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Quantum state of fields in $SU(\infty)$ quantum gravity

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Abstract

Our Universe is ruled by quantum mechanics and hence should be treated as a quantum system. $SU(\infty)$-QGR is a recently proposed quantum model for the Universe, in which gravity is associated to the $SU(\infty)$ symmetry of its Hilbert space. Fragmentation of its infinite dimensional state due to random quantum fluctuations divides the Universe into approximately isolated subsystems. In addition to the parameters of their *internal* finite rank symmetries, states and dynamics of subsystems are characterized by four continuous parameters and the perceived classical spacetime is their effective representation, reflecting quantum states of subsystems and their relative evolution. At the lowest order, the effective Lagrangian of $SU(\infty)$-QGR has the form of Yang–Mills gauge theories for both $SU(\infty)$—gravity—and internal symmetries defined on the aforementioned 4D parameter space. In the present work, we study more thoroughly some of the fundamental aspects of $SU(\infty)$-QGR. Specifically, we clarify the impact of the degeneracies of the $\mathcal{SU}(\infty)$ algebra on the construction of the model, describe mixed states of subsystems and their purification, calculate measures of their entanglement to the rest of the Universe, and discuss their role in the emergence of local gauge symmetries. We also describe the relationship between what is called *internal space* of $SU(\infty)$ Yang–Mills and the 4D parameter space, and analytically demonstrate the irrelevance of the geometry of parameter space for physical observables. Along with these topics, we demonstrate the equivalence of two sets of criteria for the compositeness of a quantum system, and show the uniqueness of the limit of various algebras leading to $\mathcal{SU}(\infty)$.

Keywords: *quantum gravity, $SU(\infty)$ symmetry, Yang–Mills theory*

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1. Introduction to $SU(\infty)$-QGR

$SU(\infty)$-QGR is a fundamentally quantum approach to cosmology and gravity. Its rationale, foundation, and essential structure and properties were investigated in earlier studies [1, 2]. It was reviewed [3, 4] and compared with several popular quantum gravity (QGR) models in other studies [5].

We begin by a nutshell description of $SU(\infty)$-QGR. The essential axiom of this model is the existence of an infinite number of mutually commuting observables in a quantum Universe. Consequently, the Hilbert space $\mathcal{H}_U$ of the Universe represents the $SU(\infty)$ symmetry group [6–16]. There is no background spacetime in this model, and the usual quantization procedure is replaced by the non-commutative $\mathcal{SU}(\infty)$ algebra of linear operators $\hat{O} \in \mathcal{B}[\mathcal{H}_U]$ acting on $\mathcal{H}_U$ [17, 18].

It was demonstrated [1] that as a whole this Universe is static and trivial, in the sense that any state can be transformed to another by a $SU(\infty)$ transformation under which observables are conserved. Nonetheless, quantum fluctuations, that is, random application of $\hat{O} \in \mathcal{B}[\mathcal{H}_U]$, lead to approximate fragmentation of its state, and thereby the Hilbert space $\mathcal{H}_U$, and an approximate division to subsystems according to the criteria defined in an ealier study [19].

We remind that the process of Hilbert space fragmentation of closed quantum systems to subsystems with their own symmetries and behavior is observed in many condensed matter contexts (see, e.g., [20] and references therein). Indeed, this topic is currently under intensive theoretical [21, 22] and experimental [20] investigation, and is reviewed in previous works [23, 24]. Due to the many-body nature of these systems and the existence of a large number of degrees of freedom, the dominance of long-range electromagnetic interactions, and the isolation of the whole system, they can be considered as *mini-universes*, which are represented by their Hilbert spaces with the $SU(N \to \infty)$ symmetry. Therefore, it is not unreasonable to assume that the Hilbert space $\mathcal{H}_U$ of the whole Universe—the only truly isolated many-body quantum system—represents the $SU(\infty)$ symmetry and becomes fragmented to subsystems by random quantum fluctuations.

The clustered and fragmented blocks in $\mathcal{B}[\mathcal{H}_U]$ represent finite rank *internal* symmetries, which we collectively call G. Moreover, the global $SU(\infty)$ becomes the symmetry of their infinite dimensional environment consisting of other subsystems entangled to them. Hence, the Hilbert space of a subsystem represents the $G \times SU(\infty)$ symmetry. As a consequence, representations of $SU(\infty)$ by subsystems become comparable, and a dimensionful area or length parameter is added to the list of parameters characterizing the quantum state of subsystems. In addition, a subsystem can play the role of a quantum clock and define a relative time and dynamics [25, 26]. Mutual entanglement of all subsystems, including the clock, with their environment ensures a synchronous relative variation and an arrow of time.

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Using the Mandelstam–Tamm inequality [27]—the quantum speed limit—it is shown that a classical spacetime emerges as an effective manifestation of the $(1+3)$-dimensional parameter space of the Hilbert space of the $SU(\infty)$ symmetry of the subsystems and their relative dynamics. Moreover, it is proved that signature of the metric of this emergent spacetime is negative. Thus, $SU(\infty)$-QGR relates the Lorentzian geometry of classical spacetime to quantum states of the content of the Universe and their dynamics. However, $SU(\infty)$-QGR does not explore the geometry of the Hilbert space per se. (Geometry has been first introduced in the Hilbert space of many-body quantum systems in order to study collective properties, such as deformation of nuclides (see [28] and references therein). Considering an Euclidean $\mathbb{R}^n$ parameter space for the quantum states, they used variation of state and its path in the Hilbert space to define a symmetric 2-tensor as metric and a 2-form as symplectic structure interpreted as the *induced geometry* of parameter space. This geometric approach helps study complicated variations of the quantum state of many-body systems; see, e.g., [29] for an application example of this method.) Rather, it relates the perceived spacetime and its geometry to the parameter space of quantum states. It is also shown that the geometry of the parameter space is irrelevant for physical observables and can be gauged out by a $SU(\infty)$ transformation.

A functional constructed from invariants of $SU(\infty)$ and G plays the role of action analogous to those of Quantum Field Theories (QFTs). At the lowest order of symmetry-invariant traces, this effective action is similar to non-Abelian Yang–Mills models for both $SU(\infty)$ and G symmetries. There is, however, an essential difference: the action is defined on the (3 + 1)D space of continuous parameters of the model, rather than a classical spacetime. Due to the Yang–Mills nature of interactions, the mediator boson of quantum gravity in $SU(\infty)$-QGR is a spin-1 field. Nonetheless, it is shown that at classical limit—that is, when quantum effects of gravity are not detectable—the gravity sector dynamics is ruled by the Einstein–Hilbert action. The non-commutative algebra of $SU(\infty)$ can be normalized such that it depends on $\hbar/M_P$, where M_P and $\hbar$ are the Planck mass and the constant, respectively. This normalization is necessary for making pure gravity—$SU(\infty)$ gauge—term of the action to have a dimensionality similar to the scalar curvature in the Einstein–Hilbert action. In the standard classical limit, i.e., $\hbar \rightarrow 0$, or in the absence of gravity, i.e., $M_P \rightarrow \infty$, the algebra becomes Abelian and the model becomes trivial.

The aim of the present work is to clarify some of the technical issues that were not addressed in the previous works (in particular [2], where the properties of $SU(\infty)$-QGR summarized in the above paragraphs are studied). Here we only remind those details that are necessary for understanding the content of this work. For complementary explanations and demonstrations, the reader should refer to [1, 2]. The topics discussed here are necessary both for reinforcing the construction and consistency of the model and for preparing the ground for its applications to topics such as black hole information puzzle and prediction of outcomes of future laboratory tests of quantum gravity. The main results reported here are summarized in Section 1.1.

In Section 2, we clarify the issue of the degeneracy of the $\mathcal{SU}(N \rightarrow \infty)$ algebra and its impact on $SU(\infty)$-QGR's parameter space. Emergence of subsystems, that is, their Hilbert spaces, and entanglement of their states with the rest of the Universe is investigated in Section 3. In order to provide an explicit and quantitative proof of Proposition 1 in an earlier study [2], we quantify the entanglement by calculating several of its measures in Section J, Supplementary material and compare the results with QGR models that try to explain gravity as a consequence of entanglement [30–32]. In Section 4, we review the dynamics of the Universe and its subsystems according to $SU(\infty)$-QGR and provide an analytical description of the relationship between the $SU(\infty)$ symmetry of the Hilbert space and the diffeomorphism of its parameter space. Irrelevance of the geometry of parameter space is the subject of Proposition 2 in an earlier study [2]. Here we investigate and demonstrate this crucial property in more detail. Finally, in Section 5, we briefly outline an experiment/observation which may be able to detect the signature of $SU(\infty)$-QGR with present facilities. We also discuss topics left for future investigations. The Supplementary material includes the details of arguments, calculations, and reviews of subjects necessary for this work.

1.1. Summary of main results

Regarding the $SU(\infty)$-QGR model, we reinforce demonstration of Propositions 1 and 2 of an earlier study [2], calculate quantum states of subsystems of the Universe, and investigate their properties:

Emergence and properties of structures in the quantum state of the Universe: In an earlier study [2], we employed an operational approach to show how structures may emerge in the quantum state of the Universe. We demonstrated that they can be approximately treated as quantum subsystems. Using those results, here we construct their states in detail. In particular, we calculate the mixed states of subsystems with finite rank symmetries and their infinite dimensional environment. We quantify their entanglement by determining their mutual information (entanglement entropy) and quantum negativity. We demonstrate the emergence of secondary parameters related to the $SU(\infty)$ symmetry and the relative dynamics of subsystem. They include two continuous parameters, namely a quantity presenting relative area or size of diffeo-surfaces of $SU(\infty)$ representations by subsystems, and a time variable. In addition, states may depend on two discrete parameters characterizing the orthogonal function basis used for construction of the generators of $\mathcal{SU}(\infty)$ algebra. The mixed states of subsystems depend on these parameters and two continuous parameters characterizing $SU(\infty)$ representations. We show that these states are similar to the fields in QFTs with a gauge symmetry, for example, those of the Standard Model (SM). Thus, $SU(\infty)$-QGR relates the origin of the observed gauge symmetries of fundamental interactions to QGR. On the other hand, the quantum mixed nature of these states and their dependence on discrete quantum numbers, unrelated to their local symmetries, deviate from pure state of the SM particles. They reflect quantum origin of the four continuous parameters, the effective values of which are shown in previous studies [1, 2] to be the perceived classical spacetime. These features might be testable in future experiments seeking signatures of quantum gravity.

