# ON RENORMALIZABILITY OF POWER-COUNTING NON-RENORMALIZABLE THEORIES

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*To my Rebeca, whose patience is greater than any of these infinities, and to Adela, who is still embedded in her mother's curvature.*

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# Abstract

Power-counting non-renormalizable theories should not be dismissed a priori as fundamental theories. The practical inconvenient of having infinitely many independent couplings can be faced in certain cases performing a *reduction of couplings*. First we study the usage of a special reduction based on the relations imposed by the renormalization group. Then, we analyze the renormalizability of a family of theories containing quantum fields interacting with a a classical gravitational field and that contain a certain class of irrelevant operators. The reduction is this case is guided by a map that also indicates that these models exhibit an acausal behavior at high energies. Finally, we investigate the renormalizability of models which, although containing irrelevant operators, are renormalizable with a finite number of couplings due to the presence of Lorentz-violating kinetic term. Along this work we consider models that can violate some principle as the Lorentz symmetry or causality, but all of them preserve unitarity. The guidelines of this thesis aim to get a better understanding of the role of renormalization as classification tool, and guide the search of a generalization of the Power-Counting criterion that allows the enlargement of the set of candidate fundamental theories.

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# Introduction

Renormalization is a central topic in Quantum Field Theory that historically has motivated opposite attitudes. Even some of its precursors like Richard Feynman showed themselves sceptical regarding the mathematical grounds and meaning of the technique designed to save Quantum Electrodynamics (QED) from infinities that appear in perturbative calculations and prevent to obtain finite results to compare to the experiment. As late as 1985 Feynman said "It's surprising that the theory still hasn't been proved self-consistent one way or the other by now; I suspect that renormalization is not mathematically legitimate"[1]. On the other hand, the impressive precision with which the experimental data was predicted in QED, for example the electron gyromagnetic ratio, was enough to many scientists to consider as validated the method used "to sweep the dust under the carpet". Moreover, the use of Power Counting (PC) as renormalizability criterion served to guide the evolution from the four-fermions model proposed by Enrico Fermi to explain the electroweak interaction, which happens to be non-renormalizable, to a renormalizable model including intermediate bosons, that lead to the discover of the W and Z bosons.

The striking success of QED, the application of renormalization concepts to statistical mechanics due mainly to Kenneth Wilson, and the development of the electroweak theory placed renormalization and in particular Power Counting in a better position. Power Counting became an essential tool to classify, and most importantly, to rule out models to be considered fundamental theories. Its strength is based in two points. First, its application is extremely simple: to determine the renormalizability of a model it is enough to perform a dimensional analysis. Nevertheless, behind PC there are very elaborated all-order demonstrations, involving combinatorics and the study of divergent diagrams, that give renormalization a "bad reputation", in particular the treatment of overlapping divergences. Second, unlike any other criterion (as symmetries for example), PC allows a *vertical* classification, that is to say it reduces the number of possible interactions (and thus theories) from infinitely many to a finite set. This classification leaves outside most models, being quantum gravity the most remarkable example.

Nevertheless, it seems clear that PC can not be the ultimate criterion for renormalizability. Certainly all models it approves are renormalizable, but some of the dismissed model could eventually renormalized. The standard example is the three dimensional four-fermion model studied by Parisi [2], where renormalization is achieved by means of a expansion of Green functions in $1/N$ where $N$ is the number of field copies, instead of the usual expansion in the coupling constant. The renormalizability of PC non-renormalizable models should motivate the search of a new criterion, an extension of PC to open the possibility of new interactions, hopefully also quantum gravity.

Renormalization, considered in a wider perspective should still serve as a vertical classification tool. That is the aim of this work.

Along these lines, we constantly keep an eye in the possible applications or implications of these ideas in a quantum theory of gravitation, one of the most important unsolved puzzles of theoretical physics.

In Chapter 1, we present some results of standard renormalization theory that will be useful in the rest of the work.

The main problem of non-renormalizable models is the need of infinitely many types of interactions and independent couplings to obtain a coherent renormalization structure when the standard renormalization program is used. In Chapter 2 we face this problem directly using the *infinite reduction of couplings*, namely the search for relations between the infinitely many couplings, guided by the consistence with the renormalization group. When some particular conditions are satisfied, the reduction is applicable and the model can be renormalized by means of redefinitions of fields, masses and a finite number of independent couplings. Dimensional regularization is used to explore the reduction from a different angle, and it provides the clearest way to obtain some of the results and all-order theorems, although the equivalence with other regularization is proved.

In chapter 3 we study the renormalizability of models containing quantum fields interaction with classical gravity. The models considered are non-renormalizable in the Power Counting sense, and contain a finite set of matter operators of dimensionality four or less coupled to purely gravitational operator of arbitrary dimensionality. The renormalization is achieved by means of a redefinition of fields, masses and a finite set of couplings, without generating higher derivatives in the kinetic term of the gravitational field, responsible of instabilities. The renormalizability is proved using a special map that relates the renormalization of a higher-derivative model, to the renormalization of a model that present causality violations.

In chapter 4, we discuss renormalization aspects of theories that contains a modified kinetic term that breaks explicitly the Lorentz-invariance and produces more convergent propagators, improving the behavior of diagrams in the ultraviolet region. Unitarity is preserved, since we add only space higher-derivatives (and no time higher-derivative) to the usual kinetic term. To prove the renormalizability of these (PC)non-renormalizable models we define a modified version of PC, adapted to theories that present this particular form of the kinetic term, called Weighted Power Counting.

Each chapter contains its own introduction, while the general conclusions are collected in a separated section. The Appendices contain some calculations and results that are used in Chapter 2 and 3. In Appendix A the analytical properties of the solutions of a certain family of differential equations are studied using perturbative expansions, while in Appendix B we present a theorem that allows to find, in certain circumstances, the perturbative version of the map that relates a higher derivative model with an acausal one.

Along all this work we make an intensive use of dimensional analysis in different circumstances. For instance, in Chapter 2 it helps to find the general form of the beta-functions and restricts the form of the reduction at renormalized and bare levels. In Chapter 4 it is useful to determine the scaling properties of Green functions under a weighted scale transformation.

# Chapter 1

# Preliminaries

This chapter is devoted to review general results in renormalization theory that are used or generalized in the rest of this work. It also serves to fix some nomenclature and terminology. It is not intended to be complete or self-contained. Here the standard renormalization program and the study of renormalizability based on power counting analysis are briefly reviewed for theories containing both fermions and bosons. Then, some special features of the renormalization group and beta-functions in dimensional regularization are exposed. Special attention is devoted to issues that are generalized in Chapter 4 to deal with Lorentz-violating models.

## 1.1 Renormalization Program

One of the first approaches to show that the divergences of quantum field theories may be absorbed into local counterterms to all orders in perturbation theory was due to Dyson [3]. We say that a theory is renormalizable if such subtractions are local, i.e. polynomials in momentum space, and implemented by counterterms through the redefinition of parameters and fields contained in the theory. Therefore, the model should have the appropriate number and type of interactions to carry out this program.

In many circumstances we will talk indistinctly of integrals and diagrams. No ambiguity should arise according to the context. For instance, the *subdivergences of a diagram* refers to the divergences of the integrals associated with subdiagrams.

Let $G$ a one-particle irreducible (1PI) divergent Feynman diagram and $\overline{G}$ the same diagram with all the subdivergences subtracted. $\overline{G}$ may still have a divergence, called the overall divergence $\overline{G}^{(\infty)}$.

The demonstration of renormalizability consists basically of three steps:

*i) Demonstrate that for all G,* $\overline{G}^{(\infty)}$ *is a polynomial in external momenta and its degree is equal to the superficial degree of divergence* $\omega(G)$ *(defined below) regardless of the number of loops of* $G$.

*ii) Prove that lower-order counterterms cancel out the subdivergences of G, i.e. the combinatoric factors have the precise value such that*

$$\operatorname{div}\left(G+\sum_{\{\gamma\}} G_{\{\gamma\}}\right)=\overline{G}^{(\infty)}$$

*with* $G_{\{\gamma\}}$ *is a diagram similar to* $G$ *where the subdiagrams* $\gamma_i$ *has been replaced by vertices representing suitable counterterms, and the sum is over all possible sets of proper divergent subdiagrams.*

*iii) Show that the lagrangian contains the appropriate interactions to provide counterterms required to cancel all the divergences. In other words, that the redefinition of coupling constants,masses and fields is enough to produce finite correlation functions to all orders.*

Statements *i)* and *ii)* are undoubtedly the hard part and its demonstration requires a very involved study of diagrams and convergence of integrals. The general analysis was begun with the Bogoliubov-Parasiuk theorem [4, 5] which rigorous demonstration was given by Hepp [6] and refined by Zimmerman [7] (commonly designated as the acronym BPHZ) using his *forest formula.* Another essential contribution was the proof of locality of counterterms, due to Weinberg [8].

In the next section we present the general analysis to deal with the point *iii).*

## 1.2 Power Counting

Power counting (PC) is a very simple and useful tool to rapidly determine the renormalizability of models. It allows to know what kind of 1PI diagrams have an overall divergence $\overline{G}^{(\infty)}$.

A fundamental diagrammatic identity used for power counting is

$$L + V - I = 1, \tag{1.1}$$

with $L$ the number of loops of the diagram, while $V$ and $I$ are the number of vertices and internal lines it has, respectively. Here we study the renormalizability of theories containing both fermion and boson fields. The subscripts $f$ and $b$ label fermionic and bosonic quantities respectively.

The *superficial degree of divergence* $\omega(G)$ of a diagram $G$ is defined in $D$ dimensions as

$$\omega(G) = DL - 2I_b - I_f + \sum_i \delta^{(i)} v_i. \tag{1.2}$$

Each vertex $i$ is characterized by the integers $N_b^{(i)}$ and $N_f^{(i)}$, the number of fields of each type it contains and $\delta^{(i)}$, the number of derivatives present in it. The number of such vertices contained in $G$ is denoted by $v_i$.

Normally, $\omega(G)$ coincides with the *dimensionality* of the integral[1] $[G]$. The dimensionality of a quantity, denoted here by square brackets, is the dimension it has in mass units. For example, masses, coordinates and momenta have dimensionalities

$$[m] = 1, \qquad [x^\mu] = -1, \qquad [p^\mu] = 1.$$

In $D$ spacetime dimensions, $[\varphi] = (D-2)/2$, $[\psi] = (D-1)/2$, where $\varphi$ and $\psi$ bosonic and fermionic fields respectively.

Calling $E$ the number of external legs of $G$, we have that

$$E_b = \sum_i N_b^{(i)} v_i - 2I_b, \qquad E_f = \sum_i N_f^{(i)} v_i - 2I_f, \tag{1.3}$$

$$I_b + I_f = I, \qquad E_b + E_f = E, \qquad \sum_i v_i = V, \tag{1.4}$$

and (1.1), allows to rewrite (1.2) as

$$\omega(G) = d_D(E_b, E_f) + \sum_i v_i \Omega_D^{(i)}, \tag{1.5}$$

[1]When the coupling constants are not considered as part of a diagram $G$.

where $\Omega_D^{(i)} = \delta^{(i)} - d_D(N_b^{(i)}, N_f^{(i)})$ is the *degree of divergence of the vertex i.* The quantity

$$
\begin{aligned}
d_D\left(X_b, X_f\right) &= \left(1 - \frac{X_b + X_f}{2}\right) D + X_b + \frac{X_f}{2} \\
&= D - \frac{X_b}{2}\left(D-2\right) - \frac{X_f}{2}\left(D-1\right)
\end{aligned}
\tag{1.6}
$$

will be very useful in the PC analysis of Lorentz-violating theories of Chapter 4 as well. In a strictly renormalizable model (see below) we have $\omega\left(G\right) = d_D\left(E_b, E_f\right)$.

To have a renormalizable model it is necessary to keep $\omega\left(G\right)$ under control to ensure that the set of divergent correlation functions remains finite. This means that the superficial degree of divergence of all diagrams that have the same external legs in number and type, should be bounded. On the other hand, there should exist a maximal number of legs $\widehat{E}$ such as all diagrams with more external legs than $\widehat{E}$ are overall convergent, in other words, we require polynomiality of counterterms. From (1.5) these requirements are translated into:

*i) All vertices should satisfy* $\Omega_D^{(i)} \leq 0$. If this condition is not fulfilled, it will be necessary in the lagrangian an infinite set of vertices with arbitrary number of derivatives, and all correlation functions would be divergent at high enough order in perturbative expansion.

*ii)* $d_D\left(E_b, E_f\right)$ *should be a decreasing function of its arguments.* Expression (1.6) tell us that this is possible only if $D > 2$ (or $D > 1$ when the theory is purely fermionic).

Note that *ii)* is not a necessary condition, but we are interested in having renormalizability as a tool to restrict the number of fields each vertex can have. If it is not fulfilled, some vertices with arbitrarily high number of legs but without derivatives satisfy *i)*. It is possible to have, for instance, renormalizable theories in $D = 2$ dimensions that contain vertices with arbitrarily high number of legs. Thus *ii)* is the requirement of polynomiality of the lagrangian.

Renormalizability, together with the restrictions imposed by symmetries, unitarity, reality of the action, etc., provides an important guide in the process of formulation of quantum field theory models. Indeed it can be regarded as the most important of the mentioned concepts because it reduces the set of allowed interactions to a finite number by limiting the number of fields. In fact, the maximal number of legs of each type that a renormalizable interaction can have is given by *i)* considering no derivatives ($\delta^{(i)} = 0$), namely $d_D(N_b^{(i)}, N_f^{(i)}) \geq 0$. Explicitly,

$$N_f(D-1) + N_b(D-2) \leq D. \tag{1.7}$$

When *ii)* holds, (1.7) shows that the maximal number of legs that a renormalizable vertex can have increases when the number of spacetime dimensions decreases.

In chapter 4, where we consider Lorentz-violating models, the same expression (1.7) limits the number of legs contained in renormalizable vertices, with $D$ representing not the physical dimension but a quantity that can take non-integer values and that is smaller than the actual physical dimension. This will open the possibility of a new set of renormalizable interactions.

Assuming *ii),* we have $D > 2$ for models where bosons are present, so the maximum number of fermion fields in a renormalizable vertex is 2.

The second part of the demonstration of renormalizability consists in showing that for each superficially divergent diagram $G$ having $E_b$ and $E_f$ external legs and superficial degree of divergence $\omega(G)$, there exist a vertex such as $N_b^{(i)} = E_b$, $N_f^{(i)} = E_f$ and $\delta^{(i)} = \omega(G)$ in the lagrangian able to provide the suitable counterterm. Of course, if there are several boson or fermion fields of different type, the structure of the external legs of $G$ and the counterterm must coincide.

If the theory is renormalizable, from (1.5) and *i)* we have that $G$ satisfies $\omega(G) - d_D(E_b, E_f) \leq 0$. Therefore, according to the previous paragraph, the vertex that absorbs the overall divergence of $G$ satisfies $\delta^{(i)} - d_D(N_b^{(i)}, N_f^{(i)}) \leq 0$, namely $\Omega_D^{(i)} \leq 0$, so also this vertex satisfies *i).* This consistence check shows us that if some vertex is not present from the beginning in a renormalizable model but it is required by renormalization, its inclusion will not spoil the renormalizability of the theory.

## 1.3 Renormalizability

Depending on the type of vertices contained in theory, (1.5) allows to classify models according to their renormalizability. It is easy to verify that for each vertex

$i$, $\Omega_D^{(i)} = -[\lambda_i]$ where $\lambda_i$ is the minimal coupling constant related to the operator $\mathcal{O}_i$ corresponding to the vertex $i$. Assuming that the kinetic terms of the fields are multiplied by unity, a coupling is called *minimal* if it is the unique coefficient of a vertex and *non-minimal* if it is expressed as the product of more than one parameter.

**Renormalizable:** $\Omega_D^{(i)} \leq 0$ (or $[\lambda_i] \geq 0$) for every vertex. Hence, $\omega(G)$ does not increase when the number of vertices increases.

**Strictly-renormalizable:** $\Omega_D^{(i)(i)} = 0$ (or $[\lambda_i] = 0$) for every vertex. $\omega(G)$ does not depend on the number of vertices. If *ii)* holds, there is a finite set of divergent correlation functions, which contain divergences at all order in perturbative expansion. If *ii)* is no fulfilled, the theory can be non renormalizable even when $\Omega_D^{(i)} = 0$ for all $i$. This is the case of the *edge renormalizability* (see section 4.5).

**Super-renormalizable:** For every vertex, $\Omega_D^{(i)} < 0$ (or $[\lambda_i] > 0$). Only a finite number of diagrams is divergent.

**Non-renormalizable**: For some vertex $\Omega_D^{(i)} > 0$ (or $[\lambda_i] < 0$). Infinitely many amplitudes are divergent at sufficiently high order in perturbation theory.

**Marginal, relevant and irrelevant couplings:** According to their dimensionalities, coupling constants can be classified in one of these 3 sets: *marginal* if it is dimensionless, *relevant* or *irrelevant* if it has positive or negative dimensionality respectively. They are related to strictly-, super- and non-renormalizable interactions, in the same order.

## 1.4 Regularization

In almost all this work *dimensional regularization* is used, in the minimal substraction scheme. The divergent integrals are regularized extending analytically the number of spacetime dimensions $d$ to complex values $D = d - \varepsilon$.

In this context, a quantity is called *evanescent* if it vanishes in the physical limit $D \to d$. It is not necessarily proportional to $\varepsilon$ (see for example the Gauss-Bonnet term for $D = 4 - \varepsilon$ in section 3.1). The *dimensionality-defect* [9] of a field, operator or coupling $\chi$ is defined in dimensional regularization as

$$p^{(\chi)} = \frac{[\chi]_D - [\chi]_d}{d - D},$$

namely the difference of dimensionality between the extended spacetime and the physical one, divided by $\varepsilon$. Normally it is a rational number. For a field, it is determined by the kinetic term and it has the value

$$p^{(\chi)} = \frac{N^{(\chi)}}{2} - 1.$$

## 1.5 Overall Divergences And Subdivergences

Let us briefly review the usual classification of divergences and the proof of locality of counterterms [10] in Lorentz symmetric theories. Consider the $L$-loop integral

$$\mathcal{I}(k) = \int \prod_{j=1}^{L} \frac{\mathrm{d}^D p_j}{(2\pi)^D} Q(p_1, ..., p_L; k)$$

with Lorentz invariant propagators $1/(p^2 + m^2)$, where $k$ denotes the external momenta. Define $q_1, \ldots, q_I$ as the momentum associated to each propagator. Clearly, each $q$ is a linear combination of the loop momenta $p$ and the external momenta $k$. The ultraviolet behavior of $\mathcal{I}(k)$ is studied letting any (sub)set of the momenta $q_1, ..., q_I$ tend to infinity with the same velocity. Proper subsets of the momenta test the presence of subdivergences, while the whole set tests the presence of overall divergences. *i*) When any subconvergence fails, counterterms corresponding to the divergent subdiagrams have to be included to subtract the subdivergences. *ii*) Once all subdivergences are removed, the subtracted integral $\mathcal{I}_{\text{sub}}(k)$ can still be overall divergent. Taking an appropriate number $M$ of derivatives with respect to the external momenta $k$ the integral $\partial_k^M \mathcal{I}_{\text{sub}}(k)$ becomes overall convergent. This proves the locality of counterterms.

The overlapping divergences can be tested sending momenta to infinity with different velocities. For example, rescale $q_1, ..., q_I$ as $\lambda q_1, ..., \lambda q_i, \lambda^2 q_{i+1}, ..., \lambda^2 q_I$. This test, however, is already covered by the previous ones, since there is always a (sub)set $s_{\text{fast}}$ of momenta tending to infinity with maximal velocity. In the example just given, $s_{\text{fast}} = (q_{i+1}, ..., q_I)$. The other momenta $s_{\text{slow}}$ grow slower, so they can be considered fixed in the first analysis and taken to infinity at a second stage. Weinberg's theorem [8, 12] ensures that when $s_{\text{fast}}$ tends to infinity the behavior of the relevant subintegral is governed by power counting and can generate logarithmic corrections depending on the momenta of $s_{\rm slow}$. Then, when $s_{\rm slow}$ tends to infinity the behavior of the integral over $s_{\rm slow}$ is still governed by power counting, because the corrections due to the integrals over $s_{\rm fast}$ do not affect the powers of the momenta $s_{\rm slow}$. The introduction of logarithms in the integral does not change the degree of the polynomials, thus it does not affect the power counting.

## 1.6 Renormalization Group

The renormalization group (RG) indicates how quantities must vary to keep bare amplitudes fixed when the scale parameter $\mu$ is shifted. In dimensional regularization, the RG has some interesting features, which we review here.

The renormalization relation of a generic coupling $\alpha$ is given by

$$\alpha_{\rm B} = \mu^{p^{(\alpha)}\varepsilon}\,(\alpha + \Delta_\alpha(\alpha_i, \varepsilon)).$$

The subscript B is used to denote bare quantities, whereas renormalized ones do not carry any special subscript in general (except in section 3.5 where the subscript R is introduced to avoid confusion). $\Delta_\alpha(\alpha_i, \varepsilon)$ is a Laurent series in $\varepsilon$ and a power series on the couplings $\alpha_i$. Its form and its relation with the Gell-Mann-Low function $\widehat{\beta}_\alpha$ (from now called simply "beta-function") is detailed below. The beta-function defines the evolution of a coupling under the RG flow,

$$\widehat{\beta}_\alpha = \mu\frac{\mathbf{d}\alpha}{\mathbf{d}\mu},$$

which is finite, and its non-evanescent part is denoted by $\beta_\alpha$, being $\widehat{\beta}_\alpha = \beta_\alpha - p^{(\alpha)}\varepsilon$. Analogously the renormalization of an operator $\mathcal{O}$ is written as $\mathcal{O}_{\rm B} = Z_{\mathcal{O}}(\alpha_i, \varepsilon)\mathcal{O}$, and its evolution is given by the "gamma-function" $\gamma_{\mathcal{O}} = \mu\frac{\mathbf{d}\ln Z_{\mathcal{O}}}{\mathbf{d}\mu}$.

In cases where the renormalization is multiplicative, i.e. all divergent diagrams are proportional to the coupling they renormalize, we can write directly

$$\alpha_{\rm B} = \mu^{p^{(\alpha)}}\alpha Z_\alpha(\alpha_i, \varepsilon)$$

and

$$\alpha_{\rm B}\mathcal{O}_{\rm B} = \mu^{p^{(\alpha)}}\alpha\mathcal{O}, \tag{1.8}$$

so $Z_{\mathcal{O}} = (Z_\alpha)^{-1}$ .Deriving (1.8) with respect to $\ln \mu$ and considering that bare quantities are $\mu$-independent, we get

$$\beta_\alpha = -\alpha \frac{\mathbf{d} \ln Z_\alpha}{\mathbf{d} \ln \mu}, \qquad \gamma_{\mathcal{O}} = -\frac{\mathbf{d} \ln Z_\alpha}{\mathbf{d} \ln \mu}, \tag{1.9}$$

that is $\beta_\alpha = \alpha \gamma_{\mathcal{O}}$.

## 1.7 Beta-function Structure

In this section two interesting features of the renormalization constants and beta-functions are shown when dimensional regularization is used. They are used for example to derive the form of beta-functions, and to simplify the pole cancellation in Chapter 2.

*i) Beta-function structure:* The structure of the beta-functions is inherited directly from divergent diagrams that renormalize the respective coupling. With "structure" we mean the particular combination of powers of couplings in each term. In general not all combinations are present (for example the term $\alpha\eta^3$ is absent in (2.4)). There exist therefore a direct correspondence between the number and type of vertices of divergent diagrams and some term in the beta-function. This allows us to know the form of beta-function simply observing the diagrams involved.

*ii) All the information of the renormalization constants is encoded in the residue of its simple pole.* This is a consequence of the RG equations. The residue of the higher poles can be determined from the residue of the simple pole. In fact, the renormalization constant and the beta-function can be completely reconstructed from it. Recall that in general the simple pole has contributions from all order in loop expansion.

Let us consider just two minimal couplings $g$ and $\rho$ of dimensionality-defects 1/2 and 1, multiplying three- and four-leg vertices respectively. The generalization to arbitrary number of couplings is straightforward and the general formulas are presented below. The bare constants can be written as

$$g_{\mathrm{B}} = \mu^{\varepsilon/2} \overline{g}_{\mathrm{B}}, \qquad \rho_{\mathrm{B}} = \mu^{\varepsilon} \overline{\rho}_{\mathrm{B}}, \tag{1.10}$$

with

$$\overline{g}_{\mathrm{B}} = g \ + \Delta_g(g, \rho, \varepsilon), \qquad \overline{\rho}_{\mathrm{B}} = \rho \ + \Delta_\rho(g, \rho, \varepsilon),$$

From the general analysis we know that $\Delta_g$ and $\Delta_\rho$ are Laurent series in $\varepsilon$

$$\Delta_g(g,\rho,\varepsilon)=\sum_{i=1}^{\infty}\frac{g^{(i)}(g,\rho)}{\varepsilon^i},\qquad \Delta_\rho(g,\rho,\varepsilon)=\sum_{i=1}^{\infty}\frac{\rho^{(i)}(g,\rho)}{\varepsilon^i}$$

where the residues $g^{(i)}(g,\rho)$ and $\rho^{(i)}(g,\rho)$ are power series in the couplings.

Deriving both expressions in (1.10) with respect to $\ln\mu$, considering that bare quantities are independent of $\mu$, and solving for $\widehat{\beta}_g$, (for $\widehat{\beta}_\rho$ is analogous), we obtain

$$\begin{aligned}\widehat{\beta}_g &= \mathcal{D}_g g^{(1)}(g,\rho)-\frac{1}{2}g\varepsilon+\text{poles that cancel out}, \\ \mathcal{D}_g &= \left(\rho\frac{\partial}{\partial\rho}+\frac{1}{2}g\frac{\partial}{\partial g}-\frac{1}{2}\right).\end{aligned}\tag{1.11}$$

The part "poles that cancel out" is a restriction imposed by the finiteness of the beta-function in the $\varepsilon\to 0$ limit, and establish relations among the residue of higher poles. From equation (1.11) we obtain two conclusions: only the residue of the simple pole $g^{(1)}(g,\rho)$ is relevant, since it is the only one appearing in the expression for $\widehat{\beta}_g$. Indeed, $\Delta_g$ can be completely reconstructed from $\widehat{\beta}_g$ or equivalently from $g^{(1)}(g,\rho)$.

Writing

$$\beta_g=\sum_{i,j}b_{ij}g^i\rho^j,\qquad g^{(1)}(g,\rho)=\sum_{i,j}G_{ij}\ g^i\rho^j,\tag{1.12}$$

from (1.11) we have

$$G_{ij}=\frac{2b_{ij}}{(i+2j-1)},$$

so $g^{(1)}(g,\rho)$ is recovered from $\beta_g$. The relation between the residues reads

$$\begin{aligned}\mathcal{D}_g\ g^{(n)} &= \left(\beta_g\frac{\partial}{\partial g}+\beta_\eta\frac{\partial}{\partial\rho}\right)g^{(n-1)} \\ &= \left(\mathcal{D}_g g^{(1)}\frac{\partial}{\partial g}+\mathcal{D}_\rho\rho^{(1)}\frac{\partial}{\partial\rho}\right)g^{(n-1)},\end{aligned}\tag{1.13}$$

for $n>1$. We can solve for $g^{(n)}$ "inverting" the differential operator $\mathcal{D}_g$ just as we did in (1.11) writing (1.12). The operator $\mathcal{D}_\rho$ is given by $\mathcal{D}_\rho=\rho\frac{\partial}{\partial\rho}+\frac{1}{2}g\frac{\partial}{\partial g}-1$.

The second conclusion refers to the structure of divergent diagrams. A diagram that renormalizes $g$ having $v_g$ and $v_\rho$ vertices of type $g$ and $\rho$ respectively, needs a counterterm proportional to $g^{v_g}\rho^{v_\rho}$. This implies that $g^{(1)}(g,\rho)$ contains a term $G_{v_g v_\rho}$

$g^{v_g}\rho^{v_\rho}$. This term represents the sum of contributions of all similar diagrams. The differential operator $\mathcal{D}_g$ in (1.11) do not rise or lower the powers of couplings, so the structure is retained also in the beta-function. It is not the case of higher residues, as seen in (1.13). Hence, we can read the structure of beta-functions directly from the respective divergent diagrams.

The previous arguments easily generalizes to $n$ couplings $\alpha_k$ with renormalization constants

$$\Delta_k\left(\alpha_1,\ldots,\alpha_n,\varepsilon\right)=\sum_{i=1}^{\infty}\frac{\alpha_k^{(i)}\left(\alpha_1,\ldots,\alpha_n\right)}{\varepsilon^i}.$$

The beta-function for $\alpha_k$ and the generalization of (1.13) are

$$\begin{aligned}\widehat{\beta}_k &= \mathcal{D}_k\alpha_k^{(1)}\left(\alpha_1,\ldots,\alpha_n\right)-p^{(k)}\alpha_k\varepsilon,\\ \mathcal{D}_k\alpha_k^{(m)} &= \left(\sum_{i=1}^{n}\mathcal{D}_i\alpha_i^{(1)}\frac{\partial}{\partial\alpha_i}\right)\alpha_k^{(m-1)},\\ \mathcal{D}_k &= \left(\sum_{j=1}^{n}p^{(j)}\alpha_j\frac{\partial}{\partial\alpha_j}\right)-p^{(k)},\end{aligned}\qquad(1.14)$$

where $p^{(i)}$ is the dimensionality-defect of $\alpha_i$. The constants of integration of these differential equations are fixed simply considering that the residues $\alpha_k^{(m)}$ are perturbative quantities, so the differential operator $\mathcal{D}_k$ can be "inverted", just as in (1.12). The previous conclusions apply directly to the general case.

This analysis is completely general since it is not assumed that the couplings are marginal or the renormalization is multiplicative. Another way to reconstruct renormalization constants from beta-functions can be found in the appendix of [11]. There, taking advantage of the the particular form of counterterms (all proportional to $\alpha$) and using the reduction of couplings, the renormalization constants are obtained directly integrating their beta-functions .

If the theory contains a single marginal constant, the renormalization constant can be easily calculated integrating in (1.9) since $Z_\alpha$ depends on $\mu$ only through $\alpha$,

$$Z_\alpha\left(\alpha,\varepsilon\right)=\exp\left(-\int_0^\alpha d\alpha'\frac{\beta_\alpha\left(\alpha'\right)}{\widehat{\beta}_\alpha\left(\alpha'\right)}\right).$$

# Chapter 2

# Infinite Reduction of Couplings

As mentioned in the introduction, the role of renormalization as a tool for discriminating which theories are appropriate to describe physical interactions has not been totally unveiled yet. The main difficulty to consider non-renormalizable models as fundamental theories is that they need, in the usual renormalization program, infinitely many independent couplings. However, we know that renormalizability as we understand it, namely Power Counting (PC) analysis, can not be an ultimate criteria to exclude models. For example, in models studied by Parisi [2], finite Green functions are obtained to all orders in the $1/N$ -expansion even when they are non-renormalizable in the PC sense. As he noticed, the perturbative series in coupling constants may not be suitable for the expansion of correlation functions, causing the appearance of divergences that are absent in other type of treatment. The idea developed in this chapter, rather than propose this kind of solutions for particular models, is to study a general framework to face directly the problem of having infinitely many couplings by considering that this infinite set is nothing but an inaccurate manner to describe a more fundamental theory which does have a finite number of parameters. Being more specific, we will try to establish relations among the couplings to regard most of them as dependent or functions of a small, finite set. This kind of reduction happens naturally in theories that present symmetries. Here instead, the reduction is based on the *running* of the couplings, i.e. the way they change under the renormalization group flow. Therefore the main feature of the reduction is that it is RG-invariant.

The RG-consistent reduction is obtained by means of the resolution of a differential equation, the *reduction equation.* Nevertheless, to actually reduce the number of independent parameters, we need to pick some special solution of this equation, using some prescriptions which are motivated by physical arguments. The reduction could not be carried on in all cases; some *invertibility conditions* will indicate whether it is possible or not depending on the parameters of the model.

The idea of the reduction of couplings was first applied by Zimmermann and Oheme [13, 14, 15] to renormalizable models and it has been used historically as an alternative to GUT theories and applied especially to supersymmetric models. The most important phenomenological results obtained using this method [16] are the masses of the top quark $m_t \simeq 81$GeV and the Higgs boson $m_h \simeq 61$GeV. Both are out range of the present knowledge [17] $m_t = 171 \pm 2.1$GeV and $m_h \gtrsim 80$GeV. These disappointing results do not invalidate the reduction of couplings as technique, only indicate that the reduction hypothesis is not applicable to this model.

A different approach to reduce the number of independent couplings in non-renormalizable theories is Weinberg's asymptotic safety [18]. Other investigations of reductions of couplings in non-renormalizable theories have been performed by Atance and Cortes [19, 20], Kubo and Nunami [21], Halpern and Huang [22, 23].

In the first two sections the method of reduction of couplings is introduced together with the criteria to select the suitable solutions to the reduction equation. Then we review in section 2.3 the Zimmermann's model to illustrate the main features of the method. There we solve exactly the leading-log approximation, and analyze the series expansion of the complete solution.

The *infinite reduction*, i.e. the process designed to establish dependence relations among couplings in non-renormalizable theories, is discussed in the rest of the chapter. There, the invertibility conditions and the reduction itself are obtained also from the *bare reduction equation*, which establishes the dependence between the bare couplings. In section 2.4.6 is shown how the invertibility condition can be refined in the case where there is no three-leg marginal vertices. Section 2.4.9 shows how relevant parameters as the masses can be included perturbatively. In section 2.4.3 we examine how the reduction is affected by the renormalization mixing in the non-renormalizable sector. In section 2.4.7 we solve explicitly the infinite reduction in the leading-log approximation, which contains enough information about the existence and uniqueness of the solution to all orders, while in section 2.4.8 the infinite reduction in the presence of several marginal couplings is analyzed.

Since the reduction criteria are formulated in terms of the extended-space parameter $\varepsilon$, we use exclusively dimensional regularization in this chapter to define divergent integrals.

Along this chapter we will find ourselves repeatedly in the situation of studying the analytic properties of solutions of certain differential equations. Although in some of them it is possible to obtain explicit closed expressions [11], we prefer to consider all at once studying generically its series solutions as explained in the Appendix A.

## 2.1 Reduction Of Couplings

Let $\alpha$ and $\eta$ be two independent coupling constants. From renormalization group analysis we know they are not actually constant but they depend on the RG scale $\mu$

$$\eta = \eta(\mu), \qquad \alpha = \alpha(\mu).$$

In some circumstances this relations can be inverted, being possible to find a dependence $\eta = \check{\eta}(\alpha)$ such as

$$\eta(\mu) = \eta(\mu\,(\alpha)) \equiv \check{\eta}(\alpha(\mu)). \tag{2.1}$$

Because $\check{\eta}(\alpha)$ does not depend explicitly on $\mu$, the main feature of the reduction is that is invariant under the RG flow, so $\eta = \check{\eta}(\alpha)$ holds for every choice of the renormalization point. This is an essential requirement because the RG parameter $\mu$ has no direct physical implication. Once the reduction $\check{\eta}(\alpha)$ is determined, we can substitute in the lagrangian $\eta$ by $\check{\eta}(\alpha)$. In this way the renormalization of the reduced theory is achieved redefining fields and the single coupling $\alpha$.

Deriving (2.1) with respect to $\ln\mu$, we obtain a differential equation, *the reduction equation*,

$$\widehat{\beta}_\alpha \frac{\mathbf{d}\check{\eta}(\alpha,\varepsilon)}{\mathbf{d}\alpha} = \widehat{\beta}_\eta. \tag{2.2}$$

Equation (2.2) does not represent by itself a reduction because the constant of integration $\xi$ is precisely the degree of freedom we want to suppress. Keeping it arbitrary amounts only to a reparametrization $(\alpha,\eta)\to(\alpha,\xi)$ . This reparametrization can be regarded as a change of scheme, with not physical consequences. The mentioned change of variables has the particularity that the new "coupling" $\xi$ is RG-invariant and this property is used in section 2.4.8, where the renormalizable sector has several marginal couplings. The core of this chapter is the study of this equation and physical arguments to select some particular solution of it to represent the reduction. Appendix A.1 will be useful for this objective.

All equations in this chapter, included (2.2) are written in continued space of $D=4-\varepsilon$ dimensions. The reduction can be carried on with other regularizations leading to similar conclusions [9]. Precisely one of the objectives of this chapter is to show the equivalence of the treatment in dimensional regularization, where some computations are in some sense more transparent and natural, respect to other regularizations. We work in four physical dimensions, but the generalization to arbitrary number of dimensions is straightforward. Recall from Chapter 1 that the hat over the beta-function indicates that it has also an evanescent part.

To select a particular solution of (2.2) some prescription or criterion is needed. Some special conditions will indicate if the criterion successfully select a single solution or not. If they are not fulfilled, either there exist no solution satisfying the criterion or there are infinitely many (i.e. the general solution satisfies the criterion). These conditions, called *invertibility conditions* depend only on the leading-log coefficients of the theory. This fact corroborates that the reduction is RG-invariant since these parameters do not depend on the substraction scheme. The special value of the integration constant that make the solution satisfy the criteria will be denoted by $\bar{\xi}$, and

$$\bar{\eta}\left(\alpha,\varepsilon\right)=\check{\eta}\left(\alpha,\varepsilon,\bar{\xi}\right)$$

is such a solution.

Similar notation will be used in the infinite reduction.

## 2.2 Reduction Criteria

Here we enunciate two prescriptions to select the particular solution of (2.2) that represents the reduction and present their physical motivations. We postpone the demonstration of the equivalence to the next section. The criteria are formulated in terms of two constants $\alpha$ and $\eta$ and the dimensionally extended parameter $\varepsilon$. For more of two couplings see the discussion of section 2.4.8. The criteria for the reduction are:

*i) The function $\breve{\eta}(\alpha, \varepsilon)$ has to be perturbatively meromorphic in $\alpha$ and analytic in $\varepsilon$.* Considering the reduction as a manifestation of a more fundamental theory, it is clear that if the relation between the two theories is perturbative, only integer powers of the independent coupling can appear. Perturbative meromorphy [9] means that negative powers can be arbitrarily high, but the maximal negative power grows linearly with the order of some expansion. As explained in section 2.4.4,. this feature allows to define an "effective Planck mass" $M_{\mathrm{Peff}}$ that gives sense to the perturbative expansion of the reduced version of a non-renormalizable theory at energies $E \ll M_{\mathrm{Peff}}$. For reductions inside a renormalizable sector (as in section 2.3.) we can impose the stronger condition of analyticity in $\alpha$ instead of perturbative meromorphy.

On the other hand, the renormalized quantum action $\Gamma[\Phi, \alpha, \eta, \varepsilon]$ is finite at physical dimensions, so $\breve{\eta}(\alpha, \varepsilon)$ should be regular at $\varepsilon \to 0$. Regular in perturbation theory means analytic.

*ii)* The fact that $\eta$ and $\alpha$ are related at renormalized level implies they are related also at bare level. This is a trivial statement in most regularizations since bare quantities can be regarded as quantities defined at the cutoff. As the reduction is RG invariant, the very same relation holds at all energies, in particular, at the cutoff. This is not the case of dimensional regularization. Thus, a different non-trivial criterion can be formulated as follows: *The reduction has to be an analytic function of $\varepsilon$ at renormalized and bare levels*. In general satisfying this requirement will be enough to complete the reduction, regardless the $\alpha$- or $\alpha_{\mathrm{B}}$-dependence.

The analyticity at $\varepsilon = 0$ of the reduction at renormalized level is justified in *i).* The bare lagrangian $\mathcal{L}(\varphi_{\rm B}, \eta_{\rm B}, \alpha_{\rm B}, \varepsilon)$, on the other hand, becomes the classical lagrangian in the "naive" limit, that is the limit $\varepsilon \to 0$ at fixed bare couplings and fields. If a reduction $\check{\eta}_{\rm B}(\alpha_{\rm B}, \varepsilon)$ is consistent, then the reduced bare action $\mathcal{L}(\varphi_{\rm B}, \alpha_{\rm B}, \check{\eta}_{\rm B}(\alpha_{\rm B}, \varepsilon), \varepsilon)$ should converge to the reduced classical lagrangian in the naive limit. Thus also $\check{\eta}_{\rm B}(\alpha_{\rm B}, \varepsilon)$ should be regular for $\varepsilon \to 0$.

In most cases any of these criteria *i)* and *ii)* will serve us to select one appropriate solution to (2.2). Indeed it will be shown explicitly that they are completely equivalent.

## 2.3 Zimmermann Model (Renormalizable Theories)

Zimmermann and Oheme [14] studied the reduction of couplings in the realm of renormalizable theories. Let us review their simpler model to illustrate the reduction mechanism and how the analyticity criteria enunciated in the previous section are equivalent, leading to the same unique reduction when it exists.

We test the criteria in the leading-log approximation first, where the general solution of the reduction equation is known, and then repeat the analysis in the complete solution using the series method of Appendix B.

