dissipation coefficient[1]

$$\Upsilon(\phi, T) = C_\Upsilon\, T^p \phi^c \mathcal{M}^{1-p-c}, \tag{19}$$

where $C_\Upsilon$ is a constant, $T$ is the temperature, and $\mathcal{M}$ is some appropriate mass scale. For specific examples of microscopic quantum field theory derivations of such dissipation coefficients in WI, see, e.g., Refs. [7,34–37].

When taking derivatives of Equation (12) such as to obtain $n_s$, $\alpha_s$ and $\beta_s$, we are naturally faced with accounting for derivatives on $Q$ and $T/H$ with respect to $N$. These expressions depend on the specific form of the dissipation coefficient used. For completeness, let us quote them here for the generic dissipation term Equation (19). We use the slow-roll approximation for the background dynamical equations for the inflaton field and for the radiation energy density in WI, e.g.,

$$3H(1+Q)\dot{\phi} \approx -V_{,\phi}\,, \tag{20}$$

$$\rho_R \equiv C_R T^4 \approx \frac{3Q}{4}\dot{\phi}^2, \tag{21}$$

$$H^2 \approx \frac{V}{3M_{\rm Pl}^2}. \tag{22}$$

From this set of equations, one can easily deduce for instance that

$$Q^7 \propto \frac{V_{,\phi}^{\,6}}{V^5}, \qquad T^7 \propto \frac{V_{,\phi}^{\,2}}{V^{1/2}}. \tag{23}$$

Thus, from Equations (20)–(22), together with Equation (19), we obtain that

$$\frac{d\ln Q}{dN} = \frac{2Q[(2+p)\epsilon_V - p\eta_V - 2c\kappa_V]}{4-p+(4+p)Q}, \tag{24}$$

and

$$\frac{d\ln\left(\frac{T}{H}\right)}{dN} = \frac{[7-(1-Q)p+5Q]\epsilon_V}{(1+Q)[4-p+(4+p)Q]} - \frac{2(1+Q)\eta_V + (1-Q)c\kappa_V}{(1+Q)[4-p+(4+p)Q]}, \tag{25}$$

where we have also introduced the quantity $\kappa_V$, defined as $\kappa_V = M_{\rm Pl}^2 V_{,\phi}/(\phi V)$. From Equations (24) and (25), we find in particular that

$$\begin{aligned} n_s &= 1 + \frac{(1+Q)}{1+Q-\epsilon_V}\left[\frac{d\ln\left(\frac{T}{H}\right)}{dN}\right. \\ &+ \frac{\frac{d\ln Q}{dN}\left(-3+Q\left\{3+2\pi\left[-1+Q\left(3+\sqrt{9+12Q\pi}\right)\right]\right\}\right)}{(1+Q)\left[3+Q\pi\left(4+\sqrt{9+12Q\pi}\right)\right]} \\ &+ \left.\frac{d\ln Q}{dN}\mathcal{A}(Q) + \frac{-6\epsilon_V+2\eta_V}{1+Q}\right], \end{aligned} \tag{26}$$

where we have defined $\mathcal{A}(Q)$ as

$$\mathcal{A}(Q) = \frac{3+2\pi Q}{3+4\pi Q} + Q\frac{d\ln G(Q)}{dQ}. \tag{27}$$

The expressions for $\alpha_s$ and $\beta_s$ are obtained by taking further derivatives of Equation (26) using Equation (18). Though this is a lengthy but straightforward exercise, the expressions are, however, too long to quote here in full. However, since we are mostly interested in models of WI in the strong dissipative regime, we can expand Equation (26) for $Q \gg 1$ and obtain, at least for $n_s$, a relatively shorter expression. Hence, expanding Equation (26) for $Q \gg 1$, we obtain that

$$n_s \simeq 1 + \frac{-7c\kappa_V - 11\epsilon_V - p\epsilon_V + 6\eta_V - 2p\eta_V}{Q(4+p)} + \frac{2\mathcal{A}(Q)(-2c\kappa_V + (2+p)\,\epsilon_V - p\eta_V)}{Q(4+p)} + \mathcal{O}(1/Q^{\frac{3}{2}}), \tag{28}$$

where we have also used Equations (24) and (25). Approximate expressions for $\alpha_s$ and $\beta_s$ can be obtained analogously. For instance, from Equation (28) we obtain for $\alpha_s$ that

$$\begin{aligned} \alpha_s &\simeq \frac{1}{(4+p)^3Q^2}\Big\{-4(4+p)\mathcal{A}(Q)(2c\kappa_V-(2+p)\epsilon_V+p\eta_V)^2 + 2(4+p)^2\mathcal{A}(Q)\Big(4(2+p)\epsilon_V^2 - 4(1+p)\epsilon_V\eta_V + p\xi_V^2\Big) \\ &+ (4+p)\Big(-2(6+p)(11+p)\epsilon_V^2 + 2\epsilon_V(c(-8+5p)\kappa_V + 56\eta_V) - 2(7c\kappa_V + 2(-3+p)\eta_V)(2c\kappa_V + p\eta_V) \\ &- 24\xi_V^2 + 2p(1+p)\xi_V^2\Big) + 4(4(-1+Q+\epsilon_V) + p(1+Q+\epsilon_V))(2c\kappa_V - (2+p)\epsilon_V + p\eta_V)^2\mathcal{A}'(Q)\Big\} + \mathcal{O}\left(\frac{1}{Q^3}\right), \end{aligned} \tag{29}$$