Purification: Quantum origin of the 4D continuous parameter space of subsystems is made transparent by purification of their mixed states. We demonstrate that states of subsystems satisfy the conditions for perfect purification according to an earlier study [33]. Moreover, using the effective classical metric mentioned earlier, we show its relationship with the quantum state of environment seen by subsystems. In turn, we show that in accordance with the results of an earlier study [34], the mixed state of environment is perspective dependent and related to the internal symmetries of the subsystem. This is a remarkable result, because in general relativity, the relation between spacetime geometry and matter is established through dynamics, that is, Einstein–Hilbert action and Einstein equation. By contrast, in $SU(\infty)$-QGR, it arises through introduction of a quantum clock and before introducing an action.

Clarification of relationship between four continuous parameters of subsystems and their $SU(\infty)$ symmetry: Association of 4D continuous parameters to the $SU(\infty)$ symmetry of subsystems seems confusing, because representations of $SU(\infty)$ depend on two continuous parameters. We show that purified states are composed of two entangled components: one representing a finite rank symmetry G and the other $SU(\infty)$. Using the fact that sine algebra represents $\mathcal{SU}(\infty) + \mathcal{U}(1)$, we demonstrate that in the presence of multiple representations of $SU(\infty)$, their individual $U(1)$ symmetry breaks (see diagrams (21) and (22)), because one can define a morphism between their algebras—in other words, between their associated 2D diffeo-surfaces. A consequence of this morphism is that the relative area of these compact surfaces becomes an observable. Moreover, with introduction of a quantum clock and time parameter, quantum states of subsystems will depend on four continuous parameters, which are indistinguishable, and quantum states are in their coherent superposition. Additionally, arbitrariness of references and clocks dictates that observables must be independent of reparameterization and diffeomorphism of the parameter space. Therefore, we conclude that 2D diffeo-surfaces of $SU(\infty)$ representations are subspaces of the 4D parameter space (see **Figure 1**).

Equivalence of $SU(\infty)$ gauge transformation and diffeomorphism of the parameter space of subsystems: We provide an analytical description of the relationship between $SU(\infty)$ symmetry of the Hilbert space and diffeomorphism of the space of its continuous parameters. We discuss both the case of the Universe as a whole, where there is only one representation of $SU(\infty)$, and the case of the ensemble of subsystems representing $SU(\infty)$ and their relative dynamics, where diffeo-surfaces of $SU(\infty)$ representations are 2D subspaces of a 4D space. We show that diffeomorphism of this parameter space can be neutralized by a $SU(\infty)$ gauge transformation. Therefore, its geometry is irrelevant for the Lagrangian functional of the model. This demonstration reinforces the proof of Proposition 2 of an ealier study [2].

Additionally, we demonstrate the following general properties of $SU(\infty)$ symmetry group and criteria for considering a quantum system as being composite. They were necessary for obtaining the results summarized in the previous paragraph.

Multiple limits of $SU(\infty)$: We show that the degeneracy of $\mathcal{SU}(N \to \infty)$ algebra demonstrated in a previous study [14] does not have impact on the construction and properties of $SU(\infty)$-QGR parameter space. Although the sine algebras (A.42) for $k \neq 2\pi/N$, where $N \in \mathbb{Z}$ are not homomorphic to each others or to $k = 2\pi/N$, they all have the same limiting algebra when $k \to 0$, namely (A.39) which is homomorphic to the algebra of area preserving diffeomorphism $ADiff(D_2)$ of 2D compact Riemannian surfaces. In the framework of $SU(\infty)$-QGR , it is this limit that we call $\mathcal{SU}(\infty)$. Thus, there is no ambiguity in the definition of parameters characterizing representations of $SU(\infty)$ in the model.

Criteria for compositeness: We show that two sets of conditions used in the literature as indicator of compositeness of a quantum system, namely factorization of density matrix or state vector [35] and factorization of the algebra of observables [19], are equivalent.

2. Multiple limits of $\mathcal{SU}(N \to \infty)$ and their consequence for $SU(\infty)$-QGR

The group $SU(\infty)$, its representations and algebra, and their properties are briefly reviewed in Section F, Supplementary material. Variables and symbols related to these entities are defined there.

A particularity of $SU(N \to \infty)$ is its non-uniqueness [14]. There are various ways to see this. For instance, if $N \to \infty$ and $N' \to \infty$ are prime with respect to each other, or more generally for irrational k's and integer vectors $(\mathbf{m}, \mathbf{n})$ in (A.42)—the case considered by previous works [12–14]—structure coefficients of the algebras would be always different, and thereby they would not be homomorphic to each other. In a study [14], it is proved that for any $k \in \mathbb{C}$, the algebra (A.43) is homomorphic to $\mathcal{SU}(\infty) + \mathcal{U}(1)$. Moreover, it is shown that for $k \in \mathbb{R}$, algebras with different values of k are not pairwise homomorphic. Consequently, it seems that $SU(N \to \infty)$ and its algebra have uncountable degeneracies. Additionally, for irrational k such that $1/k \to \infty$, these algebras do not have a shift symmetry similar to (A.48) and cannot be homomorphic to $ADiff$ of any orientable Riemann surface. We remind that the proof of $SU(\infty) \cong ADiff(D_2)$ in earlier studies [9,16] is limited to the case of rational k or equivalently countable N in (A.43) (in this work, we use the symbol $\cong$ for homomorphism, because the symbol $\simeq$ used for this purpose in the mathematics literature is used for numerical approximation in the physics literature). Furthermore, in Section G, Supplementary material, we demonstrate that the description of generators $\hat{K}_{\mathbf{m}}$ of (A.42) as functionals of $\mathbf{x}$ and $\partial_{\mathbf{x}}$ proposed in the study [12] generates this algebra only approximately, and for $k \neq 0$, no other functional of these variables can generate the algebra exactly. This finding confirms the conclusion of a previous study [14], asserting that some of (A.42) algebras are not homomorphic to $ADiff(D_2)$.

In what concerns the application of $SU(\infty)$ in string and matrix (M-)theories as candidate quantum gravity, the absence of its homomorphism with $ADiff(D_2)$ can be considered as a serious problem. Indeed, the motivation for investigating the relationship between $SU(\infty)$ and $ADiff(D_2)$ in the 1980s and 1990s was using $N \times N$ matrices as symplectic approximation of strings worldsheets or more generally membranes [9,15]. In this context, homomorphism of $SU(\infty)$ and $ADiff(D_2)$ is crucial, because quantum gravitational interactions are associated to deformation of D_2 surfaces and vice versa. Thus, the homomorphism between geometrical and algebraic operations is crucial.

By contrast, the $SU(\infty)$ symmetry of the Hilbert space in $SU(\infty)$-QGR arises because of the assumption of an infinite number of mutually commuting observables. Although the homomorphism of $SU(\infty)$ representations with $ADiff(D_2)$ is also important, especially for providing two continuous parameters, it is a subsidiary property. As discussed in detail in an earlier study [2] and in the following sections here, these parameters along with two other continuous variables, namely area or size/distance and time, characterize $SU(\infty)$ and dynamics-related aspects of the quantum states of subsystems/particles. On the other hand, it is easily seen that the $\mathcal{SU}(\infty)$ algebra always depends on two variables, irrespective of the degeneracy of algebra (A.43) and whether it is homomorphic to $ADiff(D_2)$ or not. Indeed, $SU(N), \forall N$ is a N-dim representation of $SU(2)$. As the group manifold of $SU(2)$ is the sphere $S^{(2)}$, its generators in any representation, including $SU(\infty)$, can be described as functions of angular coordinates of sphere (see [6] for details). It is easy to see that even without using association with $ADiff(D_2)$, algebras (A.43) and (A.42) also depend on two approximately continuous variables. Indeed, generators $\hat{K}_{\mathbf{m}}$ with $\sqrt{2\pi/N}\mathbf{m} \rightarrow (\infty, \infty)$ are dominant by their number. As fractional numbers are dense in $\mathbb{R}$, vectors $\sqrt{2\pi/N}\mathbf{m}$ are approximately continuous 2-vectors. Therefore, the diffeomorphism of $SU(\infty)$ and $ADiff(D_2)$ is less crucial for $SU(\infty)$-QGR.

In any case, it is important to remind that both (A.42) and (A.43) algebras are homomorphic to $\mathcal{SU}(\infty)$ only for $1/k \rightarrow \infty$. Expression (A.49) of the generators of (A.42) also becomes exact in this limit, which corresponds to algebra (A.39) and homomorphic to $ADiff(D_2)$. Therefore, although (A.42), (A.43), and (A.49) present different algebras for $k \neq 0$, they all converge to a unique limit, namely algebra (A.39), which is homomorphic to the Poisson algebra, and as demonstrated in Section F, Supplementary material to $ADiff(D_2)$. Thus, from now on and in the context of the $SU(\infty)$-QGR model by $\mathcal{SU}(\infty)$, we always mean this unique limit.

3. Order from randomness: emergence of subsystems

In previous studies [1, 2], we showed that the $SU(\infty)$-QGR Universe as a whole is static and trivial, in the sense that different states can be transformed to each other by the application of $SU(\infty)$ members, under which, according to the axioms of the model, the physics is invariant. Only when the Universe is divided to approximately isolated subsystems, structures and relative dynamics arise from their interactions and entanglement. In this section, we study in more detail the quantum state of a subsystem and its relationship with the rest of the Universe, formed by the ensemble of other subsystems.

Consider a quantum system with an N-dimensional Hilbert space representing the $SU(N)$ group. Assuming $N \gg 1$, the Cartan decomposition of $SU(N)$ [36] can be used to describe the symmetry, and thereby the Hilbert space, as the tensor product of smaller rank Lie groups:

$$\{|\psi_1\rangle\} \times \{|\psi_2\rangle\} \cdots \times \{|\psi_n\rangle\} \tag{1}$$

where the n_i-dimensional set $\{|\psi_i\rangle\}$ is a basis for the ith component of the Cartan decomposition. Notice that (1) has the same structure as the Hilbert space of an n-body system. In general, N, the number of components n, and dimensions n_i can be infinite or even innumerable. In the case of $SU(\infty)$-QGR , $N = n = \infty$.