Consider a massless Yukawa model with quartic interaction, or scalar electrodynamics

$$\begin{aligned} \mathcal{L}_{\rm Y} &= \frac{1}{2}(\partial\varphi)^2 + \overline{\psi}\partial\!\!\!/\psi + g\varphi\overline{\psi}\psi + \frac{\rho}{4!}\varphi^4, \\ \mathcal{L}_{\rm SE} &= \frac{1}{4}F_{\mu\nu}^2 + |D_\mu\varphi|^2 + \frac{\rho}{4}(\overline{\varphi}\varphi)^2, \end{aligned}$$

with $D_\mu\varphi = \partial_\mu\varphi + igA_\mu\varphi$ and $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$. In both models the reduction follows the same lines. A very convenient (but not indispensable) step that will be very useful for the infinite reduction as well, is to perform a reparametrization of couplings previous to the reduction. The idea is to choose the independent coupling such that it has no dimensionality-defect. For this goal we define $\alpha = g^2$ and $\eta = \rho/\alpha$ (so $p^{(\alpha)} = 1$ and $p^{(\eta)} = 0$). Then according to the notation of Chapter 1, we have

$$\widehat{\beta}_\rho = \eta\widehat{\beta}_\alpha + \alpha\widehat{\beta}_\eta, \qquad \widehat{\beta}_\alpha = 2g\widehat{\beta}_g. \tag{2.3}$$

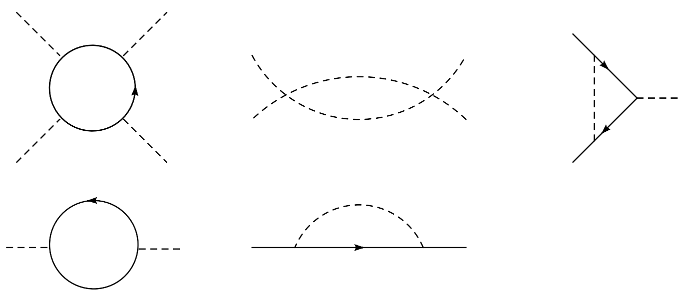

Figure 2.1: One-loop diagrams contributing to beta functions $\widehat{\beta}_g$ and $\widehat{\beta}_\rho$.

The structure of $\beta_\eta$ and $\widehat{\beta}_g$ comes[1] directly from the structure of divergent diagrams that renormalize each coupling, as explained in section 1.7. The one-loop diagrams contributing to (2.3) shown in Figure 2.1 allows us to write the leading terms of $\widehat{\beta}_\alpha$ and $\widehat{\beta}_\rho$. To all order $\beta_\alpha$ and $\beta_\eta$ have the form

$$\frac{\beta_\alpha}{\alpha} = \beta_1 \alpha + \sum_{L=2}^{\infty} \alpha^L P_L(\eta), \qquad \beta_\eta = \alpha\left(a + b\eta + c\eta^2\right) + \sum_{L=2}^{\infty} \alpha^L Q_{L+1}(\eta), \tag{2.4}$$

where $P_L(\eta)$ and $Q_L(\eta)$ are polynomials in $\eta$ of degree $L$. We assume that $\beta_1, a, b$ and $c$ are nonzero constants.

The form of (2.4) can be inferred just tracking dimensionalities in bare diagrams. First note that *a bare diagram and the bare constant it renormalizes have the same dimensionality-defect.* To visualize this, observe the contribution of a divergent diagram $G$ to the 1-PI generating functional $\Gamma$ shown in equation (4.18), where the quantity inside the parenthesis is a bare coupling. The only two sources of dimensionality-defect in a diagram[2] are the measure $d^D p$ that contributes with $-\varepsilon$ for each loop, and the coupling $\alpha_{\mathrm{B}}$ (since $p^{(\eta)} = 0$). Therefore, matching dimensionalities and recalling that renormalized and bare diagrams have the same structure, it follows that in a $L$-loop divergent diagram that renormalizes $\alpha$ there are $L+1$ powers of $\alpha$. Analogously, diagrams that renormalize $\eta$ are proportional to $\alpha^L$. In the other hand, using (1.1), (1.4) and (1.3), the number of four-legs vertices contained in a diagram is

$$v_\rho = \frac{E}{2} + L - 1 - \frac{v_g}{2}.$$

[1]Note that $\widehat{\beta}_\eta = \beta_\eta$.

[2]Here we consider the coupling constants as part of the diagram.

Thus, for diagrams renormalizing $\rho$ we have

$$v_\rho \leq L+1,$$

indicating that the $L$-loop contributions to $\beta_\rho$ is a polynomial of degree $L+1$ in $\eta$.

### 2.3.1 Leading-log Approximation

Using only the one-loop contribution of beta-functions (2.4),

$$\beta_\alpha = \beta_1 \alpha^2, \qquad \beta_\eta = \alpha\left(a + b\eta + c\eta^2\right). \tag{2.5}$$

we can solve exactly the reduction equation (2.2), obtaining

$$\begin{aligned} \check{\eta}_\pm(\alpha,\xi,\varepsilon) &= -\frac{1}{2c}\left[b \mp s\frac{1+\xi\cdot(\alpha\beta_1-\varepsilon)^{\pm s/\beta_1}}{1-\xi\cdot(\alpha\beta_1-\varepsilon)^{\pm s/\beta_1}}\right], \qquad \text{for } s\neq 0 \\ \check{\eta}_0(\alpha,\xi,\varepsilon) &= -\frac{1}{2c}\left[b + \frac{2\beta_1}{\xi+\ln(\alpha\beta_1-\varepsilon)}\right], \qquad \text{for } s=0, \end{aligned} \tag{2.6}$$

with $s$ the positive square root of $b^2-4ac$ and $\xi$ the constant of integration.

The solutions labelled with $\pm$ are actually two different ways of writing the same general solution, as can be seen substituting $\xi = 1/\xi'$.

**First Criterion** In the first criterion we look for an analytic behavior in both $\alpha$ and $\varepsilon$. Depending on the values of the beta-function coefficients, two situations could occur:

*i) The exponent $\pm s/\beta_1$ is a positive integer*

The solution is doubly-analytic for every value of the constant of integration $\xi$. As every solution is equally valid, the criterion is useless in determining the actual reduction. Keeping $\xi$ arbitrary is equivalent to a reparametrization of couplings $(\alpha,\eta)\to(\alpha,\xi)$.

*ii) The exponent $\pm s/\beta_1$ is not a positive integer,*

$$\pm\frac{s}{\beta_1} \notin \mathbb{N}_+, \tag{2.7}$$

In this case, only $\overline{\xi} = 0, \infty$ give doubly-analytic solutions,

$$\overline{\check{\eta}}_{\pm}(\alpha, \varepsilon) = -\frac{b \mp s}{2c}. \tag{2.8}$$

For $s = 0$ the only analytic solution is $\overline{\check{\eta}}_0(\alpha, \varepsilon) = -\frac{b}{2c}$ (setting $\overline{\xi} \to \infty$).

**Second Criterion** In general, the relation between $\eta_{\mathrm{B}}$ and $\alpha_{\mathrm{B}}$ can be computed using the solution of the renormalized reduction (2.6) and the inverting the relations

$$\alpha_{\mathrm{B}} = \mu^{\varepsilon} \alpha\ Z_{\alpha}(\alpha, \eta, \varepsilon), \qquad \eta_{\mathrm{B}} = \eta + \alpha \Delta_{\eta}(\alpha, \eta, \varepsilon). \tag{2.9}$$

The constants $Z_{\alpha}$ and $\Delta_{\eta}$ can be reconstructed from beta-functions as shown in Chapter 1. However, it is possible to avoid this calculation thanks to three peculiar features of the treatment made:

*i)* $\eta_{\mathrm{B}}$ has no dimensionality-defect.
*ii)* $\alpha_{\mathrm{B}}$ does have dimensionality-defect (equal to 1).
*iii)* The counterterms that renormalize $\eta$ are all proportional to $\alpha$.

When the model has two couplings we can on most cases choose a reparametrization, as we did, to have two non-minimal couplings satisfying *i)* and *ii)*. The third statement is consequence of the other two and the discussion of the paragraph below the equation (2.4).

Applying the renormalized reduction inside $\eta_{\mathrm{B}} = \eta_{\mathrm{B}}(\alpha, \eta, \varepsilon)$, we define

$$\check{\eta}_{\mathrm{B}}(\alpha, \xi, \varepsilon) \equiv \eta_{\mathrm{B}}(\alpha, \check{\eta}(\alpha, \xi, \varepsilon), \varepsilon).$$

Generally both $\check{\eta}_{\mathrm{B}}$ and $\eta_{\mathrm{B}}$ should depend explicitly on $\mu$, but they do not due to *i)*. Then, since all dependence on $\mu$ is through $\alpha$ and bare quantities are $\mu$-independent, we conclude that $\check{\eta}_{\mathrm{B}}(\alpha, \xi, \varepsilon)$ can not depend on $\alpha$ neither:

$$\mu \frac{\mathbf{d}\eta_{\mathrm{B}}}{\mathbf{d}\mu} = 0 = \frac{\mathbf{d}\check{\eta}_{\mathrm{B}}}{\mathbf{d}\alpha} \widehat{\beta}_{\alpha}.$$

Then, as $\check{\eta}_{\mathrm{B}}$ do not depend on $\alpha$,

$$\begin{aligned} \check{\eta}_{\mathrm{B}}(\alpha, \xi, \varepsilon) &= \lim_{\alpha \to 0} \check{\eta}_{\mathrm{B}}(\alpha, \xi, \varepsilon) \\ &= \lim_{\alpha \to 0} [\check{\eta}(\alpha, \xi, \varepsilon) \ + \alpha \Delta_{\eta}(\alpha, \check{\eta}(\alpha, \xi, \varepsilon), \varepsilon)] \\ &= \check{\eta}(0, \xi, \varepsilon)\ . \end{aligned} \tag{2.10}$$

Where *iii)* was used, the fact that $\Delta_\eta$ is a power series in couplings and that $\check{\eta}(0,\xi,\varepsilon)$ is finite. Therefore, without making any new calculation other than simply taking the $\alpha \to 0$ limit of (2.6) we get

$$\begin{aligned} \check{\eta}_{\pm\mathrm{B}}(\alpha_\mathrm{B},\xi,\varepsilon) &= -\frac{1}{2c}\left[b \mp s\frac{1+\xi\cdot(-\varepsilon)^{\pm s/\beta_1}}{1-\xi\cdot(-\varepsilon)^{\pm s/\beta_1}}\right], \qquad \text{for } s \neq 0, \\ \check{\eta}_{0\mathrm{B}}(\alpha_\mathrm{B},\xi,\varepsilon) &= -\frac{1}{2c}\left[b + \frac{2\beta_1}{\xi+\ln(-\varepsilon)}\right], \qquad \text{for } s = 0. \end{aligned} \tag{2.11}$$

Using this method, the $\alpha_\mathrm{B}$-dependence is trivial: $\check{\eta}_\mathrm{B}$ simply can not depend on $\alpha_\mathrm{B}$ due to dimensionality arguments.

Let us examine now the analytic properties of (2.11) with respect to $\varepsilon$. As before, we have two sensibly different cases

*I) The exponent $\pm s/\beta_1$ is a positive integer*, any $\xi$ provides a solution which is analytic in $\varepsilon$.

*II) The exponent $\pm s/\beta_1$ is not a positive integer*, the unique analytic solutions are found for $\overline{\xi} = 0, \infty$.

It is pretty clear that the criteria lead to the same conclusion, that is, for having a unique reduction the condition (2.7) must hold. In the first criterion it ensures that (2.6) will have a unique doubly-analytic (in $\alpha$ and $\varepsilon$) reduction at renormalized level. In the second one instead, it guarantees that there are only one reduction analytic in $\varepsilon$ in both renormalized and bare level.

The trick of taking the limit (2.10) for finding the bare reduction makes evident the connection between the criteria:

In the second prescription, $\varepsilon$-analyticity is required in (2.6) and in (2.11). Since the bare relation is nothing but the renormalized one in the $\alpha \to 0$ limit, the second criterion is summarized in *analyticity with respect to $\varepsilon$ of the renormalized relation* (2.6) *and its $\alpha \to 0$ limit should exist.*

In the first criterion instead, we look for analyticity in both $\alpha$ and $\varepsilon$ simultaneously in the renormalized reduction (2.6). This prescription is apparently more restrictive than the above statement, since it requires (2.6) to be not only regular in $\alpha$, but analytic. The equivalence is evident if we realize that $\alpha$ and $\varepsilon$ have exchangeable roles in the solution. This can be seen explicitly from the reduction equation using the change of variables $u = \alpha - \varepsilon/\beta_1$. More precisely, in general the solution can be expressed (see section 2.3.2) as a function of $u\,(\alpha, \varepsilon) = \alpha + \mathcal{O}\,(\varepsilon)$ which is analytic in its arguments, thus

$$\text{Analyticity in } \varepsilon \text{ (at } \alpha = 0) \Rightarrow \text{Analyticity in } u \iff \text{Analyticity in } \alpha \text{ (for all } \varepsilon).$$

because the solution depends on $\alpha$ only through $u$. Therefore a reduction which is analytic in $\varepsilon$ is automatically analytic in $\alpha$, for all $\varepsilon$.

This connection holds in the complete solution of the Zimmermann model as in the infinite reduction, and would not be apparent if the bare relation had been calculated through the inversion of the renormalization constants.

Note that if the dimensionality-defect $p^{(\eta)}$ of $\eta_{\mathrm{B}}$ were different from zero, the equation above (2.10) would read

$$\mu \frac{\mathbf{d}\eta_{\mathrm{B}}\,(\alpha, \eta, \varepsilon\,, \mu)}{\mathbf{d}\mu} = \mu \frac{\partial \check{\eta}_{\mathrm{B}}\,(\alpha, \xi, \varepsilon\,, \mu)}{\partial \mu} + \frac{\mathbf{d}\check{\eta}_{\mathrm{B}}\,(\alpha, \xi, \varepsilon, \mu)}{\mathbf{d}\alpha} \widehat{\beta}_{\alpha} = 0,$$

having the solution

$$\check{\eta}_{\mathrm{B}}\,(\alpha, \xi, \varepsilon\,, \mu) = \mu^{p^{(\eta)}\varepsilon} F\left(\xi, \varepsilon, p^{(\eta)}\right) \exp\left\{ -p^{(\eta)}\varepsilon \int^{\alpha} \frac{d\alpha'}{\widehat{\beta}_{\alpha}(\alpha', \check{\eta}(\alpha', \xi, \varepsilon))} \right\}.$$

In this manner the $\alpha$-dependence is obtained, but not the dependence on $\xi$ or $\varepsilon$. The lower limit in the integral is redundant since it can be absorbed by the $F\left(\xi, \varepsilon, p^{(\eta)}\right)$. Clearly $F\,(\xi, \varepsilon, 0)$ corresponds to (2.11).

### 2.3.2 Complete Solution

We are ready to look for $\check{\eta}(\alpha, \xi, \varepsilon)$ beyond the leading-log approximation. Since it is not possible to give a closed generic solution of the reduction equation when (2.4) are used, we study its properties using the series method explained in Appendix A, but first define for convenience the variables $u$ and $v$

$$u = \alpha - \alpha_*(\varepsilon), \qquad v = \eta - \eta_{\pm *}(\varepsilon),$$

where $\alpha_*(\varepsilon)$, $\eta_{\pm *}(\varepsilon)$ are the non-trivial RG fixed point at $\varepsilon \neq 0$, namely the solution of

$$\frac{\widehat{\beta}_{\alpha}}{\alpha}(\alpha, \eta, \varepsilon) = 0, \qquad \frac{\beta_{\eta}}{\alpha}(\alpha, \eta, \varepsilon) = 0.$$

For the expressions (2.4) they have the values

$$\alpha_*(\varepsilon) = \frac{\varepsilon}{\beta_1} + \mathcal{O}(\varepsilon^2), \qquad \eta_{\pm *}(\varepsilon) = -\frac{b \mp s}{2c} + \mathcal{O}(\varepsilon),$$

Write expansions

$$\frac{\widehat{\beta}_\alpha}{\alpha} \equiv f(u,v) = f_1 u + f_2 v + \mathcal{O}(u^2, uv, v^2), \qquad \frac{\beta_\eta}{\alpha} \equiv g(u,v) = g_1 u + g_2 v + \mathcal{O}(u^2, uv, v^2),$$

where $f_1 = \beta_1 + \mathcal{O}(\varepsilon)$, $f_2 = \mathcal{O}(\varepsilon^2)$, $g_1 = \mathcal{O}(1)$, $g_2 = \pm s + \mathcal{O}(\varepsilon)$.
The reduction of couplings is expressed by a function $v(u)$ that satisfies

$$f(u, v(u)) \frac{\mathbf{d}v(u)}{\mathbf{d}u} = g(u, v(u)).$$

This equation can be transformed into an equation with the same features as the one studied in Appendix A by performing the change of variables $w(u) \equiv v(u)/u$:

$$\left[f_1 + f_2 w + u Q_1(w) + u^2 Q_2(w) + \ldots\right] [w + u\, \frac{\mathbf{d}w(u)}{\mathbf{d}u}] = g_1 + g_2 w + u P_1(w) + u^2 P_2(w) + \ldots \tag{2.12}$$

where $P_n(w)$ and $Q_n(w)$ are polynomials of order $n$ in $w$.

Organizing the equation (2.12) properly, it has the same form as (A.1) identifying

$$A = -g_1, \qquad B = (f_1 - g_2), \qquad C = f_2, \qquad D = -(f_1 + f_2 w_{0\pm}).$$

With $w_{0\pm}$ the solution of $A + Bw + Cw^2 = 0$.

Consequently the solution for $w(u,\xi)$ can be expressed as a infinite series as in (A.5), with the relevant quantity $r$ given by

$$\begin{aligned} r &= -\frac{\sqrt{(f_1 - g_2)^2 + 4 g_1\ f_2}}{(f_1 + f_2 w_{0\pm})} = -\frac{(f_1 - g_2)}{f_1} + \mathcal{O}(\varepsilon) \\ &\equiv r^{(0)} + \mathcal{O}(\varepsilon), \end{aligned}$$

where $r^{(0)} = \pm\frac{s}{\beta_1} - 1$ is the non-evanescent part of $r$.

Multiplying the solution (A.5) for $w\,(u,\xi)$ by $u$ we recover the series solution for $v(u,\xi)$

$$v(u,\xi) = \sum_{i=0}^{\infty} v_{\pm i}\ u^{i+1} + \sum_{n=1}^{\infty}\sum_{m=0}^{\infty} v_{\pm mn}\ \xi^n\ u^{m+n\left(\pm\frac{s}{\beta_1} + \mathcal{O}(\varepsilon)\right)}$$

The coefficients $v_{\pm i}$ and $v_{\pm mn}$ are polynomial on $f_i$ and $g_i$, that are themselves analytic in $\varepsilon$. As can be seen from (A.3), when $r^{(0)}$ is a positive integer $\widehat{n}, v_{\pm\widehat{n}}$ and successive coefficients are singular at $\varepsilon \to 0$. In other words, when

$$\pm\frac{s}{\beta_1} - 1 \notin \mathbb{N}_+ \tag{2.13}$$

all coefficients $v_{\pm i}$ and $v_{\pm mn}$ are analytical in $\varepsilon$ and determinable recursively.

Finally, in terms of $\eta$ and $\alpha$,

$$\begin{aligned}\check{\eta}(\alpha,\xi,\varepsilon) = \eta_{\pm *}(\varepsilon) \quad &+ \quad \sum_{i=0}^{\infty} v_{\pm i}(\varepsilon)\ (\alpha - \alpha_*(\varepsilon))^{i+1} \\ &+ \quad \sum_{n=1}^{\infty}\sum_{m=0}^{\infty} v_{\pm mn}(\varepsilon)\ \xi^n\ (\alpha - \alpha_*(\varepsilon))^{m+n\left(\pm\frac{s}{\beta_1}+\mathcal{O}(\varepsilon)\right)}.\end{aligned} \tag{2.14}$$

**First criterion** If (2.13) holds, the solution (2.14) is analytic in $\alpha$ at $\varepsilon = 0$ only for $\overline{\xi} = 0$. Observing that $\alpha_*(\varepsilon)$ and $\eta_{\pm *}(\varepsilon)$ are analytic in $\varepsilon$, the unique doubly-analytic solution is

$$\overline{\check{\eta}}(\alpha,\xi,\varepsilon) = \eta_{\pm *}(\varepsilon) + \sum_{i=0}^{\infty} v_{\pm i}(\varepsilon)\ (\alpha - \alpha_*(\varepsilon))^{i+1}.$$

On the other hand, if (2.13) is violated, that is $r^{(0)}$ is a positive integer $\widehat{n}$, there is no choice of $\xi$ able to cancel out the singularity in $\varepsilon \to 0$ of the coefficient $v_{\pm\widehat{n}}(\varepsilon)$, so does not exist any solution analytic in $\varepsilon$.

The condition (2.13) is similar to the one obtained in the leading-log approximation. The only difference is that here also $\pm s/\beta_1 = 1$ allows a unique reduction.

**Second criterion** Since the arguments *i)*, *ii)* and *iii)* of section 2.3.1 are still valid, we can again use the limit (2.10) to obtain the bare reduction

$$\begin{aligned}\check{\eta}_{\pm\mathrm{B}}(\alpha_{\mathrm{B}},\varepsilon,\xi) = \eta_{\pm *}(\varepsilon) \quad &+ \quad \sum_{i=0}^{\infty} v_{\pm i}(\varepsilon)\ (-\alpha_*(\varepsilon))^{i+1} \\ &+ \quad \sum_{n=1}^{\infty}\sum_{m=0}^{\infty} v_{\pm mn}(\varepsilon)\ \xi^n(-\alpha_*(\varepsilon))^{m+n\left(\pm\frac{s}{\beta_1}+\mathcal{O}(\varepsilon)\right)}.\end{aligned} \tag{2.15}$$

If (2.13) holds, the bare reduction (2.15) is analytic in $\varepsilon$ only for $\overline{\xi}=0$, and this choice makes also the renormalized reduction analytic in $\varepsilon$, determining the reduction uniquely.

Summarizing, both criteria *i)* and *ii)* show that (2.13) is a sufficient condition for performing a reduction of couplings to all orders in the Zimmermann and it reads

$$\begin{aligned}\overline{\overline{\eta}}(\alpha,\varepsilon) &= \eta_{\pm *}(\varepsilon)+\sum_{i=0}^{\infty} v_{\pm i}(\varepsilon)\,(\alpha-\alpha_*(\varepsilon))^{i+1},\\ \overline{\overline{\eta}}_{\pm \mathrm{B}}(\alpha_{\mathrm{B}},\varepsilon) &= \eta_{\pm *}(\varepsilon)+\sum_{i=0}^{\infty} v_{\pm i}(\varepsilon)\,(-\alpha_*(\varepsilon))^{i+1}.\end{aligned}$$

Note that $\eta_{\pm *}(\varepsilon)$ is the leading-log analytic reduction (2.8) plus evanescent corrections.

## 2.4 Infinite Reduction

In the usual renormalization program non-renormalizable theories need an infinite number of different interactions to provide the appropriate set of counterterms to absorb all infinities generated by divergent Feynman diagrams. In this chapter we recover some results of the application of the reduction of couplings to non-renormalizable models [9] through dimensional regularization techniques where some features are more transparent.

We study a generic massless model in four dimensions although the extension to arbitrary number of dimensions is straightforward. We do not include any parameter with positive dimensionality, in particular no mass terms or relevant interactions. They can be added in a second stage, as shown in section 2.4.9. The lagrangian is divided in three pieces:

*i)* An interacting renormalizable sector denoted $\mathcal{R}$,

*ii)* the *head*, made of irrelevant operators with lowest dimensionality and

*iii)* the *queue,* made of all other irrelevant operators of higher dimensionality needed for a consistent renormalization structure.

Irrelevant operators are classified by their "level", that is, the dimensionality of their coupling constants (at $\varepsilon=0$). The dimensionality of the coupling of the head is denoted by $-\ell$, with $\ell$ a positive number, and the corresponding operator is written as $\mathcal{O}_\ell$. It is easy to verify that all vertices generated by renormalization due to the presence of the head have couplings with dimensionalities which are integer multiples of $-\ell$. The generic lagrangian then reads

$$\mathcal{L}[\varphi] = \mathcal{L}_\mathcal{R}[\varphi, \alpha] + \alpha^{p^{(\ell)}} \lambda_\ell \mathcal{O}_\ell(\varphi) + \sum_{n>1} \alpha^{p^{(n\ell)}} \lambda_{n\ell} \mathcal{O}_{n\ell}(\varphi). \qquad (2.16)$$

where $[\alpha] = 0$ and $[\lambda_{n\ell}] = -n\ell$ with $n \geq 1$. Therefore, $\mathcal{O}_{n\ell}$ is a level-$n$ operator, with $n > 1$ a positive integer.

The couplings of the head and the queue are written in a non-minimal way such that the only bare coupling with non-vanishing dimensionality-defect is $\alpha_\mathrm{B}$, with $p^{(\alpha)} = 1$. Just as in Zimmermann's model, this parametrization facilitates computations and allows to use a limit similar to (2.10) to find rapidly the bare reduction.

Although normally there are several operators with the same level, we first make the simplifying assumption that there is only one operator for any level, and subsequently study the general case where operator mixing is present, in section 2.4.3.

The reduction we are looking for is an a sense maximal: all couplings are expressed as a function of the marginal coupling $\alpha$ and a single coupling in the head $\lambda_\ell$.

As example, take the Yang-Mills model in four dimensions as the renormalizable sector $\mathcal{R}$ coupled with massless fermions, and deform it with a Pauli term [24],

$$\begin{aligned}
\mathcal{L}_\mathcal{R}[\varphi, \alpha] &= -\frac{1}{4\alpha} F^a_{\mu\nu} F^{\mu\nu a} + \overline{\psi}_i \not{D}_{ij} \psi_j, \\
\lambda_\ell \mathcal{O}_\ell &= \lambda_1 F^a_{\mu\nu} \overline{\psi}_i T^a_{ij} \sigma_{\mu\nu} \psi_j,
\end{aligned}$$

with $F^a_{\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu + f^{abc} A^b_\mu A^c_\nu$, $\not{D}_{ij}\psi_j = \gamma^\mu \left(\partial_\mu \psi_i + A^a_\mu T^a_{ij} \psi_j\right)$, $\sigma_{\mu\nu} = -i[\gamma_\mu, \gamma_\nu]/2$.

The Pauli term is the unique operator in the head; other operators of the same dimensionality as $\overline{\psi} D^2 \psi$, are equivalent to it (plus equations of motion and operator of higher levels).

The independent operators of level 2 are 11 in total,

$$\begin{array}{lll}
\alpha\lambda_2^{(1)} f^{abc} F^a_{\mu\nu} F^b_{\nu\sigma} F^c_{\sigma\mu}, & \alpha\lambda_2^{(2)} \left(\overline{\psi}_i \psi_i\right)^2, & \alpha\lambda_2^{(3)} \left(\overline{\psi}_i \gamma_5 \psi_i\right)^2, \\
\alpha\lambda_2^{(4)} \left(\overline{\psi}_i \gamma_\mu \psi_i\right)^2, & \alpha\lambda_2^{(5)} \left(\overline{\psi}_i \gamma_5 \gamma_\mu \psi_i\right)^2, & \alpha\lambda_2^{(6)} \left(\overline{\psi}_i \sigma_{\mu\nu} \psi_i\right)^2, \\
\alpha\lambda_2^{(7)} \left(\overline{\psi}_i \psi_j\right)\left(\overline{\psi}_j \psi_i\right), & \alpha\lambda_2^{(8)} \left(\overline{\psi}_i \gamma_5 \gamma_\mu \psi_j\right)\left(\overline{\psi}_j \gamma_5 \gamma_\mu \psi_i\right), & \alpha\lambda_2^{(9)} \left(\overline{\psi}_i \gamma_5 \psi_j\right)\left(\overline{\psi}_j \gamma_5 \psi_i\right), \\
\alpha\lambda_2^{(10)} \left(\overline{\psi}_i \gamma_\mu \psi_j\right)\left(\overline{\psi}_j \gamma_\mu \psi_i\right), & \alpha\lambda_2^{(11)} \left(\overline{\psi}_i \sigma_{\mu\nu} \psi_j\right)\left(\overline{\psi}_j \sigma_{\mu\nu} \psi_i\right). &
\end{array}$$

while other operators of dimensionality six can be converted in some combination of the above listed plus operator of higher dimensionality (up to equations of motion) by using the Bianchi identities.

Back to the generic lagrangian (2.16), the renormalization relations and beta-functions have the form

$$\begin{array}{ll}
\alpha_{\rm B} = \mu^\varepsilon \alpha Z_\alpha(\alpha,\varepsilon), & \beta_\alpha = \beta_\alpha^{(1)}\alpha^2 + \mathcal{O}(\alpha^3), \\
\lambda_{\ell{\rm B}} = z_\ell(\alpha,\varepsilon)\lambda_\ell, & \beta_\ell(\lambda,\alpha) = \gamma_\ell(\alpha)\lambda_\ell, \\
\lambda_{n\ell{\rm B}} = z_{n\ell}(\alpha,\varepsilon)\lambda_{n\ell} + \alpha\Delta_{n\ell}(\alpha,\lambda_{m\ell},\varepsilon), & \beta_{n\ell}(\lambda,\alpha) = \gamma_{n\ell}(\alpha)\lambda_{n\ell} + \alpha\delta_{n\ell}(\alpha,\lambda_{m\ell}),
\end{array} \tag{2.17}$$

where $\gamma_{n\ell}(\alpha)$ is the anomalous dimension of the operator $\mathcal{O}_{n\ell}$ in the theory $\mathcal{R}$.

As seen in section 1.7, the structures in (2.17) are closely related, indeed

$$\gamma_\ell(\alpha) = \alpha\frac{\partial}{\partial\alpha} z_\ell^{(1)}(\alpha), \qquad \gamma_{n\ell}(\alpha) = \alpha\frac{\partial}{\partial\alpha} z_{n\ell}^{(1)}(\alpha), \qquad \delta_{n\ell}(\alpha,\lambda) = \frac{\partial}{\partial\alpha}\left(\alpha\Delta_{n\ell}^{(1)}(\alpha,\lambda_{m\ell})\right), \tag{2.18}$$

where the superscripts refers to the residue of the simple pole, as in (1.11). The form of the expressions in (2.17) is due to:

*i)* There are no positive dimensionality constants in these models. Renormalization constants are power series in couplings, so by dimensionality matching, $\Delta_{n\ell}$ and $\delta_{n\ell}$ depend polynomially, at least quadratically on $\lambda_{m\ell}$ with $m < n$ and do not depend on $\lambda_{m\ell}$ with $m \geq n$.

*ii)* In the undeformed theory, the renormalization of the operator $\mathcal{O}_{n\ell}$ can be read from (2.17), so we have $\lambda_{n\ell{\rm B}} = z_{n\ell}(\alpha,\varepsilon)\lambda_{n\ell}$. Hence, according to the section 1.7, the coefficient of $\lambda_{n\ell{\rm B}}$ in the beta-function $\beta_{n\ell}(\lambda,\alpha)$ is the anomalous dimension $\gamma_{n\ell}(\alpha)$.

*iii)* By the same arguments explained in paragraph below the equation (2.4), $L$-loop diagrams renormalizing $\alpha_{\rm B}$ or $\lambda_{n\ell{\rm B}}$ are proportional to $\alpha^{L+1}$ and $\alpha^L$ respectively, so all counterterms are proportional to $\alpha$. This fact justify the $\alpha$-factor in front of $\Delta_{n\ell}$ and $\delta_{n\ell}$ and implies that $\gamma_{n\ell}(\alpha) = \gamma_{n\ell}^{(1)}\alpha + \mathcal{O}\left(\alpha^2\right)$.

### 2.4.1 Reduction

By dimensionality matching, the renormalized reduction relations read

$$\lambda_{n\ell}(\alpha, \lambda_\ell, \varepsilon) = \lambda_\ell^n \; f_n(\alpha, \varepsilon), \qquad n > 1. \tag{2.19}$$

As the $\lambda_\ell$-dependence is fixed, the reduction equation is an ordinary differential equation, instead of a partial differential equation. The reason why we have chosen massless theories and no relevant interactions is now evident. If some positive dimensionality parameter is present the dependence on $\lambda_\ell$ could not be fixed *a priori* as in (2.19) due to the possibility of arbitrary functions of dimensionless combination of parameters.

Deriving (2.19) with respect to $\ln\mu$ we obtain the set of differential equations

$$f_n'(\alpha, \varepsilon) \; \widehat{\beta}_\alpha = f_n(\alpha, \varepsilon) \; \tilde{\gamma}_{n_\ell}(\alpha) + \alpha\check{\delta}_{n\ell}(\alpha, \varepsilon). \tag{2.20}$$

which solutions determine the reduction. We have defined for shorten

$$\tilde{\gamma}_{n\ell}(\alpha) = \gamma_{n\ell}(\alpha) - n\gamma_\ell(\alpha), \qquad \check{\delta}_{n\ell}(\alpha, \varepsilon) = \delta_{n\ell}(\alpha, \overline{f}_m(\alpha, \varepsilon))$$

with $m < n$, and $\overline{f}_m(\alpha, \varepsilon) = f_m(\alpha, \overline{\xi}_m, \varepsilon)$ is the analytic solution for the level $m$. It is assumed in (2.20) that lower levels ($m < n$) have been already reduced by (2.19), therefore the solution can be worked out algorithmically. The constant of integration for each equation labelled with $n$ is denoted by $\xi_n$.

Using the method of the Appendix A, we study now the analyticity properties of the solutions of (2.20). Writing

$$\check{\delta}_{n\ell}(\alpha) = \check{\delta}_{n\ell}^{(0)} + \mathcal{O}(\alpha), \qquad \tilde{\gamma}_{n\ell}(\alpha) = \tilde{\gamma}_{n\ell}^{(1)}\alpha + \mathcal{O}(\alpha^2),$$

and using the change of variables $u = \alpha - \alpha_*$ as in (2.12), equation (2.20) becomes

$$\check{\delta}_{n\ell}^{(0)} + \tilde{\gamma}_{n\ell}^{(1)} \; f + u \; P(f, u) = u\frac{\mathbf{d}f}{\mathbf{d}u}\left(\beta_\alpha^{(1)} + u \; Q(f, u)\right). \tag{2.21}$$

which is like (A.1) with the replacements

$$A = \check{\delta}^{(0)}_{n\ell}\,, \qquad B = \tilde{\gamma}^{(1)}_{n\ell}\,, \qquad C = 0, \qquad D = \beta^{(1)}_{\alpha}.$$

Therefore the invertibility condition for the coupling of the level $n\ell$ is given by

$$r_n \equiv \frac{\tilde{\gamma}^{(1)}_{n\ell}}{\beta^{(1)}_{\alpha}} \notin \mathbb{N}, \qquad n > 1. \tag{2.22}$$

### 2.4.2 Integrated Bare Equations

Dimensionality matching in extended spacetime is doubly useful since it fixes two quantities: the physical and the evanescent part of the dimensionality. In cases where the bare marginal coupling has non-vanishing dimensionality-defect, this fact determines the form of the bare reduction:

$$\lambda_{n\ell\mathrm{B}}\left(\lambda_{\ell\mathrm{B}}, \alpha_{\mathrm{B}}\right) = \zeta_n\left(\varepsilon\right)\lambda^n_{\ell\mathrm{B}}, \tag{2.23}$$

where $\zeta_n$ is a dimensionless constant. Here, as in (2.11) or (2.15) the form of the bare reduction is constrained to be $\alpha_{\mathrm{B}}$-independent.

Replacing (2.17) in (2.23) we obtain an algebraic equation for $f_n(\alpha,\varepsilon)$

$$f_n(\alpha,\varepsilon) = z^{-1}_{n\ell}(\alpha,\varepsilon)\left[-\alpha\check{\Delta}_{n\ell}(\alpha,\varepsilon) + \zeta_n z^n_\ell(\alpha,\varepsilon)\right], \tag{2.24}$$

with $\check{\Delta}_{n\ell}(\alpha,\varepsilon) = \Delta_{n\ell}(\alpha, \overline{f}_m\left(\alpha,\varepsilon\right), \varepsilon)$.

**Equivalence of Criteria** The bare reduction, i.e. the determination of $\zeta_n(\varepsilon)$ can be achieved just as we did in section 2, from the renormalized reduction simply taking the $\alpha \to 0$ limit of equation (2.24) considering that $z_{n\ell}(\alpha,\varepsilon)$ and $z_\ell(\alpha,\varepsilon)$ are $1 + \mathcal{O}(\alpha)$, and that $\lim_{\alpha\to 0}\check{\Delta}_{n\ell}(\alpha,\varepsilon) < \infty$ at $\varepsilon \neq 0$ ,

$$\zeta_n\left(\varepsilon\right) = \lim_{\alpha\to 0} f_n(\alpha,\xi_n,\varepsilon). \tag{2.25}$$

The rest of the discussion about the equivalence of criteria in Zimmermann's model applies here unaltered.

**Pole Cancellation** In infinite reduction there is a new way to derive the invertibility conditions (2.22) and the analytic reduction. This method is not totally independent of studying the solutions of (2.20). Basically, equation (2.24) is the solution of (2.20), which can be verified deriving (2.24) with respect to $\ln\mu$, using (2.18) and the relation between poles as in (1.13). To make $f_n(\alpha,\varepsilon)$ analytic in $\varepsilon$, the poles present in $z_{n\ell}(\alpha,\varepsilon)$, $\check{\Delta}_{n\ell}(\alpha,\varepsilon)$ and $z_\ell(\alpha,\varepsilon)$ in the right side of (2.24) should cancel out. This requirement determines $\overline{\zeta}_n(\varepsilon)$ if the invertibility conditions are fulfilled.

As in the series treatment of Appendix A, we start writing

$$\overline{\zeta}_n(\varepsilon)=\sum_{k=0}^{\infty}\overline{\zeta}_{n,k}\varepsilon^k,$$

and look for conditions to determinate univocally the coefficients $\zeta_{n,k}$ to cancel the poles. If it is possible, criterion *ii)* is automatically satisfied: both renormalized and bare reduction are analytic in $\varepsilon$.

In cancelling the poles, it is not necessary to care about *all* the poles of (2.24) because they are related by RG to the simple pole as in (1.14). Cancelling the simple pole ensures that all the poles vanish.

In (2.24), $\zeta_{n,k}$ is multiplied by a sum of objects as

$$\varepsilon^k\left(\frac{\alpha}{\varepsilon}\right)^m\alpha^r,$$

so the simple pole has the form

$$\frac{1}{\varepsilon}\alpha^{1+k+r}. \tag{2.26}$$

The simple pole of $\check{\Delta}$ is an analytic function of $\alpha$. In total, the simple poles of (2.24) have the form

$$\frac{\alpha}{\varepsilon}\left(\sum_{s\geq 0}a_s\alpha^s+\sum_{k,r\geq 0}\overline{\zeta}_{n,k}c_{k,r}\alpha^{k+r}\right), \tag{2.27}$$

where $a_s$ and $c_{k,r}$ are known numerical factors. Thus, if the coefficients of $\overline{\zeta}_{n,j}\alpha^j$ are nonzero it is possible to determine $\overline{\zeta}_{n,j}$ iteratively in $j$ from the cancellation of the pole. The coefficient of $\overline{\zeta}_{n,j}\alpha^j$ depends only on the leading-log contributions to the wave-function renormalization constants, given by the standard formulas

$$Z_\alpha = \left(1 - \frac{\beta_\alpha^{(1)}\alpha}{\varepsilon}\right)^{-1}, \qquad z_\ell = Z_\alpha^{\gamma_\ell^{(1)}/\beta_\alpha^{(1)}}, \qquad z_{n\ell} = Z_\alpha^{\gamma_{n\ell}^{(1)}/\beta_\alpha^{(1)}}. \tag{2.28}$$

Finally, inside the parenthesis of (2.27) $\overline{\zeta}_{n,j}\alpha^j$ is multiplied by the coefficient

$$\frac{\left(-\beta_\alpha^{(1)}\right)^{j+1}}{(j+1)!} \prod_{i=0}^{j} \left(\frac{\gamma_{n\ell}^{(1)} - n\gamma_\ell^{(1)}}{\beta_\alpha^{(1)}} - i\right). \tag{2.29}$$

Assuming $\beta_\alpha^{(1)} \neq 0$, all $\overline{\zeta}_{n,k}$ can be univocally determined by recursion if and only if the factor (2.29) is not zero, or equivalently, the condition (2.22) is satisfied.

### 2.4.3 Operator Mixing

Normally each level contains more than a single operator. The reduction in this case can be worked out with the following modifications. When there is only one operator in the head of the deformation, the $n$-th level is a sum of operators of same dimensionality, labelled with $I$, $\sum_I \alpha^{p(n,I)} \lambda_{n\ell}^I \mathcal{O}_{n\ell}^I(\varphi)$ and the reduction is specified by the functions $f_n^I(\alpha,\varepsilon)$ such that

$$\lambda_{n\ell}^I = f_n^I(\alpha,\varepsilon)\,\lambda_\ell^n$$

The beta-functions are expressed as

$$\beta_{n\ell}^I = \gamma_{n\ell}^{IJ}(\alpha)\,\lambda_{n\ell}^J + \alpha\delta_{n\ell}^I(\alpha,\lambda_{m\ell}).$$

The reduction equation is then replaced by a system of coupled differential equation as (A.6) which admits a series solution only if the matrix

$$r_n^{IJ} = \frac{\left(\gamma_{n\ell}^{IJ}\right)^{(1)} - n\delta^{IJ}\gamma_\ell^{(1)}}{\beta_\alpha^{(1)}}, \tag{2.30}$$

has no non-negative integer eigenvalue, where $\delta^{IJ}$ is the identity matrix.