while for $\beta_s$ we find

$$\begin{aligned} \beta_s &\simeq \frac{2}{(4+p)^3Q^3}\Big\{8\mathcal{A}(Q)(-2c\kappa_V+(2+p)\epsilon_V-p\eta_V)^3 - 8(4+p)\mathcal{A}(Q)(-2c\kappa_V+(2+p)\epsilon_V-p\eta_V)\Big(4(2+p)\epsilon_V^2 \\ &- 4(1+p)\epsilon_V\eta_V + p\xi_V^2\Big) - 2(-2c\kappa_V+(2+p)\epsilon_V-p\eta_V)\Big(-2(6+p)(11+p)\epsilon_V^2 + 2\epsilon_V(c(-8+5p)\kappa_V+56\eta_V) \\ &- 2(7c\kappa_V+2(-3+p)\eta_V)(2c\kappa_V+p\eta_V) - 24\xi_V^2 + 2p(1+p)\xi_V^2\Big) + (-4-p)\Big(8(6+p)(11+p)\epsilon_V^3 \\ &+ 8\epsilon_V\eta_V(c(-5+4p)\kappa_V + (14+(-3+p)p)\eta_V) - 4\epsilon_V^2(c(-8+5p)\kappa_V + (150+p(17+p))\eta_V) \\ &+ c(12-11p)\kappa_V\xi_V^2 - 4\Big(-26+p+p^2\Big)\epsilon_V\xi_V^2 - (-3+p)(-4+3p)\eta_V\xi_V^2 + (-3+p)(4+p)\omega_V^3\Big) \\ &+ (4+p)^2\mathcal{A}(Q)\Big(32(2+p)\epsilon_V^3 - 8(7+5p)\,\epsilon_V^2\eta_V + 4\epsilon_V\Big(2(1+p)\eta_V^2 + \xi_V^2 + 2p\xi_V^2\Big) - p\Big(\eta_V\xi_V^2 + \omega_V^3\Big)\Big) \\ &- 4Q(-2c\kappa_V+(2+p)\epsilon_V-p\eta_V)^3\mathcal{A}'(Q) \\ &+ 2Q(4+p)(-2c\kappa_V+(2+p)\epsilon_V-p\eta_V)\Big(4(2+p)\epsilon_V^2 - 4(1+p)\epsilon_V\eta_V + p\xi_V^2\Big)\mathcal{A}'(Q) \\ &+ 4Q(-2c\kappa_V+(2+p)\epsilon_V-p\eta_V)\Big(-(2c\kappa_V-(2+p)\epsilon_V+p\eta_V)^2 + (4+p)\Big(4(2+p)\epsilon_V^2 - 4(1+p)\epsilon_V\eta_V + p\xi_V^2\Big)\Big)\mathcal{A}'(Q) \\ &+ 4Q^2(-2c\kappa_V+(2+p)\epsilon_V-p\eta_V)^3\mathcal{A}''(Q)\Big\} + \mathcal{O}\left(\frac{1}{Q^4}\right). \end{aligned} \tag{30}$$

In the above expressions, $\mathcal{A}'(Q) \equiv d\mathcal{A}/dQ$ and $\mathcal{A}''(Q) \equiv d^2\mathcal{A}/dQ^2$. The above approximate expressions for $n_s$, $\alpha_s$ and $\beta_s$ applies to any WI model with a generic dissipation coefficient of the form of Equation (19) and once the primordial potential for the inflaton is specified.

Below, as an example, we will work with two explicit models in WI, which have been shown to be of interest recently. Then, specific expressions for $n_s$, $\alpha_s$ and $\beta_s$ will be derived for those models, along with their respective analysis.

## 3. Running and Running of the Running of the Scalar Spectral Index in MWI

The minimal WI (MWI) model proposed in Ref. [21] (see also Ref. [38]) successfully realizes WI in the strong dissipative regime. In such a model, the inflaton is being treated as an axionic field with coupling to the non-Abelian gauge fields. Due to such gauge couplings, sphaleron transitions between gauge vacua at high temperatures lead to a friction term of the form

$$\Upsilon(T) = \frac{\Gamma_{sp}(T)}{2f^2 T} = \kappa(\alpha_g, N_c, N_f)\alpha_g^5 \frac{T^3}{f^2}, \tag{31}$$

where $\Gamma_{sp}$ is the sphaleron rate, $f$ is the axion decay rate, $T$ is the temperature, $\alpha_g = g^2/(4\pi)$, with $g$ being the Yang–Mills gauge coupling, and $\kappa$ is a dimensionless quantity depending on the dimension of the gauge group $(N_c)$, the representation of the fermions $(N_f)$ and on the gauge coupling $g$ through $\alpha_g$. For instance, for a quantum chromodynamics (QCD) type of axion and using typical values of parameters for QCD, we have that $\kappa(\alpha_g, N_c, N_f)\alpha_g^5 \sim 10^{-3}$.

This model is particularly attractive in the context of WI. Because of the axionic shift symmetry, the inflaton is protected from any perturbative backreaction and, hence, from acquiring a large thermal mass. Similar symmetry properties allowing the inflaton to be coupled directly to the radiation fields have also been studied in Refs. [20,37]. Models based on pseudo-Goldstone bosons for the inflaton [39] are quite reminiscent of these ideas and have gained increased attention recently in the context of WI [40].