The Cartan decomposition of the $SU(\infty)$ group is reviewed in Section F. of an earlier work [2]. It is shown that it can be decomposed to the tensor product of an infinite number of $SU(K)$, that is, $SU(\infty) \cong \bigotimes^{\infty} SU(K)$ for any K, including infinity. More generally, $SU(\infty)$ can be written as the tensor product of any finite rank compact Lie group G [16]. Thus, there is a basis similar to (1) for the Hilbert space of the Universe $\mathcal{H}_U$ with $n_1.n_2.\cdots.n_{i\rightarrow\infty}$ components, corresponding to the decomposition $SU(\infty) = G^1 \times G^2 \times \cdots$, where n_i is the dimension of representation $\{|\psi_i(k_i)\rangle\}$ for the group G^i. The d_i-dimensional set $\{k_i\}$ indicates the space of parameters characterizing the representation of G^i. With this decomposition, the state of the Universe $|\Psi_U\rangle$ can be expanded as follows:

$$|\Psi_U\rangle = \sum_{\{k_i=1,\cdots,d_i\},\ i=1,\cdots n\rightarrow\infty} \mathcal{A}(k_1k_2\ldots k_n,\cdots)\,|\psi_{k_1}\rangle \times |\psi_{k_2}\rangle \times \cdots \times |\psi_{k_n}\rangle \times \cdots \tag{2}$$

Notice that we have simplified the notation by using k_i as an index. Thus, $|\psi_{k_i}\rangle \equiv |\psi_i(k_i)\rangle$.

Description of a basis as a tensor product does not necessarily mean that the corresponding quantum system consists of subsystems [19]. In previous studies [1, 2], we used the algebraic criteria introduced in another study [19] and random quantum fluctuations to explain how the concepts of locality and approximate division to subsystems arise in the Universe. An equivalent but more explicit criterion for definition of subsystems using a tensor product basis, as in (2), is the factorization of amplitudes $A_{k_1k_2\cdots k_n}$. States for which these amplitudes can be decomposed to factors, depending only on a limited number of parameters that characterize $\otimes_i^{n\rightarrow\infty}|\psi_{k_i}\rangle$, are called *separable* or *product states* [35]. They present partial localization in the Hilbert space.

The factors $A_{k_1k_2\cdots k_n}$ depending on a single k_i expand subspaces with no entanglement. By contrast, factors depending on $n' \rightarrow \infty$ parameters expand inseparable and fully nonlocal subspaces of $\mathcal{H}_U$. Notice that unless *locality* in spacetime is explicitly mentioned, here locality means in the Hilbert space, according to the definition given in Section C, Supplementary material. In Section H, Supplementary material, we demonstrate that factorization of state vector or density matrix is equivalent to the algebraic criteria defined in a previous study [19] for the compositeness of a quantum system.

3.1. Decomposition to finite rank symmetries

In $SU(\infty)$-QGR, when dealing with the whole Universe, it is not useful to consider a tensor decomposition like (2) for its state $|\Psi_U\rangle$, because it does not survive a $SU(\infty)$ transformation, which preserves physical observables. Indeed, in earlier studies [1,2], we demonstrated that due to this global symmetry, $|\Psi_U\rangle$ is trivial—the vacuum—except for the dependence of $SU(\infty)$ representation on the topology of its diffeo-surface. Nonetheless, states like $|\Psi_U\rangle$ can have very different components. Specifically, in previous studies [1, 2], we showed that quantum fluctuations, that is, random application of operators $\hat{O} \in \mathcal{B}[\mathcal{H}_U]$ on $|\Psi_U\rangle$, induce clustering of components. In particular, a completely coherent state in an arbitrary basis changes to a less coherent state with more clustered and unbalanced components. In such state, subsets of elements with very different amplitudes from their neighbors can be approximately considered as subsystems. Hence, the state $|\Psi_U\rangle$ can be approximately factorized to a tensor product of subspaces of $\mathcal{H}_U$, where each factor represents a smaller rank symmetry.

Consider one of the factors in (2), and assume that it represents a finite rank symmetry G. The ensemble of remaining factors is still infinite dimensional and represents $SU(\infty)$ [16]. To prove this property, we use the decomposition of $SU(\infty)$ to a finite rank $SU(K)$, that is, $SU(\infty) \cong \bigotimes^{\infty} SU(K), K \in \mathbb{Z}^+ + 2$. For any finite rank compact Lie group G, one can find $K \in \mathbb{Z}^+ + 2$, such that $SU(K') \subset G \subset SU(K)$, $K' < K$, and the following relations are satisfied:

$$SU(\infty) \cong \bigotimes^{\infty} SU(K') \subseteq G \times \bigotimes^{\infty} SU(K') \subseteq G \times \bigotimes^{\infty} SU(K) \subseteq SU(K) \times \bigotimes^{\infty} SU(K) \cong SU(\infty) \quad (3)$$

Moreover, this chain shows that for any finite rank G:

$$G \times SU(\infty) \cong SU(\infty) \quad (4)$$

The difference between $SU(\infty)$s in the two extremes of (3) is similar to a line without any specific point on it—analogous to the left extreme of (3)—and a line with a finite number of selected points—analogous to the right extreme of (3). In both cases, the set of points constituting the line are the same. However, in the second case, some points are interpreted, employed, or perceived differently. A detailed demonstration of (4) is provided in Section I, Supplementary material.

We call the basis that expands states representing G symmetry $\{|\psi_G(k_G)\rangle\}$, where k_G collectively represents parameters of the representation of G by this basis. For finite rank Lie groups, the number of independent k_Gs, that is, the dimension d_G of parameter space $\{k_G\}$, is finite. For instance, for $G = SU(2)$, $k_G = (l', m')$, $l' \in \mathbb{Z}^+$, $-l' \leqslant m \leqslant l'$. For a fixed l', corresponding to a super-selected representation of $SU(2)$, the dimension $d_G = 2l' + 1$.

Following the tensor decomposition (3), the state of the Universe can be expanded as follows:

$$|\Psi_U\rangle = \sum_{\{k_G\},(\eta,\zeta,\cdots)\}} A(k_G; \eta, \zeta, \cdots)\, |\psi_G(k_G)\rangle \times |\psi_\infty(\eta, \zeta; \cdots)\rangle \quad (5)$$

and the corresponding density matrix:

$$\hat{\varrho}_U = \sum_{\substack{\{k_G, k'_G\} \\ (\eta,\zeta,\eta',\zeta',\cdots)\}}} A(k_G; \eta, \zeta, \cdots) A^*(k'_G, \eta', \zeta', \cdots)\, \hat{\rho}_G(k_G, k'_G) \times \hat{\rho}_\infty(\eta, \zeta, \eta', \zeta', \cdots) \quad (6)$$

$$\hat{\rho}_G(k_G, k'_G) \equiv |\psi_G(k_G)\rangle\langle\psi_G(k'_G)|, \quad \hat{\rho}_\infty(\eta, \zeta, \eta', \zeta', \cdots) \equiv |\psi_\infty(\eta, \zeta, \cdots)\rangle\langle\psi_\infty(\eta', \zeta', \cdots)| \quad (7)$$

$$\sum_{\substack{\{k_G\} \\ \{(\eta,\zeta,\cdots)\}}} |A(k_G; \eta, \zeta, \cdots)|^2 = 1 \quad (8)$$

where states $|\psi_\infty(\eta, \zeta; \cdots)\rangle$ constitute a basis for $SU(\infty)$ factor of $\mathcal{H}_U$ and continuous parameters (η, ζ) characterize generators of $SU(\infty)$ (see Section F., Supplementary material, for details). Extension dots indicate secondary parameters defined in the following subsection. When they are irrelevant or do not contribute to the discussion, we replace them with extension dots $''\cdots''$. Notice that we have used an arbitrary basis for G and $SU(\infty)$ factors. Consequently, in general, off-diagonal elements of the density matrix are not zero. Moreover, despite the tensor product structure of the basis in (5) and (6), amplitudes $A(k_G; \eta, \zeta, \cdots)$ are not exactly factorizable, because a global $SU(\infty)$ transformation changes the tensor product decomposition of $|\Psi_U\rangle$ and $\hat{\varrho}_U$. Therefore, isolation of a subsystem is always an approximation.

3.1.1. Secondary parameters of $SU(\infty)$ representations of subsystems

Secondary parameters in the basis $|\psi_\infty(\eta, \zeta; \cdots)\rangle$ are those that do not affect a single representation of $SU(\infty)$, but can be involved in the characterization of $SU(\infty)$ represented by Hilbert spaces of multiple quantum systems/subsystems interacting with each other. One of these parameters r indicates the relative area or a size scale for diffeo-surfaces of $SU(\infty)$ representations in the model with respect to that of a reference. As discussed in Section F, Supplementary material, the algebra $ADiff(D_2)$ is homomorphic to $\mathcal{SU}(\infty) + \mathcal{U}(1)$, where $\mathcal{U}(1)$ presents scaling of the diffeo-surface. This symmetry is irrelevant—unobservable—for the state of a single quantum system representing $SU(\infty)$ irreducibly, such as $\hat{\varrho}_U$. On the other hand, the area or size scale of diffeo-surfaces of multiple (sub)systems representing $SU(\infty)$ becomes a relative observable. This is the case when the Universe is divided to subsystems, each representing the $SU(\infty)$ symmetry.

Another secondary parameter is time t characterizing the quantum state of a clock subsystem. In earlier studies [1,2], we showed that in this way, subsystems acquire a relative dynamics [25, 26] or equivalent approaches [37]. As time measurement may be a positive operator-valued measure (PVOM) rather than projective, in $SU(\infty)$-QGR, quantum states of subsystems are in general in coherent superposition of four continuous parameters $x \equiv (\eta, \zeta, r, t)$, where (η, ζ) are parameters of $SU(\infty)$ representation. Assuming compact ranges for the last two parameters, that is, using the fact that $\mathbb{R} \cong U(1)$, or non-compact values for η, ζ, the four parameters become indistinguishable components of a 4D vector $x \equiv (x^i,\ i = 0, \cdots, 3) \in S_4(\mathbb{R}^4)$, of a compact or non-compact parameter space that we call Ξ. Logically, observables of subsystems should be invariant under reparameterization of Ξ. Moreover, Proposition 2 of the study [2] demonstrates that the geometry of Ξ is irrelevant for observables and can be modified by a $SU(\infty)$ transformation under which the physics is invariant. This property will be further discussed in Section 4.2.