A variation of the above case is presented when the head itself is made of several operators $\sum_I \alpha^{p(I)} \lambda_\ell^{(I)} \mathcal{O}_\ell^I(\varphi)$. Then the first step is to perform a reduction between these couplings, which is very similar to the Zimmermann's example. Just to illustrate, consider two operators in the head with couplings $\lambda_\ell^{(1)}$ and $\lambda_\ell^{(2)}$. Their beta-functions are

$$\beta_\ell^{(I)} = \gamma_\ell^{IJ}\ \lambda_\ell^{(J)}. \tag{2.31}$$

Taking $\lambda_\ell^{(2)}$ as independent coupling and deriving

$$\lambda_\ell^{(2)} = f\left(\alpha,\varepsilon\right)\lambda_\ell^{(1)}$$

with respect to $\ln\left(\mu\right)$, the reduction equation is obtained

$$\widehat{\beta}_\alpha f' = \gamma_\ell^{21} + (\gamma_\ell^{22} - \gamma_\ell^{11})f + \gamma_\ell^{12} f^2.$$

This equation has the same form of (A.1) with

$$A = \left(\gamma_\ell^{21}\right)^{(1)}, \qquad B = \left(\gamma_\ell^{22}\right)^{(1)} - \left(\gamma_\ell^{11}\right)^{(1)}, \qquad C = \left(\gamma_\ell^{12}\right)^{(1)}, \qquad D = \beta_\alpha^{(1)},$$

so the analytic solution is given by the coefficients (A.3) if the condition $r = \frac{\sqrt{\Delta}}{D} \notin \mathbb{N}_+$ holds. Nevertheless, It is possible to avoid the condition that involves the square of the discriminant if we perform a previous linear redefinition of couplings $\lambda_\ell^{(I)} \to M^{IJ}\lambda_\ell^{(J)}$ with $M^{IJ}$ a constant matrix to put $\left(\gamma_\ell^{IJ}\right)^{(1)}$ into its Jordan canonical form. Let us study the general case where there are $N$ operators in the head. Calling $\gamma_\ell^{(1)}$ a real eigenvalue of $\left(\gamma_\ell^{IJ}\right)^{(1)}$ with multiplicity one, we have after the redefinition is $\left(\gamma^{NN}\right)^{(1)} = \gamma_\ell^{(1)}$, $(\gamma^{N\bar{J}})^{(1)} = (\gamma^{\bar{I}N})^{(1)} = 0$. The overlined indices range from 1 to $N-1$. Take $\lambda_\ell = \lambda_\ell^{(N)}$ as the independent coupling and reduce the other couplings of the head as

$$\lambda_\ell^{(\bar{I})} = \lambda_\ell f^{\bar{I}}\left(\alpha,\varepsilon\right).$$

The level-$\ell$ beta-functions (2.31) give

$$\beta_\ell = \beta_\ell^{(N)} = \left(\gamma_\ell^{NN} + \gamma_\ell^{N\bar{I}} f^{\bar{I}}\right)\lambda_\ell, \qquad \widehat{\beta}_\alpha \frac{\mathrm{d}f^{\bar{I}}}{\mathrm{d}\alpha} = \left(\gamma_\ell^{\bar{I}\bar{J}} - \delta^{\bar{I}\bar{J}}\gamma_\ell^{NN}\right) f^{\bar{J}} + \gamma_\ell^{\bar{I}N} - f^{\bar{I}}\gamma_\ell^{N\bar{J}} f^{\bar{J}}. \tag{2.32}$$

The unique doubly analytic solution for head operators is found iteratively in $\alpha$ and $\varepsilon$ if the matrix

$$\frac{\left(\gamma_\ell^{\bar{I}\bar{J}}\right)^{(1)} - \delta^{\bar{I}\bar{J}}\gamma_\ell^{(1)}}{\beta_\alpha^{(1)}}$$

has non-negative integer eigenvalue.

This new reduction equation (2.32) has $C = 0$ since $\gamma_\ell^{N\overline{J}}$ is $\mathcal{O}(\alpha^2)$ by construction, so there is no square root in the invertibility condition.

At higher levels the reduction proceeds as usual and the invertibility conditions are still that the matrices (2.30) have no non-negative integer eigenvalue for $n > 1$. If the eigenvalue $\gamma_\ell^{(1)}$ is complex it is necessary to consider a two-head deformation involving also its complex conjugate [9].

In Zimmermann's model it is not possible to avoid the square root in the invertibility condition since there is not linear transformation of couplings able to make disappear the term $\alpha\eta^2$ from $\beta_\eta$.

The reduction with operator mixing can be worked out without new complications through bare reduction and pole cancellation as well.

### 2.4.4 Perturbative Meromorphy

The reduced theory reads

$$\mathcal{L}[\varphi] = \mathcal{L}_{\mathcal{R}}[\varphi, \alpha] + \sum_{n=1}^{\infty} \alpha^{p^{(n\ell)}} \lambda_\ell^n \overline{f}_n(\alpha, \varepsilon)\ \mathcal{O}_{n\ell}(\varphi). \tag{2.33}$$

Since $p^{(n\ell)} > 0$ every term of the irrelevant deformation is parameterized in a non-minimal way and in the $\alpha \to 0$ limit at fixed $\lambda_\ell$ the theory becomes free. In this parametrization, $\lambda_\ell = 1/M_{P\mathrm{eff}}^\ell$ defines the *effective Planck mass* $M_{P\mathrm{eff}}$ such that the perturbative expansion in powers of the energy $E$ is meaningful for $E \ll M_{P\mathrm{eff}}$. On the other hand, defining the *Planck mass* $M_P = \alpha^{-\overline{p}/\ell}\lambda_\ell^{-1/\ell}$, in such a way that the irrelevant terms with dimensionality-defect $\overline{p}$ are coupled in a minimal way, we get

$$\mathcal{L}[\varphi] = \mathcal{L}_{\mathcal{R}}[\varphi, \alpha] + \sum_{n=1}^{\infty} \alpha^{\widetilde{p}^{(n\ell)}} \overline{f}_n(\alpha, \varepsilon)\ M_P^{-n\ell}\ \mathcal{O}_{n\ell}(\varphi),$$

where $\widetilde{p}^{(n\ell)} = p^{(n\ell)} - n\overline{p}$. Most of the numbers $\widetilde{p}^{(n\ell)}$ are negative, so the $\alpha \to 0$ limit at fixed $M_P$ is singular. Nevertheless, the singularity is bounded by the order of the perturbative expansion and indeed can be reabsorbed into the effective Planck mass. For this reason, the reduction is said to be *perturbatively meromorphic* [9]. Since the $\alpha$-singularities can be reabsorbed only in a non-minimal parametrization, there is no way to turn the marginal interaction off, keeping the irrelevant interaction on. That is the reason why the renormalizable sector $\mathcal{R}$ needs to be fully interacting.

### 2.4.5 Violation Of Invertibility Conditions

When the condition (2.22) is violated for some $\overline{n}$, say $r_{\overline{n}} = \overline{r} \in \mathbb{N}$, the constant $\lambda_{\overline{n}\ell}$ can not be reduced and it must be regarded as an independent coupling. Another possibility is to write

$$\overline{f}_n(\alpha,\varepsilon) = \sum_{i=0}^{\infty} \varepsilon^i \overline{f}_{i,n}(\alpha), \qquad \check{\delta}_{i,n\ell}(\alpha,\varepsilon) = \sum_{i=0}^{\infty} \varepsilon^i \check{\delta}_{i,n\ell}(\alpha). \tag{2.34}$$

and solve the reduction equation perturbatively in $\alpha$ and $\varepsilon$. At order $\varepsilon^i$ we have

$$\beta_\alpha \frac{\mathrm{d}\overline{f}_{i,n}(\alpha)}{\mathrm{d}\alpha} = \overline{f}_{i,n}(\alpha)\tilde{\gamma}_{n\ell}(\alpha) + \alpha \frac{\mathrm{d}\overline{f}_{i-1,n}(\alpha)}{\mathrm{d}\alpha} + \alpha\check{\delta}_{i,n}(\alpha). \tag{2.35}$$

The solution $\overline{f}_{0,\overline{n}}(\alpha)$ can be worked out in power series of $\alpha$ until the order $\alpha^{\overline{r}-1}$, while the coefficient of $\alpha^{\overline{r}}$ is ill-defined. Similarly, the solutions $\overline{f}_{i,\overline{n}}(\alpha)$, $0 < i < \overline{r}$, can be worked out up to orders $\alpha^{\overline{r}-i-1}$. The coupling $\lambda_{\overline{n}\ell}$ can be written as

$$\lambda_{\overline{n}\ell} = f_{\overline{n}}(\alpha,\varepsilon)\,\lambda_\ell^{\overline{n}} + \sum_{i=0}^{\overline{r}} \alpha^{\overline{r}-i}\varepsilon^i \lambda_{\overline{n}\ell}^{(i)}, \qquad \beta_{\overline{n}\ell}^{(i)} = \gamma_{\overline{n}\ell}^{(i)}\lambda_{\overline{n}\ell}^{(i)} + \varepsilon\,(\overline{r}-i)\,\lambda_{\overline{n}\ell}^{(i)} + \alpha\ \lambda_\ell^{\overline{n}}\check{\delta}_{\overline{n}\ell}^{(i)}(\alpha,\varepsilon), \tag{2.36}$$

where $f_{\overline{n}}(\alpha,\varepsilon)$ is determined up to orders $\alpha^{\overline{r}-i-1}\varepsilon^i$, $i = 0,1,...,\overline{r}-1$ and $\lambda_{\overline{n}\ell}^{(i)}$, $i = 0,1,...,\overline{r}$ are new independent parameters introduced to have a consistent solution. However, only $\lambda_{\overline{n}\ell}^{(0)}$ is a physical coupling; all others belong to the evanescent sector of the theory, so they do not affect the physical quantities. Moreover, $\gamma_{\overline{n}\ell}^{(i)} = \gamma_{\overline{n}\ell} - (\overline{r}-i)\,\beta_a/\alpha = \mathcal{O}(\alpha)$ and $\check{\delta}_{\overline{n}\ell}^{(i)}$ are analytic in $\alpha$.

In terms of these couplings, couplings of higher levels $n > \overline{n}$ can be expressed as

$$\lambda_{n\ell} = \sum_{\{m\}} f_{n,\{m\}}(\alpha,\varepsilon)\,\lambda_\ell^{\widehat{m}} \prod_{i=i}^{\overline{r}} \left(\alpha^{\overline{r}-i}\varepsilon^i\lambda_{\overline{n}\ell}^{(i)}\right)^{m_i}$$

where $\widehat{m},\ m_i$ are integers such that $\widehat{m} + \overline{n}\sum_{i=0}^{\overline{r}} m_i = n$, and consequently

$$\delta_{n\ell}(\alpha,\lambda) = \sum_{\{m\}} \delta_{n,\{m\}}(\alpha,\varepsilon)\,\lambda_\ell^{\widehat{m}} \prod_{i=i}^{\overline{r}} \left(\alpha^{\overline{r}-i}\varepsilon^i\lambda_{\overline{n}\ell}^{(i)}\right)^{m_i}.$$

The reduction equation for $f_{n,\{m\}}$ reads

$$\widehat{\beta}_\alpha \frac{\mathrm{d}\overline{f}_{n,\{m\}}}{\mathrm{d}\alpha} = \left(\gamma_{n\ell} - \widehat{m}\gamma_\ell - \sum_{j=0}^{\overline{r}} m_j \gamma_{\overline{n}\ell}\right) f_{n,\{m\}} + \alpha \check{\delta}_{n,\{m\}}(\alpha, f, \varepsilon), \tag{2.37}$$

where $\check{\delta}_{n,\{m\}}(\alpha, f, \varepsilon)$ depends on the functions $f_{k,\{m'\}}$ with $k < n$ and $f_{n,\{m\}}$ with $\widehat{m}' < \widehat{m}$. The invertibility conditions for $n > \overline{n}$ are still (2.22) because the one-loop coefficient of the combination of anomalous dimensions inside the parenthesis in (2.37) is

$$\tilde{\gamma}_{n\ell}^{(1)} - \beta_\alpha^{(1)} \overline{r} \sum_{j=0}^{\overline{r}} m_j.$$

When the invertibility condition (2.22) is fulfilled, (2.37) can be solved recursively in $\widehat{m}$ for given $n$ and there exist unique solutions $\overline{f}_{n,\{m\}}(\alpha, \varepsilon)$ that are analytic in $\alpha$ and $\varepsilon$.

The advantage of carrying on the reduction as in (2.36) even when (2.22) is violated is a practical one. It allows low-order predictions with a relatively small number of independent couplings [9].

### 2.4.6 Absence Of Three-leg Marginal Vertices: Physical Invertibility Conditions

In some circumstances the expansion of $\overline{f}_{i,0}(\alpha)$ in powers of $\alpha$ does not start from order zero. This fact produces a modification in the invertibility conditions rendering them less restrictive. It is precisely what happens when the theory has no three-leg marginal coupling, as in the case where $\mathcal{R}$ is the theory $\varphi^4$ in four dimensions (but similar arguments apply if $\mathcal{R}$ is the theory $\varphi^6$ in three dimensions).

Define the integers

$$\widetilde{p}_n \equiv p^{(n\ell)} - np^{(\ell)}, \qquad q_n \equiv \max(-k + 1 - \widetilde{p}_n, 0). \tag{2.38}$$

According to the definition of the dimensionality-defect, the quantity $\widetilde{p}_n$ relates the number of the fields present in the vertex of level $n$ with the number of fields of the operator on the head,

$$\widetilde{p}_n = \frac{N^{(n\ell)} - nN^{(\ell)}}{2} + n - 1.$$

Here is clear that $\widetilde{p}_n$ is an integer: $N^{(n\ell)} - nN^{(\ell)}$ is an even number since $N^{(n\ell)}$ and $nN^{(\ell)}$ are both even or both odd. Recalling that operators on the queue are those "generated" by renormalization due to the presence of the head operator, we prove that

$$N^{(n\ell)} \text{ odd} \quad \Longleftrightarrow \quad n \text{ odd and } N^{(\ell)} \text{ odd.}$$

Consider first diagrams that renormalize $\lambda_{n\ell}$ made of marginal and head vertices only. Since there is no three-leg marginal vertex, a diagram can have an odd number of external legs if and only if the head has an odd number of legs and there is an odd number of such vertices in the diagram. Diagrams containing vertices $\lambda_{m\ell}$ con $m < n$ do not change the argument, since for each such diagram there exist also a diagram where the each vertices $\lambda_{m\ell}$ is replaced by a subdiagram containing only marginal and head vertices.

Using an inductive method, we will show that in the situation described,

$$\overline{f}_n(\alpha, \varepsilon) = \alpha^{q_n} \sum_{k=0}^{q_n} \overline{f}_{k,n}(\alpha, \varepsilon) \left(\frac{\varepsilon}{\alpha}\right)^k, \tag{2.39}$$

where $\overline{f}_{k,n}(\alpha, \varepsilon)$ is a power series in $\alpha$ and $\varepsilon$ that involves only non-negative powers. Moreover (as can be proved using (2.25)),

$$\overline{\zeta}_n(\varepsilon) = \varepsilon^{q_n} \sum_{k=0}^{\infty} \overline{\zeta}_{n,k} \varepsilon^k. \tag{2.40}$$

The demonstration is based in the pole cancellation of (2.24). The contributions to $\check{\Delta}(\alpha, \varepsilon)$ from the diagrams $G$ with $v_{k\ell}$ irrelevant vertices of level $k$, $L$ loops, $v_4$ marginal four-leg vertices and $V = v_4 + \sum_{k<n} v_{k\ell}$ the total number of vertices, have the form

$$\begin{aligned} \check{\Delta}_{n\ell}(\alpha, \varepsilon) &= \sum_G \frac{\alpha^{L-1}}{\varepsilon^{\min(L,V-1)-s}} \prod_{k<n} \overline{f}_k^{v_{k\ell}}(\alpha, \varepsilon) \\ &= \sum_G \frac{\alpha^{L-1+t+\sum_{k<n} v_{k\ell} q_k}}{\varepsilon^{\min(L,V-1)-s'}} \prod_{k<n} \left(\frac{\varepsilon}{\alpha}\right)^{j_k}, \end{aligned} \tag{2.41}$$

where $s, s', t, j_k$ are non-negative integers, $j_k \leq v_{k\ell} q_k$ and $\sum_{k<n} k v_{k\ell} = n$.

The factor $\alpha^{L-1}$ comes from the $\alpha$-powers attached to the vertices (considering that a $\alpha$-factor is left outside in (2.17)). According to a theorem proved in [9], the maximal order of a pole of a diagram is $\min(L, V-1)$, so $\varepsilon^{\min(L,V-1)-s}$ represents this fact and $s'$ takes care of powers of $\varepsilon$ coming from $\overline{f}_k(\alpha, \varepsilon)$. Similarly, $\alpha^t$ take into account the $\alpha$-powers coming from $\overline{f}_k(\alpha, \varepsilon)$.

Using the same argument that in paragraph below (2.4), the number of powers of $\alpha$ of a diagram that renormalizes $\alpha^{p^{(n\ell)}}\lambda_{n\ell}$ is

$$\sum_{k<n} v_{k\ell} p^{(k\ell)} + v_4 = p^{(n\ell)} + L.$$

With the definitions (2.38) this relation reads

$$\sum_{k<n} v_{k\ell} q_k = q_n - L + (V-1)\,. \tag{2.42}$$

The simple pole of (2.41) has the form

$$\frac{1}{\varepsilon}\alpha^{L-\min(V-1,L)+t+s'+\sum_{k<n} v_{k\ell} q_k}.$$

Therefore, the powers of $\alpha$ in the simple pole, using (2.42) are

$$(V-1) - \min(V-1, L) + t + s' + q_n \geq q_n,$$

so the $\alpha$-exponent of the simple pole is always $\geq q_n$. Thus the simple poles of $\check{\Delta}_{n\ell}(\alpha, \varepsilon)$ are multiplied by powers $\alpha^{q_n+s}$, $s \geq 0$.

With the ansatz (2.40) in (2.24), the coefficient $\overline{\zeta}_{n,k}$ is multiplied by a sum of objects of the form

$$\varepsilon^{q_n+k}\left(\frac{\alpha}{\varepsilon}\right)^m \alpha^r, \tag{2.43}$$

with $m, r \geq 0$. The simple pole is

$$\frac{1}{\varepsilon}\alpha^{q_n+1+k+r}.$$

In total, the simple poles of (2.24) have the form

$$\frac{\alpha^{q_n+1}}{\varepsilon}\left(\sum_{s\geq 0} a_s\alpha^s + \sum_{k,r\geq 0}\zeta_{n,k}c_{k,r}\alpha^{k+r}\right), \tag{2.44}$$

where $a_s$ and $c_{k,r}$ are known numerical factors. Thus, if the coefficients of $\overline{\zeta}_{n,j}\alpha^j$ are nonzero it is possible to determine $\overline{\zeta}_{n,j}$ iteratively in $j$ from the cancellation of the pole. Finally, using (2.28) the term $\overline{\zeta}_{n,j}\alpha^j$ inside the parenthesis of (2.44) is multiplied by the coefficient

$$\frac{\left(-\beta_\alpha^{(1)}\right)^{q_n+j+1}}{(q_n+j+1)!}\prod_{i=0}^{q_n+j}\left(\frac{\gamma_{n\ell}^{(1)}-n\gamma_\ell^{(1)}}{\beta_\alpha^{(1)}}-i\right),$$

thus the invertibility conditions are again (2.22). Nevertheless, these conditions are more restrictive than they should be.

Writing (2.39) as

$$\overline{f}_n(\alpha,\varepsilon)=\alpha^{q_n}\sum_{i=0}^{q_n}\overline{f}_{i,n}(\alpha)\left(\frac{\varepsilon}{\alpha}\right)^i+\sum_{i=q_n+1}^{\infty}\overline{f}_{i,n}(\alpha)\left(\frac{\varepsilon}{\alpha}\right)^i$$

it is clear that the first $q_n$ terms of the expansion of $\overline{f}_n(\alpha,\varepsilon)$ in $\varepsilon$ have the form (A.4), thus for the determination of the coefficients in $\overline{f}_{i,n}(\alpha), i<q_n$ it is enough to satisfy

$$\frac{\tilde{\gamma}_{n\ell}^{(1)}}{\beta_\alpha^{(1)}}-(q_n+i)\notin\mathbb{N}.$$

Since all $\overline{f}_{i,n}(\alpha)$ but $\overline{f}_{0,n}(\alpha)$ belong to the evanescent sector, the *physical invertibility* condition reads

$$\frac{\tilde{\gamma}_{n\ell}^{(1)}}{\beta_\alpha^{(1)}}-q_n\notin\mathbb{N}. \tag{2.45}$$

Therefore, the violation of (2.22) when (2.45) holds, implies the inclusion of a new coupling that affects only the evanescent sector of the theory, as explained in the previous section.

Once the poles have been cancelled out and the constants $\overline{\zeta}_{n,k}$ have been determined, collecting (2.41) and (2.43) we obtain

$$\overline{f}_n\sim\alpha^{q_n}\left[\sum_{\substack{L\geq 1,\ u,s,i_n\geq 0\\ L+i_n\leq q_n+s'+u}}\left(\frac{\varepsilon}{\alpha}\right)^{q_n-L-i_n}\alpha^t\varepsilon^{s'+u}+\sum_{\substack{m,r,j\geq 0\\ m\leq q_n+j}}\overline{\zeta}_{n,j}\varepsilon^j\alpha^r\left(\frac{\varepsilon}{\alpha}\right)^{q_n-m}\right], \tag{2.46}$$

having written $\sum_{k<n}j_k v_{k\ell}=\sum_{k<n}q_k v_{k\ell}-i_n$, $i_n\geq 0$, and $\sum_{k<n}q_k v_{k\ell}-\min(V-1,L)=q_n-L+u$, $u\geq 0$ (see (2.42)). We see that $\overline{f}_n(\alpha,\varepsilon)$ has the form (2.39), which reproduces the inductive hypothesis.

### 2.4.7 Leading-log Solution

In this section the infinite reduction is solved in the leading-log approximation. We use the minimal subtraction scheme in the unreduced theory, while the subtraction scheme of the reduced theory is the one induced by the reduction itself. We recall that the leading-log approximation is sufficient to derive the invertibility conditions for the existence of the infinite reduction to all orders.

In the leading-log approximation $\check{\delta}_{n\ell}(\alpha, \varepsilon)$ has the form

$$\check{\delta}_{n\ell}(\alpha, \varepsilon) = \alpha^{q_n} \sum_{k=0}^{q_n} d_{k,n} \left(\frac{\varepsilon}{\alpha}\right)^k$$

where $q_n$ is an integer and $d_{k,n}$ are constants. In this approximation the solution can be easily worked out and reads

$$\begin{aligned} f_n(\alpha, \xi_n, \varepsilon) &= -\frac{\alpha^{q_n}}{\beta_\alpha^{(1)}} \sum_{k=0}^{q_n} \frac{d_{k,n}\varepsilon^k}{\alpha^k(r_n - q_n + k)} \, {}_2F_1\left[1, k - q_n, r_n - q_n + k + 1, \frac{\varepsilon}{\alpha\beta_\alpha^{(1)}}\right] \\ &\quad + \xi_n \left(\alpha\beta_\alpha^{(1)} - \varepsilon\right)^{r_n}. \end{aligned} \tag{2.47}$$

with $r_n = \tilde{\gamma}_{n\ell}^{(1)}/\beta_\alpha^{(1)}$. Observe that the hypergeometric functions appearing in the sum are polynomial, since $q_n - k$ is a non-negative integer.

At the level of bare couplings, the reduction has the form (2.23). Manipulating the formulas given above and using (2.25), the formula for $\zeta_n(\xi_n, \varepsilon)$ can be derived. The result is

$$\zeta_n(\xi_n, \varepsilon) = \varepsilon^{q_n} \sum_{k=0}^{q_n} d_{k,n} \left(-\beta_\alpha^{(1)}\right)^{k-q_n-1} \frac{\Gamma\left(r_n - q_n + k\right)\Gamma\left(q_n - k + 1\right)}{\Gamma\left(r_n + 1\right)} + \xi_n(-\varepsilon)^{r_n}. \tag{2.48}$$

Both $f_n(\alpha, \varepsilon)$ and $\zeta_n(\varepsilon)$ are analytic in $\varepsilon$ (for all $\alpha$) only for $\overline{\xi}_n(\varepsilon) = 0$. This formula uniquely determines the reduction. Moreover, the invertibility conditions for the existence of the reduction to all orders can be read from (2.48) and coincide with (2.22).

**Violations of the invertibility conditions.** It is interesting to describe the appearance of new parameters, when the invertibility conditions are violated, using the pole-cancellation mechanism in the leading-log approximation. Assume that some regularized invertibility conditions (2.22) are violated, i.e. $r_n = \overline{r} \in \mathbb{N}$. To study this situation it is convenient to approach it continuously from $r_n = \overline{r} + \delta$ and then take the limit $\delta \to 0$. If $\overline{r} > q_n$ this limit is trivial in the leading-log approximation, so we just need to discuss the case $\overline{r} \leq q_n$.

Collecting the singular terms of (2.47) we get an expression of the form

$$\begin{aligned} f_n(\alpha, \xi_n, \varepsilon) &= \left(\alpha\beta^{(1)}_\alpha - \varepsilon\right)^{\overline{r}} \left\{ \frac{a}{\delta}\varepsilon^{q_n - \overline{r}} + \xi_n \left[1 + \delta \ln\left(\alpha\beta^{(1)}_\alpha - \varepsilon\right)\right] \right\} \\ &+ \alpha^{q_n} P_n(\varepsilon/\alpha) + \mathcal{O}(\delta, \xi_n\delta^2), \end{aligned}$$

where $a$ is a known numerical factor and $P_n(\alpha, \varepsilon)$ is a certain $\xi$- and $\delta$-independent polynomial of degree $q_n$. The $\delta$-singularity can be removed redefining $\xi_n$ as

$$\xi_n = -\frac{a}{\delta}\varepsilon^{q_n - \overline{r}} + \xi_n^{'},$$

thus obtaining a non-singular expression

$$f_n(\alpha, \xi_n, \varepsilon) = \left(\alpha\beta^{(1)}_\alpha - \varepsilon\right)^{\overline{r}} \left\{ \xi_n' - a\varepsilon^{q_n - \overline{r}} \ln\left(\alpha\beta^{(1)}_\alpha - \varepsilon\right) \right\} + \alpha^{q_n} P_n(\varepsilon/\alpha).$$

Finally, the relations between the bare and renormalized constants $\zeta$ and $\xi$

$$\zeta_n(\xi, \varepsilon) = \lim_{\alpha \to 0} f_n(\alpha, \xi_n, \varepsilon) = (-\varepsilon)^{\overline{r}} \left[\xi_n^{'} - a\varepsilon^{q_n - \overline{r}} \ln(-\varepsilon) + b\varepsilon^{q_n - \overline{r}}\right], \tag{2.49}$$

where $b$ is another known numerical factor, originated by $\alpha^{q_n} P_n(\varepsilon/\alpha)$.

We see that no choice of the constant $\xi_n^{'}$ is able to remove the analyticity violation in both the bare and renormalized reduction relations. The violation can be hidden in a new independent coupling, but since the $\ln(-\varepsilon)$ is multiplied by $\varepsilon^{q_n - \overline{r}}$ it is sufficient to write $\xi_n^{'} = \varepsilon^{\overline{r}}\xi_n^{''}$ and associate the new coupling with $\xi_n^{''}$.

### 2.4.8 Several Marginal Couplings

For illustrative purposes we use the Zimmermann's model in the leading-log approximation for the renormalizable sector. For $\alpha$ and $\eta$ small, the lowest-order beta-functions of $\lambda_\ell$ and $\lambda_{2\ell}$ have generically the forms

$$\beta_\ell = \lambda_\ell \alpha(d + e\eta), \qquad \beta_{2\ell} = \lambda_{2\ell}\alpha(f + g\eta) + h\lambda_\ell^2, \tag{2.50}$$

where $d, e, f, g, h$ are unspecified numerical factors. We use a trick to transform the reduction equation which is a partial differential equation into an ordinary differential one. For doing so, use the transformation (2.6) to reparametrize the couplings $(\alpha, \eta) \to (\alpha, \xi)$. Thus the reduction of the queue operator is expresses in terms of $\widetilde{f}\,(\alpha, \xi, \varepsilon)$, with

$$f(\alpha, \check{\eta}\,(\alpha, \xi, \varepsilon)\,, \varepsilon) = \widetilde{f}\,(\alpha, \xi, \varepsilon)\,,$$

where $\xi$ is the constant of integration. For instance, for $f_2$ we obtain the ordinary differential equation

$$\widehat{\beta}_1 \frac{\mathrm{d}\widetilde{f}_2(\alpha, \xi, \varepsilon)}{\mathrm{d}\alpha} + 2\alpha \widetilde{f}_2(\alpha, \xi, \varepsilon)\left(\widetilde{d} + \widetilde{e}\check{\eta}(\alpha, \xi, \varepsilon)\right) - \varepsilon \widetilde{f}_2(\alpha, \xi, \varepsilon) = h, \tag{2.51}$$

where $\widetilde{d} = d - f/2$ and $\widetilde{e} = e - g/2$. Solving for $\xi$ in (2.6)

$$\xi = (\alpha\beta_1 - \varepsilon)^{-s/\beta_1}\, z, \qquad z = \frac{b - s + 2c\eta}{b + s + 2c\eta}, \tag{2.52}$$

and replacing it in $\widetilde{f}_2\,(\alpha, \xi, \varepsilon)$, we recover $f_2(\alpha, \eta, \varepsilon)$. For definiteness we choose the positive sign in front of $s$.

The solution of (2.51) is

$$f_2(\alpha, \eta, \varepsilon) = \overline{f}_2(\alpha, \eta) + k_2\,(\xi, \varepsilon)\;\; s_2(\alpha, \eta, \varepsilon), \tag{2.53}$$

where

$$\overline{f}_2(\alpha, \eta) = \frac{h(1 - z)}{\alpha s(\gamma - 1)}\; {}_2F_1[1, \gamma - 2\widetilde{e}/c, \gamma, z], \qquad s_2(\alpha, \eta, \varepsilon) = \frac{1}{\alpha} z^{1-\gamma} (1 - z)^{2\widetilde{e}/c}, \tag{2.54}$$

with

$$\gamma = 1 + \frac{\widetilde{e}}{c} + \frac{1}{s}\left(2\widetilde{d} - \beta_1 - b\frac{\widetilde{e}}{c}\right),$$

and $k_2$ is the constant of integration, which is a function of $\varepsilon$ and $\xi$.The solution is meromorphic instead of analytic in $\alpha$ because we have used the minimal coupling in this example (note that $\beta_{2\ell}$ in (2.50) is not proportional to $\alpha$ as (2.17)).

The extension of criteria *i)* is simple: we require meromorphy in $\alpha$ and analyticity in $\eta$ and $\varepsilon$. Noting that $z$ is analytic in $\eta$ (see (2.52)), we realize that the particular solution $\overline{f}_2(\alpha, \eta)$ satisfies these requirement. Let us examine in which cases also the general solution does. Writing $z = \xi\,(\alpha\beta_1 - \varepsilon)^{s/\beta_1}$,

$$k_2\,(\xi, \varepsilon)\;\; s_2(\alpha, \eta, \varepsilon) = \frac{1}{\alpha} k_2\,(\xi, \varepsilon)\, \xi^{1-\gamma}\,(\alpha\beta_1 - \varepsilon)^{s(1-\gamma)/\beta_1}\,(1 - z)^{2\widetilde{e}/c}.$$

The only possibility to have a general solution satisfying the criterion is given by $k_2(\xi,\varepsilon) = k_2' \xi^{\gamma-1+n}\varepsilon^q$ when

$$\frac{s(1-\gamma)}{\beta_1} = n\frac{s}{\beta_1} + m \tag{2.55}$$

where $k_2'$ is a constant and $m, n, q$ are non-negative integers. In that case,

$$k_2(\xi,\varepsilon) \;\; s_2(\alpha,\eta,\varepsilon) = \frac{k_2'}{\alpha}\varepsilon^q (\alpha\beta_1-\varepsilon)^m z^n (1-z)^{2\widetilde{e}/c}$$

which is meromorphic in $\alpha$ and analytic in $\eta$ and $\varepsilon$ for every $k_2'$. Therefore, here the invertibility condition is that there should be no pair of non-negative integers $m$ and $n$ such that (2.55) holds. Since $s$ is in most cases irrational or complex and the ratios of the one-loop coefficients are rational numbers, the invertibility conditions can be rephrased as

$$-\frac{\widetilde{e}}{c} \notin \mathbb{N} \qquad \text{or} \qquad 1 + \frac{b\widetilde{e}}{c\beta_1} - \frac{2\widetilde{d}}{\beta_1} \notin \mathbb{N}. \tag{2.56}$$

It is enough to fulfill one of these conditions to fix $k_2(\xi,\varepsilon) = 0$ and uniquely determine the reduction.

We can obtain the same conclusion from the bare reduction. Matching dimensionalities,

$$\lambda_{2\ell\mathrm{B}} = \zeta_2(\eta_\mathrm{B},\varepsilon)\frac{\lambda_{\ell\mathrm{B}}^2}{\alpha_\mathrm{B}},$$

and writing the bare couplings in terms of the renormalization constants, we find that

$$\begin{aligned}\zeta_2(\eta_\mathrm{B},\varepsilon) &= \lim_{\alpha\to 0}\alpha\widetilde{f}_2(\alpha,\xi,\varepsilon)\\ &= \frac{h(1-z_\mathrm{B})}{\alpha s(\gamma-1)}\,{}_2F_1[1,\gamma-2\widetilde{e}/c,\gamma,z_\mathrm{B}] + k_2(\xi,\varepsilon)\frac{1}{\alpha}z_\mathrm{B}^{1-\gamma}(1-z_\mathrm{B})^{2\widetilde{e}/c},\end{aligned}$$

with

$$z_\mathrm{B} = \xi(-\varepsilon)^{s/\beta_1} = \frac{b-s+2c\eta_\mathrm{B}}{b+s+2c\eta_\mathrm{B}}.$$

It is natural to ask to the bare reduction to be analytical in $\eta_\mathrm{B}$ and $\varepsilon$. With a reasoning analogous to the one presented above, if one of the conditions (2.56) is fulfilled the reduction is uniquely determined setting $k_2' = 0$.

The reduction in the presence of more marginal couplings follows the same line.

### 2.4.9 Including Mass

Mass and other parameters with positive dimensionality can be included in a perturbative manner.

It will be shown how to organize the series in powers of mass to have a "triangular structure" that allows the determination of all coefficients, order by order recursively. In each new equation there is only one new unknown.

The perturbative series for the mass is equivalent to treat the mass term of the lagrangian as a two-leg vertex, which is consistent for renormalization purposes, since divergent part of diagrams is polynomial in the masses. In general, the momentum integrals should be regularized with an infrared regulator, e.g. a fictitious mass $\delta$ sent to zero right after the computation of the divergent part.

In the generic perturbed lagrangian (2.16) the unique dimensionless combination is $\lambda_\ell m^\ell$, thus the reduction relation can contain arbitrary functions of this combination in the reduction. Criterion *i)* tell us that the reduction should be analytic in couplings, so for $n > 1$ $\lambda_{n\ell}$ will be expressed as a function of $m, \alpha$ and $\lambda_\ell$:

$$\lambda_{n\ell} = \lambda_\ell^n \sum_{p=0}^{\infty} (\lambda_\ell m^\ell)^p f_n^{(p)}(\alpha, \varepsilon) \tag{2.57}$$

Replacing (2.57) in the expressions for the beta-functions $\beta_\alpha,\ \beta_{n\ell},\ \beta_\ell$ (2.17) and $\beta_m$ which are suppose to be all known, they can be written as

$$\begin{aligned} \beta_{n\ell} &= \lambda_\ell^n \sum_{q=0}^{\infty} (\lambda_\ell m^\ell)^q B_n^{(q)}, \qquad \beta_\ell = \lambda_\ell \sum_{q=0}^{\infty} (\lambda_\ell m^\ell)^q B_1^{(q)}, \\ \beta_\alpha &= \sum_{q=0}^{\infty} (\lambda_\ell m^\ell)^q B_0^{(q)}, \qquad \beta_m = m \sum_{q=0}^{\infty} (\lambda_\ell m^\ell)^q M^{(q)}, \end{aligned} \tag{2.58}$$

where $\beta_m$ is the beta-function of the "vertex" $m$. The explicit form of $B_p^{(q)}(\alpha, f_i^{(j)}(\alpha, \varepsilon))$ is not calculated, but what is important here is to know on what functions $f_i^{(j)}(\alpha, \varepsilon)$ they depend, namely those with

$$\begin{aligned} i + j &\leq p + q \\ j &\leq q. \end{aligned}$$

The function $M^{(q)}$ instead depends on $\alpha$ and $f_i^{(j)}(\alpha,\varepsilon)$ such as

$$i+j\leq q.$$

Deriving (2.57) respect $\ln(\mu)$ and replacing the beta-functions with (2.58) we get

$$\sum_{r=0}^{\infty}(\lambda_\ell m^\ell)^r B_n^{(r)}=\sum_{p,q=0}^{\infty}(\lambda_\ell m^\ell)^{p+q}f_n^{(p)}\left((n+p)\,B_1^{(q)}+(p\ell)\,M^{(q)}+\frac{\mathbf{d}\ln\left(f_n^{(p)}\right)}{\mathbf{d}\alpha}B_0^{(q)}\right).$$

Matching coefficients of the same order in $\lambda_\ell m^\ell$ starting from below, and then ascending also in $n$, all $f_n^{(p)}(\alpha,\varepsilon)$ can be computed iteratively. For example, for $r=0$ we have the massless equations (2.20), determining in this way $f_n^{(0)}$, for all $n$. Then, for $r=1$ and $n=2$ we obtain an equation that involves only $f_2^{(0)}, f_3^{(0)}$ and $f_2^{(1)}$. For $r=1$ and $n=3$, it depends on $f_2^{(0)}, f_3^{(0)}, f_4^{(0)}$, $f_2^{(1)}$ and $f_3^{(1)}$ and so on. Proceeding in this way every new equation has only one unknown.

### 2.4.10 Scheme Reduction

In the renormalization process, infinities are subtracted from the bare quantities to obtain finite renormalized quantities. There is, of course, a freedom of adding an arbitrary finite part to this subtraction. RG equation guarantees that physical quantities do not depend on the choice of the substraction point. Moreover, the behavior at any other energy is fixed by the RG to keep the physical quantities independent of the choice. In our approach, this freedom is explicitly manifested on the integration constant $\xi$, so once it is fixed by the analyticity criteria, we have lost the freedom of choosing the point of renormalization for that coupling. After the reduction the only freedom remained are those corresponding to the independent couplings.

Equivalently, a change of scheme can be regarded as a reparametrization of couplings, because a finite part is subtracted to them, so after the reduction we can reparameterize arbitrarily only the independent ones and the others will follow them coherently.

# Chapter 3

# Causality Violations of Quantum Matter Interacting With Classical Gravity

The quantization of gravity is still one of the greatest challenges of the modern theoretical physics. Although there is no definitive evidence, most of the scientific community believes that all interactions should present a quantum behavior at high energies. Precisely because there is not experimental or theoretical evidence [25, 26, 27]that gravity should be of quantum nature, it still has sense to study a coherent framework where classical gravity is coupled to quantum fields. In particular, the standard model is ready to the coupling with gravity, in the sense that anomalies still cancel when it is embedded in a curved background. Quantization in a curved background has been widely studied in the last 30 years, giving a mathematically rigorous formulation of quantum field theory in curved spacetime [28] in particular for interacting fields (for a review, see [29]). Eppley and Hannah [25] showed that the interaction of quantum matter with classical gravity, assuming the "Copenhagen" interpretation of quantum mechanics, leads to one of the following scenarios: if the gravitational interaction does not collapse the wave-function, gravity can be used to propagate information at superluminal velocity. On the other hand, if gravity collapses the wave-function, either the uncertainty principle or energy-momentum conservation can be violated.

The most natural choice to represent the gravitational interaction, the Hilbert-Einstein action using the fluctuations of the metric around flat space $g_{\mu\nu} = \eta_{\mu\nu} +$

$2\kappa\phi_{\mu\nu}$ as quantum field, happens to be renormalizable only at one loop when no matter is present. At two loops, or even at one loop when gravity is coupled with matter fields, it becomes non-renormalizable [30].

We start with the action

$$S[\phi; g] = \frac{1}{2\kappa^2} \int \mathrm{d}^4x \sqrt{-g} R + S_m[\phi; g], \tag{3.1}$$

where $S_m[\phi; g]$ is a renormalizable four-dimensional matter action embedded in a gravitational background, and $\phi$ represents generically the matter fields.

When quantized, the purely gravitational divergences of (3.1) are proportional to diffeomorphism-invariant terms of dimensionality four made of the metric, namely $R^2, R_{\mu\nu}^2$ and $(R^{\alpha}{}_{\beta\mu\nu})^2$. In four dimensions the integral of the last term can be transformed in a combination of the other two thanks to the Gauss-Bonnet identity.

$$\int \mathrm{d}^4x \sqrt{-g} G_B = \text{boundary term}, \qquad G_B = R_{\alpha\beta\mu\nu} R^{\alpha\beta\mu\nu} - 4R_{\mu\nu}R^{\mu\nu} + R^2.$$

If such terms are included in the action to provide counterterms,

$$S^{\mathrm{HD}}[\phi; g] = \frac{1}{2\kappa^2} \int \mathrm{d}^4x \sqrt{-g} R + \int \mathrm{d}^4x \sqrt{-g} \left( a R_{\mu\nu}^2 + b R^2 \right) + S_m[\phi; g], \tag{3.2}$$

where $a,\ b$ are new coupling constants, the behavior of the graviton propagator falls off at high energies rapidly enough to improve the power counting and make the theory renormalizable [31]. Nevertheless, its inclusion also allows the propagation of ghosts with the consequent loss of unitarity. Higher time-derivatives produce instabilities in classical models, so even if the gravitational field were left classical, (3.2) is not a good action. The central idea of this chapter is to absorb these purely gravitational divergences not in a redefinition of constants $(a, b)$, but in a redefinition of the classical metric tensor $g_{\mu\nu}$. This kind of redefinition defines a map that relates the renormalization of two different theories, a higher-derivative one (HD) and another which do not present higher derivative terms, but causality violations, detectable in principle at high energies. For some class of HD theories that include (3.2), there exist a theorem [32] that allows to compute iteratively the perturbative version of the map. The causality violations produced by the map differ in origin and nature from those presented by Eppley and Hannah: while the latter is intrinsic to the interaction of quantum and classical fields, the former is the effect of a particular non-local coupling between a quantum operator and a classical source.

Although the map is defined for a classical metric $g_{\mu\nu}$ and it does not extend to quantum gravity, this work could serve to motivate some lines of research in quantum gravity, if the ultimate theory of gravitation needs to be of quantum nature. One of the main features of the map is that new vertices that couple matter and gravity fields are created. In particular, the stress tensor is coupled to the Ricci tensor. Another consequence of the map is that the metric $g_{\mu\nu}$ is renormalized (so it becomes *running)* even when it is a classical field.

