The wave equation on black rings and the linear stability of slowly rotating Kerr spacetimes

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by

Gabriele Benomio

Department of Mathematics

Imperial College London

180 Queen's Gate, London SW7 2AZ

United Kingdom

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A Cristina

I certify that this thesis, and the research to which it refers, are the product of my own work, and that any ideas or quotations from the work of other people, published or otherwise, are fully acknowledged in accordance with the standard referencing practices of the discipline. None of the material presented in this thesis is the outcome of work done in collaboration.

Chapter 2 of this thesis is based on the paper

> *The Stable Trapping Phenomenon for Black Strings and Black Rings and its Obstructions on the Decay of Linear Waves* (2018), arXiv:1809.07795v1.

The content of Chapter 3 is unpublished in any form at the time of submission.

This thesis has not been submitted for any other degree or qualification.

Signed: ______________________

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Abstract

The existence of black holes is perhaps the most spectacular prediction of Einstein's classical theory of general relativity. Recent advances in both theoretical and experimental physics, such as the discovery of gravitational waves, have confirmed that black holes are stable, macroscopic objects playing a fundamental role in our universe. On the other hand, modern physical theories demand for a higher dimensional formulation of general relativity, and thus to understand the stability properties of black holes within a wider scenario than the one directly probed by the astrophysical observations. However, the *mathematical* question of whether black holes are stale as solutions to the vacuum Einstein equations

$$\mathrm{Ric}(g) = 0\,,$$

known as the *black hole stability problem*, remains, to large extent, open. The present thesis contributes to the black hole stability problem with two theorems.

Chapter 2 of the thesis considers a family of higher dimensional black holes, known as *black rings*. The potential stability of this family, and the part that physical theories should reserve to them if unstable, have been largely investigated in the physics literature. The main theorem of the chapter is the first mathematically rigorous result suggesting that these black holes are *unstable* to gravitational perturbations. In particular, we establish a logarithmic lower bound for the uniform energy decay rate of scalar linear perturbations on black ring spacetimes.

Chapter 3 of the thesis deals with the *Kerr family* of black holes, which is believed to characterise all the astrophysical stationary black holes. To agree with our physical expectation, the *Kerr stability conjecture* claims that these black holes are *stable* to gravitational perturbations. The content of the chapter represents the first part of work by the author providing the last missing ingredient towards a final proof of the conjecture for the slowly rotating members of the Kerr family. More precisely, we formulate the problem of linear stability of Kerr black holes to gravitational perturbations in a new geometric gauge.

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1

Introduction

The results established in this thesis lie at the interface between the analysis of nonlinear partial differential equations and Einstein's classical theory of general relativity. The thesis is primarily devoted to the study of the dynamical properties of the *vacuum Einstein equations*

$$\mathrm{Ric}(g) = 0\,, \tag{1.1}$$

with a particular focus on the long-time dynamics of the so-called *black hole* solutions.

The general framework underlying the thesis is that of the initial value problem for (1.1). Upon fixing a differentiable structure, the vacuum Einstein equations form a system of *nonlinear* partial differential equations with a Lorentzian manifold $(\mathcal{M}, g)$ as the unknown. The formulation of such an initial value problem, and the proof of its local well-posedness, crucially rely on understanding the system (1.1) as a system of nonlinear *wave* equations in the metric g.

The problems addressed in the thesis start from considering a well-known explicit solution $(\widetilde{\mathcal{M}}, \widetilde{g})$ to the vacuum Einstein equations, arising from initial data $\mathbb{D}$, and investigate a *nonlinear (asymptotic) stability* statement like the following:

Meta-Theorem 1 (Nonlinear stability)**.** *For all initial data sufficiently close to $\mathbb{D}$, the maximal solution $(\mathcal{M}, g)$ to the vacuum Einstein equations* (1.1) *exists globally in time and asymptotically converges to $(\widetilde{\mathcal{M}}, \widetilde{g})$ (or a nearby $(\widetilde{\mathcal{M}}', \widetilde{g}')$).*

A nonlinear stability theorem for (1.1) has been proven by the monumental work of Christodoulou–Klainerman [14] when $(\widetilde{\mathcal{M}}, \widetilde{g})$ corresponds to the trivial solution to (1.1), the Minkowski spacetime. A far more challenging problem is dealing with the case when $(\widetilde{\mathcal{M}}, \widetilde{g})$ has a more complicated geometry, as for black hole solutions. The problem is known as the *black hole stability problem* and lies at the core of this thesis.

The main difficulties in proving stability statements for the Einstein equations arise from both the nonlinear and geometric characters of the equations. As it is often the case in the analysis of partial differential equations, linear problems modelling (1.1) have proved to give insightful information on the nonlinear dynamics. The two classes of linear problems typically addressed are:

(i) Stability of solutions to the linearised Einstein equations: One considers the initial value problem for the linear system of equations obtained from linearising (1.1) relative to a reference solution $(\widetilde{\mathcal{M}}, \widetilde{g})$. The manifold $(\widetilde{\mathcal{M}}, \widetilde{g})$ is the solution to (1.1) whose stability properties are examined and is said to be *linearly stable* if a statement like the following holds:

Meta-Theorem 2 (Linear stability)**.** *Given suitable initial data for the system of linearised Einstein equations around $(\widetilde{\mathcal{M}}, \widetilde{g})$, the (unique and global in time) solution to the system decays in time and asymptotically converges to a precise reference solution (determined from $(\widetilde{\mathcal{M}}, \widetilde{g})$) solving the linearised Einstein equations.*

Understanding the linear stability of a black hole solution is a crucial step towards proving its nonlinear stability, as made clear by recent work of Dafermos–Holzegel–Rodnianski [19] for the case of the Schwarzschild solution. Meta-Theorem 2 typically relies on techniques developed in the easier study of the scalar linear wave equation on $(\widetilde{\mathcal{M}}, \widetilde{g})$, for which the tensorial nature of the Einstein equations is suppressed.

(ii) Uniform energy decay of solutions to the scalar linear wave equation: Since (1.1) is a system of nonlinear wave equations, a natural toy problem to investigate is the initial value problem for the scalar linear wave equation

$$\Box_{\widetilde{g}}\, \phi = 0 \tag{1.2}$$

on a fixed Lorentzian manifold $(\widetilde{\mathcal{M}}, \widetilde{g})$ whose stability properties we are interested in. To hope for nonlinear stability of $(\widetilde{\mathcal{M}}, \widetilde{g})$, the theorem that one aims to prove is a uniform (local) energy decay statement for solutions to (1.2):

Meta-Theorem 3 (Uniform energy decay)**.** *There exists a universal constant $C > 0$ (independent of time) such that, for any solution ϕ to* (1.2) *with smooth, compactly supported initial data prescribed at time $t = 0$, the inequality*

$$\mathcal{E}^{(1)}_{\mathrm{loc}}[\phi](t) \leq C\, \delta(t)\, \mathcal{E}^{(2)}[\phi](0)$$

holds for any $t > 0$, with $\delta(t)$ scalar function such that $\lim_{t\to+\infty} \delta(t) = 0$. The local energy $\mathcal{E}^{(1)}_{\mathrm{loc}}[\phi]$ is defined over an arbitrary compact set and the energy $\mathcal{E}^{(2)}[\phi]$ is a higher order energy involving derivatives of the solution up to second order.

A fast (polynomial) uniform energy decay rate $\delta(t)$ for solutions to (1.2) is a valid indication that one should expect nonlinear stability for $(\widetilde{\mathcal{M}}, \widetilde{g})$. In the context of the black hole stability problem, this ultimately motivates the study of the wave equation on black holes, like the one carried out by Dafermos–Rodnianski–Shlapentokh-Rothman [27] for the Kerr solution.

This thesis contributes to the mathematical study of the black hole stability problem with two theorems. The two results are versions of Meta-Theorem 2 and Meta-Theorem 3 specialised to particular choices of $(\widetilde{\mathcal{M}}, \widetilde{g})$, namely *Kerr black holes* and *black rings*.

Although the two theorems are independent results, they can be conceptually related within the following dichotomy:

STABILITY VS INSTABILITY.

In fact, one of the two results concerns black hole *stability* and constitutes Chapter 3 of the thesis. The other result deals with a particular kind of black hole *instability* and forms Chapter 2 of the thesis. The overarching theme of this thesis is that such a dichotomy can be connected to a second dichotomy:

FOUR DIMENSIONAL BLACK HOLES VS HIGHER DIMENSIONAL BLACK HOLES.

Before further elaborating on this, we remark that, throughout the thesis, we restrict our attention to the *exterior* region of black hole solutions to the vacuum Einstein equations with *zero* cosmological constant. The *interior* region of black holes represents a separate (but, to large extent, connected) story. For the vacuum Einstein equations with *positive* cosmological constant, the *nonlinear* stability of the trivial solution, de Sitter spacetime, and the slowly rotating Kerr–de Sitter black holes has been fully understood [43, 54]. In contrast, for *negative* cosmological constant, the trivial solution, Anti-de Sitter spacetime, has been conjectured to be dynamically *unstable* [16], the same being true for the Kerr–Anti-de Sitter family of black hole spacetimes [59].

1.1 Four dimensional black holes: Stability

The only known family of explicit stationary black hole solutions solving (1.1) is called the *Kerr family*. In fact, the Kerr family is conjectured to be the only family of stationary asymptotically flat black holes and thus to characterise all the possible stationary endstates of black hole dynamics. Such a conjecture, known as the *Final State Conjecture* [73], motivates the expectation that Kerr black hole solutions are nonlinearly stable.

Conjecture (Kerr Stability Conjecture)**.** *Let $(\widetilde{\mathcal{M}}, \widetilde{g})$ be a Kerr black hole solution to the Einstein equations with corresponding initial data $\mathbb{D}$. For all initial data sufficiently close to $\mathbb{D}$, the maximal solution $(\mathcal{M}, g)$ to the Einstein equations* (1.1) *exists globally in time and asymptotically converges to a nearby member of the Kerr family $(\widetilde{\mathcal{M}}', \widetilde{g}')$.*

A mathematical proof of the *Kerr stability conjecture* remains a fundamental open question in the field of hyperbolic partial differential equations. Solving such a conjecture requires a deep understanding of the nonlinear dynamics of Kerr black holes.

Recent work of Dafermos–Holzegel–Rodnianski [19] has made clear that the crucial step to approach the nonlinear stability problem for black holes is to first understand key quantitative aspects of the linear stability problem. Works [19, 20] also demonstrate that how such quantitative statements are achieved is fundamental to then address the nonlinear problem.

Work [19] considers the Schwarzschild subfamily of static black holes and opens the way towards a proof of nonlinear stability of Schwarzschild black holes. The aim of Chapter 3 of the thesis is to provide the first part of a proof of linear stability for stationary, slowly rotating Kerr black holes. This is the crucial ingredient to then be able to address the nonlinear stability problem for slowly rotating Kerr black holes.

Theorem 1.1 (Linear stability of slowly rotating Kerr black holes, Chapter 3)**.** *Given suitable initial data for the system of linearised Einstein equations around a slowly rotating member of the Kerr family* $(\widetilde{\mathcal{M}}, \widetilde{g})$*, the (unique and global in time) solution to the system decays (inverse polynomially) in time and asymptotically converges to a reference linearised Kerr solution.*

Our proof of Theorem 1.1 lies within the framework developed by Dafermos–Holzegel–Rodnianski [19] and Dafermos–Holzegel–Rodnianski–Taylor [20]. In light of the latter work [20], Theorem 1.1 can be viewed as the remaining ingredient to establish nonlinear stability for slowly rotating Kerr black holes in that framework. As a by-product, our approach circumvents some of the technical difficulties of the proof of linear stability for Schwarzschild black holes of [19].

1.2 Higher dimensional black holes: Instability

The Einstein equations can be mathematically formulated in any dimension. *Higher dimensional general relativity*, namely the theory of general relativity in a number of dimensions greater than four, is a major area of research in gravitational physics. The flourishing of this subject has been mainly motivated by the study of unifying theories, such as string theory.

Quite remarkably, and in contrast with the rigidity of the Kerr family, the higher dimensional vacuum Einstein equations admit several families of stationary black hole solutions. A natural question to ask is whether the *new families* of black holes are nonlinearly stable. As for other partial differential equations, it turns out that the stability properties of solutions to the Einstein equations heavily depend on the number of dimensions.

The black hole stability problem in higher dimensions is the subject of Chapter 2 of the thesis. In stark contrast to the conjectured stability of the Kerr family of black holes of Chapter 3, some of the higher dimensional black hole solutions are expected to be *unstable*. A well-known example of unstable solutions are *black rings*, a family of five-dimensional stationary asymptotically flat black holes discovered by Emparan–Reall [34]. Understanding their conjectured instability from the mathematically rigorous perspective would shed light on precise aspects of the Einstein equations that set the four-dimensional formulation apart from other higher dimensions.

The thesis contributes to the study of black ring instabilities with the following theorem.

Theorem 1.2 (Uniform energy decay rate on black rings, Chapter 2)**.** *Let* $(\widetilde{\mathcal{M}}, \widetilde{g})$ *be a slowly rotating black ring solution to the Einstein equations* (1.1)*. Consider solutions to the wave equation* $\Box_{\widetilde{g}}\,\phi = 0$ *arising from smooth, compactly supported initial data prescribed at time* $t = 0$*. Then, for any* $k \in \mathbb{N}$*, there exists a universal constant* $C_k > 0$ *(independent of time) such that*

$$\limsup_{t\to+\infty} \sup_{\phi\neq 0} \, [\log(2+t)]^{2k} \left(\frac{\mathcal{E}^{(1)}_{\mathrm{loc}}[\phi](t)}{\mathcal{E}^{(k+1)}[\phi](0)} \right) > C_k \, ,$$

where the local energy is defined over a suitable compact set and the energy $\mathcal{E}^{(k+1)}[\phi]$ *is a higher order energy involving derivatives of the solution up to order* $k+1$*.*

Logarithmic uniform energy decay of linear waves on slowly rotating black rings follows by a recent generalisation [89] of a classical decay result of Burq [5]. Theorem 1.2 provides a complete, sharp description of such a decay, namely it proves that no uniform energy decay with decay rate faster than logarithmic can possibly hold. Logarithmic decay at the linear level is regarded as slow decay, in that it cannot be directly applied to prove small data global existence for nonlinear wave equations. In this sense, this is the first theorem suggesting the nonlinear instability of black rings.

1.3 Guide to reading the thesis

This thesis consists of three chapters. Chapter 1 serves as an introduction. Chapter 2 investigates the decay of scalar linear waves on black rings. Chapter 3 addresses the linear stability of slowly rotating Kerr black holes to gravitational perturbations. Each of the chapters is self-contained and starts with a brief abstract, an introduction and an overview.

The main results of the thesis are Theorem 2.3 of Chapter 2 and Theorem 3.2, combined with the system of equations of Section 3.7.2, of Chapter 3.

A selective reading of the thesis should go through

* Chapter 1,
* Chapter 2: Sections 2.1, 2.5 and 2.10,
* Chapter 3: Sections 3.1, 3.2, 3.5 and 3.7.2.

2

The stable trapping phenomenon for black rings and its obstructions on the decay of linear waves

The geometry of solutions to the higher dimensional vacuum Einstein equations presents aspects that are absent in four dimensions, one of the most remarkable being the existence of stably trapped null geodesics in the exterior of asymptotically flat black holes. This chapter of the thesis investigates the stable trapping phenomenon for two families of higher dimensional black holes, namely black strings and black rings, and how this trapping structure is responsible for the slow decay of linear waves on their exterior. More precisely, we study decay properties for the energy of solutions to the scalar, linear wave equation

$$\Box_{g_{\text{ring}}} \Psi = 0 \, ,$$

where g_{ring} is the metric of a fixed black ring solution to the five-dimensional vacuum Einstein equations. For a class $\mathfrak{g}$ of black ring metrics, we prove a logarithmic lower bound for the uniform energy decay rate on the black ring exterior $(\mathcal{D}, g_{\text{ring}})$, with $g_{\text{ring}} \in \mathfrak{g}$. The proof generalizes the perturbation argument and quasimode construction of Holzegel–Smulevici [59] to the case of a non-separable wave equation and crucially relies on the presence of stably trapped null geodesics on $\mathcal{D}$. As a by-product, the same logarithmic lower bound can be established for any five-dimensional black string.

Our result is the first mathematically rigorous statement supporting the expectation that black rings are dynamically unstable to generic perturbations. In particular, we conjecture a new *nonlinear* instability for five-dimensional black strings and thin black rings which is already present at the level of scalar perturbations and clearly differs from the mechanism driven by the well-known Gregory–Laflamme instability.

2.1 Introduction

The question of the classical stability of higher dimensional black holes has motivated a lot of recent activity in the physics literature. From our perspective, what is most intriguing is that higher dimensional general relativity is not just a formal extension of the four dimensional theory, but it presents novel mathematical features, the most striking being the failure of the rigidity and exterior stability properties of stationary black hole solutions that one expects in four dimensions. See reviews [36, 60] and references therein.

The crucially new aspects of the higher dimensional theory are already manifest in *five* dimensions, for which a number of explicit families of black holes have been discovered [107, 91, 33, 34, 88, 39, 92, 77, 31]. Indeed, the geometry and classification of solutions to the five-dimensional vacuum Einstein equations is the better understood among all possible higher dimensions, and this is one of the reasons why this chapter focuses on dimension five.

For what will be later discussed, we are particularly interested in three of these families of five-dimensional black holes. The first originates from the somehow straightforward observation that one can produce new solutions by adding a flat direction to a four dimensional black hole [47]. In this way, *black strings* of the form $\mathrm{Schw}_4 \times \mathbb{R}$ and $\mathrm{Kerr}_4 \times \mathbb{R}$ can be constructed, where points along the new extended direction are usually periodically identified. Black strings suffer from a linear instability to gravitational perturbations, the so-called *Gregory–Laflamme instability* [48, 15]. The nonlinear dynamics of such instability has been largely investigated numerically and seems to suggest a possible violation of the weak cosmic censorship [80].

While black string spacetimes are not asymptotically flat, there exists a number of independent families of stationary, asymptotically flat black hole solutions in five dimensions, including *Myers–Perry black holes* [91] and *black rings* [34, 92]. The former are a five-dimensional generalization of the Kerr family to black holes with two planes of rotation and event horizon topology S^3, while black rings form a completely new family, with no analogue in four dimensions.

The most remarkable property of black rings is the non-spherical horizon topology $S^1 \times S^2$. Black rings can be *singly-spinning*, with rotation along the S^1, or *doubly-spinning*, with rotation along both the S^1 and the S^2. The mere existence of these additional solutions seriously questions the possibility to recover any rigidity result in five dimensions (see review [35]). From the stability point of view, both heuristic and numerical works show that every member of this family is affected by linear instabilities to gravitational perturbations [41, 100]. In particular, rings whose radius is much greater than the S^2-radius at the event horizon, often called *thin black rings*, resemble black strings and suffer from Gregory–Laflamme instabilities [32, 62]. The nonlinear, numerical evolution of such instabilities strongly suggests that *black rings are nonlinearly unstable to generic perturbations* and possibly lead to a violation of the weak cosmic censorship conjecture [40].

The aim of this chapter of the thesis is to provide a first mathematically rigorous result in the context of the stability problem for black rings.

2.1.1 The scalar linear wave equation

Some of the main difficulties in the mathematical study of the Einstein equations originate from the nonlinear, and together tensorial, character of the equations. Given the hyperbolicity of the Einstein equations, the scalar, linear wave equation

$$\Box_g \Psi = 0 \tag{2.1}$$

on a *fixed* Lorentzian background manifold $(\mathcal{D}, g)$ is a preliminary (and, in principle, simpler) toy model to consider. Typically, one chooses the spacetime $(\mathcal{D}, g)$ whose nonlinear stability is under investigation and studies properties of solutions to (2.1), such as *uniform boundedness* and *decay in time* of the *energy* associated to Ψ.[1] Uniform boundedness and sufficiently strong decay of the energy at the linear level are typically essential to hope for nonlinear stability of $(\mathcal{D}, g)$.

In this regard, we are ultimately interested in *uniform* energy statements of the form

> *There exists a universal constant $C > 0$ (independent of time) such that, for any given smooth, compactly supported initial data*[2] *at time $t = 0$, solutions to* (2.1) *satisfy*
>
> $$\begin{aligned} &E[\Psi](t) \leq C E[\Psi](0) && \textit{(uniform boundedness)} \\ &E_{loc}[\Psi](t) \leq C\delta(t) E[\Psi](0) && \textit{(decay)} \end{aligned} \tag{2.2}$$
>
> *for any $t > 0$, where $\delta(t) \to 0$ as $t \to \infty$.*

The *energy* $E[\Psi](t)$ is a positive definite quantity of the form

$$E[\Psi](t) \sim \sum_{|\alpha|=1} \int_{\Sigma_t} |\partial^\alpha \Psi|^2(t,x)\, dx$$

for some suitable spacelike hypersurface Σ_t.[3] A *k-th (higher) order energy* involves higher derivatives of the solution

$$E_k[\Psi](t) \sim \sum_{1\leq|\alpha|\leq k} \int_{\Sigma_t} |\partial^\alpha \Psi|^2(t,x)\, dx\,.$$

We sometimes refer to *local* energy statements, for which the *local energy* of the solution

$$E_{loc}[\Psi](t) \sim \sum_{|\alpha|=1} \int_{\Sigma_t\cap\Omega} |\partial^\alpha \Psi|^2(t,x)\, dx$$

[1] One is ultimately interested in proving pointwise decay for Ψ. The methods of proving pointwise decay by means of estimates involving positive definite, L^2-based quantities (energies) are frequently called *energy methods*. These provide a robust framework to deal with both linear and nonlinear problems.

[2] This is the class of initial data that we consider throughout the introduction, even if not specified.

[3] The reader should refer to Section 2.2 for multi-index notation and the meaning of $\sim$. In Section 2.3 we define hypersurfaces Σ_t, while in Section 2.4 we give a rigorous definition of the energy.

is considered, with $\Sigma_t \cap \Omega$ some bounded, non-empty set. We will also denote it as $E_\Omega[\Psi](t)$. The function $\delta(t)$ appearing in (2.2) is the *uniform energy decay rate*.

It is crucial that our energy statements are *uniform*, in the sense that they hold for *any solution* to the wave equation and *for all times* $t > 0$. If no uniform energy decay with decay rate faster than $\delta(t)$ can possibly hold, we say that the uniform decay rate is *sharp*.

Of particular interest for us is the case when $(\mathcal{D}, g)$ *is the exterior region of a black ring spacetime*. Before moving to that, we first briefly outline the state of the art for linear waves on some other black hole exteriors of relevance for our discussion.

A series of works [68, 4, 23, 84, 25, 24, 85, 2, 108, 109], culminating in [27], has shown that the energy of any solution to (2.1) (and of all its higher order derivatives) is uniformly bounded and, in fact, decays (fast) *polynomially* in time on the exterior of sub-extremal Kerr black holes, with a higher order energy on the right hand side of (2.2).[4] This remains true up to the event horizon, in contrast with extremal Kerr black holes, for which some derivatives do not decay along the horizon [3].

For Kerr-AdS black holes, the energy of solutions is uniformly bounded and decays *logarithmically* in time when the black hole parameters satisfy certain bounds [55, 58]. In fact, such uniform energy decay rate is *sharp* [59]. In some other parameter regime, superradiance allows linear waves to grow exponentially in time [29].

For higher dimensional spacetimes, works [78, 102] show polynomial decay on Schwarzschild black holes in any dimension, while [79] proves integrated local energy decay on five-dimensional Myers–Perry black holes with small angular momenta.

The wave equation on the exterior of black strings has been mainly investigated from the numerical point of view. For Schwarzschild black strings, the numerical analysis in [6] suggests the absence of growing mode solutions, even when the black string gets boosted. The expectation is different for Kerr black strings, for which superradiant instabilities have been numerically shown [7, 8, 99].

Scalar perturbations of black rings will be the main topic of the present chapter. To the best of the author's knowledge, there is no rigorous study of the wave equation on these spacetimes in the literature (apart from an application of a general result by Moschidis [89], we will come back to this later). In view of the fact that the near-horizon geometry of large radius, thin black rings approximates that of a boosted black string, heuristic arguments suggest a correlation between properties of linear waves on black strings and on this class of black rings [6, 28]. In this respect, one could expect uniform boundedness and decay on *singly-spinning* thin black rings, which approximate *Schwarzschild* boosted black strings, and possible instabilities on *doubly-spinning* thin black rings, because the geometry is now close to the one of a *Kerr* boosted black string. Our work exploits, to some extent, this heuristic.

[4]For reasons that we shall discuss later, uniform energy decay on black hole exteriors cannot have the same right hand side of (2.2). When we refer to decay on such spacetimes, we always implicitly consider an inequality like (2.2), but with a higher order energy on the right hand side.

It is important to note that the problem of studying the linear wave equation on a spacetime is different (but, of course, related) from investigating the linear stability of such spacetime to gravitational perturbations. In fact, the presence of an instability at the level of *gravitational* perturbations, such as the Gregory–Laflamme growing modes for thin black rings, might be an effect that is not manifest for *scalar* perturbations. This in part motivates our analysis.

2.1.2 Stable trapping and slow energy decay

One of the various geometric aspects of $(\mathcal{D}, g)$ interacting with wave propagation is *null geodesic trapping*. High frequency solutions (or high frequency components of solutions) to the wave equation approximately propagate along null geodesics for very long time. In the presence of trapped null geodesics, this causes an obstruction to decay. A manifestation of this obstruction is that uniform energy decay estimates of the form (2.2) cannot hold [94, 101].

Furthermore, the *structure* of trapping is crucial when one proves energy decay. Trapping that occurs at the Schwarzschild photon sphere $r = 3M$ and in the Kerr exterior region is *unstable*. Even though the high frequency components of a solution are trapped, the good structure of trapping allows an *integrated* local energy decay estimate of the form

$$\int_0^\infty \chi E_{\mathrm{loc}}[\Psi](s)\, ds \lesssim E[\Psi](0)\,,$$

which degenerates at the trapping set, on which $\chi = 0$.[5,6] This degeneracy can be removed by losing a derivative at initial time

$$\int_0^\infty E_{\mathrm{loc}}[\Psi](s)\, ds \lesssim E_2[\Psi](0)\,. \tag{2.3}$$

In contrast, the presence of *stable* trapping in Kerr-AdS spacetimes prevents from proving integrated energy decay for high frequencies, as first shown in [58]. This difference is what originates the different uniform energy decay rate: While integrated local energy decay (2.3) permits to establish *polynomial* decay on Kerr black holes [27, 22]

$$E_{loc}[\Psi](t) \lesssim \frac{1}{t^2}\, E_{3,w}[\Psi](0) \tag{2.4}$$

for a suitable choice of spacelike hypersurfaces Σ_t,[7] one can only prove *logarithmic* decay for Kerr-AdS spacetimes [58]

$$E_{loc}[\Psi](t) \lesssim \frac{1}{[\log(2+t)]^2}\, E_2[\Psi](0)\,.$$

[5] See Section 2.2 for the meaning of $\lesssim$.

[6] Here we are omitting another important property of Kerr black holes, namely the fact that superradiant components of solutions are not trapped. Together with the instability of trapping, this is a fundamental ingredient for the proof in [27].

[7] The energy on the right hand side contains some weights independent of t. If one restricts to compactly supported initial data, then those weights can be neglected.

Logarithmic decay is usually regarded as *slow* decay, the reason being the comparison with polynomial rates as the one appearing in (2.4) and their application to nonlinear problems (see Section 2.1.6).

From a more general point of view, these results suggest that a bad trapping structure on black hole exteriors generically leads to slow uniform energy decay rates. Furthermore, one might wonder whether arbitrarily bad trapping can produce arbitrarily slow decay or, alternatively, whether there exists a universal minimal decay rate that linear waves always satisfy. This question has been answered in the context of obstacle problems for the wave equation on Minkowski space [5], for which the local energy decays logarithmically in time without any assumption on the geometry of the trapping obstacle. Partly motivated by this result is the conjecture that, provided uniform boundedness holds, the same universal minimal decay rate should hold for waves on the exterior of any stationary black hole, again without requiring any good structure for trapping. This has been proven for a general class of stationary, asymptotically flat (black hole) spacetimes in [69] and [89]. In particular, *the class of spacetimes considered in [89] includes all black rings.* Works [38, 37, 70] show that for some non-black hole geometries this expectation fails to be true.

A separate, but equally relevant question is whether the presence of stable trapping further imposes that uniform decay rates are *sharp.* This is indeed the case for logarithmic decay on Kerr-AdS black holes [59],[8] on some static, spherically symmetric spacetimes considered in [69] and for the example constructed in [89] (which is a modification of a counterexample of Rodnianski and Tao in [98]). Sharpness of the decay rate has also been shown on certain microstate geometries [70], on more general gravitational solitons [50] and for the Dirac equation on Schwarzschild-AdS [64].

The technique employed in all these works consists in constructing approximate solutions to the wave equation, called *quasimodes*, which are localized along stably trapped null geodesics. While a proof of decay usually involves the *global* structure of a spacetime, quasimode constructions only rely on the *local* geometry in proximity of the trapping set.

Before presenting the main ideas behind this technique, let us remark that uniform energy decay rates are crucial for nonlinear applications. (Fast) polynomial decay gives hope to be able to prove stability results for nonlinear problems, whereas logarithmic decay is not enough in this regard and strongly suggests nonlinear instabilities. See Section 2.1.6.

2.1.3 Quasimodes and lower bound on the uniform energy decay rate

We give an overview of how quasimodes can be used to contradict uniform energy decay statements for solutions to the wave equation.[9]

[8] Restricted to the case of Schwarzschild-AdS black holes, the sharpness of logarithmic decay was first observed in [44].

[9] The presentation here is far from being rigorous, we leave technical details for Section 2.10. The exposition is very much inspired by [104].

Consider some stationary, $(d+1)$-dimensional Lorentzian manifold $(\mathcal{D}, g)$ and a coordinate system (t, x), with $x = (x^1, \ldots, x^d)$. An informal definition of quasimodes can be the following:

> *We define quasimodes as complex-valued, time-periodic functions of the form*
>
> $$\Psi_m(t,x) = e^{i\omega_m t} u_m(x),$$
>
> *with frequency parameters $\omega_m \in \mathbb{R}$ and $m \in \mathbb{Z}$, such that*
>
> *(i)* *Ψ_m is in some energy space,*
>
> *(ii)* *Ψ_m is localized in space, i.e. u_m is compactly supported,*
>
> *(iii)* *Ψ_m is localized in frequency, i.e. $\left\|\partial^2 \Psi_m\right\| \sim \omega_m^2 \left\|\Psi_m\right\|$ in some energy norm,*
>
> *(iv)* *Ψ_m is an approximate solution to the wave equation.*

In general, quasimodes do not satisfy the wave equation (2.1). In fact,

$$\Box_g \Psi_m = \mathfrak{Err}_m(\Psi_m)\,,$$

where $\mathfrak{Err}_m(\Psi_m)$ is the *error.* Functions Ψ_m are *approximate solutions to the wave equation* when $\mathfrak{Err}_m$ is small, in a sense that of course needs to be specified. In particular, we suppose that

$$\|\mathfrak{Err}_m\| \to 0 \quad \text{as} \quad m \to \infty$$

(again in some energy norm), i.e. Ψ_m is an approximate solution to the wave equation for high frequency m. Typically, one is also interested in the *rate* in m at which the error tends to zero. So let us further assume that

$$\|\mathfrak{Err}_m\| = \mathcal{O}\left(e^{-Cm}\right) \tag{2.5}$$

for m large, where $C > 0$ is a constant independent of m.

To see what quasimodes can tell about energy decay of solutions to the wave equation, consider the initial value problem

$$\begin{cases} \Box_g \Psi_m^H = 0 \\ \Psi_m^H|_{t=0} = \Psi_m|_{t=0} \\ \partial_t \Psi_m^H|_{t=0} = \partial_t \Psi_m|_{t=0} \end{cases}, \tag{2.6}$$

where Ψ_m^H denotes a solution to the *homogeneous* wave equation (in contrast with Ψ_m, that solves the wave equation with inhomogeneous term $\mathfrak{Err}_m$). Initial data for Ψ_m^H are the quasimodes. Duhamel's formula[10] gives

$$\Psi_m(t) = \Psi_m^H(t) + \int_0^t \xi(s; t, x)\, ds\,,$$

[10] Here we use the Duhamel's formula that holds for Minkowski space. For a general Lorentzian metric g, one needs to correct the formula with some factors that we omit.

with $\xi(t,x)$ solution of

$$\begin{cases} \Box_g \xi = 0 \\ \xi|_{t=s} = 0 \\ \partial_t \xi|_{t=s} = \mathfrak{Err}_m(\Psi_m)|_{t=s} \end{cases} . \tag{2.7}$$

From Duhamel's formula, one has

$$\left\|\Psi_m^H - \Psi_m\right\| \leq t \sup_{s\in[0,t]} \|\xi(s)\| .$$

Suppose now that *the energy of any solution to the homogeneous wave equation is uniformly bounded by the initial data for all times*, then

$$\begin{aligned} \left\|\Psi_m^H - \Psi_m\right\| &\leq Ct \left\|\mathfrak{Err}_m\right\| \qquad \text{(from (2.7))} \\ &\leq Cte^{-Cm} \end{aligned}$$

with m sufficiently large and $C > 0$ universal constant. For $t \leq e^{Cm}/2C$, the reverse triangle inequality gives

$$\begin{aligned} \left\|\Psi_m^H\right\|(t) \geq \frac{1}{2}\left\|\Psi_m\right\|(t) &= \frac{1}{2}\left\|\Psi_m\right\|(0) \\ &= \frac{1}{2}\left\|\Psi_m^H\right\|(0) , \end{aligned}$$

where the first equality holds by time-periodicity of quasimodes and the second from (2.6). We conclude

$$\left\|\Psi_m^H\right\|(t) \geq \frac{1}{2}\left\|\Psi_m^H\right\|(0) \quad \text{for } t \leq \frac{e^{Cm}}{2C} \text{ and } m \text{ large} . \tag{2.8}$$

If

$$\|\cdot\| = E_{\text{loc}}[\,\cdot\,]$$

with local energy over some bounded set containing the spatial support of any $\Psi_m|_{t=0}$, then (2.8) gives

$$E_{\text{loc}}\left[\Psi_m^H\right](t) \geq \frac{1}{2}E\left[\Psi_m^H\right](0) \quad \text{for } t \leq \frac{e^{Cm}}{2C} \text{ and } m \text{ large} , \tag{2.9}$$

where $E_{\text{loc}}\left[\Psi_m^H\right](0) = E\left[\Psi_m^H\right](0)$ by the localization in space of the quasimodes.

A sequence

$$\left\{\left(\Psi_m^H, t_m\right)\right\}_{m\in\mathbb{Z}} \tag{2.10}$$

with $t_m = e^{Cm}/2C$ contradicts a uniform energy decay statement of the form

> *There exists a universal constant $C > 0$ (independent of time) such that all smooth, compactly supported solutions Ψ to the wave equation $\Box_g \Psi = 0$ satisfy*
>
> $$E_{loc}[\Psi](t) \leq C\delta(t)E[\Psi](0)$$
>
> *for all $t > 0$, with $\delta(t) \to 0$ as $t \to \infty$.*

Proof. To see this, note that the proposition above implies that for any $\varepsilon > 0$, there exists $T > 0$ such that $E_{\text{loc}}[\Psi](t) \leq \varepsilon E[\Psi](0)$ for all $t \geq T$, this being true for any solution Ψ. We choose $\varepsilon = 1/4$. We then choose m_* sufficiently large, say $e^{Cm_*}/2C > 2T$. In view of (2.9), there exists a solution $\Psi^H_{m_*}$ to the wave equation such that $E_{\text{loc}}\left[\Psi^H_{m_*}\right](t) \geq \frac{1}{2}E\left[\Psi^H_{m_*}\right](0)$ for $T \leq t \leq 2T$. This leads to a contradiction. □

Using the frequency localization of $\Psi^H_m(0)$, i.e. the frequency localization of quasimodes, and system (2.6), inequality (2.9) gives the following

$$E_{\text{loc}}\left[\Psi^H_m\right](t) \gtrsim \frac{1}{m^2}\, E_2\left[\Psi^H_m\right](0) \quad \text{for } t \leq \frac{e^{Cm}}{2C} \text{ and } m \text{ large}\,.$$

Therefore, sequence (2.10) proves that a uniform energy decay statement of the form

> *There exists a universal constant $C > 0$ (independent of time) such that all smooth, compactly supported solutions Ψ to the wave equation $\Box_g \Psi = 0$ satisfy*
>
> $$E_{loc}[\Psi](t) \leq \frac{C}{[\log(2+t)]^2}\, E_2[\Psi](0) \tag{2.11}$$
>
> *for all $t > 0$.*

has to be *sharp*, in the sense that sequence (2.10) disproves any uniform energy decay statement of the form (2.11) (with the same loss of derivatives) with faster uniform energy decay rate.

After this brief discussion, the reader should be convinced that lower bounds for uniform energy decay rates can be proven rather easily once quasimodes are constructed. The particular bound that one is able to produce depends on the rate at which quasimodes approximate solutions to the wave equation. In fact, the hard part of the argument is proving that quasimodes with suitably small error do actually exist.

In [94], quasimodes are constructed to produce lower bounds in the context of the obstacle problem on Minkowski space. The major insight of [59] is that such construction is still possible on Kerr-AdS spacetimes and, moreover, that stable trapping lets quasimodes satisfy (2.5). Subsequent works proving lower bounds essentially follow the same ideas of [59]. Even though some additional difficulties come into play, this is also the spirit of the proof of our main result.

2.1.4 The main results

Our main theorem produces a *logarithmic* lower bound for the uniform energy decay rate of solutions to the wave equation on black rings. In particular, the theorem concerns decay on the black ring exterior $(\mathcal{D}, g_{\text{ring}})$, with g_{ring} belonging to some class $\mathfrak{g}$ of black ring metrics.

We give here a first informal version of the result. The complete statement can be found in Section 2.5. The existence of quasimodes with suitably small error is a crucial step of the proof, so we state it as an independent theorem.

Theorem 2.1 (**Quasimodes for Black Rings**). *Let $\mathfrak{g}$ be a class of black ring metrics g_{ring}, as defined in* (2.30)*, and consider the black ring exterior region $(\mathcal{D}, g_{\text{ring}})$, with $g_{\text{ring}} \in \mathfrak{g}$. Let $(t, r_*, \theta_*, \phi, \psi)$ be the coordinate system on $(\mathcal{D}, g_{\text{ring}})$ introduced in Section 2.9. Then, for δ sufficiently small, there exist non-zero functions $\Psi_m : \mathcal{D} \to \mathbb{C}$ such that*

(i) $\Psi_m \in H^k(\Sigma_{t^*})$ *for any* $k \geq 0$,

(ii) $\Psi_m(t, r_*, \theta_*, \phi, \psi) = e^{-i\,\omega_m t} e^{i(m\,\phi + \hat{J}\,\psi)} \varphi_m(r_*, \theta_*)$, *with* $\omega_m \in \mathbb{R}$ *and* $m, \hat{J} \in \mathbb{Z}$,

(iii) inequality $c \leq \omega_m^2/m^2 \leq C$ *holds, with constants* $c, C > 0$ *independent of* m,

(iv) for any $k \geq 0$, *there exists a constant* $C_k > 0$, *independent of* m, *such that*

$$\left\| \Box_{g_{\text{ring}}} \Psi_m \right\|_{H^k(\Sigma_{t^*})} \leq C_k e^{-C_k\, m} \left\| \Psi_m \right\|_{L^2(\Sigma_0)}$$

for all $t^* > 0$,

(v) the support of $\varphi_m(r_*, \theta_*)$ *is contained in* $\Omega \subset \mathcal{D}$, *where* $\Sigma_0 \cap \Omega$ *is a bounded, non-empty set and* $\Phi_{t^*}(\Sigma_0 \cap \Omega) = \Phi_{t^*}(\Sigma_0) \cap \Omega$ *remains bounded and non-empty for all* $t^* > 0$, *with* Φ_{t^*} *the one-parameter group of diffeomorphisms generated by the Killing vector field* $\partial/\partial t^*$,

(vi) the support of $\mathfrak{Err}_m(\Psi_m) := \Box_{g_{\text{ring}}} \Psi_m$ *is contained in* $\Omega_\delta \subset \Omega$, *where*

$$\Omega_\delta := \{ x \in \Omega : \exists\, y \in \partial\Omega \text{ such that } \operatorname{dist}((r_*(x), \theta_*(x)), (r_*(y), \theta_*(y))) \leq \delta \}$$

with $\operatorname{dist}(\cdot, \cdot)$ *the Euclidean distance,*

with time coordinate t^* and spacelike hypersurfaces Σ_{t^*} defined in Section 2.3.4.

Theorem 2.2 (**Lower Bound for Uniform Energy Decay Rate, First Version**). *Let $\mathfrak{g}$ be a class of black ring metrics g_{ring}, as defined in* (2.30)*. Consider smooth solutions $\Psi : \mathcal{D} \to \mathbb{C}$ to the scalar, linear wave equation*

$$\Box_{g_{\text{ring}}} \Psi = 0 \tag{2.12}$$

on the black ring exterior $(\mathcal{D}, g_{\text{ring}})$, with $g_{\text{ring}} \in \mathfrak{g}$, and assume that a uniform boundedness statement (without loss of derivatives) for the energy of solutions to (2.12) *holds. Then, there exists a universal constant $C > 0$ (independent of time) such that*

$$\limsup_{t^* \to +\infty} \sup_{\Psi \in SC_0^\infty(\Sigma_0), \Psi \neq 0} [\log(2 + t^*)]^2 \left(\frac{E_\Omega[\Psi](t^*)}{E_2[\Psi](0)} \right) > C\,,$$

where the supremum is taken over all smooth, non-zero solutions to the wave equation with compactly supported initial data on Σ_0. Furthermore, for any $k \in \mathbb{N}$, there exists a universal constant $C_k > 0$ such that

$$\limsup_{t^* \to +\infty} \sup_{\Psi \in SC_0^\infty(\Sigma_0), \Psi \neq 0} [\log(2 + t^*)]^{2k} \left(\frac{E_\Omega[\Psi](t^*)}{E_{k+1}[\Psi](0)} \right) > C_k\,.$$

The set Ω appearing in the local energy $E_\Omega[\Psi](t^*)$ is the one fixed in Theorem 2.1, with time coordinate t^* and spacelike hypersurfaces Σ_{t^*} defined in Section 2.3.4.

We present a corollary of Theorem 2.2 and some important remarks.

Corollary 2.1 (**Sharp-logarithmic uniform energy decay**)**.** *Let $\mathfrak{g}$ be a class of black ring metrics g_{ring}, as defined in* (2.30)*. Consider smooth solutions $\Psi : \mathcal{D} \to \mathbb{C}$ to the scalar, linear wave equation*

$$\Box_{g_{\text{ring}}} \Psi = 0 \tag{2.13}$$

on the black ring exterior $(\mathcal{D}, g_{\text{ring}})$, with $g_{\text{ring}} \in \mathfrak{g}$ and compactly supported initial data on Σ_0, and assume that a uniform boundedness statement (without loss of derivatives) for the energy of solutions to (2.13) *holds. Then, for any $k \in \mathbb{N}$ and $\Omega \subset \mathcal{D}$ such that $\Sigma_0 \cap \Omega$ is a bounded, non-empty set and $\Phi_{t^*}(\Sigma_0 \cap \Omega) = \Phi_{t^*}(\Sigma_0) \cap \Omega$ remains bounded and non-empty for all $t^* > 0$, there exists a universal constant $C_k(\Omega) > 0$ (independent of time) such that*

$$E_\Omega[\Psi](t^*) \leq \frac{C_k(\Omega)}{[\log(2 + t^*)]^{2k}} \, E_{k+1}[\Psi](0) \tag{2.14}$$

for all $t^* > 0$, with time coordinate t^* and spacelike hypersurfaces Σ_{t^*} defined in Section 2.3.4. Furthermore, the uniform energy decay rate of (2.14) *is sharp.*

Remark 2.1 (**Theorem 2.2 complements Moschidis [89] for $\mathfrak{g}$**)**.** *The first part of Corollary 2.1 follows by a result of Moschidis (see Theorem 2.1 in [89]) specialized to the class $\mathfrak{g}$ of black rings. The sharpness of the statement is a direct consequence of our Theorem 2.2.*

Remark 2.2 (**Uniform boundedness assumption**)**.** *Our assumption on uniform boundedness is crucial to produce the logarithmic lower bound once quasimodes are constructed. However, superradiance occurs for all black rings and obstructions to uniform boundedness might be present. As for Kerr black holes, one can successfully prove uniform boundedness for axisymmetric solutions to the wave equation and for non-superradiant, fixed-frequency modes, but no uniform boundedness statement holding for all solutions is known.*[11] *From this point of view, sharp-logarithmic energy decay appears to be the best scenario that one can hope for on the class of black rings $\mathfrak{g}$, the alternative being the failure of uniform boundedness.*

Remark 2.3 (**Lower bound for black strings**)**.** *The result of Theorem 2.2 also holds for five-dimensional static and boosted black strings.*[12] *This can be easily inferred by treating theorems in Section 2.7 and Section 2.8 as independent results and carrying out the quasimode construction*

[11] Note that, in view of the aforementioned result by Moschidis [89], in order to contradict uniform boundedness for black rings, it would be enough to disprove logarithmic uniform energy decay. For instance, it would suffice to show the existence of *real* mode solutions or to construct quasimodes whose errors decay faster than exponentially in frequency.

[12] For the latter, one needs a further assumption on the boost parameter, as we shall see later. Note also that, since string metrics are fully-separable, one does not need the whole machinery developed in this chapter to prove an analogue of Theorem 2.2 for black strings.

of Section 2.10. Note that, in the case of black strings, uniform boundedness does hold, so one does not need to assume it in Theorem 2.2. For static black strings, uniform boundedness follows from the same arguments presented in [26] to prove uniform boundedness on Schwarzschild spacetime. On the other hand, boosted black strings possess an ergoregion, but they are still not affected by superradiance. From the geometric point of view, the absence of superradiance corresponds to the existence of a Killing vector field

$$X = (\cosh\beta)\,\partial_t - (\sinh\beta)\,\partial_z\,, \tag{2.15}$$

which is causal on the whole domain of outer communication (see Section 2.3.2 for the definition of the quantities in (2.15)*). The vector field X is the linear combination of two Killing vector fields, and correctly reduces to $X = \partial_t$ in the unboosted case $\beta = 0$. As for the static black string, one can prove uniform boundedness for solutions to the wave equation in the spirit of [26]. This would give a rigorous proof to the numerical analysis in [6] and to a comment in Section 3 of [28].*

The proof of Theorem 2.2 relies on the construction of quasimodes. **The main technical difficulty arising from such construction, which does not appear in previous works, lies in that the wave equation fails to fully-separate on black rings**. This motivates our new PDE approach to the problem. See Section 2.1.7 for an overview.

2.1.5 Stable trapping in higher dimensional black holes

Black strings possess the most basic geodesic structure that one can find in higher dimensional black holes. In this regard, it is interesting to note that *null* geodesic motion on black string exteriors can be understood in terms of *timelike* geodesics in the exterior of the correspondent unextended black holes. In fact, at the level of the geodesic equation, one can treat the conserved momentum of a light ray along the extended direction as the mass of a particle orbiting the unextended object. Thus, from the existence of stable *timelike* orbits around four-dimensional Schwarzschild and Kerr black holes, one can immediately deduce the presence of stably trapped *null* geodesics in the correspondent five-dimensional (boosted) black string exteriors (see also [49]). Similarly, the absence of stable orbits for massive particles around higher dimensional Schwarzschild black holes [51] suggests that one should not expect stably trapped null geodesics for static black strings in dimensions higher than five.[13]

The class $\mathfrak{g}$ of black rings that we consider in Theorem 2.2 is morally a class of thin, singly-spinning black rings, whose near-horizon geometry resembles that of five-dimensional (boosted) black strings. It turns out that, in a sense that this chapter of the thesis shall clarify, the trapping structure of black strings locally persists for this class of black rings. In fact, we will be able to prove the existence of null geodesics whose toroidal orbits remain in a bounded region outside the event horizon (see Theorem 2.7 and Remark 2.32). **The appearance of *stable trapping* on *asymptotically flat* solutions to the vacuum Einstein equations**

[13]In this sense, one should note that Remark 2.3 concerns *five-dimensional* black strings *only*.

is a remarkable geometric property of higher dimensional black holes, which is absent in four dimensions. Stable trapping has to be regarded as the fundamental mechanism underlying the slow decay of linear waves that Theorem 2.2 determines.