As generators of $ADiff(D_2) \cong SU(\infty)$ act on continuous functions of (η, ζ), they are usually expanded with respect to a set of orthogonal functions as basis. These functions may, in turn, depend on a set of usually discrete parameters that we collectively call ℓ. For example, for a sphere basis, $\boldsymbol{\ell}$ corresponds to the spherical harmonic modes (l, m) (see appendices of the study [1] for a brief review). For a torus basis, as discussed in Section F., Supplementary material $\boldsymbol{\ell} = \mathbf{m}$, where $\mathbf{m}$ is an integer 2D vector. We emphasize that $\boldsymbol{\ell}$ is not an independent parameter of the model, because as it is shown in the study [1], the algebra $\mathcal{SU}(\infty)$ and its generators can be formally expressed with respect to two continuous parameters only.

3.2. Quantum states of subsystems

To extract from the factorized $\hat{\varrho}_U$ in (6) a state that represents only the group G, we trace out its $SU(\infty)$ representing factor:

$$\hat{\varrho}_G \equiv \mathrm{tr}_\infty\, \hat{\varrho}_U = \sum_{\substack{\{k_G, k'_G\}, \\ \{y\}}} A_G(k_G; y) A^*_G(k'_G, y)\, \hat{\rho}_G(k_G, k'_G), \quad y \equiv (\eta, \zeta, \cdots) \quad (9)$$

The amplitude A_G differs from A in (6) by a normalization factor, which may in general depends on $(\eta, \zeta, \cdots)$. We call the Hilbert space generated by $\hat{\varrho}_G$ density matrices $\mathcal{H}_G$. It represents the symmetry group G, because the basis $\hat{\rho}_G(k_G, k'_G)$ represents this symmetry. Nonetheless, amplitudes $A_G(k_G; \eta, \zeta, \cdots)$ and their complex conjugate A^*_G depend on parameters $(\eta, \zeta, \cdots)$, which are not related to the G symmetry. An observer ignoring their quantum origin interprets them as being related to a *classical* environment. On the other hand, application of any operator $\hat{O} \in \mathcal{B}[\mathcal{H}_U] \cong SU(\infty)$ changes $\hat{\varrho}_U$, and consequently the dependence of $A_G(k_G; \eta, \zeta, \cdots)$ on the *environment* parameters $(\eta, \zeta, \cdots)$. Therefore, $\hat{\varrho}_G$ presents the mixed quantum state of an open subsystem. To show explicitly the mixedness of $\hat{\varrho}_G$, it is enough to demonstrate that $\hat{\varrho}^2_G \neq \hat{\varrho}_G$. It is straightforward to show that:

$$\hat{\varrho}^2_G = \sum_{\{\{y\},\{\{y'\}} B(y, y') \sum_{\{k_G\},\{k'_G\}} A_G(k_G; y) A^*_G(k'_G; y')\, \hat{\rho}_G(k_G, k'_G) \quad (10)$$

$$B(y, y') \equiv \sum_{\{k''_G\}} A^*_G(k''_G; y) A_G(k''_G; y') \quad (11)$$

Comparison of (10) and (9) clearly shows that $\hat{\varrho}^2_G \neq \hat{\varrho}_G$, except in the limit where amplitudes A_G do not depend on the value of y. Thus, $\hat{\varrho}_G$ is in general a mixed state and the subsystem representing G is entangled to the rest of the Universe.

In a similar manner, $\hat{\rho}_G(k_G)$ factor in $\hat{\varrho}_U$ can be traced out to find the contribution of the $SU(\infty)$ symmetry in $\hat{\varrho}_U$ after separating a subsystem with the G symmetry:

$$\hat{\varrho}_\infty \equiv \text{tr}_G\, \hat{\varrho}_U = \sum_{\substack{\{k_G\},\\ \{(\eta,\zeta,\cdots)\},\{(\eta',\zeta',\cdots)\}}} A_\infty(k_G; \eta, \zeta, \cdots) A^*_\infty(k_G; \eta', \zeta', \cdots)\, \hat{\rho}_\infty(\eta, \zeta; \eta', \zeta', \cdots) \quad (12)$$

We call the Hilbert space generated by $\hat{\varrho}_\infty$ states $\mathcal{H}_\infty$. Using analogous expressions to (10–11) for $\hat{\varrho}_\infty$, it is straightforward to show that $\hat{\varrho}_\infty$ is also in a mixed state. Moreover, it depends on *external parameters* k_G, which is not related to the $SU(\infty)$ symmetry. In Section J, Supplementary material, we calculate several entanglement measures for $\hat{\varrho}_G$ and $\hat{\varrho}_\infty$.

3.2.1. Significance of entanglement

For various reasons, the above results are interesting. They show that although $\mathcal{H}_U$, and thereby $\hat{\varrho}_U$, are infinite dimensional and the resulting state $\hat{\varrho}_\infty$ is still infinite dimensional after tracing out a finite number of its components, it is no longer pure. Indeed, application of any $\hat{O} \in G \subset SU(\infty)$ to $\hat{\varrho}_U$ changes the dependence of $A_\infty(k_G; \eta, \zeta, \cdots)$ on k_G, which is the *environment* parameter for $\hat{\varrho}_\infty$. Therefore, the *environment* itself is an open subsystem of the Universe. The only *closed*—and thereby static—quantum system is the Universe itself. This observation is an explicit demonstration of Proposition 1 in the study [2] and confirms the entanglement of every subsystem with the rest of the Universe.

Another interesting aspect of $\hat{\varrho}_\infty$ is its dependence on the parameters of the traced symmetry G, which, as we have discussed in an earlier study [2] (and further in the following sections), can be considered as an approximately isolated subsystem—a matter field. In previous studies [1, 2], we related the variation of $\hat{\varrho}_\infty$ to an affine parameter and an effective metric, which we identified as the metric of the perceived classical spacetime:

$$ds^2 \equiv \Lambda ds^2_{WY} = \Lambda\, \text{tr}(\sqrt{\delta\hat{\varrho}_\infty}\sqrt{\delta\hat{\varrho}_\infty}^\dagger) \equiv g_{\mu\nu}(y) dy^\mu dy^\nu \quad (13)$$

The constant Λ is a dimensionful scaling constant, and ds^2_{WY} is the separation between $\hat{\varrho}_\infty$ and $\hat{\varrho}_\infty + \delta\hat{\varrho}_\infty$ according to the Wigner–Yanase skew information [38, 39]. Using Mandelstam-Tamm inequality [27]—also called quantum speed limit—we proved that the effective metric $g_{\mu\nu}$ is Lorentzian and has a negative signature. Thus, it is reasonable to interpret it as the perceived classical spacetime, which we call Ξ_{sp} to distinguish it from the parameter space Ξ.

The dependence of $\hat{\varrho}_\infty$ and its variation $\delta\hat{\varrho}_\infty$ on the matter parameter k_G means that in $SU(\infty)$-QGR , the relationship between the effective metric $g_{\mu\nu}$ and state of matter emerges at constructional level. By contrast, in Einstein gravity, metric tensor and affine parameter are related to matter and its state dynamically, that is, through the Einstein–Hilbert action and Einstein equation. Of course, the single equation (13) is not enough for a full determination of $g_{\mu\nu}$ tensor and dynamics is needed for this purpose.

3.3. Emergence of gauge symmetries

In quantum mechanics and quantum information, inaccessible or unobservable degrees of freedom are usually traced out. Hence, we can interpret separation/tagging of a finite dimensional subspace representing group G from $\mathcal{H}_U$ and tracing out its infinite dimensional complement subspace as isolation of a *subsystem* due to the *inaccessibility* of its *environment*.

According to (3), it is possible to factorize an infinite number of G factors from $SU(\infty)$. Each factor can be considered as one realization of a *subsystem* representing G. Moreover, $\mathcal{H}_G$ may be a reducible representation of G. In this case, $\hat{\varrho}_G$ can be interpreted as many-body quantum state of an ensemble of subsystems, each representing G irreducibly. In both cases, subsystems are not completely isolated and interact with each other through application of operators $\hat{O}_G \in \mathcal{B}[\mathcal{H}_G]$. They also interact with each other and with other types of subsystems with different internal symmetries through their shared $SU(\infty)$ symmetry/gravity.

In (9), operators $\hat{O}_G$ only affect $\hat{\rho}_G$. Therefore, they can be performed independently for every value of the *environment* parameters $(x; \boldsymbol{\ell})$. By rearranging summations in (9), the density matrix $\hat{\varrho}_G$ can be written as follows:

$$\hat{\varrho}_G = \sum_{\substack{\{k_G\},\{k'_G\}\\ \{(x;\boldsymbol{\ell})\}\}}} \hat{\mathrm{p}}_G(k_G, k'_G; x; \boldsymbol{\ell}) \quad (14)$$

$$\hat{\mathrm{p}}_G(k_G, k'_G; x; \boldsymbol{\ell}) \equiv A_G(k_G; x; \boldsymbol{\ell}) A^*_G(k'_G; x; \boldsymbol{\ell})\, \hat{\rho}_G(k_G, k'_G) \quad (15)$$

Operators $\hat{\mathrm{p}}_G$ have the structure of local quantum fields in a 4D Yang–Mills QFT with group G as their gauge symmetry, in the sense that application of the members of G symmetry projects them to the space generated by $\hat{\rho}_G$. We remind that the notion of density matrix in quantum mechanics and the notion of quantum field in QFT are related. Section H, Supplementary material reviews their relationship. We emphasize on the locality of such operations, because they can be performed independently for each set of parameters $(x; \boldsymbol{\ell})$ in a subsystem. In previous studies [2, 5], we showed that in order to have a dynamics invariant under reparameterization of Ξ, the operator $\hat{\rho}_G$ must transform similar to a spin-1 field (see also Section 4.1). This completes the demonstration of the analogy between $\hat{\mathrm{p}}_G(k_G, k'_G; x; \boldsymbol{\ell})$ and Yang–Mills QFTs. However, there are several differences between $\hat{\mathrm{p}}_G(k_G, k'_G; x; \boldsymbol{\ell})$ and fields in models without quantum gravity. In $SU(\infty)$-QGR, the locality of finite rank G symmetry arises at construction level. This is in contrast to QFTs in which the dynamics—the Lagrangian—is designed such that it be invariant under local application of symmetry operators. Additionally, in QFTs, there are no discrete parameters analogous to ℓ, which we call *mode quantum numbers* of quantum gravity.