Since the gravitational field and the map are classic while other fields are of quantum nature, in section 3.1 it is explained how the semiclassical models are formulated and in particular, how to obtain quantum-corrected equations of motion.

As should be clear from the example of the Lorentz-Abrahams force of section 3.2, the origin of the causality violation is the non-local redefinition of functions representing the map required to lower the degree of the differential equation. This motivates the treatment to lower the order of the equation of motion of the gravitational field.

In section 3.3 we study the map and its causality implications. Section 3.5 explains how to use the map to renormalize theories without generating higher-derivative kinetic terms, while in section 3.4 the perturbative map is applied explicitly to renormalize the theory (3.1). The map is explicitly applied to the acausal Einstein-Yang-Mills model in section 3.6. In sections 3.7 we prove the renormalization of more general acausal theories, using the Einstein-Yang-Mills model as prototype. In the first class, we consider models that admit some vertices that can not be generated by a map, while in the second class the matter sector contains all composite operators that have dimensionalities smaller than or equal to four. We prove the existence of consistent reductions of couplings and the renormalizability of the models obtained giving a $R$-dependence to the couplings of PC renormalizable models. The Batalin-Vilkovisky formalism is adapted to treat a curvature-dependent coupling for the Yang-Mills theory.

In the Appendix B it is shown how to compute functional derivatives of the action without working with bitensors, and the explicit perturbative map is worked out up to third order.

## 3.1 Semiclassical Models

Let us consider a generic theory characterized by a classical action $S[\phi;\varphi]$, where $\phi$ represents generically the quantum fields and $\varphi$ is a classical field.

The generating functional $Z[J;\varphi]$ depends on the sources $J_I$ and on the classical field $\varphi$ since it is not integrated

$$\begin{aligned} Z[J;\varphi] &= \int \mathcal{D}\phi \; e^{i\left(S[\phi,\varphi]-\phi^I J_I\right)} \\ &= \exp\left(iW[J;\varphi]\right) \end{aligned} \tag{3.3}$$

where $W[J;\varphi]$ is the sum of all connected vacuum-vacuum amplitudes. The 1PI-generating functional $\Gamma[\Phi;\varphi]$ is obtained through the Legendre transform of $W[J;\varphi]$ that involve only the quantum fields. Deriving $\Gamma[\Phi;\varphi]$ with respect to the fields $\Phi = \langle\phi\rangle$, correlation functions in the presence of the external field $\varphi$ are obtained.

The 1PI-generating functional $\Gamma[\Phi;\varphi]$, also called *quantum effective action* provides the quantum-corrected equations of motion for the external fields. The problem is that due to the $i\varepsilon$-prescription, the functionals $Z$, $W$, and $\Gamma$ are in general complex quantities, thus the corrections to field equations are complex. For instance, the equation of motion of $g_{\mu\nu}$ obtained from $\Gamma$ is

$$\frac{\delta\Gamma[\Phi;\varphi]}{\delta g^{\mu\nu}} = 0 \qquad \Longrightarrow \qquad G_{\mu\nu} = \langle T_{\mu\nu}\rangle \tag{3.4}$$

where $G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R$ is the Einstein tensor and $\langle T_{\mu\nu}\rangle = \frac{2}{\sqrt{-g}}\frac{\delta\Gamma_m}{\delta g^{\mu\nu}}$ is the expectation value of the canonical stress-tensor of $S_m$, which in general is complex.

This problem can be avoided at least in two ways. The expectation value in (3.4) can be replaced by the "in-in" expectation value of the stress tensor, which is another approach to semi-classical models due to Schwinger and Keldish [33, 34]. It is argued that it is a better generalization of the quantum mechanics expectation value since it is defined using the same initial and final state, while the usual QFT expectation value uses different asymptotic "in" and "out" states. Moreover, it is real and causal, and functional methods have been developed to compute it [35, 36], but it has the inconvenient that it does not provide an action.

Another possibility is to define a quantum action as

$$S_q[\phi_q;\varphi] = \mathrm{Re}(\Gamma[\Phi;\varphi]), \tag{3.5}$$

where $\phi_q = \Phi$ is real if the fields $\phi$ are real bosonic, while $\Phi$ is the conjugate of $\overline{\Phi} = \langle\overline{\phi}\rangle$ if the fields $\phi$ are fermionic or complex. The field equations are obtained functionally variating $S_q$ with respect to $\phi_q$ and $\varphi$.

Using (3.5), the equation of motion for $g_{\mu\nu}$ $\frac{\delta S_q}{\delta g^{\mu\nu}} = 0$ reads

$$G_{\mu\nu} = \mathrm{Re}\,\langle T_{\mu\nu}\rangle\,.$$

It is important to note that for the study of causality violations produced by the map, it is not relevant what real prescription is used to substitute the expectation value of the stress tensor in (3.4).

## 3.2 Motivation: The Abraham-Lorentz Force

Some of the features of the map and its consequences can be easily understood through an example extracted from classical electrodynamics: the Abraham-Lorentz force. There, the order of a differential equation is reduced at the price of introducing a violation of causality. As in Chapter 2, the suppression of a parameter requires to give to some constant a physically meaningful value. Also it is illustrated how the Green function method is idoneous to invert the differential operator and then determining the map, fixing in this way the constant of integration.

The physical situation is a particle of mass $m$ and charge $e$ driven by an external force $F_{\mathrm{ext}}$. The force produces changes in the velocity of the particle making it irradiate, thus losing kinetic energy with the consequent deceleration. The velocity of the particle is determined by a second order differential equation, or equivalently, first order for the acceleration $a(t)$. We could reduce the order of this equation pretending that the deceleration by radiation is caused by a fictitious force $F_{\mathrm{rad}}$ :

$$ma(t) = F_{\mathrm{ext}} + F_{\mathrm{rad}}.$$

The value of the force $F_{\mathrm{rad}}$ is calculated equating the work it makes and the energy lost by radiation (see for example [37]),

$$\int_{t_0}^{t_1} \mathrm{d}t \; F_{\mathrm{rad}}(t) \; v(t) = -m\tau \int_{t_0}^{t_1} \mathrm{d}t \; a(t)^2 , \tag{3.6}$$

where $v(t)$ is the velocity of the particle and $\tau = \frac{2}{3}\frac{e^2}{mc^3}$. For some velocity and acceleration conditions at the initial $t_0$ and final instant $t_1$, or for a periodic motion, the equation (3.6) is verified for $F_{\mathrm{rad}} = m\tau\dot{a}$. Note this is an effective, time-averaged representation. The equation of motion then reads

$$\left(1 - \tau\frac{\mathbf{d}}{\mathbf{d}t}\right) ma(t) = F_{\mathrm{ext}}. \tag{3.7}$$

We can symbolically invert the differential operator and define a new external force $F'_{\mathrm{ext}}$

$$\begin{aligned} ma(t) &= \left(1 - \tau\frac{\mathbf{d}}{\mathbf{d}t}\right)^{-1} F_{\mathrm{ext}} \\ &= F'_{\mathrm{ext}}. \end{aligned} \tag{3.8}$$

This operation can be made explicit using the Green function $G(t,t')$ of the operator, that satisfies

$$\left(1 - \tau\frac{\mathbf{d}}{\mathbf{d}t}\right) G(t,t') = \delta(t - t'). \tag{3.9}$$

This equation admits infinitely many solutions. Indeed, for each particular solution $G_P(t,t')$, we have also the solution $G_{P'}(t,t') = G_P(t,t') + G_0(t)$, with $G_0$ in the kernel of the operator

$$\left(1 - \tau\frac{\mathbf{d}}{\mathbf{d}t}\right) G_0(t) = 0.$$

The equation of motion is then reduced to

$$ma(t) = \int_{-\infty}^{\infty} \mathrm{d}t' \; G(t,t') \; F_{\mathrm{ext}}(t'). \tag{3.10}$$

Solving (3.9) in Fourier space we have $\tilde{G}(\omega) = 1/(1 - i\omega\tau)$, so

$$G(t,t') = \frac{1}{2\tau} e^{-(t'-t)/\tau} \; 2\theta(t' - t), \tag{3.11}$$

which gives the correct limit for $\tau \to 0^+$, a physical requirement for the inversion. That is, when $e \to 0$, ($\tau \to 0^+$) the particle does not irradiate, so we should recover the Newton equation $F_{\text{ext}}(t) = ma(t)$ from (3.7). This can verified in (3.10) realizing that the limit $\tau \to 0^{+}$of (3.11) is a representation of the Dirac delta distribution $\delta(t - t')$.

In this example the Green function found through its Fourier transform is unique. Other solutions can be written as

$$G_{P'}(t, t') = G(t, t') + \xi(t', \tau) G_0(t)$$

with $G_0(t) = \frac{1}{\tau} e^{t/\tau}$.

If we use $G_{P'}(t, t')$ in (3.10),

$$ma(t) = \int_{-\infty}^{\infty} \mathrm{d}t' \; G(t, t') \;\; F_{\text{ext}}(t') + \zeta(\tau) G_0(t),$$

with $\zeta(\tau) = \int dt' \xi(t', \tau)$, we see that *runaway solutions* are present, namely the acceleration increases monotonically with time even when there is no external force. The "$\tau \to 0^{+}$" criteria is enough to determinate the map since the $\tau \to 0^{+}$ limit of $G_{P'}(t, t')$ is infinite for $t > 0$ unless $\xi(t', \tau) = 0$. For this choice, the runaway solutions are automatically eliminated. In this case, the physical requirements that the equation should tend to Newton equation at $e \to 0$ and that there should be no runaway solutions, univocally determine the map $F_{\text{ext}}(t) \to F'_{\text{ext}}(t)$.

Writing explicitly (3.11) in (3.10),

$$ma(t) = \frac{1}{\tau} \int_{t}^{\infty} \mathrm{d}t' \; e^{(t-t')/\tau} F_{\text{ext}}(t'),$$

we realize that our solution is acausal. To know the acceleration in the instant $t_0$ we need to integrate the external force $F_{\text{ext}}$ over all future times $t > t_{0.}$ Nevertheless, this violation of causality should be observed only at scales of order of $\tau$ $(= 6.24 \times 10^{-24} s)$, where quantum effects should be taken into account. In electrodynamics this phenomenon is called *preacceleration*, and has not been experimentally observed. The fact that there is no experimental evidence of preacceleration is however not meaningful because it is a classical prediction for a situation where quantum effects predominate.

Sometimes the inverse Fourier transform requires a prescription due to the presence of poles or branch cuts over the contour of integration in the complex plane.

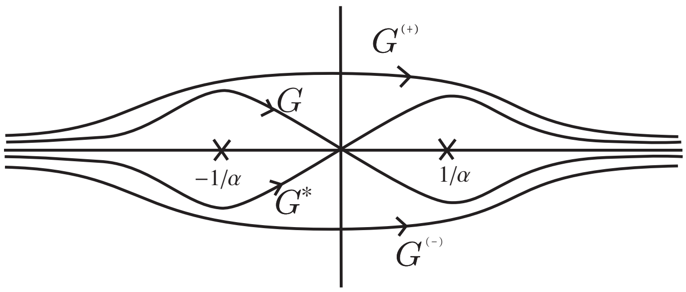


Figure 3.1: Contours for the advanced, retarded, Feynman and conjugated prescriptions for the Green function.

This gives several Green functions $G^{(i)}(t,t')$, all of which have the right limit behavior.

**Example:**

$$\tilde{G}(\omega) = \frac{1}{1-\alpha^2\omega^2},$$

with $\alpha^2 > 0$ has poles in the real axis at $\pm 1/\alpha$. The Green functions

$$\begin{aligned} G^{(+)}(t,t') &= -\frac{1}{\alpha}\sin\left(\frac{t-t'}{\alpha}\right)\theta(t'-t), \qquad G^{(-)}(t,t') = \frac{1}{\alpha}\sin\left(\frac{t-t'}{\alpha}\right)\theta(t-t') \\ G(t,t') &= -\frac{i}{2\alpha}e^{i|t-t'|/\alpha}, \qquad G^*(t,t') = \frac{i}{2\alpha}e^{-i|t-t'|/\alpha}. \end{aligned} \tag{3.12}$$

defined over the contours of the figure 3.1 satisfy

$$\left(1+\alpha^2\frac{\mathrm{d}^2}{\mathrm{d}t^2}\right)G(t,t') = \delta(t-t').$$

and they are all equally valid, since they have the correct $\alpha \to 0$ limit and do not present runaway solutions.

Summarizing, we have reduced the order of a differential equation (3.7) by means of a non-local (acausal) redefinition $F_{\text{ext}} \to F'_{\text{ext}}$ to obtain

$$ma(t) = F'_{\text{ext}}(t).$$

## 3.3 The Map

We will use a non-local redefinition similar to the one used in the Abrahams-Lorentz example to eliminate higher-derivatives in kinetic terms in lagrangian formalism.

Consider first the simpler higher-derivative lagrangian in one dimension, plus a term proportional to the square of the equation of motion of the undeformed lagrangian,

$$\mathcal{L}(q) = \frac{m}{2}\dot{q}^2 + \alpha^2 \frac{m}{2}\ddot{q}^2. \tag{3.13}$$

The equation of motion and its general solution are

$$\ddot{q} - \alpha^2 q^{(4)} = 0, \qquad q\,(t) = a + bt + ce^{t/\alpha} + de^{-t/\alpha}.$$

As we can see, it presents runaway solutions, which are absent in the case $\alpha^2 < 0$, since the above exponential functions become complex. It is our intention to reduce the order of the equation performing a redefinition of the dynamical variable $q\,(Q)$ such as

$$\int \mathrm{d}t \left(\dot{q}^2 + \alpha^2 \ddot{q}^2\right) = \int \mathrm{d}t\ \dot{Q}^2.$$

Using the Green function method, we have

$$q\,(t) = \int \mathrm{d}t' G\,(t,t')\, Q(t')$$

with

$$G\,(t,t') = \int \frac{\mathrm{d}\omega}{2\pi} \frac{e^{i\omega(t-t')}}{\sqrt{1+\alpha^2\omega^2}} = \frac{1}{\pi|\alpha|} K_0\left(\frac{|t-t'|}{|\alpha|}\right), \qquad \alpha^2 > 0. \tag{3.14}$$

This function is real and, as seen in Figure 3.2, acausal because has non zero values for $t - t' < 0$.

If $\alpha^2$ is negative, $\alpha^2 = -\overline{\alpha}^2 < 0$, a prescription is needed to avoid the branch cut in the complex plane. In that case there exist a real and causal prescription, namely the retarded Green function

$$G_{\mathrm{ret}}\,(t,t') = \int \frac{\mathrm{d}\omega}{2\pi} \frac{e^{-i\omega(t-t')}}{\sqrt{1-\overline{\alpha}^2(\omega+i\varepsilon)^2}} = \frac{\theta\,(t-t')}{\pi|\overline{\alpha}|} J_0\left(\frac{|t-t'|}{|\overline{\alpha}|}\right), \qquad \overline{\alpha}^2 > 0, \tag{3.15}$$

This kind of Green functions is the central ingredient of the map. Its form (and the prescription chosen) determines whether the model generated by the map is acausal or not

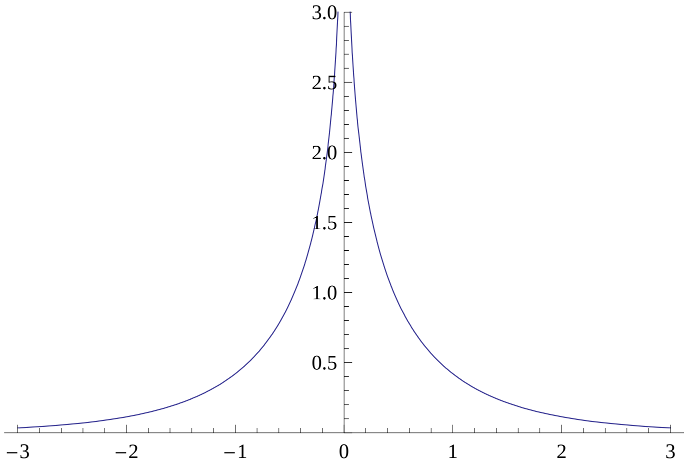


Figure 3.2: Function $K_0(|t|)$, which is non-zero at negative times, so it is acausal.

For certain class of higher-derivative theories the method of Appendix B gives directly the perturbative series corresponding to the expansion of (3.15) in $\alpha$. Clearly, all prescriptions have the same perturbative expansion, which is real.

Although the lagrangian (3.13) has higher derivatives, is still compatible with a variational treatment. In a usual lagrangian, the kinetic term is proportional to $\dot{q}^2$, producing a term $\ddot{q}^2$ in the equation of motion, which is a second order differential equation. In the variational method we need two border conditions $\delta q|_{t_1}$and $\delta q|_{t_0}$, whose number is consistent with the order of the equation of motion.

In the hypothetical case of a term $q\ddot{q}$ in the lagrangian, four conditions are needed: $\delta q$ and $(\delta\dot{q})$ should vanish in the extremes. This is not compatible with the equation of motion which is of order two, thus the variational method is not applicable.

In (3.13) this problem is absent because four conditions are required for the variational method, namely that $\delta q$ and $(\delta\dot{q})$ vanish at the ends, and the equation of motion is of order four.

### 3.3.1 Fields

The map is also applied to field without major modifications. Consider for example the following lagrangian in a four-dimensional Minkowski space

$$\mathcal{L}[\varphi] = \frac{1}{2}\partial_\mu\varphi\partial^\mu\varphi + \alpha^2\frac{1}{2}\left(\Box\varphi\right)^2 - \varphi J$$

where $J$ is a classical source. We search for a function $\varphi\left(\varphi'\right)$ such that

$$\mathcal{L}[\varphi] = \mathcal{L}'[\varphi'] = \frac{1}{2}\partial_\mu\varphi'\partial^\mu\varphi' - \varphi' J'.$$

As before, the map is expressed as

$$\begin{aligned} \varphi\left(x\right) &= \int \mathrm{d}^4x' G\left(x,x'\right)\varphi'\left(x'\right), \qquad J'\left(x\right) = \int \mathrm{d}^4x' G\left(x,x'\right) J\left(x'\right), \quad (3.16) \\ G\left(x,x'\right) &= \int \frac{\mathrm{d}^4k}{\left(2\pi\right)^4}\frac{e^{-ik\cdot\left(x-x'\right)}}{\sqrt{1+\alpha^2k^2}}. \end{aligned}$$

Therefore, all the analysis relies on the prescription chosen to evaluate $G\left(x,x'\right)$. One of them,

$$G_{\mathrm{F}}\left(x,x'\right) = \int \frac{\mathrm{d}^4k}{\left(2\pi\right)^4}\frac{e^{-ik\cdot\left(x-x'\right)}}{\sqrt{1+\alpha^2k^2+i\varepsilon}} = i\frac{\mathrm{sign}\left(\alpha^2\right)}{4\pi^2|\alpha|^4}\frac{\exp\left(-\sqrt{\frac{x^2}{\alpha^2}-i\varepsilon}\right)}{\left(\frac{x^2}{\alpha^2}-i\varepsilon\right)^{3/2}}\left(1+\sqrt{\frac{x^2}{\alpha^2}-i\varepsilon}\right)$$

tends to zero or oscillates rapidly for $|x^2| \gg |\alpha^2|$. So causality violations could be experimentally tested only at distances of order

$$\Delta x \sim 2\pi|\alpha|.$$

When $\alpha^2 = -\overline{\alpha}^2 < 0$, a real causal prescription exists

$$G_{\mathrm{ret}}\left(x,x'\right) = \int \frac{\mathrm{d}^4k}{\left(2\pi\right)^4}\frac{e^{-ik\cdot\left(x-x'\right)}}{\sqrt{1-\overline{\alpha}^2\left(k_0+i\varepsilon\right)^2+\overline{\alpha}^2\mathbf{k}^2}}.$$

The branch cuts are located in the lower half of the $k_0$-plane so $G_{\mathrm{ret}}\left(x,x'\right)$ vanish for $t<0$, as can be seen closing the contour of integration in the upper half. By Lorentz invariance, every point outside the lightcone admits a reference frame where $t<0$, thus causality is preserved. For $\alpha^2>0$ instead, no causal prescription exists.

The equation of motion for the new field $\varphi'$ is

$$\Box\varphi' = J' = \int \mathrm{d}^4x' G\left(x,x'\right) J\left(x'\right). \qquad (3.17)$$

This equation of motion violates causality if the physical source corresponds to $J$. This example is specially interesting because (3.16) is the same Green function found in the map of higher-derivative gravity.

## 3.4 Map For Gravity

The intrinsic difficulties of gravity prevent us from finding the explicit map as in (3.16) to transform away the higher-derivative terms in (3.2). However, due to the particular form of these terms, which are squarely proportional to the equation of motion of the Hilbert-Einstein action, it is possible to find the perturbative version of the map using the theorem demonstrated in Appendix B. Calling

$$\begin{aligned} S_{\mathrm{HD}}[g] &= \frac{1}{2\kappa^2}\int \sqrt{-g}\left[R(g)+aR_{\mu\nu}R^{\mu\nu}+bR^2\right], \\ S_{\mathrm{E}}[g] &= \frac{1}{2\kappa^2}\int \sqrt{-g}R(g), \end{aligned}$$

the objective is to find $g'(g)$ such as

$$S_{\mathrm{HD}}[g] = S_{\mathrm{E}}[g'].$$

To quadratic order $(a^2, b^2, ab)$ the method gives

$$\begin{aligned} g'_{\alpha\beta}(g,a,b) &= g_{\alpha\beta} - aR_{\alpha\beta} + \frac{1}{2}(a+2b)\,g_{\alpha\beta}R + \\ &\quad -\frac{ab}{2}\nabla_\alpha\nabla_\beta R - \frac{a^2}{2}\left(\nabla_\mu\nabla_\alpha + 2R_{\alpha\mu}\right)R_{\beta\mu} + \frac{a^2}{4}\left(\Box - 2R\right)R_{\alpha\beta} \\ &\quad +\frac{g_{\alpha\beta}}{8}\left[a^2\left(R^2 - 2R^2_{\mu\nu}\right) + \left(a^2+8ab+12b^2\right)\Box R\right]. \end{aligned} \tag{3.18}$$

Third-order terms are listed at the end of Appendix B. Applying the perturbative map to a generic theory (3.2) new vertices are generated due to the metric redefinition in the renormalizable matter sector $S_m(\varphi, g, \lambda)$ :

$$\begin{aligned} S_m(\varphi,g,\lambda) &= S_m(\varphi,g',\lambda) + \Delta S_m(\varphi,g',\lambda) \\ \Delta S_m(\varphi,g',\lambda) &= \int \mathrm{d}^4x\sqrt{-g'}\left[-\frac{a}{2}T^{\mu\nu}_m R_{\mu\nu}(g') + \frac{1}{4}(a+2b)R(g')\,T_m\right] + \mathcal{O}(a^2,b^2,ab), \end{aligned} \tag{3.19}$$

where $T^{\mu\nu}_m = -(2/\sqrt{-g})(\delta S_m/\delta g_{\mu\nu})$ is the (canonical) stress-tensor and $T_m$ denotes its trace.

In the expansion of the metric around the flat space, $g_{\mu\nu} = \eta_{\mu\nu} + 2\kappa\phi_{\mu\nu}$, the map is expressed as [32]

$$\phi_{\mu\nu} = \frac{1}{\sqrt{1-a\Box}}\left(\phi'_{\mu\nu} - \frac{1}{3}\eta_{\mu\nu}\phi' + \frac{1}{3\Box}\eta_{\mu\nu}\partial^\alpha\partial^\beta\phi'_{\alpha\beta}\right) + \frac{\eta_{\mu\nu}}{3\sqrt{1-b'\Box}}\left(\phi' - \frac{1}{\Box}\partial^\alpha\partial^\beta\phi'_{\alpha\beta}\right), \tag{3.20}$$

where $\phi$ is the trace of $\phi_{\mu\nu}$ and $b' = -2\,(a+3b)$. The components of $\phi'_{\mu\nu}$ in (3.20) are multiplied by $1/\sqrt{1-a\Box}$ or $1/\sqrt{1-b'\Box}$, thus the causality violation should be detectable at distances of order of $\sqrt{|a|}$, $\sqrt{|b'|}$ or smaller.

## 3.5 Renormalization Through The Map

The map can be used to relate the renormalization of two different theories. Start with a renormalizable semiclassical model which contains higher derivatives (HD) in the kinetic term of the classic field. These higher derivatives, even when correspond to a classical field are undesirable because they lead to instabilities. Then, applying the map to the bare and renormalized HD action, we obtain the bare and renormalized actions of a theory that presents an acausal (AC) behavior.

Consider a HD renormalizable model in a classical background $\varphi$ that can be written as the sum of two terms

$$S^{\mathrm{HD}} = S^{\mathrm{HD}}_{\varphi}[\alpha,\varphi] + S^{\mathrm{HD}}_{m}[\alpha,\phi,\varphi] \tag{3.21}$$

where $S^{\mathrm{HD}}_{\varphi}$ contains the (higher) kinetic term of the classical field $\varphi$ and does not depend on the matter fields $\phi$, while $S^{\mathrm{HD}}_{m}$ is a renormalizable matter action that depends on the couplings and the matter fields generically denoted by $\alpha$ and $\phi$, and the background filed $\varphi$. The superscript HD stands only to indicate that the matter action belongs to the higher-derivative model, indeed it is assumed that kinetic terms of the matter fields are not higher-derivative.

When the map is real, for example the perturbative map of the previous section, the relation between the renormalization of two models can be demonstrated applying the map directly to the classical action $S^{\mathrm{HD}}$. More explicitly, apply a map $\varphi' = F(\varphi,\alpha)$ such that $\varphi = F(\varphi,0)$ is the identity map and

$$S^{\mathrm{HD}}_{\varphi}[\alpha;\varphi] = S^{\mathrm{AC}}_{\varphi}[\alpha;\varphi']$$

where $S^{\mathrm{AC}}_{\varphi}$ contains no higher derivatives in the kinetic term of $\varphi$. This change modifies also the matter action in (3.21), namely

$$S^{\mathrm{HD}}_{m}[\alpha,\phi,\varphi] = S^{\mathrm{AC}}_{m}[\alpha,\phi,\varphi'].$$

A similar map $\varphi'' = F(\varphi, \alpha_{\rm B})$ transforms away the higher derivatives contained in the $\varphi$-kinetic term of the bare action,

$$S_{\varphi}^{\rm HD}[\alpha_{\rm B}; \varphi] = S_{\varphi}^{\rm AC}[\alpha_{\rm B}; \varphi''],$$

producing changes in the bare matter action,

$$S_{m\rm B}^{\rm HD}[\alpha_{\rm B}, \phi_{\rm B}, \varphi] = S_{m\rm B}^{\rm AC}[\alpha_{\rm B}, \phi_{\rm B}, \varphi''].$$

As always, the subscript B stands for bare quantities.

Applying the "renormalized" and the "bare" map to each side of the relation between the renormalized and bare HD action,

$$S^{\rm HD}[\alpha, \phi, \varphi] = S_{\rm B}^{\rm HD}[\alpha_{\rm B}, \phi_{\rm B}, \varphi],$$

it is possible to write

$$S_{\varphi}^{\rm AC}[\alpha_{\rm B}; \varphi_{\rm B}] + S_{m\rm B}^{\rm AC}[\alpha_{\rm B}, \phi_{\rm B}, \varphi_{\rm B}] = S_{\varphi}^{\rm AC}[\alpha; \varphi_{\rm R}] + S_{m}^{\rm AC}[\alpha, \phi, \varphi_{\rm R}], \tag{3.22}$$

namely the relation of renormalization of the bare acausal model

$$S^{\rm AC} = S_{\varphi}^{\rm AC}[\alpha, \varphi] + S_{m}^{\rm AC}[\alpha, \phi, \varphi]. \tag{3.23}$$

In (3.22) we have renamed $\varphi_{\rm B} \equiv \varphi''$ and $\varphi_{\rm R} \equiv \varphi'$.

The renormalization of the AC theory (3.23) is then achieved by the same redefinitions of quantum fields and coupling constants as in the related HD model, plus the renormalization of the classical field

$$\varphi_{\rm B} = F\left(F^{-1}\left(\varphi_{\rm R}, \alpha\right), \alpha_{\rm B}\right). \tag{3.24}$$

This redefinition is clearly non-local and in most cases acausal, as can be seen in the following example. Taking $(1 - \alpha^2\partial^2)$ as the differential operator in one dimension, the function $F$ is given schematically by

$$F\left(\varphi, \alpha\right) = \frac{1}{1 + \alpha^2\partial^2}\varphi$$

so (3.24) can be expressed as

$$\varphi_{\mathrm{B}}=\frac{1+\alpha_{\mathrm{B}}^{2}\partial^{2}}{1+\alpha^{2}\partial^{2}}\varphi_{\mathrm{R}}=\int\mathrm{d}x'G\left(x-x'\right)\varphi_{\mathrm{R}}\left(x'\right)$$

with

$$G\left(x\right)=\left(\frac{\alpha_{\mathrm{B}}^{2}}{\alpha^{2}}\right)\delta\left(x\right)+\left(1-\frac{\alpha_{\mathrm{B}}^{2}}{\alpha^{2}}\right)G_{\alpha}\left(x\right).$$

where $G_{\alpha}\left(x\right)$ is the one of the Green functions of (3.12) depending of the contour of Figure 3.1 chosen. Observe that the classical field is not renormalized in the $\alpha\rightarrow0$ limit, namely $\varphi_{\mathrm{B}}=\varphi_{\mathrm{R}}$ as expected because $\lim_{\alpha\rightarrow0^{+}}G_{\alpha}\left(x\right)=\delta\left(x\right)$. Some of the prescriptions present an acausal behavior of order of $\alpha$. The fact that the renormalization of the classical field is acausal does not represent by itself a problem, after all bare quantities are not observable. What is physically relevant is the causality violation in the equation of motion (3.17).

The equivalence of renormalizability of both theories (3.21) and (3.23) is evident since the theories HD and AC differ only by the external field.

If the map is complex, to avoid having a complex action, the map is applied to the HD quantum effective action $\Gamma^{\mathrm{HD}}$ and then the real part is taken to define an action $S_{q}$ and obtain from it real equations of motion [32], which are the classical equations plus quantum corrections.

The steps to obtain the real acausal equations from the HD theory are shown schematically in the following table,

$$\begin{array}{ccccc}
S^{\mathrm{HD}}[\varphi] & \rightarrow & \Gamma^{\mathrm{HD}}[\varphi]\in\mathbb{C} & \rightarrow & S_{q}^{\mathrm{HD}}[\varphi]=\mathrm{Re}\left\{\Gamma^{\mathrm{HD}}[\varphi]\right\}\\
 & & \updownarrow\ \varphi\left(\varphi'\right) & & \\
 & & \Gamma^{\mathrm{AC}}[\varphi']\in\mathbb{C} & \rightarrow & S_{q}^{\mathrm{AC}}[\varphi']=\mathrm{Re}\left\{\Gamma^{\mathrm{AC}}[\varphi']\right\}\rightarrow\text{Eq.Motion(AC)}
\end{array}$$

where for clarity we have written explicitly only the dependence on the classical field. Note that in general $S_{q}^{\mathrm{HD}}[\varphi]$ and $S_{q}^{\mathrm{AC}}[\varphi']$ are not related by the map.

The action $S_{q}$ contains all the information of the quantum effective action $\Gamma$, which indeed can be reconstructed perturbatively from $S_{q}$ [32]

The renormalization through the map is possible only because the field $\varphi$ is kept classic. If it is not, we should include in the functional integral (3.3) the determinant of the Jacobian of the transformation $\varphi \to \varphi'$. This implies the appearance of ghosts that no symmetry forbid their existence on final states, so unitarity is lost.

In gravity, the renormalization of the metric $g_{\mu\nu}$ is given by (3.24) is, for the perturbative map (3.18)

$$\begin{aligned} g_{\alpha\beta\mathrm{B}}\left(g,a,b\right) &= g_{\alpha\beta}+x_1\;R_{\alpha\beta}+x_2\;Rg_{\alpha\beta}+x_3\nabla_\alpha\nabla_\beta R \\ &+\left(x_4\nabla_\mu\nabla_\alpha+x_5R_{\alpha\mu}\right)R_{\beta\mu}+\left(x_6\Box+x_7R\right)R_{\alpha\beta} \\ &+\frac{g_{\alpha\beta}}{8}\left[x_8R^2+x_9R_{\mu\nu}^2+x_{10}\Box R\right]+\mathcal{O}(a^2,b^2,ab) \end{aligned} \tag{3.25}$$

where is understood that in th right side $g_{\alpha\beta}$ is the renormalized metric tensor, and the coefficients are

$$\begin{aligned} x_1 &= a-a_\mathrm{B}, & x_6 &= -\tfrac{1}{4}\left(3a^2-2aa_\mathrm{B}-a_\mathrm{B}^2\right), \\ x_2 &= -\tfrac{1}{2}\left(\left(a-a_\mathrm{B}\right)+2\left(b-b_\mathrm{B}\right)\right), & x_7 &= -\tfrac{1}{2}\left(2ab-2ab_\mathrm{B}-aa_\mathrm{B}+a_\mathrm{B}^2\right), \\ x_3 &= -\tfrac{1}{2}\left(3a^2-2aa_\mathrm{B}-a_\mathrm{B}^2\right), & x_8 &= a_\mathrm{B}^2-a^2, \\ x_4 &= \tfrac{1}{2}\left(3ab-2a_\mathrm{B}b-a_\mathrm{B}b_\mathrm{B}\right), & x_9 &= 2\left(4ab+3a^2-4ab_\mathrm{B}-2aa_\mathrm{B}-a_\mathrm{B}^2\right), \\ x_5 &= -\tfrac{1}{2}\left(a^2-a_\mathrm{B}^2\right), \end{aligned}$$

and $x_{10}=-36b^2-24ab-3a^2+24bb_\mathrm{B}+8ab_\mathrm{B}+12b_\mathrm{B}^2+8a_\mathrm{B}b+2aa_\mathrm{B}+8a_\mathrm{B}b_\mathrm{B}+a_\mathrm{B}^2$.

## 3.6 Acausal Einsten-Yang-Mills

As an example of the usage of the perturbative map, we prove the renormalization of the acausal Einstein-Yang-Mills model through the renormalization of the following HD model

$$\mathcal{L}_\mathrm{YM}^\mathrm{HD}=\sqrt{-g}\left(\frac{R}{2\kappa^2}+\xi W^2+\zeta G_B+\frac{\eta}{\left(D-1\right)^2}R^2\right)-\frac{1}{4\alpha}\sqrt{-g}F_{\mu\nu}^aF_a^{\mu\nu}, \tag{3.26}$$

where

$$W^2=R_{\alpha\beta\mu\nu}R^{\alpha\beta\mu\nu}-\frac{4}{D-2}R_{\mu\nu}R^{\mu\nu}+\frac{2}{\left(D-1\right)\left(D-2\right)}R^2,$$

is the squared Weyl tensor and $G_B$ the Gauss-Bonnet (GB) density. The relation between the constants $\xi$, $\zeta$, $\eta$ are $a,b$ from (3.2) is

$$\frac{a}{2\kappa^2}=\frac{4\left(D-3\right)}{D-2}\xi,\qquad \frac{b}{2\kappa^2}=\frac{\eta}{\left(D-1\right)^2}-\frac{D\left(D-3\right)}{\left(D-1\right)\left(D-2\right)}\xi.$$

Since the integral of the GB density in four dimensions is a total derivative, for a metric that tends to flat space at infinity rapidly enough, the term

$$\int \mathrm{d}^D x\sqrt{-g}G_B$$

in (3.26) is evanescent at 4-$\varepsilon$ dimensions. Therefore, the constant $\zeta$ belongs to the evanescent sector and have not physical consequences.

Most of the properties showed here are valid, with minor modification, to every power-counting renormalizable theory coupled with classical gravity.

The renormalization of the Yang-Mills model in a gravitational background (3.26) gives at lowest-order [38]

$$\begin{aligned}
\beta_\alpha &= -\frac{22}{3}\frac{\alpha^2 C\left(G\right)}{\left(4\pi\right)^2}+\mathcal{O}\left(\alpha^3\right),\\
\beta_\zeta &= -\frac{\dim G}{\left(4\pi\right)^2}\left(-\frac{31}{180}+\frac{17}{12}\frac{\alpha^2 C^2\left(G\right)}{\left(4\pi\right)^4}\right)+\mathcal{O}\left(\alpha^3\right),\\
\beta_\xi &= \frac{\dim G}{\left(4\pi\right)^2}\left(-\frac{1}{10}+\frac{2}{9}\frac{\alpha C\left(G\right)}{\left(4\pi\right)^2}\right)+\mathcal{O}\left(\alpha^2\right),\\
\beta_\eta &= \frac{\dim G}{\left(4\pi\right)^2}\left(\frac{187}{54}\frac{\alpha^3 C^3\left(G\right)}{\left(4\pi\right)^6}\right)+\mathcal{O}\left(\alpha^4\right).
\end{aligned}$$

Applying the map (3.18) to (3.26), we obtain the theory

$$\mathcal{L}_{\mathrm{YM}}^{\mathrm{AC}}=\frac{1}{2\kappa^2}\sqrt{-g}R-\frac{1}{4}\sqrt{-g}\left\{F_{\mu\nu}^a F^{a\mu\nu}\ H(g)+T_{\mu\nu}K^{\mu\nu}(g)+\Upsilon_{\mu\nu\rho\sigma}L^{\mu\nu\rho\sigma}(g)\right\},\quad(3.27)$$

which is renormalized by means of the redefinition of the field $A_\mu$, the constants $\alpha$, $\xi$, $\zeta$, $\eta$ and the metric (see expression (3.25)). The Newton constant $\kappa^2$ is not renormalized (if there is no cosmological term) $T_{\mu\nu}$ is the unperturbed stress tensor and $\Upsilon_{\mu\nu\rho\sigma}$ is the traceless operator $F_{\mu\nu}F_{\rho\sigma}$,

$$\begin{aligned}
T_{\mu\nu} &= -F_{\mu\alpha}^a F_\nu^{a\,\alpha}+\frac{1}{4}g_{\mu\nu}F^2, \qquad (3.28)\\
\Upsilon_{\mu\nu\rho\sigma} &= F_{\mu\nu}^a F_{\rho\sigma}^a+\frac{1}{2}\left(g_{\mu\rho}T_{\nu\sigma}-g_{\mu\sigma}T_{\nu\rho}-g_{\nu\rho}T_{\mu\sigma}+g_{\nu\sigma}T_{\mu\rho}\right)-\frac{1}{12}\left(g_{\mu\rho}g_{\nu\sigma}-g_{\mu\sigma}g_{\nu\rho}\right)F^2.
\end{aligned}$$

where $F^2 \equiv F^a_{\alpha\beta}F^{a\alpha\beta}$.

Thus, the acausal Einstein-Yang-Mills theory is just Yang-Mills theory with two composite operators, besides $F^2$, coupled with suitable external sources. Up to the second order, the external sources are

$$H(g) = 1 + \frac{1}{6}a^2 R_{\alpha\beta}R^{\alpha\beta} - \frac{1}{24}a^2 R^2, \qquad L^{\mu\nu\rho\sigma} = a^2 R^{\mu\rho}R^{\nu\sigma}, \tag{3.29}$$
$$K^{\mu\nu}(g) = 2aR^{\mu\nu} + \frac{3}{2}a^2\Box R^{\mu\nu} + a^2 RR^{\mu\nu} - 3a^2 R^{\mu\alpha\nu\beta}R_{\alpha\beta} - \frac{3a(a+2b)}{2}\nabla^\mu\nabla^\nu R.$$

Due to the renormalization of the metric $g_{\alpha\beta}$ (3.25), it becomes running even if it a classical field,

$$\mu\frac{\mathrm{d}g_{\alpha\beta\mathrm{B}}}{\mathrm{d}\mu} = 0 \quad \Rightarrow \quad \mu\frac{\mathrm{d}g_{\alpha\beta\mathrm{R}}}{\mathrm{d}\mu} = -\beta_a R_{\alpha\beta} + \frac{1}{2}(\beta_a + 2\beta_b)g_{\alpha\beta}R + \mathcal{O}\left(a\kappa^2, b\kappa^2\right).$$

## 3.7 Other Types Of Renormalizable Acausal Models

In this section we prove the renormalizability of certain families of acausal models that are not included in the classification of the previous sections. In particular, they do not have the form (3.19) so they can not be considered as the result of the application of the map to a model as (3.2). Nevertheless, the map will be useful in demonstrating the renormalizability of these models.

The first class of models we study here are the generalization of the model (3.1) with a matter action (3.19). In these models, the deformation has a head that is still proportional to the stress tensor, but now multiplied by more general functions of the metric than in (3.19). In other words, we prove the renormalizability of acausal models having actions as

$$S^{\mathrm{AC}} = \frac{1}{2\kappa^2}\int \mathrm{d}^4x\sqrt{-g}R + S_0(\phi, g, \lambda) + S_1(\phi, g, \lambda') + S_2(\phi, g, \lambda''). \tag{3.30}$$

where $S_0(\phi, g, \lambda)$ is a four dimensional renormalizable matter action in a gravitational background that depend on fields and couplings generically denoted by $\phi$ and $\lambda$. It is perturbed with two other actions $S_1$ and $S_2$ that have following special form.