Our proof of stable trapping for black rings is a by-product of the proof of Theorem 2.2 and puts in mathematically rigorous relation the trapping structure of black rings with that of black strings. This stable trapping phenomenon has already been investigated in [66], where a more computational approach was adopted. Other aspects of geodesic motion in black ring exteriors are presented in [61, 30].

2.1.6 A new instability for black strings and black rings

Theorem 2.2 suggests that one should expect nonlinear instabilities for the class $\mathfrak{g}$ of black rings and, in view of Remark 2.3, for five-dimensional black strings, the reason being that sufficiently strong decay at the linear level is necessary to prove small data global (in time) existence for nonlinear problems. It turns out that fast polynomial decay, namely faster than $1/t$, is enough to close a classical bootstrap argument, whereas logarithmic decay is not.[14]

However, **the nature of the nonlinear instability that one can conjecture from Theorem 2.2 differs from the one caused by the nonlinear dynamics of the Gregory–Laflamme modes.** To make this statement more precise, we first briefly discuss some key features of the Gregory–Laflamme instability.

Given a metric perturbation

$$g_{\mu\nu} \to g_{\mu\nu} + h_{\mu\nu}\,,$$

the original work of Gregory and Laflamme [48] shows that, if $g_{\mu\nu}$ is the metric of a static black *string* of radius r_0, then there exist exponentially growing (in time) modes of the form

$$h_{\mu\nu} \sim e^{-i(\omega t - Jz)} H_{\mu\nu}(r) \qquad \omega \in \mathbb{C}, J \in \mathbb{R}\,,$$

with $\mathrm{Im}(\omega) > 0$ and z coordinate along the extended direction, such that $g_{\mu\nu} + h_{\mu\nu}$ solves the linearised Einstein equations. For such modes to exist, the condition

$$J < J_{\mathrm{GL}} \tag{2.16}$$

needs to be satisfied, where $J_{\mathrm{GL}} \sim 1/r_0$ is a positive constant characteristic of the string, and ω has to be chosen as a function of J.

If one compactifies the string by identifying $z \sim z + L$, with L positive constant, then the frequency parameter J becomes discrete and has minimum (non-zero) value $J_{\min} = 1/L$. If $L \gg r_0$, then there exists J satisfying (2.16) and the compactified black string is *unstable*. Otherwise, for L sufficiently small, the instability disappears. Therefore, the Gregory–Laflamme

[14] Here we are thinking about nonlinearities involving both the solution and first derivatives of the solution and with some nice structure, i.e. satisfying the *null condition* of Christodoulou [12] and Klainerman [72].

instability characterises black strings whose compactified extra-dimension is, in some sense, sufficiently large compared to the horizon radius. This aspect has a meaningful connection with Kaluza–Klein theories, see [60].

Work [62] argues that the same picture essentially holds for boosted black strings and very thin, singly-spinning black *rings*. For the latter, one should understand r_0 as the radius of the S^2 at the horizon and L as the radius of the ring.

As one can see, the Gregory–Laflamme instability is a purely *gravitational* effect, which is already manifest at the *linear* level and persists in the nonlinear evolution of suitably perturbed initial data [80, 40]. On the other hand, the nonlinear instability conjectured from Theorem 2.2 would emerge in the form of time integrals of error terms (nonlinearities) becoming large in the time evolution. In this sense, it would be a genuinely *nonlinear* effect originating from slow decay at the *scalar*, linear level. Note also that our instability is ultimately connected to a *high-frequency* phenomenon, while the Gregory–Laflamme modes are *low-frequency* perturbations.

For *black strings*, our instability does not require any restrictive condition on the geometry of the spacetime, thus, in contrast with the Gregory–Laflamme instability, it would affect *any* five-dimensional black string, including those with small compactified extra-dimension $L \ll r_0$. In any dimension higher than five, black strings with $L \gg r_0$ still suffer from the Gregory–Laflamme instability, but none of them would suffer from our instability (see Section 2.1.5 for motivation).

In the case of *black rings*, it is interesting to note that both the instabilities affect (very) thin black rings. In particular, we expect that at least some of the members of the class $\mathfrak{g}$ would suffer from both the instabilities.

2.1.7 Overview of the proof of Theorem 2.2

We present an overview of the proof of Theorem 2.2. Note that each step of the proof is, to some extent, self-contained and one could jump from Step 1 or 2 to the construction of quasimodes in Step 4 (see Remark 2.3). At the beginning of each step, we report the section where the reader can find the detailed argument, so that the overview also serves as an outline of the chapter. A formal version of Theorem 2.2 is Theorem 2.3 of Section 2.5.

Step 1. (Sections 2.6 and 2.7) Our proof starts by considering five-dimensional static black strings $\mathrm{Schw}_4 \times \mathbb{R}$ in the usual Schwarzschild coordinate system. The metric reads

$$ds^2 = -f(r)\,dt^2 + f(r)^{-1}\,dr^2 + r^2\,d\Omega_2^2 + dz^2\,,$$

where $f(r) = 1 - r_0/r$, $r_0 > 0$. The event horizon lies at $r = r_0$ and the z coordinate is periodically identified by $z = z + 2\pi R$, with R some positive constant.

With an ansatz of the form

$$\Psi(t, r, \theta, \phi, z) = e^{i(-\omega t + m\,\phi + J\,z)} u(r, \theta)\,,$$

with $\omega \in \mathbb{R}$ and $m, J \in \mathbb{Z}$, and after a suitable factorization of $u(r, \theta)$, the wave equation on static black strings can be fully-separated. However, for later convenience, *we do not fully-separate the wave equation*, but we instead rewrite it as a *two-variable* Schrödinger-type partial differential equation of the form

$$-\frac{g(r)}{m^2}\,\Delta_{(r_*,\theta)} u + V(r, \theta)\,u = \frac{\omega^2}{m^2}\,u\,, \tag{2.17}$$

where $g(r)$ and $V(r, \theta)$ are smooth, positive functions *independent* of the frequency parameters ω and m. At this stage, we have set $J = b\,m$, choosing the constant b such that potential V has a local minimum on the exterior region $\{r \geq r_0\}$. We will keep this scaling of J throughout the proof.

We formulate a Dirichlet eigenvalue problem for equation (2.17) on a bounded set Ω containing the local minimum of V (see Figure 2.1). The set of eigenvalues for this problem is discrete, so eigenvalues ω^2/m^2 cannot be freely specified. However, we will be able to prove that, for any energy level E^2 satisfying $V_{\min} < E^2 < V(\partial\Omega)$ and constant $\delta > 0$ arbitrarily small, the eigenvalue problem (2.17) admits an arbitrarily large number of eigenvalues in $[E^2 - \delta, E^2 + \delta]$ for m^2 sufficiently large. This is a version of Weyl's law for the Laplacian, where $1/m^2$ has to be interpreted as a semi-classical parameter.

For this first step we essentially adapt Section 4.1 of [69] to our new two-variable PDE setting. The moral of the following two steps of the proof will be to replicate this construction for the abstract eigenvalue problems emerging from the separation of the wave equation on boosted black strings and black rings. By applying the perturbation scheme of Holzegel–Smulevici [59] *twice*, we will be able to prove an analogous asymptotic property for the eigenvalues of both problems (see Figure 2.2).

Step 2. (Section 2.8) We consider a five-dimensional *boosted* black string

$$\begin{aligned} ds^2 =& -[1 - (1 - f(r))\cosh^2\beta]\,dt^2 + 2(1 - f(r))\sinh\beta\cosh\beta\,dt\,dz \\ &+ [1 + (1 - f(r))\sinh^2\beta]\,dz^2 + f(r)^{-1}\,dr^2 + r^2\,d\Omega_2^2 \end{aligned}$$

with boost parameter $\beta \geq 0$. The corresponding Dirichlet eigenvalue problem

$$-\frac{g(r)}{m^2}\Delta_{(r_*,\theta)} u + V^\beta_{(\omega,m)}(r, \theta) u = \frac{\omega^2}{m^2} u \tag{2.18}$$

is now *nonlinear*, since the potential depends on both ω and m. Note that problem (2.18) reduces to problem (2.17) when $\beta = 0$, i.e. when the boost parameter is zero.

The analysis of potential $V^\beta_{(\omega,m)}$ represents the main technical difficulty of this part of the

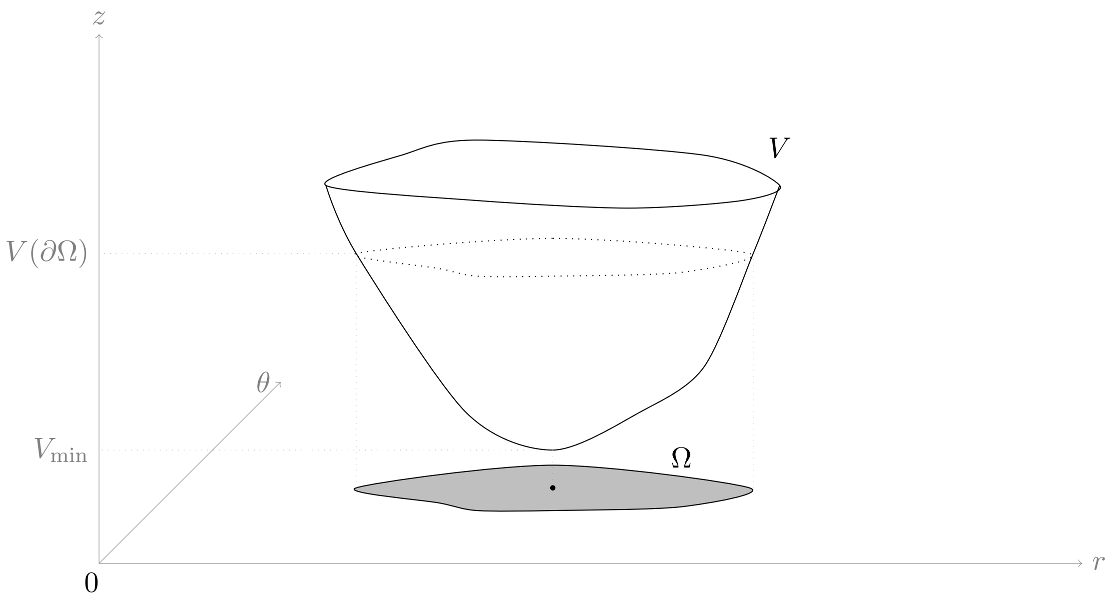

Figure 2.1: Potential V and open set Ω for the static black string eigenvalue problem (2.17).

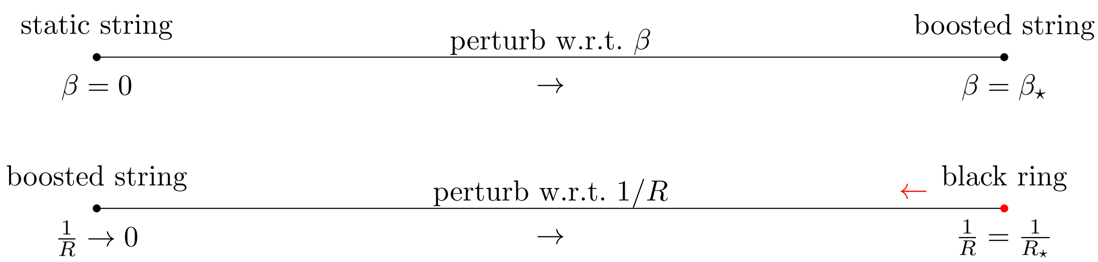

Figure 2.2: The perturbation scheme adopted in our proof can be schematically divided in two steps (two segments in figure), where for each step we apply the full perturbation argument of Holzegel–Smulevici [59]. The first time we perturb the eigenvalue problem for the static black string (boost parameter β equal to zero) towards the problem for the boosted black string with boost parameter $\beta_\star$, following the direction of the black arrow in figure. The second time we connect the problem for the boosted black string to the one for the black ring of radius $R_\star$. This second perturbation will generate extra errors that require an additional smallness parameter to be controlled, namely the inverse of the radius of the black ring $1/R_\star$. Thus, the endpoint of the second perturbation (represented by a red dot in figure) will have to be chosen sufficiently close to the starting point.

proof. The potential depends on ω both linearly and quadratically, meaning that one cannot understand the structure of $V^\beta_{(\omega,m)}$ and energy levels E independently. The idea will be to choose E such that $V^\beta_{(Em,m)}(r,\theta) - E^2$ is negative on a bounded subset of Ω and admits a local minimum in such subset. This will implicitly reproduce the setting of Step 1.

Note that no additional symmetry assumptions on solutions Ψ to the wave equation can be made to simplify the analysis of $V^\beta_{(\omega,m)}$, i.e. none of the frequency parameters can be set to zero (cf. [59], where this is possible). See Lemma 2.4 for motivation.

A perturbation argument with respect to β, based on an iterative application of the Implicit Function Theorem, will allow to use our knowledge of problem (2.17) to conclude that, for m^2 sufficiently large, eigenvalues for the eigenvalue problem (2.18) exist and accumulate in the strip $[C, E^2+\delta]$, with constants $C>0$ independent of m and $\delta>0$ arbitrarily small.

Step 3. (Section 2.9) Finally, we consider a five-dimensional, singly-spinning black ring $g_{(r_0,R)}$ belonging to the class $\mathfrak{g}$.[15] Written in coordinates $(t, r, \theta, \phi, \psi)$, [16] the eigenvalue problem for the black ring reads

$$-\frac{g_{\text{ring}}(r,\theta)}{m^2}\Delta_{(r_*,\theta_*)}u + V^{\text{ring}}_{(\omega,m)}(r,\theta)u = 0\,, \tag{2.19}$$

which is still *nonlinear* and now *non-separable* into two decoupled ODEs. On any fixed bounded set Ω, problem (2.19) reduces to problem (2.18) in the limit $R \to \infty$. This is a manifestation of the fact that the near-horizon geometry of a large radius, thin black ring resembles that of a boosted black string.

The fact that $g_{(r_0,R)} \in \mathfrak{g}$ implies that it is possible to choose Ω and E such that the potential $V^{ring}_{(Em,m)}$ has the same sign properties required for $V^{\beta}_{(Em,m)}(r,\theta) - E^2$ in Step 2. We will then iteratively apply the Implicit Function Theorem for the second time, but now perturbing problem (2.18) with respect to $1/R$ and using what we already know about the eigenvalue problem for *boosted* black strings. For m^2 sufficiently large, the eigenvalue problem (2.19) will be proven to admit a zero eigenvalue for values of ω^2/m^2 lying in a suitable strip around E^2. Our choice of the energy level E and set Ω will allow to conclude that such values of ω^2/m^2 give the potential the correct structure to construct quasimodes with exponentially small errors.

Note that this is the part of the proof where one can extract the existence of stably trapped null geodesics in the exterior of black rings $g_{(r_0,R)} \in \mathfrak{g}$. See Theorem 2.7 and related Remark 2.32.

Step 4. (Section 2.10) Once the black ring eigenvalue problem has been fully understood, we will construct quasimodes on the domain of outer communication of the form

$$\Psi_m(t, r_*, \theta_*, \phi, \psi) = e^{-i\,\omega_m t} e^{i(m\,\phi + \hat{J}\,\psi)} \chi(r_*, \theta_*)\, u_m(r_*, \theta_*)\,,$$

where u_m are eigenfunctions of problem (2.19) with associated frequencies ω_m. The function χ cuts-off close to the boundary of Ω and sets the quasimodes equal to zero outside. In this way, the quasimodes Ψ_m are *smooth* functions and solve the wave equation on the whole domain of outer communication with the exception of the cut-off region, where the error will be proved to decay exponentially in the frequency m. The exponential decay is intimately connected to the structure of the potential in the cut-off region and it is essential to prove the logarithmic lower bound of Theorem 2.2. The quantitative estimate of the error is achieved by applying Agmon distances and proving energy estimates for eigenfunctions u_m in the cut-off region. This step provides the proof of Theorem 2.1.

Step 5. (Section 2.10) The last part of the proof employs the quasimodes constructed in Step 4 to derive the lower bound of Theorem 2.2, following an argument already sketched in Section 2.1.3.

[15] The choice of this class is where we allow the radius R of the black ring to be arbitrarily large.

[16] These coordinates will be defined in Section 2.3, but the reader should already note that they differ from the coordinates adopted for the black string metrics.

2.1.8 Applications of our method and outlook

Our work provides a framework to prove a version of Theorem 2.2 for spacetimes on which the wave equation is not fully-separable. In particular, our approach could generalize the lower bound of Keir [69] to a more general class of static, *axisymmetric* spacetimes exhibiting stable trapping.[17] Our method (as the one in [69]) only relies on the local geometry of the spacetime in a neighbourhood the trapping set, so the more general class of spacetimes would not require any specific asymptotic structure.

Furthermore, the stable trapping phenomenon observed for black rings is likely to characterise some other higher dimensional black hole spacetimes, including

(i) Kerr (possibly boosted) black strings ($\mathrm{Kerr}_4 \times \mathbb{R}$), Schwarzschild p-branes ($\mathrm{Schw}_4 \times \mathbb{R}^p$) and Kerr p-branes ($\mathrm{Kerr}_4 \times \mathbb{R}^p$),

(ii) Thin, doubly-spinning black rings in five dimensions,

(iii) Some ultraspinning objects, such as ultraspinning Myers–Perry black holes in six dimensions [65].

In the case of *stationary* black holes, to apply our method and produce an analogue of Theorem 2.2, one would have to go through some sort of perturbation argument. Even though, for some of these spacetimes, a more complicated perturbation argument might be needed, some of them are easier to treat in that the wave equation fully-separates.[18] In any case, the presence of stable trapping would already suggest that a result like Theorem 2.2 might hold and, therefore, nonlinear instabilities of similar nature to the ones discussed in Section 2.1.6 might be conjectured.

As another potential direction, it would be certainly interesting to further investigate uniform boundedness of the energy of linear waves on black rings, inside or outside our class $\mathfrak{g}$. One of the main conceptual difficulties in doing this originates from the fact that the black ring family does not possess a static black hole. Therefore, even the (in principle) easier analysis on slowly rotating black rings cannot be, at first glance, understood in a perturbative fashion. Singly-spinning black rings (especially the thin ones) seem to be good candidates for uniform boundedness to hold, while doubly-spinning (thin) black rings might exhibit a superradiant instability and exponentially growing modes could be rigorously constructed.

2.2 Notation and conventions

We collect here some basic notation and conventions adopted in this chapter.

[17] Note that without the spherical symmetry assumption of [69], one cannot a priori conclude that the wave equation fully-separates.

[18] This is, for instance, the case of Myers–Perry black holes.

- **(Signature).** The signature of a Lorentzian metric g is $(-,+,\dots,+)$. We denote by $\Box_g$ the usual Laplace-Beltrami operator with respect to g. With our signature convention, $\Box_{g_{\mathrm{Mink}}} = -\partial^2_{x^0} + \sum_i \partial^2_{x^i}$ for the Minkowski metric g_{Mink} in rectangular coordinates $(x^0, x^1, \dots, x^n)$, where $\partial_{x^i} := \partial/\partial x^i$. We use the notation $\Delta_{(x^1,\dots,x^n)} := \sum_i \partial^2_{x^i}$ to denote the Cartesian Laplacian in coordinates $(x^1, \dots, x^n)$.

- **(Indices).** Given a coordinate system $(x^0, x^1, \dots, x^n)$, we will follow the convention for which Greek indices take on all values, e.g. $\mu = 0, 1, \dots, n$, and Latin indices only the spatial ones, e.g. $a = 1, \dots, n$. We adopt the usual upper/lower indices conventions and the summation convention. For instance, $g_{\mu\nu}$ are components of the metric tensor g, while $g^{\mu\nu} \equiv (g^{-1})^{\mu\nu}$ are components of the inverse metric. We will avoid the abstract index notation.

- **(Volume form).** For a given system of coordinates $(x^0, x^1, \dots, x^n)$, we denote by $dvol_g$ the volume form associated to g, i.e. $dvol_g = \sqrt{-\det g}\, dx^0 \cdots dx^n$. On any spacelike hypersurface Σ, $\overline{dvol_g}$ denotes the volume form associated to the Riemannian metric induced by g on Σ. The volume form will be often omitted in the integrals.

- **(Inequalities).** When we write $f \lesssim h$, we implicitly mean that there exists a constant $C > 0$ such that $f \leq Ch$. The notation $f \gtrsim h$ is analogous. We have $f \sim h$ when $f \lesssim h$ and $f \gtrsim h$. The notation $f \ll h$ means that there exists a sufficiently small constant $c > 0$ such that $|f/h| \leq c$.

- **(Constants).** We will not keep track of the constants appearing on the right hand side of the estimates. We adopt the notation C_k to denote a constant C which depends on some parameter k.

- **(Multi-index).** Given coordinates $(x^0, x^1, \dots, x^n)$, we will adopt the multi-index notation $\partial^\alpha := \partial^{\alpha_0}_{x^0} \cdots \partial^{\alpha_n}_{x^n}$, with $\alpha = (\alpha_1, \dots, \alpha_n)$ multi-index of order $|\alpha| = \alpha_1 + \cdots + \alpha_n$.

- **(Function spaces).** Function spaces $H^k(\Sigma)$ are the usual Sobolev spaces $W^{k,2}(\Sigma)$, with the standard Sobolev norm. With $C^\infty_0(\Sigma)$ we denote the space of smooth, compactly supported functions on Σ. We will write $SC^\infty_0(\Sigma)$ when we refer to the space of solutions of the wave equation which are C^∞_0 on Σ.

- **(Black ring metric).** We will use the notation g_{ring}, $g_{(r_0,R)}$ or $g_{(\nu,R)}$ for the black ring metric. The first denotes a black ring metric in general, while the second and third one refer to the black ring metric as a two-parameter family of metrics (these are not components of the metric). The definition of the parameters will depend on the particular coordinate system considered.

2.3 Metrics

Before introducing the relevant metrics, we define the *domain of outer communication* of a black hole spacetime $(\mathcal{M}, g)$ as a subset $\mathcal{D} \subset \mathcal{M}$ such that

$$\mathcal{D} := \mathrm{clos}(J^-(\mathcal{I}^+) \cap J^+(\mathcal{I}^-)) \,,$$

where $\mathcal{I}^+$ and $\mathcal{I}^+$ are *future* and *past null infinity* respectively. The event horizon is defined as $\mathcal{H} := \partial\mathcal{D}$. We understand $(\mathcal{D}, g)$ as a Lorentzian manifold with boundary.

Note that we will often refer to $(\mathcal{D}, g)$ as the *black hole exterior*.

2.3.1 Static black string

The metric of a five-dimensional *static (Schwarzschild) black string* is

$$ds^2 = -f(r)\, dt^2 + f(r)^{-1}\, dr^2 + r^2\, d\Omega_2^2 + dz^2 \,, \tag{2.20}$$

where (t, r, θ, ϕ) are the Schwarzschild coordinates and $d\Omega_2^2$ the round metric on the unit two-sphere. We have

$$f(r) = 1 - \frac{r_0}{r}$$

with $r = r_0 > 0$ corresponding to the event horizon $\mathcal{H}$. For each fixed time t, the z-coordinate is periodically identified so that

$$z = z + 2\pi R$$

with $R > 0$ constant independent of t.

The spacetime is static and possesses a Killing vectorfield $\partial/\partial z$ in addition to the symmetries of four-dimensional Schwarzschild. In view of the compactification along the z-direction, static black strings are asymptotically Kaluza–Klein. The domain of outer communication corresponds to $\mathcal{D} = \mathcal{M} \cap \{r \geq r_0\}$.

2.3.2 Boosted black string

The five-dimensional *boosted (Schwarzschild) black string* is obtained by the change of coordinates (Lorentz-boost in the z-direction)

$$t \to (\cosh\beta)t + (\sinh\beta)z \qquad\qquad z \to (\sinh\beta)t + (\cosh\beta)z$$

applied to the static black string (2.20), with $\beta > 0$ boost parameter. The metric is

$$ds^2 = -\left[1 - (1-f)\cosh^2\beta\right] dt^2 + 2(1-f)\sinh\beta\cosh\beta\, dt\, dz + \left[1 + (1-f)\sinh^2\beta\right] dz^2 \\ + f^{-1}\, dr^2 + r^2\, d\Omega_2^2\,, \tag{2.21}$$

where $f = f(r)$.

The spacetime is stationary (but not static) with respect to the boosted coordinates. The Killing vectorfield $\partial/\partial t$ becomes null at $r = r_0\cosh^2\beta$ and spacelike for $r < r_0\cosh^2\beta$. Boosted black strings do *not* present superradiance. In fact, there exists a Killing vectorfield of the form

$$X = (\cosh\beta)\,\frac{\partial}{\partial t} - (\sinh\beta)\,\frac{\partial}{\partial z} \qquad\qquad g(X,X) = -f(r)\,,$$

which is causal on the whole domain of outer communication $\mathcal{D} = \mathcal{M} \cap \{r \geq r_0\}$. As for the static black string, this spacetime is asymptotically Kaluza–Klein.

2.3.3 Singly-spinning black ring

Ring coordinates

We construct ring coordinates following the exposition in [35]. Consider five-dimensional Minkowski space with coordinates

$$(t, r_1, r_2, \phi, \psi)\,,$$

where (r_1, ϕ) and (r_2, ψ) are polar coordinates on two independent rotation planes. The Minkowski metric in these coordinates reads

$$ds^2 = -dt^2 + dr_1^2 + r_1^2\, d\phi^2 + dr_2^2 + r_2^2\, d\psi^2\,.$$

We define *ring coordinates of radius R* as coordinates

$$(t, y, x, \phi, \psi)$$

such that

$$y = -\frac{R^2 + r_1^2 + r_2^2}{\sqrt{(r_1^2 + r_2^2 + R^2)^2 - 4R^2 r_2^2}} \qquad\qquad x = \frac{R^2 - r_1^2 - r_2^2}{\sqrt{(r_1^2 + r_2^2 + R^2)^2 - 4R^2 r_2^2}}$$

with $R > 0$ constant. The coordinate ranges are

$$-\infty \leq y \leq -1 \qquad\qquad -1 \leq x \leq 1$$

and the Minkowski metric in ring coordinates of radius R reads

$$ds^2 = -dt^2 + \frac{R^2}{(x-y)^2}\left[(y^2-1)d\psi^2 + \frac{dy^2}{y^2-1} + \frac{dx^2}{1-x^2} + (1-x^2)d\phi^2\right]. \tag{2.22}$$

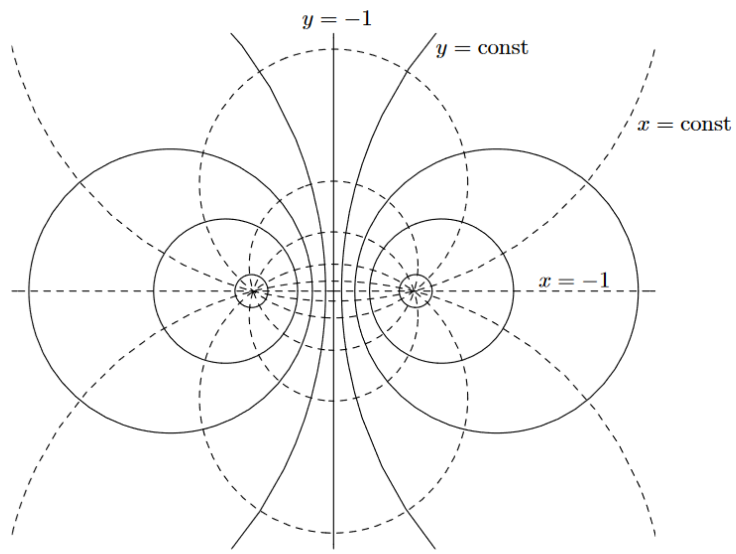

Figure 2.3: Lines of constant y and x on the (r_1, r_2)-plane. The set of points with $y = -1$ and $x \neq -1$ are points on the ψ-axis of rotation (vertical axis in figure). The equatorial plane is divided into an inner region, for $x = 1$, and an outer region, for $x = -1$. The rotational coordinate ϕ has to be considered with respect to this plane. The figure is taken from [35], courtesy of R. Emparan and H. Reall.

In these coordinates, surfaces of constant y (for fixed t) have topology $S^1 \times S^2$. To see this, we change coordinates to

$$(t, r, \theta, \phi, \psi) \tag{2.23}$$

such that

$$r = -\frac{R}{y} \qquad\qquad \cos\theta = x$$

with ranges

$$0 \leq r \leq R \qquad\qquad 0 \leq \theta \leq \pi\,.$$

The Minkowski metric becomes

$$ds^2 = -dt^2 + \frac{1}{\left(1 + \frac{r\cos\theta}{R}\right)^2}\left[\left(1 - \frac{r^2}{R^2}\right) R^2 d\psi^2 + \frac{dr^2}{1 - \frac{r^2}{R^2}} + r^2\left(d\theta^2 + \sin^2\theta d\phi^2\right)\right].$$

It is now manifest that surfaces of constant r (for fixed t), which correspond to surfaces of constant y (for fixed t), have topology $S^1 \times S^2$.

Black ring metrics

Consider the smooth manifold with boundary

$$\mathcal{D} = \text{clos}(\mathbb{R} \times \Sigma)$$

with

$$\Sigma = \mathbb{R}^4 \setminus (S^1 \times B^3)$$

and differential structure given by ring coordinates (t, y, x, ϕ, ψ), where B^3 is the closed 3-ball. The *black ring exterior* is the Lorentzian manifold with boundary $(\mathcal{D}, g_{\text{ring}})$, where g_{ring} is the five-dimensional, singly-spinning black ring metric[19] with line element [34, 35]

$$ds^2 = -\frac{F(y)}{F(x)}\left(dt - CR\frac{(1+y)}{F(y)}\,d\psi\right)^2 + \frac{R^2}{(x-y)^2}F(x)\left[-\frac{G(y)}{F(y)}\,d\psi^2 - \frac{dy^2}{G(y)} + \frac{dx^2}{G(x)} + \frac{G(x)}{F(x)}\,d\phi^2\right] \tag{2.24}$$

on $\mathcal{D}$. Functions $F(\cdot)$ and $G(\cdot)$ are defined as

$$F(\xi) = 1 + \lambda\xi \qquad\qquad G(\xi) = (1-\xi^2)(1+\nu\xi)$$

and the positive constant C is

$$C = \sqrt{\lambda(\lambda-\nu)\frac{1+\lambda}{1-\lambda}}\,.$$

The coordinate ranges are

$$t \in (-\infty, +\infty) \qquad y \in (-\infty, -1] \qquad x \in [-1, 1]$$

$$\phi \in \left[0, 2\pi\sqrt{\frac{1}{1+\nu^2}}\right) \qquad \psi \in \left[0, 2\pi\sqrt{\frac{1}{1+\nu^2}}\right)\,.$$

The parameters ν and λ satisfy

$$\nu, \lambda \in \mathbb{R} \qquad\qquad 0 < \nu \leq \lambda < 1$$

and the equilibrium condition

$$\lambda = \frac{2\nu}{1+\nu^2}\,. \tag{2.25}$$

Condition (2.25) is necessary to avoid conical singularities in addition to those introduced by degenerations of our coordinate system. As a result, parameters λ and ν are *not* independent, leaving us with only two independent parameters ν and R. The black ring metric can therefore be understood as a *two-parameter family* (ν, R) *of metrics.*

Black rings are *stationary*, *bi-axially symmetric*, *asymptotically flat* spacetimes. The event horizon corresponds to the surface $y = -1/\nu$ and it is a Killing horizon with associated positive surface gravity. For each fixed time t, sections of the horizon have topology $S^1 \times S^2$. The rotation of the singly-spinning black ring can be interpreted as a rotation along the S^1. See [34] for further details.

The metric admits three independent Killing vector fields, corresponding to stationarity and two rotational symmetries. The stationary Killing vector field $\partial/\partial t$ is timelike for $y > -1/\lambda$, null at $y = -1/\lambda$ and spacelike for $-1/\nu \leq y < -1/\lambda$. The surface $y = -1/\lambda$ is an ergosurface

[19] From now on, we will be frequently referring to singly-spinning black rings as *black rings*. Note that there exist *doubly*-spinning black rings [92].

and has sectional topology $S^1 \times S^2$.

Spacelike infinity lies at $x = y = -1$, while the limit $y \to -\infty$ corresponds to a curvature (spacelike) singularity. The ring metric (2.24) presents coordinate singularities along the axes (including at spacelike infinity) and at the event horizon. The metric is regular at the ergosurface, where $F(y) \to 0$.

For future convenience, we are interested in re-writing metric (2.24) in coordinates (2.23). After changing from (ν, λ) to new parameters (r_0, β) such that

$$\nu = \frac{r_0}{R} \qquad\qquad \lambda = \frac{r_0 \cosh^2 \beta}{R}\,,$$

the ring metric in coordinates (2.23) becomes [35]

$$\begin{aligned} ds^2 = &-\frac{\hat{f}}{\hat{g}} \left(dt - r_0 \sinh\beta \cosh\beta \sqrt{\frac{R + r_0 \cosh^2 \beta}{R - r_0 \cosh^2 \beta}}\, \frac{\frac{r}{R} - 1}{r \hat{f}}\, R\, d\psi \right)^2 \\ &+ \frac{\hat{g}}{(1 + \frac{r\cos\theta}{R})^2} \left[\frac{f}{\hat{f}} \left(1 - \frac{r^2}{R^2}\right) R^2\, d\psi^2 + \frac{dr^2}{(1 - \frac{r^2}{R^2}) f} + \frac{r^2}{g}\, d\theta^2 + \frac{g}{\hat{g}}\, r^2 \sin^2\theta\, d\phi^2 \right] \end{aligned} \tag{2.26}$$

with

$$\begin{aligned} f &= 1 - \frac{r_0}{r} & \hat{f} &= 1 - \frac{r_0 \cosh^2\beta}{r} \\ g &= 1 + \frac{r_0}{R}\cos\theta & \hat{g} &= 1 + \frac{r_0 \cosh^2\beta}{R}\cos\theta\,. \end{aligned}$$

The event horizon now corresponds to $r = r_0 < R$, the ergosurface to $r = r_0 \cosh^2 \beta$ and spacelike infinity to $(r, \theta) = (R, \pi)$. The equilibrium condition (2.25) becomes

$$\cosh\beta = \sqrt{\frac{2R^2}{r_0^2 + R^2}}\,. \tag{2.27}$$

As before, we understand black rings as a two-parameter family (r_0, R).

The metric in form (2.26) shows how the parameter ν can be interpreted. This is the ratio between the radius r_0 of the S^2 at the event horizon and the radius R of the ring. We refer to *thin black rings* when ν is *small.* We sometimes use the expression *large radius, thin black rings* to emphasise that we are considering black rings whose parameter ν is small because R is very large (in contrast to thin black rings for which $R \sim 1$ and r_0 very small).

The event horizon has angular velocity

$$\Omega_{\mathcal{H}} \sim \frac{1}{R}\sqrt{\frac{\lambda - \nu}{\lambda(1+\lambda)}}\,.$$

In view of the equilibrium condition (2.25), $\Omega_{\mathcal{H}}$ can be re-written as a quantity depending on ν and R only. This shows that the rotational velocity of the ring cannot be freely specified, but it is univocally determined by its "thickness". *Fat* black rings ($\nu \sim 1$) need more rotation than thin black rings to prevent the ring from collapsing.

The boosted black string limit

Consider the black ring metric (2.26). In the limit

$$r,\, r_0,\, r_0 \cosh^2 \beta \ll R\,, \tag{2.28}$$

the black ring metric reduces to the one of a boosted black string with boost parameter

$$\beta = \cosh^{-1} \sqrt{2}\,.$$

This can be seen by fixing r and r_0 in (2.26) and sending R to infinity (see also [32]). The boost parameter is fixed in the limit by relation (2.27). The z-coordinate of the string is defined as $z = R\,\psi$, with periodic identification $z = z + R\,\Delta\psi$. In limit (2.28), $\Delta\psi \to 2\pi$ and one correctly obtains the boosted string metric (2.21).

From a more heuristic point of view, limit (2.28) corresponds to large radius, thin black rings. Condition $r \ll R$ implies that thin black rings approximate the geometry of boosted black strings in a region sufficiently close to the black ring horizon. Note that such a limit is not uniform in r.[20]

2.3.4 Foliation

We define Σ_t as the hypersurface of constant t and denote by Σ_0 the hypersurface $\Sigma_{t=0}$.

Given any of the black hole spacetimes $(\mathcal{M}, g)$ introduced in the previous sections, we fix a Cauchy hypersurface Σ_{Cauchy} of $\mathcal{M}$.[21] We define a time function $t^* : J^+(\Sigma_{\text{Cauchy}}) \cap \mathcal{D} \to \mathbb{R}$ such that

$$t^*|_{\Sigma_{\text{Cauchy}} \cap \mathcal{D}} = 0$$

and $T(t^*) = 1$, where T is the stationary Killing vector field. We define the hypersurface

$$\Sigma_0 := \Sigma_{\text{Cauchy}} \cap \mathcal{D}$$

and, for any $\tau > 0$,

$$\Sigma_\tau := \Phi_{t^*}(\Sigma_0)\,,$$

where Φ_{t^*} is the one-parameter group of diffeomorphisms generated by the Killing vector field T from 0 to τ. We have that $\{\Sigma_\tau\}_{\tau \geq 0}$ is a regular spacelike foliation of $J^+(\Sigma_{\text{Cauchy}}) \cap \mathcal{D}$. For black rings, hypersurfaces Σ_τ are asymptotically flat.

[20] In fact, the two spacetimes have different asymptotic structure.

[21] This is possible because $(\mathcal{M}, g)$ is globally hyperbolic.

2.4 Vectorfield Method and energy currents

In the context of the energy methods, the idea of obtaining positive definite quantities (energies) by contracting the energy momentum tensor (defined below) with suitable vector fields is an application of the *vectorfield method*. The construction of vector field multipliers captures the geometry of the problem and its interplay with the analysis. In this section, we illustrate how energies can be rigorously defined via such vector field multipliers.

Given a scalar field $\Psi : \mathcal{D} \to \mathbb{C}$ on the black ring exterior $(\mathcal{D}, g_{\text{ring}})$, we define the associated *energy-momentum tensor* by

$$\mathbb{T}_{\mu\nu}[\Psi] := \text{Re}\left(\nabla_\mu \Psi \cdot \overline{\nabla_\nu \Psi}\right) - \frac{1}{2}\, g_{\mu\nu} |\nabla \Psi|^2 \,,$$

where $g_{\mu\nu}$ are components of the ring metric g_{ring}. If $\Box_{g_{\text{ring}}} \Psi = 0$, then

$$\nabla^\mu \mathbb{T}_{\mu\nu}[\Psi] = 0 \,.$$

Consider a regular, future directed, everywhere *timelike* vector field N on $\mathcal{D}$ such that $[T, N] = 0$ on $\mathcal{D}$. We define the associated *N-energy current* as

$$\mathbb{J}^N_\mu[\Psi] := \mathbb{T}_{\mu\nu}[\Psi] N^\nu \,.$$

The (non-degenerate) *N-energy* associated to Ψ at time t^* is defined as[22]

$$\mathcal{E}^N[\Psi](t^*) := \int_{\Sigma_{t^*}} \mathbb{J}^N_\mu[\Psi] n^\mu_{\Sigma_{t^*}} \,\overline{dvol}_g \,,$$

where $n^\mu_{\Sigma_{t^*}}$ is the future directed, unit normal to the spacelike hypersurface Σ_{t^*} (as defined in Section 2.3.4) and the integration is with respect to the volume form associated to the Riemannian metric induced by g_{ring} on Σ_{t^*} (the volume form will be often dropped). In the way it is defined, the scalar quantity $\mathbb{J}^N_\mu[\Psi] n^\mu_{\Sigma_{t^*}}$ is *positive definite.*[23]

The *local N-energy* associated to Ψ at time t^* reads

$$\mathcal{E}^N_\Omega[\Psi](t^*) := \int_{\Sigma_{t^*} \cap \Omega} \mathbb{J}^N_\mu[\Psi] n^\mu_{\Sigma_{t^*}} \,\overline{dvol}_g \,,$$

where $\Sigma_{t^*} \cap \Omega$ is some bounded set. The *k-th (higher) order N-energy* is

$$\mathcal{E}^N_k[\Psi](t^*) := \sum_{0 \le |\alpha| \le k-1} \int_{\Sigma_{t^*}} \mathbb{J}^N_\mu[\partial^\alpha \Psi] n^\mu_{\Sigma_{t^*}} \,\overline{dvol}_g \,,$$

with ∂^α in multi-index notation. Note that, for us, $\mathcal{E}^N_1[\Psi](t^*) = \mathcal{E}^N[\Psi](t^*)$.

[22] More appropriately, this is the N-energy *flux* through Σ_{t^*} associated to the scalar field Ψ.

[23] This is true by the so-called *positivity property*: If X,Y are future directed timelike vectors, then $\mathbb{T}_{\mu\nu}[\Psi] X^\mu Y^\nu > 0$. In particular, $\mathbb{T}_{\mu\nu}[\Psi] X^\mu Y^\nu \gtrsim \sum_{|\alpha|=1} |\partial^\alpha \Psi|^2$.

Remark 2.4 ($\mathcal{E}_k^N[\Psi]$ **controls derivatives of** Ψ **up to order** k**).** *If* $N = \partial_{t^*}$ *in a neighbourhood of spacelike infinity and hypersurfaces* Σ_{t^*} *are asymptotically flat, then one can easily show that*

$$\mathcal{E}^N[\Psi](t^*) \sim \sum_{|\alpha|=1} \int_{\Sigma_{t^*}} |\partial^\alpha \Psi|^2 \, \overline{dvol}_g \, ,$$

which means that $\mathcal{E}^N[\Psi](t^*)$ *is morally the* $\dot{H}^1$ *(semi-)norm of* Ψ *on* Σ_{t^*}*. Note also that* $\mathcal{E}^N[\,\cdot\,] \sim E[\,\cdot\,]$*, where* $E[\,\cdot\,]$ *are the informal energies adopted in the Introduction. For higher-order energies, we have*

$$\mathcal{E}_k^N[\Psi](t^*) = \sum_{0 \le |\alpha| \le k-1} \mathcal{E}^N[\partial^\alpha \Psi](t^*) \, ,$$

so $\mathcal{E}_k^N[\Psi](t^*)$ *corresponds to the sum of the* $\dot{H}^s$ *(semi-)norms of* Ψ *on* Σ_{t^*} *for* $1 \le s \le k$*.*

2.5 The main theorem

We give a more precise statement of Theorem 2.2 and two additional remarks.

Theorem 2.3 (**Lower Bound for Uniform Energy Decay Rate, Second Version).** *Consider the black ring exterior* $(\mathcal{D}, g_{(r_0,R)})$*, with* $g_{(r_0,R)}$ *the metric of a singly-spinning black ring* (2.26)*. Then, for any* $r_0 > 0$*, there exists a constant* $\mathcal{R} > r_0$ *such that the following statement holds for all metrics* $g_{(r_0,R)}$ *with* $R \ge \mathcal{R}$*. Fix* $t_0^* \ge 0$ *and let* N *be a regular, future directed, everywhere timelike vector field on* $J^+(\Sigma_{t_0^*})$ *such that* $[T, N] = 0$ *on* $J^+(\Sigma_{t_0^*})$ *and* $N = \partial_{t^*}$ *in a neighbourhood of spacelike infinity. Consider smooth solutions* $\Psi : J^+(\Sigma_{t_0^*}) \to \mathbb{C}$ *to the linear wave equation*

$$\Box_{g_{(r_0,R)}} \Psi = 0 \tag{2.29}$$

and assume that there exists a universal constant $B > 0$ *(independent of time) such that, for any smooth solution* Ψ *to* (2.29)*, the inequality*

$$\mathcal{E}^N[\Psi](t^*) \le B \, \mathcal{E}^N[\Psi](t_0^*)$$

holds for all $t^* \ge t_0^*$*. Then, there exists a set* $\Omega \subset J^+(\Sigma_{t_0^*})$ *and a universal constant* $C > 0$ *(independent of time) such that* $\Sigma_{t_0^*} \cap \Omega$ *is non-empty and bounded,* $\Phi_{t^*}(\Sigma_{t_0^*} \cap \Omega) = \Phi_{t^*}(\Sigma_{t_0^*}) \cap \Omega$ *remains non-empty and bounded for all* $t^* \ge t_0^*$ *and*

$$\limsup_{t^* \to +\infty} \sup_{\Psi \in SC_0^\infty(\Sigma_{t_0^*}), \Psi \neq 0} [\log(2 + t^*)]^2 \left(\frac{\mathcal{E}_\Omega^N[\Psi](t^*)}{\mathcal{E}_2^N[\Psi](t_0^*)} \right) > C \, ,$$

where the supremum is taken over the space of smooth, non-zero solutions to the wave equation (2.29) *with compactly supported initial data on* $\Sigma_{t_0^*}$ *and* Φ_{t^*} *is the one-parameter group of diffeomorphisms generated by the stationary Killing vector field* ∂_{t^*}*. Moreover, for any* $k \in \mathbb{N}$*, there exists a universal constant* $C_k > 0$ *(independent of time) such that*

$$\limsup_{t^* \to +\infty} \sup_{\Psi \in SC_0^\infty(\Sigma_{t_0^*}), \Psi \neq 0} [\log(2 + t^*)]^{2k} \left(\frac{\mathcal{E}_\Omega^N[\Psi](t^*)}{\mathcal{E}_{k+1}^N[\Psi](t_0^*)} \right) > C_k \, .$$

Remark 2.5 (**Class $\mathfrak{g}$ is not optimal**). *The class of black rings considered in the theorem can be defined as*

$$\mathfrak{g} := \big\{ g_{(r_0,R)} \text{ singly-spinning black ring metric (2.26) such that } \nu = r_0/R < \nu_0, \text{ for some suitably small constant } 0 < \nu_0 < 1 \big\} \,. \tag{2.30}$$

As we shall see later in the chapter, the constant ν_0 will be treated as a smallness parameter. For ν_0 sufficiently small but not explicitly identified, we will be able to close the proof of Theorem 2.3. In this sense, the class $\mathfrak{g}$ that our proof selects is neither explicitly constructed nor necessarily includes all black rings for which Theorem 2.3 holds.

Remark 2.6 (**Function space and Σ_{t^*}-foliation**). *Theorem 2.3 remains true if one takes the supremum over a suitably larger function space. We stated the result for $\Psi \in SC_0^\infty(\Sigma_{t_0^*})$ because this is the space that naturally emerges from the quasimode construction. Note also that the statement of the theorem does not depend on the particular choice of time coordinate t^*. In fact, the coordinate t^* can be constructed so that it agrees with t in the region $r \geq r_0 \cosh^2 \beta$. Since our analysis will be mainly carried out on a bounded set contained in such region, we are free to equivalently refer to coordinate t instead of t^* in the proof of Theorem 2.3.*

2.6 Separation of the wave equation and reduction

This section serves as a preliminary discussion to illustrate the logic behind the abstract eigenvalue problems that we are about to consider. In fact, these problems emerge as *reduced equations* when one separates the wave equation

$$\Box_g \Psi = \frac{1}{\sqrt{-\det g}}\, \partial_\mu \left(\sqrt{-\det g}\, g^{\mu\nu} \partial_\nu \Psi \right) = 0 \,. \tag{2.31}$$

The wave equation fully-separates for all the four-dimensional explicit black hole solutions, including all members of the Kerr family $g_{a,M}$ (see [9]). However, in our work we will encounter a non-fully-separable wave equation on higher dimensional black holes, namely the wave equation on black rings. The only partial separation will necessarily leave us with a two-variable PDE, which can be reduced to a *Schrödinger-type equation.*

To see this, consider a coordinate system (t, r, θ, ϕ, z), where the particular meaning of the coordinates is not important at this stage, and assume that the metric g depends on r and θ (but not on t, ϕ and z) and has zero $g_{r\theta}$ components. A formal computation to check the full-separability of the wave equation can be carried out by introducing an ansatz of the form

$$\Psi_{\text{ansatz}}(t, r, \theta, \phi, z) = e^{i(-\omega\, t + m\, \phi + J\, z)} u^{(\omega)}_{m,J}(r, \theta) \,, \tag{2.32}$$

where $\omega \in \mathbb{R}$ and $m, J \in \mathbb{Z}$ and the choice of the complex exponential factor is determined by the symmetries of the metric g. Any solution Ψ to the wave equation (2.31) can be formally written as an infinite sum of solutions of the form (2.32) as follows

$$\Psi(t,r,\theta,\phi,z) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} \sum_{m,J} e^{-i\omega t} e^{i(m\,\phi + J\,z)}\, u^{(\omega)}_{m,J}(r,\theta)\, d\omega\,, \tag{2.33}$$

where the Fourier transform in time and Fourier series decompositions in ϕ and z have been taken.[24,25] Therefore, if equation (2.31) separates for the ansatz (2.32), then it separates for any solution Ψ.