3.4. Purified states of subsystems

In previous studies [1,2], we used the global $SU(\infty)$ symmetry and how it affects subsystems to demonstrate that the full symmetry of subsystems is $G \times SU(\infty)$. In this section, we use purification of the mixed state $\hat{\varrho}_G$ to obtain this symmetry from a quantum point of view, rather than algebraic properties of the model.

Although tracing of $SU(\infty)$-related factors in (6) demonstrates how a local Yang–Mills structure associated to a finite rank symmetry arises for subsystems, it erases the quantum origin of $SU(\infty)$ and dynamics-related parameters $x; \ell$. Nonetheless, with the prior knowledge that the origin of these parameters in $\hat{\varrho}_G$ is the global $SU(\infty)$ symmetry of the Universe, we can extend $\mathcal{H}_G$ by a Hilbert space representing $SU(\infty)$. The new Hilbert space $\mathcal{H}_{G_\infty}$ represents $G \times SU(\infty)$, and its states can be expanded as follows:

$$|\Psi_{G_\infty}\rangle \equiv \sum_{\{k_G\};(\eta,\zeta,\cdots)} A_{G_\infty}(k_G; \eta, \zeta, \cdots) \, |\psi_G(k_G)\rangle \times |\psi_\infty(\eta, \zeta, \cdots)\rangle \quad (16)$$

where $|\psi_\infty(\eta, \zeta, \cdots)$ is the same as in (5). Notice that $|\Psi_{G_\infty}\rangle$ can also be considered as purification of $\hat{\varrho}_\infty$. Although the state $|\Psi_{G_\infty}\rangle$ looks like the state of the Universe $|\Psi_U\rangle$ in (5), they do not present exactly the same quantum system for the following reasons.

In (16), the group $SU(\infty)$ acts only on the $|\psi_\infty(\eta, \zeta, \cdots)\rangle$ factor of the basis. Thus, in comparison with the state $|\Psi_U\rangle$ in (5), or equivalently its density matrix $\hat{\varrho}_U$ in (6), the global $SU(\infty)$ symmetry is broken. The difference between $|\Psi_{G_\infty}\rangle$ and $|\Psi_U\rangle$ becomes more transparent if we express $SU(\infty)$ as $SU(N \to \infty)$. Indeed, in $|\Psi_{G_\infty}\rangle$, the $SU(\infty)$ symmetry of the appended Hilbert space should be read as $SU((N - d_G) \to \infty)$. In addition, the basis of the appended subspace $|\psi_\infty(\eta, \zeta, \cdots)\rangle$ is not necessarily the same as that used for obtaining $\hat{\varrho}_G$. Therefore, amplitudes $A_{G_\infty}(k_G; \eta, \zeta, \cdots)$ are arbitrary up to the unitarity constraint:

$$\sum_{\substack{\{k_G\} \\ \{(\eta,\zeta,\cdots)\}}} |A_{G_\infty}(k_G; \eta, \zeta, \cdots)|^2 = 1 \quad (17)$$

and *faithfulness* constraint [33], that is, tracing over extended subspace must recover $\hat{\varrho}_G$:

$$\mathrm{tr}_\infty \hat{\varrho}_{G_\infty} = \hat{\varrho}_G, \qquad \hat{\varrho}_{G_\infty} \equiv |\Psi_{G_\infty}\rangle\langle\Psi_{G_\infty}| \quad (18)$$

These conditions are not sufficient to uniquely relate amplitudes A_{G_∞}s to A_Gs. This is not a surprise as the degeneracy of purification is well known through the Schödinger–HJW theorem [40–42], asserting that any mixed state can be purified to many unitarily equivalent pure states. This is a consequence of the arbitrariness of both the *auxiliary* space used for the extension of the Hilbert space, and the basis chosen for this subspace. Although here we have chosen a fixed auxiliary Hilbert space based on our prior knowledge about how $\hat{\varrho}_G$ was obtained, as mentioned earlier, the basis remains arbitrary. It is well known that the lack of information about the basis—the reference frame—leads to decoherence [43]. Specifically, because the basis $|\psi_\infty(\eta, \zeta, \cdots)\rangle$ is orthogonal to the G-related factor and can be chosen or modified arbitrarily, the global $SU(N \to \infty)$ coherence symmetry is broken to $G \times SU((N - d_G) \to \infty)$. Nonetheless, if we treat the two subspaces representing G and $SU((N - d_G) \to \infty)$ as subsystems, they remain entangled because the constraint (18) ensures a perfect and faithful purification. Details of conditions for faithful purification and their satisfaction by $\hat{\varrho}_G$ and $\hat{\varrho}_\infty$ are discussed in Section K, Supplementary material.

3.4.1. Physical interpretation of purified states

Using linearity of the map Λ and the last equality in (A.70), which is the condition for faithfulness, it is straightforward to show that either Λ maps all $\hat{\varrho}_G$ to the same pure density matrix, or it cannot be a universal purifier for an ensemble of closely related mixed density matrices [33]. (Here by closeness, we mean density matrices that can be possible states of a given quantum system or subsystem. If states $\hat{\rho}_1$ and $\hat{\rho}_2$ are in the set of possible states for a system, their sum can also be a plausible state. The proof of Theorem 1 in the study [33] uses this superposition property.) Thus, when it is a universal purifier, its outcome would be a pure quantum state independent of details of $\hat{\varrho}_G$. For $\hat{\varrho}_G$ in a given representation of G symmetry, amplitudes $A_{G_\infty}(k_G; \eta, \zeta, \cdots)$ would be independent of $A_G(k_G; \eta, \zeta, \cdots)$ up to a unitary transformation of $\hat{\rho}_G \times \hat{\rho}_\infty$ basis. Physically, this means that when subsystems with the same internal symmetry are approximately disentangled from the rest of the Universe, they see the same environment, and therefore are indistinguishable. This conclusion is also consistent with the global equivalence of all $\hat{\varrho}_U$ states up to a physically irrelevant unitary transformation. By contrast, subsystems with different internal symmetries or different representations of the same symmetry see the rest of the Universe differently. Consequently, their purifying map Λ would be different, because their projections into the extended Hilbert space would be different.

In quantum information and quantum technologies, purification of the states of quantum systems is necessary for compensating noise and decoherence induced by interaction with an uncontrollable environment. Here, the purification of states $\hat{\varrho}_G$ and $\hat{\varrho}_\infty$ has rather a conceptual purpose. It shows that in a Universe with an infinite number of mutually commuting observables, pure quantum states can be associated to subsystems and their environment, at least intermittently. Indeed, the random action of global operators $\hat{O} \in SU(\infty)$, which can be considered to be part of the environment, affects both. On the other hand, similarity of the purified subsystem and environment states demonstrates that they are not really separable, and irrespective of what happens to them and the way their states are presented, their entanglement is retained.

3.5. Emergence of area/size parameter

In Section 3.4 , we explicitly showed that in $SU(\infty)$-QGR, subsystems of the Universe represent $G \times SU(\infty)$. This means that the Hilbert space $\mathcal{H}_{G_\infty}$ of each subsystem is a representation of $SU(\infty)$. In previous studies [1,2], we argued that in the presence of multiple subsystems representing $SU(\infty)$, the area of their diffeo-surfaces becomes a relative observable and should be added to the list of parameters characterizing their states. Here, we investigate this property further.

The Poisson algebra (A.37) of *ADiff*(D_2), which is homomorphic to the algebra $\mathcal{SU}(\infty)$ [6,12], is invariant under scaling. Thus, (A.37) and associated algebras, that is, the Poisson algebra of spherical harmonics in sphere basis [6], algebras (A.42) and (A.43) in torus basis, and their limit (A.39) are homomorphic to the Lie algebra $\mathcal{SU}(\infty) + \mathcal{U}(1)$ and represent the symmetry group $SU(\infty) \times \mathbb{R} \cong SU(\infty) \times U(1) = U(\infty)$. The scaling symmetry $\mathbb{R} \cong U(1)$ reflects the irrelevance of the area of compact diffeo-surfaces D_2 in the limit where these algebras are homomorphic to $ADiff(D_2)$.

It is easy to prove that $\otimes^n SU(\infty) \cong SU(\infty)$, $\forall n$ (see, e.g., appendices of [2]). This relation can also be expressed as follows:

$$\otimes^n SU(\infty) \cong \bigcup_{i=1}^{n} ADiff(D_2^{(i)}) \cong ADiff(D_2), \qquad D_2 \equiv \bigcup_{i=1}^{n} D_2^{(i)} \quad (19)$$

In isolation, the area of each diffeo-surface $D_2^{(i)}$ is arbitrary and its variation does not affect the homomorphism of $ADiff(D_2^{(i)})$ with $SU(\infty)$. However, once diffeo-surfaces are stuck together, only the area of their ensemble D_2 can be arbitrarily scaled. Thus, scaling diffeo-surface of one subsystem must be necessarily compensated by scaling of others. For this reason, the algebra becomes:

$$\otimes^n(\mathcal{SU}(\infty) + \mathcal{U}(1)) \rightarrow \otimes^n \mathcal{SU}(\infty) + \mathcal{U}(1) \cong \mathcal{SU}(\infty) + \mathcal{U}(1) \quad (20)$$

The consequence of these properties is the dependence of quantum states of subsystems to an additional continuous parameter $r > 0$ that indicates relative area (or its square root) of compact diffeo-surfaces of subsystems. Notice that $r = 0$ is equivalent to trivial representation of $SU(\infty)$ and is excluded by axioms of the model. The following diagrams summarize the relationship of $SU(\infty)$ and $ADiff$ for single and multiple representations:

Single subsystem

$$\begin{array}{ccccc} SU(\infty) & \longrightarrow & \mathcal{SU}(\infty) & \xrightarrow{\text{area irrel.}} & \mathcal{SU}(\infty) + \mathcal{U}(1) \\ \downarrow\cong & & & & \downarrow\cong \\ ADiff(D_2) & & \xrightarrow[\text{area irrel.}]{} & & ADiff(D_2) \times U(1) \end{array} \quad (21)$$

Multiple subsystems

$$\begin{array}{ccc} SU(\infty) \times \cdots \times SU(\infty) \cong SU(\infty) \longrightarrow \mathcal{SU}(\infty) & \xrightarrow{\text{area irrel.}} & \mathcal{SU}(\infty) + \mathcal{U}(1) \\ \downarrow \text{area irrel.} & & \downarrow \text{area irrel.} \\ (ADiff(D_2^{(1)}) \times U(1)) \times \cdots \times (ADiff(D_2^{(n)}) \times U(1)) & \xrightarrow[\text{symm. break}]{\text{area preserv.}} & ADiff(D_2) \times U(1) \end{array} \quad (22)$$