They both are diffeomorphism invariant and have matter operators of dimensionality less than or equal to 4, are gauge invariant, covariant under diffeomorphisms, invariant under the global symmetries of the theory, not necessarily scalar, and can be contracted with tensors constructed with the metric in a non-trivial way. $S_1$ represents the head of the perturbation and is proportional to the stress-tensor of $S_0$ denoted $T_0^{\mu\nu}$, and linearly proportional to Ricci tensor or its covariant derivatives[1]. Explicitly,

$$S_1 = \int \mathrm{d}^4x\sqrt{-g} f_{\mu\nu}(g_{\rho\sigma})T_0^{\mu\nu}, \tag{3.31}$$

with

$$f_{\mu\nu}(g_{\rho\sigma}) = \lambda_1' R_{\mu\nu} + \lambda_2' R g_{\mu\nu} + \mathcal{O}(\nabla^2 R_{\rho\sigma}).$$

On the other hand, $S_2$ is the queue that renormalizes coherently with $S_1$ and it is proportional to the square of the Ricci tensor or superior $S_2 = \mathcal{O}(R^2_{\mu\nu}, \phi)$.

The renormalizability of (3.30) is proved in two steps:

*I)* *Demonstrating the renormalizability of this HD theory*

$$\begin{aligned} S^{\mathrm{HD}} &= S_g[g] + S_0[\phi, g, \lambda] + S_1[\phi, g, \lambda'] + S_2[\phi, g, \lambda''], \\ S_g\,[g] &= \frac{1}{2\kappa^2}\int \mathrm{d}^4x\sqrt{-g}\left(R + R_{\mu\nu}\mathcal{T}^{\mu\nu\alpha\beta}\left(\lambda', \lambda''\right)R_{\alpha\beta}\right), \end{aligned} \tag{3.32}$$

where $S_0\,(\phi, g, \lambda)$ is the same as above, instead $S_1$ and $S_2$ have the same restrictions as in (3.30), so we use the same names. The tensor $\mathcal{T}^{\mu\nu\alpha\beta}$ is a differential operator that can depend on the metric, covariant derivatives and the couplings contained in $S_1$ and $S_2$

*II)* *Demonstrating* (3.30) *and* (3.32) *are related by a perturbative map as* (3.18) .

We basically demonstrate that the action (3.32) keeps its form under renormalization, even if it contains an infinite number of constants. Working inductively, let $\Gamma^{(n)}$ be the generating functional of 1PI diagrams which has been renormalized up to order $n$ in loop expansion[2]. Its divergent part $\Gamma^{(n)}_{\infty}$ which is local and of order

[1] To be more specific, an even number of derivatives to be contracted with the Ricci tensor, the stress-tensor or themselves.

[2] Or equivalently, of order $\hbar^n$.

$(n+1)$ has the form

$$\Gamma_\infty^{(n)} = \underbrace{(\delta S_0)^{(n)} + \int \left(\delta R^2\right)^{(n)}}_{i)} + \underbrace{\int f_{\mu\nu}(g)\, \langle T_0^{\mu\nu}\rangle_\infty^{(n)}}_{ii)} + \underbrace{\int R_{\mu\nu}\left(\mathcal{T}^{\mu\nu\alpha\beta}\right)^{(n)} R_{\alpha\beta} + (\delta S_2)^{(n)}}_{iii)} \tag{3.33}$$

plus BRST exact terms treated in the next section. Here $\int$ denotes $\int \mathrm{d}^4x\sqrt{-g}$ for shorten. $\Gamma_\infty^{(n)}$ is separated in three parts according to the type of divergent diagrams that contribute to each part. First we show explicitly the form of these parts, and then we explain how the divergences are absorbed in suitable redefinitions of couplings and fields.

The terms collected in $i)$ come from divergent diagrams that do not contain any $\lambda'$ or $\lambda''$ vertices. They are the divergences of the unperturbed theory, and can be written as

$$\begin{aligned} (\delta S_0)^{(n)} &= \frac{\partial S_0}{\partial\lambda}\Delta_\lambda^{(n)} + \phi\frac{\delta S_0}{\delta\phi}\Delta_\phi^{(n)}, \\ \left(\delta R^2\right)^{(n)} &= \Delta_a^{(n)} R_{\mu\nu}R^{\mu\nu} + \Delta_b^{(n)} R^2, \end{aligned} \tag{3.34}$$

where $\Delta_\lambda^{(n)}, \Delta_\phi^{(n)}, \Delta_a^{(n)}, \Delta_b^{(n)}$ are divergent coefficients corresponding to the order-$(n+1)$ part of the renormalization constant of $\lambda, \phi, a$ and $b$.

The part denoted *ii)* is made of contributions from diagrams with only one insertion of a $S_1$ vertex and no $S_2$-vertex insertion. They are collected in

$$\int \mathrm{d}^4x\sqrt{-g} f_{\mu\nu}(g_{\rho\sigma})\, \langle T_0^{\mu\nu}\rangle\,, \tag{3.35}$$

with

$$\langle T_0^{\mu\nu}\rangle = -\frac{2}{\sqrt{-g}}\left.\frac{\delta\Gamma[\Phi, g_{\mu\nu}]}{\delta g_{\mu\nu}(x)}\right|_{\lambda'=\lambda''=0} + \frac{2}{\sqrt{-g}}\frac{\delta S_g[g_{\mu\nu}]}{\delta g_{\mu\nu}(x)} \tag{3.36}$$

The couplings $\lambda'$ and $\lambda''$ are set to zero in (3.36) to extract the unperturbed part of the stress-tensor. The divergent part of (3.35) of order $n+1$ is obtained substituting $\Gamma[\Phi, g_{\mu\nu}]$ by $\Gamma_\infty^{(n)}$ in (3.36). Hence, from (3.36), (3.33), (3.34) and recalling that $T_0^{\mu\nu} = -\frac{2}{\sqrt{-g}}\frac{\delta S_0[g_{\mu\nu}]}{\delta g_{\mu\nu}(x)}$, we have

$$\langle T_0^{\mu\nu}\rangle_\infty^{(n)} = \frac{\partial T_0^{\mu\nu}}{\partial\lambda}\Delta_\lambda^{(n)} + \phi\frac{\delta T_0^{\mu\nu}}{\delta\phi}\Delta_\phi^{(n)} + \frac{\mathrm{d}\left(\delta R^2\right)^{(n)}}{\mathrm{d}g_{\mu\nu}}. \tag{3.37}$$

The divergences coming from diagrams that contain at least two vertices of type $S_1$ or at least one vertex $S_2$ are all proportional to the square of Ricci tensor, so those purely gravitational are contained in $R_{\mu\nu}\left(\mathcal{T}^{\mu\nu\alpha\beta}\right)^{(n)}R_{\alpha\beta}$, while those that have a matter operator are contained in $(\delta S_2)^{(n)}$, so they are all collected in *iii)*.

In this classification, we have considered all possible diagrams and verified that (3.33) contains all possible contribution of order $n$.

The elimination of the divergences of order $n+1$ proceeds as follows. Those collected in *i)* are the divergences of the unperturbed theory $S_0$, absorbed by redefinitions

$$\lambda \rightarrow \lambda - \Delta_\lambda^{(n)}, \qquad \phi \rightarrow \phi - \phi\Delta_\phi^{(n)}, \qquad a \rightarrow a - \Delta_a^{(n)}, \qquad b \rightarrow b - \Delta_b^{(n)}. \qquad (3.38)$$

These redefinition cancel the divergences *i)* because

$$\begin{aligned} S_0 &\rightarrow S_0 - (\delta S_0)^{(n)} + \mathcal{O}\left(\hbar^{2(n+1)}\right), \\ aR_{\mu\nu}R^{\mu\nu} + bR^2 &\rightarrow aR_{\mu\nu}R^{\mu\nu} + bR^2 - \left(\delta R^2\right)^{(n)}, \end{aligned}$$

but also affect the rest of the action $S^{\mathrm{HD}}$ (3.32). Inside the action $S_1$, the unperturbed stress tensor changes as

$$T_0^{\mu\nu} \rightarrow T_0^{\mu\nu} - \frac{\partial T_0^{\mu\nu}}{\partial\lambda}\Delta_\lambda^{(n)} - \phi\frac{\delta T_0^{\mu\nu}}{\delta\phi}\Delta_\phi^{(n)} + \mathcal{O}\left(\hbar^{2(n+1)}\right)$$

cancelling out exactly the non-purely gravitational part of *ii)* (see (3.37)). The redefinitions (3.38) applied to $S_2$ generate divergent terms of the same form as $S_2$, which are regrouped in $(\delta S_2)^{(n)}$.

All divergences remaining in (3.33) are squarely proportional to the Ricci tensor. The purely gravitational ones are absorbed in redefinitions of constants in $\mathcal{T}^{\mu\nu\alpha\beta}$ while the rest have the form as $S_2$ so they are eliminated through a redefinition of the couplings $\lambda''$. The constants $\lambda'$ are not renormalized; this is consequence of the finiteness of the stress tensor[3].

Removing the $(n+1)$-divergences we obtain $\Gamma^{(n+1)}$, and remove the $(n+2)$ divergences as explained and proceed inductively. This prove the renormalizability of (3.32) to all order.

[3]Deriving $\Gamma[\Phi, g]$ with respect to $g_{\alpha\beta}$ produces an insertion of the operator $T_0^{\alpha\beta}$ in a renormalized correlation function, which remains finite.

As far as the point *II)* is concerned, the theorem enunciated in Appendix B ensures that there exist a map $g'(g)$ such that

$$\frac{1}{2\kappa^2}\int \mathrm{d}^4x\sqrt{-g}\left[R+R_{\mu\nu}\mathcal{T}^{\mu\nu\alpha\beta}R_{\alpha\beta}\right]=\frac{1}{2\kappa^2}\int \mathrm{d}^4x\sqrt{-g'}R\left(g'\right)$$

This map relates the HD theory (3.32) to the acausal model (3.30). Under this perturbative redefinition of the metric the actions $S_0$, $S_1$ and $S_2$ are mapped into terms that can be classified again in $S_0$, $S_1$ and $S_2$. To first order in $\Delta g_{\alpha\beta}$ we have,

1) the terms generated varying $S_0$ are proportional to the energy-momentum tensor of $S_0$, $T_0^{\mu\nu}$ and proportional to the Ricci tensor, so they fall in the class $S_1$.

2) the terms coming from the variation of $S_1$ are of two types:

a) those obtained varying the metric outside $T_0^{\mu\nu}$ are still proportional to $T_0^{\mu\nu}$ and at least linearly proportional to the Ricci tensor, so they fall in the class $S_1$ or $S_2$

b) those obtained varying the metric inside $T_0^{\mu\nu}$ are not necessarily proportional to $T_0^{\mu\nu}$, but they are at least squarely proportional to the Ricci tensor, so they fall in the class $S_2$

3) the terms generated varying $S_2$ are necessarily in the class $S_2$.

The second and higher orders in $\Delta g_{\alpha\beta}$ are quadratic in the Ricci tensor, or its covariant derivatives, so they all fall in the class $S_2$.

We have therefore proved that the theory (3.30) of classical gravity coupled with quantum matter is renormalizable in the form (3.30). The matter action $S_0+S_1+S_2$ is non-polynomial and $S_1+S_2$ contains infinitely many independent couplings. The set of independent couplings, however, is considerably smaller than the set of independent couplings of quantum gravity, since $S_1+S_2$ contains only lagrangian terms of the specified form.

**More general acausal models** The non-renormalizable perturbation $S_1$ (3.31) of the theories considered in the previous section has a special form. Precisely, it contains a unique matter operator, the energy-momentum tensor $T_0^{\mu\nu}$ of the unperturbed matter action $S_0$. In this section we prove the renormalizability of more general theories. Specifically, we study the renormalizability of a class of theories that have the same form as (3.32) but where $S_1$ and $S_2$ are more general, namely their only restriction is that they can contain matter operators of dimensionality four or less. As consequence, the purely gravitational sector can have terms that are not squarely proportional to Ricci tensor. The action of these models can be written as

$$\begin{aligned} S^{\mathrm{HD}} &= \frac{1}{2\kappa^2}\int \mathrm{d}^4x\sqrt{-g}\left[R+R_{\mu\nu}\mathcal{T}^{\mu\nu\alpha\beta}R_{\alpha\beta}+V\left(g\right)\right] \\ &+ S_0+\int \mathrm{d}^4x\sqrt{-g}\sum O_I\left(\phi,g\right)K^I\left(g\right), \end{aligned} \tag{3.39}$$

where $V\left(g\right)$ collects all purely gravitational terms not squarely proportional to the Ricci tensor. $S_0$ is a power-counting renormalizable matter action embedded in curved space and $O_I$ is a basis of covariant gauge- (or BRST-) invariant local operator, not necessary scalar, of dimensionality smaller than or equal to four. The sources $K^I$ are arbitrary tensorial functions of the metric.

For instance, we can obtain a generalization of the Yang-Mills model using

$$S_0=-\frac{1}{4\alpha}\int \mathrm{d}^4x\sqrt{-g}F_{\mu\nu}^aF_a^{\mu\nu},$$

$$\sum O_I\left(\phi,g\right)K^I\left(g\right)=-\frac{1}{4}\left\{F_{\mu\nu}^aF^{a\mu\nu}\ H(g)+T_{\mu\nu}K^{\mu\nu}(g)+\Upsilon_{\mu\nu\rho\sigma}L^{\mu\nu\rho\sigma}(g)\right\},$$

where the operators $T_{\mu\nu}$ and $\Upsilon_{\mu\nu\rho\sigma}$ are defined in (3.28) and $H(g)$, $K^{\mu\nu}(g)$, $L^{\mu\nu\rho\sigma}(g)$ are unconstrained functions of the metric.

The kinetic term for the gravitational field in the action (3.39), namely the term quadratic in $\phi_{\alpha\beta}$ when the metric is expanded around flat space $g_{\alpha\beta}\left(x\right)=\eta_{\alpha\beta}+\kappa\phi_{\alpha\beta}\left(x\right)$, comes exclusively from the curvature scalar $R$ and from terms squarely proportional to equation of motion [39, 40]. Thus, the higher derivatives contained in $V\left(g\right)$ do not need to be transformed away since $V\left(g\right)$ contains only vertices. We can obtain an acausal renormalizable model from (3.39) simply applying a map

$$\int \mathrm{d}^4x\sqrt{-g}\left[R+R_{\mu\nu}\mathcal{T}^{\mu\nu\alpha\beta}R_{\alpha\beta}\right]=\int \mathrm{d}^4x\sqrt{-g'}R[g']. \tag{3.40}$$

The terms in $V\left(g\right)$ are mapped into terms that could be eventually squarely proportional to the equations of motion. It is possible to apply the map iteratively to eliminate every newly-generated term of this kind [40], but it is not necessary since the terms coming from the transformation of $V(g)$ are certainly vertices. The discussion about violations of causality is exactly the same as before, since the map $g(g')$ is the same.

The theory (3.39) is renormalizable in the sense "closed under renormalization". Its renormalizability can be easily proved realizing that the gravity content is the more general at all and no matter operator with dimensionality greater than four can be generated by renormalization. This can verified noting that divergent diagrams are the same as those found in flat space, except that there are external fields attached to the vertices. The same PC analysis of Chapter 1 works here, so the same matter operators that are needed as counterterms in flat space are needed here, but coupled to purely gravitational tensors to form diffeomorphism-invariant combinations.

Taking (3.39) as reference, we can construct models with a reduced number of couplings. For instance, starting with a renormalizable theory and letting the couplings depend on the curvature of the spacetime. In a second stage we will apply the perturbative map to obtain an renormalizable acausal model. To be more specific, once again consider our prototype HD Yang-Mills in curved space, plus the special head operator $F^a_{\mu\nu}F^{a\mu\nu}\ R$, which is not proportional to the stress tensor of $S_0$,

$$S^{\mathrm{HD}} = \int \mathrm{d}^D x\sqrt{-g}\left[\frac{R}{2\kappa^2} + \xi W^2 + \zeta G_B + \frac{\eta}{(D-1)^2}R^2 - \frac{1}{4\alpha}F^a_{\mu\nu}F^{a\mu\nu} - \frac{\theta}{4}F^a_{\mu\nu}F^{a\mu\nu}\ R\right]. \tag{3.41}$$

The deformation could be absorbed in a Weyl rescaling of $g_{\alpha\beta}$

$$g'_{\mu\nu} = g_{\mu\nu}\mathrm{e}^{2\phi}, \qquad \phi = \frac{1}{D-4}\ln(1+\theta R).$$

In four dimension, the Yang-Mills (YM) model is invariant under conformal transformations, so (3.41) can not be obtained from YM through a redefinition of the metric, as can be verified observing the singularity of the Weyl scaling factor $\phi$ at $D = 4$. The new vertex is therefore not trivial.

Analyzing divergent diagrams we will include progressively all the terms required by renormalization. Let us study them as in the previous section, grouping the diagrams by the number of $\theta$- vertex insertions. Diagrams without $\theta$ insertions are those of the undeformed YM, so their divergences are absorbed in redefinitions of $\zeta, \xi, \eta, \alpha$, $A_\mu$, and BRST sources.

Divergent diagrams with only one insertion of $\theta$ are proportional to the renormalization of the operator $F^a_{\mu\nu}F^{a\mu\nu}$ in curved space,

$$\begin{aligned} -\frac{1}{4\alpha}[F^a_{\mu\nu}F^{a\mu\nu}] &= -\frac{\varepsilon\alpha\mu^\varepsilon}{4\widehat{\beta}_\alpha\alpha_{\mathrm{B}}}F^a_{\mathrm{B}\mu\nu}F^{a\mu\nu}_{\mathrm{B}} + \frac{\alpha}{\widehat{\beta}_\alpha}\left[(\varepsilon\Delta_\xi - \beta_\xi)W^2 + (\varepsilon\Delta_\zeta - \beta_\zeta)G_B\right. \\ &\quad \left. + (\varepsilon\Delta_\eta - \beta_\eta)\frac{R^2}{(D-1)^2} + \frac{4}{(D-1)}\left(\varepsilon\Delta_\eta - \frac{\alpha\beta_\eta}{\beta_\alpha}\right)\Box R\right] \\ &\quad +\sigma\mathcal{X}, \end{aligned} \tag{3.42}$$

This result is formally identical to the one found by Hathrell [41] for quantum electrodynamics, except by the $\sigma$-exact terms, specified in the next section. Here we can read the renormalization of $F^2$ in flat space

$$Z_{F^2} = Z_\alpha\left(1 - \frac{\beta_a}{\varepsilon\alpha}\right).$$

The renormalization of $\theta$ is then

$$\theta_{\mathrm{B}} = \mu^{-\varepsilon}\theta Z_{F^2}^{-1}, \qquad \beta_\theta = \theta\left(\frac{\mathrm{d}\beta_\alpha}{\mathrm{d}\alpha} - 2\frac{\beta_\alpha}{\alpha}\right).$$

To absorb the purely gravitational divergences of (3.42) we should add to (3.41)

$$\int \mathrm{d}^4x\sqrt{-g}\left[\gamma RW^2 + \rho RG_B + \frac{\upsilon}{(D-1)^2}R^3 + \frac{\tau}{(D-1)}R\Box R\right].$$

The renormalization of these new constants can be read from (3.42),

$$\begin{aligned} \gamma_{\mathrm{B}} &= \mu^{-\varepsilon}\left(\gamma - \theta\frac{\alpha^2}{\widehat{\beta}}(\varepsilon\Delta_\xi - \beta_\xi)\right), \qquad \rho_{\mathrm{B}} = \mu^{-\varepsilon}\left(\rho - \theta\frac{\alpha^2}{\widehat{\beta}}(\varepsilon\Delta_\zeta - \beta_\zeta)\right) \\ \upsilon_{\mathrm{B}} &= \mu^{-\varepsilon}\left(\upsilon - \theta\frac{\alpha^2}{\widehat{\beta}}(\varepsilon\Delta_\eta - \beta_\eta)\right), \qquad \tau_{\mathrm{B}} = \mu^{-\varepsilon}\left(\tau - 4\theta\frac{\alpha^2}{\widehat{\beta}}\left(\Delta_\eta - \frac{\alpha\beta_\eta}{\beta_\alpha}\right)\right). \end{aligned}$$

And their beta-functions (see section 1.7) read

$$\begin{aligned}\widehat{\beta}_\gamma &= \varepsilon\gamma - \theta\alpha^2\frac{\mathbf{d}\beta_\xi}{\mathbf{d}\alpha}, \qquad \widehat{\beta}_\rho = \varepsilon\rho - \theta\alpha^2\frac{\mathbf{d}\beta_\zeta}{\mathbf{d}\alpha}, \\ \widehat{\beta}_\upsilon &= \varepsilon\upsilon - \theta\alpha^2\frac{\mathbf{d}\beta_\eta}{\mathbf{d}\alpha}, \qquad \widehat{\beta}_\tau = \varepsilon\tau - 4\theta\alpha\frac{\mathbf{d}}{\mathbf{d}\alpha}\left(\frac{\alpha^2\beta_\eta}{\beta}\right).\end{aligned}$$

Now consider Feynman diagrams of higher order in $\theta$, obtained with multiple insertions of $RF^2$ and its BRST completion $R\sigma\mathcal{X}$. Since these insertions are proportional to $R$, diagrams with no gauge fields, ghost and BRST sources on the external legs, purely gravitational counterterms with multiple insertions are certainly squarely proportional to the Ricci tensor and can be absorbed in a term $R_{\mu\nu}\mathcal{T}^{\mu\nu\alpha\beta}R_{\alpha\beta}$.BRST invariance, parity invariance and power counting, ensure that divergent diagrams that carry gauge, ghost or BRST sources on the external legs can give only counterterms proportional to $F^2$ plus $\sigma$-exact contributions, and carry a power of $R$ at least equal to the number of insertions. As shown below with the Batalin-Vilkovisky formalism, this means that $\alpha$ and the gauge field $A_\mu^a$ renormalizes as

$$\begin{aligned}\alpha &\rightarrow L(R)\,\alpha \\ A_\mu^a &\rightarrow L_A(R)\,A_\mu^a\end{aligned}$$

where $L, L_A$ are function of the curvature scalar $R$. Therefore, completing (3.41), we study the renormalizability of

$$\begin{aligned}S^{\mathrm{HD}} &= \int \mathrm{d}^D x\sqrt{-g}\left[\frac{R}{2\kappa^2} + (\xi+\gamma R)\,W^2 + (\zeta+\rho R)\,G_B + \frac{(\eta+\upsilon R)}{(D-1)^2}R^2\right. \\ &\left.+\frac{\tau}{(D-1)}R\Box R + R_{\mu\nu}\mathcal{T}^{\mu\nu\alpha\beta}R_{\alpha\beta} - \frac{1}{4\alpha L}F_{\mu\nu}^a\,(L_A A)\,F^{a\mu\nu}\,(L_A A)\right] \qquad (3.43)\end{aligned}$$

For certain functions $\mathcal{T}^{\mu\nu\alpha\beta}, L, L_A$, containing a set of new couplings, the action (3.43) is renormalizable, and we examine its renormalizability using the Batalin-Vilkovisky formalism in the next section. This theory can be mapped into an acausal one, which is renormalizable, using the map (3.40). The only term not squarely proportional to Ricci tensor multiplies $\rho$, and corresponds to a vertex, collected in $V(g)$ in (3.39), and as explained above, it is mapped into vertices so they do not present instability problems.

**Batalin-Vilkovisky** In the renormalization of gauge theories the terms in bare lagrangian do no have the appropriate form to absorb the divergences simply redefining couplings or scaling fields. What is needed there is a transformation of fields that redefines the symmetry transformation. The Batalin-Vilkovisky formalism [42] provides general tools to face this problem. In the following we briefly review it and generalize it to prove the renormalizability of (3.43).

In this section and the deWitt notation is used. Each index represents a set of indices, discrete or continuous and sum[4] over repeated indices is understood. The symbol $\phi^I$ represents a generic field, that can be scalar, spinor, gauge, ghost, etc. Also suitable $\sqrt{-g}$ factors are understood in appropriated places to form diffeomorphism invariants. For instance,

$$\sum_{\mu,\nu} \int \mathrm{d}^4x \sqrt{-g} \left( \frac{1}{\sqrt{-g}} \frac{\delta S}{\delta g_{\mu\nu}(x)} \right) X^{\mu\nu}(x) = \frac{\delta S}{\delta g_I} X^I$$

Let $s$ be the BRST operator, being $s\phi^I$ the infinitesimal transformation of $\phi^I$. The extended action $\mathcal{S}$ is defined adding the gauge-fixing term $s\Psi$ and the BRST source terms to the action $S$, namely

$$\mathcal{S} = S + s\Psi + \left(s\phi^I\right) K_I,$$

Over the matter and gauge fields, the action of $s$ corresponds to an infinitesimal gauge transformation with the ghost field used as the parameter of the transformation. The action of the operator $s$ over the ghost and auxiliary fields is defined to be nilpotent $s^2 = 0$.

$$sA^a_\mu = \partial_\mu C^a + f^{abc} A^b_\mu C^c, \qquad sC^a = -\frac{1}{2} f^{abc} C^b C^c, \qquad s\overline{C}^a = B^a, \qquad sB^a = 0,$$

where $A^a_\mu, C^a, \overline{C}^a, B^a$ are the gauge, ghost, antighost and Lagrange multiplier fields.

The new extended action $\mathcal{S}$ is not gauge invariant, but invariant under the BRST transformation, since $S$ is gauge-invariant and do not contain ghosts, thus $sS = 0$. The gauge-fixing term is invariant by the nilpotence of $s$.

[4] Integrals over continuous indices, like spacetime coordinates.

For two functionals $\mathcal{X}$ and $\mathcal{Y}$ depending on $\phi^I$ and $K_I$, define the antiparenthesis

$$(\mathcal{X},\mathcal{Y}) = \frac{\delta_R \mathcal{X}}{\delta\phi^I}\frac{\delta_L \mathcal{Y}}{\delta K_I} - \frac{\delta_R \mathcal{X}}{\delta K_I}\frac{\delta_L \mathcal{Y}}{\delta\phi^I}.$$

It is necessary to distinguish between the left and right derivatives, because functionals, fields and sources can be fermionic or bosonic. Note if the functionals $\mathcal{B}_1, \mathcal{B}_2$ are bosonic,

$$(\mathcal{B}_1,\mathcal{B}_2) = 2\frac{\delta \mathcal{B}_1}{\delta\phi^I}\frac{\delta \mathcal{B}_2}{\delta K_I},$$

since each BRST source has the opposite statistic respect to its corresponding field.

A canonical transformation is a redefinition of functionals depending on fields and sources, that preserves the antiparenthesis. It can be generated by a generating functional $\mathcal{F}(\phi, K')$ such as

$$\phi'^I = \frac{\delta \mathcal{F}}{\delta K'_I}, \qquad K_I = \frac{\delta \mathcal{F}}{\delta \phi^I}.$$

The (classical) invariance of $\mathcal{S}$ with respect to BRST transformations is expressed as

$$(\mathcal{S},\mathcal{S}) = 2\frac{\delta \mathcal{S}}{\delta\phi^I}\left(s\phi^I\right) = 2s\mathcal{S} = 0. \tag{3.44}$$

and is called the *master equation*. In regularizations that preserves the BRST invariance and the functional measure, as dimensional regularization, the expression (3.44) implies that

$$(\Gamma,\Gamma) = 0, \tag{3.45}$$

where $\Gamma$ is the 1PI generating functional of the completed action $\mathcal{S}$, including BRST sources and ghost terms

Now, using the antiparenthesis define the generalized BRST operator $\sigma$ such as

$$\sigma\mathcal{X} = (\mathcal{S},\mathcal{X})$$

which is nilpotent $\sigma^2 = 0$ due to the master equation (3.44) and the Jacobi identity satisfied by the antiparenthesis.

Proceeding inductively, order by order in loop expansion as in the previous section, call $\Gamma^{(n)}$ the generating functional of 1PI diagrams which has been renormalized up to order $n$ in loop expansion and separate it as

$$\Gamma^{(n)} = \Gamma^{(n)}_{\text{fin}} + \Gamma^{(n)}_{\infty} \tag{3.46}$$

where $\Gamma^{(n)}_{\text{fin}}$ is the finite part of $\Gamma^{(n)}$ and $\Gamma^{(n)}_{\infty}$ is the divergent part, which is of order $n+1$.

Writing (3.46) in (3.45), we conclude that the order-$n$ divergent part of the $\Gamma$ is $\sigma$-closed $\sigma\Gamma^{(n)}_{\infty} = 0$. Then, it can be expressed as the sum of a gauge invariant part plus a $\sigma$-exact term

$$\Gamma^{(n)}_{\infty} = \mathcal{G}_n + \sigma\mathcal{R}_n,$$

where $\mathcal{G}$ is gauge invariant, therefore these divergences can be absorbed in redefinitions of couplings and fields present in $S$. The $\sigma$-exact part can be absorbed through a canonical transformation of fields and BRST sources.

The canonical transformation required in our case to eliminate the $\sigma\mathcal{R}_n$ term is given by

$$\mathcal{F}\left(\phi, K'\right) = \phi^I K'_I + \mathcal{R}_n, \qquad \phi^{I\prime} = \phi^I + \frac{\delta\mathcal{R}}{\delta K'_I}, \qquad K'_I = K_I - \frac{\delta\mathcal{R}}{\delta\phi^I}.$$

Before studying the model (3.43) where the coupling constant depend on $R$, let us review the Batalin-Vilkovisky for the Yang-Mills model in a curved background,

$$\begin{aligned} \mathcal{S} \;=\; & \int \mathrm{d}^D x\sqrt{-g}\left[\frac{R}{2\kappa^2} + \xi W^2 + \zeta G_B + \frac{\eta}{(D-1)^2}R^2 - \frac{1}{4\alpha}F^a_{\mu\nu}F^{a\mu\nu}\right] \\ & + s\Psi\left(\phi, g\right) - \int \mathrm{d}^D x\sqrt{-g}\left[\left(sA^a_\mu\right)K^\mu_a + \left(s\overline{C}^a\right)K^a_{\overline{C}} + \left(sC^a\right)K^a_C + \left(sB^a\right)K^a_B\right]. \end{aligned}$$

We choose a gauge fixing term that break only gauge invariance and not general covariance, $\nabla^\mu A^a_\mu = 0$.

$$\Psi\left(\phi, g\right) = \int \mathrm{d}^4 x\sqrt{-g}\left[-\frac{\lambda}{2}\overline{C}^a B^a + \overline{C}^a\nabla^\mu A^a_\mu\right].$$

Let us examine the $\sigma$-exact terms. Simply considering that $\mathcal{R}_n$ should be covariant combination of fields and sources with ghost number equal to $-1$ and of dimensionality 3 or less (because they can be coupled to functions of the metric), there are 17 candidates,

$$\begin{aligned}
&K_a^\mu A_\nu^a, \qquad K_C^a C^a, \qquad K_{\overline{C}}^a \overline{C}^a, \qquad K_B^a B^a, \qquad \overline{C}^a B^a, \qquad \overline{C}^a \nabla_\mu A_\nu^a, \\
&\nabla_\mu \overline{C}^a A_\nu^a, \qquad \overline{C}^a A_\mu^a, \qquad f^{abc} \overline{C}^a A_\mu^b A_\nu^c, \qquad f^{abc} \overline{C}^a \overline{C}^b C^c, \\
&f^{abc} K_B^a A_\mu^b A_\nu^c, \qquad f^{abc} K_B^a \overline{C}^b C^c, \qquad f^{abc} K_B^a K_B^b C^c, \qquad K_B^a K_{\overline{c}}^a, \\
&K_B^a \nabla_\mu A_\nu^a, \qquad \nabla_\mu K_B^a A_\nu^a, \qquad K_B^a A_\mu^a. \qquad (3.47)
\end{aligned}$$

Note that in the case where there is no functions of $R$ inside the integral, some terms with covariant derivatives listed above are equivalent. Since the action provides no vertices with $B^a, K_B^a$ or $K_{\overline{C}}^a$ fields on the external legs, $\sigma X_i$ should not contain $B^a, K_B^a$ or $K_{\overline{C}}^a$, where $X_i$ is some linear combination of the above terms. There are only three such combinations, namely

$$\begin{aligned}
X_1 &= K_C^a C^a, \\
(X_2)_\nu^\mu &= \nabla^\mu \overline{C}^a A_\nu^a + K_a^\mu A_\nu^a, \\
X_3 &= \lambda \overline{C}^a B^a + K_B^a B^a + K_{\overline{C}}^a \overline{C}^a - \overline{C}^a \nabla^\mu A_\mu^a.
\end{aligned}$$

Actually, it is enough to consider only two of them, since $X_3$ is $\sigma$-closed, indeed it is exact $X_3 = \sigma \left( K_B^a \overline{C}^a \right)$.

In the unperturbed theory it is convenient to use $X_1$ and

$$\begin{aligned}
X_4 &= X_3 - (X_2)_\mu^\mu + \nabla^\mu (\overline{C}^a A_\mu^a) \\
&= \lambda \overline{C}^a B^a + K_B^a B^a + K_{\overline{C}}^a \overline{C}^a - K_a^\mu A_\mu^a
\end{aligned}$$

as a basis. Note that $\sigma X_4$ contains a $B^a$-field, but it is a total derivative. With this choice,

$$\mathcal{R}_n = \int \mathrm{d}^4 x \sqrt{-g} \left[ \delta_C^{(n)} X_1 - \delta_A^{(n)} X_4 \right],$$

where $\delta_C^{(n)}$ and $\delta_A^{(n)}$ are divergent constants. This basis is such that $\sigma$-exact counterterms are reabsorbed by a renormalization $\lambda' = \lambda Z_{n\ \lambda}$ of the gauge-fixing parameter $\lambda$ and the canonical transformation

$$\phi^i = Z_{n\ i}^{1/2} \phi^i, \qquad K_i' = Z_{n\ i}^{-1/2} K_i,$$

generated by

$$\mathcal{F}\left(\phi,K'\right)=\int \mathrm{d}^4x\sqrt{-g}\sum_i Z_{n\ i}^{1/2}\phi^i K_i' = \phi^I K_I' - \mathcal{R}_n\left(\phi,K'\right) + \text{higher orders},$$

where

$$Z_{n\ \overline{C}}=Z_{n\ B}=Z_{n\ A}^{-1}=Z_{n\ \lambda}^{-1},\quad Z_{n\ K_i}=Z_{n\ i}^{-1},\quad Z_{n\ A}^{1/2}=1-\delta_A^{(n)},\quad Z_{n\ C}^{1/2}=1-\delta_C^{(n)}$$

In the perturbed model (3.43), the matter operators in (3.47) are coupled to functions of $R$. If we choose

$$J_1=X_1,\qquad J_2=X_2$$

as a basis, then

$$\mathcal{R}_n=\int \mathrm{d}^4x\sqrt{-g}\left[\delta_C^{(n)}J_1+\delta_A^{(n)}J_2\right],$$

where $\delta_C^{(n)}$, and $\delta_A^{(n)}$ are now functions $R$ with divergent coefficients. The generating functional reads

$$\begin{aligned}\mathcal{F}\left(\phi,K'\right) &= \int \mathrm{d}^4x\sqrt{-g}\left[Z_{n\ A}^{1/2}A_\mu^a K_a'^\mu+\overline{C}^a K_{\overline{C}}'^a+B^a K_B'^a\right.\\ &\quad\left.+Z_{n\ C}^{1/2}C^a K_C'^a+\left(Z_{n\ A}^{1/2}-1\right)(\nabla^\mu\overline{C}^a)A_\mu^a\right]\end{aligned}$$

where $Z_{n\ C}$ and $Z_{n\ A}$ are defined above. Here $\overline{C}^a$, $B^a$, $K_B^a$ and $\lambda$ are not renormalized, and the unique non-trivial redefinitions are

$$\begin{aligned}A_\mu^a &\to Z_{n\ A}^{1/2}A_\mu^a,\qquad C^a\to Z_{n\ C}^{1/2}C^a,\qquad K_C^a\to Z_{n\ C}^{-1/2}K_C^a\\ K_{\overline{C}}^a &\to K_{\overline{C}}^a-\nabla^\mu A_\mu^a+\nabla^\mu\left(Z_{n\ A}^{1/2}A_\mu^a\right),\\ K_a^\mu &\to Z_{n\ A}^{-1/2}K_a^\mu+\nabla^\mu\overline{C}^a\left(Z_{n\ A}^{-1/2}-1\right),\end{aligned}$$

besides the renormalization of the gauge coupling.

Now define a map $\Sigma_{\mathcal{L}}$, $(L,L_A,L_C)$ made of a redefinition

$$\alpha\to\alpha\left(\theta R\right)=\alpha L\left(\theta R\right)\tag{3.48}$$

of the gauge coupling, plus the addition of purely gravitational terms (described below), plus the canonical transformation generated by

$$\begin{aligned}\mathcal{F}(\phi, K') &= \int \mathrm{d}^4x\sqrt{-g}\,\big[L_A A_\mu^a K_a^{\prime\mu} + \overline{C}^a K_{\overline{C}}^{\prime a} + B^a K_B^{\prime a} \\ &\quad + L_C C^a K_C^{\prime a} + (L_A - 1)\,(\nabla^\mu \overline{C}^a) A_\mu^a\big]\end{aligned}$$

where the functions on $\mathcal{L}$ depend on $\theta R$. The unique non-trivial redefinition of fields and BRST sources are

$$\begin{aligned}A_\mu^a &\to L_A A_\mu^a, \qquad C^a \to L_C C^a, \qquad K_C^a \to L_C^{-1} K_C^a \\ K_{\overline{C}}^a &\to K_{\overline{C}}^a - \nabla^\mu A_\mu^a + \nabla^\mu\left(L_A A_\mu^a\right), \\ K_a^\mu &\to L_A^{-1} K_a^\mu + \nabla^\mu \overline{C}^a\left(L_A^{-1} - 1\right).\end{aligned} \tag{3.49}$$

We assert that the perturbed theory obtained applying the map $\Sigma_{\mathcal{L}}$ to the unperturbed one is renormalizable and that the subtraction of divergences is again a map $\Sigma_{\mathcal{L}}$, namely a renormalization of the gauge coupling of the form (3.48), plus a canonical transformation of the form (3.49), plus a suitable renormalization of the pure gravitational terms.

The action of the perturbed Yang-Mills theory in curved space (3.43) for the Batalin-Vilkovisky formalism is

$$\begin{aligned}\mathcal{S}_{\mathrm{YM}-\theta}^{\mathrm{HD}} &= \Sigma_{\mathcal{L}}\mathcal{S}_{\mathrm{YM}}^{\mathrm{HD}} = \int \mathrm{d}^4x\sqrt{-g}\left[\frac{R}{2\kappa^2} - \frac{1}{4\alpha L}F_{\mu\nu}^a\left(L_A A\right)F^{a\mu\nu}\left(L_A A\right) - \frac{\lambda}{2}B^a B^a\right. \\ &\quad + B^a \nabla^\mu A_\mu^a + L_A^{-1}\left(K_a^\mu + \nabla^\mu \overline{C}^a\right)\left[\partial_\mu\left(L_C C^a\right) + f^{abc} L_A L_C A_\mu^b C^c\right] \\ &\quad \left. + \frac{L_C}{2} f^{abc} K_C^a C^b C^c - K_{\overline{C}}^a B^a\right] + \Delta S_g,\end{aligned} \tag{3.50}$$

where $\Delta S_g$ denotes pure gravitational terms, so far unspecified. The Batalin-Vilkovisky analysis has to be applied with the perturbed $\sigma$-operator $\sigma_\theta$, defined by

$$\sigma_\theta \mathcal{X} = \left(\mathcal{S}_{\mathrm{YM}-\theta}^{\mathrm{HD}}, \mathcal{X}\right).$$

It easy to verify that $\sigma_\theta$ is nilpotent, namely

$$\left(\mathcal{S}_{\mathrm{YM}-\theta}^{\mathrm{HD}}, \mathcal{S}_{\mathrm{YM}-\theta}^{\mathrm{HD}}\right) = 0.$$

Calling $\Gamma_{\text{YM}-\theta}$ the generating functional of the one-particle irreducible diagrams, we have also

$$(\Gamma_{\text{YM}-\theta}, \Gamma_{\text{YM}-\theta}) = 0.$$

Thus, analogously to the unperturbed model,

$$\Gamma^{(n)}_{\text{YM}-\theta,\infty} = \mathcal{G}_{n\ \theta} \ + \sigma_\theta \mathcal{R}_{n\ \theta}. \tag{3.51}$$

Let us analyze the vertices of the theory (3.50). We can write

$$\mathcal{S}^{\text{HD}}_{\text{YM}-\theta} = \mathcal{S}^{\text{HD}}_{\text{YM}} + \int \mathrm{d}^4x\sqrt{-g}\left[\overline{K}_1 O_1 + \overline{K}_{2\mu} O^\mu_2 + \overline{K}_{3\mu\nu} O^{\mu\nu}_3\right] + \Delta S_g,$$

where $O_1$, $O^\mu_2$ and $O^{\mu\nu}_3$ are $\theta$-independent operators with dimensionality 4, 3 and 2 respectively, constructed with the fields, the BRST sources and their derivatives, while the gravitational sources read

$$\overline{K}_1 = P_1(\theta R), \qquad \overline{K}_{2\mu} = \theta P_2(\theta R)\nabla_\mu R, \qquad \overline{K}_{3\mu\nu} = \theta^2 P_1(\theta R)\nabla_\mu R \nabla_\nu R.$$

Every $\theta$-dependence is contained in the $\overline{K}$'s. The counterterms (3.51) are local, covariant, have dimensionality four and are constructed with the $\overline{K}$'s, the matter fields $\phi^I$, the BRST sources $K_I$, the curvature tensor and their covariant derivatives.

Let us study the $\sigma_\theta$-cohomology. The $\sigma_\theta$-closed terms of type $\mathcal{G}_{n\ \theta}$ can contain $F^2$, $T_{\mu\nu}$ and $\Upsilon_{\mu\nu\alpha\beta}$, with $A^a_\mu$ replaced by $L_A A^a_\mu$. However, PC excludes both $T_{\mu\nu}$ and $\Upsilon_{\mu\nu\alpha\beta}$ because they have dimensionality four and the only dimensionless $\overline{K}$ is scalar. Therefore, only $F^2$ remains. The functional $\mathcal{R}_{n\ \theta}$ is a linear combination of the terms listed in (3.47), with coefficients constructed with the sources $\overline{K}$'s, the curvature tensors and their covariant derivative, such that $\sigma_\theta \mathcal{R}_{n\ \theta}$ does not contain $B^a$, $K^a_B$ or $K^a_{\overline{C}}$. There no $\sigma_\theta$-exact term with dimensionality two or less, so $\overline{K}_{3\mu\nu}$ can be dropped. We can drop also $\overline{K}_{2\mu}$ together with the terms $\overline{C}^a A^a_\mu$ and $K^a_B A$ of (3.47), because the counterterms constructed with these objects can be easily converted, by means of partial integration, into products of a scalar function times a combination of the other terms of (3.47).