In order for the ansatz (2.32) to be a solution to the wave equation, functions $u^{(\omega)}_{m,J}$ must satisfy[26]

$$\begin{aligned} &\frac{1}{\sqrt{-\det g}}\, \partial_r \left(\sqrt{-\det g}\, g^{rr} \partial_r u(r,\theta) \right) \\ &\qquad + \frac{1}{\sqrt{-\det g}}\, \partial_\theta \left(\sqrt{-\det g}\, g^{\theta\theta} \partial_\theta u(r,\theta) \right) + V_{(\omega,m,J)}(r,\theta) u(r,\theta) = 0 \end{aligned} \tag{2.34}$$

for some *real* function $V_{(\omega,m,J)}(r,\theta)$. If by factorizing

$$u(r,\theta) = R(r)\,\Theta(\theta)$$

equation (2.34) gives a system of two decoupled ODEs, one for $R(r)$ and one for $\Theta(\theta)$, then the wave equation is fully-separable. If such factorization does not separate (2.34), then the wave equation is not fully-separable.

Assuming the latter scenario, we aim to reduce the PDE (2.34) to a Schrödinger-type equation of the form

$$-\Delta_{(r_*,\theta_*)} \tilde{u}(r_*,\theta_*) + \tilde{V}_{(\omega,m,J)}(r,\theta) \tilde{u}(r_*,\theta_*) = 0\,,$$

with $\Delta_{(r_*,\theta_*)} = \partial^2_{r_*} + \partial^2_{\theta_*}$ and some real function $\tilde{V}_{(\omega,m,J)}(r,\theta)$. To do this, we need to implicitly define new coordinates (r_*,θ_*) as follows

$$\frac{dr_*}{dr} = a(r) \qquad\qquad \frac{d\theta_*}{d\theta} = b(\theta)$$

and define a new function $\tilde{u}(r_*,\theta_*)$ such that

$$u(r,\theta) = h(r,\theta)\, \tilde{u}(r_*,\theta_*)\,,$$

[24] To take the Fourier transform in time

$$\Psi(t,\cdot) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} e^{-i\omega t}\, \hat{\Psi}(\omega,\cdot)\, d\omega\,,$$

one should first know that the solution Ψ is in L^2_t. However, this cannot be established a priori, so a cut-off in time is usually needed. This technical issue will not occur in our problem.

[25] The functions $u^{(\omega)}_{m,J}(r,\theta)$ have to be thought as Fourier coefficients.

[26] Equation (2.34) is obtained by simply plugging the ansatz in (2.31). For simplicity, we write u instead of $u^{(\omega)}_{m,J}$. If one plugs-in the full solution Ψ in the form (2.33), then equation (2.34) has to be understood in an integral sense, i.e. as an equality in $L^2_\omega l^2_{m,J}$ for each fixed (r,θ). Furthermore, some further integrability conditions on Ψ would be needed.

for some real functions $a(r), b(\theta)$ and $h(r,\theta)$. If the system of equations

$$\begin{cases} \partial_r \left(\sqrt{-\det g}\, g^{rr}\, a(r)\right) h(r,\theta) + 2\left(\sqrt{-\det g}\, g^{rr}\, a(r)\right) \partial_r h(r,\theta) = 0 \\ \partial_\theta \left(\sqrt{-\det g}\, g^{\theta\theta}\, b(\theta)\right) h(r,\theta) + 2\left(\sqrt{-\det g}\, g^{\theta\theta}\, b(\theta)\right) \partial_\theta h(r,\theta) = 0 \\ a^2(r) g^{rr} = b^2(\theta) g^{\theta\theta} \end{cases} \tag{2.35}$$

holds, then the wave equation (2.34) correctly reduces to the Schrödinger-type equation wanted. Note that the last equation is key, since it gives a necessary condition for the reduction to be possible. Indeed, we have

$$a^2(r) = \frac{g^{\theta\theta}}{g^{rr}}\, b^2(\theta)\,,$$

which holds if and only if there exists $b(\theta)$ such that $\left(g^{\theta\theta}/g^{rr}\right) b^2(\theta)$ is independent of θ, i.e. the function $g^{\theta\theta}/g^{rr}$ needs to be separable. Note that $g^{\theta\theta}/g^{rr}$ is consistently positive because the (r,θ)-block of g is Riemannian and $g_{r\theta} = 0$.

If such condition is satisfied,[27] then the first two equations of the system give an explicit expression for $h(r,\theta)$, which reads

$$h(r,\theta) = \left(\sqrt{-\det g}\, \sqrt{g^{rr} g^{\theta\theta}}\right)^{-\frac{1}{2}}\,,$$

while the third equation determines $a(r)$ and $b(\theta)$ up to multiplication by a real factor.

Remark 2.7 (**Isothermal coordinates**). *For any two dimensional Riemannian manifold, there always exists a system of locally isothermal coordinates, for which the metric is conformal to the Euclidean metric (see, for instance, [10]). In view of this abstract result, the reduction that we have just discussed is, in fact, always possible locally.*

In what follows, we will consider abstract eigenvalue problems without further referring to the separation of the wave equation.

2.7 Eigenvalue problem for the static black string

In this section we discuss the first part of the proof of Theorem 2.3. The structure of the exposition and the formulation of the results closely follow Section 4 in [69].

With coordinates as in Section 2.3.1 and ansatz (2.32), equation (2.34) for static black strings gives the eigenvalue problem

$$-\frac{f(r)}{r^2}\partial_r(r^2 f(r)\partial_r u) - \frac{f(r)}{r^2\sin\theta}\partial_\theta(\sin\theta\partial_\theta u) + \left[\frac{f(r)}{r^2\sin^2\theta}m^2 + f(r)J^2\right] u = \omega^2 u\,,$$

[27] This will be the case for black string and black ring metrics.

with $u = u(r, \theta)$, $f(r) = 1 - r_0/r$, $\omega \in \mathbb{R}$ and $m, J \in \mathbb{Z}$. For any m, J such that

$$m^2 > 3J^2 r_0^2 \,, \tag{2.36}$$

the everywhere positive potential

$$\frac{f(r)}{r^2 \sin^2 \theta} m^2 + f(r) J^2$$

has a local maximum and a local minimum at $(r_{\max}, \pi/2)$ and $(r_{\min}, \pi/2)$ respectively, with $r_0 < r_{\max} < r_{\min}$.

Let us now *fix* the scaling between m and J by setting

$$J = b\, m \,, \qquad\qquad b^2 < \frac{1}{3r_0^2} \,,$$

with $b \in \mathbb{Q}$ positive constant.[28] In this way, condition (2.36) is satisfied and, therefore, *the potential has a local minimum.*

The eigenvalue problem becomes

$$-\frac{f(r)}{r^2} \partial_r (r^2 f(r) \partial_r u) - \frac{f(r)}{r^2 \sin\theta} \partial_\theta (\sin\theta \partial_\theta u) + m^2 \left[\frac{f(r)}{r^2 \sin^2\theta} + b^2 f(r) \right] u = \omega^2 u \,.$$

We now change coordinates to (r_*, θ) and define a function $h(r, \theta)$ as follows

$$h(r, \theta) := (r \sin\theta)^{-\frac{1}{2}} f(r)^{-\frac{1}{4}} \,, \qquad\qquad \frac{dr_*}{dr} = \frac{1}{r\sqrt{f(r)}} \,.$$

Let $u(r, \theta) = h(r, \theta)\, \tilde{u}(r_*, \theta)$. With the abuse of notation $\tilde{u}(r_*, \theta) = u(r_*, \theta)$, the eigenvalue problem reduces to

$$g(r)(-\partial^2_{r_*} u - \partial^2_\theta u) + [V_j(r, \theta) + m^2 V(r, \theta)] u = \omega^2 u \,,$$

where

$$\begin{aligned} g(r) &:= \frac{f(r)}{r^2} \,, \\ V_j(r, \theta) &:= -\frac{(r - r_0)}{4r^3 \sin^2\theta} - \frac{r_0^2}{16 r^4} \,, \\ V(r, \theta) &:= \frac{f(r)}{r^2 \sin^2\theta} + b^2 f(r) \,. \end{aligned}$$

Note that these are smooth, real-valued functions which are bounded away from the axis $r = 0$.

[28] The sign of b does not matter at this stage, since m and J always appear squared.

We define

$$h^{-2} := m^2 \,,$$
$$V_{\text{eff}}^h(r,\theta) := h^2 V_j(r,\theta) + V(r,\theta)\,,$$
$$\kappa := \omega^2 h^2 \,,$$

and write

$$-h^2 g(r)\Delta_{(r_*,\theta)} u + V_{\text{eff}}^h(r,\theta) u = \kappa u\,, \tag{2.37}$$

where the Laplacian is the Cartesian Laplacian and, again, $u = u(r_*, \theta)$.

For the first part of our discussion, we will be considering the eigenvalue problem (2.37) on a bounded domain Ω, for which Dirichlet boundary conditions on $\partial\Omega$ will be imposed. Our interest is to determine the existence of eigenvalues arbitrarily close to some suitably fixed energy level.

2.7.1 Continuity of the potential

By continuity of the potential V, we have the following lemma, which defines the domain Ω on which our eigenvalue problem will be formulated and suitable energy levels E. We use the notation $x = (r, \theta)$ for a point in $[r_0, \infty) \times [0, \pi)$.

Lemma 2.1 (adapted from [69] Lemma 4.1)**.** *Define $V_{\min}$ to be the minimum of V and $x_{\min} \in (r_0, \infty) \times (0, \pi)$ such that $V(x_{\min}) = V_{\min}$. From a previous observation, $r(x_{\min}) > r_0$. Let $c > 0$ be a sufficiently small constant such that there exists $\Omega \subset [r_0, \infty) \times [0, \pi)$ for which*

$$x_{\min} \in \Omega$$

and satisfying

(i) $V(x) = V_{\min} + c$ for $x \in \partial\Omega$,

(ii) there are no local maxima of V in Ω.

Fix some energy level $E > V_{\min}$ such that $E - V_{\min} < c$. Then, for any sufficiently small constants $\delta, \delta' > 0$, there exists some constant $c' > 0$ such that

$$\text{dist}(x, \partial\Omega) < \delta' \implies V(x) - \kappa > c'$$

for all $\kappa \in [E-\delta, E+\delta]$, with $\text{dist}(\cdot,\cdot)$ the Euclidean distance. Note that one can reformulate the lemma for V_{eff}^h instead of V, where all the constants appearing in the statement are independent of h for h sufficiently small.

Remark 2.8 (Ω **has smooth boundary**)**.** *The boundary of Ω is defined as a level set of the smooth function V. Since, by definition of Ω, the gradient of V is non zero at each point of*

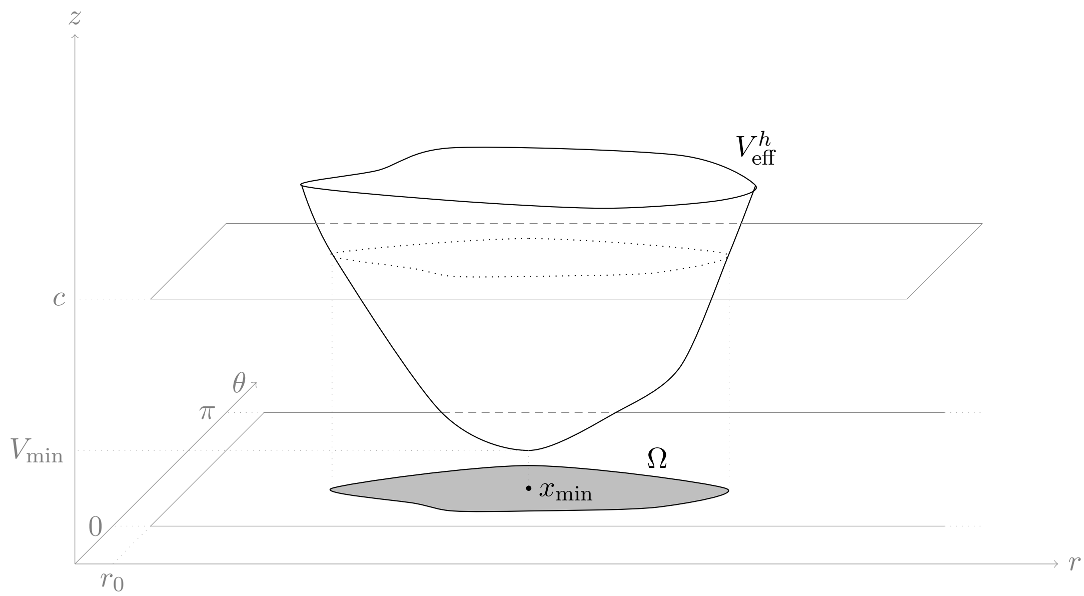

Figure 2.4: The potential in figure is illustrative, it is not the graph of V_{eff}^h. The shaded region corresponds to Ω.

the level set, one can conclude that the level set (and therefore $\partial\Omega$) is a smooth curve (this is an application of the Implicit Function Theorem). The set Ω is therefore a compact set with smooth boundary.

The abstract eigenvalue problem that we consider is

$$\boxed{\begin{array}{rlll} -h^2 g(r)\Delta_{(r_*,\theta)}u + V_{\text{eff}}^h(r,\theta)u = & \kappa\, u & \text{on} & \Omega \\ u = & 0 & \text{on} & \partial\Omega\,. \end{array}} \tag{2.38}$$

2.7.2 Weyl's law for (2.38)

In this section we prove the following theorem, which is a version of Weyl's law for the eigenvalue problem (2.38). When the operator is a pure Laplacian on a bounded domain, this result is a standard asymptotic property of the eigenvalues. In our case, the presence of a potential requires more work, but without major additional difficulties. Our argument follows the lines of [106, 69] and review [105].

Theorem 2.4 (**Weyl's law**, adapted from [69] Lemma 4.2)**.** *Consider the abstract eigenvalue problem* (2.38)*. Let $E > V_{\min}$ be an energy level such that $E - V_{\min}$ is sufficiently small and fix some positive constant $\delta < E - V_{\min}$ such that $E + \delta < c$, with $c > 0$ the constant introduced in Lemma 2.1. Then, the number of eigenvalues of* (2.38) *in the interval $[E-\delta, E+\delta]$, denoted by $N[E-\delta, E+\delta]$, tends to infinity as $h \to 0$. In particular,*

$$N[E-\delta, E+\delta] \sim \frac{1}{\pi h^2}\int_\Omega (E - V)\,\chi_{\{V\le E\}}dr_* d\theta\,,$$

as $h \to 0$, with $\chi_{\{\cdot\}}$ indicator function.

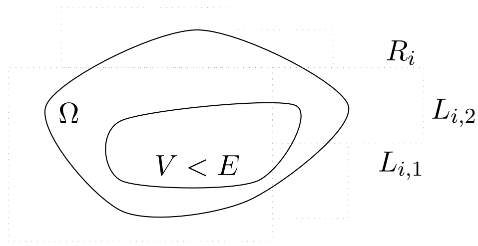

Figure 2.5: Finite family of recatngles R_i covering Ω.

Remark 2.9. *The scaling in $N \sim h^{-2}$ of Theorem 2.4 is different from the one obtained in [69], where $N \sim h^{-1}$. In fact, from Weyl's law, one should expect the number of eigenvalues N to scale like h^{-d}, where d is the dimension of the compact domain Ω. For the eigenvalue problem considered in [69], Ω is one-dimensional (a closed interval on the real line), while for our eigenvalue problem Ω is a two-dimensional compact set.*

Let $\sigma(\Omega)$ be the set of all possible finite families σ of open sets covering Ω up to a set of measure zero. For any family $\sigma \in \sigma(\Omega)$, we require that

(i) each open set of σ is an open rectangle R_i of area $A_i = L_{i,1} \times L_{i,2}$,

(ii) $R_i \cap R_j = \emptyset$ for any $i \neq j$, with $i, j \in \mathcal{I}$,

(iii) $R_i \cap \Omega \neq \emptyset$ for any $i \in \mathcal{I}$.

Note that, in case condition (ii) is not satisfied for some family of open rectangles satisfying (i) and (iii), one can always consider a refinement of such family for which condition (ii) holds (this being true because the intersection $R_i \cap R_j$ of two rectangles is still a rectangle).

Let P be an eigenvalue problem. We denote by $N_{\leq E}(P)$ the number of eigenvalues of the problem P which are less or equal to some fixed energy value E.

Let us define the problem $\tilde{P}_D^i$ as

$$\begin{aligned} -C_\Omega^+ h^2 \Delta_{(r_*,\theta)} u + V_+^i u = &\quad \kappa\, u \qquad \text{on} \quad R_i \cap \Omega \\ u = &\quad 0 \qquad \text{on} \quad \partial(R_i \cap \Omega)\,, \end{aligned}$$

where

$$C_\Omega^+ := \sup_\Omega \left[2r_*^2 + \frac{1}{2} |\partial_{r_*} g(r)|^2 + g(r) \right] , \qquad\qquad V_+^i := \sup_{R_i \cap \Omega} \left[V_{\text{eff}}^h(r, \theta) \right] .$$

Note that the constant $C_\Omega^+ > 0$ is finite.

The two following lemmas, Lemma 2.2 and Lemma 2.3, are the key technical ingredients to prove Theorem 2.4. We will only sketch their proofs, further details can be found in the proof of Lemma 4.2 in [69].

Lemma 2.2 (**Weyl's lower bound**). *For any family $\sigma \in \sigma(\Omega)$, we have*

$$\sum_{i\in\mathcal{I}} N_{\leq E}(\tilde{P}_D^i) \leq N_{\leq E}(P_D(\Omega))\,, \tag{2.39}$$

where $P_D(\Omega)$ is the eigenvalue problem (2.38).

Remark 2.10 (**Rectangles with no eigenvalues**). *By fixing the value E as in the statement of Theorem 2.4, the bounded domain Ω gets separated into two disjoint regions, namely a region where $V \leq E$ and a region where $V > E$ (see Figure 2.5). The latter region extends up to the boundary $\partial\Omega$. The problem $\tilde{P}_D^i$ formulated on a rectangle R_i intersecting the latter region will involve a constant potential $V_+^i > E$, so $N_{\leq E}(\tilde{P}_D^i) = 0$ by classical arguments. In particular, this remains true for each $R_i \in \sigma$ such that $R_i \not\subset \Omega$, i.e. for each $R_i \in \sigma$ exceeding Ω. For this reason, when we consider a particular family $\sigma \in \sigma(\Omega)$, the R_i exceeding Ω (as well as any R_i intersecting the region where $V > E$) do not contribute to the sum appearing on the left hand side of* (2.39).

Proof. From min-max theory, the n-th eigenvalue of $P_D(\Omega)$ is given by

$$\kappa_n = \inf_{\substack{(f_1,\dots,f_n),\, f_k\in H_0^1(\Omega) \\ \|f_k\|_{L^2}\neq 0,\, \langle f_k,f_j\rangle=0\,\forall k\neq j}} \max_{k\leq n} Q_\Omega[f_k]$$

$$Q_\Omega[f_k] := \frac{\int_\Omega \left[h^2 f_k(\partial_{r_*} f_k)(\partial_{r_*} g(r)) + h^2 g(r)\left(|\partial_{r_*} f_k|^2 + |\partial_\theta f_k|^2\right) + V_{\text{eff}}^h(r,\theta)|f_k|^2\right] dr_* d\theta}{\|f_k\|^2_{L^2(\Omega)}}\,.$$

Similarly, the n-th eigenvalues for the Dirichlet problem P_D^i, defined as

$$\begin{aligned} -h^2 g(r)\Delta_{(r_*,\theta)} u + V_{\text{eff}}^h(r,\theta) u &= \kappa u \qquad \text{on} \quad R_i \cap \Omega \\ u &= 0 \qquad \text{on} \quad \partial(R_i\cap\Omega)\,, \end{aligned}$$

is determined by the formula

$$\lambda_n^i = \inf_{\substack{(f_1,\dots,f_n),\, f_k\in H_0^1(R_i\cap\Omega) \\ \|f_k\|_{L^2}\neq 0,\, \langle f_k,f_j\rangle=0\,\forall k\neq j}} \max_{k\leq n} Q_\Omega^i[f_k]$$

$$Q_\Omega^i[f_k] := \frac{\int_{R_i\cap\Omega} \left[h^2 f_k(\partial_{r_*} f_k)(\partial_{r_*} g(r)) + h^2 g(r)\left(|\partial_{r_*} f_k|^2 + |\partial_\theta f_k|^2\right) + V_{\text{eff}}^h(r,\theta)|f_k|^2\right] dr_* d\theta}{\|f_k\|^2_{L^2(R_i\cap\Omega)}}\,.$$

We order the eigenvalues λ_n^i in a non-decreasing sequence $\lambda_1 < \lambda_2 < \dots$ *for all i*. For each n, one has

$$\kappa_n \leq \lambda_n\,.$$

The proof of this fact essentially relies on the variational definition of the eigenvalues. The reader interested in the details should refer to Sublemma 4.2.1 in [69] and realise that the argument therein can be easily adapted to our PDE setting.

We observe that, for any $f_k \in H_0^1(R_i \cap \Omega)$, we have

$$\begin{aligned}
&\int_{R_i\cap\Omega} h^2 \left[f_k(\partial_{r_*} f_k)(\partial_{r_*} g(r)) + g(r)\left(|\partial_{r_*} f_k|^2 + |\partial_\theta f_k|^2\right)\right] dr_* d\theta \\
&\leq \int_{R_i\cap\Omega} h^2 \left[\frac{1}{2}|f_k|^2 + \frac{1}{2}|\partial_{r_*} g(r)|^2 |\partial_{r_*} f_k|^2 + g(r)\left(|\partial_{r_*} f_k|^2 + |\partial_\theta f_k|^2\right)\right] dr_* d\theta \\
&\leq \int_{R_i\cap\Omega} h^2 \left[\left(2r_*^2 + \frac{1}{2}|\partial_{r_*} g(r)|^2 + g(r)\right) |\partial_{r_*} f_k|^2 + g(r)\,|\partial_\theta f_k|^2\right] dr_* d\theta \\
&\leq \int_{R_i\cap\Omega} C_\Omega^+ h^2 \left(|\partial_{r_*} f_k|^2 + |\partial_\theta f_k|^2\right) dr_* d\theta\,,
\end{aligned} \tag{2.40}$$

where we used Young's and Hardy's (Poincaré's) inequalities and introduced the positive constant

$$C_\Omega^+ = \sup_\Omega \left[2r_*^2 + \frac{1}{2}|\partial_{r_*} g(r)|^2 + g(r)\right] .$$

In particular, note that C_Ω^+ does not depend on i. Let us now define

$$\tilde{\lambda}_n^i := \inf_{\substack{(f_1,\ldots,f_n),\, f_k \in H_0^1(R_i\cap\Omega) \\ \|f_k\|_{L^2} \neq 0,\, \langle f_k, f_j\rangle = 0\, \forall k\neq j}} \max_{k\leq n} \frac{\int_{R_i\cap\Omega} \left[C_\Omega^+ h^2 \left(|\partial_{r_*} f_k|^2 + |\partial_\theta f_k|^2\right) + V_+^i |f_k|^2\right] dr_* d\theta}{\|f_k\|^2_{L^2(R_i\cap\Omega)}}\,, \tag{2.41}$$

where

$$V_+^i = \sup_{R_i\cap\Omega} \left[V_{\text{eff}}^h(r,\theta)\right] .$$

From estimate (2.40), combined with the variational definition of the eigenvalues, we have $\lambda_n^i \leq \tilde{\lambda}_n^i$ for each i because the Rayleigh quotient gets increased. Therefore, after ordering $\tilde{\lambda}_n^i$ in a non-decreasing sequence for all i, we have

$$\kappa_n \leq \lambda_n \leq \tilde{\lambda}_n\,. \tag{2.42}$$

Recall now the definition of the problem $\tilde{P}_D^i$. Formula (2.41) gives the n-th eigenvalue of $\tilde{P}_D^i$. So, from (2.42), we can conclude

$$\sum_{i\in\mathcal{I}} N_{\leq E}(\tilde{P}_D^i) \leq N_{\leq E}(P_D(\Omega))\,,$$

where the result is independent of the family $\sigma \in \sigma(\Omega)$ chosen.

□

Lemma 2.3 (Weyl's upper bound). *For any family $\sigma \in \sigma(\Omega)$, we have*

$$N_{\leq E}(P_D(\Omega)) \leq \sum_{i\in\mathcal{I}} N_{\leq E}(\tilde{P}_N^i)\,,$$

where $\tilde{P}_N^i$ is the problem

$$-C_\Omega^- h^2 \Delta_{(r_*,\theta)} u + V_-^i u = \kappa u \quad \text{on} \quad R_i \cap \Omega\,,$$

with Neumann boundary conditions imposed on $\partial(R_i \cap \Omega)$ *and*

$$C_\Omega^- := \frac{1}{2}\inf_\Omega g(r)\,, \qquad\qquad V_-^i := \inf_{R_i\cap\Omega}\left[V_{\text{eff}}^h(r,\theta)\right] - h^2\frac{C_\Omega^+}{4C_\Omega^-}\,.$$

Proof. Consider the variational definition of the eigenvalues

$$\mu_n^i = \inf_{\substack{(f_1,\ldots,f_n),\, f_k\in H^1(R_i\cap\Omega)\\ \|f_k\|_{L^2}\neq 0,\, \langle f_k,f_j\rangle=0\,\forall k\neq j}} \max_{k\leq n} Q_\Omega^i[f_k]$$

and order them in a non-decreasing sequence $\mu_1 < \mu_2 < \ldots$ for all i. These are eigenvalues of the problem P_N^i, defined as the problem P_D^i but now with Neumann boundary conditions imposed on $\partial(R_i\cap\Omega)$. Note that the space of test functions is now H^1 instead of H_0^1. Following the argument in [69], we define the space

$$Y := \Big\{ f \in H^1(\Omega) \text{ such that } \|f\|^2_{L^2(\Omega)} \neq 0 \text{ and } f \text{ is in the closure in } H^1 \text{ of piecewise } C^2 \\ \text{functions, which are } C^2 \text{ on each } R_i\cap\Omega \Big\}$$

and eigenvalues

$$\hat{\mu}_n = \inf_{\substack{(f_1,\ldots,f_n),\, f_k\in Y\\ \|f_k\|_{L^2}\neq 0,\, \langle f_k,f_j\rangle=0\,\forall k\neq j}} \max_{k\leq n} Q_\Omega[f_k]\,.$$

These are eigenvalues of the problem $\hat{P}_N(\Omega)$, defined as $P_D(\Omega)$ but with Neumann boundary conditions on $\partial\Omega$ and test function space Y. Since $H_0^1(\Omega) \subset Y$, we can easily conclude that

$$\hat{\mu}_n \leq \kappa_n\,.$$

The delicate part of the argument is now proving that $\hat{\mu}_n = \mu_n$ for each n (when counted with multiplicity). See the proof of Sublemma 4.2.1 in [69] and textbook [106] for more details on this. Once the equality $\hat{\mu}_n = \mu_n$ has been established, one has

$$\mu_n \leq \kappa_n\,.$$

Similar considerations to the ones presented in the proof of Lemma 2.2 allow to conclude that

$$\tilde{\mu}_n \leq \mu_n \leq \kappa_n\,,$$

where $\tilde{\mu}_n$ are eigenvalues of $\tilde{P}_N^i$. The estimate for the Rayleigh quotient now reads

$$\begin{aligned}
&\int_\Omega \left[h^2 f_k(\partial_{r_*} f_k)(\partial_{r_*} g(r)) + h^2 g(r)\left(|\partial_{r_*} f_k|^2 + |\partial_\theta f_k|^2\right) + V_{\text{eff}}^h(r,\theta)|f_k|^2\right] dr_* d\theta \\
&\geq \int_{R_i\cap\Omega}\left[-h^2\varepsilon|\partial_{r_*} f_k|^2 - \frac{h^2}{4\varepsilon}|\partial_{r_*} g(r)|^2|f_k|^2| + h^2 g(r)\left(|\partial_{r_*} f_k|^2 + |\partial_\theta f_k|^2\right) + V_{\text{eff}}^h(r,\theta)|f_k|^2\right] dr_* d\theta \\
&\geq \int_{R_i\cap\Omega}\left[h^2(g(r)-\varepsilon)\left(|\partial_{r_*} f_k|^2 + |\partial_\theta f_k|^2\right) + \left(V_{\text{eff}}^h(r,\theta) - \frac{h^2}{4\varepsilon}|\partial_{r_*} g(r)|^2\right)|f_k|^2\right] dr_* d\theta\,,
\end{aligned}$$

where one chooses $\varepsilon = C_\Omega^-$ and obtains that the last line is greater or equal to

$$\int_{R_i\cap\Omega}\left[C_\Omega^- h^2\left(|\partial_{r_*}f_k|^2+|\partial_\theta f_k|^2\right)+\left(V_{\text{eff}}^h(r,\theta)-h^2\frac{C_\Omega^+}{4C_\Omega^-}\right)|f_k|^2\right]dr_*d\theta\,.$$

This proves the lemma.

□

Remark 2.11. *In what follows we will show how we can derive an explicit lower bound for $N_{\leq E}(P_D(\Omega))$ from Lemma 2.2. In the same way, Lemma 2.3 provides an upper bound that can be explicitly computed. Since the calculations are almost identical, we only present the computation for the former.*

The key point of inequality (2.39) is that we can derive an explicit expression for the sum appearing on the left hand side, which will provide an explicit lower bound for $N_{\leq E}(P_D(\Omega))$.

Recall that the problem $\tilde{P}_D^i$ reads

$$-C_\Omega^+ h^2\Delta_{(r_*,\theta)}u + V_+^i u = \kappa u\,,$$

with Dirichlet boundary conditions imposed on $\partial(R_i\cap\Omega)$. The open set $R_i\cap\Omega$ is an open *rectangle* if $R_i\cap\Omega\equiv R_i$, otherwise it has a more general shape. Remember that, in the latter case where $R_i\cap\Omega\not\equiv R_i$, we have $V_+^i > E$, so

$$N_{\leq E}(\tilde{P}_D^i) = 0\,.$$

Therefore, these R_i do not enter the computation of the sum in (2.39), for which it is enough to only consider the R_i such that $R_i\cap\Omega$ are rectangles.

For such R_i, the eigenvalue problem $\tilde{P}_D^i$ admits eigenfunctions

$$u_{n_{i,1},n_{i,2}}(r_*,\theta) = \sin\left(\frac{n_{i,1}\pi}{L_{i,1}}\,r_*\right)\sin\left(\frac{n_{i,2}\pi}{L_{i,2}}\,\theta\right)$$

with eigenvalues

$$\tilde{\lambda}^i_{n_{i,1},n_{i,2}} = \left(\frac{n_{i,1}\pi}{L_{i,1}}\right)^2+\left(\frac{n_{i,2}\pi}{L_{i,2}}\right)^2,$$

where $n_{i,1},n_{i,2}\in\mathbb{N}$. Therefore, we have that $N_{\leq E}(\tilde{P}_D^i)$ is the number of eigenvalues $\tilde{\lambda}^i_{n_{i,1},n_{i,2}}$ satisfying

$$\left(\frac{n_{i,1}}{L_{i,1}}\right)^2+\left(\frac{n_{i,2}}{L_{i,2}}\right)^2\leq\left(\frac{E-V_+^i}{C_\Omega^+\pi^2h^2}\right)\chi_{\{V_+^i\leq E\}}\,,$$

which corresponds to the number of integer lattice points $(n_{i,1},n_{i,2})\in\mathbb{N}\times\mathbb{N}$ contained in the portion of the first quadrant defined by

$$\mathcal{E}_E = \left\{ (x,y) \in \mathbb{R}^2 : \left(\frac{x}{\sqrt{\frac{E-V_+^i}{C_\Omega^+ \pi^2 h^2}\, \chi_{\{V_+^i \leq E\}}}\, L_{i,1}} \right)^2 + \left(\frac{y}{\sqrt{\frac{E-V_+^i}{C_\Omega^+ \pi^2 h^2}\, \chi_{\{V_+^i \leq E\}}}\, L_{i,2}} \right)^2 \leq 1, x \geq 0, y \geq 0 \right\}$$

This is a region delimited by an ellipse. By constructing unit-area squares $[n_{i,1}-1, n_{i,1}] \times [n_{i,2} - 1, n_{i,2}]$, one can easily see that the number of such squares contained in $\mathcal{E}_E$ equals the number of integer lattices points in $\mathcal{E}_E$ (see [105] for more details). Since the squares have unit area, we have

$$N_{\leq E}(\tilde{P}_D^i) = \sum \text{Area(squares)} \leq \text{Area}(\mathcal{E}_E)\,,$$

where

$$\begin{aligned} \text{Area}(\mathcal{E}_E) &= \frac{\pi}{4} \left(\frac{E - V_+^i}{C_\Omega^+ \pi^2 h^2} \right) \chi_{\{V_+^i \leq E\}}\, L_{i,1} L_{i,2} \\ &= \frac{1}{4\pi} \left(\frac{E - V_+^i}{C_\Omega^+ h^2} \right) \chi_{\{V_+^i \leq E\}}\, \text{Area}(R_i)\,. \end{aligned}$$

By an analogous construction (again, see [105] for details), it is possible to obtain the lower bound

$$N_{\leq E}(\tilde{P}_D^i) \geq \frac{1}{2\pi} \left(\frac{E - V_+^i}{C_\Omega^+ h^2} \right) \chi_{\{V_+^i \leq E\}}\, \text{Area}(R_i) - \frac{\text{Perimeter}(R_i)}{2\pi} \sqrt{\frac{E - V_+^i}{C_\Omega^+ h^2}\, \chi_{\{V_+^i \leq E\}}}\,.$$

Therefore, for h sufficiently small, we have

$$N_{\leq E}(\tilde{P}_D^i) \sim \frac{\text{Area}(R_i)}{\pi h^2} \left(E - V_+^i \right) \chi_{\{V_+^i \leq E\}}\,.$$

Note that the scaling in h is different from the one obtained in [69]. This comes from the fact that we are considering an eigenvalue problem on a rectangle, instead of a problem on a segment of the real line.

From (2.39), we can conclude that

$$\sum_{i \in \mathcal{I}} \frac{\text{Area}(R_i)}{\pi h^2} \left(E - V_+^i \right) \chi_{\{V_+^i \leq E\}} \lesssim N_{\leq E}(P_D(\Omega))\,. \tag{2.43}$$

We now refine the family σ by taking the supremum over all the possible families $\sigma \in \sigma(\Omega)$. In other words, we take a limit which sends the Riemann sum to an integral. From (2.43), we have

$$\sup_{\sigma \in \sigma(\Omega)} \sum_{i \in \mathcal{I}} \frac{\text{Area}(R_i)}{\pi h^2} \left(E - V_+^i \right) \chi_{\{V_+^i \leq E\}} \lesssim N_{\leq E}(P_D(\Omega))\,,$$

which gives

$$N_{\leq E}(P_D(\Omega)) \gtrsim \frac{\text{Area}(\Omega)}{\pi h^2} \int_\Omega (E - V)\, \chi_{\{V \leq E\}} dr_* d\theta\,. \tag{2.44}$$

Remark 2.12. *As already observed, we are sure that* $R_i \cap \Omega$ *is a rectangle only when* $R_i \cap \Omega \equiv R_i$*. For problems* $\tilde{P}^i_D$*, rectangles for which* $R_i \cap \Omega \not\equiv R_i$ *did not contribute to the sum of the eigenvalues, so we only had to compute the number of eigenvalues for problems formulated on actual rectangles, which is what we are able to do explicitly. However, this is not true for problems* $\tilde{P}^i_N$*, where one could have a rectangle intersecting both the allowed region and the boundary of* Ω *on which* $V^i_- < E$*. This issue can be avoided by refining the family* σ *after the energy value* E *has been fixed, in a way that the* R_i *never intersect both the region where* $V < E$ *and the boundary* $\partial\Omega$*.*

In view of Lemma 2.3, an analogous calculation gives

$$N_{\leq E}(P_D(\Omega)) \lesssim \frac{\text{Area}(\Omega)}{\pi h^2} \int_\Omega (E - V)\, \chi_{\{V \leq E\}} dr_* d\theta$$

for h sufficiently small. Combined with (2.44), this provides the relation

$$N_{\leq E}(P_D(\Omega)) \sim \frac{\text{Area}(\Omega)}{\pi h^2} \int_\Omega (E - V)\, \chi_{\{V \leq E\}} dr_* d\theta\,. \tag{2.45}$$

Let us now denote by $N[E-\delta, E+\delta]$ the number of eigenvalues in $[E-\delta, E+\delta]$ for $P_D(\Omega)$. By applying (2.45) to

$$N[E-\delta, E+\delta] = N_{\leq E+\delta}(P_D(\Omega)) - N_{\leq E-\delta}(P_D(\Omega))$$

we obtain the version of Weyl's law of Theorem 2.4.

Remark 2.13. *Since the result is obtained by considering* h *sufficiently small, Weyl's law can be equivalently expressed in terms of* V *instead of* V^h_{eff}*. This motivates the presence of* V *in Theorem 2.4 and in the limit of the Riemann sum considered in our proof.*

2.8 Eigenvalue problem for the boosted black string

The aim of this section is to prove Theorem 2.6. To do that, we will mainly follow a perturbation argument of [59].

The eigenvalue problem for the boosted black string is *nonlinear*. From (2.34), we have

$$\begin{aligned}
&-\frac{f(r)}{r^2}\partial_r(r^2 f(r)\partial_r u) - \frac{f(r)}{r^2\sin\theta}\partial_\theta(\sin\theta\partial_\theta u)\\
&+\left\{\left(\frac{f(r)}{r^2\sin^2\theta}\right)m^2 - \left[(1-f(r))\sinh^2\beta\right]\omega^2 - \left[2(1-f(r))\sinh\beta\cosh\beta\right]\omega\, J\right.\\
&\left.+[1-(1-f(r))\cosh^2\beta]\, J^2\right\} u = \omega^2 u\,,
\end{aligned}$$

which reduces to

$$-g(r)\Delta_{(r_*,\theta)}u + [V_j(r,\theta) + V^\beta_{(\omega,m,J)}(r,\theta)]u = \omega^2 u\,,$$

with

$$V^{\beta}_{(\omega,m,J)}(r,\theta) := \left(\frac{f(r)}{r^2\sin^2\theta}\right) m^2 - \left[(1-f(r))\sinh^2\beta\right]\omega^2 - \left[2(1-f(r))\sinh\beta\cosh\beta\right]\omega\, J \\ + \left[1-(1-f(r))\cosh^2\beta\right] J^2 \tag{2.46}$$

and $V_j(r,\theta)$ independent of the frequency parameters (ω, m, J). The eigenvalue problem is

$$\boxed{\begin{array}{rlll} -g(r)\Delta_{(r_*,\theta)}u + [V_j(r,\theta) + V^{\beta}_{(\omega,m,J)}(r,\theta)]u = & \omega^2 u & \text{on} & \Omega \\ u = & 0 & \text{on} & \partial\Omega \end{array}} \tag{2.47}$$

for some *fixed* open set Ω to be later specified. Remember that $u = u(r_*,\theta)$, with the radial coordinate r_* previously defined. The eigenvalue problem (2.47) is *nonlinear* in the sense that the potential $V^{\beta}_{(\omega,m,J)}$ depends on the eigenvalue ω^2.

Remark 2.14 (**Comparison with Holzegel–Smulevici [59]**)**.** *Potential* (2.46) *has one quadratic term and one linear term in ω. In the nonlinear eigenvalue problem considered in [59] for Kerr-AdS black holes, ω only appears quadratically in the potential, as an effect of the assumption that solutions to the wave equation are axisymmetric (in fact, this kills the cross-term linear in ω). This symmetry assumption cannot be reproduced in our setting, because the vanishing of J (which is the frequency parameter that generates the cross term $\omega\, J$) would imply that, at the geodesic level, no stable trapping possibly occurs (see Lemma 2.4). We are therefore left with a more complicated analysis of the potential and, in turn, a more involved perturbation argument than the ones presented in [59]. Note that the terms involving ω_0 in the application of the Implicit Function Theorem (see, for instance, inequality* (2.64)*), which are absent in [59], are a manifestation of this fact.*

As in [59], we consider the two-parameter family of eigenvalue problems

$$\boxed{\begin{array}{rlll} Q(b,\omega)u = & \Lambda(b,\omega)u & \text{on} & \Omega \\ u = & 0 & \text{on} & \partial\Omega\,, \end{array}} \tag{2.48}$$

where the operator $Q(b,\omega)$ has the form

$$Q(b,\omega)u := -g(r)\Delta_{(r_*,\theta)}u + [V_j(r,\theta) + V^{b\beta}_{(\omega,m,J)}(r,\theta) - \omega^2]u\,,$$

with $b \in \mathbb{R}\cap[0,1]$ dimensionless parameter.

Remark 2.15 (**Notation**)**.** *In what follows, we will provide statements about ω in the context of problem* (2.48)*. In most of the cases, ω will necessarily depends on the parameters appearing in* (2.48)*, namely b, β, m and J. Since β will be considered fixed and the other parameters allowed to vary, we shall introduce the notation $\omega_{b,m,J}$. However, to this notation we prefer $\omega_{b,m}$, because the frequency parameter J will be treated as a rescaled version of m. To sum up,* *we denote ω in (2.48) by $\omega_{b,m}$ and adopt*

$$\omega_{\text{lin}} \equiv \omega_{\text{lin},m} := \omega_{0,m} \,, \qquad\qquad \omega \equiv \omega_m := \omega_{1,m} \,.$$

We aim to prove that, for $b = 1$ and m sufficiently large, there exists $\omega_{1,m}$ such that the abstract eigenvalue problem (2.48) admits a zero eigenvalue. This, in turn, would prove that the nonlinear eigenvalue problem (2.47) admits eigenvalues ω^2.

The main idea is to make use of what we have already proven for the linear eigenvalue problem (2.38) in Theorem 2.4 of Section 2.7, namely that, for $b = 0$ and m sufficiently large, there exists $\omega_{0,m}$ such that $Q(0, \omega_{0,m})$ admits a zero eigenvalue, say the n-th eigenvalue $\Lambda_n(0, \omega_{0,m})$ is zero (where $n = n(m)$). In view of the Implicit Function Theorem, we will be able to prove that this statement remains true for any $b \in [0, 1]$, i.e. for any $b \in [0, 1]$ and m sufficiently large, there exists $\omega_{b,m}$ such that the eigenvalue $\Lambda_n(b, \omega_{b,m})$ of $Q(b, \omega_{b,m})$ is equal to zero. In particular, this statement will hold for $b = 1$, giving us the existence result for (2.47).

2.8.1 Preliminaries

Analysis of the potential $V^{\beta}_{(\omega,m,J)} - \omega^2$

We collect here some properties of the potential $V^{\beta}_{(\omega,m,J)} - \omega^2$, where $V^{\beta}_{(\omega,m,J)}$ was defined in (2.46). Let us define

$$\begin{aligned}
\hat{V}^{\beta}_{(\omega,m,J)}(r,\theta) &:= V^{\beta}_{(\omega,m,J)}(r,\theta) - \omega^2 \\
&= \left(\frac{f(r)}{r^2 \sin^2\theta}\right) m^2 - \left[(1 - f(r)) \sinh^2\beta\right] \omega^2 - \left[2(1 - f(r)) \sinh\beta \cosh\beta\right] \omega\, J \\
&\quad + \left[1 - (1 - f(r)) \cosh^2\beta\right] J^2 - \omega^2 \\
&= \left(\frac{f(r)}{r^2 \sin^2\theta}\right) m^2 - (1 - f(r))(\omega \sinh\beta + J \cosh\beta)^2 + J^2 - \omega^2 \,.
\end{aligned} \tag{2.49}$$

When we do not need to specify any particular dependence on the frequency parameters, we denote $\hat{V}^{\beta}_{(\omega,m,J)}$ by $\hat{V}$.

Remark 2.16 (Fix $\boldsymbol{\theta = \pi/2}$). *For most of this section we will be fixing $\theta = \pi/2$ and the potential will be considered as a function of r only. Note that $\partial_\theta \hat{V}\,(r, \pi/2) = 0$ for any $r \geq r_0$ and $\theta = \pi/2$ is a local minimum (and the only stationary point) in the θ-direction. Furthermore, for any fixed $r = \tilde{r}$ and (ω, m, J), there exists θ_0 sufficiently close to 0 (or π) such that $\hat{V}^{\beta}_{(\omega,m,J)}(\tilde{r}, \theta) > 0$ for $0 \leq \theta \leq \theta_0$ (or $\pi - \theta_0 \leq \theta \leq \pi$). Indeed, for θ small (or close to π), i.e. $\sin^2\theta$ small, the potential gains more and more positivity from the first term in (2.49). The constant θ_0 depends on both $\tilde{r}$ and (ω, m, J).*

Theorem 2.5 (Analysis of $\boldsymbol{\hat{V}^{\beta}_{(\omega,m,J)}}$). *Consider the potential $\hat{V}$, with $\omega \in \mathbb{R}$ and $m, J \in \mathbb{Z}$. Assume $\cosh^2\beta < 3$, where $\beta \in \mathbb{R}_{\geq 0}$ is fixed. Then, the following statements hold.*

1. *The potential $\hat{V}$ admits at most two stationary points. It has both one local maximum and one local minimum if and only if*

$$m^2 > 3r_0^2(\omega \sinh\beta + J\cosh\beta)^2 . \tag{2.50}$$

For any (ω, m, J) satisfying (2.50), we have

$$r_0 < r_{\max} < 3r_0 < r_{\min} ,$$

where $r_{\max}$ and $r_{\min}$ are the radial coordinates of the local maximum and local minimum respectively. For $m^2 = 3r_0^2(\omega \sinh\beta + J\cosh\beta)^2$, the potential $\hat{V}$ has an inflection point at $r = 3r_0$, while for $m^2 < 3r_0^2(\omega \sinh\beta + J\cosh\beta)^2$ there are no stationary points.

2. *The potential $\hat{V}$ vanishes as a cubic in r. In particular, the potential has (i) one real root, or (ii) three distinct real roots, or (iii) three real roots where one is a multiple root. Without any condition on the frequency parameters (ω, m, J), the roots lie in general on $(-\infty, \infty)$.*

3. *If $\hat{V}^{\beta}_{(\omega_0,m_0,J_0)}$ has three distinct real roots in (r_0, ∞) for some triple (ω_0, m_0, J_0), then $J_0^2 > \omega_0^2$ must hold.*

4. *If $\omega = 0$, then $\hat{V}^{\beta}_{(0,m,J)}$ does not admit three distinct real roots in (r_0, ∞) for any m, J.*

5. *There exists a triple (ω_0, m_0, J_0) such that the potential $\hat{V}^{\beta}_{(\omega_0,m_0,J_0)}$ admits three distinct real roots in (r_0, ∞). In view of (1),(3) and (4), for any such triple (ω_0, m_0, J_0), we have (i) $0 < |\omega_0| < |J_0|$ and (ii) condition (2.50) holds. In particular, there exists such a triple (ω_0, m_0, J_0) which also satisfies the condition $\omega_0 J_0 > 0$.*

6. *Consider a triple (ω_0, m_0, J_0) such that the potential $\hat{V}^{\beta}_{(\omega_0,m_0,J_0)}$ admits three distinct real roots $r_1^{\omega_0}, r_2^{\omega_0}, r_3^{\omega_0} \in (r_0, \infty)$, with $r_1^{\omega_0} < r_2^{\omega_0} < r_3^{\omega_0}$ and $\omega_0 J_0 > 0$. Then, there exist $\mathcal{E}^-, \mathcal{E}^+$, with $0 < \mathcal{E}^- < \mathcal{E}^+ < |J_0|$, satisfying the following properties:*

   (i) $|\omega_0| \in (\mathcal{E}^-, \mathcal{E}^+)$.

   (ii) for any $|\omega| \in (\mathcal{E}^-, \mathcal{E}^+)$, $\mathrm{sign}(\omega) = \mathrm{sign}(\omega_0)$, the potential $\hat{V}^{\beta}_{(\omega,m_0,J_0)}$ admits three distinct real roots in (r_0, ∞).

   (iii) For any ω_1, ω_2 such that $|\omega_1|, |\omega_2| \in (\mathcal{E}^-, \mathcal{E}^+)$, $|\omega_1| < |\omega_2|$, $\mathrm{sign}(\omega_1) = \mathrm{sign}(\omega_2) = \mathrm{sign}(\omega_0)$, we have $r_2^{\omega_1} > r_2^{\omega_2}$ and $r_3^{\omega_1} < r_3^{\omega_2}$, where $r_i^{\omega_j}$ denotes the i-th root of $\hat{V}^{\beta}_{(\omega_j,m_0,J_0)}$.