The scalar power spectrum in the MWI model that was considered in Ref. [21] was given by

$$\Delta_{\mathcal{R}} \approx \frac{\sqrt{3}}{4\pi^{\frac{3}{2}}} \frac{H^3 T}{\dot{\phi}^2} \left(\frac{Q}{Q_3}\right)^9 Q^{\frac{1}{2}}, \tag{32}$$

where $Q_3 \approx 7.3$. At the end of this section and in the next one, we will discuss a more accurate form for the scalar spectrum in this model for a dissipation coefficient of the form $\Upsilon \propto T^3$ as it has been considered here. However, for now, let us use the above expression Equation (32) as considered by the authors in Ref. [21].

Since we are also deriving expressions in the high dissipative regime, $Q \gg 1$, let us note that in this case, using $p = 3$ and $c = 0$ in Equation (19), that

$$\frac{d \ln H}{dN} \approx -\frac{\epsilon_V}{Q}, \tag{33}$$

$$\frac{d \ln Q}{dN} \approx \frac{10\epsilon_V - 6\eta_V}{7Q}, \tag{34}$$

$$\frac{d \ln T}{dN} \approx \frac{\epsilon_V - 2\eta_V}{7Q}, \tag{35}$$

$$\frac{d \ln \dot{\phi}}{dN} \approx -\frac{3\epsilon_V + \eta_V}{7Q}. \tag{36}$$

Hence, the scalar spectral index, in the strong dissipative regime can be calculated as

$$n_s - 1 = \frac{3}{7Q}(27\epsilon_V - 19\eta_V). \tag{37}$$

For a detailed derivation of the scalar spectral index see, e.g., Ref. [22].

It is evident from Equation (37) that to obtain the observed red tilt of the scalar spectral index $(n_s < 1)$ in MWI, one requires a potential which yields $\epsilon_V < \eta_V$. In Ref. [21], for this objective a potential of a hybrid inflation model was considered. The effective potential of hybrid inflation contains the inflaton field $\phi$ along with a waterfall field $\sigma$, which can be written as[2]

$$V(\phi,\sigma) = \frac{1}{4\lambda}(M^2 - \lambda\sigma)^2 + \frac{1}{2}m^2\phi^2 + \frac{1}{2}g^2\phi^2\sigma^2. \tag{38}$$

The $\sigma$ field has a squared mass term as $-M^2 + g^2\phi^2$ in this potential. When $\phi > M/g$, which is the region where inflation takes place, $\sigma$ has only one minimum at $\sigma = 0$, and the effective potential becomes[3]

$$V_{\rm eff}(\phi) = \frac{M^4}{4\lambda} + \frac{1}{2}m^2\phi^2. \tag{39}$$

During this stage, the constant term $\frac{M^4}{4\lambda}$ ($\gg \frac{1}{2}m^2\phi^2$) drives the expansion. Below the threshold $\phi = M/g$, the waterfall field $\sigma$ quickly rolls down to its minimum $\sigma(\phi) = M_\sigma(\phi)/\sqrt{\lambda}$ and puts an end to inflation. The effective potential, given in Equation (39), yields the slow-roll parameters as[4]

$$\begin{aligned} \epsilon_V &\approx 8\lambda^2 \frac{M_{\rm Pl}^2 m^4 \phi^2}{M^8}, \\ \eta_V &\approx 4\lambda \frac{M_{\rm Pl}^2 m^2}{M^4}, \end{aligned} \tag{40}$$

which shows that

$$\frac{\epsilon_V}{\eta_V} \sim \frac{\frac{1}{2}m^2\phi^2}{\frac{M^4}{4\lambda}} \ll 1. \tag{41}$$

Therefore, such a potential is suitable to produce a red-tilted scalar spectral index in MWI. Hence, with the hybrid inflation potential given in Equation (39), the scalar spectral index in Equation (37) can be approximated as

$$n_s - 1 \sim -\frac{57\eta_V}{7Q}. \tag{42}$$

Above all, as we can see from Equation (40) that $\eta_V$ is effectively constant in this model, we denote $\eta_V = k$, where $k \equiv 4\lambda\frac{M_{\rm Pl}^2 m^2}{M^4}$ is a constant. Hence, to yield the observed value of the red-tilt, $1 - n_s \sim 0.04$, we require

$$\frac{k}{Q} \sim 0.005. \tag{43}$$

We now estimate the running $(\alpha_s)$ and the running of the running $(\beta_s)$ of the scalar spectral index in this model. The running of the scalar spectral index, $\alpha_s$, can be calculated as

$$\alpha_s \simeq \frac{dn_s}{dN} = \frac{57}{7}\frac{k}{Q}\frac{d\ln Q}{dN}, \tag{44}$$

where derivative has been taken with respect to the number of $e$-foldings $N$, and in MWI we have [22]

$$\frac{d\ln Q}{dN} = \frac{10\epsilon_V - 6\eta_V}{7Q}. \tag{45}$$

However, as the effective hybrid potential given in Equation (39) yields $\epsilon_V \ll \eta_V$, we can approximate the above equation as

$$\frac{d\ln Q}{dN} \approx -\frac{6k}{7Q}. \tag{46}$$

This yields the running of the scalar spectral index in MWI as

$$\alpha_s \approx -\frac{342}{49}\frac{k^2}{Q^2}. \tag{47}$$

Now, the value of $k/Q$, which yields the observed red-tilt of the scalar spectral index given in Equation (43), produces a running of the scalar spectral index as

$$\alpha_s \sim -1.7 \times 10^{-4}. \tag{48}$$

The running of the running of the scalar spectral index then turns out to be

$$\beta_s \simeq \frac{d\alpha_s}{dN} = \frac{684}{49}\frac{k^2}{Q^2}\frac{d\ln Q}{dN} \approx -\frac{4104}{441}\frac{k^3}{Q^3}. \tag{49}$$