Using the canonical transformation (3.49) to relate the $\sigma$- and $\sigma_\theta$-cohomologies, we demonstrate that the generic functional $\mathcal{R}_{n\ \theta}$ can be written in terms of the basis $J_1$, and $J_2$. Start applying the canonical transformation to

$$\sigma\mathcal{X} = \left(\mathcal{S}^{\text{HD}}_{\text{YM}}, \mathcal{X}\right) \equiv \mathcal{Y}.$$

Denoting the transformed functionals with a tilde and using the invariance of the antiparenthesis, we have

$$\left(\widetilde{\mathcal{S}}_{\mathrm{YM}}^{\mathrm{HD}},\widetilde{\mathcal{X}}\right)\equiv\widetilde{\mathcal{Y}}. \tag{3.52}$$

The transformed action $\widetilde{\mathcal{S}}_{\mathrm{YM}}^{\mathrm{HD}}$ differs from $\mathcal{S}_{\mathrm{YM}-\theta}^{\mathrm{HD}}$ because of the coupling redefinition:

$$\widetilde{\mathcal{S}}_{\mathrm{YM}}^{\mathrm{HD}}=\mathcal{S}_{\mathrm{YM}-\theta}^{\mathrm{HD}}+\int\mathrm{d}^4x\sqrt{-g}\frac{1-L}{4\alpha L}F_{\mu\nu}^{a}\left(L_AA\right)F^{a\mu\nu}\left(L_AA\right)\equiv\mathcal{S}_{\mathrm{YM}-\theta}^{\mathrm{HD}}+\widetilde{\Delta}_L,$$

therefore (3.52) can be written as

$$\sigma_\theta\widetilde{\mathcal{X}}=\widetilde{\mathcal{Y}}-\left(\widetilde{\Delta}_L,\widetilde{\mathcal{X}}\right)=\widetilde{\mathcal{Y}}-\widetilde{(\Delta_L,\mathcal{X})}.$$

Assume that $\mathcal{X}$ is a combination of the terms (3.47). Since $\Delta_L$ depends only on $A_\mu^a$, only the term proportional to $K_a^\mu A_\mu^a$ in $\mathcal{X}$ contributes to $(\Delta_L,\mathcal{X})$. So, $(\Delta_L,\mathcal{X})$ does not contain $B^a$, $K_B^a$ or $K_{\overline{C}}^a$. Moreover, the canonical transformation (3.49) is such that functionals (not) containing $B^a$, $K_B^a$ or $K_{\overline{C}}^a$ are mapped into functional (not) containing $B^a$, $K_B^a$ or $K_{\overline{C}}^a$. Thus, an $\mathcal{X}$ such that $\sigma\mathcal{X}$ does (not) contain $B^a$, $K_B^a$ or $K_{\overline{C}}^a$ is mapped into a $\widetilde{\mathcal{X}}$ such that $\sigma_\theta\widetilde{\mathcal{X}}$ does (not) contain $B^a$, $K_B^a$ or $K_{\overline{C}}^a$, and viceversa. Having dropped both $\overline{K}_{2\mu}$ and $\overline{K}_{3\mu\nu}$, we can focus on scalars functionals $\mathcal{X}$, $\widetilde{\mathcal{X}}$.

These properties ensures that the most general $\widetilde{\mathcal{X}}$ can be obtained applying the canonical transformation (3.49) to the most general $\mathcal{X}$. Since the latter is a linear combination of $J_1$, and $J_2$, and

$$\widetilde{J}_1=J_1,\qquad\widetilde{J}_2=J_2,$$

also the former is a linear combination of $J_1$ and $J_2$.

On the other hand, the pure gravitational counterterms,which are trivially $\sigma_\theta$-closed, can be constructed with the $\overline{K}$'s , the curvature tensors and their covariant derivatives. The list of independent terms is

$$Q_1R_{\mu\nu\alpha\beta}R^{\mu\nu\alpha\beta},\qquad Q_2R_{\mu\nu}R^{\mu\nu},\qquad Q_3R^2,\qquad Q_4\Box R,\qquad \theta^2Q_5R^{\mu\nu}\nabla_\mu R\nabla_\nu R,$$
$$\theta^3Q_6\nabla_\mu R\nabla^\mu R\Box R,\qquad \theta^4Q_7\left(\nabla_\mu R\nabla^\mu R\right)^2,\qquad \theta^2Q_8\left(\Box R\right)^2,$$

where $Q_i$, $i = 1, ..., 8$ are functions of $\theta R$. Thus $\Delta S_g$ is a linear combination of such terms. We see that there is only one vertex, $RR_{\mu\nu\alpha\beta}R^{\mu\nu\alpha\beta}$ that is not squarely proportional to the Ricci tensor.

Finally, we have

$$\Gamma^{(n)}_{\text{YM}-\theta,\infty} = U_n F^a_{\mu\nu}\left(L_A A\right) F^{a\mu\nu}\left(L_A A\right) + \sigma_\theta\left(V_n J_1 + W_n J_2\right) + \text{pure gravitational terms},$$

where $U_n, V_n$, and $W_n$ are functions of $\theta R$. The divergences are inductively subtracted by a map of the form of $\Sigma_{\mathcal{L}}$ with $\mathcal{L} = (1 - 4\alpha L U_n, 1 - V_n, 1 - W_n)$

# Chapter 4

# Renormalizable Lorentz-Violating Theories

While in Chapter 2, we studied how to give sense to (PC) non-renormalizable models as fundamental theories by means of a RG-consistent reduction of couplings, in this chapter we use a modified version of PC, adapted to a particular class models that have only higher spatial derivatives in the kinetic term, to prove its renormalizability.

The inclusion of higher-derivative terms improves the power counting making the propagator fall faster at high energies [47]. However, (as in Chapter 3) higher time-derivatives should be avoided since they produce instabilities (in classical models) or loss of unitarity in QFT (see for example [31] for higher-derivative gravity).

Recalling that the loss of unitarity is due to the presence of higher time-derivatives, in this chapter we study the renormalizability of models where higher derivatives are present, but only the spatial ones, thus the PC is improved without implying a loss of unitarity. In particular it is shown that higher time-derivatives are not turned on by renormalization in these models. Thus, we obtain a renormalizable and unitarity model but at the price of losing the Lorentz invariance.

In the usual PC, the number of types of vertices allowed in a renormalizable theory increases when the number of spacetime dimensions decreases, because the maximal number of legs that a vertex can have increases (see the inequality (1.7)). The net effect of including higher spatial-derivatives in the quadratic part of the lagrangian, is that the spacetime dimension $D$ is replaced in some equations of PC analysis (in particular in (1.5) and (1.7) by a fractional number which is smaller than the physical number of dimensions. Therefore, the use of spatial higher-derivatives opens the possibility of a new set of interactions for renormalizable theories. The set of consistent theories is still very restricted, yet considerably larger than the set of Lorentz invariant theories. Renormalizable models exist in arbitrary spacetime dimensions.

Renormalizability follows from a modified Power-Counting criterion, which weights time and space differently. In this weighted PC, some concepts and quantities of normal PC (Chapter 1) are suitably generalized. With the intention of making the analysis clearer, we suggest a parallelism between Lorentz-invariant models and spatial higher-derivative (SHD) Lorentz-violating theories, where concepts as scale transformation, dimensionality, superficial degree of divergence, etc., are mapped into *weighted scale transformation, weighted dimensionality, weighted superficial degree of divergence*, etc.

The mathematical framework used to the study the divergent integrals coming from Feynman diagrams with Lorentz-violating propagators is based on the scaling properties of weighted polynomials defined in section 4.1 and section 4.9.

In the weighted PC analysis we consider models containing fermion and boson fields. In a recent work, the weighted PC is applied to generalized gauge theories [48]. It is not evident how to extend this kind of Lorentz-violating terms to gravity without losing the diffeomorphism invariance. Lorentz-violating models with higher space derivatives might be useful to define the ultraviolet limit of theories that are otherwise non-renormalizable, including quantum gravity, removing the divergences through the redefinition of a finite number of independent couplings. Other domains where the models of this paper might find applications are Lorentz-violating extensions of the Standard Model [49, 50], effective field theory [51], RG methods for the search of asymptotically safe fixed points [52, 53], non-relativistic quantum field theory for nuclear physics [54], condensed matter physics and the theory of critical phenomena [55]. Effects of Lorentz and CPT violations on stability and microcausality have been studied [56], as well as the induction of Lorentz violations by the radiative corrections [57, 58, 59, 60, 61, 62, 63, 64, 65]. The renormalization of gauge theories containing Lorentz violating terms has been studied in [48, 66, 67, 68].

For a recent review on astrophysical constraints on the Lorentz violation at high energy see ref. [69].

In section 4.1 the concept of weighted polynomial is defined, and used in section 4.2 to extend the power counting analysis to treat this kind of Lorentz violating theories. Also the dimensional regularization technique has to be adapted to extended separately different subspaces, as explained in 4.4. In section 4.3 a series of examples are displayed, where special emphasis is put on homogeneous models, a class of renormalizable models which present a classical invariance. This symmetry is anomalous at quantum level; the anomaly is calculated explicitly in section 4.8. The concept of *edge renormalizability* is introduced in section 4.5 to name a special type of theories that although require an infinite number of interactions, the structure in derivatives of such vertices is preserved. In section 4.7 we analyze the renormalization structure and the renormalization group. The renormalizability of non-relativistic theories with higher space-derivatives is studied in section 4.10. In fermionic theories, a new set of invariant vertices are allowed by the remaining symmetries, which are not generated by renormalization if they are not included from the beginning. How the weighted PC gives correctly the weighted degree of divergence and the extension of the prove of locality of counterterms, is shown in section 4.9. In that section is explained also the relation between the (weighted) scaling properties of polynomials and the determination of the zone of the multi-dimensional momentum space where the integrals has the most divergent behavior. This analysis is what motivate the definition of weighted polynomial in the weighted PC.

## 4.1 Weighted Polynomials

For the study of the Lorentz violation, the spacetime manifold is split into two submanifolds. The first one, that contains the time coordinate and some or none of the spatial coordinates, is denoted by hat $\widehat{\mu}$ over the indices, while the second submanifold, containing all other coordinates, is denoted by a bar $\overline{\mu}$. Later, in section 4.4 the analysis is generalized to the case where the spacetime is split into an arbitrary number of submanifolds (a positive integer number less than or equal to the number of spacetime dimensions). The coordinates of a vector $p$ are then written in one of the following forms

$$p^{\mu} = (\widehat{p}^{\mu}, \overline{p}^{\mu}) = \left(p^{\widehat{\mu}}, p^{\overline{\mu}}\right).$$

We use also the symbols

$$\widehat{\triangle} = \widehat{\partial}_{\mu}\widehat{\partial}^{\mu} = \partial_{\widehat{\mu}}\partial^{\widehat{\mu}}, \qquad \overline{\triangle} = \overline{\partial}_{\mu}\overline{\partial}^{\mu} = \partial_{\overline{\mu}}\partial^{\overline{\mu}}.$$

for the contracted derivatives (d'Alembertian / Laplacian) in each subspace. We call $P_{k,n}(\widehat{p}, \overline{p})$ a weighted polynomial of degree $k$ and weight $n$ if scaling separately both sets of coordinates, $P_{k,n}(\xi\widehat{p}, \xi^{1/n}\overline{p})$ is a polynomial of degree $k$ in $\xi$.

In a similar way, the polynomial $H_{k,n}(\widehat{p}, \overline{p})$ is called *homogeneous* if $H_{k,n}(\xi\widehat{p}, \xi^{1/n}\overline{p}) = \xi^{k} H_{k,n}(\widehat{p}, \overline{p})$. Clearly the product of weighted polynomials (of same weight) is a weighted polynomial with degree equal to the sum of the degree of the factors,

$$P_{k_1,n}(\widehat{p}, \overline{p}) \;\; Q_{k_2,n}(\widehat{p}, \overline{p}) = R_{k_1+k_2,n}(\widehat{p}, \overline{p}).$$

If the factors are homogeneous, so its product also is.

We use the concept of weighted degree for some non-polynomials functions as well, for example the quotient of two polynomials

$$f_{k_1-k_2,n}(\widehat{p}, \overline{p}) = \frac{P_{k_1,n}(\widehat{p}, \overline{p})}{Q_{k_2,n}(\widehat{p}, \overline{p})}.$$

All these definitions based in the scaling properties are motivated by the study of divergent integrals required to extend the PC analysis to Lorentz-violating theories.

## 4.2 Weighted Power-Counting Analysis

We consider bosonic and fermionic models where the kinetic term has been replaced by

$$\begin{aligned}
\mathcal{L}_{\varphi} &= \frac{1}{2}(\widehat{\partial}\varphi)^2 + \frac{1}{2}\frac{1}{\Lambda_L^{2n-2}}\left(\overline{\partial}^{n}\varphi\right)^2, \qquad (4.1)\\
\mathcal{L}_{\psi} &= \overline{\psi}\widehat{\partial\!\!\!/}\psi + \frac{1}{\Lambda_L^{n-1}}\overline{\psi}\overline{\partial\!\!\!/}^{\,n}\psi,
\end{aligned}$$

respectively, where $\Lambda_L$ is a constant with unit mass dimension and the bosonic and fermionic fields are represented by $\varphi$ and $\psi$ respectively. It is possible to include also other quadratic terms as

$$\frac{1}{2}\frac{a_m}{\Lambda_{L^{2m-2}}}\left(\overline{\partial}^m\varphi\right)^2, \qquad m < n, \tag{4.2}$$

or a mass term. They do not carry the leading divergent behavior in momentum integrals, hence for renormalization purposes is more convenient to treat them perturbatively as two-leg vertices without loss of generality. As explained in section 4.9, a fictitious mass that is set to zero in the end, can be introduced in momentum integrals to avoid infrared divergences.

To denote a particular spacetime splitting, we use the following nomenclature: $(\widehat{d},\overline{d})_n$ indicates that the spacetime is divided into two subspaces of $\widehat{d}$ and $\overline{d}$ dimensions respectively, and that all polynomials of the theory are considered as of weight $n$.

In Fourier space, the bosonic propagator is the inverse of a weighted polynomial of degree 2 and weight $n$. Similarly, the fermionic propagator is the quotient between a weighted polynomial of degree 1 and a weighted polynomial of degree 2, i.e. a weighted function of degree -1. Explicitly the propagators are

$$\frac{1}{\widehat{p}^2 + \frac{(\overline{p}^2)^n}{\Lambda_L^{2n-2}}}, \qquad \frac{-i\widehat{\not{p}} + (-i)^n \frac{\overline{\not{p}}^n}{\Lambda_L^{n-1}}}{\widehat{p}^2 + \frac{(\overline{p}^2)^n}{\Lambda_L^{2n-2}}}.$$

Now we extend the concepts and quantities of section 1.2 for weighted polynomials. Consider a generic vertex, with $N_b$ and $N_f$ bosonic and fermionic fields, with $p_1$ and $p_2$ derivatives of each type,

$$\left[\widehat{\partial}^{p_1}\overline{\partial}^{p_2}\varphi^{N_b}\left(\overline{\psi}\psi\right)^{N_f/2}\right]. \tag{4.3}$$

The square bracket represents all possible fully contracted combination of fields and derivatives. Other elements, different from fields or derivatives as gamma matrices,which are not relevant for the renormalizability analysis, are omitted. For renormalization analysis it is enough to consider only one field of each type.

Define the *weighted number of derivatives* $\not{\delta}^{(i)}$ of the vertex labelled with $i$, in analogy to $\delta^{(i)}$ of (1.2) as the weighted degree of the Fourier polynomial associated to the vertex (4.3)

$$\not{\delta} = p_1 + \frac{p_2}{n}.$$

To find the generalization of (1.5) let us examine the scaling properties of divergent integrals.

The integral associated to a $L$-loop diagram $G$ with $I$ propagators and $V$ vertices has the form

$$\mathcal{I}_G(k) = \int \frac{\mathrm{d}^{L\widehat{D}}\widehat{p}}{(2\pi)^{L\widehat{D}}} \int \frac{\mathrm{d}^{L\overline{D}}\overline{p}}{(2\pi)^{L\overline{D}}} \prod_{i=1}^{I_b} \mathcal{B}^{(i)}_{-2,n}(p,k) \prod_{j=1}^{I_f} \mathcal{F}^{(j)}_{-1,n}(p,k) \prod_{\ell=1}^{V} \mathcal{V}^{(\ell)}_{\not{\delta},n}(p,k),$$

where $\mathcal{B}^{(i)}_{-2,n}(p,k)$ and $\mathcal{F}^{(j)}_{-1,n}(p,k)$ are the bosonic and fermionic propagators, weighted functions of degree $-2$ and $-1$ respectively, while $\mathcal{V}^{(\ell)}_{\not{\delta},n}$ is a weighted polynomial of degree $\not{\delta}$ . The weight of all of these quantities is considered to be $n$.

The scaling of $\mathcal{I}_G(k)$ is tested letting $(\widehat{k}, \overline{k}) \to (\xi\widehat{k}, \xi^{1/n}\overline{k})$ and making a similar change of variables for the internal momenta generically called $p$. The degree of $\mathcal{I}_G(k)$, namely the generalization of (1.2) is then

$$\not{D}L - 2I_b - I_f + \sum_{i}^{V} \not{\delta}^{(i)}\, v_i, \tag{4.4}$$

where we define $\not{D} = \widehat{D} + \overline{D}/n$, the *effective dimension.*

In dimensional regularization, as explained in 4.4 we extend separately the dimension of each submanifold,

$$\widehat{D} = \widehat{d} - \varepsilon_1, \qquad \overline{D} = \overline{d} - \varepsilon_2,$$

where $d = \widehat{d} + \overline{d}$ is the physical dimension.

Once the subtraction of subdivergences is made, by locality of counterterms the overall divergence is local and corresponds to a weighted polynomial of degree[1]

$$\omega\,(G) = \not{d}L - 2I_b - I_f + \sum_{i} \not{\delta}^{(i)}\, v_i, \tag{4.5}$$

[1]It is simply the non evanescent part of (4.4).

and weight $n$, where $\not{d} = \widehat{d} + \overline{d}/n$ is the *physical effective dimension.* The effective dimension and the physical effective dimension are only names to relate equations of the PC analysis in the ultraviolet region to those of the Lorentz invariant theories. Indeed, the treatment of infrared divergences is the same as in the Lorentz invariant theories, using the actual spacetime dimension $d$.

If the theory contains several fields, all kinetic terms should be weighted polynomial of the same weight $n$, otherwise the naive PC analysis gives erroneous conclusions, as discussed in the Appendix. The expression (4.5) is exactly the same as (1.5) if we interpret $\not{d}$ as the physical dimension and $\not{\delta}^{(i)}$ as the degree of divergence of vertices. Thus all the conclusions about renormalizability of section 1.3 apply here directly. As before we can write

$$\omega\left(G\right) = d_{\not{d}}\left(E_b, E_f\right) + \sum_i v_i \not{\Omega}_{\not{d}}^{(i)}, \tag{4.6}$$

with $\not{\Omega}_{\not{d}}^{(i)} = \not{\delta}^{(i)} - d_{\not{d}}(N_b^{(i)}, N_f^{(i)})$. According to (1.7), since the effective dimension $\not{d}$ is smaller than the physical dimension, renormalizable Lorentz-violating models admit vertices with more legs than Lorentz-invariant models, opening the possibility for new interactions, some of them studied in the next section. To avoid unitarity problems, it is important to prove that when these new vertices are included, *no counterterms with time derivatives are generated* by renormalization, other than those proportional to kinetic terms.

For having a renormalizable model, just as in section 1.3, $\not{\Omega}_{\not{d}}^{(i)} \leq 0$ for all $i$ is required. Thus, from (4.6) we have

$$\omega\left(G\right) \leq d_{\not{d}}(E_b, E_f) \leq E_b + \frac{E_f}{2},$$

where the inequality of right hand comes from (1.6) realizing that each diagram has at least two external legs[2]. Therefore, $\omega\left(G\right) \leq 2$ if the model contain bosons, or $\omega\left(G\right) \leq 1$ if there are only fermions. The equality holds only for two-point correlation functions, namely the renormalization of kinetic terms. Since time-derivatives are weighted monomials of degree one, renormalization can generate terms with

[2] It is enough to consider only correlation functions of two or more fields, since the expectation value of a field can be set to zero by renormalization conditions.

two time-derivatives in bosonic models or one time-derivative in the fermionic case. These terms are quadratic in fields, so they correspond to kinetic terms. The only possibility of presence of time derivative other than in kinetic terms, is having vertices with only one time derivative in a bosonic theory, but they can be ruled out imposing time reversal symmetry.

All other divergent diagrams have $\omega(G) < 2$ (or $\omega(G) < 1$ for purely fermionic theories) and thus renormalize vertices that can have only spatial derivatives. The absence of kinetic terms with higher time derivatives ensures perturbative unitarity.

## 4.3 Examples

### 4.3.1 Homogeneous Models

Homogeneous models are those where propagators and all vertices are homogeneous weighted polynomials (which degree depends on the number and type of legs, see (4.8)). In general, besides the continuous spacetime symmetries remaining of the Lorentz breaking, we usually ask parity, time reversal and the *U-parity* that transforms every field into its opposite $\varphi \to -\varphi$, $\psi \to -\psi$.

Homogeneous theories can be defined as models that present classically a *weighted scale invariance.* A weighed scaling is defined by the transformations

$$\widehat{x} \to \widehat{x}\ \mathrm{e}^{-\Omega}, \qquad \overline{x} \to \overline{x}\ \mathrm{e}^{-\Omega/n}, \qquad \varphi \to \varphi\ \mathrm{e}^{\frac{\Omega}{2}(\text{đ}-2)}, \qquad \psi \to \psi\ \mathrm{e}^{\frac{\Omega}{2}(\text{đ}-1)}, \tag{4.7}$$

where $\Omega$ is a scaling factor. Due to this invariance, it is convenient in these models to use the following trick: define the *weighted dimensionality* denoted by $[X]_*$ such that the constant $\Lambda_L$ is dimensionless while for the hatted coordinates coincides with the dimensionality $[\widehat{x}]_* = [\widehat{x}]$. Other quantities have weighted dimensionalities assigned consistently

$$[\Lambda_L]_* = 0, \qquad [\overline{p}]_* = 1/n, \qquad [\overline{x}]_* = -1/n, \qquad [\varphi]_* = (\text{đ}-2)/2 \qquad [\psi]_* = (\text{đ}-1)/2.$$

The suffix remind us that the square brackets does not represent the actual dimensionality. This interpretation is not necessary but it makes easier and clearer the expressions. In this way we get the following simplifications and analogies with Lorentz invariant models:

*I) All quantities scale according to its weighted dimensionality.* This will be very useful in the study of the anomaly of the scaling symmetry.

*II) The operators and couplings can be classified by its weighted dimensionalities,* because $\not{\Omega}^{(i)} = -\left[\lambda_i\right]_*$ , giving sense to the terms *weighted marginal*, *relevant* or *irrelevant* couplings.

*III) The renormalizability analysis is simplified* because $\omega\left(G\right) = \left[G\right]_*$.

Note that the fields in (4.7) scale according to their weighted dimensionality, which is equal to the usual dimensionality replacing $d$ by $\not{d}$.

In homogeneous models all couplings are dimensionless [3] as can be easily verified considering that these models are scale invariant and using *I)*. Moreover, *II)* tells us that they are strictly renormalizable, because $\not{\delta}^{(i)} = d_{\not{d}}(N_b^{(i)}, N_f^{(i)})$. In particular they are massless. Thus,

Homogeneous Model $\Longleftrightarrow$ Weighted scale invariance $\Longleftrightarrow$ Strictly Renormalizable.

This invariance is present in Feynman diagrams and in the 1PI generating functional and also in the counterterms in a appropriate subtraction scheme. Thus, dimensional regularization in the minimal subtraction scheme, no non-homogeneous counterterms can be generated by renormalization.

In theories having fermionic fields, there exist other homogeneous renormalizable interactions allowed by all symmetries present, but they not included in models above. Renormalization do not generate them if they are not in the bare lagrangian, as discussed in 4.11.

The general form of a homogeneous bosonic-fermionic theory is

$$\begin{aligned} \mathcal{L} &= \overline{\psi}\widehat{\not{\partial}}\psi + \frac{\eta}{\Lambda_L^{n-1}}\overline{\psi}\overline{\not{\partial}}^{\,n}\psi + \frac{1}{2}(\widehat{\partial}\varphi)^2 + \frac{1}{2\Lambda_L^{2n-2}}(\overline{\partial}^n\varphi)^2 \\ &\quad + \sum_i \lambda_i\left[H^{(i)}_{\not{\delta},n}\left(\widehat{\partial},\overline{\partial}\right)\varphi^{N_b^{(i)}}\left(\overline{\psi}\psi\right)^{N_f^{(i)}/2}\right], \end{aligned} \tag{4.8}$$

where $H^{(i)}_{\not{\delta}}\left(\widehat{\partial},\overline{\partial}\right)$ is a homogeneous weighted polynomial with $\not{\delta}^{(i)} = d_{\not{d}}(N_b^{(i)}, N_f^{(i)})$ and $\eta$ a dimensionless constant.

[3] In this section the word “weighted” is implicitly understood for concepts as dimensionality, scaling transformation, marginal couplings, etc.

Let us start with a purely bosonic model. Requiring $d_D\left(N_b,0\right)=0$, we obtain the maximal number of legs that strictly renormalizable vertex can have, as function of the dimension $D$.

There are only three possibilities: $\varphi^6$ in $D=2, \varphi^4$ in $D=4$ and $\varphi^3$ in $D=6$.

Vertices that satisfy $d_D\left(N_b,0\right)>0$ are super-renormalizable, and can be transformed into renormalizable ones adding a suitable number of derivatives as in (4.8).

A $\varphi^4$-vertex is marginal in 4 dimensions. Interpreting it as the effective dimension, Đ$=4$ can be obtained from

*i)* $(1,3n)_n$ splitting in $3n+1$ dimensions,

*ii)* $(2,2n)_n$ splitting in $2(n+1)$ dimensions,

*iii)* $(3,n)_n$ splitting in $3+n$ dimensions.

That is, in every dimension greater than or equal to 3 there exist at least one splitting that makes the $\varphi^4$-interaction marginal. For instance, using *ii)* in six dimensions with $n=2$, we have

$$\mathcal{L}_{(2,4)_2}=\frac{1}{2}(\widehat{\partial}\varphi)^2+\frac{1}{2\Lambda_L^2}(\overline{\triangle}\varphi)^2+\frac{\lambda}{4!}\varphi^4. \tag{4.9}$$

This model is used to study the critical behavior at Lifshitz points [70, 71, 72].

On the other hand, for every strictly renormalizable Lorentz-invariant model in four dimensions there exist a family of homogeneous renormalizable Lorentz-violating models in higher dimensions, using some of the splittings *i), ii)* or *iii)*

Let us focus our attention in models in four dimensions. The effective dimension đ has a value between 2 and 4 for renormalizable models, due to *ii)* of section 1.2. and the definition of đ. For theories without bosons, $1<$ đ $<4$. Analyzing one by one the possibilities, we have the splittings

$(0,4)_n$: It contains higher time derivatives, so it is not unitary

$(4,0)_n$: It represents the usual Lorentz-invariant splitting (no splitting at all).

$(1,3)_n$: When bosons are present, đ $>2$. This implies $n<3$ and the unique non-trivial splitting has $n=2$. In this case, the vertices with maximal number of fields are those that satisfy $d_{1+3/2}(N_b,N_f)=0$, namely $(N_b,N_f)=(4,2)$ and $(N_b,N_f)=(10,0)$. Other marginal vertices, are those that satisfy

$$d_{1+3/2}(N_b, N_f) = \not{\delta}^{(i)} = \frac{k}{2}$$

and contain $k$ spatial derivatives $\left(\overline{\partial}\right)$, where $k$ is a positive integer. The only combination allowed by the symmetries are $(N_b, N_f, k) = (6, 0, 2)$ and $(N_b, N_f, k) = (2, 2, 1)$.

The model then reads

$$\begin{aligned}\mathcal{L}_{(1,3)_2} &= \overline{\psi}\widehat{\partial\!\!\!/}\psi + \frac{\eta}{\Lambda_L}\overline{\psi\Delta}\psi + \frac{1}{2}(\widehat{\partial}\varphi)^2 + \frac{1}{2\Lambda_L^2}(\overline{\Delta}\varphi)^2 + \frac{\lambda_2}{2}\varphi^2(\overline{\psi}\overleftrightarrow{\overline{\partial\!\!\!/}}\psi) + \frac{\lambda_2'}{2}\varphi^2\overline{\partial}\cdot(\overline{\psi}\overline{\gamma}\psi) \\ &\quad + \frac{\lambda_4}{4!}\varphi^4\overline{\psi}\psi + \frac{\lambda_6}{6!}\varphi^4(\overline{\partial}\varphi)^2 + \frac{\lambda_{10}}{10!}\varphi^{10}. \end{aligned} \tag{4.10}$$

Note that the couplings $\lambda_2, \lambda_2', \lambda_4, \lambda_6, \lambda_{10}$ in (4.10) have all zero weighted dimensionality as expected. Their actual dimensionalities are non zero, for instance

$$[\lambda_{10}]_* = 0, \qquad [\lambda_{10}] = -6.$$

In purely fermionic models, many other vertices are allowed, because we require only $1 < \not{d}$. Homogeneous vertices in the $(1,3)_n$ splitting satisfy $d_{1+3/n}(0, N_f) = 0$, namely

$$\frac{n}{3} = \frac{N_f}{2} - 1,$$

so taking $n = 3m$ with $m$ a positive integer, the operator with the maximal number of fields is $\left(\overline{\psi}\psi\right)^{m+1}$. For example, the first two models corresponding to $m = 1, 2$ are

$$\mathcal{L}'_{(1,3)_3} = \overline{\psi}\widehat{\partial\!\!\!/}\psi + \frac{1}{\Lambda_L^2}\overline{\psi\Delta\overline{\partial\!\!\!/}}\psi + \lambda_i\left[\left(\overline{\psi}\psi\right)^2\right]_i, \tag{4.11}$$

$$\mathcal{L}''_{(1,3)_6} = \overline{\psi}\widehat{\partial\!\!\!/}\psi + \frac{1}{\Lambda_L^5}\overline{\psi\Delta}^3\psi + \lambda_i\left[\overline{\partial}^3\left(\overline{\psi}\psi\right)^2\right]_i + \lambda_i'\left[\left(\overline{\psi}\psi\right)^3\right]_i, \tag{4.12}$$

where the square bracket has the same meaning as in (4.3), labelling with $i$ vertices that are different combinations of fields and derivatives inside the brackets.

$(2,2)_n$ : Here the restriction $\not{d} > 2$ does not add any information because is trivially satisfied, so $n$ is an arbitrary positive integer. The first one, corresponding to $n = 2$, has marginal vertices $(N_b, N_f) = (6, 0)$, $(N_b, N_f) = (2, 2)$, and $(N_b, N_f) = (4, 0)$ with two spatial derivatives. The corresponding lagrangian reads

$$\mathcal{L}_{(2,2)_2} = \overline{\psi}\widehat{\partial\!\!\!/}\psi + \frac{\eta}{\Lambda_L}\overline{\psi\Delta}\psi + \frac{1}{2}(\widehat{\partial}\varphi)^2 + \frac{1}{2\Lambda_L^2}(\overline{\Delta}\varphi)^2 + \frac{\lambda_2}{2}\varphi^2\overline{\psi}\psi + \frac{\lambda_4}{4!}\varphi^2(\overline{\partial}\varphi)^2 + \frac{\lambda_6}{6!}\varphi^6. \tag{4.13}$$

Another possible term is $\varphi\overline{\psi\bar{\partial}}\psi$, but can be excluded by the $U$-parity $\varphi \to -\varphi$.As mentioned above, the interaction $\varphi^6$ is marginal in $D = 3$, which is the effective dimension for $(2,2)_2$. In the same sense there is an infinite family of homogeneous models with the $\varphi^4$ vertex, $\varphi^6$ models are homogeneous in $2(n+1)$ dimensions with a $(2,2n)_n$ splitting ($\not{d} = 3$).

$(3,1)_n$ : For $n = 2$ there is no marginal vertex without derivatives. Only $\overline{\partial}\varphi^4$ is homogeneous, but there is no way of contracting the derivative without losing the parity invariance and $SO(\widehat{d})$.

For $n = 3$, the model reads

$$\begin{aligned}\mathcal{L}_{(3,1)_3} &= \overline{\psi}\widehat{\partial\!\!\!/}\psi + \frac{\eta}{\Lambda_L}\overline{\psi\Delta\bar{\partial\!\!\!/}}\psi + \frac{1}{2}(\widehat{\partial}\varphi)^2 + \frac{1}{2\Lambda_L^4}\left(\overline{\partial\triangle}\varphi\right)^2 + \frac{\lambda_3'}{3!}\varphi^2\overline{\triangle}^2\varphi + \frac{\lambda_3}{3!}\varphi\left(\overline{\triangle}\varphi\right)^2 \\ &+ \frac{\lambda_4}{4!}\varphi^2(\overline{\partial}\varphi)^2 + \frac{\lambda_5}{5!}\varphi^5 + \frac{\lambda'}{2}\varphi\overline{\psi\bar{\partial\!\!\!/}}\psi,\end{aligned}$$

which is clearly unstable. Imposing the $U$-parity we have the modified $\varphi^4$-theory

$$\mathcal{L}^{\rm even}_{(3,1)_3} = \overline{\psi}\widehat{\partial\!\!\!/}\psi + \frac{\eta}{\Lambda_L}\overline{\psi\Delta\bar{\partial\!\!\!/}}\psi + \frac{1}{2}(\widehat{\partial}\varphi)^2 + \frac{1}{2\Lambda_L^4}\left(\overline{\partial\triangle}\varphi\right)^2 + \frac{\lambda_4}{4!}\varphi^2(\overline{\partial}\varphi)^2,$$

### 4.3.2 Non-Homogeneous Models

Non homogeneous models can be constructed simply adding to an homogeneous lagrangian some super-renormalizable vertices, kinetic terms with a number space derivatives smaller than $n$ or a mass term. For example, keeping the $U$-parity, the non-homogeneous extension of (4.9) is just

$$\mathcal{L}^{\rm nh}_{(2,4)_2} = \frac{1}{2}(\widehat{\partial}\varphi)^2 + \frac{a}{2}(\overline{\partial}\varphi)^2 + \frac{m^2}{2}\varphi^2 + \frac{1}{2\Lambda_L^2}(\overline{\triangle}\varphi)^2 + \frac{\lambda}{4!}\varphi^4$$

and the non-homogeneous one of the bosonic part of (4.13) is

$$\mathcal{L}^{\rm nh}_{(2,2)_2} = \frac{1}{2}(\widehat{\partial}\varphi)^2 + \frac{a}{2}(\overline{\partial}\varphi)^2 + \frac{m^2}{2}\varphi^2 + \frac{1}{2\Lambda_L^2}\left(\overline{\triangle}\varphi\right)^2 + \frac{\lambda_4}{4!}\varphi^2(\overline{\partial}\varphi)^2 + \frac{\lambda_4'}{4!}\varphi^4 + \frac{\lambda_6}{6!}\varphi^6.$$

## 4.4 Splitting The Spacetime In More Sectors

In a similar way, the spacetime could be split into more submanifolds, eventually one for each coordinate. Calling $M_d$ the spacetime manifold, it can be considered as the tensor product of the submanifolds $M_{\widehat{d}}$ and $M_{\overline{d}_i}$,

$$M_d = M_{\widehat{d}} \otimes \prod_{i=1}^{\ell} M_{\overline{d}_i},$$

corresponding to a splitting into $\ell + 1$ subspaces. The kinetic term (for bosonic fields) for such splitting is

$$\mathcal{L}_{\text{kin}} = \frac{1}{2}(\widehat{\partial}\varphi)^2 + \frac{1}{2}\varphi P_2(\overline{\partial}_i, \Lambda_L)\varphi,$$

where $P_2(\overline{\partial}_i, \Lambda_L)$ is the most general weighted homogeneous polynomial of degree 2 in the spatial derivatives, $P_2(\xi^{1/n_i}\overline{\partial}_i, \Lambda_L) = \xi^2 P_2(\overline{\partial}_i, \Lambda_L)$, invariant under rotations in the subspaces $M_{\overline{d}_i}$. The $\Lambda_L$-dependence is arranged so that $P_2$ has dimensionality 2.

The usage of dimensional regularization requires the analytic continuation of the dimension of each subspace separately $\widehat{D} = \widehat{d} - \varepsilon_1, \overline{D}_i = \overline{d}_i - \varepsilon_{i+1}$. The quantity called effective dimension is extended to

$$đ = \widehat{d} + \sum_{i=1}^{\ell} \frac{\overline{d}_i}{n_i}$$

and has the same role as in normal weighted PC analysis. In this scheme, the divergence in renormalization constants is due exclusively to poles in $\varepsilon$, with

$$\varepsilon = \varepsilon_1 + \sum_{i=1}^{\ell} \frac{\varepsilon_{i+1}}{n_i}.$$

This can be verified with the same dimensional argument presented in section 4.7, in the paragraph below 4.21. However, the residues of such poles could depend on $\varepsilon_i$ separately.

## 4.5 Edge Renormalizability

In some theories, renormalization generates vertices that preserve the number of derivatives, but the number of fields in each vertex is not restricted. This situation is called *edge renormalizability* because although the number of vertices and couplings is infinity, not all possible vertices are admitted.

In a purely bosonic theory, equations (4.6) and (1.6) tell us that if $\bar{d}=2$ and all vertices are marginal, the degree of divergence $\omega(G)$ is always 2, independently of the number of external legs of $G$. Hence all correlation functions are divergent, even when all couplings are marginal.

Something similar occurs when the Einstein-Hilbert term is expanded around the flat metric: it is a sum of infinitely many terms each one with two derivatives and an arbitrary number of fields $\phi_{\mu\nu}(x)$. Due to diffeomorphism invariance, there is only one way to sum all this terms to form a scalar quantity[4].

Using the Lorentz splitting, we have the same effect in every physical dimension for a suitable $n$. In four dimensions is $(1,3)_3$ and the general form is

$$\mathcal{L}=\mathcal{L}_{\text{free}}+\mathcal{L}_{\text{I}}, \tag{4.14}$$

where

$$\mathcal{L}_{\text{free}}=\frac{1}{2}(\widehat{\partial}\varphi)^2+\frac{1}{2\Lambda_L^4}\left(\overline{\partial}\overline{\triangle}\varphi\right)^2.$$

The interaction lagrangian can be written generically applying the derivatives in all possible combinations,

$$\begin{aligned}\mathcal{L}_{\text{I}} &= V_1(\varphi)(\widehat{\partial}\varphi)^2+V_2(\varphi)[(\partial_i\varphi)^2]^3+V_3(\varphi)\overline{\triangle}\varphi(\partial_i\varphi)^2(\partial_j\varphi)^2\\ &+ V_4(\varphi)(\partial_i\partial_j\varphi)(\partial_i\partial_j\overline{\triangle}\varphi)+V_5(\varphi)\overline{\triangle}^2\varphi(\partial_i\varphi)^2+V_6(\varphi)(\overline{\triangle}\varphi)^3\\ &+ V_7(\varphi)(\partial_i\overline{\triangle}\varphi)^2+V_8(\varphi)(\partial_i\partial_j\partial_k\varphi)^2+V_9(\varphi)\overline{\triangle}^3\varphi,\end{aligned} \tag{4.15}$$

where the $V_i$'s are unspecified functions of $\varphi$ with $V_1(\varphi)=\mathcal{O}(\varphi)$, $V_4(\varphi),V_7(\varphi),V_8(\varphi),V_9(\varphi)=\mathcal{O}(\varphi^2)$. Note from (4.7) that the dimensionality of the field $\varphi$ is zero, thus the theory is scale invariant even when the vertices have arbitrarily many fields.

The lagrangian of the most general non-homogeneous theory is (4.14) with

$$\mathcal{L}_{\text{free}}=\frac{1}{2}(\widehat{\partial}\varphi)^2-\frac{1}{2}\varphi\left(a\overline{\triangle}+b\frac{\overline{\triangle}^2}{\Lambda_L^2}+\frac{\overline{\triangle}^3}{\Lambda_L^4}\right)\varphi$$

and $\mathcal{L}_{\text{I}}$ equal to (4.15) plus

$$V_{10}(\varphi)+V_{11}(\varphi)\overline{\triangle}\varphi+V_{12}(\varphi)\overline{\triangle}^2\varphi+V_{13}(\varphi)(\overline{\triangle}\varphi)^2+V_{14}(\varphi)[(\partial_i\varphi)^2]^2,$$

[4]It can considered as a reduction of couplings, but differently from chapter 2, here the reduction is originated by a symmetry.

with $V_{11}(\varphi), V_{12}(\varphi) = \mathcal{O}(\varphi^2)$, $V_{13}(\varphi) = \mathcal{O}(\varphi)$.

If fermions are included in this theory, correlation functions with more than four external fermions are finite. The reason is that $d_2(E_b, E_f) < 0$ if $E_f > 4$, thus $\omega(G) < 0$, indicating that the diagram $G$ is overall convergent.

In a purely fermionic model, the effective dimension for edge renormalization is 1. The only splitting possible for it in four dimension is $(0,4)_4$, which is ruled out because it is HD.

## 4.6 Remaining Symmetries

Due to the particular way in which Lorentz symmetry is broken by the kinetic term, it is clear that each subspace keeps a reduced version of Lorentz or rotation symmetry. Here we study what kind of symmetry involving all coordinates can still remain after the splitting.