7. *For any fixed β_0, there exists a triple (ω_0, m_0, J_0), with $\omega_0 J_0 > 0$, such that $\hat{V}^{\beta}_{(\omega_0,m_0,J_0)}$ admits three distinct real roots in (r_0, ∞) for any $\beta \in [0, \beta_0]$. In particular, $\hat{V}^{0}_{(\omega_0,m_0,J_0)}$ admits three distinct real roots in (r_0, ∞) and we have $r_2^0 > r_2^{\beta_0}$ and $r_3^0 < r_3^{\beta_0}$, with the notation r_i^{β} for the i-th root of $\hat{V}^{\beta}_{(\omega_0,m_0,J_0)}$.*

Remark 2.17 ($\boldsymbol{\beta}$ is fixed). *The boost parameter β has to be considered fixed. The statement of the theorem holds for any fixed β satisfying $\cosh^2\beta < 3$. In particular, it holds true for $\beta = 0$, i.e. for the static black string.*

Proof. We start by observing two properties of $\hat{V}$:

$$\hat{V}\,(r_0, \theta) = -(\omega \cosh \beta + J \sinh \beta)^2 < 0 \tag{2.51}$$

$$\lim_{r \to +\infty} \hat{V}\,(r, \theta) = J^2 - \omega^2 \tag{2.52}$$

for any (ω, m, J). We present the proof divided in parts, accordingly to the statements of the theorem.

1. We compute the derivative

$$\frac{\partial \hat{V}}{\partial r}\left(r, \frac{\pi}{2}\right) = \frac{1}{r^4}\left[m^2(3r_0 - 2r) + r_0(J\cosh\beta + \omega\sinh\beta)^2 r^2\right] .$$

Note that this is positive at $r = r_0$. By imposing $\frac{\partial \hat{V}}{\partial r}(r, \pi/2) = 0$, we find that the potential $\hat{V}$ has at most two stationary points with radial coordinate

$$\begin{aligned} r_{\max} &= \frac{m^2 - \sqrt{m^2\left[m^2 - 3r_0^2(\omega\sinh\beta + J\cosh\beta)^2\right]}}{r_0(\omega\sinh\beta + J\cosh\beta)^2} \\ r_{\min} &= \frac{m^2 + \sqrt{m^2\left[m^2 - 3r_0^2(\omega\sinh\beta + J\cosh\beta)^2\right]}}{r_0(\omega\sinh\beta + J\cosh\beta)^2} , \end{aligned} \tag{2.53}$$

with the condition

$$m^2 > 3r_0^2(\omega\sinh\beta + J\cosh\beta)^2 \tag{2.54}$$

for the stationary points to exist and be distinct. Using (2.54), we have

$$r_{\min} > 3r_0 + \text{positive term} > 3r_0 .$$

Furthermore, we clearly have $r_{\max} < r_{\min}$ and

$$\begin{aligned} r_{\max} &= \frac{3m^2}{(\omega\sinh\beta + J\cosh\beta)^2 r_{\min}} \\ &\geq \frac{3m^2}{(\omega\sinh\beta + J\cosh\beta)^2}\left[\frac{r_0(\omega\sinh\beta + J\cosh\beta)^2}{2m^2}\right] = \frac{3}{2}r_0 \\ &\geq r_0 . \end{aligned}$$

Since $\frac{\partial \hat{V}}{\partial r}\,(r_0, \pi/2) > 0$ and the two stationary points are in (r_0, ∞), we can conclude that $r_{\max}$ is really the radial coordinate of a local *maximum* and $r_{\min}$ of a local *minimum*. For $m^2 = 3r_0^2(\omega\sinh\beta + J\cosh\beta)^2$, we have $r_{\max} = r_{\min} = 3r_0$ and $\frac{\partial^2 \hat{V}}{\partial r^2}\,(3r_0, \pi/2) = 0$, so $r = 3r_0$ is the radial coordinate of an inflection point.

2. We set $\sin\theta = 1$ and rewrite $\hat{V}$ as

$$\frac{1}{r^3}\left[(J^2 - \omega^2)r^3 - r_0(\omega\sinh\beta + J\cosh\beta)^2 r^2 + m^2 r - m^2 r_0\right] , \tag{2.55}$$

which vanishes as a cubic in r.

3. From (2.51), if $\hat{V}$ has three distinct real roots in (r_0, ∞), then we must have $\lim_{r\to+\infty} \hat{V} > 0$, which implies $J^2 > \omega^2$ from (2.52).

4. Suppose that $\hat{V}$ admits three distinct real roots in (r_0, ∞), with $r_1 < r_2 < r_3$. Then the largest root must satisfy $r_3 > 3r_0$. This is true because, if $\hat{V}$ admits three roots in (r_0, ∞), then it must have both one local maximum and one local minimum, with $r_{\rm min} > 3r_0$. In view of (2.51) and point (3), we need to have $r_3 > r_{\rm min} > 3r_0$. However, if we set $\omega = 0$, the potential becomes

$$\hat{V}^{\beta}_{(0,m,J)}(r,\theta) = \left(\frac{f(r)}{r^2 \sin^2\theta}\right) m^2 + [1 - (1 - f(r))\cosh^2\beta] J^2 \, .$$

Using assumption $\cosh^2\beta < 3$, potential $\hat{V}^{\beta}_{(0,m,J)}(r,\theta)$ is positive for any $r > 3r_0$, contradicting $r_3 > 3r_0$. This implies that we need to have $\omega \neq 0$. In particular, for any fixed m, J, the frequency parameter ω has to be bounded away from zero, where the lower bound depends on m and J.

5. From point (2), we know that the potential is a cubic in r, whose discriminant

$$\begin{aligned} D = & -4\left(J^2-\omega^2\right)m^4 \\ & + \Big[-27r_0^2\left(J^2-\omega^2\right)^2 + 18r_0^2\left(J^2-\omega^2\right)(\omega\sinh\beta + J\cosh\beta)^2 \\ & + r_0^2(\omega\sinh\beta + J\cosh\beta)^4\Big] m^2 - 4r_0^4(\omega\sinh\beta + J\cosh\beta)^6 \end{aligned}$$

is positive if and only if the potential admits three distinct real roots in $(-\infty, +\infty)$. We claim that, for the triple (ω, m, J) with $J^2 = \omega^2 + \varepsilon$ and $m^2 = 5r_0^2(\omega\sinh\beta + J\cosh\beta)^2$, the discriminant is indeed positive for sufficiently small $\varepsilon > 0$.[29] With such a triple, D becomes

$$\begin{aligned} D = & \varepsilon\Big[-4m^4 + 18r_0^2(\omega\sinh\beta + J\cosh\beta)^2 m^2\Big] - 27r_0^2\varepsilon^2 m^2 \\ & + r_0^2(\omega\sinh\beta + J\cosh\beta)^4 m^2 - 4r_0^4(\omega\sinh\beta + J\cosh\beta)^6 \, . \end{aligned}$$

The expression for m^2, which correctly satisfies (2.50), makes the second line positive. By choosing ε sufficiently small, we conclude $D > 0$. This proves the existence of a triple (ω, m, J) for which $\hat{V}$ admits three distinct real roots. In particular, we do not have to impose a sign for ωJ, so the existence of the triple is compatible with ωJ being positive. We now need to check that the roots all lie in (r_0, ∞). To do that, note that the potential $\hat{V}$ is everywhere negative for $0 < r \leq r_0$, $\lim_{r\to0^\pm} \hat{V} = \mp\infty$ and $\hat{V}$ is everywhere positive for $r < 0$ because sum of positive terms (using $J^2 - \omega^2 = \varepsilon > 0$). Thus, the potential can only vanish for $r \in (r_0, \infty)$.

6. Part (i) uses point (5), while part (ii) comes from the continuity of $\hat{V}^{\beta}_{(\omega,m_0,J_0)}$ in ω (note that m_0 and J_0 are fixed). For part (iii), we note the following monotonicity property

[29] To ensure that $m, J \in \mathbb{Z}$, one can choose $\omega = \omega(\varepsilon)$ such that $\omega^2 + \varepsilon \in \mathbb{Z}$ and take the integer part of $5r_0^2(\omega\sinh\beta + J\cosh\beta)^2$.

of $\hat{V}^{\beta}_{(\omega,m_0,J_0)}$. If $\omega\, J_0 > 0$, then, for any fixed $r = \tilde{r}$, (2.49) shows that $\hat{V}^{\beta}_{(\omega,m_0,J_0)}\,(\tilde{r}, \pi/2)$ increases when ω decreases, because all the terms involving ω are negative.

7. We look at the discriminant D and suppose $\omega_0, J_0 > 0$ (the case $\omega_0, J_0 < 0$ is equivalent). We again define $J_0^2 = \omega_0^2 + \varepsilon$ and $m_0^2 = 5r_0^2(\omega_0 \sinh\beta_0 + J_0 \cosh\beta_0)^2$. As proved in point (5), this gives the existence of three distinct real roots for $\hat{V}^{\beta_0}_{(\omega_0,m_0,J_0)}$ when ε is small enough. Note that, for any $\beta \in [0, \beta_0)$, we have $m_0^2 > 5r_0^2(\omega_0 \sinh\beta + J_0 \cosh\beta)^2$, which implies $D > 0$ for any $\beta \in [0, \beta_0)$ and ε sufficiently small (ε depends on β_0, but uniform in β). The monotonicity property of the roots derives form the fact that, for (ω_0, m_0, J_0) with $\omega_0 J_0 > 0$, all the terms of $\hat{V}^{\beta}_{(\omega_0,m_0,J_0)}$ depending on β are negative and decrease in absolute value when β decreases.

□

The frequency parameters m and J

Lemma 2.4. *Let $m, J \in \mathbb{Z}$. If $m = 0$, then $\hat{V}$ does not admit three distinct real roots in (r_0, ∞). Analogously, if $J = 0$, then $\hat{V}$ does not admit three distinct real roots in (r_0, ∞).*

Proof. If $m = 0$, then condition (2.50) of Theorem 2.5 implies that $\hat{V}$ has a local maximum and minimum in (r_0, ∞) if and only if ω and J satisfy

$$3r_0^2(\omega \sinh\beta + J \cosh\beta)^2 < 0\,.$$

Therefore, $\hat{V}$ does not admit stationary points in (r_0, ∞), so it cannot have three distinct real roots in such interval. If $J = 0$ and $\hat{V}$ has three distinct real roots in (r_0, ∞), then we know from point (3) of Theorem 2.5 that $J^2 > \omega^2$ must hold, which gives a contradiction. □

Motivated by Lemma 2.4, we will assume

$$m \neq 0,\ m > 0 \qquad\qquad J \neq 0 \tag{2.56}$$

throughout the discussion. In fact, the sign of m does not play any role because m only appears squared in both eigenvalue problems (2.38) and (2.47). As done for the eigenvalue problem (2.38), we further assume

$$J = c\,m \qquad\qquad c^2 < \frac{1}{3r_0^2} \tag{2.57}$$

for some *fixed*, *positive* constant $c > 0$. We will not make a substitution $J \to c\,m$ in our equations, but the reader should keep in mind that, from now on, J is really a rescaled version of the frequency parameter m. In view of (2.56), we have $J > 0$. We summarize all the assumptions on m and J in the following:

Claim 2.1 (Assumptions on m and J). *Let $m, J \in \mathbb{Z}$. We assume*

$$m \neq 0 \qquad\qquad J = c\,m,\ J > 0,\ c \in \mathbb{Q}$$

for some fixed constant c such that $0 < c^2 < 1/3r_0^2$.

Remark 2.18. *Note that:*

(i) In the linear problem (2.38) the sign of the frequency parameters (ω, m, J) is not relevant, because they all appear squared in the potential. For the nonlinear problem (2.47), the sign ambiguity becomes relevant when one has to deal with the cross-term $\omega\,J$ in potential $\hat{V}$. See, for instance, the proof of Lemma 2.6.

(ii) What we really want to assume is $J > 0$, while parameters m and J do not need to have the same sign (one could equivalently assume $m < 0$ and $c < 0$). Most of the statements in the following sections will require m^2, J^2 sufficiently large. Claim 2.1 assumes that, in the limit $J^2 \to \infty$, we have $J \to +\infty$ (and not $J \to -\infty$).

(iii) The results of Theorem 2.5 did not require any assumption on the sign of J. Therefore, all the statements therein remain true after assuming $J > 0$.

Remark 2.19 (The choice of sign for J does not matter). *One could equivalently assume $J < 0$ in Claim 2.1 and go through the same arguments that we are about to present. In other words, one only has to eliminate the sign ambiguity of J from the problem, but the particular choice of sign is not important.*

How we choose Ω for (2.47)

The choice of Ω and suitable energy levels E for the nonlinear eigenvalue problem (2.47) is specified by the following proposition, which has to be seen as the nonlinear analogue of Lemma 2.1.

Proposition 2.1. *Define $\hat{V}^{\beta}_{\min}$ to be the minimum of $\hat{V}^{\beta}_{(\omega,m,J)}$ and $x_{\min} \in (r_0, \infty) \times (0, \pi)$ such that $\hat{V}^{\beta}_{(\omega,m,J)}(x_{\min}) = \hat{V}^{\beta}_{\min}$. For suitable $m, J \in \mathbb{Z}$ (as in Claim 2.1) and $\beta \in \mathbb{R}_+$ (as assumed in Theorem 2.5), consider some constant $\mathcal{E} > 0$ such that $\hat{V}^{\beta}_{(\mathcal{E}m,m,J)}$ has a local minimum and such that there exists $\Omega \subset [r_0, \infty) \times [0, \pi)$ satisfying*

(i) $x_{\min} \in \Omega$,

(ii) $\hat{V}^{\beta}_{(\mathcal{E}m,m,J)}(x) = 0$ for $x \in \partial\Omega$,

(iii) there are no local maxima of $\hat{V}^{\beta}_{(\mathcal{E}m,m,J)}$ in Ω,

(iv) $x \in \Omega \implies r(x) > 3\,r_0$.

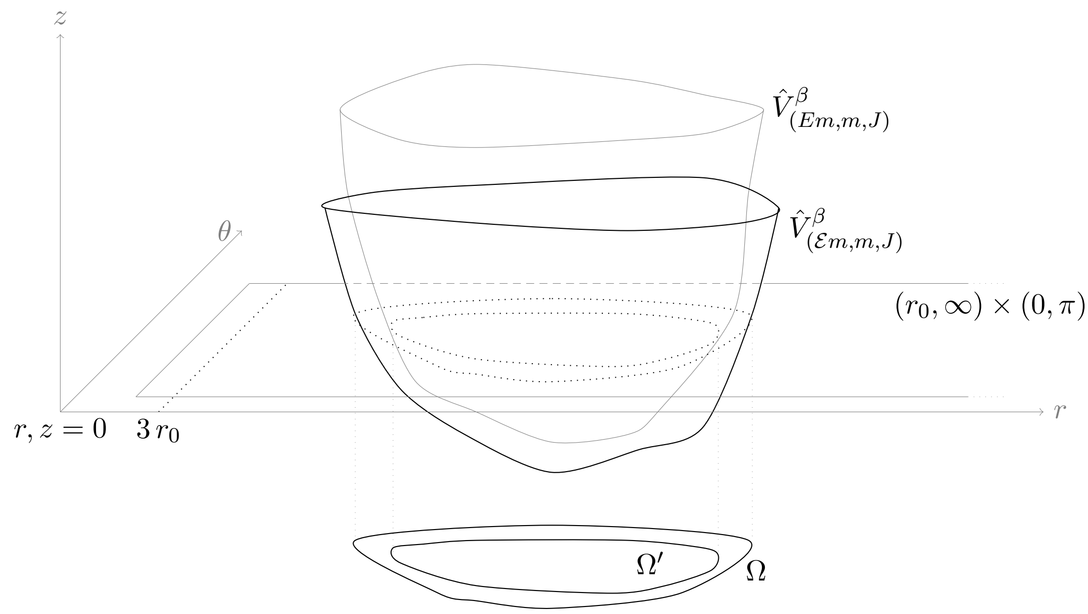

Figure 2.6: Conditions (i)-(iv) on Ω imply that the potential $\hat{V}^{\beta}_{(\mathcal{E}m,m,J)}$ is negative on Ω, the same being true for $\hat{V}^{\beta}_{(Em,m,J)}$ on Ω'. Note also that condition (iv) implies condition (iii) because $r_{\max} < 3\, r_0$, so the latter is in this sense redundant. Similarly, condition (iv) also ensures that $\hat{V}^{\beta}_{(Em,m,J)}$ (and $\hat{V}^{0}_{(Em,m,J)}$) has no local maxima on $\Omega \setminus \Omega'$ (and $\Omega \setminus \Omega''$). They have therefore positive sign on such sets.

Fix now some energy level $E > 0$ such that

(a) $\hat{V}^{\beta}_{(Em,m,J)}$ has a local minimum and there exists $\Omega' \subset \Omega$ satisfying the same properties (i)-(iv) of Ω, but now with respect to $\hat{V}^{\beta}_{(Em,m,J)}$,

(b) $\hat{V}^{0}_{(Em,m,J)}$ has a local minimum and there exists $\Omega'' \subset \Omega$ satisfying the same properties (i)-(iv) of Ω, but now with respect to $\hat{V}^{0}_{(Em,m,J)}$.

Then, the final part of Lemma 2.1 holds for $\frac{1}{m^2}\hat{V}^{0}_{(Em,m,J)}$ with respect to the open set Ω. Furthermore, for any sufficiently small constants $\delta, \delta' > 0$, there exists some constant $c > 0$ such that

$$\operatorname{dist}(x, \partial\Omega) < \delta' \implies \frac{1}{m^2}\hat{V}^{\beta}_{(\kappa\, m,m,J)}(x) > c$$

for all $\kappa \in \mathbb{R}$ satisfying $|\kappa^2 - E^2| \leq \delta$, with $\operatorname{dist}(\cdot, \cdot)$ the Euclidean distance.

Remark 2.20. *Note that, a priori, it is not obvious that constants $\mathcal{E}$ and E with the properties required by Proposition 2.1 do exist. The existence of such constants is the content of parts (5)-(6)-(7) of Theorem 2.5. In particular, we have $E < \mathcal{E}$ in view of (6)-(iii) in Theorem 2.5. Note also that condition (iv) of Proposition 2.1 is always realizable, since by (6)-(iii) in Theorem 2.5 one can continuously vary ω and obtain roots of the potential arbitrarily close to the local minimum, for which $r_{\min} > 3\, r_0$ holds.*

Remark 2.21. *As one can see from Proposition 2.1, the choice of Ω and the energy level E for the nonlinear problem is slightly more delicate than the one introduced in Lemma 2.1 for* *the linear problem. One reason for this is the non-linearity of the problem, namely the fact that the potential varies when we vary the energy level E. On the other hand, our formulation of Proposition 2.1 (in particular point (b)) aims to fix an energy level E which also agrees with the assumptions of Theorem 2.4 for the linear eigenvalue problem with potential $\hat{V}^0_{(Em,m,J)}$. This is because we will need to apply Weyl's law of Theorem 2.4 in our perturbation argument. By part (7) of Theorem 2.5, a choice of E with these properties is possible.*

Remark 2.22 (**A condition on β**). *Boosted black strings present an ergosurface at $r = r_0 \cosh^2 \beta$. From now on, we assume that the real parameter $\beta \geq 0$ satisfies*

$$\cosh^2 \beta < 3 \,. \tag{2.58}$$

In view of condition (iv) of Proposition 2.1, our assumption ensures that the open set Ω is disjoint from the ergoregion in physical space. This is key for Lemma 2.5 and, even more importantly, for the proof of Lemma 2.6.

The open set Ω is uniquely determined by the choice of the constant $\mathcal{E}$. This concludes the formulation of the eigenvalue problem (2.47). In what follows, the set Ω will be considered *fixed* throughout.

2.8.2 A lower bound for ω^2

In this section we prove that the claim that the n-th eigenvalue $\Lambda_n(b,\omega)$ of $Q(b,\omega)$ is zero determines some compatibility conditions on $\omega_{b,m}$.

Let $\psi_n(b,\omega)$ be an eigenfunction associated to the eigenvalue $\Lambda_n(b,\omega) = 0$. Assuming that $\psi_n(b,\omega)$ is normalized on Ω, i.e. $\int_\Omega |\psi_n(b,\omega)|^2 \, dr_* d\theta = 1$, we have

$$\Lambda_n(b,\omega) = \int_\Omega \psi_n(b,\omega) Q(b,\omega) \psi_n(b,\omega) \, dr_* d\theta = 0 \,. \tag{2.59}$$

It turns out that $\omega_{b,m}$ needs to satisfy some a priori conditions for (2.59) to be possible. The following lemma states that we certainly need to have $\omega_{b,m} \neq 0$.

Lemma 2.5. *Let $\psi_n(b,\omega)$ be a non identically zero eigenfunction of* (2.48)*, normalized on Ω. If β satisfies* (2.58) *and $m^2, J^2 > M$, for some positive constant M, then the implication*

$$\omega_{b,m} = 0 \implies \int_\Omega \psi_n(b,\omega) Q(b,\omega) \psi_n(b,\omega) \, dr_* d\theta \neq 0$$

holds for any $b \in [0,1]$.

Remark 2.23. *In Lemma 2.5 and Corollary 2.2 we require m^2 large. This assumption finds motivation later in our discussion, when we will be looking at certain energy estimates in a high frequency limit. Note also that, in view of* (2.57)*, assuming m^2 large implies J^2 large as well, so the hypothesis on J^2 is in some sense redundant.*

Proof. Setting $\omega_{b,m} = 0$, we have

$$\begin{aligned}
&\int_\Omega \psi_n(b,0)Q(b,0)\psi_n(b,0)\, dr_* d\theta \\
&= \int_\Omega -\psi_n(b,0)g(r)\Delta_{(r_*,\theta)}\psi_n(b,0) + \left[V_j(r,\theta) + V^{b\beta}_{(0,m,J)}(r,\theta)\right]\psi_n^2(b,0)\, dr_* d\theta \\
&= \int_\Omega g(r)\left(\left|\frac{\partial\psi_n(b,0)}{\partial r_*}\right|^2 + \left|\frac{\partial\psi_n(b,0)}{\partial\theta}\right|^2\right) + \psi_n(b,0)\partial_{r_*}\psi_n(b,0)\partial_{r_*}g(r) \\
&+ \left[V_j(r,\theta) + V^{b\beta}_{(0,m,J)}(r,\theta)\right]\psi_n^2(b,0)\, dr_* d\theta \\
&\geq \int_\Omega \left(g(r) - \varepsilon|\partial_{r_*}g(r)|^2\right)\left|\frac{\partial\psi_n(b,0)}{\partial r_*}\right|^2 + g(r)\left|\frac{\partial\psi_n(b,0)}{\partial\theta}\right|^2 \\
&+ \left[V_j(r,\theta) + V^{b\beta}_{(0,m,J)}(r,\theta) - \frac{1}{4\varepsilon}\right]\psi_n^2(b,0)\, dr_* d\theta\,,
\end{aligned}$$

where we have integrated by parts, used the boundary condition on $\psi_n(b,\omega)$ and Young's inequality. We first set $b = 1$ and we look at

$$V^{\beta}_{(0,m,J)}(r,\theta) = \left(\frac{f(r)}{r^2\sin^2\theta}\right)m^2 + \left[1 - (1 - f(r))\cosh^2\beta\right]J^2\,, \tag{2.60}$$

where the first term is always positive, while the second term becomes negative inside the ergoregion $r < r_0\cosh^2\beta$. We now recall that we are integrating over Ω, which is an open set satisfying the conditions of Proposition 2.1. Furthermore, β satisfies condition (2.58). This implies that the second term of (2.60) is positive on Ω. Note that $\cosh^2 b\beta \leq \cosh^2\beta < 3$, so this gives us a positive sign for the second term of (2.60) on Ω for any $b \in [0,1]$.

By choosing ε sufficiently small (independently of m, J) and m^2, J^2 sufficiently large, the integral involves only positive terms ($V_j(r,\theta)$ and $1/4\varepsilon$ can be both absorbed by $V^{b\beta}_{(0,m,J)}$ when m and J are large). The integral is therefore positive for m^2 and J^2 sufficiently large, unless $\psi_n = 0$ identically on Ω. In particular, the integral is nonzero.

□

We now derive a second condition for $\omega_{b,m}$, namely a lower bound. The proof follows the same idea of that of Lemma 2.5. We state the result as a corollary.

Corollary 2.2. *Let $\psi_n(b,\omega)$ be a non identically zero eigenfunction of* (2.48)*, normalized on Ω. If β satisfies* (2.58)*, then the implication*

$$\begin{aligned}
&\omega^2_{b,m} = o(m^2) \text{ as } m^2, J^2 \text{ are sufficiently large} \\
&\implies \int_\Omega \psi_n(b,\omega)Q(b,\omega)\psi_n(b,\omega)\, dr_* d\theta \neq 0
\end{aligned} \tag{2.61}$$

holds for any $b \in [0,1]$.

Proof. The case $\omega_{b,m}^2 = 0$ is the content of Lemma 2.5. We therefore discuss the case $\omega_{b,m}^2 > 0$. Note that, in view of (2.57), condition (2.61) still holds with m replaced by J.

Condition (2.61) implies that we can always choose a constant $C > 0$ sufficiently small and m^2, J^2 sufficiently large such that $V^{b\beta}_{(\omega,m,J)}(r,\theta) - \omega_{b,m}^2$ is positive on Ω for any $b \in [0,1]$. This is true because

$$\begin{aligned} V^{b\beta}_{(\omega,m,J)}(r,\theta) - \omega_{b,m}^2 \geq & \left(\frac{f(r)}{r^2 \sin^2\theta}\right) m^2 + \left[1 - (1-f(r))\cosh^2 b\beta\right] J^2 \\ & - C[1 + (1-f(r))\sinh^2 b\beta + 2(1-f(r))\sinh b\beta \cosh b\beta] m^2 \end{aligned}$$

remains positive when C is sufficiently small. The rest of the argument is analogous to the one presented in the proof of Lemma 2.5.

□

In view of Corollary 2.2, we conclude that, if the eigenvalue problem (2.47) admits eigenvalues ω_m^2, then, for m^2 sufficiently large, there exists a positive constant $C_{r_0,\beta}$, independent of m, such that

$$\omega_m^2 \geq C_{r_0,\beta}\, m^2 \,. \tag{2.62}$$

In the following section, by an application of the Implicit Function Theorem, we will be able to prove the existence of such eigenvalues and produce an upper bound of the type $\omega_m^2 \leq C_{r_0,\beta}\, m^2$ when m^2 is sufficiently large.

2.8.3 An application of the Implicit Function Theorem

We now state the key lemma for the nonlinear eigenvalue problem (2.47). In doing this, we will make an assumption on the sign of $\omega_{b_0,m}$, for some $b_0 \in [0,1]$. In the application of the lemma to our problem, this sign assumption on $\omega_{b_0,m}$ will become an assumption on the sign of $\omega_{\mathrm{lin},m}$, where $\omega_{\mathrm{lin},m}^2$ are the eigenvalues of the *linear* eigenvalue problem (2.38). Since $\omega_{\mathrm{lin},m}$ only appears squared in the linear problem, this choice of sign is free and does not determine any loss of generality.

Lemma 2.6 (adapted from [59] Lemma 4.3)**.** *Let $b_0 \in [0,1]$ and $\omega_{b_0,m} > 0$ be such that the n-th eigenvalue of $Q(b_0, \omega_{b_0,m})$ is zero. We further assume $m, J \in \mathbb{Z}$ as in Claim 2.1. Then, for m^2, J^2 sufficiently large, there exists a constant $\varepsilon > 0$ (independent of b_0) such that the following property holds. There exists a differentiable function $\omega_{b,m}(b)$ such that the n-th eigenvalue of $Q(b, \omega_{b,m})$ is zero for any $b \in (\max(0, b_0 - \varepsilon), b_0 + \varepsilon)$.*

Proof. Recall that the n-th eigenvalue of $Q(b, \omega_{b,m})$ follows the formula

$$\Lambda_n(b,\omega) = \int_\Omega \psi_n(b,\omega) Q(b,\omega) \psi_n(b,\omega)\, dr_* d\theta \,,$$

where the associated eigenfunction $\psi_n(b,\omega)$ has been renormalized on Ω. Our assumption on the n-th eigenvalue of $Q(b_0, \omega_{b_0,m})$ claims that $\Lambda_n(b_0, \omega_{b_0,m}) = 0$.

The idea of the proof is to apply the Implicit Function Theorem to the equation $\Lambda_n(b,\omega) = 0$ in a neighbourhood of the point $(b_0, \omega_{b_0,m})$, where Λ_n is a function of b and $\omega_{b,m}$. To apply the Implicit Function Theorem, we first need to compute the derivatives of Λ_n with respect to $\omega_{b,m}$ and b at the point $(b_0, \omega_{b_0,m})$ and check that they are non-zero. Note that, at this stage, we are interested in proving the existence of a solving function $\omega_{b,m}(b)$, so only the derivative of Λ_n with respect to $\omega_{b,m}$ needs to be checked. We have

$$\frac{\partial \Lambda_n}{\partial \omega}(b_0, \omega_{b_0,m}) = \int_\Omega \psi_n(b_0, \omega_{b_0,m}) \left(\frac{\partial V^{b_0\beta}_{(\omega,m,J)}}{\partial \omega}(b_0, \omega_{b_0,m}) - 2\omega_{b_0,m} \right) \psi_n(b_0, \omega_{b_0,m})\, dr_* d\theta \quad (2.63)$$

$$\frac{\partial \Lambda_n}{\partial b}(b_0, \omega_{b_0,m}) = \int_\Omega \psi_n(b_0, \omega_{b_0,m}) \left(\frac{\partial V^{b_0\beta}_{(\omega,m,J)}}{\partial b}(b_0, \omega_{b_0,m}) \right) \psi_n(b_0, \omega_{b_0,m})\, dr_* d\theta\,.$$

If integral (2.63) is non-zero, then an application of the Implicit Function Theorem allows us to solve for $\omega_{b,m}$ as a function of b in a neighbourhood of the point $(b_0, \omega_{b_0,m})$ and close the argument. To check this, we compute

$$\begin{aligned} \frac{\partial V^{b_0\beta}_{(\omega,m,J)}}{\partial \omega}(b_0, \omega_{b_0,m}) - 2\omega_{b_0,m} =& - 2\left[(1-f(r))\sinh^2 b_0\beta + 1\right]\omega_{b_0,m} \\ & - \left[2(1-f(r))\sinh b_0\beta \cosh b_0\beta\right] J \\ \leq& - 2\,\omega_{b_0,m}\,, \end{aligned} \quad (2.64)$$

where the estimate holds because $\omega_{b_0,m} > 0$ (by assumption of the theorem) and $J > 0$ (by Claim 2.1).

From Corollary 2.2, we have $\omega_{b_0,m} \geq C_{b_0,r_0,\beta}\, m$ for some constant $C_{b_0,r_0,\beta} > 0$ (independent of m) and m sufficiently large. In particular, there exists a constant $B_{r_0,\beta} := \inf_{b_0\in[0,1]} C_{b_0,r_0,\beta} > 0$ independent of b_0 such that $\omega_{b_0,m} \geq B_{r_0,\beta}\, m$. We deduce

$$\frac{\partial V^{b_0\beta}_{(\omega,m,J)}}{\partial \omega}(b_0, \omega_{b_0,m}) - 2\omega_{b_0,m} \leq -B_{r_0,\beta}\, m\,. \quad (2.65)$$

Therefore, (2.64) is bounded away from zero *uniformly in* b_0. This implies that integral (2.63) is bounded away from zero uniformly in b_0 as well. We can therefore conclude the proof by Implicit Function Theorem. Note that the fact that estimate (2.65) is uniform in b_0 implies that the constant ε is independent of b_0.

□

Remark 2.24. *The Implicit Function Theorem also provides a formula for the derivative of* *the function $\omega_{b,m}(b)$ at $b = b_0$, namely*

$$\frac{d\omega_{b,m}}{db}(b_0) = -\frac{\frac{\partial \Lambda_n}{\partial b}(b_0, \omega_{b_0,m})}{\frac{\partial \Lambda_n}{\partial \omega}(b_0, \omega_{b_0,m})}\,. \tag{2.66}$$

By computing

$$\frac{\partial V^{b\beta}_{(\omega,m,J)}}{\partial b}(b_0, \omega_{b_0,m}) = -\beta(\omega^2_{b_0,m} + J^2)(1 - f(r))\sinh(2b_0\beta) - 2\beta\,\omega_{b_0,m}\,J(1 - f(r))\cosh(2b_0\beta)\,,$$

we conclude that, under the assumptions of Lemma 2.6, we have

$$-\,C_{r_0,\beta}\,\omega_{b_0,m} \leq \frac{d\omega}{db}(b_0) < 0 \tag{2.67}$$

for some constant $C_{r_0,\beta} > 0$ and any $b_0 \in [0, 1]$.

2.8.4 The main theorem for the eigenvalue problem (2.47)

This is the main theorem for the nonlinear eigenvalue problem (2.47). Previous sections provided all the preliminary results that we need to prove the following:

Theorem 2.6 (Eigenvalues boosted black string). *Consider the fixed energy levels $\mathcal{E}, E > 0$ and the bounded set Ω introduced in Proposition 2.1. Let $\beta \in \mathbb{R}_+$ satisfy (2.58) and $m, J \in \mathbb{Z}$ as in Claim 2.1. Given eigenvalues $\omega^2_{\mathrm{lin},m}$ for the linear eigenvalue problem (2.38) on Ω, we further assume $\omega_{\mathrm{lin},m} > 0$. Then, there exists a constant $M > 0$ such that the following statement holds for any $m^2 > M$. There exists an eigenvalue ω^2_m and an associated smooth eigenfunction u_m to the nonlinear eigenvalue problem (2.47). Furthermore, we have $\omega_m > 0$ and*

$$C_{r_0,\beta} \leq \frac{\omega_m^2}{m^2} \leq E^2 + \frac{\mathcal{E}^2 - E^2}{10} \tag{2.68}$$

for some constant $C_{r_0,\beta} > 0$ independent of m.

Remark 2.25. *In view of Proposition 2.1, the energy level E agrees with the assumptions of Theorem 2.4 for the linear eigenvalue problem (2.38) on Ω (see also Remark 2.21). Therefore, Theorem 2.4 ensures that eigenvalues $\omega^2_{\mathrm{lin},m}$ do exist and, in particular, $\omega^2_{\mathrm{lin},m}/m^2$ accumulate in an arbitrarily small strip around E^2 for m^2 sufficiently large. This last accumulation property of $\omega^2_{\mathrm{lin},m}$ will be crucial to prove inequality (2.68).*

Proof. The spirit of this proof is to repeatedly apply Lemma 2.6 to prove the existence of eigenvalues for the nonlinear eigenvalue problem (2.47).

Let us start by choosing $b_0 = 0$. Then, we know by our study of the linear problem (2.38) that, for m^2 sufficiently large, there exists an $\omega_{0,m}$ such that $Q(0, \omega_{0,m})$ admits a zero eigenvalue. By Lemma 2.5, we necessarily have $\omega_{0,m} \neq 0$.

By assumption in the statement of the theorem, we have $\omega_{0,m} > 0$. Lemma 2.6 now ensures that there exists $\varepsilon > 0$ such that there exists a continuous function $\omega_{b,m}(b)$ such that, for any $b \in [0, \varepsilon)$, the eigenvalue problem (2.48) admits a zero eigenvalue $\Lambda_n(b, \omega_{b,m}) = 0$. In particular, in view of Lemma 2.5, the function $\omega_{b,m}(b)$ cannot vanish for $b \in [0, \varepsilon)$, so $\omega_{b,m}$ has to remain *positive* on $[0, \varepsilon)$. From (2.67), we also have $\omega_{b,m} < \omega_{0,m}$ for all $b \in [0, \varepsilon)$.

For any $b_0 \in [0, \varepsilon)$, the assumptions of Lemma 2.6 are still satisfied, so we can apply the lemma again and deduce the existence of a function $\omega_{b,m}(b)$ such that $\Lambda_n(b, \omega_{b,m}) = 0$ in an enlarged interval of b. The fact that ε is independent of b_0 is crucial here and ensures that a *finite* number of applications of Lemma 2.6 covers the whole interval $b \in [0, 1]$.

Therefore, problem (2.48) admits a zero eigenvalue for any $b \in [0, 1]$. In particular, for $b = 1$, this proves the existence of eigenvalues ω_m^2 for the nonlinear eigenvalue problem (2.47).

From (2.67), for m^2 sufficiently large, we have

$$\omega_m^2 = \omega_{b,m}^2(1) \leq \omega_{b,m}^2(0) \leq C\, m^2 \,,$$

where the second inequality comes from Weyl's law for the linear problem. Combining with (2.62), we have that, for m^2 sufficiently large, the eigenvalues ω_m^2 for the nonlinear problem satisfy

$$c_{r_0,\beta} \leq \frac{\omega_m^2}{m^2} \leq C_{r_0}$$

for some constants $c_{r_0,\beta}, C_{r_0} > 0$ independent of m.

We now prove inequality (2.68). Consider the problem

$$\overline{Q}_b u = E_n^2(b) u \,,$$

where the operator $\overline{Q}_b$ and the n-th eigenvalue $E_n^2(b)$ are defined as

$$\overline{Q}_b u := -\frac{g(r)}{m^2}\Delta_{(r_*,\theta)} u + \frac{1}{m^2}[V_J(r,\theta) + V^{b\beta}_{(\omega,m,J)}(r,\theta)]u$$
$$E_n^2(b) := \frac{\omega_{b,m}^2(b)}{m^2}$$

with $b \in [0, 1]$. *From part (b) of Proposition 2.1*, which allows the application of Weyl's law (see Remark 2.25), we have

$$E_n^2(0) \in [E^2 - \delta, E^2 + \delta]$$

for some $\delta > 0$ arbitrarily small and m^2 sufficiently large. Furthermore, we have the estimate

$$\begin{aligned} 0 \leq \int_\Omega \left[u(\overline{Q}_0 - \overline{Q}_b)u\right] dr_* d\theta = \int_\Omega \Big\{ & c^2 f(r) + \left[(1 - f(r)) \sinh^2 b\beta\right] \frac{\omega_{b,m}^2}{m^2} \\ & + 2\, c\, (1 - f(r)) \sinh b\beta \cosh b\beta \, \frac{\omega_{b,m}}{m} \\ & - c^2 [1 - (1 - f(r)) \cosh^2 b\beta] \Big\} \, |u|^2 dr_* d\theta \end{aligned}$$

for any $b \in [0, 1]$, where the constant c was defined in Claim 2.1 and the factor $c\,\omega_{b,m}/m$ is positive. Note that $f(r) \geq 1-(1-f(r))\cosh^2 b\beta$, so all the terms in the integral are positive. Therefore,

$$\int_\Omega \left[u(\overline{Q}_b u)\right] dr_* d\theta \leq \int_\Omega \left[u(\overline{Q}_0 u)\right] dr_* d\theta \,,$$

which implies, recalling the min-max definition of the eigenvalue $E_n(b)$, that

$$E_n^2(b) \leq E_n^2(0)$$

for any $b \in [0,1]$. In particular,

$$E_n^2(1) \leq E_n^2(0) \leq E^2 + \delta$$

with δ arbitrarily small. The lower bound in inequality (2.68) has already been proven in (2.62).

□

Remark 2.26 (**$\hat{V}^\beta_{(\omega_m,m,J)}$ has a good structure**). *Inequality* (2.68) *ensures that the eigenvalues ω_m^2 for the nonlinear eigenvalue problem determine a potential $\hat{V}^\beta_{(\omega_m,m,J)}$ with the same sign properties of $\hat{V}^\beta_{(Em,m,J)}$ in Figure 2.6. When $\omega_m^2/m^2 \in [E^2-\delta, E^2+\delta]$, this is rigorously motivated by the final part of Proposition 2.1. For $C \leq \omega_m^2/m^2 \leq E^2-\delta$, one can invoke part (6) of Theorem 2.5 or, equivalently, remember that $\partial\hat{V}^\beta_{(\omega,m,J)}/\partial\omega < 0$. Indeed, the potential gains positivity for lower values of ω_m^2/m^2, thus the sign properties deriving from Proposition 2.1 still hold. This structure of the potential will be crucial in our quasimode construction.*

Remark 2.27 (**The choice of sign for J does not matter**). *The reader should now go back to Remark 2.19 and realise that all our arguments can be repeated if $J<0$ is assumed. One only needs to be careful with the proof of Lemma 2.6, where the estimate on*

$$\frac{\partial V^{b\beta}_{(\omega,m,J)}}{\partial\omega}(b_0,\omega_{b_0,m}) - 2\omega_{b_0,m}$$

was possible because $\omega_{b_0,m}$ and J were both positive. Note that the estimate still holds if $\omega_{b_0,m}$ and J are both negative, now in the form

$$\frac{\partial V^{b\beta}_{(\omega,m,J)}}{\partial\omega}(b_0,\omega_{b_0,m}) - 2\omega_{b_0,m} \geq 2\,\omega_{b_0,m} \,.$$

In other words, what we really need is $\omega_{b_0,m}J>0$. Therefore, if we assume $J<0$ at the level of Claim 2.1, we then need to assume $\omega_{b_0,m}<0$ in Lemma 2.6. In terms of our problem, this means that we are assuming $\omega_{lin,m}<0$ for $\omega^2_{lin,m}$ eigenvalues of the linear eigenvalue problem.

2.9 Eigenvalue problem for the black ring

The aim of this section is to prove Theorem 2.8.

Consider the coordinate system $(t, r, \theta, \phi, \psi)$ and the ring metric $g_{(r_0,R)}$ introduced in (2.26). Recall that

$$r \in [r_0, R] \qquad\qquad \theta \in [0, \pi)\,,$$

where $r = r_0$ corresponds to the event horizon and (R, π) to spacelike infinity.

Equation (2.34) for the black ring becomes

$$-g_{\text{ring}}(r,\theta)\Delta_{(r_*,\theta_*)}u + \left[V_j^{\text{ring}}(r,\theta) + V_{(\omega,m,\hat{J})}^{\text{ring}}(r,\theta)\right]u = 0\,,$$

where $u = u(r_*, \theta_*)$, the coordinates (r_*, θ_*) are implicitly defined by

$$\frac{dr_*}{dr} = \sqrt{\frac{R^2}{r(r-r_0)(R^2-r^2)}} \qquad\qquad \frac{d\theta_*}{d\theta} = \sqrt{\frac{R}{r_0\cos\theta + R}}$$

and the function $g_{\text{ring}}(r,\theta)$ is

$$g_{\text{ring}}(r,\theta) := \frac{(r-r_0)(R+r\cos\theta)^2}{r^3 R\left(R + r_0\cosh^2\beta\cos\theta\right)}\,.$$

The potential $V_{(\omega,m,\hat{J})}^{\text{ring}}(r,\theta)$ is given by

$$V_{(\omega,m,\hat{J})}^{\text{ring}}(r,\theta) = f_{1,R}(r,\theta)\,m^2 + f_{2,R}(r,\theta)\,\omega^2 + f_{3,R}(r,\theta)\,\omega\,\hat{J} + f_{4,R}(r,\theta)\,\hat{J}^2\,, \tag{2.69}$$

where $\omega \in \mathbb{R}$, $m, \hat{J} \in \mathbb{Z}$ and the functions $f_{i,R}(r,\theta)$ are defined as follows:

$$\begin{aligned}
f_{1,R}(r,\theta) &:= \frac{(r-r_0)(R+r\cos\theta)^2}{R\,r^3(r_0\cos\theta + R)\sin^2\theta}\,,\\
f_{2,R}(r,\theta) &:= \frac{r_0^2(R-r)(R+r\cos\theta)^2(R\sinh^2 2\beta + 4r_0\cosh^4\beta\sinh^2\beta)}{4rR(r+R)(R-r_0\cosh^2\beta)(r-r_0\cosh^2\beta)(R+r_0\cosh^2\beta\cos\theta)}\\
&\quad - \frac{(r-r_0)(R+r_0\cosh^2\beta\cos\theta)}{R(r-r_0\cosh^2\beta)}\,,\\
f_{3,R}(r,\theta) &:= \frac{2r_0\sinh\beta\cosh\beta(R+r\cos\theta)^2}{rR(R+r)\left(r_0\cosh^2\beta\cos\theta + R\right)}\sqrt{\frac{R+r_0\cosh^2\beta}{R-r_0\cosh^2\beta}}\,,\\
f_{4,R}(r,\theta) &:= \frac{(r-r_0\cosh^2\beta)(R+r\cos\theta)^2}{rR\,(R^2-r^2)\left(r_0\cosh^2\beta\cos\theta + R\right)}\,.
\end{aligned}$$

Potential $V_j^{\text{ring}}(r,\theta)$ is a smooth, real-valued function and does not involve any frequency parameter. We also recall that for a black ring in equilibrium, the following condition must hold:

$$\cosh^2\beta = \frac{2R^2}{r_0^2 + R^2}\,.$$

Remark 2.28 ($\boldsymbol{\beta}$ is a function of $\boldsymbol{r_0}$ and $\boldsymbol{R}$). *The reader should keep in mind that β is really a function of the parameters r_0 and R. Furthermore, note that* $\cosh^2\beta$ *is an increasing* *function of R and*

$$1 \leq \cosh^2\beta \leq 2$$

for any r_0 *and* R *such that* $r_0 \leq R$. *Crucially, we have*

$$\cosh^2\beta < 3$$

for any $r_0 \leq R$, *which agrees with condition* (2.58) *introduced for boosted strings if we interpret* β *as a boost parameter (see later why this is important). We also define*

$$\beta_0 := \lim_{R\to\infty} \beta = \cosh^{-1}\sqrt{2}\,,$$

which satisfies $\cosh^2\beta_0 < 3$. *In what follows, we sometimes denote* β *by* β_R *to emphasise the dependence of* β *on* R.

Remark 2.29 ($f_{i,R}$ are smooth). *Functions* $f_{i,R}$, $i = 1, \ldots, 4$, *are smooth for* $(r,\theta) \in (r_0, R) \times (0,\pi)$. *This is not obvious for* $f_{2,R}$, *for which the denominator goes to zero at the ergosurface* $r = r_0 \cosh^2\beta$. *However, a more careful analysis shows that such limit is finite.*

Without loss of generality, we fix $r_0 = 2$ throughout.

2.9.1 A local property of the ring potential $V^{\mathrm{ring}}_{(\omega,m,\hat{J})}$

As

$$r,\ 2,\ 2\cosh^2\beta_R \ll R\,,$$

one can rewrite potential (2.69) in the form

$$\begin{aligned}
V^{\mathrm{ring}}_{(\omega,m,\hat{J})}(r,\theta) =& (1+\mathcal{O}_1(R^{-1}))\left(1-\frac{2}{r}\right)\frac{m^2}{r^2\sin^2\theta} + (1+\mathcal{O}_2(R^{-1}))\left(-1-\frac{2}{r}\sinh^2\beta_R\right)\omega^2 \\
&- (1+\mathcal{O}_3(R^{-1}))\left(2\,\frac{2}{r}\sinh\beta_R\cosh\beta_R\right)\omega\,\frac{\hat{J}}{R} + (1+\mathcal{O}_4(R^{-1}))\left(1-\frac{2}{r}\cosh^2\beta_R\right)\frac{\hat{J}^2}{R^2} \\
=&(1+\mathcal{O}_1(R^{-1}))\left[\left(1-\frac{2}{r}\right)\frac{m^2}{r^2\sin^2\theta}\right] + (1+\mathcal{O}_2(R^{-1}))\left[-1-\frac{2}{r}\sinh^2\beta_0+\frac{16}{r(4+R^2)}\right]\omega^2 \\
&- (1+\mathcal{O}_3(R^{-1}))\left[2\,\frac{2}{r}\sinh\beta_0\cosh\beta_0 - \frac{4\sqrt{2}}{r}\left(1-\frac{\sqrt{R^2(R^2-4)}}{4+R^2}\right)\right]\omega\,\frac{\hat{J}}{R} \\
&+ (1+\mathcal{O}_4(R^{-1}))\left[1-\frac{2}{r}\cosh^2\beta_0+\frac{16}{r(4+R^2)}\right]\frac{\hat{J}^2}{R^2}
\end{aligned}$$

with β_0 as in Remark 2.28 and $\mathcal{O}_i$ in the usual Bachmann-Landau notation (the label i simply denotes different terms). The following lemma has to be regarded as a formal statement about functions of the same real variables r and θ.[30]

[30] Coordinates (r,θ) in the black ring metric (2.26) actually differ from the ones that we used for black strings. In the limit $R\to\infty$, the two different coordinate systems do locally coincide.