Again, the value of $k/Q$ given in Equation (43), which yields the observed red-tilt of the scalar spectral index, gives rise to a running of the running of the scalar spectral index as

$$\beta_s \approx -1.16 \times 10^{-6}. \tag{50}$$

Therefore, we note that the MWI model with hybrid-type potential generates negative $\alpha_s$ and $\beta_s$, and the running of the running is two orders of magnitude smaller than the running of the scalar spectral index. These results come from the simplified analysis made in particular with the approximated scalar power spectrum given by Equation (32).

Let us now check whether an improved form for the scalar power spectrum might somehow improve on the estimates made above for $n_s$, $\alpha_s$ and $\beta_s$. Using the previously derived Equations (28)–(30), we obtain for instance that

$$n_s \simeq 1 + \frac{2\mathcal{A}(Q)\left(\frac{5m^4\phi^2 M_{\rm Pl}^2}{2V_{\rm eff}^2} - \frac{3m^2 M_{\rm Pl}^2}{V_{\rm eff}}\right)}{7Q} - \frac{m^4\phi^2 M_{\rm Pl}^2}{QV_{\rm eff}^2}, \tag{51}$$

$$\begin{aligned}\alpha_s \simeq\ & \frac{1}{686Q^2V_{\rm eff}^6}\Big\{7\mathcal{A}'(Q)m^8M_{\rm Pl}^6\left(5m^2\phi^3 - 6\phi V_{\rm eff}\right)^2 + 2m^4M_{\rm Pl}^4V_{\rm eff}^2\Big(m^4(-441 + 315\mathcal{A}(Q) + 25\mathcal{A}'(Q)(-1+7Q))\phi^4 \\ +\ & 4V_{\rm eff}\left(m^2(98 - 91\mathcal{A}(Q) + 15\mathcal{A}'(Q)(1-7Q))\,\phi^2 - 9(7\mathcal{A}(Q) + \mathcal{A}'(Q) - 7\mathcal{A}'(Q)Q)V_{\rm eff}\right)\Big)\Big\},\end{aligned} \tag{52}$$

and

$$\begin{aligned}\beta_s \simeq\ & \frac{m^6M_{\rm Pl}^6}{343Q^3V_{\rm eff}^6}\Big\{m^6(-1134 + 810\mathcal{A}(Q) + 25Q[32\mathcal{A}'(Q) + 5\mathcal{A}''(Q)Q])\phi^6 \\ -\ & 2m^4(-812 + 646\mathcal{A}(Q) + 15Q(68\mathcal{A}'(Q) + 15\mathcal{A}''(Q)Q))\phi^4V_{\rm eff} \\ +\ & 4m^2(-28 - 10\mathcal{A}(Q) + 9Q(26\mathcal{A}'(Q) + 15\mathcal{A}''(Q)Q))\phi^2V_{\rm eff}^2 - 216(2\mathcal{A}(Q) + Q(-2\mathcal{A}'(Q) + \mathcal{A}''(Q)Q))V_{\rm eff}^3\Big\},\end{aligned} \tag{53}$$

where $V_{\rm eff}$ is given by Equation (39), and the function $\mathcal{A}(Q)$ is derived using the growth function $G(Q)$ as defined in the next section and given by Equation (57).

We solve the complete system of dynamical equations for the hybrid inflation model and compare the numerical results with the analytical ones coming from Equations (51)–(53). As in Ref. [21], we consider the case of the model consisting of a pure $SU(3)$, with relativistic degrees of freedom $g_* = 17$, consisting of two polarizations per eight gauge bosons plus one for the axion. The gauge coupling is assumed to be $\alpha_g = 0.1$ and $\kappa(\alpha_g, N_c, N_f)\alpha_g^5 = 10^{-3}$ for the coefficient in the dissipation coefficient in Equation (31). As also shown in Ref. [21], the strong dissipative regime ($Q \gg 1$) in the model with the hybrid type of potential Equation (38) is favored when the parameters $M$, $g$ and axion term $f$ satisfies $gf/M \lesssim 10^{-8}$.

We also recall that when assuming a QCD axion, astrophysical constraints [43] put a lower bound $f \gtrsim 10^9$ GeV on the QCD axion decay constant. In our numerical experiments, we have checked that both the strong dissipative regime and acceptable values for $f$ satisfying the astrophysical bounds can be achieved by taking, for example, $M \gtrsim 2.5 \times 10^{17}\, g$ GeV and choosing appropriate values for $g$ and for the other constant, $\lambda$, appearing in the potential Equation (38).

We have studied two representative cases, and in the Table 1 we summarize the relevant parameters and quantities obtained from the numerical analysis. Note that for the parameters considered, we have the temperature at the end of the WI slightly smaller than the axion decay constant, $f > T_{\rm end}$, which indicates an axion symmetry breaking still in the inflationary regime and, hence, prior to the end of WI. As an aside regarding the hybrid inflation model studied here, we note that the inflaton field excursions for examples shown in Table 1 are all sub-Planckian, $|\Delta\phi| \sim 0.04 M_{\rm Pl}$. Hence, the hybrid WI model studied here satisfies the swampland distance conjecture, while it marginally satisfies the (refined) de Sitter conjecture [10,11] by having $\epsilon_{V_*} \ll 1$, but $\eta_V \gtrsim 1$.