For the sake of simplicity we consider the kinetic term of a bosonic model in a $(1,1)_2$ splitting in a two-dimensional Euclidean space,

$$\mathcal{L}_{\text{free}} = \left(\frac{\partial\varphi(x)}{\partial \mathrm{x}}\right)^2 + \left(\frac{\partial^2\varphi(x)}{\partial \mathrm{y}^2}\right)^2,$$

with $x = (\mathrm{x}, \mathrm{y})$.

Under an arbitrary infinitesimal coordinate transformation, the coordinates and the field changes as

$$\mathrm{x}' = \mathrm{x} + \varepsilon\ \delta\mathrm{x}(x), \qquad \mathrm{y}' = \mathrm{y} + \varepsilon\ \delta\mathrm{y}(x), \qquad \varphi'(x') = \varphi(x) + \varepsilon\ \delta\varphi(x).$$

The action $S = \int \mathrm{d}^2x\ \mathcal{L}_{\text{free}}$ transforms into

$$\begin{aligned} S' &= \int \mathrm{d}^2x \left[\left(\frac{\partial\varphi'(x)}{\partial \mathrm{x}}\right)^2 + \left(\frac{\partial^2\varphi'(x)}{\partial \mathrm{y}^2}\right)^2\right] \\ &= \int \mathrm{d}^2x \left[1 + \varepsilon\left(\frac{\partial(\delta\mathrm{x})}{\partial \mathrm{x}} + \frac{\partial(\delta\mathrm{y})}{\partial \mathrm{y}}\right)\right] \\ &\quad \times \left[\left(\frac{\partial\varphi(x)}{\partial \mathrm{x}}\right)^2 + 2\varepsilon\frac{\partial\varphi}{\partial \mathrm{x}}\left(\frac{\partial(\delta\varphi)}{\partial \mathrm{x}} - \frac{\partial\varphi}{\partial \mathrm{x}}\frac{\partial(\delta\mathrm{x})}{\partial \mathrm{x}} - \frac{\partial\varphi}{\partial \mathrm{y}}\frac{\partial(\delta\mathrm{y})}{\partial \mathrm{x}}\right) + \left(\frac{\partial^2\varphi'(x)}{\partial \mathrm{y}^2}\right)^2\right] \end{aligned}$$

where the square bracket in the second line corresponds to the determinant of the Jacobian of the transformation, and $S'$ is written up to order $\varepsilon$. To have an invariant action, the factor coming from the Jacobian has to be cancelled out. The only possibility is

$$\delta\varphi = C + \frac{\lambda}{4}\varphi, \qquad \delta \mathrm{x} = A + \lambda \mathrm{x}, \qquad \delta \mathrm{y} = B + \frac{\lambda}{2}\mathrm{y}, \tag{4.16}$$

where $A$, $B$,$C$ are constants. The transformation (4.16) corresponds to an infinitesimal translation plus a weighted scaling, as we already knew. When the action contains fields $\varphi$ without derivatives, for example a mass term, the constant $C$ is set to zero to have $SO(\widehat{d})$ or $SO(\overline{d})$ invariance. The scaling factor of the field $\varphi$ is its weighted dimensionality $[\varphi]_* = \text{đ}/2-1$

Summarizing, there exist no continuous symmetry that mixes the coordinates of different subspaces.

## 4.7 Renormalization Group Structure

The flux of renormalization group is closely related to the dilatation transformation when the theory is scale invariant. This symmetry, which requires a lagrangian having no dimensionful parameters, is broken at quantum level by renormalization by the introduction of the energy scale $\mu$. In the models studied, we already have an energy scale $\Lambda_L$ at classical level (and dimensionful couplings $\lambda_i$ ) but they do not spoil the invariance because they all have zero weighted dimensionality, what is actually relevant for the weighted dilatation invariance.

The connection between a dilatation and a change of renormalization scale in models that present classical weighted dilatation, as the homogeneous models of section 4.3.1 can be made explicit giving to the RG parameter $\mu$ scaling properties according to its dimensionality. Using again the weighted dimensionalities, the quantities scale under a weighted dilatation in $D$ spacetime dimensions as

$$\begin{aligned} \widehat{x} &\rightarrow \widehat{x}\, \mathrm{e}^{-\Omega}, \quad \overline{x} \rightarrow \overline{x}\, \mathrm{e}^{-\Omega/n}, \quad \varphi \rightarrow \varphi\, \mathrm{e}^{\Omega(\text{Đ}-2)/2}, \qquad \psi \rightarrow \psi\, \mathrm{e}^{\frac{\Omega}{2}(\text{Đ}-1)}, \\ \Lambda_L &\rightarrow \Lambda_L, \quad \mu \rightarrow \mu \mathrm{e}^{\Omega}. \end{aligned} \tag{4.17}$$

This transformation leaves invariant the renormalized action, but it does not constitute a symmetry; it only specifies how a change on $\mu$ compensates a dilatation, or in other words, how such change is equivalent to a dilatation.

Consider a generic vertex as (4.8). The weighted dimensionality of its bare coupling is evanescent since it is marginal in physical dimensions

$$[\lambda_{i\mathrm{B}}]_* = -\Omega_{D}^{(i)} = -\delta^{(i)} + d_{D}(N_b^{(i)}, N_f^{(i)}) = \varepsilon\left(\frac{N^{(i)}}{2} - 1\right) = \varepsilon p^{(i)},$$

with $N^{(i)} = N_b^{(i)} + N_f^{(i)}$ the total number of fields of the vertex $i$.

Omitting the $\Lambda_L$ dependence for a while, the contributions of diagrams to the 1PI-generating functional $\Gamma$ have the form

$$\mathcal{I} = \int \mathrm{d}^D x \prod_j \left(\lambda_j \mu^{\varepsilon\left(\frac{N^{(j)}}{2}-1\right)}\right)^{v_j} G\, \varphi^{E_b} \left(\overline{\psi}\psi\right)^{E_f/2}. \tag{4.18}$$

As other quantities, the scaling of $G$ corresponds to its weighted dimensionality,

$$\begin{aligned}[G]_* &= d_D(E_b, E_f) + \sum_j \Omega_D^{(j)}\ v_j \\ &= \omega(G) - \varepsilon L,\end{aligned}$$

which is easily verified. In physical dimensions, $[G]_* = \omega(G)$, and the only source of dimensionality defect[5] in $G$ is the measure $\mathrm{d}^D p$, contributes with $-\varepsilon$ for each integral, or what is the same, for each loop.

By locality of counterterms and the scaling properties under (4.17), $\overline{G}_\infty$ is an homogeneous weighted polynomial of degree $\omega(G)$ and weight $n$ in derivatives,

$$\overline{G}_\infty = \mu^{-\varepsilon L} H_{\omega(G),n}(\widehat{\partial}, \overline{\partial})$$

where $H_{\omega(G),n}(\widehat{\partial}, \overline{\partial})$ contains divergent coefficients. Hence the divergent contributions to $\Gamma$ are of the form

$$\mathcal{I}_\infty = -\int \mathrm{d}^D x \prod_j (\lambda_j)^{v_j} \mu^{\varepsilon\left(\frac{E}{2}-1\right)} [H_{\omega(G),n}(\widehat{\partial}, \overline{\partial}))\varphi^{E_b} \left(\overline{\psi}\psi\right)^{E_f/2}], \tag{4.19}$$

where we have used $\sum_i \left(\frac{N^{(i)}}{2} - 1\right) v_i - L = \frac{E}{2} - 1$,

Summing the contributions of the same kind,

[5]The coupling constants are not considered as part the diagram.

$$\lambda_{i\mathrm{B}} = \mu^{\varepsilon\left(\frac{N^{(i)}}{2}-1\right)} \left( \lambda_i + \Lambda_L^{[\lambda_i]} \sum_k c_k \prod_j \left( \lambda_j \Lambda_L^{-[\lambda_j]} \right)^{v_j} \right), \tag{4.20}$$

with $\left(\frac{N^{(i)}}{2} - 1\right)$ the *weighted dimensionality defect*, which coincides with the definition in section 1.4. In (4.20) we have restored the $\Lambda_L$-dependence matching the physical dimensionality. The dimensionality of the coupling $\lambda_i$ is

$$[\lambda_i] = (1-n)\, d_d(N_b^{(i)}, N_f^{(i)}). \tag{4.21}$$

The renormalization expression (4.20) has the same form as in section 1.7 so the same arguments and properties of beta-function apply. In particular, something that is not trivial at all, is that *all poles in renormalization constants are in one special combination of* $\varepsilon_1$ *and* $\varepsilon_2$*, namely* $\varepsilon = \varepsilon_1 + \varepsilon_2/n$. Similar arguments are used to demonstrate that also non-homogeneous models presents poles only in $\varepsilon$ in their renormalization constants. In the action, the dimensionality of integration measure $\mathrm{d}^D x$ has an evanescent part proportional to $\varepsilon$. Looking at the kinetic terms, also the fields will have an evanescent part proportional to $\varepsilon$, and thus the bare constants. Therefore its renormalization, as in (4.20) will be proportional to $\mu^{\varepsilon p^{(i)}}$. By finiteness of their beta-functions, all poles are in $\varepsilon$.

The constant $\Lambda_L$ does renormalize, as can be seen from divergences of two-point correlation function. The divergent part of diagrams quadratic in fields are polynomials $H_{2,n}(\widehat{\partial}, \overline{\partial})$ which coefficients multiplying $\widehat{\partial}$ and $\overline{\partial}^n$ are not constrained to have the same value, so these divergences are absorbed in general by a redefinition of the field $\varphi$ *and* $\Lambda_L$. The form of the relation (4.20) could be guessed from the beginning simply matching the dimensionalities and the scale properties (or what is the same, the weighted dimensionality). Similarly,

$$\begin{aligned} \Lambda_{L\mathrm{B}} &= \Lambda_L Z_L, \qquad Z_\Lambda = 1 + \sum_k d_k \prod_j \left( \lambda_j \Lambda_L^{-[\lambda_j]} \right)^{v_j}, \\ \eta_L &= \frac{1}{\Lambda_L} \frac{\mathbf{d}\Lambda_L}{\mathbf{d}\ln\mu} = -\frac{\mathbf{d}\ln Z_\Lambda}{\mathbf{d}\ln\mu}. \end{aligned}$$

The Callan-Symanzik equation has the same form as usual. For instance, for the model (4.9), we have

$$G_k(\widehat{x}_1, \cdots, \widehat{x}_k; \overline{x}_1, \cdots, \overline{x}_k; \lambda, \Lambda_L, \mu) = \langle \varphi(x_1) \cdots \varphi(x_k) \rangle\,,$$

we have

$$\left(\mu\frac{\partial}{\partial\mu}+\widehat{\beta}_\lambda\frac{\partial}{\partial\lambda}+\eta_L\Lambda_L\frac{\partial}{\partial\Lambda_L}+k\gamma_\varphi\right)G_k(\widehat{x}_1,\cdots,\widehat{x}_k;\overline{x}_1,\cdots,\overline{x}_k;\lambda,\Lambda_L,\mu)=0. \tag{4.22}$$

The equation can be immediately integrated to give

$$G_k(\widehat{x}_1,\cdots,\widehat{x}_k;\overline{x}_1,\cdots,\overline{x}_k;\lambda,\Lambda_L,\xi\mu)=z^{-k}(t)G_k(\widehat{x}_1,\cdots,\widehat{x}_k;\overline{x}_1,\cdots,\overline{x}_k;\lambda(t),\Lambda_L(t),\mu),$$

where $t=\ln\xi$ and

$$z(t)=\exp\left(\int_0^t\gamma_\varphi(\lambda(t'))\mathrm{d}t'\right),\ \frac{\mathrm{d}\lambda(t)}{\mathrm{d}t}=-\widehat{\beta}_\lambda(\lambda(t)),\Lambda_L(t)=\Lambda_L\exp\left(-\int_0^t\eta_L(\lambda(t'))\mathrm{d}t'\right),$$

with $\lambda(0)=\lambda$. Now the renormalization-group flow specifies how the correlation functions changes under a weighted overall rescaling. Indeed, the weighted scale invariance (4.17) tells us that

$$G_k(\widehat{x}_1,\cdots,\widehat{x}_k;\overline{x}_1,\cdots,\overline{x}_k;\lambda,\Lambda_L,\xi\mu)=\xi^{\omega(G)}G_k(\xi\widehat{x}_1,\cdots,\xi\widehat{x}_k;\xi^{1/n}\overline{x}_1,\cdots,\xi^{1/n}\overline{x}_k;\lambda,\Lambda_L,\mu).$$

A one-loop calculation for the model as (4.9) in a $(2,2n)_n$ splitting gives

$$\widehat{\beta}_\lambda=-\varepsilon\lambda+\frac{3\lambda^2}{(4\pi)^{n+1}n!}+\mathcal{O}(\lambda^3),\qquad\gamma_\varphi=\mathcal{O}(\lambda^2),\qquad\eta_L=\mathcal{O}(\lambda^2),$$

so these models are IR free. Only the beta-function has a non-vanishing one-loop contribution. Indeed, using the dimensional-regularization technique tadpoles vanish in homogeneous models, so $\gamma_\varphi$ and $\eta_L$ start from two loops.

Let us now consider the model (4.13) without fermionic fields. The bare lagrangian reads

$$\mathcal{L}_{(2,2)_2\mathrm{B}}=\frac{1}{2}(\widehat{\partial}\varphi_\mathrm{B})^2+\frac{1}{2\Lambda_{L\mathrm{B}}^2}\left(\overline{\triangle}\varphi_\mathrm{B}\right)^2+\frac{\lambda_{4\mathrm{B}}}{4!}\varphi_\mathrm{B}^2(\overline{\partial}\varphi_\mathrm{B})^2+\frac{\lambda_{6\mathrm{B}}}{6!}\varphi_\mathrm{B}^6,$$

where

$$\varphi_\mathrm{B}=Z_\varphi^{1/2}\varphi,\qquad\Lambda_{L\mathrm{B}}=Z_\Lambda\Lambda_L,\qquad\lambda_{4\mathrm{B}}=\mu^\varepsilon\left(\lambda_4+\Delta_4\right),$$
$$\lambda_{6\mathrm{B}}=\mu^{2\varepsilon}\left(\lambda_6+\Delta_6\right),\qquad\varepsilon\equiv\varepsilon_1+\frac{\varepsilon_2}{2}.$$

The theory is invariant under the scale transformation (4.17) with $n=2$. At one-loop we find $Z_\varphi=1$, $Z_\Lambda=1$ and

$$\Delta_4=\frac{5\lambda_4^2}{2(12\pi)^2\varepsilon},\qquad\Delta_6=\frac{5\lambda_4\lambda_6}{(8\pi)^2\varepsilon}-\frac{5\lambda_4^3}{(48\pi)^2\varepsilon},$$

so the beta-functions read

$$\widehat{\beta}_4 = -\varepsilon\lambda_4 + \frac{5\lambda_4^2}{2(12\pi)^2}, \qquad \widehat{\beta}_6 = -2\varepsilon\lambda_6 + \frac{5\lambda_4\lambda_6}{(8\pi)^2} - \frac{5\lambda_4^3}{(48\pi)^2}.$$

The asymptotic solutions of the RG flow equations are

$$\lambda_4 \sim \frac{2(12\pi)^2}{5t}, \qquad \lambda_6 \sim \frac{1}{20}\lambda_4^2,$$

where $t = \ln|x|\mu$ and $|x|$ is a typical weighted scale of the process. Since $\lambda_4$ and $\lambda_6$ must be non-negative, the theory is IR free.

## 4.8 Weighted Trace Anomaly

The weighted scale invariance (4.7) of the homogeneous models can be anomalous due to the radiative corrections. In this section we calculate the weighted trace anomaly, following [41, 43] . For definiteness, we work with the model (4.9), but the discussion generalizes immediately to the other models.

Before going forward with the study of the anomaly, we should take into consideration some issues about the variational treatment of HD theories. The lagrangian does not depend on the field and its first derivatives only $\mathcal{L} = \mathcal{L}(\varphi, \partial_\mu\varphi)$ as usual, but also on their successive derivatives, thus Euler-Lagrange equations and the Noether current will have a different form.

The variation of the action $S$ with respect to a variation $\delta\varphi$ of the fields is

$$\begin{aligned} \delta S \;=\; & \int \mathrm{d}^4x \left[\frac{\partial\mathcal{L}}{\partial\varphi} - \widehat{\partial}_\mu \frac{\partial\mathcal{L}}{\partial\left(\widehat{\partial}_\mu\varphi\right)} + \left(-\overline{\partial}\right)^n \frac{\partial\mathcal{L}}{\partial\left(\overline{\partial}^n\varphi\right)}\right]\delta\varphi \qquad (4.23) \\ & + \int \mathrm{d}^4x\, \partial_\mu \left[\frac{\partial\mathcal{L}}{\partial\left(\widehat{\partial}_\mu\varphi\right)}\delta\varphi + \sum_{i=0}^{n-1}\left(-\overline{\partial}\right)^i \left(\frac{\partial\mathcal{L}}{\partial\left(\overline{\partial}^n\varphi\right)}\right) \left(\overline{\partial}\right)^{n-2-i}\, \overline{\partial}^\mu\delta\varphi\right], \end{aligned}$$

The square bracket of the first line represents the equation of motion of the field $\varphi$. If $\delta\varphi$ is the infinitesimal variation due to a symmetry transformation, the variation of the lagrangian can be written as $\delta\mathcal{L} = \partial_\mu K^\mu$. The Noether current related to the symmetry transformation is the difference between $K^\mu$ and the square brackets of the second line of (4.23),

$$J^\mu = \frac{\partial \mathcal{L}}{\partial\left(\widehat{\partial}_\mu \varphi\right)}\delta\varphi + \sum_{i=0}^{n-1}\left(-\overline{\partial}\right)^i \left(\frac{\partial \mathcal{L}}{\partial\left(\overline{\partial}^n \varphi\right)}\right) \left(\overline{\partial}\right)^{n-2-i} \overline{\partial}^\mu \delta\varphi - K^\mu, \tag{4.24}$$

and is conserved $\partial_\mu J^\mu = 0$ when the equation of motion is satisfied.

Continuous symmetry transformations related to spacetime as translations, rotations, boosts and dilatations have conserved currents expressible in terms of the canonical energy-momentum tensor $T_{\mu\nu}$. For instance, the Noether current $S^\nu$ for a continuous dilatation is

$$S^\nu = x^\mu \tilde{T}_\mu{}^\nu, \tag{4.25}$$

where $\tilde{T}_{\mu\nu}$ is the improved energy-momentum tensor, namely the canonical stress tensor plus conserved terms that makes its trace vanish. Therefore the divergence of the current $S^\nu$ (4.25) is

$$\partial_\nu S^\nu = \tilde{T}^\mu_\mu$$

in Lorentz invariant theories. In the models with Lorentz splitting, the dilatation current is

$$S^\nu = \left(\widehat{x}^\mu + \frac{1}{n}\overline{x}^\mu\right)\tilde{T}_\mu{}^\nu,$$

so its divergence is the *weighted trace* of the improved energy momentum tensor

$$\Theta = \partial_\nu S^\nu = \tilde{T}^{\widehat{\mu}}_{\widehat{\mu}} + \frac{1}{n}\tilde{T}^{\overline{\mu}}_{\overline{\mu}},$$

which is explicitly calculated in the next section.

**Weighted dilatation.** In the case of the model (4.9), write the lagrangian as $\mathcal{L}(\varphi, \widehat{\partial}_\mu\varphi, \overline{\triangle}\varphi)$. The infinitesimal version of the transformation (4.7) reads

$$\delta\varphi = \Omega\left(1 + \widehat{x}\cdot\widehat{\partial} + \frac{1}{2}\overline{x}\cdot\overline{\partial}\right)\varphi \equiv \Omega\check{D}\varphi, \tag{4.26}$$

with $\Omega \ll 1$.The "1" in the parenthesis is the weighted dimensionality of $\varphi$ in physical dimensions and the rest, the weighted trace of the operator $\mathcal{D}_{\mu\nu} = x_\mu\partial_\nu$. The conserved Noether current $S^\mu = (\widehat{S}^\mu, \overline{S}^\mu)$ according to (4.24)

$$\widehat{S}^\mu = -\widehat{x}^\mu\mathcal{L} + \frac{\partial\mathcal{L}}{\partial(\widehat{\partial}_\mu\varphi)}\check{D}\varphi, \qquad \overline{S}^\mu = -\frac{1}{2}\overline{x}^\mu\mathcal{L} + \frac{\partial\mathcal{L}}{\partial(\overline{\triangle}\varphi)}\overleftrightarrow{\overline{\partial}}^\mu\check{D}\varphi.$$

We continue the spacetime dimensions to complex values as explained in 4.4. The continued transformation $\delta\varphi'$ and the continued current $S'^{\mu}$ are obtained replacing $\check{D}\varphi$ in $\delta\varphi$ and $S^{\mu}$ with the extension of (4.26) to $D$ dimensions

$$\check{D}'\varphi = \left(\frac{\text{Đ}}{2} - 1 + \widehat{x}\cdot\widehat{\partial} + \frac{1}{2}\overline{x}\cdot\overline{\partial}\right)\varphi \tag{4.27}$$

(see (4.17)), where $\text{Đ}= 4-\varepsilon$. At the bare level, the anomaly of (4.27) is expressed by the divergence of $S'^{\mu}$. We find

$$\partial_{\mu}S'^{\mu} = -\varepsilon\frac{\lambda_{\mathrm{B}}}{4!}\varphi_{\mathrm{B}}^{4}. \tag{4.28}$$

**Improved energy-momentum tensor and its weighted trace.** The anomaly of the weighted dilatation is encoded also in the energy-momentum tensor, precisely in its "weighted trace". Let us start from the energy-momentum tensor given by the Noether method[6]. For the model (4.9), equation (4.24) gives

$$T_{\mu\nu} = \frac{\partial\mathcal{L}}{\partial(\widehat{\partial}_{\mu}\varphi)}\partial_{\nu}\varphi + \frac{\partial\mathcal{L}}{\partial(\overline{\triangle}\varphi)}\overleftrightarrow{\overline{\partial}}_{\mu}\partial_{\nu}\varphi - \delta_{\mu\nu}\mathcal{L}. \tag{4.29}$$

This tensor is not symmetric, but conserved: it is easy to check that $\partial_{\mu}T_{\mu\nu} = 0$, using the field equations. Next, define the improved energy-momentum tensor

$$\begin{aligned}\widetilde{T}_{\mu\nu} \;=\;& \widehat{\partial}_{\mu}\varphi\partial_{\nu}\varphi - \frac{1}{\Lambda_L^2}\partial_{\nu}\varphi\overleftrightarrow{\overline{\partial}}_{\mu}\overline{\triangle}\varphi - \delta_{\mu\nu}\mathcal{L} + \frac{3\text{Đ}-2\overline{D}\text{Đ}+3\overline{D}-5}{(\overline{D}-1)\Lambda_L^2}\overline{\pi}_{\mu\nu}\left(\varphi\overline{\triangle}\varphi\right)\\ &+\frac{3-2\text{Đ}}{2(\overline{D}-1)\Lambda_L^2}\overline{\pi}_{\mu\nu}\left(\overline{\partial}_{\alpha}\varphi\right)^2 + \frac{3-2\text{Đ}}{\Lambda_L^2}\overline{\pi}_{\mu\alpha}\left(\varphi\overline{\pi}_{\alpha\nu}\varphi\right) - \frac{\text{Đ}-2}{4(\widehat{D}-1)}\widehat{\pi}_{\mu\nu}\varphi^2\end{aligned} \tag{4.30}$$

where $\widehat{\pi}_{\mu\nu} = \widehat{\partial}_{\mu}\widehat{\partial}_{\nu} - \widehat{\delta}_{\mu\nu}\widehat{\partial}^2$ and $\overline{\pi}_{\mu\nu} = \overline{\partial}_{\mu}\overline{\partial}_{\nu} - \overline{\delta}_{\mu\nu}\overline{\partial}^2$. The first three terms of (4.30) correspond to the Noether tensor (4.29), while the rest collects the improvement terms, identically conserved.

Using the field equations, it is easy to show that $\widetilde{T}_{\mu\nu}$ is conserved and that its weighted trace $\Theta$ vanishes in the physical spacetime dimension $d = \widehat{d}+\overline{d}$. Moreover, $\widetilde{T}_{\mu\nu}$ is conserved also in the continued spacetime dimension. The coefficients of the improvement terms are chosen so that in the free-field limit $\Theta$ vanishes also in the continued dimension $D = \widehat{D}+\overline{D}$. Finally, it is immediate to check that the weighted trace $\Theta$ coincides with the divergence (4.28) of the current $S'^{\mu}$.

[6]The Noether current related to the infinitesimal translation $\delta\varphi = a^{\mu}\partial_{\mu}\varphi$.

**Anomaly.** We need to write $\Theta$ in terms of renormalized operators. When we differentiate a renormalized correlation function with respect to $\lambda$ or $\Lambda_L$ we obtain a renormalized correlation function containing additional insertions of $-\partial S/\partial\lambda$ or $-\partial S/\partial\Lambda_L$, respectively. Thus, $-\partial S/\partial\lambda$ and $-\partial S/\partial\Lambda_L$ are renormalized operators. Following a standard procedure [43] we can find which operators $\mathcal{O}$ they are the renormalized versions of. In the minimal subtraction scheme, it is sufficient to express the renormalized operators as bare operators $\mathcal{O}_{\mathrm{B}}$ plus poles. Schematically,

$$\text{finite} = \mathcal{O}_{\mathrm{B}} + \text{poles} \qquad \Rightarrow \qquad \text{finite} = [\mathcal{O}].$$

where $[\mathcal{O}]$ denotes the renormalized version of the operator $\mathcal{O}$. We find

$$\begin{aligned} \frac{\partial S}{\partial\lambda} &= \text{finite} = \frac{1}{\widehat{\beta}_\lambda}\left(\gamma_\varphi[E_\varphi] - \Lambda_L\eta_L\frac{\partial S}{\partial\Lambda_L} - \varepsilon\frac{\lambda_{\mathrm{B}}}{4!}\int\varphi_{\mathrm{B}}^4\right) = \frac{\mu^\varepsilon}{4!}\int[\varphi^4], \\ -\Lambda_L\frac{\partial S}{\partial\Lambda_L} &= \text{finite} = \frac{1}{\Lambda_{\mathrm{B}L}^2}\int(\overline{\triangle}\varphi_{\mathrm{B}})^2 = \frac{1}{2\Lambda_L^2}\int[(\overline{\triangle}\varphi)^2], \end{aligned}$$

where $[E_\varphi] = \int\varphi(\delta S/\delta\varphi)$ is the operator that counts the number of $\varphi$-insertions. Thus,

$$\int\Theta = -\int\varepsilon\frac{\lambda_{\mathrm{B}}}{4!}\varphi_{\mathrm{B}}^4 = \frac{\mu^\varepsilon}{4!}\widehat{\beta}_\lambda\int[\varphi^4] - \frac{\eta_L}{\Lambda_L^2}\int[(\overline{\triangle}\varphi)^2] - \gamma_\varphi[E_\varphi].$$

The result agrees with the Callan-Symanzik equation (4.22), which can be expressed as

$$\left\langle\int\Theta\ \varphi(x_1)\cdots\varphi(x_k)\right\rangle = \mu\frac{\partial}{\partial\mu}\left\langle\varphi(x_1)\cdots\varphi(x_k)\right\rangle.$$

Indeed,

$$\int\Theta = -\mu\frac{\partial S}{\partial\mu} = \widehat{\beta}_\lambda\frac{\partial S}{\partial\lambda} + \eta_L\Lambda_L\frac{\partial S}{\partial\Lambda_L} - \gamma_\varphi[E_\varphi].$$

## 4.9 Renormalization

In this section we study the structure of Feynman diagrams, their divergences and subdivergences and the locality of counterterms. For definiteness, we work with scalar fields, but the conclusions are general.

**One loop** Consider the most general one-loop Feynman diagram $G$, with $E$ external legs, $I$ internal legs and $v_N^{(\alpha)}$ vertices of type $(N,\alpha)$ and weighted degree $\delta_N^{(\alpha)}$.

Collectively denote the external momenta by $k$. The divergent part of $G$ can be calculated expanding the integral in powers of $k$. We obtain a linear combination of contributions of the form

$$\mathcal{I}^{(I,n)}_{\mu_1\cdots\mu_{2r}|j_1\cdots j_{2s}}\widehat{k}_{\nu_1}\cdots\widehat{k}_{\nu_u}\ \overline{k}_{i_1}\cdots\overline{k}_{i_v}, \tag{4.31}$$

where

$$\mathcal{I}^{(I,n)}_{\mu_1\cdots\mu_{2r}|j_1\cdots j_{2s}} = \int\frac{\mathrm{d}^{\widehat{D}}\widehat{p}}{(2\pi)^{\widehat{D}}}\int\frac{\mathrm{d}^{\overline{D}}\overline{p}}{(2\pi)^{\overline{D}}}\frac{\widehat{p}_{\mu_1}\cdots\widehat{p}_{\mu_{2r}}\ \overline{p}_{j_1}\cdots\overline{p}_{j_{2s}}}{\left(\widehat{p}^2+(\overline{p}^2)^n/\Lambda_L^{2(n-1)}+m^2\right)^I}.$$

To avoid infrared problems we insert a mass $m$ in the denominators. For the purposes of renormalization, it is not necessary to think of $m$ as the real mass. It can be considered as a fictitious parameter, introduced to calculate the divergent part of the integral and set to zero afterwards. The real mass, as well as the other parameters $a_m$ of (4.2), can be treated perturbatively, so they are included in the set of "vertices".

From the weighted power-counting analysis of section 2 we know that the numerator of (4.31), namely

$$\widehat{p}_{\mu_1}\cdots\widehat{p}_{\mu_{2r}}\ \overline{p}_{j_1}\cdots\overline{p}_{j_{2s}}\ \widehat{k}_{\nu_1}\cdots\widehat{k}_{\nu_u}\ \overline{k}_{i_1}\cdots\overline{k}_{i_v},$$

is a weighted monomial $P_{q,n}(\widehat{p},\widehat{k};\overline{p},\overline{k})$ of weight $n$ and degree

$$q=u+2r+\frac{v}{n}+\frac{2s}{n}=\sum_i\delta^{(i)}v_i.$$

If the theory is PC renormalizable, according to (4.7), $\delta^{(i)}\leq d_{\not{D}}(N_b,0)$, thus

$$u+\frac{v}{n}\leq 2\left(I-r-\frac{s}{n}\right)+E\left(1-\frac{\not{d}}{2}\right). \tag{4.32}$$

By symmetric integration we can write

$$\begin{aligned}\mathcal{I}^{(I,n)}_{\mu_1\cdots\mu_{2r}|j_1\cdots j_{2s}} &= \delta^{(1)}_{\mu_1\cdots\mu_{2r}}\delta^{(2)}_{j_1\cdots j_{2s}}\mathcal{I}^{(I,n)}_{r,s},\quad \mathcal{I}^{(I,n)}_{r,s}\\ &= \int\frac{\mathrm{d}^{\widehat{D}}\widehat{p}}{(2\pi)^{\widehat{D}}}\int\frac{\mathrm{d}^{\overline{D}}\overline{p}}{(2\pi)^{\overline{D}}}\frac{(\widehat{p}^2)^r(\overline{p}^2)^s}{\left(\widehat{p}^2+(\overline{p}^2)^n/\Lambda_L^{2(n-1)}+m^2\right)^I},\end{aligned} \tag{4.33}$$

where $\delta^{(1)}_{\mu_1\cdots\mu_{2r}}$ and $\delta^{(2)}_{j_1\cdots j_{2s}}$ are appropriately normalized completely symmetric tensors constructed with the Kronecker tensors of $M^{\widehat{D}}$ and $M^{\overline{D}}$, respectively. Performing the change of variables

$$\overline{p}_i = \overline{p}'_i \left(\frac{\Lambda_L^2}{\overline{p}'^2}\right)^{(n-1)/(2n)}, \tag{4.34}$$

the integral $\mathcal{I}^{(I,n)}_{r,s}$ can be calculated using the standard formulas of the dimensional-regularization technique. We obtain

$$\begin{aligned}\mathcal{I}^{(I,n)}_{r,s} &= \frac{1}{n}\Lambda_L^{(2s+\overline{D})(n-1)/n}\int\frac{\mathrm{d}^{\widehat{D}}\widehat{p}}{(2\pi)^{\widehat{D}}}\int\frac{\mathrm{d}^{\overline{D}}\overline{p}'}{(2\pi)^{\overline{D}}}\frac{(\widehat{p}^2)^r(\overline{p}'^2)^{(2s+\overline{D}-n\overline{D})/(2n)}}{(\widehat{p}^2+\overline{p}'^2+m^2)^I}\\ &= \frac{\Lambda_L^{(2s+\overline{D})(n-1)/n}(m^2)^{r-I+s/n+\rlap{-}D/2}\Gamma\left(\frac{2s+\overline{D}}{2n}\right)\Gamma\left(\frac{2r+\widehat{D}}{2}\right)\Gamma\left(I-r-\frac{s}{n}-\frac{\rlap{-}D}{2}\right)}{n(4\pi)^{D/2}\Gamma(\widehat{D}/2)\Gamma\left(\overline{D}/2\right)\Gamma\left(I\right)}.\end{aligned}$$

The factor $1/n$ is due to the Jacobian determinant of the transformation (4.34). The singularities occur[7] for

$$I \leq r + \frac{s}{n} + \frac{\rlap{-}d}{2} \tag{4.35}$$

Combining this inequality with (4.32) we find that the divergent contributions satisfy

$$u + \frac{v}{n} \leq \rlap{-}d + E\left(1 - \frac{\rlap{-}d}{2}\right) = d_{\rlap{-}d}(E,0), \tag{4.36}$$

that is, the consistence condition (see the final paragraph of section 1.2

$$w\left(G\right) \leq d_{\rlap{-}d}(E,0),$$

which ensures that divergent contributions can be absorbed by counterterms. The counterterms are a $P_{u+v/n,n}(\widehat{k},\overline{k})$:

$$\frac{1}{\varepsilon}\widehat{k}_{\nu_1}\cdots\widehat{k}_{\nu_u}\,\overline{k}_{i_1}\cdots\overline{k}_{i_v}, \qquad \text{where } \varepsilon = \rlap{-}d - \rlap{-}D = \varepsilon_1 + \frac{\varepsilon_2}{n}.$$

The poles are in $\varepsilon$ as expected from the discussion of the paragraph below (4.21). The residues instead, depend on $\varepsilon_1,\ \varepsilon_2$ separately.We know that taking a sufficient

[7] Since the gamma function $\Gamma\left(x\right)$ is singular only for non-positive integers, that is

$$2\left(s+nr\right)+n\widehat{d}+\overline{d}=2nl,$$

with $l$ a non-negative integer. This means that some divergent integrals are mapped to finite values by dimensional regularization. This is similar to what happens when this regularization is used in Lorentz-invariant theories in odd-dimensions: no one-loop divergences appear.

number of derivatives with respect to the masses, the external momenta and the parameters $a_m$ of (4.2), the integral becomes convergent. Therefore, the finite parts are regular in the limits $\varepsilon_1, \varepsilon_2 \to 0$, which can be safely taken in any preferred order. Objects such as $\varepsilon_1/\varepsilon$ and $\varepsilon_2/\varepsilon$ are finite regardless of the path we choose to approach to the origin of the $(\varepsilon_1, \varepsilon_2)$ plane. Moreover, they multiply only local terms, so they parametrize different scheme choices and never enter the physical quantities. We define the minimal subtraction schemes as the schemes where

$$\varepsilon_1 = \alpha\varepsilon, \qquad \varepsilon_2 = n(1-\alpha)\varepsilon,$$

with $\alpha$=constant, and only the pure poles in $\varepsilon$ are subtracted away, with no finite contributions.

**Overall divergences and subdivergences.** Generalizing the analysis of section 1.5 to Lorentz violating theories, we say that the components $\widehat{p}$ and $\overline{p}$ of each momentum are rescaled with the same "weighted velocity" when

$$\widehat{p} \to \lambda\widehat{p}, \qquad \overline{p} \to \lambda^{1/n}\overline{p}.$$

Step $i)$ is modified studying the convergence when any subset of momenta tend to infinity with the same weighted velocity. Whenever a subconvergence fails the counterterms associated with the divergent subdiagrams have to be included. Once the subdivergences are subtracted away, step $ii)$ consists of taking an appropriate number of "weighted derivatives" (see below) with respect to the external momenta, to eliminate the overall divergences. It is easy to check that this procedure automatically takes care of the overlapping divergences.

**Weighted Taylor expansion.** Every Taylor expansion

$$f(\widehat{k}, \overline{k}) = \sum_{u=0}^{\infty}\sum_{v=0}^{\infty} \frac{f_{\nu_1\cdots\nu_u, i_1\cdots i_v}}{u!v!}\widehat{k}_{\nu_1}\cdots\widehat{k}_{\nu_u}\ \overline{k}_{i_1}\cdots\overline{k}_{i_v}$$

can be rearranged into a "weighted Taylor expansion"

$$f(\widehat{k}, \overline{k}) = \sum_{\ell=0}^{\infty}\frac{1}{\ell!}f^{(\ell)}(\widehat{k}, \overline{k}),$$

where

$$f^{(\ell)}(\widehat{k},\overline{k})=\sum_{u=0}^{[\ell/n]}\frac{\ell!}{u!(\ell-nu)!}f_{\nu_1\cdots\nu_u,i_1\cdots i_{\ell-nu}}\widehat{k}_{\nu_1}\cdots\widehat{k}_{\nu_u}\,\overline{k}_{i_1}\cdots\overline{k}_{i_{\ell-nu}}$$

is a weighted homogeneous polynomial of degree $\ell/n$:

$$f^{(\ell)}(\lambda\widehat{k},\lambda^{1/n}\overline{k})=\lambda^{\ell/n}f^{(\ell)}(\widehat{k},\overline{k}).$$

The $\ell$-th weighted derivatives with weight $n$ are the coefficients $f_{\nu_1\cdots\nu_u,i_1\cdots i_{\ell-nu}}$.

In the BPHZ subtraction scheme no regulator is needed since the integral is rendered finite subtracting from its integrand the $\omega$ first terms of its Taylor expansion around zero momentum, where $\omega$ is the superficial degree of divergence of the integral. In dimensional regularization, this procedure corresponds to a suitable schema of subtraction which does not coincide with the minimal subtraction schema.

In Lorentz-violating models, analogous subtraction can be made to carry on the renormalization process, but using the weighted Taylor expansion instead normal Taylor expansion. The overall-subtracted version of an integral whose weighted degree of divergence is $\omega$ reads

$$\int\frac{\mathrm{d}^{L\widehat{D}}\widehat{p}}{(2\pi)^{L\widehat{D}}}\frac{\mathrm{d}^{L\overline{D}}\overline{p}}{(2\pi)^{L\overline{D}}}\left[Q(\widehat{p},\overline{p};\widehat{k},\overline{k})-\sum_{\ell=0}^{n\omega}\frac{1}{\ell!}Q^{(\ell)}(\widehat{p},\overline{p};\widehat{k},\overline{k})\right],$$

where $Q^{(\ell)}$ denotes the $\ell$-th homogeneous polynomial of the weighted Taylor expansion of $Q$ in $\widehat{k},\overline{k}$. In this procedure, subdivergences are systematically subtracted from integrals using a suitable subtraction algorithm.

## 4.10 Non-Relativistic Theories

Non-relativistic theories can be studied along the same lines. The action contains only a single time derivative $\widehat{\partial}$,

$$\mathcal{L}=\overline{\varphi}\left(i\widehat{\partial}+\frac{\overline{\triangle}}{2m}+\xi\frac{\overline{\triangle}^2}{m^2}+\cdots\right)\varphi+\zeta\overline{\varphi}^2\overline{\triangle}\varphi^2+\cdots+\lambda(\overline{\varphi}\varphi)^2+\cdots$$

so the theory is more divergent. The dimensional-regularization is not easy to use, since there is no simple way to continue the single-derivative term $\overline{\varphi}\widehat{\partial}\varphi$ to complex dimensions. Thus we assume an ordinary cut-off regularization.

The propagator is defined by the term $\overline{\varphi}\widehat{\partial}\varphi$ plus the lagrangian quadratic term with the highest number of $\overline{\partial}$-derivatives, say $n$,

$$\mathcal{L}_{\text{free}} = \overline{\varphi}\left(i\widehat{\partial} + \frac{\overline{\partial}^n}{\Lambda_L^{n-1}}\right)\varphi,$$

with $n$ a even positive integer (to have rotational invariance). For the purposes of renormalization, the other quadratic terms, if present, can be treated perturbatively, as explained in section 4.2. Thus the non-relativistic propagator is the inverse of a homogeneous weighted polynomial of degree 1 and weight $n$. Hence, all the PC analysis is the same as purely fermionic theories, being possible to map each Lorentz-violating purely fermionic theory into a non-relativistic model. For instance, in for $\text{đ} = 2$ we have a family of homogeneous models in $d = n + 1$ dimensions,

$$\mathcal{L}_{(1,n)_n} = \overline{\varphi}i\widehat{\partial}\varphi + \frac{1}{\Lambda_L^{n-1}}\overline{\varphi}\overline{\partial}^n\varphi + \frac{\lambda}{4}(\overline{\varphi}\varphi)^2, \tag{4.37}$$

which is analogous to the four-fermionic model with $(1,3)_3$ splitting, (4.11).

The generalization of (4.12) to non-relativistic theories is, for $\text{đ} = 3/2$ in $d = m + 1$ dimensions reads

$$\mathcal{L}_{(1,m)_{2m}} = \overline{\varphi}i\widehat{\partial}\varphi + \frac{1}{\Lambda_L^{2m-1}}\overline{\varphi}\overline{\partial}^{2m}\varphi + \frac{\lambda_6}{36}(\overline{\varphi}\varphi)^3.$$

if $m$ is odd.