Lemma 2.7. *Let $2 < r_1 < r_2$ and $0 < \theta_1 < \theta_2 < \pi$. Consider the function $\hat{V}^{\beta_0}_{(\omega,m,J)}(r,\theta)$ as defined in (2.49), with $\cosh^2\beta_0 = 2$. Then, for any frequency triple (ω, m, J), with $\omega \in \mathbb{R}$ and $m, J \in \mathbb{Z}$, and constant $\varepsilon > 0$, there exist a frequency parameter $\hat{J}$ and a constant $\mathcal{R}$, with $\mathcal{R} > r_2$, such that*

$$\left\| V^{\text{ring}}_{(\omega,m,\hat{J})} - \hat{V}^{\beta_0}_{(\omega,m,J)} \right\|_{L^\infty([r_1,r_2]\times[\theta_1,\theta_2])} \leq \varepsilon$$

for all $R \geq \mathcal{R}$. The frequency parameter $\hat{J}$ satisfies

$$\hat{J} = R\,J \tag{2.70}$$

for each R.

Proof. Consider the difference

$$\begin{aligned} V^{\text{ring}}_{(\omega,m,\hat{J})}(r,\theta) - \hat{V}^{\beta_0}_{(\omega,m,J)}(r,\theta) &= \mathcal{O}_1(R^{-1}) \left[\left(1 - \frac{2}{r}\right) \frac{m^2}{r^2 \sin^2\theta} \right] + (1 + \mathcal{O}_2(R^{-1})) \left(\frac{16}{r(4+R^2)} \right) \omega^2 \\ &\quad + \mathcal{O}_2(R^{-1}) \left(-1 - \frac{2}{r} \sinh^2\beta_0 \right) \omega^2 + (1 + \mathcal{O}_3(R^{-1})) \left[\frac{4\sqrt{2}}{r} \left(1 - \frac{\sqrt{R^2(R^2-4)}}{4+R^2} \right) \right] \omega\, J \\ &\quad - \mathcal{O}_3(R^{-1}) \left(2\, \frac{2}{r} \sinh\beta_0 \cosh\beta_0 \right) \omega\, J + (1 + \mathcal{O}_4(R^{-1})) \left(\frac{16}{r(4+R^2)} \right) J^2 \\ &\quad + \mathcal{O}_4(R^{-1}) \left(1 - \frac{2}{r} \cosh^2\beta_0 \right) J^2\,, \end{aligned}$$

where $r \in [r_1, r_2]$, $\theta \in [\theta_1, \theta_2]$ and R is large. Note that $\hat{V}^{\beta_0}_{(\omega,m,J)}(r,\theta)$ is independent of R. For any fixed triple (ω, m, J) and $\varepsilon > 0$, one proves the lemma by choosing R sufficiently large.

□

Remark 2.30 (Lemma 2.7 is local). *It is important to note that Lemma 2.7 only holds* <u>*locally*</u>*, i.e. on a fixed compact region satisfying $r \ll R$. This is a manifestation of the fact that the local geometry close to the horizon of large radius, thin black rings resembles that of suitably boosted black strings.*

Remark 2.31 (C^k-formulation of Lemma 2.7). *The same statement of Lemma 2.7 can be proven to hold for the C^k-norm of $V^{\text{ring}}_{(\omega,m,\hat{J})} - \hat{V}^{\beta_0}_{(\omega,m,J)}$, for any $k \in \mathbb{N}$. However, one does not need such improvement to prove Theorem 2.7, which will be the main application of Lemma 2.7.*

2.9.2 Black rings admit stable trapping

We now want to prove the analogue of part (7) of Theorem 2.5 for potential $V^{\text{ring}}_{(\omega,m,\hat{J})}$. At the geodesic level, the following statement can be seen as claiming the existence of stably trapped null geodesics for a class of black ring spacetimes.

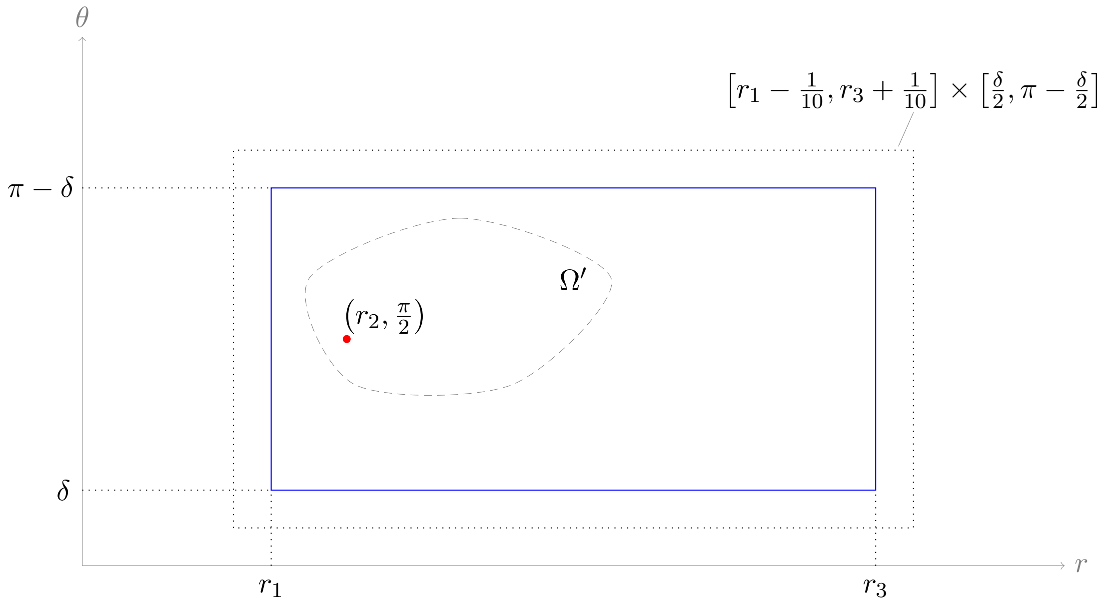

Figure 2.7: Construction for the proof of Theorem 2.7. The existence of the open set Ω' is deduced by continuity of the potential.

Theorem 2.7. *Consider the black ring metric $g_{(r_0,R)}$ defined in (2.26) and fix $r_0 = 2$. Then, there exists a constant $\mathcal{R} > 2$ such that, for any $R \geq \mathcal{R}$, the metric $g_{(2,R)}$ satisfies the following property. There exists a frequency triple $(\omega, m, \hat{J})$, with $\omega \in \mathbb{R}$ and $m, \hat{J} \in \mathbb{Z}$, and bounded sets $\Omega' \Subset \Omega \subset [2, R] \times [0, \pi)$ such that the potential $V^{\text{ring}}_{(\omega,m,\hat{J})}$ is negative on Ω', vanishes on $\partial\Omega'$ and is positive on $\Omega \setminus \overline{\Omega'}$.*

Proof. Consider the function $\hat{V}^{\beta_0}_{(\omega,m,J)}$ with $r_0 = 2$. Fix for $\hat{V}^{\beta_0}_{(\omega,m,J)}$ a triple (ω, m, J) such that there exist $2 < r_1 < r_2 < r_3$ satisfying

$$\hat{V}^{\beta_0}_{(\omega,m,J)}\left(r_1, \frac{\pi}{2}\right) > 0\,, \qquad \hat{V}^{\beta_0}_{(\omega,m,J)}\left(r_2, \frac{\pi}{2}\right) < 0\,, \qquad \hat{V}^{\beta_0}_{(\omega,m,J)}\left(r_3, \frac{\pi}{2}\right) > 0\,.$$

Such a triple exists by Theorem 2.5. For any $r_1 \leq r \leq r_3$, we have

$$\hat{V}^{\beta_0}_{(\omega,m,J)}\,(r, 0 + \delta) > 0 \qquad\qquad \hat{V}^{\beta_0}_{(\omega,m,J)}\,(r, \pi - \delta) > 0$$

for $\delta < \pi/4$ sufficiently small. Moreover, in view of the fact that in the θ direction the only stationary point of $\hat{V}^{\beta_0}_{(\omega,m,J)}$ is a local minimum at $\theta = \pi/2$, we have

$$\begin{aligned}
\hat{V}^{\beta_0}_{(\omega,m,J)}\left(r_1, \frac{\pi}{2}\right) > 0 &\implies \hat{V}^{\beta_0}_{(\omega,m,J)}\,(r_1, \theta) > 0 \quad \text{for any } \delta < \theta < \pi - \delta \\
\hat{V}^{\beta_0}_{(\omega,m,J)}\left(r_3, \frac{\pi}{2}\right) > 0 &\implies \hat{V}^{\beta_0}_{(\omega,m,J)}\,(r_3, \theta) > 0 \quad \text{for any } \delta < \theta < \pi - \delta\,.
\end{aligned}$$

By Lemma 2.7, we can deduce that, for R sufficiently large, $V^{\text{ring}}_{(\omega,m,\hat{J})}$ preserves the sign properties of $\hat{V}^{\beta_0}_{(\omega,m,J)}$ for a frequency triple $(\omega, m, \hat{J})$ such that $\hat{J} = R\,J$ (ω and m remain fixed). The lemma has to be applied in a compact region containing the points examined above, say on $[r_1 - 1/10, r_3 + 1/10] \times [\delta/2, \pi - \delta/2]$.

To conclude, note that $V^{\text{ring}}_{(\omega,m,\hat{J})}$ has been proven to be positive along the sides of the rectangle $[r_1, r_3] \times [\delta, \pi - \delta]$, while negative at the point $(r_2, \pi/2)$, which lies inside the rectangle. By continuity, $V^{\text{ring}}_{(\omega,m,\hat{J})}$ has to be negative on a bounded region inside the rectangle.

□

Remark 2.32 (Stable trapping). *Theorem 2.7 implies the presence of at least one local minimum in the bounded region Ω' where potential $V^{\text{ring}}_{(\omega,m,\hat{J})}$ is negative. In this sense, Theorem 2.7 proves the existence of stably trapped null geodesics. The orbit of the null geodesic corresponding to the local minimum of potential $V^{\text{ring}}_{(\omega,m,\hat{J})}$ lies on the torus $\mathbb{T}^2$ generated by ∂_ϕ and ∂_ψ.*

Remark 2.33. *After possibly choosing a larger $\mathcal{R}$, the following inequalities*

$$\frac{16}{r(4+R^2)} < \left| -1 - \frac{2}{r} \sinh^2 \beta_0 \right|$$
$$\frac{16}{r(4+R^2)} < \left| 1 - \frac{2}{r} \cosh^2 \beta_0 \right|$$
$$\frac{4\sqrt{2}}{r} \left(1 - \frac{\sqrt{R^2(R^2-4)}}{4+R^2} \right) < 2\,\frac{2}{r} \sinh \beta_0 \cosh \beta_0$$
$$\left| \mathcal{O}_i(R^{-1}) \right| < 1$$

hold for $R \geq \mathcal{R}$ on the compact region Ω of Theorem 2.7, with $i = 1, \ldots, 4$.

From now on, the constant $\mathcal{R}$ will be allowed to be arbitrarily large and such that both Theorem 2.7 and the inequalities of Remark 2.33 hold for $R \geq \mathcal{R}$. In turn, $1/R$ can be thought as a smallness parameter.

The class of black rings that will prove Theorem 2.8 is

$$\mathfrak{g} := \left\{ g_{(r_0,R)} \text{ such that } r_0 = 2 \text{ and } R \geq \mathcal{R} \right\} .$$

We will not be able to identify the minimum value of $\mathcal{R}$ such that Theorem 2.8 holds, so there might exist black rings metrics $g_{(r_0,R)} \notin \mathfrak{g}$ but still satisfying the theorem. In this sense, the class $\mathfrak{g}$ is not optimal.

2.9.3 Formulation of the eigenvalue problem

In analogy with (2.48), we define the two-parameter family of eigenvalue problems for black rings as

$$\boxed{\begin{aligned} Q_{\text{ring}}(b,\omega)u &= \Lambda(b,\omega)u && \text{on} \quad \Omega \\ u &= 0 && \text{on} \quad \partial\Omega\,, \end{aligned}} \tag{2.71}$$

where the operator $Q_{\text{ring}}(b,\omega)$ has the form

$$Q_{\text{ring}}(b,\omega) := -g^b_{\text{ring}}(r,\theta)\Delta_{(r_*,\theta_*)}u + \left[V_j^{b,\text{ring}}(r,\theta) + V^{b,\text{ring}}_{(\omega,m,\hat{J})}(r,\theta)\right]u\,,$$

with $b \in \mathbb{R} \cap [0,1]$, and Ω is a bounded set.

Remark 2.34 (Notation). *In the same spirit of Remark 2.15, we denote by $\omega_{b,m}$ the frequency appearing in $Q_{\text{ring}}(b,\omega)$ and*

$$\omega_{\text{boost}} \equiv \omega_{\text{boost},m} := \omega_{0,m}\,, \qquad\qquad \omega \equiv \omega_m := \omega_{1,m}\,.$$

Note that ω_m now refers to the black ring eigenvalue problem, while in Remark 2.15 it was referring to the boosted black string problem. The ω_m in Remark 2.15 is now $\omega_{\text{boost},m}$.

We define the function $g^b_{\text{ring}}(r,\theta)$ as

$$g^b_{\text{ring}}(r,\theta) := \frac{r-2}{r^3} + b(r-2)\frac{(R+r\cos\theta)^2 - R(R+2\cosh^2\beta\cos\theta)}{r^3R(R+2\cosh^2\beta\cos\theta)}\,,$$

while the potential $V^{b,\text{ring}}_{(\omega,m,\hat{J})}(r,\theta)$ reads

$$\begin{aligned}
&V^{b,\text{ring}}_{(\omega,m,\hat{J})}(r,\theta) := \\
&(1+b\,\mathcal{O}_1(R^{-1}))\left[\left(1-\frac{2}{r}\right)\frac{m^2}{r^2\sin^2\theta}\right] + (1+b\,\mathcal{O}_2(R^{-1}))\left[-1-\frac{2}{r}\sinh^2\beta_0 + b\,\frac{16}{r(4+R^2)}\right]\omega^2_{b,m} \\
&-(1+b\,\mathcal{O}_3(R^{-1}))\left[2\,\frac{2}{r}\sinh\beta_0\cosh\beta_0 - b\,\frac{4\sqrt{2}}{r}\left(1-\frac{\sqrt{R^2(R^2-4)}}{4+R^2}\right)\right]\omega_{b,m}\,\frac{\hat{J}}{R} \\
&+(1+b\,\mathcal{O}_4(R^{-1}))\left[1-\frac{2}{r}\cosh^2\beta_0 + b\,\frac{16}{r(4+R^2)}\right]\frac{\hat{J}^2}{R^2}
\end{aligned}$$

as $r \ll R$. When $b=0$, we have

$$\begin{aligned}
g^0_{\text{ring}}(r,\theta) &= g(r,\theta) \\
V_j^{0,\text{ring}}(r,\theta) &= V_j(r,\theta) \\
V^{0,\text{ring}}_{(\omega,m,\hat{J})}(r,\theta) &= \hat{V}^{\beta_0}_{(\omega,m,J)}(r,\theta)
\end{aligned}$$

with $J = \hat{J}/R$ and the quantities on the right hand side previously defined. Therefore, $Q_{\text{ring}}(0,\omega) = Q(1,\omega)$, with $\beta = \beta_0$ in $Q(1,\omega)$.

We present here the ring analogue of Lemma 2.1 and Proposition 2.1. This defines the set Ω and suitable energy levels E, completing the formulation of the eigenvalue problem.

Proposition 2.2. *Consider a black ring metric $g_{(r_0,R)}$ and fix (r_0,R) such that $r_0 = 2$ and $R \geq \mathcal{R}$. Let $m, \hat{J} \in \mathbb{Z}$ such that*

$$m \neq 0 \qquad\qquad \hat{J} = R(c\,m),\ \ \hat{J} > 0$$

for some fixed constant c, with $0 < c^2 < \frac{1}{12}$. Define $V^{\text{ring}}_{\min}$ to be the minimum of $V^{\text{ring}}_{(\omega,m,\hat{J})}$ and $x_{\min} \in (2,R)\times(0,\pi)$ such that $V^{\text{ring}}_{(\omega,m,\hat{J})}(x_{\min}) = V^{\text{ring}}_{\min}$. Consider some constant $\mathcal{E} > 0$ such that $V^{\text{ring}}_{(\mathcal{E}m,m,\hat{J})}$ has a local minimum and such that there exists $\Omega \subset [2,R]\times[0,\pi)$ satisfying

(i) $x_{\min} \in \Omega\,,$

(ii) $V^{\text{ring}}_{(\mathcal{E}m,m,\hat{J})}(x) = 0$ *for* $x \in \partial\Omega\,,$

(iii) there are no local maxima of $V^{\text{ring}}_{(\mathcal{E}m,m,\hat{J})}$ *in* $\Omega\,,$

(iv) $x \in \Omega \implies r(x) > 6\,,$

(v) Remark 2.33 holds on $\Omega\,.$

Fix now some energy level $E > 0$ such that

(a) $V^{\text{ring}}_{(Em,m,\hat{J})}$ has a local minimum and there exists $\Omega' \subset \Omega$ satisfying the same properties (i)-(v) of Ω, but now with respect to $V^{\text{ring}}_{(Em,m,\hat{J})}$,

(b) $\hat{V}^{\beta_0}_{(Em,m,J)}$ has a local minimum and there exists $\Omega'' \subset \Omega$ satisfying the same properties (i)-(v) of Ω, but now with respect to $\hat{V}^{\beta_0}_{(Em,m,J)}$,

(c) $\hat{V}^{0}_{(Em,m,J)}$ has a local minimum and there exists $\Omega''' \subset \Omega$ satisfying the same properties (i)-(v) of Ω, but now with respect to $\hat{V}^{0}_{(Em,m,J)}$.

Then, the final parts of Lemma 2.1 and Proposition 2.1 hold for $\frac{1}{m^2}\hat{V}^{0}_{(Em,m,J)}$ and $\frac{1}{m^2}\hat{V}^{\beta_0}_{(Em,m,J)}$ with respect to the open set Ω. Furthermore, for any sufficiently small constants $\delta, \delta' > 0$, there exists some constant $c' > 0$ such that

$$\mathrm{dist}(x,\partial\Omega) < \delta' \implies \frac{1}{m^2}V^{\text{ring}}_{(\kappa m,m,\hat{J})}(x) > c'$$

for all $\kappa \in \mathbb{R}$ satisfying $|\kappa^2 - E^2| \leq \delta$, with $\mathrm{dist}(\cdot,\cdot)$ the Euclidean distance.

Proof. Combining Proposition 2.1, Lemma 2.7 and the proof of Theorem 2.7, one can show that a choice of $\mathcal{E}$ and E with such properties is possible. In particular, to ensure that Ω satisfies property (iii), one would need a stronger version of Lemma 2.7 (see Remark 2.31). The result of the theorem follows by continuity arguments. □

Analogous statements of those of Lemma 2.5 and Corollary 2.2 hold for our class of black rings.

Lemma 2.8. *Let $\psi_n(b,\omega)$ be a non identically zero eigenfunction of (2.71), normalized on Ω. If the assumptions of Proposition 2.2 are satisfied and $m^2, J^2 > M$, for some positive constant M, then the implication*

$$\omega_{b,m} = 0 \implies \int_\Omega \psi_n(b,\omega) Q_{\text{ring}}(b,\omega)\psi_n(b,\omega)\, dr_* d\theta_* \neq 0$$

holds for any $b \in [0,1]$.

Proof. One follows the same argument of the proof of Lemma 2.5. In particular, properties (iv) (combined with $\cosh^2\beta_0 < 3$) and (v) of Ω in Proposition 2.2 are crucial to prove that the potential is everywhere positive on Ω when $\omega_{b,m} = 0$. □

Lemma 2.9. *Let* $\psi_n(b,\omega)$ *be a non identically zero eigenfunction of* (2.71), *normalized on* Ω. *If the assumptions of Proposition 2.2 are satisfied, then the implication*

$$\begin{aligned} &\omega_{b,m}^2 = o(m^2) \text{ as } m^2, J^2 \text{ are sufficiently large} \\ &\Longrightarrow \int_\Omega \psi_n(b,\omega) Q_{\text{ring}}(b,\omega)\psi_n(b,\omega)\, dr_* d\theta_* \neq 0 \end{aligned}$$

holds for any $b \in [0,1]$.

Proof. See proof of Corollary 2.2, combined with considerations in the proof of Lemma 2.8. □

2.9.4 A second application of the Implicit Function Theorem

The perturbation argument to prove Theorem 2.8 is almost identical to the one presented for Lemma 2.6 and Theorem 2.6, so we only sketch it. As already discussed, to apply the Implicit Function Theorem to $\Lambda(b,\omega) = 0$ at $(b_0, \omega_{b_0,m})$, we need

$$\frac{\partial \Lambda}{\partial \omega}(b_0, \omega_{b_0,m}) = \int_\Omega \psi_n(b_0, \omega_{b_0,m}) \left(\frac{\partial V^{b,\text{ring}}_{(\omega,m,\hat{J})}}{\partial \omega}(b_0, \omega_{b_0,m}) \right) \psi_n(b_0, \omega_{b_0,m})\, dr_* d\theta_*$$

bounded away from zero. It is therefore enough to show

$$\begin{aligned} \frac{\partial V^{b,\text{ring}}_{(\omega,m,\hat{J})}}{\partial \omega}(b_0, \omega_{b_0,m}) &= 2(1 + b_0\, \mathcal{O}_2(R^{-1})) \left[-1 - \frac{2}{r}\sinh^2\beta_0 + b_0 \frac{16}{r(4+R^2)} \right] \omega_{b_0,m} \\ &\quad - (1 + b_0\, \mathcal{O}_3(R^{-1})) \left[2\,\frac{2}{r}\sinh\beta_0\cosh\beta_0 - b_0 \frac{4\sqrt{2}}{r} \left(1 - \frac{\sqrt{R^2(R^2-4)}}{4+R^2} \right) \right] \frac{\hat{J}}{R} \end{aligned}$$

bounded away from zero. Note that, in view of point (v) of Proposition 2.2, we have

$$\begin{aligned} b_0 \frac{16}{r(4+R^2)} &< \left| -1 - \frac{2}{r}\sinh^2\beta_0 \right| \\ b_0 \frac{4\sqrt{2}}{r}\left(1 - \frac{\sqrt{R^2(R^2-4)}}{4+R^2} \right) &< 2\,\frac{2}{r}\sinh\beta_0\cosh\beta_0 \\ \left| b_0\, \mathcal{O}_2(R^{-1}) \right| &< 1 \\ \left| b_0\, \mathcal{O}_3(R^{-1}) \right| &< 1 \end{aligned}$$

on Ω, for any $b_0 \in [0,1]$. With an assumption on the sign of $\omega_{b_0,m}$ as the one we formulated for Lemma 2.6, i.e. $\omega_{b_0,m} > 0$, the derivative becomes the sum of two negative terms and can be bounded away from zero *uniformly in* b_0 as

$$\begin{aligned}\frac{\partial V^{b,\text{ring}}_{(\omega,m,\hat{J})}}{\partial \omega}(b_0,\omega_{b_0,m}) &\leq -C_R\,\omega_{b_0,m}\\ &\leq -C_R\,m\,,\end{aligned}$$

with constant $C_R > 0$ independent of m, where the first inequality follows from the same argument used in the proof of Lemma 2.6, while the second one makes use of the lower bound on $\omega_{b,m}$ provided by Lemma 2.9.

Furthermore, we have

$$\begin{aligned}&\frac{\partial \Lambda}{\partial b}(b_0,\omega_{b_0,m})\\ &= \int_\Omega \psi_n(b_0,\omega_{b_0,m})\left(-\frac{\partial g^b_{\text{ring}}}{\partial b}(b_0)\Delta_{(r_*,\theta_*)} + \frac{\partial V^{b,\text{ring}}_{\hat{J}}}{\partial b}(b_0) + \frac{\partial V^{b,\text{ring}}_{(\omega,m,\hat{J})}}{\partial b}(b_0,\omega_{b_0,m})\right)\psi_n(b_0,\omega_{b_0,m})\,dr_* d\theta_*\,,\end{aligned}$$

with

$$\begin{aligned}\frac{\partial V^{b,\text{ring}}_{(\omega,m,\hat{J})}}{\partial b}(b_0,\omega_{b_0,m}) &= \mathcal{O}_1(R^{-1})\left[\left(1-\frac{2}{r}\right)\frac{m^2}{r^2\sin^2\theta}\right]\\ &\quad + \left[\mathcal{O}_2(R^{-1})\left(-1-\frac{2}{r}\sinh^2\beta_0\right) + \left(1+2b_0\mathcal{O}_2(R^{-1})\right)\left(\frac{16}{r(4+R^2)}\right)\right]\omega^2_{b_0,m}\\ &\quad - \left[\mathcal{O}_3(R^{-1})\left(2\,\frac{2}{r}\sinh\beta_0\cosh\beta_0\right) - \frac{4\sqrt{2}}{r}\left(1+2b_0\mathcal{O}_3(R^{-1})\right)\left(1-\frac{\sqrt{R^2(R^2-4)}}{4+R^2}\right)\right]\omega_{b_0,m}\,\frac{\hat{J}}{R}\\ &\quad + \left[\mathcal{O}_4(R^{-1})\left(1-\frac{2}{r}\cosh^2\beta_0\right) + \left(1+2b_0\mathcal{O}_4(R^{-1})\right)\left(\frac{16}{r(4+R^2)}\right)\right]\frac{\hat{J}^2}{R^2}\,.\end{aligned}$$

In view of Lemma 2.9, the bound

$$\left|\frac{\partial V^{b,\text{ring}}_{(\omega,m,\hat{J})}}{\partial b}(b_0,\omega_{b_0,m})\right| \leq C_R\,\omega^2_{b_0,m}$$

holds. Therefore, we have

$$\left|\frac{d\omega_{b,m}}{db}(b_0)\right| \leq C_R\,\omega_{b_0,m}\,.$$

This gives a bound $\omega_{b,m} \leq C_R\,\omega_{0,m}$ for any $b \in (0,1]$, with constant $C_R > 0$ independent of b and m.

Remark 2.35. *All the relations derived so far, as well as the proof of Theorem 2.8, are independent of the choice of* r_0. *In fact, all the arguments can be repeated for any* $r_0 > 0$. *Theorem 2.8 is stated relaxing the assumption* $r_0 = 2$.

The main theorem for the black ring eigenvalue problem is the following.

Theorem 2.8 (**Eigenvalues black ring**). *Consider the fixed energy levels $\mathcal{E}, E > 0$ and the open set Ω, according to Proposition 2.2. Let $m, \hat{J} \in \mathbb{Z}$ as in Proposition 2.2. Given eigenvalues $\omega_{\text{boost},m}$ for the eigenvalue problem (2.47) on Ω, we further assume $\omega_{\text{boost},m} > 0$. Then, there exists a constant $M > 0$ such that the following statement holds for any $m^2 > M$. There exists an eigenvalue ω_m^2 and an associated smooth eigenfunction u_m to the black ring eigenvalue problem. Furthermore, we have $\omega_m > 0$ and*

$$C_{r_0,R} \leq \frac{\omega_m^2}{m^2} \leq E^2 + \frac{\mathcal{E}^2 - E^2}{10} \tag{2.72}$$

for some constant $C_{r_0,R} > 0$ independent of m.

Proof. The proof follows the same lines of the proof of Theorem 2.6. We only show how to prove inequality (2.72), while the first part of the statement is left to the reader.

Recall that we have already concluded that

$$c_{r_0,R} \leq \frac{\omega_m^2}{m^2} \leq C_{r_0,R} \tag{2.73}$$

for some constants $c_{r_0,R}, C_{r_0,R} > 0$ independent of m, where ω_m^2 is such that the n-th eigenvalue $\Lambda_n(1, \omega)$ of problem (2.71) is zero. Define

$$E_n(b) := \frac{\omega_{b,m}(b)}{m}$$

with $b \in [0, 1]$. Then,

$$\begin{aligned} |E_n(1) - E_n(0)| &= \left| \frac{1}{m} \int_0^1 \frac{d\omega_{b,m}(b)}{db} db \right| \\ &\leq \frac{1}{m} \int_0^1 \left| \frac{d\omega_{b,m}(b)}{db} \right| db \\ &\leq \delta'_{r_0,R} , \end{aligned}$$

with $\delta'_{r_0,R} > 0$ arbitrarily small constant for R/r_0 sufficiently large. Crucially, $\delta'_{r_0,R}$ is independent of m. The last inequality holds as a consequence of inequality (2.73) and the fact that, for any fixed r_0, on the bounded set Ω one has

$$\frac{1}{m} \left| \frac{d\omega_{b,m}(b)}{db} \right| \to 0 \quad \text{as} \quad R \to \infty$$

uniformly in b and m.

From Theorem 2.6, combined with part (b) and (c) of Proposition 2.2, we have

$$E_n^2(0) \in \left[C_{r_0}, E^2 + \delta\right]$$

for m^2 sufficiently large, where $\delta > 0$ is an arbitrarily small constant. We obtain

$$C_{r_0} - \delta'_{r_0,R} \leq E_n^2(0) - \delta'_{r_0,R} \leq E_n^2(1) \leq E_n^2(0) + \delta'_{r_0,R} \leq E^2 + \delta + \delta'_{r_0,R} \,.$$

This concludes the proof of inequality (2.72).

□

Remark 2.36 (**$V^{\text{ring}}_{(\omega_m,m,\hat{J})}$ has a good structure**)**.** *In view of the same considerations of Remark 2.26, potential $V^{\text{ring}}_{(\omega_m,m,\hat{J})}(r,\theta)$ has the structure that we need to construct quasimodes with exponentially small errors. Quasimodes will be localised around the minimum of the potential, which is approximately located at*

$$\begin{aligned} r(x_{min}) &\sim \frac{1+\sqrt{\left[1-12(\omega_m/m+\sqrt{2}\,c)^2\right]}}{2(\omega_m/m+\sqrt{2}\,c)^2}\,, \\ \theta(x_{min}) &\sim \frac{\pi}{2} \end{aligned}$$

with $c^2 \leq 1/12$, as one can infer from (2.53) *and Proposition 2.2.*

2.10 Proof of Theorem 2.3

The aim of this section is to construct quasimodes and prove Theorem 2.3. We will crucially apply Theorem 2.8. The structure of the presentation and the formulation of the main results closely follow Section 4 of [69].

2.10.1 A lemma for the energy estimate

We need a preliminary lemma, which will be used to prove an energy estimate for solutions to the black ring eigenvalue problem. We give a statement for a generic open set and smooth functions.

Lemma 2.10 (adapted from [69] Lemma 4.3)**.** *Let $\Omega \subset \mathbb{R}^2$ be a bounded set and $h > 0$ a real constant. Given smooth functions $u, W, \phi : \Omega \to \mathbb{R}$ such that $u|_{\partial\Omega} = 0$, we have*

$$\int_\Omega \left(\left| \frac{\partial}{\partial r_*}\left(e^{\phi/h}u\right)\right|^2 + \left| \frac{\partial}{\partial \theta_*}\left(e^{\phi/h}u\right)\right|^2 + h^{-2}\left(W - \left(\frac{\partial\phi}{\partial r_*}\right)^2 - \left(\frac{\partial\phi}{\partial\theta_*}\right)^2\right) e^{2\phi/h}|u|^2 \right) dr_* d\theta_*$$
$$= \int_\Omega \left(-\frac{\partial^2 u}{\partial r_*^2} - \frac{\partial^2 u}{\partial \theta_*^2} + h^{-2} W u \right) u\, e^{2\phi/h} dr_* d\theta_* \,.$$

Proof. The proof is an integration by parts. We have

$$\begin{aligned}
&\int_\Omega \left(-\frac{\partial^2 u}{\partial r_*^2} - \frac{\partial^2 u}{\partial \theta_*^2} + h^{-2} W u\right) u\, e^{2\phi/h}\, dr_* d\theta_* \\
&= \int_\Omega \frac{\partial u}{\partial r_*}\, \partial_{r_*}\left(u\, e^{2\phi/h}\right) + \frac{\partial u}{\partial \theta_*}\, \partial_{\theta_*}\left(u\, e^{2\phi/h}\right) + h^{-2} W |u|^2 e^{2\phi/h}\, dr_* d\theta_* \\
&= \int_\Omega \left(\frac{\partial u}{\partial r_*}\right)^2 e^{2\phi/h} + \frac{\partial u}{\partial r_*} u \frac{2}{h}\frac{\partial \phi}{\partial r_*} e^{2\phi/h} + \frac{\partial u}{\partial \theta_*}\, \partial_{\theta_*}\left(u\, e^{2\phi/h}\right) + h^{-2} W |u|^2 e^{2\phi/h}\, dr_* d\theta_* \\
&= \int_\Omega \left(\frac{\partial u}{\partial r_*} e^{\phi/h}\right)^2 + \frac{\partial u}{\partial r_*} u \frac{2}{h}\frac{\partial \phi}{\partial r_*} e^{2\phi/h} + \left(h^{-1}\frac{\partial \phi}{\partial r_*} e^{\phi/h} u\right)^2 - \left(h^{-1}\frac{\partial \phi}{\partial r_*} e^{\phi/h} u\right)^2 \\
&\quad + \frac{\partial u}{\partial \theta_*}\, \partial_{\theta_*}\left(u\, e^{2\phi/h}\right) + h^{-2} W |u|^2 e^{2\phi/h}\, dr_* d\theta_* \\
&= \int_\Omega \left|\frac{\partial}{\partial r_*}\left(e^{\phi/h} u\right)\right|^2 + \frac{\partial u}{\partial \theta_*}\, \partial_{\theta_*}\left(u\, e^{2\phi/h}\right) + h^{-2}\left(W - \left(\frac{\partial \phi}{\partial r_*}\right)^2\right) e^{2\phi/h} |u|^2\, dr_* d\theta_* \\
&= \int_\Omega \left|\frac{\partial}{\partial r_*}\left(e^{\phi/h} u\right)\right|^2 + \left|\frac{\partial}{\partial \theta_*}\left(e^{\phi/h} u\right)\right|^2 + h^{-2}\left(W - \left(\frac{\partial \phi}{\partial r_*}\right)^2 - \left(\frac{\partial \phi}{\partial \theta_*}\right)^2\right) e^{2\phi/h} |u|^2\, dr_* d\theta_*\,,
\end{aligned}$$

where we make use of the fact that u vanishes on $\partial\Omega$ in the first equality and we omit the identical calculation for ∂_{θ_*} in the last equality. □

2.10.2 Agmon distance

Consider

$$V_{\text{eff}}^{h,E} := h^2\left[V_j^{\text{ring}} + V_{(E\,m,m,\hat{J})}^{\text{ring}}\right]$$
$$h^2 = m^{-2}\,,$$

where the energy level E is fixed as in Proposition 2.2. The *Agmon distance* between two points $x, y \in \mathbb{R}^2$ associated to the energy level E and potential $V_{\text{eff}}^{h,E}$ is

$$d(x,y) := \inf_{\gamma \in C^{1,pw}([0,1];x,y)} \int_0^1 \left(\sqrt{\frac{V_{\text{eff}}^{h,E}(\gamma(t))}{g_{\text{ring}}(\gamma(t))}}\right) |\gamma'(t)|\, \chi_{\left\{V_{\text{eff}}^{h,E} \geq 0\right\}} dt\,,$$

where

$$C^{1,pw}([0,1];x,y) = \left\{\gamma \in C^{1,pw}([0,1];\mathbb{R}^2), \gamma(0) = x, \gamma(1) = y\right\}$$

with $C^{1,pw}([0,1];\mathbb{R}^2)$ set of piecewise C^1 curves in $\mathbb{R}^2$ (see [53]). Recall that $g_{\text{ring}}(x) > 0$. The Agmon distance satisfies

$$|\nabla_x d(x,y)|^2 \leq \max\left\{\frac{V_{\text{eff}}^{h,E}(x)}{g_{\text{ring}}(x)}, 0\right\}\,. \tag{2.74}$$

For any fixed energy level E, we define the distance from the classically allowed region as

$$d_E(x) := \inf_{y \in \left\{V_{\text{eff}}^{h,E} \leq 0\right\}} d(x,y)$$

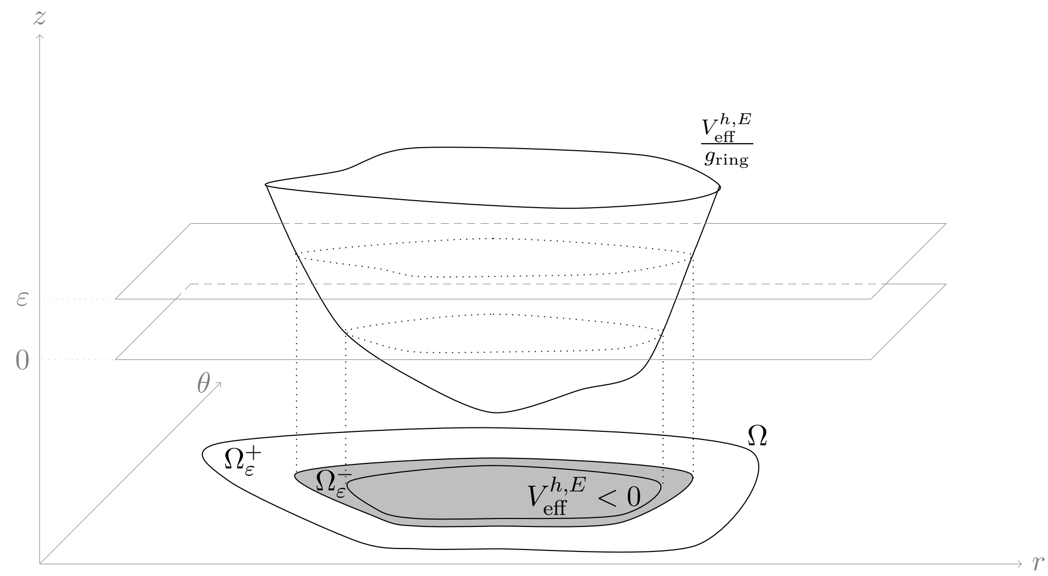

Figure 2.8: The smallest set in the figure is the region where $V_{\text{eff}}^{h,E} < 0$. The shaded region corresponds to $\Omega_\varepsilon^-(E)$, while the largest set is Ω.

with x, y points in $\mathbb{R}^2$. We will also make use of the two following sets

$$\Omega_\varepsilon^+(E) := \left\{ x : \frac{V_{\text{eff}}^{h,E}(x)}{g_{\text{ring}}(x)} > \varepsilon \right\} \cap \Omega$$

$$\Omega_\varepsilon^-(E) := \left\{ x : \frac{V_{\text{eff}}^{h,E}(x)}{g_{\text{ring}}(x)} \leq \varepsilon \right\} \cap \Omega$$

with constant $\varepsilon > 0$. Note that $\Omega_\varepsilon^+(E) \cap \Omega_\varepsilon^-(E) = \emptyset$ and $\Omega_\varepsilon^+(E) \cup \Omega_\varepsilon^-(E) = \Omega$.

2.10.3 The key energy estimate

For any $\varepsilon \in (0, 1)$, we define

$$\phi_{E,\varepsilon}(x) := (1 - \varepsilon) d_E(x)$$

and

$$a_E(\varepsilon) := \sup_{\Omega_\varepsilon^-(E)} d_E\,.$$

Consider u smooth eigenfunction to the black ring eigenvalue problem, with associated eigenvalue $\kappa := h\,\omega_m$ such that $|\kappa^2 - E^2| \leq \delta$, with constant $\delta > 0$. Then, the following key energy estimate holds.

Claim 2.2 (adapted from [69] Lemma 4.4)**.** *For any sufficiently small constants $\varepsilon, h > 0$, and*

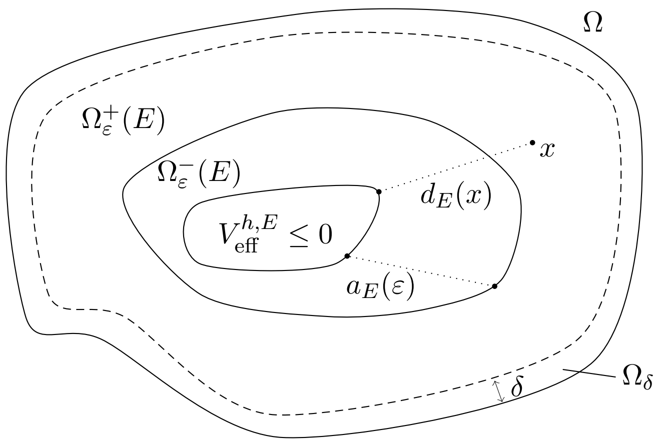

Figure 2.9: The Agmon distance differs from the Euclidean distance between two points in that it has to be weighted by the square root of $V_{\text{eff}}^{h,E}(\gamma(t))/g_{\text{ring}}(\gamma(t))$. To give some intuition, the figure shows the various quantities introduced so far as if they were defined in the Euclidean sense, i.e. with the weights set equal to one. Although this gives an idea of the construction, the reader should keep in mind that $d_E(x)$ and $a_E(\varepsilon)$ do not necessarily look like the ones in figure.

for sufficiently small constant $\delta_{\varepsilon,h} > 0$, we have

$$\int_\Omega h^2 \left(\left| \frac{\partial}{\partial r_*}(e^{\phi_{E,\varepsilon}/h}u) \right|^2 + \left| \frac{\partial}{\partial \theta_*}(e^{\phi_{E,\varepsilon}/h}u) \right|^2 \right) dr_* d\theta_* + \frac{1}{2}\varepsilon^2 \int_{\Omega_\varepsilon^+(E)} e^{2\phi_{E,\varepsilon}/h}|u|^2 dr_* d\theta_*$$
$$\le C(\kappa^2 + \frac{1}{2}\varepsilon)e^{2a_E(\varepsilon)/h} \|u\|^2_{L^2(\Omega)}$$

for some constant $C > 0$ depending only on Ω.

Proof. By the assumption that u is a smooth solution to the black ring eigenvalue problem, we have

$$-h^2\Delta_{(r_*,\theta_*)}u + \frac{1}{g_{\text{ring}}(r,\theta)} V_{\text{eff}}^{h,\kappa}(r,\theta)u = 0\,. \tag{2.75}$$

With the notation of Lemma 2.10, we define

$$W(r,\theta) := \frac{1}{g_{\text{ring}}(r,\theta)} V_{\text{eff}}^{h,\kappa}(r,\theta) \qquad\qquad \phi := \phi_{E,\varepsilon}$$

Together with equation (2.75), Lemma 2.10 gives

$$\int_\Omega h^2 \left(\left| \frac{\partial}{\partial r_*}(e^{\phi_{E,\varepsilon}/h}u) \right|^2 + \left| \frac{\partial}{\partial \theta_*}(e^{\phi_{E,\varepsilon}/h}u) \right|^2 \right) dr_* d\theta_*$$
$$+ \int_{\Omega_\varepsilon^+(E)} \left(\frac{V_{\text{eff}}^{h,\kappa}(r,\theta)}{g_{\text{ring}}(r,\theta)} - \left(\frac{\partial \phi_{E,\varepsilon}}{\partial r_*}\right)^2 - \left(\frac{\partial \phi_{E,\varepsilon}}{\partial \theta_*}\right)^2 \right) e^{2\phi_{E,\varepsilon}/h}|u|^2 dr_* d\theta_*$$
$$= \int_{\Omega_\varepsilon^-(E)} \left(\frac{-V_{\text{eff}}^{h,\kappa}(r,\theta)}{g_{\text{ring}}(r,\theta)} + \left(\frac{\partial \phi_{E,\varepsilon}}{\partial r_*}\right)^2 + \left(\frac{\partial \phi_{E,\varepsilon}}{\partial \theta_*}\right)^2 \right) e^{2\phi_{E,\varepsilon}/h}|u|^2 dr_* d\theta_*\,,$$

where we used $\Omega_\epsilon^+(E) \cup \Omega_\epsilon^-(E) = \Omega$. For h sufficiently small, we can estimate the right hand side as

$$\int_{\Omega_\varepsilon^-(E)} \left(\frac{-V_{\text{eff}}^{h,\kappa}(r,\theta)}{g_{\text{ring}}(r,\theta)} + \left(\frac{\partial \phi_{E,\varepsilon}}{\partial r_*}\right)^2 + \left(\frac{\partial \phi_{E,\varepsilon}}{\partial \theta_*}\right)^2 \right) e^{2\phi_{E,\varepsilon}/h} |u|^2 dr_* d\theta_*$$
$$\leq C\left(\kappa^2 + \varepsilon(1-\varepsilon)\right) e^{2a_E(\varepsilon)/h} \|u\|^2_{L^2(\Omega)}$$

with C positive constant independent of h. To see this, first recall that $g_{\text{ring}}(r,\theta)$ is strictly positive and bounded on Ω and note that, by definition of $a_E(\varepsilon)$, one has $\phi_{E,\varepsilon}|_{\Omega_\varepsilon^-(E)} \leq a_E(\varepsilon)$. Using property (2.74) of the Agmon distance combined with $V_{\text{eff}}^{h,E}(x)/g_{\text{ring}}(x) \leq \varepsilon$ on $\Omega_\varepsilon^-(E)$, we also have

$$\begin{aligned}
\left(\frac{\partial \phi_{E,\varepsilon}}{\partial r_*}\right)^2 + \left(\frac{\partial \phi_{E,\varepsilon}}{\partial \theta_*}\right)^2 &= |\nabla \phi_{E,\varepsilon}|^2 \\
&= (1-\varepsilon)^2 |\nabla d_E|^2 \\
&\leq (1-\varepsilon)^2 \varepsilon \\
&\leq (1-\varepsilon)\varepsilon
\end{aligned}$$

on $\Omega_\varepsilon^-(E)$, where the last inequality holds because $\varepsilon \in (0,1)$. Note the obvious $\|u\|^2_{L^2(\Omega_\varepsilon^-(E))} \leq \|u\|^2_{L^2(\Omega)}$.

On $\Omega_\varepsilon^+(E)$, one has

$$\begin{aligned}
\left(\frac{\partial \phi_{E,\varepsilon}}{\partial r_*}\right)^2 + \left(\frac{\partial \phi_{E,\varepsilon}}{\partial \theta_*}\right)^2 &= (1-\varepsilon)^2 |\nabla d_E|^2 \\
&\leq (1-\varepsilon)^2 \frac{V_{\text{eff}}^{h,E}(r,\theta)}{g_{\text{ring}}(r,\theta)} \,.
\end{aligned}$$

This implies

$$\begin{aligned}
\frac{V_{\text{eff}}^{h,\kappa}(r,\theta)}{g_{\text{ring}}(r,\theta)} - \left(\frac{\partial \phi_{E,\varepsilon}}{\partial r_*}\right)^2 - \left(\frac{\partial \phi_{E,\varepsilon}}{\partial \theta_*}\right)^2 &\geq (1-(1-\epsilon)^2)\left(\frac{V_{\text{eff}}^{E,\kappa}(r,\theta)}{g_{\text{ring}}(r,\theta)}\right) - \hat{\delta} \\
&\geq \varepsilon^2 - \hat{\delta}\,,
\end{aligned}$$

where the constant $\hat{\delta} > 0$ is such that

$$\frac{V_{\text{eff}}^{h,\kappa}(r,\theta)}{g_{\text{ring}}(r,\theta)} \geq \frac{V_{\text{eff}}^{E,\kappa}(r,\theta)}{g_{\text{ring}}(r,\theta)} - \hat{\delta}\,,$$

on $\Omega_\varepsilon^+(E)$ and continuously depends on δ, with $\hat{\delta} \to 0$ when $\delta \to 0$.

To conclude the proof, we choose $\varepsilon \leq 1/2$ and $\hat{\delta} \leq \varepsilon^2/2$, which can be achieved by choosing δ sufficiently small.

□

We now define the set

$$\Omega_\delta := \{x \in \Omega : \text{dist}(x, \partial\Omega) \leq \delta\}\,.$$

The inequality of Claim 2.2 gives control on each term appearing on the left hand side (since they are both positive), i.e.

$$\int_{\Omega} h^2 \left(\left| \frac{\partial}{\partial r_*}(e^{\phi_{E,\varepsilon}/h}u) \right|^2 + \left| \frac{\partial}{\partial \theta_*}(e^{\phi_{E,\varepsilon}/h}u) \right|^2 \right) dr_* d\theta_* \leq C(\kappa^2 + \frac{1}{2}\varepsilon) e^{2a_E(\varepsilon)/h} \|u\|^2_{L^2(\Omega)} \tag{2.76}$$

$$\frac{1}{2}\varepsilon^2 \int_{\Omega_\varepsilon^+(E)} e^{2\phi_{E,\varepsilon}/h} |u|^2 dr_* d\theta_* \leq C(\kappa^2 + \frac{1}{2}\varepsilon) e^{2a_E(\varepsilon)/h} \|u\|^2_{L^2(\Omega)} . \tag{2.77}$$

Inequality (2.77) implies that, for any sufficiently small constants $\delta, \delta' > 0$, we have

$$\int_{\Omega_\delta} |u|^2 dr_* d\theta_* \leq C e^{-C/h} \|u\|^2_{L^2(\Omega)}$$

for all $\kappa^2 \in [E^2 - \delta', E^2 + \delta']$ and some *positive* constant $C > 0$ *independent* of h. To see this, remember that E is fixed as in Proposition 2.2 and note that there exists a constant $c > 0$ such that $\phi_{E,\varepsilon} \geq c$ for any $x \in \Omega_\delta$ and $\kappa^2 \in [E^2 - \delta', E^2 + \delta']$, with c *uniform* in ε (this is simply by definition of $\phi_{E,\varepsilon}$). Furthermore, we claim that there exists $\varepsilon > 0$ such that $a_E(\varepsilon) \leq c/2$ for any h sufficiently small, which gives a negative exponent on the right hand side. In fact, $a_E(\varepsilon) \to 0$ as $\varepsilon \to 0$ uniformly in h when h is sufficiently small. The inequality follows after choosing δ small enough such that $\Omega_\delta \subset \Omega_\varepsilon^+(E)$.