**Table 1.** Model parameters and relevant quantities obtained from the MWI model with the hybrid inflaton potential. In all cases, we have considered $M \gtrsim 2.5 \times 10^{17}\, g$ GeV and $\lambda = 10^{-2}$. The number of e-folds $N_*$ at Hubble radius crossing is found from Equation (54), and it is $N_* \simeq 49$ for the cases shown here.

| $g$ | $Q_*$ | $m/M_{\rm Pl}$ | $\epsilon_{V_*}$ | $\eta_{V_*}$ | $\phi_*/M_{\rm Pl}$ | $T_{\rm end}$ [GeV] | $f$ [GeV] |
|---|---|---|---|---|---|---|---|
| $5 \times 10^{-8}$ | 200 | $1.41 \times 10^{-16}$ | 0.01 | 1.13 | 0.15 | $1.25 \times 10^{9}$ | $4.75 \times 10^{9}$ |
| $10^{-8}$ | 350 | $7.25 \times 10^{-18}$ | 0.03 | 1.86 | 0.14 | $2.77 \times 10^{8}$ | $1.85 \times 10^{9}$ |

The number of $e$-folds $N_*$ at Hubble radius crossing is found from the relation obtained in WI (see, e.g., Ref. [22] for details)

$$\frac{k_*}{a_0 H_0} = e^{-N_*} \left[ \frac{43}{11 g_s(T_{\rm end})} \right]^{1/3} \frac{T_0}{T_{\rm end}} \frac{H_*}{H_0}, \tag{54}$$

where $T_{\rm end}$ is the temperature at the end of WI, $H_0$ is the Hubble parameter today and for which we assume the *Planck* result, $H_0 = 67.66\,{\rm km\ s^{-1} Mpc^{-1}}$ [from the *Planck* Collaboration [44], TT,TE,EE-lowE+lensing+BAO 68% limits, $H_0 = (67.66 \pm 0.42)\,{\rm km\ s^{-1}Mpc^{-1}}$], $T_0$ is the CMB temperature today, $T_0 = 2.725\,{\rm K} = 2.349 \times 10^{-13}\,{\rm GeV}$, while the for the pivot scale $k_*$ we take the *Planck* value $k_* = 0.05/{\rm Mpc}$, and we also use the convention $a_0 = 1$. For $g_s(T_{\rm end})$, we assume the MWI value, $g_s(T_{\rm end}) = 17$.

The numerical and analytical results obtained for $n_s$, $\alpha_s$ and $\beta_s$ are given in Table 2. We note from the results from Table 2 that the analytical and numerical results for $n_s$ and $\alpha_s$ agree quite well. There are discrepancies between the values for $\beta_s$, especially for the lower dissipative case studied, $Q_* = 200$, but that can be attributed to the lack of numerical precision in the determination of $\beta_s$ in that case, recalling that to obtain $\beta_s$ numerically, it requires three derivatives of the scalar power spectrum with respect to the number of e-folds. In any case, these results also corroborate the previous ones using the approximated form of the power spectrum, with $\alpha_s$ and $\beta_s$ having the same order of magnitudes as obtained previously, Equations (48) and (50), respectively. We have not quoted the results for the tensor-to-scalar ratio $r$, but as typical for WI in the strong dissipative regime, $r$ is too small to be accessible by any future observation. For the dissipation values quoted in Table 1, $r \lesssim 10^{-25}$.

We now study another type of WI models that can allow for large dissipation and are still consistent with the *Planck* observations. The objective is to check whether with these models we can break the large hierarchy between $\alpha_s$ and $\beta_s$.

**Table 2.** Analytical estimation of $n_s$, $\alpha_s$ and $\beta_s$, along also with their full numerical determination, for the 2 cases considered in Table 1.

| $Q_*$ | $n_{s_*,\text{analytical}}$ | $n_{s_*,\text{numerical}}$ | $\alpha_{s_*,\text{analytical}}$ | $\alpha_{s_*,\text{numerical}}$ | $\beta_{s_*,\text{analytical}}$ | $\beta_{s_*,\text{numerical}}$ |
|---|---|---|---|---|---|---|
| 200 | 0.9611 | 0.9612 | $-2.0 \times 10^{-4}$ | $-1.9 \times 10^{-4}$ | $-1.8 \times 10^{-6}$ | $-2.0 \times 10^{-6}$ |
| 350 | 0.9642 | 0.9643 | $-1.9 \times 10^{-4}$ | $-1.9 \times 10^{-4}$ | $-1.5 \times 10^{-6}$ | $-1.1 \times 10^{-6}$ |

## 4. Running and Running of the Running of the Scalar Spectral Index in MWI with Exponential Potentials

In CI, steep exponential potentials of the form

$$V(\phi) = V_0 e^{-\alpha\phi/M_{\rm Pl}}, \tag{55}$$

lead to power-law type inflation [45], where inflation does not exit gracefully in standard general relativity. However, it was shown in Ref. [46] that WI with dissipative coefficient of the form $\Upsilon(T) \propto T^p$, with $p > 2$, can gracefully exit inflation with such an exponential potential. The MWI model [21] is, thus, such a model where the dissipative coefficient scales with the cubic power of the temperature of the thermal bath, $\Upsilon(T) \propto T^3$.