3.5.1. Consistency of the two perceptions of the quantum Universe

Once the quantum state of the Universe is self-clustered to approximately isolated subsystems and a clock parameter t is chosen, the purified state of the ensemble of subsystems $\hat{\varrho}_{U_s}$ can be written as follows:

$$\begin{aligned} \hat{\varrho}_{U_s} &= \bigotimes_{i=1}^{\infty} \hat{\varrho}_{G^{(i)}_\infty} = \bigotimes_{i=1}^{\infty} \sum_{\substack{\{k_{G^{(i)}}, k'_{G^{(i)}}\} \\ (x_i; \ell_i; x'_i; \ell'_i)\}}} A(k_{G^{(i)}}; x_i; \ell_i) A^*(k'_{G^{(i)}}; x'_i; \ell'_i) \\ & \left(\hat{\rho}_{G^{(i)}}(k_{G^{(i)}}, k'_{G^{(i)}}) \times \hat{\rho}_\infty(x_i; \ell; x'_i; \ell'_i) \right) \\ &= \prod_i \left(\sum_{\substack{\{k_{G^{(i)}}, k'_{G^{(i)}}\} \\ (x_i; \ell; x'_i; \ell'_i)\}}} A(k_{G^{(i)}}; x_i; \ell_i) A^*(k'_{G^{(i)}}, x'_i; \ell'_i) \right) \\ & \bigotimes_{i=1}^{\infty} \hat{\rho}_{G^{(i)}}(k_{G^{(i)}}, k'_{G^{(i)}}) \times \bigotimes_{i=1}^{\infty} \hat{\rho}_\infty(x_i; \ell; x'_i; \ell'_i) \end{aligned} \quad (23)$$

Here we have assumed that each subsystem i may represent a different internal finite rank symmetry $G^{(i)}$. However, from cosmological and laboratory observations, we know that many subsystems/particles share the same internal symmetry. Therefore, many of $G^{(i)}$s and their representations in (23) are the same. In addition, the number of subsystems does not need to be countable.

In appendices of the study [2], we demonstrated that tensor products $\otimes_{n\rightarrow\infty} G \cong SU(\infty)$ for any G, including $G = SU(\infty)$ (see also [16]). Hence, both factors in the tensor product in the last equality of (23) are density matrices representing $SU(\infty)$. Even if the time parameter t is fixed despite the apparent dependence of $\hat{\varrho}_{U_s}$ on time, it belongs to the same Hilbert space $\mathcal{H}_U$ as the state $\hat{\varrho}_U$ in (6), namely the quantum state of the entire Universe without breaking it to subsystems. Specifically, due to $\otimes_{i=1}^{\infty} G_i \cong SU(\infty)$, the state $\hat{\varrho}_{U_s}(t) \in \mathcal{H}_{U(t)} \subset \mathcal{H}_U$, and can be transformed to any other state of $\mathcal{H}_U$ by application of $SU(\infty)$ group, which, according to axioms of $SU(\infty)$-QGR , should not affect observables. Nonetheless, the state of the clock for this outcome should be included in $\hat{\varrho}_{U_s}$, because the clock may have other observables commuting with that used as time. For instance, if t corresponds to the phase of an oscillating spin of an atom, its uncorrelated (commuting) degrees of freedom to spin, such as kinetic energy, would be in general in coherent superposition and full quantum state of the clock would be similar to (16). The state of the Universe can be evolved only if the clock is approximately isolated from other subsystems. In this case, as explained in Section (3.2), it will remain entangled to the rest of the Universe and can evolve by both internal (self-) interaction and interaction with other subsystems through both shared internal symmetries and $SU(\infty)$—gravity.

In conclusion, considering the Universe as the ensemble of its infinite number of subsystems is not in contradiction with treating it as a single quantum system with an infinite number of mutually commuting observables. The difference of perception is on whether concentration or access is on the local features of the Universe's state or its global—topological—properties. In what concerns our present observational capabilities, we can only detect its local aspects.

4. Dynamics over a parameter space

Quantum state of the ensemble of subsystems $\hat{\varrho}_{U_s}(t)$ calculated in (23) includes all information about them. However, it is not suitable for applications, because it does not show how subsystems interact with each other and evolve with respect to the clock. In quantum mechanics and QFT, the evolution is usually formulated through definition of an action functional or a Hamiltonian in non-relativistic cases, which are usually taken or inspired from classical limit of the model. However, $SU(\infty)$-QGR is built as a quantum theory and it does not have a classical formulation, or even a true classical limit (see the Introduction and [2]). Therefore, dynamics must be constructed solely based on the axioms of the model and properties concluded from them.

In an earlier study [2], we introduced dynamics by constructing a Lagrangian functional over parameter space Ξ. It consists of symmetry invariant traces of $SU(\infty)$ generators, and those of internal symmetry G when the Universe is divided to subsystems. After imposing invariance under reparameterization of Ξ, we showed that at the lowest order in traces, it has the form of a Yang–Mills theory for both symmetries. Traces of multiple generators of Lie groups are functions of their rank and structure coefficients. Moreover, in the Lagrangian functional, such terms are proportional to higher orders of coupling constant. Therefore, trace terms with more than two generators can be considered as perturbative corrections to the lowest order effective Lagrangian. Consequently, methodology and techniques of QFTs such as path integral and Feynman diagrams are applicable to $SU(\infty)$-QGR. Importantly, as the Yang–Mills models are known to be renormalizable, $SU(\infty)$-QGR does not have any issue in this regard.

In this section, after a brief reminding of the $SU(\infty)$-QGR action, we elaborate some of the details that were not addressed in previous works.

4.1. $SU(\infty)$ as a Yang–Mills QFT on the parameter space of subsystems

The effective Lagrangian of $SU(\infty)$-QGR is analogous to the effective Lagrangian of QFTs, with the difference that it is defined on the parameter space Ξ, rather than an external spacetime. Reparameterization and symmetry invariance lead, at the lowest order, to a Yang–Mills- type Lagrangian functional on Ξ. Specifically, for the whole Universe, that is, when the Universe is treated as a single isolated quantum system, the lowest order effective Lagrangian $\mathcal{L}_U$ is a 2D Yang–Mills QFT on the diffeo-surface $\Xi = D_2$ (see **Figure 1** for a schematic depiction):

$$\mathcal{L}_U = \int d^2\Omega \left[\frac{1}{2}\,\mathrm{tr}(F^{\mu\nu}F_{\mu\nu}) + \frac{1}{2}\mathrm{tr}(\not{\mathcal{D}}\hat{\varrho}_U) \right], \quad d^2\Omega \equiv d^2x\sqrt{|\eta^{(2)}|} \tag{24}$$

$$F_{\mu\nu} \equiv F^a_{\mu\nu}\hat{L}^a \equiv [D_\mu, D_\nu], \quad D_\mu = (\partial_\mu - \Gamma_\mu) + \sum_a i\lambda A^a_\mu(x)\hat{L}^a(x) \tag{25}$$

where $\hat{L}^a(x)$ are generators of the $SU(\infty)$ symmetry, Γ_μ is a suitable connection for Ξ, and reparameterization invariant derivative $\not{\mathcal{D}}$ depends on how $\hat{\varrho}_U$ is transformed under diffeomorphism of Ξ. See the next paragraph for the definition of $\not{\mathcal{D}}$. Notice that both the gauge field A^a_μ and generators $\hat{L}^a$ depend on the coordinates of the parameter space—the diffeo-surface. For instance, for the sphere basis [6] (reviewed in [1]), $x \equiv (\theta, \phi)$ are angular coordinates on Ξ, index $a = (l, m)$, and $\hat{L}_{lm}(\theta, \phi)$ are generators of $SU(\infty)$. In torus basis reviewed in Section F, Supplementary material, x is the 2D Cartesian coordinates of a point on Ξ, index $a = \mathbf{m}$, and $\hat{L}_{\mathbf{m}}$ is defined in (A.38). In $SU(\infty)$ Yang–Mills on a background spacetime of any dimension, in addition to the *internal coordinates* x, the gauge field depends on the spacetime (see Section L, Supplementary material for a brief review of these models). The Lagrangian $\mathcal{L}_U$ is clearly static, because it does not include any time parameter. Moreover, it has been shown that it is topological [1, 2] and its value does not depend on the 2D metric $\eta_{\mu\nu},\ \mu, \nu = 1, 2$.

When the Universe is considered as an ensemble of its subsystems representing $G \times SU(\infty)$, as explained in Section 3, their purified states $\hat{\varrho}_{G_\infty}$ depend, in addition to coordinates of the diffeo-surface, on an area/size parameter and on time. The symmetry and reparameterization invariant Lagrangian $\mathcal{L}_{U_s}$ is defined on the 4D parameter space Ξ (depicted in **Figure 1**) and has the following expression:

$$\mathcal{L}_{U_s} = \int d^4x\sqrt{|\eta|} \left[\mathrm{tr}(F^{\mu\nu}F_{\mu\nu}) + \frac{1}{4}\mathrm{tr}(G^{\mu\nu}G_{\mu\nu}) + \frac{1}{2}\sum_s \mathrm{tr}(\not{\mathcal{D}}\hat{\varrho}_{G_\infty}) \right] \tag{26}$$

$$F_{\mu\nu}(x) \equiv [D_\mu, D_\nu], \quad D_\mu = (\partial_\mu - \Gamma_\mu) - i\lambda A_\mu, \quad A_\mu \equiv \sum_a A^a_\mu(x; \eta, \zeta)\hat{L}^a(\eta, \zeta) \tag{27}$$

$$G_{\mu\nu}(x) \equiv [D'_\mu, D'_\nu], \quad D'_\mu = (\partial_\mu - \Gamma_\mu - i\lambda A_\mu)\mathbb{1}_G - i\lambda_G B_\mu, \quad B_\mu \equiv \sum_b B^b_\mu(x)\hat{T}^a \tag{28}$$