By the same argument, combined with Young's inequality, inequality (2.76) gives

$$\int_{\Omega_\delta} \left(\left| \frac{\partial u}{\partial r_*} \right|^2 + \left| \frac{\partial u}{\partial \theta_*} \right|^2 \right) dr_* d\theta_* \leq C h^{-2} e^{-C/h} \|u\|^2_{L^2(\Omega)} .$$

For h sufficiently small, one can absorb the h^{-2} factor, i.e. there exists a constant $C > 0$ independent of h such that

$$\int_{\Omega_\delta} \left(\left| \frac{\partial u}{\partial r_*} \right|^2 + \left| \frac{\partial u}{\partial \theta_*} \right|^2 \right) dr_* d\theta_* \leq C e^{-C/h} \|u\|^2_{L^2(\Omega)} .$$

Putting everything together, we get the key estimate

$$\int_{\Omega_\delta} \left(\left| \frac{\partial u}{\partial r_*} \right|^2 + \left| \frac{\partial u}{\partial \theta_*} \right|^2 + |u|^2 \right) dr_* d\theta_* \leq C e^{-C/h} \|u\|^2_{L^2(\Omega)} . \tag{2.78}$$

In the following statement, we sum up what we have obtained so far, combining Theorem 2.8 with estimate (2.78).

Theorem 2.9 (adapted from [69] Lemma 4.5)**.** *Consider the assumptions of Theorem 2.8 and let $\delta' > 0$ be a sufficiently small constant such that $\delta' < (\mathcal{E}^2 - E^2)/10$. Then, for any sufficiently small constant $\delta > 0$, there exists a constant $M > 0$ and a sequence of solutions $\{u_m\}_{m \geq M}^{\infty}$ to the black ring eigenvalue problem such that the associated eigenvalues κ_m satisfy*

$\kappa_m^2 \in [C'_{r_0,R}, E^2 + \delta']$ *and*

$$\int_{\Omega_\delta} \left(\left| \frac{\partial u_m}{\partial r_*} \right|^2 + \left| \frac{\partial u_m}{\partial \theta_*} \right|^2 + |u_m|^2 \right) dr_* d\theta_* \le C e^{-Cm} \|u_m\|^2_{L^2(\Omega)}$$

for some constants $C, C'_{r_0,R} > 0$ *independent of* m.

In the next sections we aim to apply this result to the quasimode construction, which ultimately leads to the proof of Theorem 2.3.

2.10.4 Construction of quasimodes

Define a smooth, real valued cut-off function $\chi : \mathcal{D} \to \mathbb{R}$ as

$$\chi(r_*, \theta_*) = \begin{cases} 1 & \text{if } (r_*, \theta_*) \in \Omega \setminus \Omega_\delta \\ 0 & \text{if } (r_*, \theta_*) \notin \Omega \end{cases} .$$

The quasimodes are defined as functions $\Psi_m : \mathcal{D} \to \mathbb{C}$ such that

$$\Psi_m(t, r_*, \theta_*, \phi, \psi) := e^{-i\,\omega_m\, t} e^{i(m\,\phi + \hat{J}\,\psi)} \chi(r_*, \theta_*) \left[(-\det g_{\text{ring}})\, g^{rr}_{\text{ring}}\, g^{\theta\theta}_{\text{ring}} \right]^{-\frac{1}{4}} u_m(r_*, \theta_*) \quad (2.79)$$

with ω_m, m, $\hat{J}$ and $u_m(r_*, \theta_*)$ as in Theorem 2.9 and $g_{\text{ring}} \in \mathfrak{g}$ in form (2.26). Quasimodes do not solve the wave equation on the whole domain of outer communication $\mathcal{D}$, but the error is exponentially small for high frequency m and supported on a bounded region for each fixed time t. This is proven by the following lemma.

Lemma 2.11. *Consider the quasimodes* Ψ_m *defined in* (2.79)*, which satisfy*

$$\Box_{g_{\text{ring}}} \Psi_m = \mathfrak{Err}_m(\Psi_m) ,$$

where $\mathfrak{Err}_m(\Psi_m)$ *is the error. Then, for* m *sufficiently large, we have the following estimate*

$$\left\| \Box_{g_{\text{ring}}} \Psi_m \right\|_{H^k(\Sigma_{t^*})} \le C_k e^{-C_k m} \|\Psi_m\|_{L^2(\Sigma_0)} ,$$

with constant $C_k > 0$ *independent of* m. *Furthermore, the error is supported on* Ω_δ.

Remark 2.37. *Note the abuse of notation in Lemma 2.11 and in what follows, where* $\Omega_\delta \equiv \Omega_\delta \times [0, \infty) \times [0, \Delta\phi) \times [0, \Delta\psi)$ *and* $\Omega \equiv \Omega \times [0, \infty) \times [0, \Delta\phi) \times [0, \Delta\psi)$*, with* $[0, \infty)$ *time domain and* $\Delta\phi, \Delta\psi$ *the periods of the angular coordinates* ϕ, ψ.

Proof. The part of the statement concerning the support of the error immediately follows from the definition of the quasimodes Ψ_m. To prove the inequality, first note that given functions χ

and f, we have

$$\Box_{g_{\text{ring}}}(\chi f) = \chi(\Box_{g_{\text{ring}}} f) + 2g^{\mu\nu}_{\text{ring}}(\partial_\mu \chi)(\partial_\nu f) + f(\Box_{g_{\text{ring}}}\chi)\,.$$

Since

$$\Box_{g_{\text{ring}}}\left(e^{-i\,\omega_m\,t} e^{i(m\,\phi+\hat{J}\,\psi)}\,\left[(-\det g_{\text{ring}})\, g^{rr}_{\text{ring}}\, g^{\theta\theta}_{\text{ring}}\right]^{-\frac{1}{4}}\, u_m(r_*,\theta_*)\right) = 0$$

on Ω when u_m is solution to the black ring eigenvalue problem, using the formula above we can deduce

$$\begin{aligned}\left\|\Box_{g_{\text{ring}}}\Psi_m\right\|_{L^2(\Sigma_{t^*}\cap\Omega_\delta)} &\lesssim \|u_m\|_{H^1(\Sigma_{t^*}\cap\Omega_\delta)}\\ &\lesssim \|u_m\|_{H^1(\Sigma_0\cap\Omega_\delta)}\ ,\end{aligned}$$

where we used the fact that χ is a smooth function and therefore can be controlled in L^∞ on Ω, together with all its derivatives. The second inequality holds because the H^1 norm of u_m is constant in time. Note that the H^1 norm of u_m involves the L^2 norm of u_m and its first derivatives only in the variables r_* and θ_*, so, for m sufficiently large, Theorem 2.9 gives

$$\left\|\Box_{g_{\text{ring}}}\Psi_m\right\|_{L^2(\Sigma_{t^*}\cap\Omega_\delta)} \leq Ce^{-C\,m}\,\|\Psi_m\|_{L^2(\Sigma_0\cap\Omega)}\ ,$$

with constant $C > 0$ independent of m. In view of the spatial localization of the quasimodes and of the error, the inequality can be rewritten as

$$\left\|\Box_{g_{\text{ring}}}\Psi_m\right\|_{L^2(\Sigma_{t^*})} \leq Ce^{-C\,m}\,\|\Psi_m\|_{L^2(\Sigma_0)}\ .$$

To estimate the H^k norm of $\Box_{g_{\text{ring}}}\Psi_m$, one commutes the wave equation and controls second derivatives of u_m in r_* and θ_* using the equation satisfied by u_m. Higher order derivatives in t, ϕ and ψ are controlled by the L^2 norm of u_m.

□

2.10.5 Lower bound for the uniform energy decay rate

Define by Ψ^H_m the solution to the homogeneous wave equation

$$\begin{cases}\Box_{g_{\text{ring}}}\Psi^H_m = 0\\ \Psi^H_m(0) = \Psi_m(0)\\ \partial_t\Psi^H_m(0) = \partial_t\Psi_m(0)\end{cases}, \tag{2.80}$$

where the $\Psi_m(0)$ are the quasimodes defined in (2.79) at time $t = 0$.

By construction, the quasimodes Ψ_m satisfy

$$\Box_{g_{\text{ring}}}\Psi_m = \mathfrak{Err}_m(\Psi_m)$$

with the same initial data of (2.80), where $\mathfrak{Err}_m$ is the error that we estimated in Lemma 2.11. By Duhamel's formula for curved spacetimes (see Proposition 6.4 in [70]), we can write Ψ_m at time t as

$$\Psi_m(t) = \Psi_m^H(t) + \int_0^t \xi(t,s)\, ds \tag{2.81}$$

with

$$\begin{cases} \Box_{g_{\text{ring}}} \xi = 0 \\ \xi(s) = 0 \\ \partial_t \xi(s) = \left(\frac{1}{g_{\text{ring}}^{tt}} \mathfrak{Err}_m(\Psi_m) \right)(s) \end{cases}, \tag{2.82}$$

where $\xi(s)$ is $\xi(t=s)$.

We now consider the local energy $\mathcal{E}_\Omega^N[\Psi](t)$, as defined in Section 2.4. By (2.81), we have

$$\left(\mathcal{E}_\Omega^N\left[\Psi_m - \Psi_m^H\right](t)\right)^{\frac{1}{2}} \leq t \sup_{s\in[0,t]} \left(\mathcal{E}_\Omega^N[\xi](t)\right)^{\frac{1}{2}} .$$

Using the assumption of uniform boundedness for solutions to the wave equation, we can bound the energy of the solution to (2.82) as follows

$$\left(\mathcal{E}_\Omega^N\left[\Psi_m - \Psi_m^H\right](t)\right)^{\frac{1}{2}} \leq Ct \left(\mathcal{E}_\Omega^N[\xi](0)\right)^{\frac{1}{2}} \tag{2.83}$$

for some constant $C > 0$ independent of time. Note that one can replace the global initial energy $\mathcal{E}^N[\xi](0)$ with the local energy $\mathcal{E}_\Omega^N[\xi](0)$ on the right hand side of (2.83) because ξ has initial data compactly supported on Ω. Since

$$\left(\mathcal{E}_\Omega^N[\xi](0)\right)^{\frac{1}{2}} \sim \left\| \frac{1}{g_{\text{ring}}^{tt}} \mathfrak{Err}_m(\Psi_m) \right\|_{L^2(\Omega)} (0)\,,$$

we have

$$\begin{aligned} \left(\mathcal{E}_\Omega^N\left[\Psi_m - \Psi_m^H\right](t)\right)^{\frac{1}{2}} &\leq Ct \left\| \frac{1}{g_{\text{ring}}^{tt}} \mathfrak{Err}_m(\Psi_m) \right\|_{L^2(\Omega)} (0) \\ &\leq Cte^{-Cm} \|\Psi_m\|_{L^2(\Omega)}(0) \\ &\leq Cte^{-Cm} \left(\mathcal{E}_\Omega^N[\Psi_m](0)\right)^{\frac{1}{2}} \end{aligned}$$

for m sufficiently large, where we have used Lemma 2.11 and Poincaré inequality for the second and third inequality respectively.

Note that $\left(\mathcal{E}_\Omega^N[\Psi_m](t)\right)^{\frac{1}{2}} = \left(\mathcal{E}_\Omega^N[\Psi_m](0)\right)^{\frac{1}{2}}$ for the quasimodes. Therefore, for any time

$$0 < t \leq \frac{e^{Cm}}{2C}\,,$$

we have

$$\left(\mathcal{E}_\Omega^N\left[\Psi_m^H\right](t)\right)^{\frac{1}{2}} \geq \frac{1}{2} \left(\mathcal{E}_\Omega^N[\Psi_m](0)\right)^{\frac{1}{2}}$$

by reverse triangle inequality. Since the Ψ_m are localised in space, the local energy $\mathcal{E}^N_\Omega[\Psi_m]$ is equal to the total energy $\mathcal{E}^N[\Psi_m]$, which gives

$$\begin{aligned}\left(\mathcal{E}^N_\Omega\left[\Psi^H_m\right](t)\right)^{\frac{1}{2}} &\geq \frac{1}{2}\left(\mathcal{E}^N\left[\Psi_m\right](0)\right)^{\frac{1}{2}}\\ &\geq \frac{C}{m}\left(\mathcal{E}^N_2\left[\Psi_m\right](0)\right)^{\frac{1}{2}}\end{aligned}$$

for m sufficiently large, where $\mathcal{E}^N_2[\Psi_m]$ is the second order energy defined in Section 2.4. For the last inequality we made use of the localisation in frequency of Ψ_m and exchanged derivatives in r_* and θ_* for derivatives in the other variables via the wave equation. Using $\Psi^H_m(0) = \Psi_m(0)$ from system (2.80), we conclude

$$\left(\mathcal{E}^N_\Omega\left[\Psi^H_m\right](t)\right)^{\frac{1}{2}} \geq \frac{C}{m}\left(\mathcal{E}^N_2\left[\Psi^H_m\right](0)\right)^{\frac{1}{2}} \tag{2.84}$$

for any time $0 < t_m \leq e^{C\,m}/2C$. By controlling higher order energies $\mathcal{E}^N_{k>2}\left[\Psi^H_m\right](0)$ on the right hand side of (2.84), one gains powers of $1/m$.

Inequality (2.84) shows that a sequence $\left\{t_m, \Psi^H_m\right\}^\infty_{m\geq M}$, where Ψ^H_m is a solution to the homogeneous wave equation with initial data prescribed as in (2.80) and $M > 0$ a sufficiently large constant, proves Theorem 2.3.

3

The linear stability of the slowly rotating Kerr solution to gravitational perturbations: The construction of a new geometric gauge

The present chapter of the thesis is the first part of work by the author proving uniform boundedness and uniform, inverse polynomial decay for solutions to the full system of linearised vacuum Einstein equations around a slowly rotating Kerr solution. The uniform boundedness statement is an orbital stability result, thus providing uniform bounds for solutions in terms of the size of the initial data, and does not lose derivatives.

A key element that makes our proof of linear stability directly adaptable to nonlinear applications is the construction of a new geometric gauge for *nonlinear* perturbations of a sub-extremal Kerr solution. The gauge construction, together with the formal derivation of the nonlinear and linearised vacuum Einstein equations in the new gauge, pose a number of novel technical difficulties. Such difficulties, which include the inevitable divorce of the frame from the spacetime foliation and the correct formulation of tensor perturbations, will be the primary focus of the present chapter.

As in Dafermos–Holzegel–Rodnianski [19], our proof of linear stability is structured in two parts. The former exploits the well-known decoupling of two gauge invariant quantities of the linear system and builds on previous work by Dafermos–Holzegel–Rodnianski [18]. The latter introduces a novel scheme to prove stability for the gauge dependent quantities in the system. In particular, we prove *both* boundedness and decay relying (almost) exclusively on transport equations in the outgoing null direction and adopting only *one* gauge normalisation, performed at the level of the initial data. This two-part approach to the proof makes fundamental use of our nonlinear gauge construction, which resolves the tension between the special algebraic properties required, at the linear level, for the decoupling of the gauge invariant quantities and the geometric properties necessary to produce a hierarchical structure in the equations and prove stability for the gauge dependent part of the system.

Our result may be viewed as the remaining ingredient to establish nonlinear stability for slowly rotating Kerr black holes in the geometric framework of Dafermos–Holzegel–Rodnianski–Taylor [20].

3.1 Introduction

The two-parameter Kerr family of Lorentzian manifolds [71] is widely believed to be the unique family of stationary, asymptotically flat black hole solutions to the vacuum Einstein equations

$$\mathrm{Ric}(g) = 0 \tag{3.1}$$

and thus to characterise all the possible stationary endstates of black hole dynamics [73].

A fundamental open problem in general relativity is to formulate a mathematical proof of nonlinear stability for the Kerr *exterior* region $(\mathcal{M}, g_{a,M})$, with $|a| < M$, as a solution to the Cauchy problem [11] for (3.1):

Open Problem ([17]). *For all vacuum Cauchy data sufficiently close to the data corresponding to a Kerr exterior solution with parameters* (a_0, M_0), $|a_0| < M_0$, *the maximal solution to* (3.1) *asymptotically settles down to a nearby member of the Kerr exterior family with parameters* $a \approx a_0$ *and* $M \approx M_0$.

The nonlinear stability of the trivial solution to (3.1), the Minkowski space, has first been proven in [14], while the first nonlinear stability result outside symmetry for a black hole solution to (3.1), namely the Schwarzschild solution, is due to [20]. See also some recent progress for the Kerr solution in [75, 76, 46] and related works [86, 83]. Nonlinear stability (and instability) statements have also been obtained for the Einstein equations with positive and negative cosmological constant, both in vacuum and for various matter models. See, for instance, [43, 54] and [90].

Proving nonlinear stability for a solution to (3.1) poses a number of conceptual and technical difficulties. Most of the key conceptual difficulties, however, already appear, and can be understood, in the context of *linear stability*, namely the study of stability of solutions to the *linearised Einstein equations*. This fact has motivated a lot of recent activity around the mathematical study of linear stability of black holes, although the subject has a long (and to some extent independent from the interest in nonlinear problems) tradition in the physics literature. For a careful account of previous works in the subject, the reader should refer to the introduction of [19] and the references therein.

An important milestone in the mathematically rigorous approach to linear stability was the work [19] proving the full linear stability of the Schwarzschild solution, and ultimately leading to the nonlinear stability result in [20]. Alternative proofs of this result [67, 63] and extensions to other static solutions [45] to (3.1) have recently appeared. For results addressing the linear stability of the Kerr family, see [52, 1].

The purpose of the present work is to address the linear stability of the Kerr solution as a direct building block in a proof of nonlinear stability. Such an intent imposes the formulation of a precise notion of linear stability and, maybe even more importantly, to develop a geometric framework that can be suitably and directly transferred to the nonlinear problem. An approach in this spirit can be summarised in the following *four steps*:

1. ***Nonlinear gauge***: One has to construct a gauge with respect to which the *nonlinear* vacuum Einstein equations (3.1) are well-posed.

2. ***Linearisation***: One has to develop a linearisation procedure to linearise the vacuum Einstein equations in the gauge chosen in step 1. The linear system obtained should come with a natural notion of solutions and data and inherit well-posedness from the nonlinear equations.

   Ideally, from this point onwards, one wants to formulate everything in terms of the linear system and never refer back to the nonlinear problem.

3. ***Orbital stability***: An essential stability result to be proven should be a uniform boundedness statement for solutions, and derivatives of solutions, to the linear system *in terms of the size of the initial data*. For nonlinear applications, it is important that such a statement does *not* lose derivatives.[1]

4. ***Asymptotic stability***: The core of the linear stability result should be a uniform decay statement for all solutions to the linear system. The decay statement will necessarily lose derivatives, but the loss should be quantifiable. One also has to ensure that the decay statement obtained is quantitative and, at least in principle, sufficient for nonlinear applications. In this sense, the decay rates of certain quantities need to be sufficiently fast.

Two additional aspects distinguish our approach: Its geometric nature and the manifest relation with the treatment of black hole stability problems in the physics literature. In fact, in line with the physics tradition in the subject (and with [19]), we structure the proof of linear stability of the Kerr solution into two parts:

[1] Orbital stability is a weaker notion of stability than asymptotic stability, in that it does not require that solutions to the linearised system decay. However, the reader should note that, in general, one does not expect to be able to achieve an orbital stability result without proving decay for *at least some* of the quantities in the system.

(i) ***Gauge invariant part***: If the nonlinear gauge of step 1 is constructed appropriately, two of the curvature components in the system of linearised Einstein equations are *gauge invariant*, *decouple* from the rest of the linear system and satisfy two decoupled equations, known as the *spin* ± 2 *Teukolsky equations*. Quite remarkably, one can transform such decoupled quantities into two new (higher order) gauge invariant quantities satisfying a *wave-like equation*. In the context of [19], this latter equation is known as the *Regge–Wheeler equation*.

Relying on previous knowledge on the scalar linear wave equation

$$\Box_g \varphi = 0 \tag{3.2}$$

on black hole spacetimes, the asymptotic properties of the derived quantities (and, in turn, of the two original decoupled quantities) can therefore be addressed independently from the other quantities in the system. At the most fundamental level, this first part of the proof should be seen as the inheritance of a series of works, culminating in [27, 22], developing important robust methods in the study of (3.2) on Kerr.

The proof of stability for the gauge invariant quantities thus captures an essential feature of the dynamics of the problem, namely the *hyperbolic* character of the system of (linearised) Einstein equations, and, in fact, already controls *all* the dynamical degrees of freedom independent of the choice of gauge. In view of this, it is a common (mis)belief in the physics literature that a stability result for the gauge invariant quantities constitutes a *full* linear stability result for the Kerr solution.

(ii) ***Gauge dependent part***: To successfully apply the linear result to the nonlinear stability problem, one needs to prove *full* linear stability, namely stability for the *full* system of linearised Einstein equations. The remaining quantities in the system are *gauge dependent*. For this reason, a proof of stability for the remaining part of the system is very much sensitive to the geometric properties of the gauge constructed.

This second part of the proof was first resolved by [19] in the case of the Schwarzschild solution in a *double-null gauge*. After gaining control over the gauge invariant quantities, the proof of [19] exploits a *hierarchical structure* in the *transport* equations of the system to prove decay for the gauge dependent quantities. In the Kerr case, it is not a priori clear whether an analogous hierarchy in the equations can be found. In fact, the transport equations of the system present a *stronger coupling*, ultimately related to the fact that the Kerr solution is only stationary. Nonetheless, the linear system *in our gauge* exhibits a hierarchical structure that allows to estimate the gauge depend quantities.

While this two-part approach to the problem has the advantage of exploiting the decoupling of the gauge invariant quantities, it also poses a new technical difficulties in setting up the gauge. Heuristically, one might wish to adopt a double-null gauge and follow the scheme of the proof of [19]. However, *the curvature components of part (i) do not decouple from the rest of the system in a double-null gauge*. The decoupling, in fact, requires some delicate *algebraic properties* for the gauge and forces the gauge to be *non-integrable*.[2]

Understanding the geometry of a non-integrable gauge is the main technical challenge of our proof. In particular, as a manifestation of the non-integrability of the gauge, *the frame adopted cannot be tied to the spacetime foliation*. The divorce of the frame from the spacetime foliation makes the geometry of the problem significantly more complicated to handle.

To implement our two-part approach to the problem, one needs to resolve the tension between a gauge with the desired algebraic properties for the decoupling of part (i) (which, as we said, force certain undesired but necessary geometric properties) and a gauge with a convenient geometric structure to perform the estimates. The reader should think of this tension as the subject of the present chapter. We resolve the tension by constructing a gauge with the necessary algebraic properties for the gauge invariant quantities to decouple and, at the same time, sufficiently nice geometric properties to produce a hierarchical structure in the equations and prove stability for the gauge dependent quantities.

Stability results for the Teukolsky equation on Kerr have been established in [18, 103]. We are thus able, in our gauge, to exploit such results in part (i) of the proof and, by addressing part (ii) *in the case* $|a| \ll M$, to close a ***complete proof of linear stability for the slowly rotating Kerr solution***. Crucially, our proof includes all the steps 1-4. With regard to step 1, our nonlinear gauge construction does *not* rely on any smallness assumption, and can therefore be complemented by [103] and employed to address the full linear (and nonlinear) stability problem for the *full sub-extremal range* $|a| < M$.

3.2 Overview

The present chapter of the thesis primarily focuses on step 1 and step 2 of our proof of linear stability, namely on the construction of a new nonlinear gauge for perturbations of a Kerr solution and the formulation of the system of linearised Einstein equations in the new gauge. Step 3 and step 4 will be the subject of forthcoming work by the author, but a preview is included in the last part of this overview. With our linearised equations at hand, the experienced reader will already be able to appreciate the new hierarchical structure of the quantities in the system and anticipate the resolution of the main difficulties of the analysis.

The body of the chapter is structured in three self-contained blocks:

* **Section 3.3** introduces the necessary geometric preliminaries. In particular, it addresses

[2]Loosely speaking, a *non-integrable gauge* can be viewed as a choice of a system of coordinates and a null frame

$$\mathcal{N} = (e_1, e_2, e_3, e_4)$$

for which the distribution generated by the spacelike frame vectors (e_1, e_2) is *not* tangent to the two-dimensional leaves of *any* spacetime foliation (or, more synthetically, the distribution (e_1, e_2) is *non-integrable*). This notion will be discussed at length in Section 3.3.1 of this chapter. The reader can already find some details in Section 3.2.3 of the overview.

the geometry of non-integrable null frames and formulates the nonlinear vacuum Einstein equations relative to a general, non-integrable null frame.

We give an overview to this part of the chapter in Section 3.2.1.

* **Sections 3.4-3.5-3.6** discuss some important properties of the Kerr exterior manifold, consider nonlinear perturbations around the Kerr solution in a new gauge and treat the nonlinear vacuum Einstein equations in the new gauge respectively. Section 3.5 contains the main result of the chapter, namely the construction of the new nonlinear gauge. In a sense that will be rigorously specified, the new gauge is a non-integrable gauge. This fact motivates, a posteriori, the formalism developed in Section 3.3.

  An overview to these sections can be found in Sections 3.2.2-3.2.3.

* **Section 3.7** presents our linearisation procedure and the system of linearised vacuum Einstein equations around a Kerr solution in the new gauge.

  Section 3.2.4 provides an overview to this part.

The three systems of equations can be found in Section 3.3.5, Section 3.6 and Section 3.7.2 of the chapter respectively. A formal derivation of the nonlinear equations of Section 3.3.5 appears in Appendix A.

Each of the three blocks can be consulted independently from the rest of the chapter. The first two blocks combined yield step 1 of our proof of linear stability, the third block yields step 2.

The following table may be used by the reader as a guide:

Step of proof	**Section**	**Overview**
1: Nonlinear gauge	Sections 3.3-3.4-3.5-3.6	Sections 3.2.1-3.2.2-3.2.3
2: Linearisation	Section 3.7	Section 3.2.4
3: Orbital stability	Forthcoming work	End of Section 3.2.4
4: Asymptotic stability	Forthcoming work	End of Section 3.2.4

The overview that we are about to start is designed to give a full account of the logic and content of the chapter. The reader should note that in the overview we give all the necessary context and motivation to our results, while the style adopted in the body of the chapter is less pedagogical and more devoted to clarifying the technical aspects of the problem. The relation between the present chapter and forthcoming work by the author is discussed in the overview only.

Note on notation: As in [19], all quantities that refer to perturbations of a Kerr solution will be written in bold. We already adopt this notation in the present overview.

3.2.1 Geometric preliminaries

This first section serves as an overview of Section 3.3 of the chapter and deals with some of the geometric preliminaries needed for the problem. The main aspects that we address are the definition and geometry of a *non-integrable* null frame and the formulation of the nonlinear vacuum Einstein equations relative to it.

Non-integrable null frames

Let $(\boldsymbol{\mathcal{M}}, \boldsymbol{g})$ be a $(3+1)$-dimensional, smooth, orientable Lorentzian manifold. We define a *null frame* as a frame

$$\boldsymbol{\mathcal{N}} = (\boldsymbol{e_1}, \boldsymbol{e_2}, \boldsymbol{e_3}, \boldsymbol{e_4}) \tag{3.3}$$

on $(\boldsymbol{\mathcal{M}}, \boldsymbol{g})$ such that

$$\begin{aligned} \boldsymbol{g}(\boldsymbol{e_A}, \boldsymbol{e_B}) &= \boldsymbol{\delta_{AB}}\,, & \boldsymbol{g}(\boldsymbol{e_A}, \boldsymbol{e_3}) = \boldsymbol{g}(\boldsymbol{e_A}, \boldsymbol{e_4}) &= 0\,, \\ \boldsymbol{g}(\boldsymbol{e_3}, \boldsymbol{e_3}) = \boldsymbol{g}(\boldsymbol{e_4}, \boldsymbol{e_4}) &= 0\,, & \boldsymbol{g}(\boldsymbol{e_3}, \boldsymbol{e_4}) &= -2\,, \end{aligned}$$

with $\boldsymbol{A}, \boldsymbol{B} = \{\boldsymbol{1}, \boldsymbol{2}\}$.[3]

The null frame $\boldsymbol{\mathcal{N}}$ is said to be *integrable* if and only if *both* the identities

$$\boldsymbol{g}([\boldsymbol{e_A}, \boldsymbol{e_B}], \boldsymbol{e_4}) = 0\,, \qquad \boldsymbol{g}([\boldsymbol{e_A}, \boldsymbol{e_B}], \boldsymbol{e_3}) = 0$$

hold on $(\boldsymbol{\mathcal{M}}, \boldsymbol{g})$. If *at least one* of the identities does not hold, then the frame is said to be *non-integrable*. In what follows, we do *not* assume the integrability of the frame $\boldsymbol{\mathcal{N}}$.

To capture some of the geometric difficulties arising in the problem, it is important to note that the frame $\boldsymbol{\mathcal{N}}$ induces a *distribution*

$$\mathfrak{D}_{\boldsymbol{\mathcal{N}}} := \big\{(\boldsymbol{p}, \boldsymbol{X_p}) \in T\boldsymbol{\mathcal{M}} \,|\, \boldsymbol{X_p} \in \mathrm{span}_{\boldsymbol{p}}\{\boldsymbol{e_1}, \boldsymbol{e_2}\}\big\}$$

on $\boldsymbol{\mathcal{M}}$. The distribution $\mathfrak{D}_{\boldsymbol{\mathcal{N}}}$ is *integrable* if and only if the frame $\boldsymbol{\mathcal{N}}$ is integrable.

We denote by $\boldsymbol{\nabla}$ the Levi-Civita connection and by $\boldsymbol{R}$ the Riemann curvature tensor with respect to $\boldsymbol{g}$. We define the connection coefficients

$$\begin{gathered} \hat{\boldsymbol{\omega}}, \underline{\hat{\boldsymbol{\omega}}}, \\ \boldsymbol{\eta}, \underline{\boldsymbol{\eta}}, \boldsymbol{Y}, \underline{\boldsymbol{Y}}, \boldsymbol{\zeta}, \\ \boldsymbol{\chi}, \boldsymbol{\chi} \end{gathered}$$

[3] Note that our notion of null frame will always require that $\boldsymbol{e_A}$ and $\boldsymbol{e_B}$ are *orthonormal*.

relative to the frame $\mathcal{N}$ as tensor fields *on* $\mathfrak{D}_{\mathcal{N}}$ such that

$$\chi_{AB} = g(\nabla_A e_4, e_B)\,, \qquad \underline{\chi}_{AB} = g(\nabla_A e_3, e_B)\,,$$

$$\eta_A = \frac{1}{2}\, g(\nabla_3 e_4, e_A)\,, \qquad \underline{\eta}_A = \frac{1}{2}\, g(\nabla_4 e_3, e_A)\,,$$

$$Y_A = \frac{1}{2}\, g(\nabla_4 e_4, e_A)\,, \qquad \underline{Y}_A = \frac{1}{2}\, g(\nabla_3 e_3, e_A)\,,$$

$$\hat{\omega} = \frac{1}{2}\, g(\nabla_4 e_3, e_4)\,, \qquad \underline{\hat{\omega}} = \frac{1}{2}\, g(\nabla_3 e_4, e_3)\,,$$

$$\zeta_A = \frac{1}{2}\, g(\nabla_A e_4, e_3)\,.$$

We define the curvature components

$$\rho, \sigma,$$
$$\beta, \underline{\beta},$$
$$\alpha, \underline{\alpha}$$

relative to the frame $\mathcal{N}$ as tensor fields *on* $\mathfrak{D}_{\mathcal{N}}$ such that

$$\alpha_{AB} = R(e_A, e_4, e_B, e_4)\,, \qquad \underline{\alpha}_{AB} = R(e_A, e_3, e_B, e_3)\,,$$

$$\beta_A = \frac{1}{2}\, R(e_A, e_4, e_3, e_4)\,, \qquad \underline{\beta}_A = \frac{1}{2}\, R(e_A, e_3, e_3, e_4)\,,$$

$$\rho = \frac{1}{4}\, R(e_4, e_3, e_4, e_3)\,, \qquad \sigma = \frac{1}{4}\, {}^\star R(e_4, e_3, e_4, e_3)\,,$$

where ${}^\star R$ is the Hodge dual of R on $(\mathcal{M}, g)$. We refer to such tensors as $\mathfrak{D}_{\mathcal{N}}$ *tensors*. The notion of $\mathfrak{D}_{\mathcal{N}}$ tensor is rigorously defined in Section 3.3.1 and should be seen as a generalisation of the notion of *S-tensors* of [14, 13] that allows $\mathcal{N}$ to be non-integrable.

Note that, if $\mathcal{N}$ is non-integrable, then at least one of the identities

$$\chi_{AB} \neq \chi_{BA}\,, \qquad \underline{\chi}_{AB} \neq \underline{\chi}_{BA}$$

holds on $\mathcal{M}$. One can thus define the symmetric traceless, trace and *antitrace* parts[4]

$$\hat{\chi}_{\bigcirc}\,, (\mathrm{tr}\chi_{\bigcirc})\,, (\not\epsilon \cdot \chi_{\square})\,,$$
$$\underline{\hat{\chi}}_{\bigcirc}\,, (\mathrm{tr}\underline{\chi}_{\bigcirc})\,, (\not\epsilon \cdot \underline{\chi}_{\square})\,,$$

of χ and $\underline{\chi}$ respectively and decompose

$$\chi = \hat{\chi}_{\bigcirc} + \frac{1}{2}(\mathrm{tr}\chi_{\bigcirc})\not g + \frac{1}{2}(\not\epsilon \cdot \chi_{\square})\not\epsilon\,,$$
$$\underline{\chi} = \underline{\hat{\chi}}_{\bigcirc} + \frac{1}{2}(\mathrm{tr}\underline{\chi}_{\bigcirc})\not g + \frac{1}{2}(\not\epsilon \cdot \underline{\chi}_{\square})\not\epsilon\,,$$

[4] We denote by $\chi_{\bigcirc}, \underline{\chi}_{\bigcirc}$ the symmetric parts of $\chi, \underline{\chi}$ and by $\chi_{\square}, \underline{\chi}_{\square}$ the antisymmetric parts of $\chi, \underline{\chi}$.

where we denote by $\not{g}$ and $\not{\varepsilon}$ the metric and volume form induced by the spacetime metric $\boldsymbol{g}$ and volume form $\boldsymbol{\varepsilon}$ on $\mathfrak{D}_{\mathcal{N}}$. See Section 3.3.2 for further details on the properties of $\boldsymbol{\chi}$ and $\underline{\boldsymbol{\chi}}$ and their decompositions.

For future convenience, we also define, for $\boldsymbol{A}, \boldsymbol{B} = \{\boldsymbol{1}, \boldsymbol{2}\}$, the smooth scalar functions

$$\boldsymbol{\Gamma}^{\boldsymbol{B}}_{\boldsymbol{4A}} = \boldsymbol{g}\left(\boldsymbol{\nabla}_{\boldsymbol{4}} \boldsymbol{e}_{\boldsymbol{A}}, \boldsymbol{e}_{\boldsymbol{C}}\right) \boldsymbol{g}^{\boldsymbol{BC}}\,, \qquad\qquad \boldsymbol{\Gamma}^{\boldsymbol{B}}_{\boldsymbol{3A}} = \boldsymbol{g}\left(\boldsymbol{\nabla}_{\boldsymbol{3}} \boldsymbol{e}_{\boldsymbol{A}}, \boldsymbol{e}_{\boldsymbol{C}}\right) \boldsymbol{g}^{\boldsymbol{BC}}$$

on $(\boldsymbol{\mathcal{M}}, \boldsymbol{g})$.

One can define products of $\mathfrak{D}_{\mathcal{N}}$ tensors and differential operators acting on $\mathfrak{D}_{\mathcal{N}}$ tensors, which generalise the products and differential operators for S-tensors of [14, 13]. In particular, the Levi-Civita connection $\boldsymbol{\nabla}$ induces, in a suitable sense, a connection $\not{\boldsymbol{\nabla}}$ on $\mathfrak{D}_{\mathcal{N}}$ compatible with the metric $\not{g}$. All the relevant definitions can be found in Section 3.3.3 and Section 3.3.4 respectively.

The vacuum Einstein equations

Let us now assume that $(\boldsymbol{\mathcal{M}}, \boldsymbol{g})$ solves the vacuum Einstein equations

$$\mathbf{Ric}(\boldsymbol{g}) = 0\,.$$

Then, the vacuum Einstein equations can be written as a nonlinear system of equations for the frame vectors of $\boldsymbol{\mathcal{N}}$ and the connection coefficients and curvature components relative to $\boldsymbol{\mathcal{N}}$. These equations are the *null frame equations*, the *null structure equations* and the *Bianchi equations*. The full system of equations is presented in Section 3.3.5, where its main properties are also discussed, and derived in Appendix A. We remark that the system of Section 3.3.5 differs from the systems of Einstein equations presented in [14, 13], the reason being that we do not assume the integrability of $\boldsymbol{\mathcal{N}}$. This fact will play a fundamental role in the present chapter. See Section 3.2.3 of the overview.

We now outline some selected properties of the system of equations of Section 3.3.5, assuming that the reader has some familiarity with the equations of [14, 13].

We have a set of equations for the frame

$$\begin{aligned}
\boldsymbol{\nabla}_{\boldsymbol{A}}\, \boldsymbol{e}_{\boldsymbol{B}} - \boldsymbol{\nabla}_{\boldsymbol{B}}\, \boldsymbol{e}_{\boldsymbol{A}} &= \left(\boldsymbol{\Gamma}^{\boldsymbol{C}}_{\boldsymbol{AB}} - \boldsymbol{\Gamma}^{\boldsymbol{C}}_{\boldsymbol{BA}}\right) \boldsymbol{e}_{\boldsymbol{C}} + \boldsymbol{\chi}_{\square\boldsymbol{AB}} \boldsymbol{e}_{\boldsymbol{3}} + \underline{\boldsymbol{\chi}}_{\square\boldsymbol{AB}} \boldsymbol{e}_{\boldsymbol{4}}\,,\\
\boldsymbol{\nabla}_{\boldsymbol{3}}\, \boldsymbol{e}_{\boldsymbol{A}} - \boldsymbol{\nabla}_{\boldsymbol{A}}\, \boldsymbol{e}_{\boldsymbol{3}} &= \left(\boldsymbol{\Gamma}^{\boldsymbol{B}}_{\boldsymbol{3A}} - \underline{\boldsymbol{\chi}}^{\sharp_2}{}^{\boldsymbol{B}}_{\boldsymbol{A}}\right) \boldsymbol{e}_{\boldsymbol{B}} + \left(\boldsymbol{\eta}_{\boldsymbol{A}} - \boldsymbol{\zeta}_{\boldsymbol{A}}\right) \boldsymbol{e}_{\boldsymbol{3}} + \underline{\boldsymbol{Y}}_{\boldsymbol{A}}\, \boldsymbol{e}_{\boldsymbol{4}}\,,\\
\boldsymbol{\nabla}_{\boldsymbol{4}}\, \boldsymbol{e}_{\boldsymbol{A}} - \boldsymbol{\nabla}_{\boldsymbol{A}}\, \boldsymbol{e}_{\boldsymbol{4}} &= \left(\boldsymbol{\Gamma}^{\boldsymbol{B}}_{\boldsymbol{4A}} - \boldsymbol{\chi}^{\sharp_2}{}^{\boldsymbol{B}}_{\boldsymbol{A}}\right) \boldsymbol{e}_{\boldsymbol{B}} + \boldsymbol{Y}_{\boldsymbol{A}} \boldsymbol{e}_{\boldsymbol{3}} + \left(\underline{\boldsymbol{\eta}}_{\boldsymbol{A}} + \boldsymbol{\zeta}_{\boldsymbol{A}}\right) \boldsymbol{e}_{\boldsymbol{4}}\,,\\
\boldsymbol{\nabla}_{\boldsymbol{3}}\, \boldsymbol{e}_{\boldsymbol{4}} - \boldsymbol{\nabla}_{\boldsymbol{4}}\, \boldsymbol{e}_{\boldsymbol{3}} &= \left(2\boldsymbol{\eta}^{\boldsymbol{A}} - 2\underline{\boldsymbol{\eta}}^{\boldsymbol{A}}\right) \boldsymbol{e}_{\boldsymbol{A}} + \hat{\boldsymbol{\omega}}\, \boldsymbol{e}_{\boldsymbol{3}} - \underline{\hat{\boldsymbol{\omega}}}\, \boldsymbol{e}_{\boldsymbol{4}}\,,
\end{aligned}$$

which coincide with the commutators for the frame vectors. The role of the *second fundamental forms* of [14, 13] is now played by the symmetric two-tensors $\boldsymbol{\chi}_{\bigcirc}$ and $\underline{\boldsymbol{\chi}}_{\bigcirc}$. We have, for instance, the first variational formulae

$$\not{\mathcal{L}}_{e_4}\not{g} = 2\hat{\chi}_{\bigcirc} + (\mathbf{tr}\chi_{\bigcirc})\not{g}\,,$$
$$\not{\mathcal{L}}_{e_3}\not{g} = 2\hat{\underline{\chi}}_{\bigcirc} + (\mathbf{tr}\underline{\chi}_{\bigcirc})\not{g}\,.$$

Another interesting aspect of the system of equations is that the usual elliptic equations for the curl part of the connection coefficients $\boldsymbol{Y}$, $\underline{\boldsymbol{Y}}$, $\boldsymbol{\eta}$ and $\underline{\boldsymbol{\eta}}$ can now be seen as transport equations for the antitraces of χ and $\underline{\chi}$. We have

$$\not{\nabla}_4(\not{\in}\cdot\chi_{\square}) + (\mathbf{tr}\chi_{\bigcirc})(\not{\in}\cdot\chi_{\square}) - \hat{\omega}(\not{\in}\cdot\chi_{\square}) = 2(\underline{\eta}+2\zeta)\wedge\boldsymbol{Y} + 2\boldsymbol{Y}\wedge\eta + 2\,\mathbf{c\not{u}rl}\,\boldsymbol{Y}\,,$$
$$\not{\nabla}_3(\not{\in}\cdot\underline{\chi}_{\square}) + (\mathbf{tr}\,\underline{\chi}_{\bigcirc})(\not{\in}\cdot\underline{\chi}_{\square}) - \hat{\underline{\omega}}(\not{\in}\cdot\underline{\chi}_{\square}) = 2(\eta-2\zeta)\wedge\underline{\boldsymbol{Y}} + 2\underline{\boldsymbol{Y}}\wedge\underline{\eta} + 2\,\mathbf{c\not{u}rl}\,\underline{\boldsymbol{Y}}\,,$$

and

$$\not{\nabla}_4(\not{\in}\cdot\underline{\chi}_{\square}) + \frac{1}{2}(\mathbf{tr}\chi_{\bigcirc})(\not{\in}\cdot\underline{\chi}_{\square}) + \hat{\omega}(\not{\in}\cdot\underline{\chi}_{\square}) = -\hat{\chi}_{\bigcirc}\wedge\hat{\underline{\chi}}_{\bigcirc} - \frac{1}{2}(\mathbf{tr}\,\underline{\chi}_{\bigcirc})(\not{\in}\cdot\chi_{\square}) + 2\boldsymbol{Y}\wedge\underline{\boldsymbol{Y}}$$
$$+ 2\sigma + 2\,\mathbf{c\not{u}rl}\,\underline{\eta}\,,$$
$$\not{\nabla}_3(\not{\in}\cdot\chi_{\square}) + \frac{1}{2}(\mathbf{tr}\,\underline{\chi}_{\bigcirc})(\not{\in}\cdot\chi_{\square}) + \hat{\underline{\omega}}(\not{\in}\cdot\chi_{\square}) = -\hat{\chi}_{\bigcirc}\wedge\hat{\underline{\chi}}_{\bigcirc} - \frac{1}{2}(\mathbf{tr}\chi_{\bigcirc})(\not{\in}\cdot\underline{\chi}_{\square}) + 2\underline{\boldsymbol{Y}}\wedge\boldsymbol{Y}$$
$$- 2\sigma + 2\,\mathbf{c\not{u}rl}\,\eta\,.$$

The Codazzi equations read

$$\mathbf{d\not{i}v}\ \hat{\chi}_{\bigcirc} = -\frac{1}{2}\not{\in}^{\sharp_2}\cdot\not{\nabla}(\not{\in}\cdot\chi_{\square}) - \hat{\chi}^{\sharp}_{\bigcirc}\cdot\zeta - \frac{1}{2}(\not{\in}\cdot\chi_{\square})^{\star}\zeta + \frac{1}{2}(\mathbf{tr}\,\chi_{\bigcirc})\zeta + \frac{1}{2}\not{\nabla}(\mathbf{tr}\,\chi_{\bigcirc}) - (\not{\in}\cdot\chi_{\square})^{\star}\boldsymbol{Y}$$
$$- (\not{\in}\cdot\chi_{\square})^{\star}\eta - \beta\,,$$
$$\mathbf{d\not{i}v}\ \hat{\underline{\chi}}_{\bigcirc} = -\frac{1}{2}\not{\in}^{\sharp_2}\cdot\not{\nabla}(\not{\in}\cdot\underline{\chi}_{\square}) + \hat{\underline{\chi}}^{\sharp}_{\bigcirc}\cdot\zeta + \frac{1}{2}(\not{\in}\cdot\underline{\chi}_{\square})^{\star}\zeta - \frac{1}{2}(\mathbf{tr}\,\underline{\chi}_{\bigcirc})\zeta + \frac{1}{2}\not{\nabla}(\mathbf{tr}\,\underline{\chi}_{\bigcirc}) - (\not{\in}\cdot\underline{\chi}_{\square})^{\star}\underline{\boldsymbol{Y}}$$
$$- (\not{\in}\cdot\underline{\chi}_{\square})^{\star}\underline{\eta} + \underline{\beta}\,,$$

where one should note the new terms depending on the antitraces of χ and $\underline{\chi}$. The Gauss equation reads

$$\not{K} = \frac{1}{2}(\hat{\chi}_{\bigcirc},\hat{\underline{\chi}}_{\bigcirc}) - \frac{1}{4}(\mathbf{tr}\chi_{\bigcirc})(\mathbf{tr}\,\underline{\chi}_{\bigcirc}) - \frac{1}{4}(\not{\in}\cdot\chi_{\square})(\not{\in}\cdot\underline{\chi}_{\square}) + (\boldsymbol{\Gamma}_3,\chi_{\square}) + (\boldsymbol{\Gamma}_4,\underline{\chi}_{\square}) - \rho\,.$$

The smooth scalar function $\not{K}$ is defined in Section 3.3.4 and coincides with the Gauss curvature of the integral manifold of $\mathfrak{D}_{\mathcal{N}}$ when $\mathcal{N}$ is an integrable frame.

3.2.2 The Kerr exterior background

This section overviews the content of Section 3.4 of the chapter, where the Kerr exterior manifold and its key properties are discussed.

In this section, we will introduce the Kerr exterior manifold $(\mathcal{M}, g_{a,M})$ with its double-null differentiable structure. We then consider a particular null frame of $(\mathcal{M}, g_{a,M})$, namely the *algebraically special frame*, and outline some of its geometric properties. We conclude the section by constructing a new differentiable structure on $(\mathcal{M}, g_{a,M})$ that will play an important role in the definition of the new nonlinear gauge.