A MWI model with such an exponential potential was studied in Ref. [47][5]. Though inflation ends gracefully in such a model, it was shown there that such a combination yields way too large red-tilt in the scalar spectral index to be compliant with the current observation. Therefore, such models are not viable models of inflation.

A generalized form of the exponential potential, given as

$$V(\phi) = V_0 e^{-\alpha(\phi/M_{\rm Pl})^n}, \tag{56}$$

with $n > 1$, was then studied in Ref. [22], which shows that this model not only exits inflation gracefully, but also fully satisfies all the observational constraints (comply with the observed values of scalar spectral index $n_s$ and the tensor-to-scalar ratio $r$) and is in tune with the swampland conjectures (see, e.g., Table 1 of Ref. [22]). This same model was also considered in the context of WI for quintessential inflation [49], displaying interesting features as far as the late Universe physics is concerned. Therefore, here we will further analyze such a model to determine the running and running of the running of the scalar spectral index that are yielded by this model.

In the model that we are presently considering, the explicit form for the function $G(Q_*)$ appearing in the scalar of curvature power spectrum in WI, Equation (12), has been derived in Ref. [22], and it is given as[6]

$$\begin{aligned} G(Q_*) &= \frac{1+6.12Q_*^{2.73}}{(1+6.96Q_*^{0.78})^{0.72}} \\ &\quad + \frac{0.01Q_*^{4.61}(1+4.82\times10^{-6}Q_*^{3.12})}{(1+6.83\times10^{-13}Q_*^{4.12})^2}. \end{aligned} \tag{57}$$

We will first derive the scalar spectral index $n_s$, its running $\alpha_s$ and running of its running $\beta_s$ analytically in this model. Note that, although the values of $n_s$ in this model have been reported previously in Table 1 of Ref. [22], they have been obtained numerically. To the best of our knowledge, this is the first time anyone has determined $n_s$, $\alpha_s$ and $\beta_s$ in any WI model analytically.

If we ignore the thermal distribution factor $(1 + 2n_*)$ of the inflaton field in Equation (13), which can be justified when working in the strong dissipative regime $Q \gg 1$ (see, e.g., Ref. [28]), then the scalar power spectrum simplifies to the form

$$\Delta_{\mathcal{R}}(k) = \frac{\sqrt{3}}{4\pi}\frac{H^3 T}{\dot\phi^2}\frac{Q}{\sqrt{3+4\pi Q}}G(Q). \tag{58}$$

This is the form of the power spectrum that we will use to calculate the spectral index and its running and running of its running analytically. Such an approximation of the power spectrum will later be justified by obtaining $n_s$, $\alpha_s$ and $\beta_s$ numerically and matching them with the analytically obtained results.

First, we calculate the scalar spectral index $n_s$. Using the scalar spectrum given in Equation (58), we obtain

$$\begin{aligned} n_s - 1 &= 3\frac{d\ln H}{dN} + \frac{d\ln T}{dN} - 2\frac{d\ln\dot{\phi}}{dN} \\ &+ \left(\frac{3+2\pi Q}{3+4\pi Q} + Q\frac{d\ln G(Q)}{dQ}\right)\frac{d\ln Q}{dN}. \end{aligned} \tag{59}$$

From the equations in Equation (36), after some straightforward algebra, we obtain the scalar spectral index as

$$n_s \approx 1 - \frac{(14 - 10\mathcal{A}(Q))\epsilon_V + 6\mathcal{A}(Q)\eta_V}{7Q}, \tag{60}$$

where $\mathcal{A}(Q)$ has been defined in Equation (27).

We now analytically determine $\alpha_s$. Using the form of $n_s$ from Equation (60), we obtain

$$\begin{aligned} \alpha_s &= \frac{1}{7Q}\Bigg\{-(14 - 10\mathcal{A}(Q))\frac{d\epsilon_V}{dN} - 6\mathcal{A}(Q)\frac{d\eta_V}{dN} \\ &+[(14 - 10\mathcal{A}(Q))\epsilon_V + 6\mathcal{A}(Q)\eta_V \\ &+(10\epsilon_V - 6\eta_V)Q\mathcal{A}'(Q)]\frac{d\ln Q}{dN}\Bigg\}. \end{aligned} \tag{61}$$

Using now that

$$\frac{d\epsilon_V}{dN} = \frac{4\epsilon_V^2 - 2\epsilon_V\eta_V}{Q}, \tag{62}$$

$$\frac{d\eta_V}{dN} = \frac{2\epsilon_V\eta_V - \xi_V^2}{Q}, \tag{63}$$

the expression for the running of the scalar index can be computed to be

$$\begin{aligned} \alpha_s &= \frac{1}{49Q^2}\Big[4\big(-63 + 45\mathcal{A}(Q) + 25Q\mathcal{A}'(Q)\big)\epsilon_V^2 \\ &- 8\big(-14 + 13\mathcal{A}(Q) + 15Q\mathcal{A}'(Q)\big)\epsilon_V\eta_V \\ &+36\big(-\mathcal{A}(Q) + Q\mathcal{A}'(Q)\big)\eta_V^2 + 42\mathcal{A}(Q)\xi_V^2\Big]. \end{aligned} \tag{64}$$