Definitions of quantities in this Lagrangian are as follows. The symmetric tensor $\eta_{\mu\nu}$ is the metric of the parameter space Ξ. It should not be confused with the metric $g_{\mu\nu}$ of the emergent classical spacetime Ξ_{sp}. According to Proposition 2 of the study [2], $\eta_{\mu\nu}$ is arbitrary, because the geometry of Ξ is not a physical observable. We further discuss this property in Section 4.2. The first and second terms in (26) are the Lagrangian density for the $SU(\infty)$ and internal symmetry G gauge fields A^{lm}_μ and B^a_μ, respectively. The $SU(\infty)$ generators $\hat{L}^{lm}$ and the range of (l, m) are defined in Section F, Supplementary material. Γ_μ is the geometric connection of the parameter space Ξ. Operators T^a are generators of the internal symmetry G of subsystems. Their number is determined by the range of index a, which must be finite because the rank of G is finite. The density matrix $\hat{\rho}_s$ is the quantum state of a subsystem. The covariant operator $\not{\mathcal{D}}$ is a differential operator, and its exact definition depends on the representation of the symmetries of Ξ by the states of subsystems. For instance, for spinors, $\not{\mathcal{D}} \equiv \gamma^0\gamma^i e^\mu_i(\partial_\mu - \Gamma_\mu)\mathbb{1} - \sum_{lm} i\lambda_s A^{lm}_\mu \hat{L}^{lm} - \sum_a i\lambda_G B^a_\mu \hat{T}^a$, where $\gamma^\mu, \mu = 0, \cdots, 3$ are Dirac matrices and $e^\mu_i, i = 0, \cdots, 3$ are tetrads (see [2] for more details). The reason for the presence of $SU(\infty)$ field in (28) and $\not{\mathcal{D}}$ is the fact that all subsystems represent $SU(\infty)$. In Section 4.2 , we demonstrate that geometry connection terms in $D'_\mu, \not{\mathcal{D}}$ and field equations can be neutralized by a $SU(\infty)$ gauge transformation. Therefore, as mentioned earlier, the Lagrangian (26) does not depend on $\eta_{\mu\nu}$. Finally, notice that the interaction of G gauge field (amplitude) B_μ with $SU(\infty)$ is implicit, because it depends on the parameters $x \in \Xi$ of the $SU(\infty)$ symmetry of subsystems.

Apart from including terms for Yang–Mills fields representing G, the main difference between the $SU(\infty)$ gauge field in $\mathcal{L}_{U_s}$ and $\mathcal{L}_U$ is its dependence on additional parameters. In $\mathcal{L}_U$, the gauge field depends only on two continuous parameters characterizing the $\mathcal{SU}(\infty)$ algebra. By contrast, A^a_μ in $\mathcal{L}_{U_s}$ depends on 4D vectors $x \in \Xi$ and on $(\eta, \zeta) \in D_2$, where D_2 is the diffeo-surface of $SU(\infty)$ symmetry it represents. Although the $SU(\infty)$ gauge field (A.79) defined on a background spacetime has apparently a similar structure, there is an important difference. In contrast to coordinates of the so-called *internal space* in (A.79), the induced coordinates (η, ζ) of the diffeo-surface D_2 are functions of x. As **Figure 1** shows, D_2 is immersed in Ξ, that is, $D_2 \subset \Xi$. By contrast, in (A.79), the internal space is independent of the background spacetime. It is the reason for using the name *diffeo-surface* rather than *internal space* for a compact 2D surface that its *ADiff* group is homomorphic to $SU(\infty)$.

4.1.1. Classical spacetime from quantum gravity action

The perturbation theory applied to $\mathcal{L}_{U_s}$ allows to calculate components of $\hat{\varrho}_{G_\infty}$. Tracing out the contribution of G symmetry parameters gives $\hat{\varrho}_\infty$, which can be then used to determine the effective classical metric in (13). However, this procedure only determines ds^2 rather than the metric $g_{\mu\nu}$ of the emergent classical spacetime Ξ_{sp}. For determining components of $g_{\mu\nu}$, we have to use the classical limit of $\mathcal{L}_{U_s}$ as explained in an earlier study [2]. The procedure consists of treating pure $SU(\infty)$ term in (26) as the classical scalar curvature, other terms as *matter*, and identifying the metric of parameter space $\eta_{\mu\nu}$ with the effective metric $g_{\mu\nu}$ defined in (13). However, the $SU(\infty)$ gauge field A_μ must be included in the connection Γ_μ, because in classical gravity, it is the connection that convoys gravitational interaction. The effective energy-momentum tensor should be determined from effective Lagrangian for matter part of (26), which can include quantum corrections in a curved spacetime (see, e.g., [44–46] for examples). Then, the Einstein equation obtained from applying variational principle to this approximation can be solved to calculate components of the effective metric. Of course, the Lagrangian $\mathcal{L}_{U_s}$ can be used to calculate Green's functions and other measurables of both quantum matter and quantum gravity—the $SU(\infty)$ gauge field—without having to consider the classical limit.

4.2. Equivalence of diffeomorphism of parameter space and $SU(\infty)$ gauge transformation

The equivalence of reparameterization of the parameter space Ξ and transformation of its 2D subspaces under the action of $SU(\infty)$ group is necessary for proving that the geometry of Ξ is irrelevant for physics (Proposition 2 in the study [2]). It is straightforward to verify that the connection form Γ_μ does not affect gauge invariant field intensity $F_{\mu\nu}$. However, it appears in its covariant derives as well as in the covariant derivative $\not{\mathcal{D}}$ of the density matrix. In the Einstein gravity, metric, connection, and curvature tensors of the background spacetime are related to the energy-momentum tensor of matter through the Einstein–Hilbert action and Einstein equation. By contrast, in the $SU(\infty)$-QGR geometry of parameter space, Ξ is arbitrary and should not affect observables. In an earlier study [2], we used algebraic arguments to prove this property. Here we demonstrate it analytically.

Equivalence of the diffeomorphism of Ξ and $SU(\infty)$ gauge transformation in the Lagrangian $\mathcal{L}_U$ of the whole Universe is trivial, because Ξ is the same as the diffeo-surface D_2. Thus, area preserving diffeomorphisms of Ξ are simply $SU(\infty)$ transformations, under which $\mathcal{L}_U$ is invariant. To see this, consider a change of $\Gamma_\mu \rightarrow \Gamma'_\mu$ that preserves the area of $\Xi = D_2$. We remind that vectors in $\mathcal{H}_U$ are 2D complex–valued functions. Nonetheless, in an earlier study [2], we showed that the phase of gauge fields is irrelevant, and without the loss of generality, these fields can be considered to be real. Therefore, in analogy with A_μ, components of Γ_μ can be considered as vectors of $\mathcal{H}_U$. Using, for instance, the sphere basis for $SU(\infty)$, both Γ_μ and A_μ can be expanded with respect to spherical harmonic functions $Y_{lm}(\theta, \phi)$. As $\hat{L}_{lm} Y_{l'm'} = -if^{l''m''}_{lm,l'm'} Y_{l''m''}$ (for $\hbar = 1$) where $f^{l''m''}_{lm,l'm'}$ are structure coefficients of $SU(\infty)$ [6], a $SU(\infty)$ gauge transformation of A_μ changes coefficients of its expansion with respect to $Y_{lm}(\theta, \phi)$. Thus, a gauge transformation such that:

$$(\Gamma'_\mu - \Gamma_\mu) = \Omega(x) A_\mu(x) \Omega^{-1}(x) - i\Omega(x)\partial_\mu \Omega^{-1}(x) - A_\mu(x) \quad (29)$$

$$\Omega(x) \equiv e^{i \int_{D_2} d^2\Omega' \, \epsilon_{lm}(\theta,\phi,\eta',\zeta') \hat{L}^{lm}(\eta',\zeta')} \quad (30)$$

can compensate the change of Γ_μ. A general diffeomorphism of any manifold $\mathcal{M}$ can be decomposed to:

$$Diff(\mathcal{M}) \cong ADiff(\mathcal{M}) \times \Lambda(\mathcal{M}), \quad \forall \mathcal{M} \quad (31)$$

where $\Lambda(\mathcal{M})$ presents the operation of a global scaling of the manifold $\mathcal{M}$. Such transformation changes $\mathcal{L}_U \rightarrow \Lambda \mathcal{L}_U$. In other words, it changes the volume of Ξ by rescaling its coordinates. However, due to the $U(1)$ symmetry discussed in Section 3.5, such a rescaling is not an observable.

Demonstration of the invariance of Lagrangian $\mathcal{L}_{U_s}$ of subsystems and the corresponding field equations under diffeomorphism of

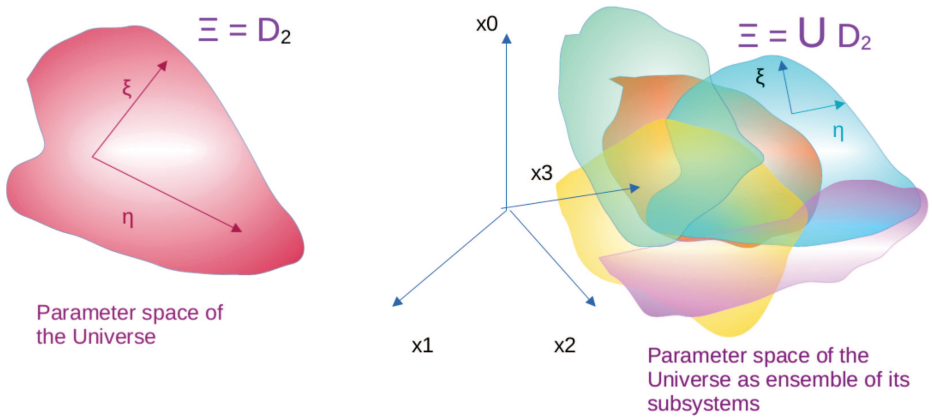

Figure 1 • Schematic presentation of parameter spaces Ξ of the $SU(\infty)$ symmetry. Left: A single 2D compact surface corresponding to diffeo-surface of the $SU(\infty)$ symmetry of the Universe. Its area is arbitrary and irrelevant. Axes (η, ζ) are local coordinates. Right: The 4D parameter space generated by an infinite number of 2D diffeo-surfaces of subsystems and time parameter of a quantum clock. Axes x^i, $i = 0, \cdots, 3$ are coordinates at each point $x \in \Xi$, and axes $\eta(x), \zeta(x)$ are induced coordinates on one of the diffeo-surfaces passing through x.