The Kerr metric in double-null form

For any real parameters a, M, with $M > 0$ and $0 < |a| < M$, we identify the Lorentzian manifold $(\boldsymbol{\mathcal{M}}, \boldsymbol{g})$ of Section 3.2.1 with the *(sub-extremal) Kerr exterior manifold* $(\mathcal{M}, g_{a,M})$. We introduce the differentiable structure

$$(u, v, \theta^1, \theta^2) \tag{3.4}$$

on $\mathcal{M}$,

$$u, v \in (-\infty, \infty)\,, \quad (\theta^1, \theta^2) \in S^2\,,$$

such that the *(sub-extremal) Kerr metric* $g_{a,M}$ takes the form

$$g_{a,M} = -4\,\Omega^2_{\text{Kerr}} du\, dv + \not{\gamma}_{\text{Kerr}\,\theta^A\theta^B}(d\theta^A - b^{\theta^A}_{\text{Kerr}} dv)(d\theta^B - b^{\theta^B}_{\text{Kerr}} dv) \tag{3.5}$$

on $\mathcal{M}$, with $\Omega_{\text{Kerr}} \in C^\infty(\mathcal{M})$, $\not{\gamma}_{\text{Kerr}}$ a symmetric two-tensor on the two-spheres

$$S^2_{u,v} \equiv \{u, v\} \times S^2$$

and b_{Kerr} a vector field tangent to $S^2_{u,v}$.[5] Such a *global*[6] differentiable structure will be called *double-null* differentiable structure on $(\mathcal{M}, g_{a,M})$ and the Kerr metric (3.5) is said to be in *double-null form*. In fact, the level sets

$$C_{u_0} := \{u = u_0\}\,, \qquad\qquad \underline{C}_{v_0} := \{v = v_0\}$$

are (outgoing and ingoing respectively) *null* hypersurfaces of $(\mathcal{M}, g_{a,M})$.

The *future event horizon* $\mathcal{H}^+$ of $(\mathcal{M}, g_{a,M})$ corresponds to the (asymptotic)[7] null hypersurface

$$\mathcal{H}^+ = (\infty, v, \theta^1, \theta^2)\,, \tag{3.6}$$

with $v \in \mathbb{R}$, while *future null infinity* $\mathcal{I}^+$ corresponds to the (asymptotic) null hypersurface

$$\mathcal{I}^+ = (u, \infty, \theta^1, \theta^2)\,, \tag{3.7}$$

with $u \in \mathbb{R}$. The *past event horizon* $\mathcal{H}^-$ and *past null infinity* $\mathcal{I}^-$ correspond to the (asymptotic)

[5]The Kerr metric coefficients in (3.5) will be rigorously defined in Section 3.4 of this chapter.

[6]The existence of a *global* system of coordinates on $(\mathcal{M}, g_{a,M})$ such that $g_{a,M}$ takes the form (3.5) is a non-trivial observation due to [93].

[7]We note that the system of double-null coordinates $(u, v, \theta^1, \theta^2)$ breaks down at the future event horizon $\mathcal{H}^+$. However, one can still formally parametrise $\mathcal{H}^+$ as in (3.6).

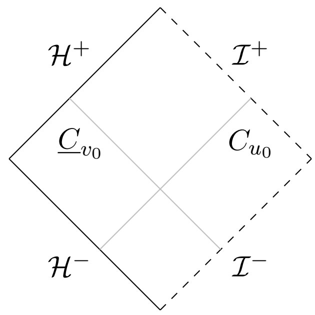

Figure 3.1: Penrose diagram of the Kerr exterior manifold $(\mathcal{M}, g_{a,M})$.

null hypersurfaces

$$\mathcal{H}^- = (u, -\infty, \theta^1, \theta^2)\,, \qquad\qquad \mathcal{I}^- = (-\infty, v, \theta^1, \theta^2)$$

respectively, with $u, v \in \mathbb{R}$. See Figure 3.1. We note that, in line with footnote 7, we have

$$\Omega_{\text{Kerr}}(\infty, v, \theta^A) = \Omega_{\text{Kerr}}(u, -\infty, \theta^A) = 0\,.$$

The algebraically special frame

In this section we define the *algebraically special frame* of $(\mathcal{M}, g_{a,M})$ by giving its coordinate form. Although this can be done relative to the differentiable structure (u, v, θ^A), we introduce a second differentiable structure (known as the *Boyer–Lindquist coordinates*) on $(\mathcal{M}, g_{a,M})$ that makes the coordinate expression of the frame simpler (and explicit). Let us note that the algebraically special frame can also be introduced *geometrically* via what we will call *property (i)* of the frame later in this section.[8]

It is important that the reader realises that the properties of the algebraically special frame discussed in this section are all *geometric*. The Boyer–Lindquist differentiable structure plays, at this point (and in fact throughout our discussion), no role apart from simplifying some of the coordinate expressions.

We introduce the *Boyer–Lindquist* differentiable structure

$$(t_{bl}, r_{bl}, \theta_{bl}, \phi_{bl})$$

on $(\mathcal{M}, g_{a,M})$, which can be implicitly related to the differentiable structure (u, v, θ^A) as discussed in Section 3.4. In Boyer–Lindquist coordinates, the Kerr metric takes the form

$$g_{a,M} = -\frac{\Delta}{\Sigma}(dt_{bl} - a\sin^2\theta_{bl} d\phi_{bl})^2 + \frac{\Sigma}{\Delta}dr_{bl}^2 + \Sigma d\theta_{bl}^2 + \frac{\sin^2\theta_{bl}}{\Sigma}(a dt_{bl} - (r_{bl}^2 + a^2)d\phi_{bl})^2\,,$$

[8]Strictly speaking, the definition of the algebraically special frame via property (i) leaves some residual freedom in rescaling the null frame vectors and in performing orthogonal rotations of the spacelike frame vectors, while our coordinate definition fixes the frame completely. The rescaling along the null directions that we choose will be dictated by the fact that we desire a frame that extends regularly to $\mathcal{H}^+$.

where the standard functions $\Delta = \Delta(r_{bl})$ and $\Sigma = \Sigma(r_{bl}, \theta_{bl})$ are defined in Section 3.4.

We define the *algebraically special frame*

$$\mathcal{N}_{\mathrm{as}} = (e_1^{\mathrm{as}}, e_2^{\mathrm{as}}, e_3^{\mathrm{as}}, e_4^{\mathrm{as}})$$

of $(\mathcal{M}, g_{a,M})$ as the null frame with coordinate form

$$\begin{aligned}
e_4^{\mathrm{as}} &= \partial_{t_{bl}} + \frac{\Delta}{r_{bl}^2 + a^2}\,\partial_{r_{bl}} + \frac{a}{r_{bl}^2 + a^2}\,\partial_{\phi_{bl}}\,, \\
e_3^{\mathrm{as}} &= \frac{(r_{bl}^2 + a^2)^2}{\Sigma\Delta}\,\partial_{t_{bl}} - \frac{r_{bl}^2 + a^2}{\Sigma}\,\partial_{r_{bl}} + \frac{a(r_{bl}^2 + a^2)}{\Sigma\Delta}\,\partial_{\phi_{bl}}\,, \\
e_1^{\mathrm{as}} &= \frac{a^2 \sin\theta_{bl}\cos\theta_{bl}}{\Sigma}\,\partial_{t_{bl}} + \frac{r_{bl}}{\Sigma}\,\partial_{\theta_{bl}} + \frac{a\cot\theta_{bl}}{\Sigma}\,\partial_{\phi_{bl}}\,, \\
e_2^{\mathrm{as}} &= \frac{a r_{bl}\sin\theta_{bl}}{\Sigma}\,\partial_{t_{bl}} - \frac{a\cos\theta_{bl}}{\Sigma}\,\partial_{\theta_{bl}} + \frac{r_{bl}\csc\theta_{bl}}{\Sigma}\,\partial_{\phi_{bl}}
\end{aligned}$$

on $(\mathcal{M}, g_{a,M})$. One can easily check that $\mathcal{N}_{\mathrm{as}}$ extends *regularly* to $\mathcal{H}^+$.[9]

The connection coefficients of the Kerr metric $g_{a,M}$ relative to $\mathcal{N}_{\mathrm{as}}$ are the $\mathfrak{D}_{\mathcal{N}_{\mathrm{as}}}$ tensors[10]

$$\chi^{\mathrm{as}}_{\mathrm{Kerr}\,11} = \chi^{\mathrm{as}}_{\mathrm{Kerr}\,22} = \frac{r_{bl}\Delta}{\left(r_{bl}^2 + a^2\right)\Sigma}\,, \qquad \chi^{\mathrm{as}}_{\mathrm{Kerr}\,12} = \frac{a\Delta\cos\theta_{bl}}{\left(r_{bl}^2 + a^2\right)\Sigma}\,, \qquad \chi^{\mathrm{as}}_{\mathrm{Kerr}\,21} = -\chi^{\mathrm{as}}_{\mathrm{Kerr}\,12}\,,$$

$$\underline{\chi}^{\mathrm{as}}_{\mathrm{Kerr}\,11} = \underline{\chi}^{\mathrm{as}}_{\mathrm{Kerr}\,22} = -\frac{r_{bl}\left(r_{bl}^2 + a^2\right)}{\Sigma^2}\,, \quad \underline{\chi}^{\mathrm{as}}_{\mathrm{Kerr}\,12} = \frac{a\left(r_{bl}^2 + a^2\right)\cos\theta_{bl}}{\Sigma^2}\,, \quad \underline{\chi}^{\mathrm{as}}_{\mathrm{Kerr}\,21} = -\underline{\chi}^{\mathrm{as}}_{\mathrm{Kerr}\,12}\,,$$

$$\begin{aligned}
\hat{\omega}^{\mathrm{as}}_{\mathrm{Kerr}} &= \frac{2M\left(r_{bl}^2 - a^2\right)}{\left(r_{bl}^2 + a^2\right)^2}\,, &\qquad \underline{\hat{\omega}}^{\mathrm{as}}_{\mathrm{Kerr}} &= \frac{2a^2 r_{bl}\sin^2\theta_{bl}}{\Sigma^2}\,, \\
\eta^{\mathrm{as}}_{\mathrm{Kerr}\,1} &= 0\,, & \eta^{\mathrm{as}}_{\mathrm{Kerr}\,2} &= \frac{2a\sin\theta_{bl}}{\Sigma}\,, \\
\underline{\eta}^{\mathrm{as}}_{\mathrm{Kerr}\,1} &= -\frac{a^2 r_{bl}\sin(2\theta_{bl})}{\Sigma^2}\,, & \underline{\eta}^{\mathrm{as}}_{\mathrm{Kerr}\,2} &= \frac{a\sin\theta_{bl}\left(a^2\cos^2\theta_{bl} - r_{bl}^2\right)}{\Sigma^2}\,, \\
Y^{\mathrm{as}}_{\mathrm{Kerr}\,A} &= 0\,, & \underline{Y}^{\mathrm{as}}_{\mathrm{Kerr}\,A} &= 0\,, \\
\zeta^{\mathrm{as}}_{\mathrm{Kerr}\,1} &= \frac{a^2 r_{bl}\sin(2\theta_{bl})}{\Sigma^2}\,, & \zeta^{\mathrm{as}}_{\mathrm{Kerr}\,2} &= -\frac{a\sin\theta_{bl}\left(a^2\cos^2\theta_{bl} - r_{bl}^2\right)}{\Sigma^2}\,,
\end{aligned}$$

with

$$\begin{aligned}
\hat{\chi}_{\circ}{}^{\mathrm{as}}_{\mathrm{Kerr}\,AB} &= 0\,, &\qquad \underline{\hat{\chi}}_{\circ}{}^{\mathrm{as}}_{\mathrm{Kerr}\,AB} &= 0\,, \\
(\not{\epsilon}\cdot\chi_{\square})^{\mathrm{as}}_{\mathrm{Kerr}} &= \frac{2a\Delta\cos\theta_{bl}}{\left(r_{bl}^2 + a^2\right)\Sigma}\,, & (\not{\epsilon}\cdot\underline{\chi}_{\square})^{\mathrm{as}}_{\mathrm{Kerr}} &= \frac{2a\left(r_{bl}^2 + a^2\right)\cos\theta_{bl}}{\Sigma^2}\,, \\
\mathrm{tr}\chi_{\circ}{}^{\mathrm{as}}_{\mathrm{Kerr}} &= \frac{2r_{bl}\Delta}{\left(r_{bl}^2 + a^2\right)\Sigma}\,, & \mathrm{tr}\,\underline{\chi}_{\circ}{}^{\mathrm{as}}_{\mathrm{Kerr}} &= -\frac{2r_{bl}\left(r_{bl}^2 + a^2\right)}{\Sigma^2}\,.
\end{aligned}$$

[9]Note, however, that the Boyer–Lindquist coordinates are *not* regular along $\mathcal{H}^+$.

[10]For $\mathfrak{D}_{\mathcal{N}_{\mathrm{as}}}$ covariant tensors, we give the coordinate expression (relative to the Boyer–Lindquist differentiable structure) of their frame components. Note that these are functions of both coordinates r_{bl} and θ_{bl}, as opposed to the spherically symmetric connection coefficients and curvature components of the Schwarzschild metric.

Note that

$$\underline{\eta}^{\text{as}}_{\text{Kerr}} + \zeta^{\text{as}}_{\text{Kerr}} = 0$$

and

$$\Gamma^{B\ \text{as}}_{4A\text{Kerr}} = 0\,, \qquad\qquad \Gamma^{C\ \text{as}}_{3A\text{Kerr}}\, g_{CB} = 2\,\underline{\chi}_{\square}{}^{\text{as}}_{\text{Kerr}AB}\,.$$

The curvature components are

$$\begin{aligned}
\alpha^{\text{as}}_{\text{Kerr}AB} &= 0\,, & \underline{\alpha}^{\text{as}}_{\text{Kerr}AB} &= 0\,,\\
\beta^{\text{as}}_{\text{Kerr}A} &= 0\,, & \underline{\beta}^{\text{as}}_{\text{Kerr}A} &= 0\,,\\
\rho^{\text{as}}_{\text{Kerr}} &= \frac{2Mr_{bl}\left(3a^2\cos^2\theta_{bl} - r_{bl}^2\right)}{\Sigma^3}\,, & \sigma^{\text{as}}_{\text{Kerr}} &= \frac{aM\cos\theta_{bl}\left(3r_{bl}^2 - a^2\cos^2\theta_{bl}\right)}{\Sigma^3}\,.
\end{aligned}$$

All the connection coefficients and curvature components are *regular* quantities along $\mathcal{H}^+$.

The algebraically special frame possesses several important properties, which are carefully discussed in Section 3.4. The crucial ones for our problem can be summarised as follows:

(i) ***Special algebraic property***: The curvature components

$$\alpha^{\text{as}}_{\text{Kerr}}\,,\ \underline{\alpha}^{\text{as}}_{\text{Kerr}}\,,\ \beta^{\text{as}}_{\text{Kerr}}\,,\ \underline{\beta}^{\text{as}}_{\text{Kerr}}$$

all *vanish* identically on $(\mathcal{M}, g_{a,M})$. This is often considered as the *defining* (geometric) property of the frame $\mathcal{N}_{\text{as}}$ on $(\mathcal{M}, g_{a,M})$.

(ii) ***Relation to double-null gauge***: The frame $\mathcal{N}_{\text{as}}$ does *not* coincide with the geometric frame arising from the double-null differentiable structure (u, v, θ^A) of $(\mathcal{M}, g_{a,M})$, i.e.

$$\mathcal{N}_{\text{as}} \not\equiv \widehat{\mathcal{N}}_{\text{ad}}\,,$$

with

$$\widehat{\mathcal{N}}_{\text{ad}} = \left(\partial_v + b^{\theta^A}_{\text{Kerr}}\partial_{\theta^A}, \Omega^{-2}_{\text{Kerr}}\partial_u, \gamma\!\!\!/_{\text{Kerr}\theta^A\theta^A}^{-1/2}\partial_{\theta^A}\right).$$

(iii) ***Non-integrability***: The frame $\mathcal{N}_{\text{as}}$ is *non-integrable*. This, in particular, implies

$$(\not\epsilon\cdot\chi_{\square})^{\text{as}}_{\text{Kerr}} \neq 0\,, \qquad\qquad (\not\epsilon\cdot\underline{\chi}_{\square})^{\text{as}}_{\text{Kerr}} \neq 0$$

on $(\mathcal{M}, g_{a,M})$.

(iv) ***Integral curves***: Away from the future event horizon $\mathcal{H}^+$, the null frame vectors e_4^{as} and e_3^{as} generate *timelike* hypersurfaces.

(v) ***Geometric properties along*** $\mathcal{H}^+$: The generators of the future event horizon $\mathcal{H}^+$ of $(\mathcal{M}, g_{a,M})$ coincide with the frame vector field e_4^{as} on $\mathcal{H}^+$. The frame vector fields e_1^{as} and

$e_2^{\rm as}$ are *tangent* to $\mathcal{H}^+$. These two properties imply, for instance, that

$$e_4^{\rm as}({\rm tr}\chi_\circ)^{\rm as}_{\rm Kerr} = 0\,, \qquad\qquad e_A^{\rm as}({\rm tr}\chi_\circ)^{\rm as}_{\rm Kerr} = 0$$

along $\mathcal{H}^+$, the same holding with $({\rm tr}\chi_\circ)^{\rm as}_{\rm Kerr}$ replaced by any constant scalar function along $\mathcal{H}^+$. Furthermore, the frame vector fields

$$(e_1^{\rm as}, e_2^{\rm as}, e_4^{\rm as})$$

generate an *integrable* distribution along $\mathcal{H}^+$.

Note that the properties (ii), (iii) and (iv) do *not* hold for the *Schwarzschild* exterior manifold, for which the algebraically special frame *coincides* with the frame of a (integrable) double-null gauge and thus the null frame vectors generate *null* hypersurfaces. Properties (i) and (v) should be regarded as convenient (non-trivial) geometric properties of the algebraically special frame of $(\mathcal{M}, g_{a,M})$ that we will exploit.

A new differentiable structure

We now define a new differentiable structure on (a subregion of) $(\mathcal{M}, g_{a,M})$. The definition relies on the geometry of the algebraically special frame of $(\mathcal{M}, g_{a,M})$. As we said, the Boyer–Lindquist coordinates introduced in Section 3.2.2 have no conceptual role in our construction (and can, in principle, be avoided), while the differentiable structure discussed in this section will be truly important for the formulation of our new nonlinear gauge. In particular, later in our discussion, we will make crucial use of the properties of the Kerr metric and the algebraically special frame *relative to the new differentiable structure* (see the identities (3.15)-(3.16) and (3.17)).

We consider the union of two null hypersurfaces

$$C_{u_0, v\geq v_0} \cup \underline{C}_{u\geq u_0, v_0} \tag{3.8}$$

on $(\mathcal{M}, g_{a,M})$, with u_0 and v_0 finite constants,[11] and define the manifold[12]

$$\mathcal{M}^+ := [u_0,\infty)\times[v_0,\infty)\times S^2\,. \tag{3.9}$$

We define a new differentiable structure

$$(\tau, s, \vartheta, \psi)\,,$$

[11] At the end of the section, the reader should note that the construction of the new differentiable structure breaks down if $u_0 = v_0 = -\infty$.

[12] Here there is a technical subtlety due to the fact that the double-null coordinates (u, v, θ^A) break down along the future event horizon $\mathcal{H}^+$ of $(\mathcal{M}, g_{a,M})$. The manifold $\mathcal{M}^+$ as define in (3.9) has $\mathcal{M}^+ \cap \mathcal{H}^+ = \emptyset$. The definition of the desired $\mathcal{M}^+$ requires to first introduce the Kruskal coordinates on $(\mathcal{M}, g_{a,M})$. See the defining identity (3.120) in Section 3.4.

on $\mathcal{M}^+$, with[13]

$$\tau \in [0,\infty)\,, \qquad\qquad s \in (-\infty,\infty)\,, \qquad\qquad (\vartheta,\psi) \in S^2\,,$$

such that

$$(\tau,s,\vartheta,\psi) = \begin{cases} (0, v_0-u, \theta^1,\theta^2) & \text{along } \underline{C}_{u\geq u_0,v_0} \\ (0, v-u_0, \theta^1,\theta^2) & \text{along } C_{u_0,v\geq v_0} \end{cases} \tag{3.10}$$

and

$$e_4^{\text{as}}(\tau) = 1 \tag{3.11}$$

$$e_4^{\text{as}}(s) = e_4^{\text{as}}(\vartheta) = e_4^{\text{as}}(\psi) = 0 \tag{3.12}$$

on $(\mathcal{M}^+, g_{a,M})$. The new system of coordinates can be seen as the unique solution to the linear system of ODEs (3.11)-(3.12) along the integral curves of e_4^{as} (with initial data (3.10) prescribed on (3.8)) and is therefore well-defined *globally* on $(\mathcal{M}^+, g_{a,M})$. The future event horizon $\mathcal{H}^+$ is formally parametrised by

$$\mathcal{H}^+ = (\tau_{\geq 0}, -\infty, \vartheta, \psi)\,,$$

while future null infinity is formally parametrised by

$$\mathcal{I}^+ = (\infty, s, \vartheta, \psi)\,.$$

Coordinates (τ,s,ϑ,ψ) induce a *global* foliation of the region $(\mathcal{M}^+, g_{a,M})$ of the Kerr exterior manifold $(\mathcal{M}, g_{a,M})$. Hypersurfaces of constant τ and s are depicted in Figure 3.2. The two-spheres

$$S^2_{\tau,s} = \{\tau,s\}\times S^2 \tag{3.13}$$

coincide with the double-null spheres $S^2_{u,v}$ along (3.8) (but *not* globally), i.e. for all $s\in(-\infty,\infty)$,

$$S^2_{0,s} \equiv S^2_{u,v} \tag{3.14}$$

for some (u,v). Let us also note that

$$C_{u_0,v\geq v_0} \cup \underline{C}_{u\geq u_0,v_0} \equiv \{\tau = 0\}\ , \qquad\qquad \mathcal{M}^+ \equiv \{\tau \geq 0\}\ .$$

We refer to the Kerr metric $g_{a,M}$ in coordinates (τ,s,ϑ,ψ) and associated algebraically special frame $\mathcal{N}_{\text{as}}$ on $\mathcal{M}^+$ as the Kerr metric in an *outgoing frame-calibrated gauge* (or *OFC-gauge*).[14,15]

The new coordinate form of the metric can (only implicitly) be related to its double-null form

[13]As usual, at least two coordinate charts are needed to cover the sphere S^2. We do not discuss this issue here.

[14]To the best of the author's knowledge, this name appears for the first time in this thesis. We will elaborate on the choice of the name for the new gauge later in the overview.

[15]We will occasionally refer to the outgoing frame-calibrated gauge as the *new gauge*, where the name *new gauge* now has a precise definition.

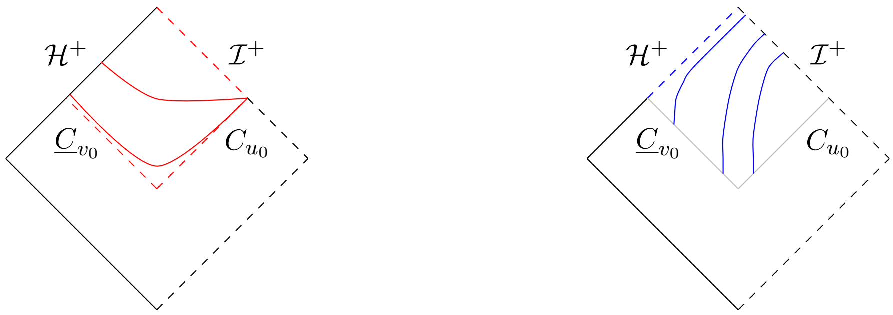

Figure 3.2: The two Penrose diagrams of the Kerr exterior manifold $(\mathcal{M}, g_{a,M})$ show the foliation of the region $(\mathcal{M}^+, g_{a,M})$ induced by the new differentiable structure $(\tau, s, \vartheta, \psi)$. Each level set of τ is a *spacelike* hypersurface (in solid red), with the exception of the hypersurface (3.8) and future null infinity $\mathcal{I}^+$, which coincide with the $\tau = 0$ and $\tau = \infty$ *null* hypersurfaces respectively (in dashed red). Each level set of s is a *timelike* hypersurface (in solid blue) which becomes asymptotically null towards future null infinity $\mathcal{I}^+$, with the exception of the future event horizon $\mathcal{H}^+$, which coincides with the $s = -\infty$ *null* hypersurface (in dashed blue).

(3.5). The Kerr metric in the OFC-gauge satisfies

$$g_{a,M\,\tau\tau} = 0\,, \tag{3.15}$$

$$\Gamma^s_{\tau\tau} = \Gamma^\vartheta_{\tau\tau} = \Gamma^\psi_{\tau\tau} = 0 \tag{3.16}$$

on $(\mathcal{M}^+, g_{a,M})$.

The integral curves of

$$e_4^{\mathrm{as}} = \partial_\tau \tag{3.17}$$

are curves of constant s, thus *tangent* to the hypersurfaces depicted as blue curves in Figure 3.2, and with parameter τ along the curve.

Additional properties of the Kerr exterior manifold with its new differentiable structure can be found in Section 3.4.

3.2.3 The nonlinear gauge

This section serves as an overview to the core of the chapter, corresponding to the introduction of a one-parameter family of nonlinear metric perturbations around Kerr in an *outgoing frame-calibrated gauge* (Section 3.5) and the formulation of the vacuum Einstein equations for such a family of metrics (Section 3.6).

The fixed manifold and differentiable structure

We start by considering the manifold $\mathcal{M}^+$ defined by (3.9) and its differentiable structure $(\tau, s, \vartheta, \psi)$ introduce in Section 3.2.2. These will remain fixed throughout our discussion.

At this point, it is important to note that one has some *residual freedom* in constructing the differentiable structure $(\tau, s, \vartheta, \psi)$ on $\mathcal{M}^+$, ultimately related to a residual freedom in choosing the differentiable structure (u, v, θ^A) on $\mathcal{M}^+$.[16] In fact, the differentiable structure relative to which the Kerr metric $g_{a,M}$ takes the double-null form (3.5) on $\mathcal{M}^+$ is not unique. In particular, there exist (infinitely many) one-parameter families of coordinates

$$(u_{\widetilde{\epsilon}}, v_{\widetilde{\epsilon}}, \theta^1_{\widetilde{\epsilon}}, \theta^2_{\widetilde{\epsilon}}) \tag{3.18}$$

on $\mathcal{M}^+$, with $\widetilde{\epsilon} \geq 0$ and

$$(u_0, v_0, \theta^1_0, \theta^2_0) \equiv (u, v, \theta^1, \theta^2)\,,$$

for which the Kerr metric $g_{a,M}$ takes the double-null form (3.5) *for all* $\widetilde{\epsilon} \geq 0$. In turn, coordinates $(\tau, s, \vartheta, \psi)$ on $\mathcal{M}^+$ are not uniquely defined. One can define one-parameter families of coordinates

$$(\tau_{\widetilde{\epsilon}}, s_{\widetilde{\epsilon}}, \vartheta_{\widetilde{\epsilon}}, \psi_{\widetilde{\epsilon}}) \tag{3.19}$$

on $\mathcal{M}^+$, with $\widetilde{\epsilon} \geq 0$ and

$$(\tau_0, s_0, \vartheta_0, \psi_0) \equiv (\tau, s, \vartheta, \psi)\,,$$

such that

$$(\tau_{\widetilde{\epsilon}}, s_{\widetilde{\epsilon}}, \vartheta_{\widetilde{\epsilon}}, \psi_{\widetilde{\epsilon}}) = \begin{cases} (0, v_0 - u_{\widetilde{\epsilon}}, \theta^1_{\widetilde{\epsilon}}, \theta^2_{\widetilde{\epsilon}}) & \text{along } \underline{C}_{u_{\widetilde{\epsilon}} \geq u_0, v_0} \\ (0, v_{\widetilde{\epsilon}} - u_0, \theta^1_{\widetilde{\epsilon}}, \theta^2_{\widetilde{\epsilon}}) & \text{along } C_{u_0, v_{\widetilde{\epsilon}} \geq v_0} \end{cases}$$

and

$$e_4^{\text{as}}(\tau_{\widetilde{\epsilon}}) = 1 \tag{3.20}$$

$$e_4^{\text{as}}(s_{\widetilde{\epsilon}}) = e_4^{\text{as}}(\vartheta_{\widetilde{\epsilon}}) = e_4^{\text{as}}(\psi_{\widetilde{\epsilon}}) = 0 \tag{3.21}$$

on $\mathcal{M}^+$ for all $\widetilde{\epsilon} \geq 0$, which preserve the form of the Kerr metric $g_{a,M}$ in the OFC-gauge *for all* $\widetilde{\epsilon} \geq 0$. Note that

$$\tau_{\widetilde{\epsilon}} \equiv \tau \tag{3.22}$$

on $\mathcal{M}^+$ for all $\widetilde{\epsilon} \geq 0$.

As an aside, we also note that, in view of the transport equations (3.20)-(3.20), to fully resolve the residual gauge freedom for the coordinates $(\tau, s, \vartheta, \psi)$ on $\mathcal{M}^+$, it is sufficient to resolve it along the hypersurface $\{\tau = 0\}$.

In what follows, the fixed differentiable structure $(\tau, s, \vartheta, \psi)$ on $\mathcal{M}^+$ should be understood as the $\epsilon = 0$-member of *one* of the (infinitely many) one-parameter families of coordinates (3.19). The choice of the particular one-parameter family will be discussed in forthcoming work by the author and will play a crucial role in what is called *gauge normalisation*.

The one-parameter family of metric perturbations

In this section we consider a one-parameter family of nonlinear metric perturbations around a Kerr solution. The family of metrics is introduced in an *outgoing frame-calibrated gauge*. What

[16] Recall that, in Section 3.2.2, the differentiable structure (u, v, θ^A) was involved in the definition of the coordinates $(\tau, s, \vartheta, \psi)$ on $\mathcal{M}^+$.

we really mean by this is the core of the present section of the overview and of Section 3.5 of the body of the chapter.

We consider a one-parameter family of smooth Lorentzian metrics $\boldsymbol{g}(\epsilon)$ on $\mathcal{M}^+$

$$\boldsymbol{g}(\epsilon) = \boldsymbol{g}_{x^\mu x^\nu}(\epsilon)\, dx^\mu\, dx^\nu \,,$$

with $x^\mu = (\tau, s, \vartheta, \psi)$, and the associated one-parameter family of null frames

$$\boldsymbol{\mathcal{N}}(\epsilon) = (\boldsymbol{e_1}(\epsilon), \boldsymbol{e_2}(\epsilon), \boldsymbol{e_3}(\epsilon), \boldsymbol{e_4}(\epsilon))$$

on $\mathcal{M}^+$ such that

(i) The metric component

$$\boldsymbol{g}_{\tau\tau}(\epsilon)$$

and the Christoffel symbols

$$\boldsymbol{\Gamma}^s_{\tau\tau}(\epsilon)\,,\ \boldsymbol{\Gamma}^\vartheta_{\tau\tau}(\epsilon)\,,\ \boldsymbol{\Gamma}^\psi_{\tau\tau}(\epsilon)$$

are fixed on $\mathcal{M}^+$ such that

$$\boldsymbol{g}_{\tau\tau}(\epsilon) = 0$$

and

$$\begin{aligned} \boldsymbol{\Gamma}^s_{\tau\tau}(\epsilon) &= 0\,, \\ \boldsymbol{\Gamma}^\vartheta_{\tau\tau}(\epsilon) &= 0\,, \\ \boldsymbol{\Gamma}^\psi_{\tau\tau}(\epsilon) &= 0 \end{aligned}$$

for all $\epsilon \geq 0$.

(ii) The connection coefficients

$$\hat{\boldsymbol{\omega}}(\epsilon)\,,\ \boldsymbol{Y}(\epsilon)\,,\ \underline{\boldsymbol{\eta}}(\epsilon)\,,\ \boldsymbol{\Gamma}^{\boldsymbol{B}}_{\boldsymbol{4A}}(\epsilon)$$

are fixed on $\mathcal{M}^+$ such that

$$\hat{\boldsymbol{\omega}}(\epsilon) = \hat{\omega}^{\,\text{as}}_{\,\text{Kerr}}\,, \tag{3.23}$$

$$\boldsymbol{Y}(\epsilon) = 0\,, \tag{3.24}$$

$$\underline{\boldsymbol{\eta}}(\epsilon) = \underline{\eta}^{\,\text{as}}_{\,\text{Kerr}}\,, \tag{3.25}$$

$$\boldsymbol{\Gamma}^{\boldsymbol{B}}_{\boldsymbol{4A}}(\epsilon) = 0 \tag{3.26}$$

for all $\epsilon \geq 0$, where the tensors $\hat{\omega}^{\text{as}}_{\text{Kerr}}$ and $\underline{\eta}^{\text{as}}_{\text{Kerr}}$ are known quantities relative to the fixed differentiable structure on $\mathcal{M}^+$.[17,18]

[17] The tensors $\hat{\omega}^{\text{as}}_{\text{Kerr}}$ and $\underline{\eta}^{\text{as}}_{\text{Kerr}}$ have been defined in Section 3.2.2 of the overview relative to the Boyer–Lindquist coordinates on $\mathcal{M}^+$. One can (although implicitly) define them relative to the coordinates $(\tau, s, \vartheta, \psi)$ on $\mathcal{M}^+$.

[18] The reader should note that the *tensorial* identity (3.25) is comparing covariant tensors living in different tensor bundles on $\mathcal{M}^+$, namely a $\mathfrak{D}_{\boldsymbol{\mathcal{N}}(\epsilon)}$ one-tensor and a $\mathfrak{D}_{\mathcal{N}_{\text{as}}}$ one-tensor, and is therefore to be understood as an *informal* identity. This issue, and the formally correct version of the identity, will be briefly discussed later in the overview (see, for instance, the expression (3.38)) and rigorously treated in Section 3.5.2 of this chapter,

(iii) The frame vector field $\boldsymbol{e_4}(\epsilon)$ is fixed on $\mathcal{M}^+$ such that

$$\boldsymbol{e_4}(\epsilon) = \partial_\tau \tag{3.27}$$

for all $\epsilon \geq 0$, where the vector field ∂_τ is a known quantity relative to the fixed differentiable structure on $\mathcal{M}^+$.

(iv) For any $\epsilon \geq 0$, the frame $\boldsymbol{\mathcal{N}}(\epsilon)$ extends to a regular frame for $s \to -\infty$,

and such that

$$\boldsymbol{g}(0) \equiv g_{a,M}$$

in the new gauge of Section 3.2.2, with associated frame

$$\boldsymbol{\mathcal{N}}(0) \equiv \mathcal{N}_{\text{as}}$$

on $\mathcal{M}^+$.[19]

We refer to the family of metrics $\boldsymbol{g}(\epsilon)$ as a one-parameter family of metric perturbations around the Kerr solution in an *outgoing frame-calibrated gauge.*

The family of metrics $\boldsymbol{g}(\epsilon)$ in our OFC-gauge exhibits a number of important properties. For reasons that we shall discuss in Section 3.2.4 of the overview, it is a fundamental fact that $\boldsymbol{\mathcal{N}}(0)$ coincides with the algebraically special frame of Kerr $\mathcal{N}_{\text{as}}$ and

$$\begin{gathered}
\boldsymbol{\chi}(0) \equiv \chi^{\text{as}}_{\text{Kerr}} \,,\, \underline{\boldsymbol{\chi}}(0) \equiv \underline{\chi}^{\text{as}}_{\text{Kerr}} \,,\, \boldsymbol{\eta}(0) \equiv \eta^{\text{as}}_{\text{Kerr}} \,,\, \underline{\boldsymbol{\eta}}(0) \equiv \underline{\eta}^{\text{as}}_{\text{Kerr}} \,, \\
\hat{\boldsymbol{\omega}}(0) \equiv \hat{\omega}^{\text{as}}_{\text{Kerr}} \,,\, \underline{\hat{\boldsymbol{\omega}}}(0) \equiv \underline{\hat{\omega}}^{\text{as}}_{\text{Kerr}} \,,\, \boldsymbol{\zeta}(0) \equiv \zeta^{\text{as}}_{\text{Kerr}} \,,\, \boldsymbol{\Gamma}^{\boldsymbol{B}}_{\boldsymbol{3A}}(0) \equiv \Gamma^{B\ \text{as}}_{3A\text{Kerr}} \,, \\
\boldsymbol{Y}(0) \equiv \underline{\boldsymbol{Y}}(0) \equiv 0 \,, \\
\boldsymbol{\Gamma}^{\boldsymbol{B}}_{\boldsymbol{4A}}(0) \equiv 0 \,, \\
\boldsymbol{\alpha}(0) \equiv \underline{\boldsymbol{\alpha}}(0) \equiv \boldsymbol{\beta}(0) \equiv \underline{\boldsymbol{\beta}}(0) \equiv 0 \,, \\
\boldsymbol{\rho}(0) \equiv \rho^{\text{as}}_{\text{Kerr}} \,, \\
\boldsymbol{\sigma}(0) \equiv \sigma^{\text{as}}_{\text{Kerr}} \,.
\end{gathered}$$

Two additional crucial properties of $\boldsymbol{g}(\epsilon)$, which introduce new geometric difficulties, are also discussed later in this section.

However, before considering the properties of the new gauge, we address a key, preliminary question:

Question. *Is an outgoing frame-calibrated gauge a well-posed gauge? Namely, given a (suitably) general Lorentzian manifold $(\boldsymbol{\mathcal{M}}, \boldsymbol{g})$, do there always exist a system of local coordinates*

where we will provide the geometric formalism to understand (3.25) as an identity between *tensors living in the same tensor bundle.* For the present statement, the reader can pretend that $\underline{\boldsymbol{\eta}}(\epsilon)$ and $\underline{\eta}^{\text{as}}_{\text{Kerr}}$ live in the same bundle.

[19]Note that the Kerr metric $g_{a,M}$ in coordinates $(\tau, s, \vartheta, \psi)$, and with the associated null frame $\mathcal{N}_{\text{as}}$, does satisfy the gauge conditions for the family of metrics $\boldsymbol{g}(\epsilon)$.

$(\boldsymbol{\tau}, \boldsymbol{s}, \boldsymbol{\vartheta}, \boldsymbol{\psi})$ *and a local null frame* $\boldsymbol{\mathcal{N}}$ *on* $(\boldsymbol{\mathcal{M}}, \boldsymbol{g})$ *such that all the gauge conditions above hold locally?*

The main result of the chapter answers this question in the affirmative. An overview of the statement and its proof can be found in the next two sections.

The statement of the main result

A key ingredient in our proof of linear stability is the introduction of the one-parameter family of Lorentzian metrics $\boldsymbol{g}(\epsilon)$ on $\mathcal{M}^+$ in a new nonlinear gauge. We consider here the question of whether the new gauge is a *well-posed gauge* for a general Lorentzian manifold $(\boldsymbol{\mathcal{M}}, \boldsymbol{g})$. The question is fundamental for both the consistency of our approach and for future nonlinear applications.

We remark that, in this section, one can forget that we will be ultimately dealing with perturbations of the Kerr solution. In fact, the well-posedness question for the new gauge is a geometric question that can be addressed independently from our problem. The answer to this question for a double-null gauge corresponds to the claim that any (suitably) general Lorentzian manifold $(\boldsymbol{\mathcal{M}}, \boldsymbol{g})$ can be locally put in a double-null gauge. Our main result states that the same holds for our outgoing frame-calibrated gauge, clarifying that a one-parameter family of metrics $\boldsymbol{g}(\epsilon)$ satisfying the gauge conditions above can indeed be considered.

Examples of gauges that have proved successful in the nonlinear analysis of the vacuum Einstein equations are *double-null gauges* [13, 110, 21, 95, 96, 20], *harmonic gauges* [81, 82], *time-transported gauges* [97, 42] and *Bondi gauges* [74].

A fully detailed account of what we discuss in this section of the overview can be found in Section 3.5 of the chapter.

Let $(\boldsymbol{\mathcal{M}}, \boldsymbol{g})$ be a $(3+1)$-dimensional, smooth, orientable Lorentzian manifold[20] with topology

$$\boldsymbol{\mathcal{M}} \cong \mathbb{R}^2 \times \boldsymbol{S}^2 .$$

We start by recalling that any such Lorentzian manifold $(\boldsymbol{\mathcal{M}}, \boldsymbol{g})$ can be locally put in a double-null gauge, i.e. there exist local coordinates

$$(\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{\theta^1}, \boldsymbol{\theta^2})$$

on $(\boldsymbol{\mathcal{M}}, \boldsymbol{g})$ such that $\boldsymbol{g}$ takes the double-null form

$$\boldsymbol{g} = -4\,\boldsymbol{\Omega}^2 \boldsymbol{du}\,\boldsymbol{dv} + \boldsymbol{\gamma\!\!\!/}_{\boldsymbol{\theta^A}\boldsymbol{\theta^B}}(\boldsymbol{d\theta^A} - \boldsymbol{b^{\theta^A}}\boldsymbol{dv})(\boldsymbol{d\theta^B} - \boldsymbol{b^{\theta^B}}\boldsymbol{dv}) \tag{3.28}$$

[20] Note that we do *not* assume any special algebraic property for $(\boldsymbol{\mathcal{M}}, \boldsymbol{g})$. We also do *not* assume that $(\boldsymbol{\mathcal{M}}, \boldsymbol{g})$ is a solution to the vacuum Einstein equations. These two facts can be contrasted to the *linear* gauge adopted by [1].

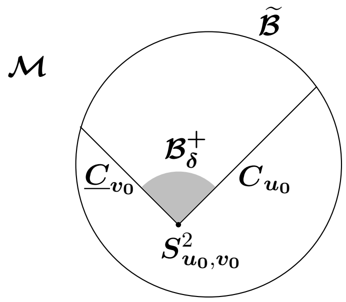

Figure 3.3: We define the set $\mathcal{B}_\delta$ to be a sufficiently small neighbourhood of $S^2_{u_0,v_0}$ of the form $\mathcal{B}_\delta := \bigcup_{(u-u_0)^2+(v-v_0)^2<\delta^2} S^2_{u,v}$, with $\delta > 0$ and $\mathcal{B}_\delta \subset \widetilde{\mathcal{B}}$. We define the set $\mathcal{B}_\delta^+ := \mathcal{B}_\delta \cap \{v \geq v_0\} \cap \{u \geq u_0\}$ corresponding to the shaded region in the figure.

on a sufficiently small neighbourhood $\widetilde{\mathcal{B}} \subseteq \mathcal{M}$.

Relying on the double-null differentiable structure of $\widetilde{\mathcal{B}}$, we consider the union of two null hypersurfaces

$$\widetilde{\mathcal{B}} \cap (C_{u_0, v \geq v_0} \cup \underline{C}_{u \geq u_0, v_0}) \tag{3.29}$$

on $\widetilde{\mathcal{B}}$, with $S^2_{u_0,v_0} \subset \widetilde{\mathcal{B}}$, and a *sufficiently small* neighbourhood

$$\mathcal{B}_\delta^+ \, ,$$

with $\mathcal{B}_\delta^+ \subset \widetilde{\mathcal{B}}$ corresponding to the shaded region in Figure 3.3.

Without loss of generality, one can state the well-posedness of the new gauge as the local existence of the new gauge on $\mathcal{B}_\delta^+$. The following theorem is the main result of the chapter.

Theorem 3.1 (Outgoing frame-calibrated gauge)**.** *Consider the Lorentzian manifold* $(\mathcal{M}, g)$, *its local differentiable structure* (u, v, θ^A) *and the sufficiently small set* $\mathcal{B}_\delta^+ \subset \mathcal{M}$ *as above. Then, for any* $\widetilde{\omega}, \mathfrak{f}_1, \mathfrak{f}_2$ *smooth scalar functions of the coordinates* (u, v, θ^A) *on* $\mathcal{B}_\delta^+$, *there exist a system of coordinates*

$$(\tau, s, \vartheta, \psi) \tag{3.30}$$

on $\mathcal{B}_\delta^+$, *with* $(\vartheta, \psi) \in S^2$, *and a null frame*

$$\mathcal{N} = (e_1, e_2, e_3, e_4) \tag{3.31}$$

on $\mathcal{B}_\delta^+$ *such that*

(i) ***Anchoring of the gauge****: The restriction of the coordinates* (3.30) *to*

$$\mathcal{B}_\delta^+ \cap (C_{u_0, v \geq v_0} \cup \underline{C}_{u \geq u_0, v_0}) \tag{3.32}$$

satisfies

$$(\tau, s, \vartheta, \psi) = \begin{cases} (0, v_0 - u, \theta^1, \theta^2) & \text{along } \mathcal{B}_\delta^+ \cap \underline{C}_{u \geq u_0, v_0} \\ (0, v - u_0, \theta^1, \theta^2) & \text{along } \mathcal{B}_\delta^+ \cap C_{u_0, v \geq v_0} \end{cases} .$$

In particular, each of the two-spheres

$$\boldsymbol{S}^2_{0,\boldsymbol{s}} \equiv \{0, \boldsymbol{s}\} \times \boldsymbol{S}^2$$

on (3.32) *coincides with a double-null sphere* $\boldsymbol{S}^2_{\boldsymbol{u},\boldsymbol{v}}$ *for some* $(\boldsymbol{u}, \boldsymbol{v})$.

(ii) ***Metric conditions**: The metric identity*

$$\boldsymbol{g}_{\boldsymbol{\tau}\boldsymbol{\tau}} = 0$$

and the identities for the Christoffel symbols

$$\begin{aligned} \boldsymbol{\Gamma}^{\boldsymbol{s}}_{\boldsymbol{\tau}\boldsymbol{\tau}} &= 0\,, \\ \boldsymbol{\Gamma}^{\boldsymbol{\vartheta}}_{\boldsymbol{\tau}\boldsymbol{\tau}} &= 0\,, \\ \boldsymbol{\Gamma}^{\boldsymbol{\psi}}_{\boldsymbol{\tau}\boldsymbol{\tau}} &= 0 \end{aligned}$$

hold on $\boldsymbol{\mathcal{B}}^+_{\boldsymbol{\delta}}$.

(iii) ***Connection conditions**: The connection coefficients*

$$\hat{\boldsymbol{\omega}}\,,\, \boldsymbol{Y}\,,\, \underline{\boldsymbol{\eta}}\,,\, \boldsymbol{\Gamma}^{\boldsymbol{B}}_{\boldsymbol{4A}}$$

relative to $\boldsymbol{\mathcal{N}}$ *are such that*

$$\begin{aligned} \hat{\boldsymbol{\omega}} &= \widetilde{\hat{\boldsymbol{\omega}}}\,, \\ \boldsymbol{Y} &= 0\,, \\ \underline{\boldsymbol{\eta}} &= \widetilde{\underline{\boldsymbol{\eta}}}\,, \\ \boldsymbol{\Gamma}^{\boldsymbol{B}}_{\boldsymbol{4A}} &= 0 \end{aligned}$$

on $\boldsymbol{\mathcal{B}}^+_{\boldsymbol{\delta}}$, *where* $\widetilde{\underline{\boldsymbol{\eta}}}$ *is the* $\mathfrak{D}_{\boldsymbol{\mathcal{N}}}$ *one-tensor such that* $\widetilde{\underline{\boldsymbol{\eta}}}(\boldsymbol{e_1}) = \mathfrak{f}_{\boldsymbol{1}}$ *and* $\widetilde{\underline{\boldsymbol{\eta}}}(\boldsymbol{e_2}) = \mathfrak{f}_{\boldsymbol{2}}$ *on* $\boldsymbol{\mathcal{B}}^+_{\boldsymbol{\delta}}$.

(iv) ***Frame condition**: The coordinates* $(\boldsymbol{s}, \boldsymbol{\vartheta}, \boldsymbol{\psi})$ *are transported along the integral curves of* $\boldsymbol{e_4}$, *meaning that the identity*

$$\boldsymbol{e_4} = \partial_{\boldsymbol{\tau}} \tag{3.33}$$

holds on $\boldsymbol{\mathcal{B}}^+_{\boldsymbol{\delta}}$.

(v) ***Regularity**: The frame* $\boldsymbol{\mathcal{N}}$ *is a regular frame on* $\boldsymbol{\mathcal{B}}^+_{\boldsymbol{\delta}}$.

The proof of the main result and the name of the gauge

In the present section, we sketch the main ideas of the proof of Theorem 3.1. We include two different arguments, one that will appear in forthcoming work by the author and one corresponding to the proof of the theorem in Section 3.5.1 of this chapter. We will briefly comment on the aspect that discriminates between the two. The choice of the name for the new gauge is also addressed.

In forthcoming work by the author, the proof of Theorem 3.1 goes as follows. The gauge construction starts by considering $(\boldsymbol{\mathcal{M}}, \boldsymbol{g})$ in the local double-null gauge (3.28), with associated *regular* null frame

$$\widehat{\boldsymbol{\mathcal{N}}}_{\text{ad}} = (\partial_{\boldsymbol{v}} + \boldsymbol{b}^{\boldsymbol{\theta^A}}\partial_{\boldsymbol{\theta^A}}, \boldsymbol{\Omega}^{-2}\partial_{\boldsymbol{u}}, \boldsymbol{\gamma\!\!/}^{-1/2}_{\boldsymbol{\theta^A\theta^A}}\partial_{\boldsymbol{\theta^A}}) \,.$$

We then claim the existence of a *null frame transformation*[21] $\mathcal{F}$ such that

$$\widehat{\boldsymbol{\mathcal{N}}}_{\text{ad}} \xrightarrow{\mathcal{F}} \boldsymbol{\mathcal{N}} \,,$$

with the connection coefficients relative to $\boldsymbol{\mathcal{N}}$ satisfying

$$\begin{aligned} \hat{\boldsymbol{\omega}} &= \widetilde{\hat{\boldsymbol{\omega}}} \,, \\ \boldsymbol{Y} &= 0 \,, \\ \underline{\boldsymbol{\eta}} &= \widetilde{\underline{\boldsymbol{\eta}}} \,, \\ \boldsymbol{\Gamma}^{\boldsymbol{B}}_{\boldsymbol{4A}} &= 0 \end{aligned}$$

locally on $(\boldsymbol{\mathcal{M}}, \boldsymbol{g})$, as in Theorem 3.1.