To determine $\beta_s$, one requires

$$\frac{d\xi_V^2}{dN} = \frac{1}{Q}[(4\epsilon_V - \eta_V)\xi_V^2 - \omega_V^3]. \tag{65}$$

Using the form of $\alpha_s$ from Equation (64), $\beta_s$ can be written as

$$\beta_s = \frac{1}{343Q^2}\Big[8\Big(-1764 + 1260\mathcal{A}(Q) + 1050Q\mathcal{A}'(Q) + 125Q^2\mathcal{A}''(Q)\Big)\epsilon_V^3 -24\Big(-490 + 392\mathcal{A}(Q) + 490Q\mathcal{A}'(Q) + 75Q^2\mathcal{A}''(Q)\Big)\epsilon_V^2\eta_V +8\Big(-196 + 56\mathcal{A}(Q) + 504Q\mathcal{A}'(Q) + 135Q^2\mathcal{A}''(Q)\Big)\epsilon_V\eta_V^2 +28(-28 + 68\mathcal{A}(Q) + 45Q\mathcal{A}'(Q))\epsilon_V\xi_V^2 - 216Q^2\mathcal{A}''(Q)\eta_V^3 +42(5\mathcal{A}(Q) - 18Q\mathcal{A}'(Q))\eta_V\xi_V^2 - 294\mathcal{A}(Q)\omega_V^3\Big] - \frac{2(10\epsilon_V - 6\eta_V)}{7Q}\alpha_s. \quad (66)$$

The equations for $n_s$, $\alpha_s$ and $\beta_s$ can also be expressed explicitly in terms of the inflaton field $\phi$ using the potential Equation (56), but that only complicates the form of the expressions more, and we do not need to perform that explicitly here. In Table 3, we will explicitly furnish the numerical values for the required slow-roll parameters.

**Table 3.** Values for the parameters considered in the analytical estimation of $n_s$, $\alpha_s$ and $\beta_s$ for 4 sets of the chosen model parameters. The number of e-folds $N_*$ is obtained from Equation (54), and it is $N_* \simeq 48$ for all the cases shown here [22].

| Model | $V_0$ (GeV$^4$) | $V_*^{1/4}/M_{\rm Pl}$ | $\epsilon_{V_*}$ | $\eta_{V_*}$ | $\xi^2_{V_*}$ | $\omega^3_{V_*}$ | $T_{\rm end}$ [GeV] | $f$ [GeV] |
|---|---|---|---|---|---|---|---|---|
| $n=2$, $\alpha=9.6$, $Q_*=850.96$ | $3.07\times10^{38}$ | $1.14\times10^{-9}$ | 31.7 | 44.2 | 367.72 | $-1.38\times10^{5}$ | $2.1\times10^{7}$ | $7.3\times10^{9}$ |
| $n=3$, $\alpha=2.5$, $Q_*=740.15$ | $1.82\times10^{39}$ | $9.55\times10^{-10}$ | 25.9 | 37.1 | 507.73 | $-4.16\times10^{4}$ | $3.4\times10^{7}$ | $8.8\times10^{9}$ |
| $n=4$, $\alpha=0.45$, $Q_*=719.68$ | $3.59\times10^{39}$ | $1.34\times10^{-9}$ | 25.0 | 36.56 | 603.87 | $-2.6\times10^{4}$ | $3.9\times10^{7}$ | $9.2\times10^{9}$ |
| $n=5$, $\alpha=0.06$, $Q_*=699.53$ | $6.01\times10^{39}$ | $1.69\times10^{-9}$ | 24.1 | 35.5 | 612.63 | $-1.89\times10^{4}$ | $4.3\times10^{7}$ | $9.6\times10^{9}$ |

We now estimate the values of $n_s$, $\alpha_s$ and $\beta_s$ using the analytical forms of these quantities that we derived above. To obtain these values, we need to fix $n$, $\alpha$ and $Q_*$, the values of which have been taken from Table 1 of Ref. [22] which produce $n_s$ and $r$ that are in tune with the observations[7]. In the table, we have also quoted the temperature at the end of WI, $T_{\rm end}$ and the value obtained for the axion decay constant. To obtain $f$, we made the same considerations regarding the gauge sector coupled to the scalar field as in the previous section for the MWI model with the hybrid potential. We have considered QCD like values for the parameters. With relativistic degrees of freedom $g_* = 17$, the gauge coupling is assumed to be $\alpha_g = 0.1$ and $\kappa(\alpha_g, N_c, N_f)\alpha_g^5 = 10^{-3}$ for the coefficient in the dissipation coefficient in Equation (31). Note that in all the cases considered here, we have that $T_{\rm end} < f$, pointing to cases where the axion is already in a symmetry broken phase at the end of WI.

In Table 3, we give the values of the required slow-roll parameters required in the derivation of $n_s$, $\alpha_s$ and $\beta_s$. We also give the normalization $V_0$ of the potential for each of the models and the scale of the inflaton potential at Hubble radius crossing, $V_*$. The analytically estimated values for $n_s$, $\alpha_s$ and $\beta_s$ that are obtained for each of those models are then shown in Table 4. In the same table, the numerically obtained values for $n_s$, $\alpha_s$ and $\beta_s$ are given, where the full scalar power spectrum, given in Equation (12), has been used to determine them.