their 4D parameter space Ξ is more complicated, but it follows the same line of reasoning. As discussed earlier, the 2D diffeo-surface Σ characterizing $SU(\infty)$ generators is immersed in Ξ, that is, $\Sigma \subset \Xi$. Thus, the induced coordinates (η, ζ) of Σ are functions of the 4D parameter space Ξ. To demonstrate the irrelevance of the geometry of Ξ for physical observables, we show that gauge transformations involving all parameters can restore curvatures, geometrical connection, and covariant derivatives of deformed or reparameterized Ξ. Riemann and Ricci curvatures depend on the derivatives of the Levi–Civita connection. For this reason, calculating connection as a function of curvatures amounts to solving a partial differential equation, which is not a trivial task. Therefore, we take the inverse approach and prove that curvatures are restored by $SU(\infty)$ gauge transformation. Specifically, Riemann and Ricci curvature tensors—hence the connection at a point x of a Riemannian manifold—can be determined from sectional curvatures of 2D surfaces passing through that point [47, 48]. There are exactly six such pairs of axes at each point x of the 4D parameter space Ξ leading to the following expression for sectional curvatures:

$$K(\Pi_{ij}(x)) \equiv K(\hat{x}^i, \hat{x}^j) = \frac{R_p(\hat{x}^i, \hat{x}^j, \hat{x}^i, \hat{x}^j)}{\langle \hat{x}^i, \hat{x}^i \rangle \langle \hat{x}^j, \hat{x}^j \rangle - \langle \hat{x}^i, \hat{x}^j \rangle^2}, \quad i, j = 0, \cdots 3, \quad i \neq j \tag{32}$$

The 4D unit vectors $\hat{x}^i \in T\Xi(x)$, where $T\Xi(x)$ is the tangent space of Ξ at $x \in \Xi$, and the plane $\Pi_{ij}(x)$ passes through vectors $\hat{x}^i$ and $\hat{x}^j$. As the 2D compact surfaces used for the definition of $\hat{L}^a(\eta, \zeta)$ in (27) are arbitrary, we can locally identify them with surfaces generated by a pair of unit axes $(\hat{x}^i, \hat{x}^j) \in T\Xi(x)$ at x. Moreover, as $SU(\infty)^n \cong SU(\infty)$, the union of diffeo-surfaces is again a diffeo-surface associated to a representation of $SU(\infty)$. Thus, $SU(\infty)$ gauge transformations, such as:

$$A_\mu(x; \eta, \zeta) \to A'_\mu(x; \eta, \zeta) = \Omega(x; \eta, \zeta) A_\mu(x; \eta, \zeta) \Omega^{-1}(x; \eta, \zeta) - i\Omega(x; \eta, \zeta)\partial_\mu \Omega^{-1}(x; \eta, \zeta) \tag{33}$$

$$\Omega(x; \eta, \zeta) \equiv \exp(i\epsilon_{lm}(x; \eta, \zeta)\hat{L}^{lm}(\eta, \zeta)), \quad x \in \Xi \tag{34}$$

can be performed using $(\eta, \zeta) \in \text{Span}(\hat{x}^i, \hat{x}^j)$, $i, j = 0, \cdots 3, i \neq j$, which means that (η, ζ) are chosen such that the surface passing through them corresponds to $\Pi_{ij}(x)$ at x. Consequently, by properly choosing a set of $SU(\infty)$ gauge transformation (33), Riemann and Ricci curvatures of the deformed or reparameterized Ξ can be restored. Hence, even in the case of Lagrangian $\mathcal{L}_{U_s}$ that depends on four parameters, a $SU(\infty)$ gauge transformation is homomorphic to an *ADiff*(Ξ) and vice versa. In other words, a change in the field space that preserves the Lagrangian $\mathcal{L}_{U_s}$ is equivalent to a diffeomorphism of its parameter space. This demonstration finalizes analytic proof of Proposition 2 in the study [2] and shows that the geometry of Ξ is irrelevant for physical observables of subsystems. Thus, without the loss of generality, we can consider a fixed geometry for Ξ in both the Universe as a whole, and when it is perceived through the ensemble of its subsystem. This choice is equivalent to fixing $SU(\infty)$ gauge.

5. Conclusions

5.1. Perspectives for test and applications of $SU(\infty)$-QGR

Currently, there is no observed evidence of quantum behavior of gravity. However, the existence of a quantum description for the universal interaction called gravity seems inevitable [49–51]. Therefore, the absence of evidence is most probably due to the insufficiency of precision and resolution of current experiments. Nonetheless, with the huge progress of quantum technologies in the past few decades, there is a strong hope that in the near future, tests of the quantumness of gravity and eventually discrimination between various QGR proposals will become achievable. Indeed, there is already some progress in this direction. Specifically, cosmological and astro-particle observations put stringent constraint on the modification of dispersion relation of high-energy photons due to the *micro-structure* of the spacetime [52] predicted by some QGR models such as loop quantum gravity; gravitational waves constrain graviton mass [53] and Lorentz invariance violation [54] predicted by some string theories; and large extra dimensions are constraint by various analyses and observations, for example [55–57].

In what concerns the test of $SU(\infty)$-QGR, as mentioned earlier, its most discriminative signature is a spin-1 boson as mediator of QGR. However, observation of this attribute in laboratory experiments—usually based on the detection of a change in coherence [58–60] or entanglement [61, 62] of the state of a quantum system by gravitational interaction—is much harder (there are many proposals for the test of QGR in the literature, and the citations here are only examples and far from being exclusive). Meanwhile and until the limited capabilities of laboratory experiments become available, cosmological observations may be a faster road to detection of a QGR signal, in particular that of $SU(\infty)$-QGR. Here we outline a proposal for the detection of a QGR signature using coherent astronomical sources.

5.2. Difference in the coherence of astronomical masers as a signature of quantum gravity

Astronomical masers have short duration quantum coherence [63] and can be used as a coherent source for testing QGR. The change of their polarization may also be useful for estimating the spin of particles which led to their decoherence [62]. On their path from the place of their formation—usually in molecular clouds of galaxies at cosmological distances—to the Earth, maser photons may be affected by the gravitational field of another galaxy or galaxy cluster and their dark matter halos and being lensed [64].

In a quantum view, the deflection of photons is the result of their interaction with gravitons. In general, such process changes the coherence of photons. Specifically, a quantum process (channel) affecting a quantum system can be described by a set of Krauss operators: $\hat{\rho}_f = \sum_i \hat{K}_i \hat{\rho}_i \hat{K}_i^\dagger$, where $\hat{\rho}_i$ and $\hat{\rho}_f$ are the initial and final states of the system, respectively, and Krauss operators $\hat{K}_i$, satisfying $\sum_i \hat{K}_i \hat{K}_i^\dagger = \mathbb{1}$, define the quantum process. Using the sum of off-diagonal components $C \equiv \sum_{i \neq j} \hat{\rho}_{ij}$ as a measure of the coherence of a state $\hat{\rho}$ [65], it is evident that in general $C_f \neq C_i$.

In the framework of $SU(\infty)$-QGR, the $SU(\infty)$ gauge-invariant Krauss operators for the process of photon deflection by a gravitational field can be written as follows: $\hat{K}^a_{\mu\nu} \equiv \mathcal{N} F^a_{\mu\nu} \hat{L}^a$, where $\mathcal{N}$ is a normalization constant, $a \equiv (l, m)$ defined in Section F, Supplementary material is the $SU(\infty)$ graviton *color*, and $F^a_{\mu\nu}$ is the spin-1 graviton field defined in (27). Similarly, the $U(1)$ gauge invariant photon state can be expanded with respect to $G_{\mu\nu}$ defined in (28): $\hat{\rho} = \mathcal{N}' \hat{G}_{\mu\nu} \hat{G}^{\mu\nu}$, where $\mathcal{N}'$ is a normalization constant. A hat is added to emphasize that the field should be considered as an operator in $\mathcal{B}[\mathcal{H}_{G=U(1)}]$.

Due to the deflection of maser's photons by the gravitational field of a galaxy or cluster playing the role of a gravitational lens, the optical paths of photons in different images are not the same. Consequently, the Krauss operators affecting them would be different. Thus, the coherence of the maser emission in the images would not be the same. Therefore, detection of coherence difference in lensed masers—after taking into account other sources of coherence distortion—would be a signature of QGR. Another measure of the quantum effect of gravity on the lensed photons is entanglement fidelity [66] (also called channel fidelity [67]) defined as $F_e \equiv \sum_i \text{tr}(\hat{\rho}\hat{K}_i)\text{tr}(\hat{\rho}\hat{K}_i^\dagger)$. This quantity measures how much a quantum channel changes the state of a signal. Assuming that before interacting with gravitational field of the lens, the maser photons have the same single-photon state, the difference between channel fidelity of images would be due to the different Krauss operators of the channel (path) taken by photons in separate images. Nevertheless, photons of a maser beam are entangled by their phase. A priori the difference between entanglement entropy of images could be used as a complementary signature of QGR. However, different optical path of images, short coherence time of the maser photons, and different arrival time of the images mean that their entanglement before getting lensed cannot be used. In contrast to the measurement of coherence that only depends on the quantum state of detected photons, channel fidelity depends also on the properties of the channel, in particular the spin of gravity mediator. However, it needs modeling of the lens and is more sensitive to its uncertainties. Nonetheless, a priori it should make possible to discriminate between different QGR models, in particular to test the prediction of a spin-1 intermediate boson in $SU(\infty)$-QGR.

This test of QGR has the advantage to be intrinsically multi-messenger. Incoherent radiation from the galaxy or cluster can be used to model the lens and its gravitational field, and various baryonic effects on the maser photons, which would be the main source of uncertainty for the observation of a tiny QGR signal. The large volume and depth of future radio surveys should be able to detect with better resolution more lensed galaxies with maser sources.

5.3. Perspective for further investigations

Apart from seeking the best methods for testing $SU(\infty)$-QGR, future works should investigate formulation of dynamics of subsystems as open quantum systems, because there is no truly isolated quantum system in the Universe. An open system approach would be particularly necessary for the design of experimental setups for the test of $SU(\infty)$-QGR and other QGR models, for example, the experiment proposed in an earlier study [68]. Other topics to be studied, both for verifying the capability of $SU(\infty)$-QGR to solve puzzles of QGR and for determining its predictions include physics of quantum black holes, their evaporation, and puzzles of apparent information loss and singularity; QGR effects in the early Universe and inflation; nature of dark energy and whether it has a QGR origin; particle physics in the framework of $SU(\infty)$-QGR—in other words, how the Universe was fragmented to subsystems—and the emergence of local symmetries which have ultimately led to the Standard Model at low energies; and the potential role of QGR in matter—antimatter symmetry breaking.

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