In particular, we see that there exist four-dimensional ($m = 3$) non-relativistic renormalizable $\varphi^6$-theories. If $m$ is even we must include additional vertices,

$$\mathcal{L}_{(1,m)_{2m}} = \overline{\varphi}i\widehat{\partial}\varphi + \frac{1}{\Lambda_L^{2m-1}}\overline{\varphi}\overline{\partial}^{2m}\varphi + \sum_i \frac{\lambda_i}{4}[\overline{\partial}^m\overline{\varphi}^2\varphi^2]_i + \frac{\lambda_6}{36}(\overline{\varphi}\varphi)^3.$$

## 4.11 Invariants

When a symmetry is broken, the theory admits a new set of invariants. For example, a Lorentz invariant made of the contraction of two vectors generates two new invariants $A_\mu B^\mu \to A_{\widehat{\mu}}B^{\widehat{\mu}}, A_{\overline{\mu}}B^{\overline{\mu}}$ in a Lorentz-violating model, namely the scalar product defined in each subspace. Only a particular combination of them is Lorentz invariant. With spinor fields is not simple to make such splitting since their components do not correspond to a specific subspace. In other words, every spinor changes under any Lorentz transformation. This is not the case of vectors; in the above example the vectors $A^{\widehat{\mu}}$ and $B^{\widehat{\mu}}$ do not change under a rotation in the $M_{\overline{D}}$ submanifold.

We can find the invariants made of spinor or vector fields studying the remaining symmetries. In Lorentz-invariant models we can form invariants thanks to the matrices $\eta$ and $\gamma_0$ such that

$$\Lambda^T \eta \Lambda = \eta, \qquad \Lambda^\dagger_{1/2} \gamma_0 \Lambda_{1/2} = \gamma_0, \tag{4.38}$$

where $\Lambda$ and $\Lambda_{1/2}$ are representations of Lorentz transformations of spin 1 and 1/2 respectively. For instance, scalar bilinear quantities are

$$A_\mu B^\mu = A_\mu B_{\mu'} \eta^{\mu\mu'}, \qquad \overline{\psi}\psi = \psi^\dagger \gamma_0 \psi.$$

We look for two matrices $\tilde{\eta}$ and $\tilde{\gamma}_0$, with $\tilde{\eta}$ real and symmetric and $\tilde{\gamma}_0$ hermitian such that they satisfy relations as (4.38) but only for the remaining symmetries. For example, for a $(2,2)_n$ splitting, $\Lambda$ should be replaced by a (t-x)- boost or a (y-z) -rotation, or a combination of them. Clearly, for vectors this analysis leads to the invariants mentioned above.

Define as usual $\overline{\psi} = \psi^\dagger \gamma_0$, and $X_{(\widehat{d},\overline{d})} = \gamma_0 \tilde{\gamma}_0$, recalling that $(\gamma_0)^2 = 1$. The fermionic scalar bilinear in $d = \widehat{d} + \overline{d}$ dimensions is $\overline{\psi} X_{(\widehat{d},\overline{d})} \psi$. In the $(2,2)_n$ splitting, such matrix is

$$X_{(2,2)} = a \begin{pmatrix} c^* I & 0 \\ 0 & c\, I \end{pmatrix} + a \begin{pmatrix} d^* \sigma_1 & 0 \\ 0 & d\, \sigma_1 \end{pmatrix},$$

where $a$ , $c$ , $d$ are constants. In order to normalize $\left(X_{(2,2)}\right)^2 = 1$, we require

$$cd = 0, \quad (ac)^2 = 1, \quad (ac^*)^2 = 1, \quad (ad)^2 = 1, \quad (ad^*)^2 = 1.$$

Without loss of generality we choose $|a| = 1$. Then $a,\ c,\ d$ are all real or all imaginary. The possible independent values for $X_{(2,2)}$ are

$$\begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix}, \begin{pmatrix} -I & 0 \\ 0 & I \end{pmatrix}, \begin{pmatrix} -\sigma_1 & 0 \\ 0 & \sigma_1 \end{pmatrix}, \begin{pmatrix} \sigma_1 & 0 \\ 0 & \sigma_1 \end{pmatrix},$$

where $\sigma_i$ are the Pauli matrices and $I$ is the 2x2 identity matrix. The first two matrices are the 4x4 identity matrix and $\gamma_5 = i\gamma_0\gamma_1\gamma_2\gamma_3$, which form Lorentz-invariant bilinears. The other two matrices correspond to

$$\widehat{\gamma}_5 = \gamma_0\gamma_1 = \begin{pmatrix} -\sigma_1 & 0 \\ 0 & \sigma_1 \end{pmatrix}, \qquad \overline{\gamma}_5 = i\gamma_2\gamma_3 = \begin{pmatrix} \sigma_1 & 0 \\ 0 & \sigma_1 \end{pmatrix},$$

namely the matrices $\gamma_5$ of each subspace. Some of the fermionic scalar invariants constructed with these matrix violate some of the discrete symmetries: parity ($P$), time reversal ($T$) or charge conjugation ($C$), as showed in the following table

| | $P$ | $T$ | $C$ | $CPT$ |
|---|---|---|---|---|
| $i\overline{\psi}\widehat{\gamma}_5\psi$ | $-1$ | $+1$ | $-1$ | $+1$ |
| $\overline{\psi}\overline{\gamma}_5\psi$ | $+1$ | $-1$ | $-1$ | $+1$ |

as can be verified using the commutators and anticonmmutators

$$\begin{aligned} [\widehat{\gamma}_5, \gamma_{\overline{\mu}}] &= 0, \qquad \{\widehat{\gamma}_5, \gamma_{\widehat{\mu}}\} = 0, \\ [\overline{\gamma}_5, \gamma_{\widehat{\mu}}] &= 0, \qquad \{\overline{\gamma}_5, \gamma_{\overline{\mu}}\} = 0. \end{aligned}$$

Analogously, in the $(1,3)_n$ splitting, new invariants can be constructed using the matrices

$$\widehat{\gamma}_5 = \gamma_0 = \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix}, \qquad \overline{\gamma}_5 = \gamma_1\gamma_2\gamma_3 = i\begin{pmatrix} 0 & -I \\ I & 0 \end{pmatrix}.$$

The scalar fermionic Lorentz invariants behave under the discrete symmetries as

| | $P$ | $T$ | $C$ | $CPT$ |
|---|---|---|---|---|
| $\overline{\psi}\widehat{\gamma}_5\psi$ | $+1$ | $+1$ | $-1$ | $-1$ |
| $\overline{\psi}\overline{\gamma}_5\psi$ | $-1$ | $+1$ | $+1$ | $-1$ |

while in the $(3,1)_n$ splitting, the $\widehat{\gamma}_5$ and $\overline{\gamma}_5$ matrices are

$$\widehat{\gamma}_5 = i\gamma_0\gamma_1\gamma_2 = \begin{pmatrix} 0 & \sigma_3 \\ \sigma_3 & 0 \end{pmatrix}, \qquad \overline{\gamma}_5 = i\gamma_3 = i\begin{pmatrix} 0 & \sigma_3 \\ -\sigma_3 & 0 \end{pmatrix},$$

and the discrete symmetries table reads

| | $P$ | $T$ | $C$ | $CPT$ |
|---|---|---|---|---|
| $\overline{\psi}\widehat{\gamma}_5\psi$ | $+1$ | $-1$ | $+1$ | $-1$ |
| $i\overline{\psi}\overline{\gamma}_5\psi$ | $-1$ | $-1$ | $+1$ | $-1$ |

For the splitting $(1,3)_n$ or $(3,1)_n$ the commutation relations are

$$\begin{aligned} [\widehat{\gamma}_5, \gamma_{\widehat{\mu}}] &= 0, \qquad \{\widehat{\gamma}_5, \gamma_{\overline{\mu}}\} = 0, \\ [\overline{\gamma}_5, \gamma_{\overline{\mu}}] &= 0, \qquad \{\overline{\gamma}_5, \gamma_{\widehat{\mu}}\} = 0. \end{aligned}$$

The values in the tables are calculated [44], for an interaction of type $\overline{\psi}X\psi$, observing if $X$, $X^*, X^T$ satisfy the following commutators or anticommutators,

$$\begin{aligned} P &: \qquad [X, \gamma_0]_\pm = 0 \\ T &: \qquad [X^*, \gamma_1\gamma_3]_\pm = 0 \\ C &: \qquad [X^T, \gamma_0\gamma_2]_\pm = 0 \end{aligned} \tag{4.39}$$

where $X^*$ and $X^T$ represent the complex conjugate and the transpose of $X$. The signs $(+)$ and $(-)$ represent the anticommutator and the commutator respectively. The change under each symmetry corresponds to the opposite sign for which the respective equality in (4.39) is verified. For example, if $X$ commutates with $\gamma_0$, namely $[X, \gamma_0]_- = 0$, the interaction $\overline{\psi}X\psi$ is even under parity.

The results obtained for the CPT symmetry can be verified using the CPT theorem [45] that states that an hermitian interaction changes under the CPT symmetry as $(-1)^s$, where $s$ is the number of Lorentz indices in the operator. Assuming that the theory is CPT-symmetric, the bilinears with a odd number of Lorentz indices can not be generated by renormalization. Other interactions with an even number of Lorentz indices could be generated if they respect the other symmetries present in the theory.

It is possible to construct other tensor quantities using the Levi-Civita tensor of each subspace, defined as

$$\begin{aligned} \widehat{\varepsilon}^{\widehat{\mu}_1 \ldots \widehat{\mu}_{\widehat{d}}} &= \begin{cases} +1 & \text{if } \widehat{\mu}_1 \ldots \widehat{\mu}_{\widehat{d}} \text{ is an even permutation of } \{0, 1, \ldots \widehat{d} - 1\}, \\ -1 & \text{if } \widehat{\mu}_1 \ldots \widehat{\mu}_{\widehat{d}} \text{ is an odd permutation of } \{0, 1, \ldots \widehat{d} - 1\}, \end{cases} \\ \overline{\varepsilon}^{\overline{\mu}_1 \ldots \overline{\mu}_{\overline{d}}} &= \begin{cases} +1 & \text{if } \overline{\mu}_1 \ldots \overline{\mu}_{\overline{d}} \text{ is an even permutation of } \{\widehat{d}, \ldots \widehat{d} + \overline{d}\}, \\ -1 & \text{if } \overline{\mu}_1 \ldots \overline{\mu}_{\overline{d}} \text{ is an odd permutation of } \{\widehat{d}, \ldots \widehat{d} + \overline{d}\}, \end{cases} \end{aligned}$$

For instance, in the $(1,3)_n$ splitting,

$$\overline{\gamma}_5 = \frac{1}{3!}\overline{\varepsilon}^{\mu\nu\sigma}\gamma_\mu\gamma_\nu\gamma_\sigma, \qquad \overline{\gamma}^{\mu\nu} \equiv \widehat{\varepsilon}^{\mu\nu\sigma}\gamma_\sigma\gamma_{5.}$$

Since the transformation of the combination $\overline{\psi}X_{(\widehat{d},\overline{d})}$ is $\overline{\psi}X_{(\widehat{d},\overline{d})} \to \overline{\psi}X_{(\widehat{d},\overline{d})}\left(\Lambda_{1/2}\right)^{-1}$, other (vector, tensor) invariants of higher dimensionality can be easily constructed, for instance,

$$\overline{\psi}X_{(\widehat{d},\overline{d})}\gamma^\mu\psi, \qquad \overline{\psi}X_{(\widehat{d},\overline{d})}\sigma^{\mu\nu}\psi, \qquad \overline{\psi}X_{(\widehat{d},\overline{d})}\gamma^\mu\psi, \qquad \overline{\psi}X_{(\widehat{d},\overline{d})}\gamma^\mu\gamma_5\psi, \qquad \overline{\psi}X_{(\widehat{d},\overline{d})}\gamma_5\psi.$$

# Conclusions

Non-renormalizable models are normally excluded as valid candidates to represent physical interactions even when they do not imply a violation of any fundamental physical principle. They can be certainly used as effective models which are good for most practical purposes, but unable to suggest new physics beyond them. The adjective "non-renormalizable" is not absolute, it only indicates our incapacity to remove all the infinities that appear in perturbative calculations in quantum field theories, and usually refers to Power Counting. It is a fact that some models considered as non-renormalizable by the PC criteria could be renormalized through some special procedure. On the other hand, Power Counting is commonly trusted because it had guided the construction of the Standard Model, indicating for example that the non-renormalizable four-fermions interaction was only an effective description of the weak interaction, and leading to the discover of the intermediate vector bosons. In view of these elements, it is almost mandatory to direct some efforts in the search of a criterion to extend or supersede Power Counting as classification tool. The final version of this principle should leave room also to quantum gravity and new physics beyond the Standard Model.

In this work we have first examined a general framework in which a wide class of PC non-renormalizable models can be renormalized by a redefinition of fields, masses and a finite set of couplings by means of a RG-consistent reduction of couplings. The infinitely many terms in the lagrangian could be regarded as consequence of writing it in "a wrong way", for example in an inappropriate expansion or basis. It is remarkable that the conditions that indicate which theories are reducible are not too restrictive. Moreover, even in the cases where the reduction is not doable because of the failure of some conditions, it is useful to introduce a new independent constant for each reduction failure, because low-order calculations can be made with a relatively small number of couplings. All-order theorems and the infinite reduction can be carried out completely using some criteria based on dimensional regularization. In this scheme, it becomes necessary to perform also the bare reduction, namely the relation between bare couplings, which is not trivial. The equivalence with other regularizations is proved.

It is showed that the invertibility conditions can be made more precise in certain circumstances, for example in the absence of three-leg marginal couplings. The leading-log approximation is solved explicitly and contains sufficient information for the existence and uniqueness of the reduction to all orders. One of the main features of the models where the infinite reduction can be applied is that the strictly-renormalizable subsector of the theory must be fully interacting, because the reduction is perturbatively meromorphic in the marginal coupling. In a first approach we have considered massless models without relevant parameters, but it is also shown how to include them in a perturbative manner. The reduction can be applied also to theories with several marginal couplings without important modifications. In quantum gravity, an infinite reduction of couplings could be tested, but differently from the cases studied, dimensional analysis does not constrain the form of the reduction. This means that the reduction contains an arbitrary function of the dimensionless combination of the Newton constant and the cosmologic constant.

In Chapter 3 we have studied the renormalizability of a more specific class of (PC) non-renormalizable theories. In these models, where quantum fields interacts with classical gravity field, the lagrangian contains a finite set of matter operators of dimensionality equal to or less than four coupled with purely gravitational operators of dimensionality arbitrarily high. These theories are characterized by an acausal behavior at high energies, which is not a problem in principle, since semiclassical theories are known to have this kind of problems intrinsically. The renormalizability is proved using a map that relates its own renormalization with the renormalization of a physically different theory that presents no causality violation (other than the one relative to semiclassical models) but instabilities due to higher time-derivative in its kinetic terms. As consequence of the map, the metric, although classical, is renormalized and thus it is running.

The renormalization is achieved redefining a finite number of couplings plus field redefinitions, without introducing higher-derivative kinetic terms in the gravitational sector. We have studied as a specific example, the Yang-Mills model with an $R$-dependent gauge coupling. The perturbation induces extra gravitational terms, one of which, $RR_{\alpha\beta\mu\nu}R^{\alpha\beta\mu\nu}$ is not squarely proportional to the Ricci tensor. General formulas for the beta-functions of the vertices of dimensionality six are derived. They are expressed in terms of the trace-anomaly coefficients of the matter sector embedded in curved background. The renormalization-group flow depends on the scalar curvature of the spacetime. These results can be extended to all PC renormalizable theories with $R$-dependent coupling constants coupled to classical gravity. Since the map is classical, the conclusions apply only if it is verified that gravity is not a quantum interaction, but could motivate research pointing to find a quantum version of the map or causality violations in quantum gravity.

In Chapter 4, we considered theories that contain irrelevant operators, but are certainly renormalizable thanks to a modified kinetic term that renders the propagators more convergent at high energies. This is achieved raising the order of a subset of space derivatives, with the consequent breaking of the Lorentz symmetry. Perturbative unitarity is preserved, since no higher time-derivative is generated by renormalization. For this kind of theories, we propose a modified version of Power Counting called *weighted Power Counting*, which resembles usual power Counting but with some quantities redefined. The set of renormalizable theories is enlarged, but is still finite, so the weighted PC is useful as classification criterion. The spacetime manifold could be split eventually in many submanifolds. Some of the theories studied present classically a weighted scale-symmetry (where different coordinates are weighted differently) which is anomalous at quantum level. Lorentz-violating models could find applications in high-energy physics, effective field theory, nuclear physics and the theory of critical phenomena. An interesting generalization to be investigated is the application of the spacetime splitting to improve the renormalizability of quantum gravity.

# Appendix A Analytic properties of solutions of a differential equation

For the discussion of Chapter 2, it is convenient to have at our disposal some tools to study systematically the analytic behavior of the solutions of some differential equations. There, the analysis of certain solutions plays a central role in the reduction of couplings in renormalizable models as in the infinite reduction as well . Although some of the equations found can be solved in a closed expression (see [11]), the same conclusions can be obtained in a unified frame that include both renormalizable and non-renormalizable models in physical or extended spacetime. The method explained below is simply a series expansion of the solution of a generic differential equation for $f(x)$:

$$A + B\ f + C\ f^2 + x\ P(f,x) = x\frac{\mathbf{d}f}{\mathbf{d}x}\left(D + x\ Q(f,x)\right), \tag{A.1}$$

with $A, B, C, D$ non-vanishing constants. $P(f,x)$ and $Q(f,x)$ are polynomials or series in $f$ and $x$.

The strategy starts writing a particular solution of (A.1)

$$f_s(x) = \sum_{i=0}^{\infty} c_i x^i \tag{A.2}$$

as a series and solve for $c_i$ matching the coefficients of the same power of $x$ when placing (A.1) in (A.1). If it is possible to find univocally the value of all of them, the equation admits an unique analytic particular solution. If it is not, it is a signal that there is no analytic solution or there are infinitely many (the general solution is analytic). Proceeding in this way, we found that in principle two series $f_{s+}$ and $f_{s-}$ exist, which coefficients are

$$\begin{aligned} c_{0\pm} &= \frac{-B \pm \sqrt{\Delta}}{2C}, \qquad \text{with} \qquad \Delta = B^2 - 4AC, \\ c_{n\pm} &= \frac{K_n}{nD \mp \sqrt{\Delta}} = \frac{K_n/D}{n \mp r}, \qquad \text{for} \qquad n > 0, \qquad r \equiv \frac{\sqrt{\Delta}}{D}. \end{aligned} \tag{A.3}$$

$K_n$ is a polynomial in $c_m$ with $m < n$ and the other constants appearing in $P(f,x)$ and $Q(f,x)$. The discriminant $\Delta$ must be greater than zero because only real solutions are physically meaningful. If the denominator in the expression for $c_{n\pm}$ does not vanish for any $n$, both $f_{s+}$ and $f_{s-}$ are the unique analytic solutions. Therefore, depending on the value of $r$, three cases can occur:

*i)* $r > 0$:

The series $f_{s-}$ exists (the denominator can not be zero). If $r$ is a positive integer $r = \widehat{n} \in \mathbb{N}_+$, the coefficient $c_{\widehat{n}+}$ remains undetermined. Instead of $f_{s+}$ there is a solution

$$\sum_{i=0}^{\widehat{n}} c_{i+} x^i + \sum_{j=\widehat{n}}^{\infty} d_{j+}\ x^j\ \ln^{j-\widehat{n}+1}(x)\,,$$

where $c_{\widehat{n}+}$ is arbitrary and the coefficients $d_{j+}$ depend on $c_{i+}$ and $d_{k+}$, with $k < j$. In the exceptional case where also $K_{\widehat{n}} = 0$, $f_{s+}$ will exist but with $c_{\widehat{n}+}$ arbitrary, namely the general solution is analytic. On the other hand, if $r \notin \mathbb{N}_+$, the denominator never cancels and therefore also a unique $f_{s+}$ exists.

*ii)* $r < 0$:

Analogously to the previous case, $f_{s+}$ exists (its denominator can not cancel). If $-r = \widehat{n} \in \mathbb{N}_+$, the coefficient $c_{\widehat{n}-}$ is undetermined. Instead of $f_{s-}$ we have the non-analytic solution

$$\sum_{i=0}^{\widehat{n}} c_{i-} x^i + \sum_{j=\widehat{n}}^{\infty} d_{j-}\ x^j\ \ln^{j-\widehat{n}+1}(x)\,.$$

where $c_{\widehat{n}-}$ is arbitrary and $d_{j-}$ depends on $c_{i-}$ and $d_{k-}, k < j$. If also $K_{\widehat{n}} = 0$, $f_{s-}$ exists but with $c_{\widehat{n}-}$ arbitrary, thus the general solution is analytic.

If $-r \notin \mathbb{N}_+$, the denominators never cancels, so we have both $f_{s+}$ and $f_{s-}$.

*iii)* $r = 0$:

From (A.3) it is evident that both series coincides. The denominator never cancels, thus the series represents the unique analytic solution $f_s = f_{s+} = f_{s-}$.

The following table summarizes the above conclusions:

| | Conditions | | Analytic solutions |
|---|---|---|---|
| $r>0$ ($D>0$) | $r=\widehat{n}\in\mathbb{N}_+$ | $K_{\widehat{n}}=0$ | $f_{s-}$ and $f_{s+}$ but with $c_{\widehat{n}+}$ arbitrary. |
| | | $K_{\widehat{n}}\neq 0$ | $f_{s-}$ |
| | $r\notin\mathbb{N}_+$ | | $f_{s-}$ and $f_{s+}$. |
| $r<0$ ($D<0$) | $-r=\widehat{n}\in\mathbb{N}_+$ | $K_{\widehat{n}}=0$ | $f_{s+}$ and $f_{s-}$ but with $c_{\widehat{n}-}$ arbitrary. |
| | | $K_{\widehat{n}}\neq 0$ | $f_{s+}$ |
| | $-r\notin\mathbb{N}_+$ | | $f_{s-}$ and $f_{s+}$. |
| $r=0$ ($B^2=4AC$) | | | $f_{s+}\,(=\ f_{s-})\,.$ |

In the infinite reduction, the equations have $C=0$, which simplifies expressions to

$$c_0=-\frac{A}{B},\qquad\qquad c_n=\frac{K_n}{nD-B}=\frac{K_n/D}{n-r},$$

with $r=B/D$. Note that now there is only one series.

Repeating the analysis, we focus on the denominator of $c_n$. Two cases can happen

*i)* $r>0$:

If $r=\widehat{n}\in\mathbb{N}_+$, the denominator is zero, thus there is no solution $f_s$, but

$$\sum_{i=0}^{\widehat{n}} c_i x^i+\sum_{j=\widehat{n}}^{\infty} d_j\ x^j\ \ln(x)^{j-\widehat{n}+1}\,,$$

which is non analytic ($c_{\widehat{n}}$ is arbitrary and $d_j$ depends on $c_i$ and $d_k, k<j$). Again, if $K_{\widehat{n}}=0$, there is one-parameter family of analytic solutions. If $r\notin\mathbb{N}_+$, the series solution $f_s$ is unique.

*ii)* $r<0$:

For all $n>0$ the denominator is non-vanishing, thus $f_s$ always exists. This is the situation where $B$ and $D$ have opposite signs.

If the expansion of $f(x)$ starts at some power $q$, the condition of existence of the series are modified. Writing

$$f(x) = x^q \tilde{f}(x), \qquad \tilde{f}(x) = \sum_{i=0}^{\infty} c_i x^i, \tag{A.4}$$

replacing in (A.1),

$$A + x^q (B - qD)\ \tilde{f} + x^{q+1}\ P'(\ \tilde{f}, x) = x^{q+1} \frac{\mathbf{d}\tilde{f}}{\mathbf{d}x} \Big( D + x^{q+1}\ Q'(\ \tilde{f}, x) \Big),$$

it is clear that the coefficients are

$$c_n = \frac{K_n}{(n+q)\,D - B} = \frac{K_n/D}{n - r + q}, \qquad n \geq 0.$$

hence the invertibility condition changes into

$$r - q \notin \mathbb{N}$$

It is worth mentioning that in the cases where there is an unique analytic solution, as (A.2), the general solution has the form

$$\begin{aligned} f_{\pm}(x) &= \sum_{i=0}^{\infty} c_{i\pm} x^i + \sum_{n=1}^{\infty} \sum_{m=0}^{\infty} d_{mn\pm}\ \xi^n\ x^{m \pm nr}, \ \text{ with } d_{01\pm} = 1, \text{ and } r = \frac{\sqrt{\Delta}}{D}, \\ f(x) &= \sum_{i=0}^{\infty} c_i x^i + \sum_{n=1}^{\infty} \sum_{m=0}^{\infty} d_{mn}\ \xi^n\ x^{m+nr}, \ \text{ with } d_{01} = 1 \qquad \text{and} \qquad r = \frac{B}{D}, \end{aligned} \tag{A.5}$$

for $C \neq 0$ and $C = 0$ respectively. $\xi$ is arbitrary and all $d_{mn\pm}$ and $d_{mn}$ are iteratively calculable. From (A.5) we see that when $\pm r$ is a positive integer, the **general solution** is analytic, this is the reason why the coefficients in (A.2) are not uniquely determined.

Normally the non-renormalizable models have several operators by level. When operator mixing occurs, the equation (A.1) is replaced by a system of equations that can be studied in the same manner. Adding indices, we consider a set of functions $f(x)^J, P^I(f,x), Q^{IJ}(f,x)$ and constants $A^J, B^{IJ}$ and $D^{IJ}$ such as

$$A^I + B^{IJ}\ f^J + x\ P^I(f,x) = x \frac{\mathbf{d}f^J}{\mathbf{d}x} \left( D^{IJ} + x\ Q^{IJ}(f,x) \right), \tag{A.6}$$

with $I, J = 1 \dots N_0$. $N_0$ is the number of operators mixed at the level. We look for analytic solutions

$$f_s^J(x) = \sum_{i=0}^{\infty} c_i^J x^i.$$

Now the coefficients are obtained from the inversion of certain matrices

$$c_0^I = -\left(B^{-1}\right)^{IJ} A^J, \qquad c_n^I = \left[(nD - B)^{-1}\right]^{IJ} K_n^J,$$

with $K_n^J$ defined analogously to $K_n$.The invertibility condition in this case comes from requiring $B^{IJ}$ and $(nD - B)^{IJ}$ to be invertible matrices. The latter is equivalent to $\det(n\delta - r) \neq 0$, where $\delta$ is the identity matrix and $r \equiv BD^{-1}$. If the matrix $r$ has eigenvalues $r_{i,}$ the matrix $(n\delta - r)$ has eigenvalues $(n - r_i)$, therefore the invertibility conditions are translated to $\det(B) \neq 0$ and $r$ having no positive integer eigenvalues.

# Appendix B Explicit perturbative map

It is a known fact that terms in the lagrangian proportional to the equations of motion can be removed with a field transformation, at least to first order. For some class of theories, the HD kinetic term is proportional to the square of the equation of motion of the unperturbed (low-derivative) action . In these cases the perturbative transformation can be carried to all orders [9].

Write the HD action $S^{\mathrm{HD}}[\phi]$ as the sum of two parts

$$S^{\mathrm{HD}}[\phi] = S[\phi] + S_i F_{ij} S_j,$$

where $S_i = \frac{\delta S}{\delta \phi_i}$ is the equation of motion for $\phi_i$, and $F_{ij}$ is in general a differential operator that can depend on $\phi$. $S_{i_1 \dots i_n}$ represents the $n-th$ functional derivative of $S$ with respect to $\phi$. For instance, $S_{ijk} = \frac{\delta^3 S}{\delta\phi_i \delta\phi_j \delta\phi_k}$. These indices have the deWitt meaning explained in section 3.7. We are looking for a redefinition $\phi' = \phi'(\phi)$ such that

$$S^{\mathrm{HD}}[\phi] = S[\phi']. \tag{B.1}$$

Writing

$$\phi_i' = \phi_i + \Delta_{ij} S_j,$$

replacing it in (B.1), and performing a Taylor expansion, we get

$$\begin{aligned} S[\phi'] &= S[\phi + \Delta S] \\ S[\phi] + SFS &= S[\phi] + S\Delta S + \frac{1}{2} S\Delta S\Delta S + \frac{1}{3!} S_{klm} \left(\Delta S\right)_k \left(\Delta S\right)_l \left(\Delta S\right)_m + ..., \end{aligned} \tag{B.2}$$

where for clarity some indices have been omitted using the matrix notation. Both $F$ and $\Delta$ are assumed symmetric without loss of generality.

One solution for $\Delta$ can be obtained solving recursively the equation

$$\Delta_{ij} = F_{ij} - \frac{1}{2} \left(\Delta S \Delta\right)_{ij} - \frac{1}{3!} S_{klm} \Delta_{ki} \Delta_{lj} \left(\Delta S\right)_m + ... \tag{B.3}$$

Obviously this solution is not unique. Other solutions can be obtained replacing $F_{ij}$ by $F_{ij} + \tilde{F}_{ij}$ in (B.3), where $\tilde{F}_{ij}$ satisfies

$$S_i \tilde{F}_{ij} S_j = 0.$$

Expressed in orders of $F$, $\Delta = \Delta^{(1)} + \Delta^{(2)} + \Delta^{(3)} + \Delta^{(4)} + ...$ the solution of (B.3) reads

$$\begin{aligned} \Delta_{ij}^{(1)} &= F_{ij} \\ \Delta_{ij}^{(2)} &= -\frac{1}{2} \left(FSF\right)_{ij} \\ \Delta_{ij}^{(3)} &= -\frac{1}{3!} S_{klm} F_{ki} F_{lj} \left(FS\right)_m + \frac{1}{2} \left(FSFSF\right)_{ij} . \end{aligned} \tag{B.4}$$

A practical problem found in this approach is that being $S$ an integral, only its first functional derivative is a tensor. All successive derivatives appearing in (B.2) are bi-, tri-tensor densities and so on. They have several Dirac delta distribution, and although this fact is not an impediment, it is cumbersome working with them directly [73]. In the situation where we use the map, there is no need of doing so, because in all our expressions they are integrated. Thus, a clearer and simpler way to treat the functional derivatives of $S$ is to change them into variations as follows. We illustrate the method with functions and normal derivatives. For small $a_i$,

$$\frac{\mathrm{d}f}{\mathrm{d}x_i}a_i = \delta f|_a ,$$

where $\delta(f)|_a$ represents the variation of $f(x)$ under the variation $\delta x_i = a_i$. In this way, a Taylor expansion can be written in terms of variations,

$$\begin{aligned} f(x+a) &= f(x) + \frac{\mathrm{d}f}{\mathrm{d}x_i}a_i + \frac{1}{2}\frac{\mathrm{d}^2 f}{\mathrm{d}x_i \mathrm{d}x_j}a_i a_j + ... \\ &= f(x) + \delta f|_a + \frac{1}{2}\,\delta\left(\delta f|_a\right)|_a + ... \end{aligned}$$

However, in the case we are interested in, the variation itself depends on $x$, $a(x)$. Taking as example the second-order term,

$$\begin{aligned} \frac{\mathrm{d}^2 f}{\mathrm{d}x_i \mathrm{d}x_j}a_i(x)\,a_j(x) &= \frac{\mathrm{d}}{\mathrm{d}x_j}\left(\frac{\mathrm{d}f}{\mathrm{d}x_i}a_i(x)\right)a_j(x) - \frac{\mathrm{d}f}{\mathrm{d}x_i}\frac{\mathrm{d}a_i}{\mathrm{d}x_j}a_j(x) \qquad \text{(B.5)} \\ &= \delta\left(\delta f|_{a(x)}\right)\Big|_{a(x)} - \delta f|_{\delta a|_{a(x)}} , \end{aligned}$$

and similarly for all orders. It is possible to arrange terms in the Taylor expansion to facilitate the computation of variations. Write

$$f(x+a(x)) = E_0 + E_1 + E_2 + E_3 + ...$$

where

$$\begin{aligned} E_0 &= f, \\ E_1 &= \delta E_0|_{a(x)} , \\ E_2 &= \frac{1}{2}\left[\delta E_1|_{a(x)} - \delta E_0|_{\delta a|_{a(x)}}\right], \\ E_3 &= \frac{1}{3}\left[\delta E_2|_{a(x)} - \delta E_1|_{\delta a|_{a(x)}} + \delta E_0|_{\delta a|_{\delta a|_{a(x)}}}\right], \text{ etc.} \end{aligned}$$

As we can see, at each order only one new variation must be calculated, the variation of the previous term.

To obtain the perturbative map we can use this kind of Taylor expansion in (B.2), or directly transform the derivatives into variations as in (B.5) at the final stage (B.4).

**Gravity.** A minor consideration should be made to apply the method to gravity. Due to its diffeomorphism invariance, the action is the spacetime integral of a scalar quantity times the density scalar $\sqrt{-g}$, thus the functional derivative $S_i = \frac{\delta S}{\delta g_i}$ is not a tensor. Neither $\Delta_{ij}, F_{ij}$ or $S_{ij}$ are bitensor. But is easy to define tensorial quantities from them

$$\tilde{S}_i = \frac{S_i}{\sqrt{-g}}, \qquad \tilde{\Delta}_{ij} = \sqrt{-g}\Delta_{ij}, \qquad \tilde{F}_{ij} = \sqrt{-g}F_{ij}, \qquad \tilde{S}_{ij} = \frac{S_{ij}}{\sqrt{-g}}.$$

Consider the Einstein-Hilbert term plus terms quadratic in Ricci tensor

$$S^{\mathrm{HD}} = \frac{1}{2\kappa^2}\int \sqrt{-g}\left(R + aR_{\mu\nu}R^{\mu\nu} + bR^2\right),$$

According to the above nomenclature, the equation of motion for $g_{\mu\nu}$ is

$$S_i = \frac{\delta S}{\delta g_i} = \frac{\delta S}{\delta g_{\mu\nu}(x)} = -\frac{\sqrt{-g}}{2\kappa^2}\left(R^{\mu\nu} - \frac{R}{2}g^{\mu\nu}\right) + \text{boundary term.} \tag{B.6}$$

The boundary term arises because the action contains second derivatives. We consider it vanishing.

The perturbative map is given by (B.4) with the bitensor density

$$F_{ij} = F_{\mu\nu\alpha\beta}(x,y) = \frac{2\kappa^2}{\sqrt{-g}}\left[ag_{\alpha(\mu}\, g_{\nu)\beta} + bg_{\alpha\beta}g_{\mu\nu}\right]\delta(x-y).$$

The orders $a, b, a^2, b^2$,$ab$ is displayed in 3.4. Using Bianchi identities, the number of operators of third order $(a^3, a^2b, ab^2, b^3)$ can be reduced to a minimal basis of 43 operators, 29 of which are not proportional to $g_{a\beta.}$ Organizing them according to the number of derivatives they contain, the third order of the perturbative map is

1/48 times the sum of all 43 terms

| | | |
|---|---|---|
| $x_{4,1}\ \Box^2 R_{\alpha\beta}$ | $x_{2,1}\nabla_\alpha R \nabla_\beta R$ | $x_{0,1}R_{\alpha\beta}R^2_{\mu\nu}$ |
| $x_{4,2}\ \nabla_\alpha\nabla_\beta\Box R$ | $x_{2,2}R\nabla_\alpha\nabla_\beta R$ | $x_{0,2}R^2 R_{\alpha\beta}$ |
| $x_{4,3}\ \Box\nabla_\alpha\nabla_\beta R$ | $x_{2,3}R\nabla_\gamma\nabla_\alpha R_{\gamma\beta}$ | $x_{0,3}RR_\alpha{}^\epsilon R_{\epsilon\beta}$ |
| $x_{4,4}\ \nabla_\gamma\nabla_\alpha\Box R_{\gamma\beta}$ | $x_{2,4}\nabla_\gamma R\nabla_\gamma R_{\alpha\beta}$ | $x_{0,4}R_\alpha{}^\epsilon R_{\epsilon\delta}R_\beta{}^\delta$ |
| $x_{4,5}\Box\nabla_\gamma\nabla_\alpha R_{\gamma\beta}$ | $x_{2,5}R_{\gamma\beta}\nabla_\gamma\nabla_\alpha R$ | |
| $x_{4,6}R_{\alpha\beta}\Box R$ | $x_{2,6}\nabla_\varepsilon R_{\gamma\beta}\nabla_\gamma R_{\alpha\varepsilon}$ | |
| $x_{4,7}R\Box R_{\alpha\beta}$ | $x_{2,7}\nabla_\gamma R_{\varepsilon\beta}\nabla_\gamma R_{\alpha\varepsilon}$ | |
| $x_{4,8}R_{\delta\beta}\Box R_{\alpha\delta}$ | $x_{2,8}\nabla_\gamma R_{\beta\varepsilon}\nabla_\alpha R_{\gamma\varepsilon}$ | |
| $x_{4,9}\nabla_\gamma\nabla_\alpha\nabla_\gamma\nabla_\beta R$ | $x_{2,9}\nabla_\alpha R_{\delta\varepsilon}\nabla_\beta R_{\delta\varepsilon}$ | |
| $x_{4,10}\nabla_\gamma\nabla_\alpha\nabla_\varepsilon\nabla_\gamma R_{\varepsilon\beta}$ | $x_{2,10}R_{\gamma\beta}\nabla_\varepsilon\nabla_\gamma R_{\alpha\varepsilon}$ | |
| $x_{4,11}\nabla_\gamma\nabla_\alpha\nabla_\varepsilon\nabla_\beta R_{\gamma\varepsilon}$ | $x_{2,11}R_{\gamma\beta}\nabla_\varepsilon\nabla_\alpha R_{\gamma\varepsilon}$ | |
| | $x_{2,12}R_{\gamma\delta}\nabla_\delta\nabla_\gamma R_{\alpha\beta}$ | |
| | $x_{2,13}R_{\gamma\delta}\nabla_\delta\nabla_\alpha R_{\gamma\beta}$ | |
| | $x_{2,14}R_{\mu\nu}\nabla_\alpha\nabla_\beta R_{\mu\nu}$ | |

where $\alpha - \beta$ symmetrization is understood. The terms proportional to $g_{\alpha\beta}$ are

| | | |
|---|---|---|
| $y_{4,1}\Box^2 R$ | $y_{2,1}\Box R^2$ | $y_{0,1}R^3$ |
| $y_{4,2}\nabla_\gamma\nabla_\delta\Box R_{\gamma\delta}$ | $y_{2,2}R\Box R$ | $y_{0,2}RR_{\mu\nu}{}^2$ |
| $y_{4,3}\nabla_\gamma\nabla_\delta\nabla_\varepsilon\nabla_\delta R_{\gamma\varepsilon}$ | $y_{2,3}\Box R_{\mu\nu}{}^2$ | $y_{0,3}R_\gamma{}^\epsilon R_{\epsilon\delta}R_\gamma{}^\delta$ |
| $y_{4,4}\nabla_\gamma\nabla_\delta\nabla_\varepsilon\nabla_\gamma R_{\delta\varepsilon}$ | $y_{2,4}R_{\gamma\delta}\Box R_{\gamma\delta}$ | |
| | $y_{2,5}\nabla_\delta R_{\gamma\varepsilon}\nabla_\gamma R_{\delta\varepsilon}$ | |
| | $y_{2,6}R_{\gamma\delta}\nabla_\delta\nabla_\gamma R$ | |
| | $y_{2,7}R_{\gamma\delta}\nabla_\varepsilon\nabla_\delta R_{\gamma\varepsilon}$ | |

The respective factors are

$$
\begin{aligned}
x_{4,1} &= -12a^3 & x_{2,1} &= 2a\left(11a^2+4ab+12b^2\right) & x_{0,1} &= 44a^3\\
x_{4,2} &= -12a\left(a^2+6ab+12b^2\right) & x_{2,2} &= 4a^2\left(5a-6b\right) & x_{0,2} &= -46a^3\\
x_{4,3} &= 24a^2b & x_{2,3} &= -64a^3 & x_{0,3} &= 136a^3\\
x_{4,4} &= 24a^3 & x_{2,4} &= 4a^2\left(11a-2b\right) & x_{0,4} &= -176a^3\\
x_{4,5} &= 24a^3 & x_{2,5} &= -8a^2\left(5a-12b\right) & &\\
x_{4,6} &= 4a^2\left(5a-8b\right) & x_{2,6} &= -8a^3 & &\\
x_{4,7} &= 32a^3 & x_{2,7} &= -88a^3 & &\\
x_{4,8} &= -136a^3 & x_{2,8} &= 96a^3 & &\\
x_{4,9} &= -48a^2b & x_{2,9} &= -44a^3 & &\\
x_{4,10} &= -28a^3 & x_{2,10} &= 32a^3 & &\\
x_{4,11} &= -20a^3 & x_{2,11} &= 144a^3 & &\\
& & x_{2,12} &= 8a^3 & &\\
& & x_{2,13} &= 80a^3 & &\\
& & x_{2,14} &= -40a^3 & &
\end{aligned}
$$

and

$$
\begin{aligned}
y_{4,1} &= 12\left(a^3+11a^2b+36ab^2+36b^3\right) & y_{2,1} &= -\tfrac{11a^3}{2}+5a^2b+2ab^2-12b^3 & y_{0,1} &= a^2\left(11a-2b\right)\\
y_{4,2} &= -12a^2\left(a+2b\right) & y_{2,2} &= a^3+2a^2b-12ab^2+24b^3 & y_{0,2} &= -6a^2\left(7a-2b\right)\\
y_{4,3} &= 10a^2\left(a+2b\right) & y_{2,3} &= 3a^2\left(7a+6b\right) & y_{0,3} &= 8a^2\left(5a-2b\right)\\
y_{4,4} &= 14a^2\left(a+2b\right) & y_{2,4} &= 2a^2\left(5a-2b\right) & &\\
& & y_{2,5} &= -44a^2\left(a+2b\right) & &\\
& & y_{2,6} &= 16ab\left(a+5b\right) & &\\
& & y_{2,7} &= -16a^2\left(4a+5b\right) & &
\end{aligned}
$$

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