**Table 4.** Analytical estimation of $n_s$, $\alpha_s$ and $\beta_s$, along with their full numerical determination, for 4 sets of chosen model.

| Model | $n_{s*,\text{analytical}}$ | $n_{s*,\text{numerical}}$ | $\alpha_{s*,\text{analytical}}$ | $\alpha_{s*,\text{numerical}}$ | $\beta_{s*,\text{analytical}}$ | $\beta_{s*,\text{numerical}}$ |
|---|---|---|---|---|---|---|
| $n = 2$<br>$\alpha = 9.6$<br>$Q_* = 850.96$ | 0.9651 | 0.9648 | $-5.9 \times 10^{-3}$ | $-6.1 \times 10^{-3}$ | $-8.6 \times 10^{-6}$ | $-2.7 \times 10^{-5}$ |
| $n = 3$<br>$\alpha = 2.5$<br>$Q_* = 740.15$ | 0.9697 | 0.9689 | $-4.2 \times 10^{-3}$ | $-4.4 \times 10^{-3}$ | $-1.2 \times 10^{-4}$ | $-1.3 \times 10^{-4}$ |
| $n = 4$<br>$\alpha = 0.45$<br>$Q_* = 719.68$ | 0.9662 | 0.9655 | $-3.7 \times 10^{-3}$ | $-3.8 \times 10^{-3}$ | $-1.3 \times 10^{-4}$ | $-1.4 \times 10^{-4}$ |
| $n = 5$<br>$\alpha = 0.06$<br>$Q_* = 699.53$ | 0.9654 | 0.9645 | $-3.4 \times 10^{-3}$ | $-3.5 \times 10^{-3}$ | $-1.3 \times 10^{-4}$ | $-1.4 \times 10^{-4}$ |

From the results shown in Table 4, we notice that the analytical results match well with the numerical results. Therefore, the approximation we made to the scalar power spectrum in Equation (58) is justified. We also note that the MWI model with generalized exponential potentials also produces negative $\alpha_s$ and $\beta_s$. However, in comparison to the MWI model with hybrid potential, it produces one order higher $\alpha_s$ and two-order higher $\beta_s$. These results are promising in the sense that they indicate that the large hierarchy observed between $\alpha_s$ and $\beta_s$, which appears in the vanilla CI models, can be broken in the context of WI.

## 5. Discussion and Conclusions

There is a hint in the final data released by *Planck* that the running and the running of the running of the scalar spectral index might be larger than predicted in standard cold inflation models, with $\alpha_s$ of the order of $10^{-2}$, and the running of the running might be larger than the running itself [1]. Vanilla cold inflationary models preferred by *Planck* data fail to produce such large (and also positive) running and running of the running [4,5]. Thus, it is of importance to seek inflationary models that can follow these trends for the runnings hinted by the *Planck* data.

In this paper, we analyzed a variant inflationary scenario, namely warm inflation, in a strong dissipative regime ($Q \gg 1$). Previously, the running and the running of the running of the scalar spectral index was studied in Ref. [19], where WI was realized in a weak dissipative regime ($Q \ll 1$). To analyze WI in the strong dissipative regime, we chose a particular model of WI, namely the minimal warm inflation model first studied in Ref. [21]. In this model, the dissipative term is proportional to the cubic power of the temperature of the thermal bath ($\Upsilon \propto T^3$). MWI was first studied with a hybrid potential in Ref. [21] to obtain a red-tilted scalar spectrum. However, later in Ref. [22], MWI was studied with generalized forms of the exponential potential, and it was shown in that reference that not only is this model in accordance with the current observations, but it can also accommodate the swampland conjectures. We determine $\alpha_s$ and $\beta_s$ in both of these MWI models. Explicit analytical expressions for the running and for the running of the running were derived in the present paper. To the best of our knowledge, this is the first time that the running and the running of the running of scalar spectral index in the context of WI have been analyzed analytically, especially in the strong dissipative regime of WI. We believe that this work will motivate further exploration of other WI models along the same line to see whether WI can produce the *Planck*-hinted running and running of the running. This is of particular importance, given that the next generation of cosmological observations might further improve on the determination for the values for these cosmological parameters.

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**Conflicts of Interest:** The authors declare no conflict of interest.

## Notes

1. For earlier studies considering this functional form for the dissipation coefficient in WI, see, e.g., Refs. [32,33].
2. Note that there are many variations of the hybrid inflation model, including the effect of possible quantum correction terms (see, e.g., Ref. [41]). In the present work, we only consider the potential for the hybrid model in its traditional form, given by Equation (38), since our intention is to compared our results with those from Ref. [21].
3. In this work we are not concerned with possible issues related to formation of topological defects that might result in hybrid inflation associated with the symmetry breaking of the $\sigma$ field and which are related to the post-inflationary dynamics regime. For a discussion about topological defects in the context of hybrid inflation, see, e.g., Ref. [42].
4. Note that in Ref. [21], $\epsilon_V$ was defined as $\epsilon_{\rm WI}$.
5. Recently such exponential potential is also studied in the context of power-law warm inflation [48]
6. See Appendix B of Ref. [22] for the full derivation of the scalar power spectrum in this model.
7. Note that the values of $V_*^{1/4}/M_{\rm Pl}$ have been wrongly quoted in Table 1 of [22]. Furthermore, in the same reference we have incorrectly quoted the estimate for the axion decay constant for this model, due to the use of an underestimated value for the gauge field coupling constant $\alpha_g$ used in the numerical code in that work. We have now corrected both of these values in the present work.

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