The most technical part of this proof is to formulate the notion of a null frame transformation appropriately and to show the *local* existence of the transformation $\mathcal{F}$.[22] Once the frame $\boldsymbol{\mathcal{N}}$ has been constructed, the coordinates $(\boldsymbol{\tau}, \boldsymbol{s}, \boldsymbol{\vartheta}, \boldsymbol{\psi})$ are obtained by solving the linear system of ODEs

$$\begin{aligned} \boldsymbol{e_4}(\boldsymbol{\tau}) &= 1 \\ \boldsymbol{e_4}(\boldsymbol{s}) &= 0 \\ \boldsymbol{e_4}(\boldsymbol{\vartheta}) &= 0 \\ \boldsymbol{e_4}(\boldsymbol{\psi}) &= 0 \end{aligned}$$

along the integral curves of $\boldsymbol{e_4}$, with initial data for $(\boldsymbol{\tau}, \boldsymbol{s}, \boldsymbol{\vartheta}, \boldsymbol{\psi})$ prescribed on the hypersurface (3.32) according to the *anchoring gauge condition (i)* of Theorem 3.1.

The construction of such a frame and coordinates guarantees that the remaining gauge conditions of the theorem automatically hold. The regularity of the frame $\boldsymbol{\mathcal{N}}$ is inherited from the regularity of $\widehat{\boldsymbol{\mathcal{N}}}_{\text{ad}}$ once the transformation $\mathcal{F}$ is *regular* in a suitable sense.

In this thesis, we achieve the gauge construction in a more *direct* fashion by solving a system of transport equations for the final frame (instead of obtaining the frame $\boldsymbol{\mathcal{N}}$ via a null frame transformation). See Section 3.5.1. This approach to the proof has the disadvantage that it makes the residual gauge freedom for the frame (see Section 3.2.3) more difficult to handle. However, the issue of resolving the residual freedom of the new gauge will not be addressed in the present thesis, and therefore this latter approach is more convenient.

[21] A *null frame transformation* on $(\boldsymbol{\mathcal{M}}, \boldsymbol{g})$ will be a transformation that transforms a null frame of $(\boldsymbol{\mathcal{M}}, \boldsymbol{g})$ into a new null frame of $(\boldsymbol{\mathcal{M}}, \boldsymbol{g})$.

[22] Note that $\mathcal{F}$ will be obtained as a local solution to a *nonlinear* system of evolution equations. Here is where the *local* nature of the gauge arises.

As it becomes evident from the proof of Theorem 3.1, the *frame* has the most prominent role in the gauge construction and determines, via its *outgoing* null frame vector, the differentiable structure of the manifold. In particular, by choosing the functions $\widetilde{\boldsymbol{\omega}}$, $\mathfrak{f}_{\mathbf{1}}$ and $\mathfrak{f}_{\mathbf{2}}$, one can suitably *calibrate* the frame in the *outgoing* null direction,[23] and thus the entire gauge. This motivates the name *outgoing frame-calibrated gauge.* For $(\boldsymbol{\mathcal{M}}, \boldsymbol{g})$ corresponding to the Kerr exterior manifold $(\mathcal{M}^+, g_{a,M})$ of Section 3.2.2, the frame-calibration of the gauge corresponds to the choice of the algebraically special frame of Kerr. The calibration of the one-parameter family of frames $\boldsymbol{\mathcal{N}}(\epsilon)$ associated to $\boldsymbol{g}(\epsilon)$ is fixed, for all $\epsilon \geq 0$, to coincide with the Kerr one, meaning that the functions $\widetilde{\boldsymbol{\omega}}(\epsilon)$, $\mathfrak{f}_{\mathbf{1}}(\epsilon)$ and $\mathfrak{f}_{\mathbf{2}}(\epsilon)$ coincide, in a suitable sense, with their Kerr values for all $\epsilon \geq 0$.

The divorce of the frame from the spacetime foliation

In this section we discuss the first of two important geometric properties of the one-parameter family of metrics $\boldsymbol{g}(\epsilon)$ in the new gauge. We have that

$$\mathfrak{D}_{\boldsymbol{\mathcal{N}}(\epsilon)} \text{ is a } \underline{\text{non-integrable}} \text{ distribution for all } \epsilon \geq 0 \,,$$

which we refer to by saying that *our outgoing frame-calibrated gauge is non-integrable.* In particular, $\mathfrak{D}_{\boldsymbol{\mathcal{N}}(0)}$ corresponds to the non-integrable distribution $\mathfrak{D}_{\mathcal{N}_{\text{as}}}$ induced by the algebraically special frame of Kerr on $\mathcal{M}^+$.

As mentioned in Section 3.2.1, a geometric manifestation of the non-integrability of the gauge is that $\boldsymbol{\chi}(\epsilon)$ and $\underline{\boldsymbol{\chi}}(\epsilon)$ are *not* symmetric tensors, and at least one of the identities

$$\boldsymbol{\chi}_{\boldsymbol{AB}}(\epsilon) \neq \boldsymbol{\chi}_{\boldsymbol{BA}}(\epsilon) \qquad\qquad \underline{\boldsymbol{\chi}}_{\boldsymbol{AB}}(\epsilon) \neq \underline{\boldsymbol{\chi}}_{\boldsymbol{BA}}(\epsilon)$$

holds on $\mathcal{M}^+$ for all $\epsilon \geq 0$. One can therefore define the *antitraces*

$$(\not\in \cdot\, \boldsymbol{\chi}_{\square})(\epsilon)\,,\ (\not\in \cdot\, \underline{\boldsymbol{\chi}}_{\square})(\epsilon)$$

of $\boldsymbol{\chi}(\epsilon)$ and $\underline{\boldsymbol{\chi}}(\epsilon)$, which, together with the symmetric traceless parts

$$\hat{\boldsymbol{\chi}}_{\bigcirc}(\epsilon)\,,\ \hat{\underline{\boldsymbol{\chi}}}_{\bigcirc}(\epsilon)\,,$$

and traces

$$\mathbf{tr}\boldsymbol{\chi}_{\bigcirc}(\epsilon)\,,\ \mathbf{tr}\underline{\boldsymbol{\chi}}_{\bigcirc}(\epsilon)\,,$$

[23] In the proof of Section 3.5.1, the reader will appreciate how the choice of the functions $\widetilde{\boldsymbol{\omega}}$, $\mathfrak{f}_{\mathbf{1}}$ and $\mathfrak{f}_{\mathbf{2}}$ determines the system of transport equations in the *outgoing* null direction

$$\begin{aligned}
&\boldsymbol{\nabla}_{\boldsymbol{e_4}}\boldsymbol{e_4} = \widetilde{\boldsymbol{\omega}}\,\boldsymbol{e_4}\,,\\
&\boldsymbol{\nabla}_{\boldsymbol{e_4}}\boldsymbol{e_3} = -\widetilde{\boldsymbol{\omega}}\,\boldsymbol{e_3} + 2\,\mathfrak{f}_{\mathbf{1}}\,\boldsymbol{e_1} + 2\,\mathfrak{f}_{\mathbf{2}}\,\boldsymbol{e_2}\,,\\
&\boldsymbol{\nabla}_{\boldsymbol{e_4}}\,\boldsymbol{e_1} = \mathfrak{f}_{\mathbf{1}}\,\boldsymbol{e_4}\,,\\
&\boldsymbol{\nabla}_{\boldsymbol{e_4}}\,\boldsymbol{e_2} = \mathfrak{f}_{\mathbf{2}}\,\boldsymbol{e_4}
\end{aligned}$$

used to construct the frame.

give the decomposition

$$\chi(\epsilon) = \hat{\chi}_{\circ}(\epsilon) + \frac{1}{2}(\mathrm{tr}\chi_{\circ}(\epsilon))\not{g}(\epsilon) + \frac{1}{2}((\not\in \cdot \chi_{\square})(\epsilon))\not\in(\epsilon)\,,$$
$$\underline{\chi}(\epsilon) = \hat{\underline{\chi}}_{\circ}(\epsilon) + \frac{1}{2}(\mathrm{tr}\underline{\chi}_{\circ}(\epsilon))\not{g}(\epsilon) + \frac{1}{2}((\not\in \cdot \underline{\chi}_{\square})(\epsilon))\not\in(\epsilon)\,,$$

where $\not{g}(\epsilon)$ and $\not\in(\epsilon)$ are the metric and volume form induced by $\boldsymbol{g}(\epsilon)$ and $\boldsymbol{\varepsilon}(\epsilon)$ on $\mathfrak{D}_{\mathcal{N}(\epsilon)}$.

As another effect of the non-integrability of the gauge, $\mathfrak{D}_{\mathcal{N}(\epsilon)}$ cannot be identified with the tangent bundle of the leaves of any spacetime foliation. In this sense, the frame and the spacetime foliation cannot be tied one to the other. The foliation induced by the fixed differentiable structure on $\mathcal{M}^+$ is indeed only *adapted to the integral curves of* $\boldsymbol{e_4}(\epsilon)$, meaning that $\boldsymbol{e_4}(\epsilon)$ is everywhere *tangent* to the s-constant leaves of the spacetime foliation for all $\epsilon \geq 0$. Further details on the relation between the frame and the spacetime foliation in our gauge can be found in Section 3.5.3.

The reader should note that, in a *double-null gauge*, $\mathfrak{D}_{\mathcal{N}(\epsilon)}$ is an *integrable* distribution and $\chi(\epsilon)$ and $\underline{\chi}(\epsilon)$ are *symmetric* tensors. The spacetime foliation and the null frame are naturally tied, giving nice geometric properties to the gauge that one typically exploits.

The formulation of tensor perturbations

In this section we discuss the second of two important geometric properties of the one-parameter family of metrics $\boldsymbol{g}(\epsilon)$ in the new gauge. We have that

$$\mathfrak{D}_{\mathcal{N}(\epsilon)} \textit{ is } \underline{\textit{not fixed}} \textit{ for all } \epsilon \geq 0\ ,$$

which we refer to by saying that *the horizontal structure of our outgoing frame-calibrated gauge is variable.*[24] In general,

$$\mathfrak{D}_{\mathcal{N}(\epsilon)} \not\equiv \mathfrak{D}_{\mathcal{N}(0)} \tag{3.34}$$

for all $\epsilon \geq 0$.

Relation (3.34) gives rise to the following difficulty, which will be rigorously addressed in Section 3.5.2. Let us consider, for instance, the one-parameter family of one-tensors $\underline{\eta}(\epsilon)$ on $\mathcal{M}^+$. For any $\epsilon > 0$, the one-tensor $\underline{\eta}(\epsilon)$ and the one-tensor $\underline{\eta}(0)$ are covariant tensors on $\mathcal{M}^+$ which, in view of (3.34), live in two different tensor bundles, namely the bundle of $\mathfrak{D}_{\mathcal{N}(\epsilon)}$ *one-tensors* on $\mathcal{M}^+$ and the bundle of $\mathfrak{D}_{\mathcal{N}(0)}$ *one-tensors* on $\mathcal{M}^+$. This fact becomes problematic when we want to compare such tensors by taking the difference

$$\text{“}\,\underline{\eta}(\epsilon) - \underline{\eta}(0)\,\text{”} \tag{3.35}$$

on $\mathcal{M}^+$. In particular, note that expressions of the form (3.35) appeared when we introduced the family of metrics $\boldsymbol{g}(\epsilon)$ in the new gauge (see, for instance, the identity (3.25)).

[24] Equivalently, we have that the spacelike frame vectors $\boldsymbol{e_A}(\epsilon)$ are not fixed for all $\epsilon \geq 0$.

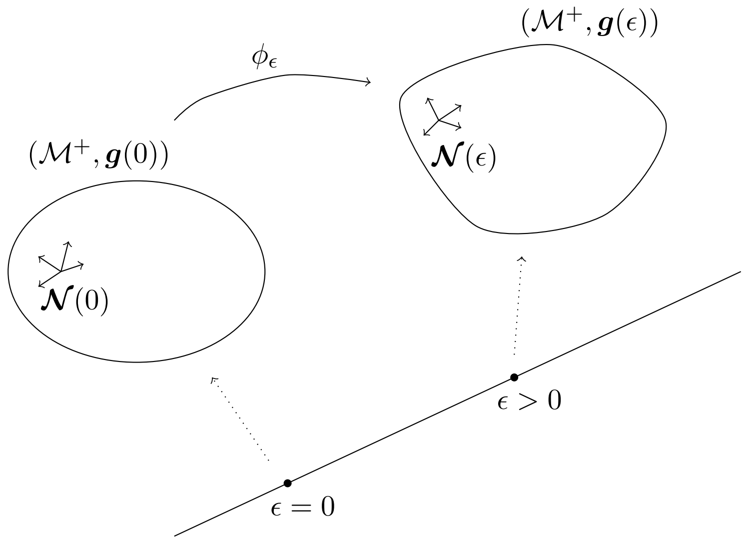

Figure 3.4: The one-parameter family of metric perturbations $\boldsymbol{g}(\epsilon)$ on $\mathcal{M}^+$ induces a one-parameter family of Lorentzian manifolds $(\mathcal{M}^+, \boldsymbol{g}(\epsilon))$. The family of frames $\boldsymbol{\mathcal{N}}(\epsilon)$ defines, for each $\epsilon \geq 0$, a null frame on $(\mathcal{M}^+, \boldsymbol{g}(\epsilon))$. One can define a diffemorphism ϕ_ϵ between $(\mathcal{M}^+, \boldsymbol{g}(0))$ and $(\mathcal{M}^+, \boldsymbol{g}(\epsilon))$.

In this section, and later in Section 3.5.2 of the chapter, we take an *active* point of view relative to the family of metrics $\boldsymbol{g}(\epsilon)$ on $\mathcal{M}^+$ and regard $\boldsymbol{g}(\epsilon)$ as inducing a one-parameter family of *distinct* Lorentzian manifolds $(\mathcal{M}^+, \boldsymbol{g}(\epsilon))$. Following this approach, *even if the horizontal structure of the new gauge was not variable*, the tensors $\underline{\boldsymbol{\eta}}(\epsilon)$ and $\underline{\boldsymbol{\eta}}(0)$ result defined on two different Lorentzian manifolds, namely $(\mathcal{M}^+, \boldsymbol{g}(\epsilon))$ and $(\mathcal{M}^+, \boldsymbol{g}(0))$, making the formal meaning of the expression (3.35) already unclear. See Figure 3.4.

To give a formal meaning to the expression (3.35), we implement the following four-step procedure:

1. ***Extension***: We extend both $\underline{\boldsymbol{\eta}}(\epsilon)$ and $\underline{\boldsymbol{\eta}}(0)$ to one-tensors in $T^0_1\mathcal{M}^+$ on $(\mathcal{M}^+, \boldsymbol{g}(\epsilon))$ and $(\mathcal{M}^+, \boldsymbol{g}(0))$ respectively. The extension is defined by imposing

$$\underline{\boldsymbol{\eta}}(\epsilon)(\boldsymbol{e_3}(\epsilon)) := 0\,, \qquad \underline{\boldsymbol{\eta}}(\epsilon)(\boldsymbol{e_4}(\epsilon)) := 0\,,$$
$$\underline{\boldsymbol{\eta}}(0)(\boldsymbol{e_3}(0)) := 0\,, \qquad \underline{\boldsymbol{\eta}}(0)(\boldsymbol{e_4}(0)) := 0\,.$$

2. ***Pullback***: We define a suitable diffemorphism ϕ_ϵ between $(\mathcal{M}^+, \boldsymbol{g}(0))$ and $(\mathcal{M}^+, \boldsymbol{g}(\epsilon))$ and consider the pullback one-tensor

$$\phi^*_\epsilon(\underline{\boldsymbol{\eta}}(\epsilon)) \tag{3.36}$$

in $T^0_1\mathcal{M}^+$ on $(\mathcal{M}^+, \boldsymbol{g}(0))$. Since, in general, the pullback distribution does *not* coincide with the distribution $\mathfrak{D}_{\boldsymbol{\mathcal{N}}(0)}$ on $(\mathcal{M}^+, \boldsymbol{g}(0))$, i.e.

$$\phi^*_\epsilon(\mathfrak{D}_{\boldsymbol{\mathcal{N}}(\epsilon)}) \not\equiv \mathfrak{D}_{\boldsymbol{\mathcal{N}}(0)}\,, \tag{3.37}$$

we possibly have that (3.36) *is not an horizontal one-tensor* and at least one of the following relations

$$(\phi_\epsilon^*(\underline{\boldsymbol{\eta}}(\epsilon)))(\boldsymbol{e_3}(0)) \neq 0\,, \qquad\qquad (\phi_\epsilon^*(\underline{\boldsymbol{\eta}}(\epsilon)))(\boldsymbol{e_4}(0)) \neq 0$$

possibly holds.

3. ***Projection***: We therefore project the pullback tensor $\phi_\epsilon^*(\underline{\boldsymbol{\eta}}(\epsilon))$ and consider the *projected* pullback one-tensor

$$\Pi\phi_\epsilon^*(\underline{\boldsymbol{\eta}}(\epsilon))$$

such that

$$(\Pi\phi_\epsilon^*(\underline{\boldsymbol{\eta}}(\epsilon)))(\boldsymbol{e_3}(0)) = 0\,, \qquad\qquad (\Pi\phi_\epsilon^*(\underline{\boldsymbol{\eta}}(\epsilon)))(\boldsymbol{e_4}(0)) = 0\,.$$

The tensor $\Pi\phi_\epsilon^*(\underline{\boldsymbol{\eta}}(\epsilon))$ is an *horizontal* tensor in $T_1^0\mathcal{M}^+$ on $(\mathcal{M}^+, \boldsymbol{g}(0))$.

4. ***Restriction***: We conclude by restricting $\Pi\phi_\epsilon^*(\underline{\boldsymbol{\eta}}(\epsilon)))$ to a $\mathfrak{D}_{\boldsymbol{\mathcal{N}}(0)}$ *one-tensor* on $(\mathcal{M}^+, \boldsymbol{g}(0))$.

The expression

$$\Pi\phi_\epsilon^*(\underline{\boldsymbol{\eta}}(\epsilon))) - \underline{\boldsymbol{\eta}}(0) \tag{3.38}$$

is now a difference of $\mathfrak{D}_{\boldsymbol{\mathcal{N}}(0)}$ one-tensors on $(\mathcal{M}^+, \boldsymbol{g}(0))$ and should be seen as the formally correct version of the expression (3.35). We refer to (3.38) as the (nonlinear) *tensor perturbation* of $\underline{\boldsymbol{\eta}}(0)$.

In the more rigorous formulation of the family of metrics $\boldsymbol{g}(\epsilon)$ of Section 3.5, we replace all the informal gauge conditions (like, for instance, (3.25)) with tensor perturbations in the correct form (3.38).

For what concerns the definition of the diffeomorphism ϕ_ϵ, our choice of ϕ_ϵ will be such that it identifies points of $(\mathcal{M}^+, \boldsymbol{g}(0))$ and $(\mathcal{M}^+, \boldsymbol{g}(\epsilon))$ having the same coordinate values relative to the fixed differentiable structure on $\mathcal{M}^+$. To correctly interpret this choice, we will consider the Lorentzian manifolds $(\mathcal{M}^+, \boldsymbol{g}(\epsilon))$ as leaves of a larger manifold in which they naturally embed, as it is standard practice in geometric perturbation theories. See Section 3.5.2.

In a *double-null gauge*, the distribution $\mathfrak{D}_{\boldsymbol{\mathcal{N}}(\epsilon)}$ is *fixed* on $\mathcal{M}^+$ for all $\epsilon \geq 0$, i.e.

$$\phi_\epsilon^*(\mathfrak{D}_{\boldsymbol{\mathcal{N}}(\epsilon)}) \equiv \mathfrak{D}_{\boldsymbol{\mathcal{N}}(0)}$$

for all $\epsilon \geq 0$,[25] which should be contrasted with (3.37). The formulation of tensor perturbations does not pose any of the technical difficulties involved in the projection part of our procedure, although one still needs to consider pulled-back tensors.

As a concluding remark, we note that one can, in principle, avoid to introduce pulled-back objects and compare *coordinate components* of tensors relative to the fixed differentiable structure

[25] In the case of [19], the frame vector fields $\boldsymbol{e_A}(\epsilon)$ are *coordinate* vector fields. The statement that the distribution $\mathfrak{D}_{\boldsymbol{\mathcal{N}}(\epsilon)}$ is fixed for all $\epsilon \geq 0$ is, in that context, equivalent to the statement that the differentiable structure of the ambient manifold remains fixed for all $\epsilon \geq 0$.

on $\mathcal{M}^+$.[26] However, our approach is more *geometric* in that it compares *tensors* (as opposed to tensor components) on $\mathcal{M}^+$. Nonetheless, we still secretly rely on the fixed differentiable structure of $\mathcal{M}^+$ to formulate our geometric tensor perturbations, namely to define the diffeomorphism ϕ_ϵ.

Residual gauge freedom for the frame

Our outgoing frame-calibrated gauge comes with some *residual gauge freedom.* This is, in part, connected to fixing the differentiable structure on $\mathcal{M}^+$, as we already discussed. We will refer to such a residual freedom as the *residual gauge freedom for the coordinates.*

On the other hand, the new gauge also leaves some *residual gauge freedom for the frame.* In fact, the one-parameter family of null frames $\boldsymbol{\mathcal{N}}(\epsilon)$ on $\mathcal{M}^+$ associated to $\boldsymbol{g}(\epsilon)$ is not fully determined. In other words, given the family of metrics $\boldsymbol{g}(\epsilon)$, there exist infinitely many one-parameter families of null frames $\boldsymbol{\mathcal{N}}(\epsilon)$ on $\mathcal{M}^+$ such that all the gauge conditions hold. This latter residual freedom is treated in Section 3.5.1 of the chapter. Here we give an informal overview of its main aspects.

The key observation is the following. The connection conditions of Theorem 3.1 should be thought as conditions for the covariant derivative $\boldsymbol{\nabla}_{\boldsymbol{e_4}(\epsilon)}$ of each of the families of frame vectors $\boldsymbol{e_1}(\epsilon)$, $\boldsymbol{e_2}(\epsilon)$, $\boldsymbol{e_3}(\epsilon)$ and $\boldsymbol{e_4}(\epsilon)$. In analogy with the freedom of specifying a constant of integration, the residual gauge freedom for the frame can be understood as the freedom to freely prescribe a one-parameter family of frames

$$\boldsymbol{\mathcal{N}}(\epsilon) = (\partial_\tau, \boldsymbol{e_3}(\epsilon), \boldsymbol{e_A}(\epsilon))$$

along the hypersurface $\{\tau = 0\}$.[27,28] Upon making such a prescription <u>along $\{\tau = 0\}$</u>, the gauge conditions yield a *fully-determined* family of frames $\boldsymbol{\mathcal{N}}(\epsilon)$ (with no residual gauge freedom left) <u>on $\mathcal{M}^+$</u>.

We remark that the residual gauge freedom for the frame is conceptually *distinct* from the one for the coordinates $(\tau, s, \vartheta, \psi)$. The double nature of the residual freedom of our gauge should be contrasted to the residual gauge freedom arising in double-null gauges, where the residual gauge freedom for the coordinates is *equivalent* to the residual gauge freedom for the frame.[29]

As for the residual gauge freedom for the coordinates, the residual gauge freedom for the frame will be exploited in forthcoming work by the author. The precise way in which the residual gauge freedom is resolved contributes to what is called *gauge normalisation.*[30]

[26] This is the approach adopted by [19].

[27] Strictly speaking, this free choice is limited to the $\epsilon > 0$ members of the family, since the frame $\boldsymbol{\mathcal{N}}(0)$ is already fully determined by the requirement $\boldsymbol{\mathcal{N}}(0) \equiv \mathcal{N}_{\text{as}}$.

[28] Recall that the new gauge fixes the frame vector $\boldsymbol{e_4}(\epsilon)$ for all $\epsilon \geq 0$ relative to the fixed differentiable structure on $\mathcal{M}^+$. Nonetheless, there exist infinitely many one-parameter families of null frames $\boldsymbol{\mathcal{N}}(\epsilon)$ on $\{\tau = 0\}$ that embed the fixed $\boldsymbol{e_4}(\epsilon) = \partial_\tau$ as the outgoing null frame vector.

[29] This is also the case for the Bondi gauge of [74].

[30] How one chooses to fix the residual gauge freedom is what we refer to as *gauge normalisation.* As we observed

The vacuum Einstein equations in the new gauge

We now assume that the family of metrics $\boldsymbol{g}(\epsilon)$ satisfies the vacuum Einstein equations

$$\mathbf{Ric}(\boldsymbol{g}(\epsilon)) = 0 \,.$$

Since our outgoing frame-calibrated gauge is *non-integrable*, the system of vacuum Einstein equations in the new gauge has to be seen as a special case of the *general* system of equations briefly discussed in Section 3.2.1 of the overview and presented in Section 3.3.5 of the chapter. This fact ultimately motivates the formalism of Section 3.3.

A key aspect of the system of Einstein equations in our gauge is that *the transport equations in the $\boldsymbol{e_4}(\epsilon)$-direction take a particularly convenient form.* For the connection coefficients

$$\hat{\boldsymbol{\chi}}_{\circ}(\epsilon)\,,\; (\mathbf{tr}\boldsymbol{\chi}_{\circ})(\epsilon)\,,\; (\not\in\cdot\boldsymbol{\chi}_{\square})(\epsilon)\,,$$

we have the set of transport equations

$$\boldsymbol{\nabla\!\!\!/}_{\tau}\hat{\boldsymbol{\chi}}_{\circ}(\epsilon) + (\mathbf{tr}\boldsymbol{\chi}_{\circ})\,\hat{\boldsymbol{\chi}}_{\circ}(\epsilon) - \hat{\omega}\,{}^{\text{as}}_{\text{Kerr}}\,\hat{\boldsymbol{\chi}}_{\circ}(\epsilon) = -\boldsymbol{\alpha}(\epsilon)\,, \tag{3.39}$$

$$\partial_{\tau}(\mathbf{tr}\boldsymbol{\chi}_{\circ})(\epsilon) + \frac{1}{2}\,(\mathbf{tr}\boldsymbol{\chi}_{\circ})^2(\epsilon) - \hat{\omega}\,{}^{\text{as}}_{\text{Kerr}}\,(\mathbf{tr}\boldsymbol{\chi}_{\circ})(\epsilon) = -(\hat{\boldsymbol{\chi}}_{\circ},\hat{\boldsymbol{\chi}}_{\circ})(\epsilon) + \frac{1}{2}(\not\in\cdot\boldsymbol{\chi}_{\square})^2(\epsilon)\,, \tag{3.40}$$

$$\partial_{\tau}(\not\in\cdot\boldsymbol{\chi}_{\square})(\epsilon) + (\mathbf{tr}\boldsymbol{\chi}_{\circ})(\not\in\cdot\boldsymbol{\chi}_{\square})(\epsilon) - \hat{\omega}\,{}^{\text{as}}_{\text{Kerr}}\,(\not\in\cdot\boldsymbol{\chi}_{\square})(\epsilon) = 0 \tag{3.41}$$

on $\mathcal{M}^+$, which only couples with the rest of the system via the curvature component $\boldsymbol{\alpha}(\epsilon)$. Furthermore, for the connection coefficient

$$\hat{\boldsymbol{\chi}}_{\circ}(\epsilon)\,,$$

we have the transport equation

$$\begin{aligned}\boldsymbol{\nabla\!\!\!/}_{\tau}\hat{\boldsymbol{\chi}}_{\circ}(\epsilon) + \frac{1}{2}\,(\mathbf{tr}\boldsymbol{\chi}_{\circ})\,\hat{\boldsymbol{\chi}}_{\circ}(\epsilon) + \frac{1}{2}\,(\not\in\cdot\boldsymbol{\chi}_{\square})\,{}^{\star}\hat{\boldsymbol{\chi}}_{\circ}(\epsilon) + \hat{\omega}\,{}^{\text{as}}_{\text{Kerr}}\,\hat{\boldsymbol{\chi}}_{\circ}(\epsilon) &= (-2\,\boldsymbol{\mathcal{D}\!\!\!/}^{\star}_{\boldsymbol{2}}\,\underline{\boldsymbol{\eta}} + \underline{\boldsymbol{\eta}}\,\hat{\otimes}\,\underline{\boldsymbol{\eta}})(\epsilon) \\ &\quad - \frac{1}{2}\,(\mathbf{tr}\,\boldsymbol{\chi}_{\circ})\,\hat{\boldsymbol{\chi}}_{\circ}(\epsilon) + \frac{1}{2}\,(\not\in\cdot\boldsymbol{\chi}_{\square})\,{}^{\star}\hat{\boldsymbol{\chi}}_{\circ}(\epsilon)\end{aligned} \tag{3.42}$$

on $\mathcal{M}^+$. In view of the gauge condition (3.25), the right hand side of the equation (3.42) is independent of the *vanishing* (informal) tensor perturbation

$$\underline{\boldsymbol{\eta}}(\epsilon) - \underline{\eta}\,{}^{\text{as}}_{\text{Kerr}}$$

on $\mathcal{M}^+$ for all $\epsilon \geq 0$.

Note that, for simplicity, in the equations (3.39)-(3.42) we simply replaced some bold connection

before, to fully resolve the residual gauge freedom of the new gauge on $\mathcal{M}^+$, it is sufficient to resolve it along the hypersurface $\{\tau = 0\}$. In forthcoming work by the author, we will discuss how the hypersurface $\{\tau = 0\}$ can be interpreted as the hypersurface on which the initial data for the linearised system of vacuum Einstein equations (in the new gauge) are prescribed. Our gauge normalisation will be an *initial data gauge normalisation*, in that it will *only* depend on such initial data.

coefficients with the corresponding Kerr connection coefficients (relative to the algebraically special frame) to stress how our gauge enters in the equations. As already noted, this substitution is rather informal. The formally correct version of these equations can be found in Section 3.6 and ultimately motivates the rigorous formulation of *nonlinear* tensor perturbations in Section 3.5.2 of the chapter.

3.2.4 The linearised Einstein equations

This section serves as an overview to the derivation and *properties* of the linearised vacuum Einstein equations around a Kerr solution in our outgoing frame-calibrated gauge. Section 3.7 of the chapter will *only* treat the *formal derivation* of the linear system. A detailed account of the properties of the linearised equations will be part of forthcoming work by the author.

The system of linearised Einstein equations that we present in this thesis can be employed to investigate a wide class of problems that go beyond our proof of linear stability. As particularly interesting examples, one can derive *energy conservation laws* for the linearised system, revisiting work [56] and the ideas developed therein, or construct a *scattering theory* in the same vain as [87].

The structure of this section of the overview is divided into two parts. We first introduce the main ideas involved in the linearisation procedure. We then select some of the linearised Einstein equations and discuss their key features.

The linearisation procedure

The rigorous linearisation procedure of the nonlinear vacuum Einstein equations for $\boldsymbol{g}(\epsilon)$ in our gauge is discussed in Section 3.7.1. As a general observation, we note that all the *new* technical difficulties arising in the linearisation procedure are ultimately due to the presence of one-parameter families of connection coefficients, such as $\boldsymbol{\eta}(\epsilon)$, which are $\mathfrak{D}_{\boldsymbol{\mathcal{N}}(\epsilon)}$ *covariant* tensors and do *not* vanish for $\epsilon = 0$. [31]

The linearisation procedure proceeds by linearising each of the terms appearing in the nonlinear vacuum Einstein equations for $\boldsymbol{g}(\epsilon)$ around its Kerr value. To do that, we first pull-back the whole nonlinear system via $\Pi\phi^*_\epsilon$ and renormalise each of the pulled-back nonlinear equations by subtracting its correspondent Kerr equation. This allows to exploit our rigorous formulation of nonlinear tensor perturbations (3.38) and consider, for instance, terms of the form

$$\Pi\phi^*_\epsilon(\boldsymbol{\eta}(\epsilon)) - \eta^{\text{as}}_{\text{Kerr}} \tag{3.43}$$

on $(\mathcal{M}^+, g_{a,M})$. We linearise the one-parameter family of $\mathfrak{D}_{\mathcal{N}_{\text{as}}}$ *tensors*[32] (3.43) around $\epsilon = 0$

[31] Note that, in [19], all the connection coefficients that do not vanish for $\epsilon = 0$ are smooth *scalar functions*. This fact is connected to the spherical symmetry of the Schwarzschild background and makes the linearisation procedure much simpler.

[32] Note that we linearise *tensors*, as opposed to linearising tensor *components*.

by neglecting the $\mathcal{O}(\epsilon^2)$ terms in the formal expansion

$$\Pi\phi_\epsilon^*(\boldsymbol{\eta}(\epsilon)) - \eta_{\text{Kerr}}^{\text{as}} = \epsilon \cdot \overset{(1)}{\not\eta} + \mathcal{O}(\epsilon^2)\,,$$

where $\overset{(1)}{\not\eta}$ is a $\mathfrak{D}_{\mathcal{N}_{\text{as}}}$ one-tensor on $(\mathcal{M}^+, g_{a,M})$ and $\mathcal{O}(\epsilon^2)$ denotes a $\mathfrak{D}_{\mathcal{N}_{\text{as}}}$ one-tensor whose components are $\mathcal{O}(\epsilon^2)$ scalar functions.[33]

Crucially, in our outgoing frame-calibrated gauge, we have

$$\overset{(1)}{\hat{\omega}} \equiv 0 \tag{3.44}$$

$$\overset{(1)}{\Upsilon} \equiv 0 \tag{3.45}$$

$$\overset{(1)}{\not\xi} \equiv 0 \tag{3.46}$$

$$\overset{(1)}{\Gamma_{4A}^{B}} \equiv 0 \tag{3.47}$$

and

$$\overset{(1)}{e_4^\tau} \equiv \overset{(1)}{e_4^s} \equiv \overset{(1)}{e_4^\vartheta} \equiv \overset{(1)}{e_4^\psi} \equiv 0 \tag{3.48}$$

on $(\mathcal{M}^+, g_{a,M})$ by the gauge conditions (3.23)-(3.27), where the linearisation of the scalar functions $\hat{\boldsymbol{\omega}}(\epsilon)$, $\boldsymbol{\Gamma_{4A}^{B}}(\epsilon)$ and $\boldsymbol{e_4^\mu}(\epsilon)$ is obtained, for instance, by considering the formal expansion

$$\phi_\epsilon^*(\hat{\boldsymbol{\omega}}(\epsilon)) - \hat{\omega}_{\text{Kerr}}^{\text{as}} = \epsilon \cdot \overset{(1)}{\hat{\omega}} + \mathcal{O}(\epsilon^2)$$

on $(\mathcal{M}^+, g_{a,M})$ and neglecting the $\mathcal{O}(\epsilon^2)$ terms.

We note that the *identities* (3.44)-(3.48) *are responsible for the simplification of the linearised transport equations in the outgoing null direction introduced by our gauge.* See the next section for further details on this.

The most challenging part of the linearisation procedure is linearising some of the *projected* covariant derivatives of $\mathfrak{D}_{\boldsymbol{\mathcal{N}}(\epsilon)}$ one-tensors, such as[34]

$$(\boldsymbol{\not\nabla_3}\,\boldsymbol{\zeta})(\epsilon)\,. \tag{3.49}$$

To achieve that, we introduce the projection tensor $\boldsymbol{\Pi}(\epsilon)$ on $\mathcal{M}^+$ associated to $\mathfrak{D}_{\boldsymbol{\mathcal{N}}(\epsilon)}$ and

---

[33]We adopt the slashed notation $\overset{(1)}{\not\eta}$ to stress the fact that $\overset{(1)}{\not\eta}$ is a $\mathfrak{D}_{\mathcal{N}_{\text{as}}}$ tensor. In Section 3.7.1 we will also consider the following linearisation

$$\phi_\epsilon^*(\boldsymbol{\eta}(\epsilon)) - \eta_{\text{Kerr}}^{\text{as}} = \epsilon \cdot \overset{(1)}{\eta} + \mathcal{O}(\epsilon^2)\,,$$

where the pulled-back tensor $\phi_\epsilon^*(\boldsymbol{\eta}(\epsilon))$ is not projected and thus $\overset{(1)}{\eta}$ is not necessarily a $\mathfrak{D}_{\mathcal{N}_{\text{as}}}$ tensor.

[34]The reader should note that (3.49) is actually the most technically difficult covariant derivative to linearise in the whole system of nonlinear equations. The linearisation of all the other covariant derivatives gets simplified by either our gauge identities or by the fact that the tensor vanishes on the Kerr background.

linearise the $\mathfrak{D}_{\mathcal{N}_{\rm as}}$ one-tensor

$$\Pi\phi_\epsilon^*((\boldsymbol{\Pi}\cdot\boldsymbol{\nabla}_{\boldsymbol{e_3}}\boldsymbol{\zeta})(\epsilon))-\slashed{\nabla}_{e_3^{\rm as}}\,\zeta^{\rm as}_{\rm Kerr}$$

on $(\mathcal{M}^+,g_{a,M})$. The linearisation formulae for (3.49), and all the covariant derivatives of tensors, are the formulae (3.185)-(3.190) of Section 3.7.1. Note that these formulae involve the *linearised Christoffel symbols*

$$\overset{(1)}{\Gamma}\,,\;\overset{(1)}{\slashed{\Gamma}}\,, \tag{3.50}$$

which are $(1,2)$-*tensors* on $(\mathcal{M}^+,g_{a,M})$ such that[35]

$$(\phi^*_\epsilon\,\boldsymbol{\nabla}(\epsilon)-\nabla)_\mu\,\partial_\nu=\epsilon\cdot\overset{(1)}{\Gamma}{}^\sigma_{\mu\nu}\,\partial_\sigma+\mathcal{O}(\epsilon^2)\,,$$
$$\Pi^\alpha_\mu\,\Pi^\beta_\nu\,(\phi^*_\epsilon\,\boldsymbol{\nabla}(\epsilon)-\nabla)_\alpha\,\partial_\beta=\epsilon\cdot\overset{(1)}{\slashed{\Gamma}}{}^\sigma_{\mu\nu}\,\partial_\sigma+\mathcal{O}(\epsilon^2)\,,$$

and the *linearised projection tensor*

$$\overset{(1)}{\slashed{\Pi}}$$

such that

$$\Pi\phi^*_\epsilon\,\boldsymbol{\Pi}(\epsilon)-\Pi^{\rm as}_{\rm Kerr}=\epsilon\cdot\overset{(1)}{\slashed{\Pi}}+\mathcal{O}(\epsilon^2)\,.$$

The linearisation formula for (3.49) reads

$$\begin{aligned}\Pi\phi^*_\epsilon((\slashed{\boldsymbol{\nabla}}_{\boldsymbol{e_3}}\,\boldsymbol{\zeta})(\epsilon))-\slashed{\nabla}_{e_3^{\rm as}}\,\zeta^{\rm as}_{\rm Kerr}=&\,\epsilon\cdot(\slashed{\nabla}_{\overset{(1)}{e_3}}\,\zeta^{\rm as}_{\rm Kerr}+\slashed{\nabla}_{e_3^{\rm as}}\,\overset{(1)}{\slashed{\zeta}}-\overset{(1)}{\slashed{\Gamma}}{}_3\cdot\zeta^{\rm as}_{\rm Kerr}\\&-(\overset{(1)}{\zeta}(e_3^{\rm as}))\eta^{\rm as}_{\rm Kerr}+\overset{(1)}{\slashed{\Pi}}\cdot(\nabla_{e_3^{\rm as}}\zeta^{\rm as}_{\rm Kerr}))+\mathcal{O}(\epsilon^2)\,.\end{aligned}\tag{3.51}$$

Through the linearisation of the covariant derivatives of tensors, the reader will be able to appreciate, in part, how the transport equations in the outgoing null direction gets simplified in our gauge. Indeed, as a general principle, all the covariant derivatives in the $\boldsymbol{e_4}(\epsilon)$-direction will be easier to linearise than the ones in the $\boldsymbol{e_3}(\epsilon)$-direction. In fact, the analogue of formula (3.51) for the covariant derivative

$$(\slashed{\boldsymbol{\nabla}}_{\boldsymbol{4}}\,\boldsymbol{\eta})(\epsilon)$$

appearing in the transport equation

$$(\slashed{\boldsymbol{\nabla}}_{\boldsymbol{4}}\,\boldsymbol{\eta})(\epsilon)=(\slashed{\boldsymbol{\nabla}}_{\boldsymbol{3}}\,\boldsymbol{Y})(\epsilon)-\boldsymbol{\chi}^{\sharp_1}\cdot(\boldsymbol{\eta}-\underline{\boldsymbol{\eta}})(\epsilon)+2\underline{\hat{\boldsymbol{\omega}}}\,\boldsymbol{Y}(\epsilon)-\boldsymbol{\beta}(\epsilon)$$

more easily reads

$$\Pi\phi^*_\epsilon((\slashed{\boldsymbol{\nabla}}_{\boldsymbol{e_4}}\,\boldsymbol{\eta})(\epsilon))-\slashed{\nabla}_{e_4^{\rm as}}\,\eta^{\rm as}_{\rm Kerr}=\epsilon\cdot(\slashed{\nabla}_{e_4^{\rm as}}\,\overset{(1)}{\slashed{\eta}}-\overset{(1)}{\slashed{\Gamma}}{}_4\cdot\eta^{\rm as}_{\rm Kerr}+\overset{(1)}{\slashed{\Pi}}\cdot(\nabla_{e_4^{\rm as}}\eta^{\rm as}_{\rm Kerr}))+\mathcal{O}(\epsilon^2)\,,$$

[35] The connection ∇ is the Levi-Civita connection with respect to the Kerr metric $g_{a,M}$ on $\mathcal{M}^+$, while $\boldsymbol{\nabla}(\epsilon)$ is the Levi-Civita connection with respect to the metric $\boldsymbol{g}(\epsilon)$ on $\mathcal{M}^+$. The pulled-back connection $\phi^*_\epsilon\,\boldsymbol{\nabla}(\epsilon)$ is a connection on $(\mathcal{M}^+,g_{a,M})$. The linearised Christoffel symbols are *tensors* because they arise as the difference of two connections on $(\mathcal{M}^+,g_{a,M})$.

where we exploit the identity (3.48) of our gauge.[36]

It is also worth noticing how the linearisation of the covariant derivatives of scalar functions is performed. For example, we have the linearisation of the scalar function

$$\Pi\phi_\epsilon^*(\boldsymbol{\nabla\!\!\!/}_{\boldsymbol{e_4}}(\mathbf{tr}\boldsymbol{\chi}_\circ)(\epsilon)) - \nabla\!\!\!/_{e_4^{\rm as}}(\mathrm{tr}\chi_\circ)^{\rm as}_{\rm Kerr} = \epsilon\cdot\nabla\!\!\!/_{e_4^{\rm as}}(\overset{(1)}{\mathrm{tr}\chi_\circ}) + \mathcal{O}(\epsilon^2)$$

in the Raychaudhuri equation

$$\begin{aligned}\boldsymbol{\nabla\!\!\!/}_{\mathbf{4}}(\mathbf{tr}\boldsymbol{\chi}_\circ)(\epsilon) + \frac{1}{2}(\mathbf{tr}\boldsymbol{\chi}_\circ)^2(\epsilon) - \hat{\boldsymbol{\omega}}\,\mathbf{tr}\boldsymbol{\chi}_\circ(\epsilon) = &-(\hat{\boldsymbol{\chi}}_\circ,\hat{\boldsymbol{\chi}}_\circ)(\epsilon) + \frac{1}{2}(\boldsymbol{\not\in}\cdot\boldsymbol{\chi}_\square)^2(\epsilon)\\ &+2\,(\mathbf{d\!\!\!/iv}\,\boldsymbol{Y})(\epsilon) + 2(\boldsymbol{\eta}+\underline{\boldsymbol{\eta}}+2\boldsymbol{\zeta},\boldsymbol{Y})(\epsilon)\,,\end{aligned}$$

where, again relying on the identity (3.48) of our gauge, one does not see the term

$$\nabla\!\!\!/_{\overset{(1)}{e_4}}(\mathrm{tr}\chi_\circ)^{\rm as}_{\rm Kerr}$$

in the linearisation. The linearisation of the angular derivatives of scalar functions has to be understood as the linearisation of a $\mathfrak{D}_{\mathcal{N}_{\rm as}}$ one-tensor and reads

$$\Pi\phi_\epsilon^*(\boldsymbol{\nabla\!\!\!/}(\mathbf{tr}\boldsymbol{\chi}_\circ)(\epsilon)) - \nabla\!\!\!/(\mathrm{tr}\chi_\circ)^{\rm as}_{\rm Kerr} = \epsilon\cdot(\nabla\!\!\!/(\overset{(1)}{\mathrm{tr}\chi_\circ}) + \overset{(1)}{\Pi\!\!\!/}\cdot(\nabla(\mathrm{tr}\chi_\circ))) + \mathcal{O}(\epsilon^2)\,.$$

This term appears in the Codazzi equation

$$\begin{aligned}\mathbf{d\!\!\!/iv}\;\hat{\boldsymbol{\chi}}_\circ(\epsilon) = &-\frac{1}{2}\boldsymbol{\not\in}^{\sharp_2}\cdot\boldsymbol{\nabla\!\!\!/}(\boldsymbol{\not\in}\cdot\boldsymbol{\chi}_\square)(\epsilon) - \hat{\boldsymbol{\chi}}^\sharp_\circ\cdot\boldsymbol{\zeta}(\epsilon) - \frac{1}{2}(\boldsymbol{\not\in}\cdot\boldsymbol{\chi}_\square)^\star\boldsymbol{\zeta}(\epsilon) + \frac{1}{2}(\mathbf{tr}\,\boldsymbol{\chi}_\circ)\boldsymbol{\zeta}(\epsilon) + \frac{1}{2}\boldsymbol{\nabla\!\!\!/}(\mathbf{tr}\,\boldsymbol{\chi}_\circ)(\epsilon)\\ &-(\boldsymbol{\not\in}\cdot\boldsymbol{\chi}_\square)^\star\boldsymbol{Y}(\epsilon) - (\boldsymbol{\not\in}\cdot\boldsymbol{\chi}_\square)^\star\boldsymbol{\eta}(\epsilon) - \boldsymbol{\beta}(\epsilon)\,.\end{aligned}$$

To conclude the section, we remark that the presence of the *linearised Christoffel symbols* in the linearisation formulae for the covariant derivatives is a manifestation of the fact that some of the connection coefficients of the Kerr metric are non-vanishing covariant tensors. These terms would appear in the linearised Einstein equations around Kerr in *any gauge*, including in a double-null gauge. On the other hand, the terms depending on the *linearised projection tensor* represent a difficulty which is *specific to our gauge*. They are connected to the geometric property that the horizontal structure of our gauge is variable, and would not appear in the equations if linearised in a double-null gauge.

The linear system of equations

The system of linearised Einstein equations of Section 3.7.2 exhibits several remarkable properties. In this section of the overview we survey the crucial ones. A more detailed discussion will be included in forthcoming work by the author. However, this section will already be enough

[36] In Section 3.4 we will point out that $\nabla_{e_4^{\rm as}}\eta^{\rm as}_{\rm Kerr}\equiv 0$ along $\mathcal{H}^+$, introducing a further simplification for the formula when one restricts to the future event horizon.

for the experienced reader to appreciate how the new structure of the system can be exploited in the analysis and anticipate that no further major difficulties will need to be addressed.

For this section, as well as throughout Section 3.7, we adopt the de-bolded notation

$$e_I = \boldsymbol{e_I}(0)\,,\ \eta = \boldsymbol{\eta}(0)\,,\ \rho = \boldsymbol{\rho}(0)\ \ldots$$

The de-bolded frame vectors, connection coefficients and curvature components thus coincide, or are relative to, the algebraically special frame of Kerr.

We start by noting that the symmetric traceless $\mathfrak{D}_{\mathcal{N}_{\text{as}}}$ two-tensors

$$\overset{(1)}{\alpha}\,,\ \overset{(1)}{\underline{\alpha}}$$

are *gauge invariant* and satisfy the two linearised Bianchi equations

$$\nabla\!\!\!/_3 \overset{(1)}{\alpha} + \frac{1}{2}(\operatorname{tr}\chi_\circ)\overset{(1)}{\alpha} + 2\,\underline{\hat{\omega}}\,\overset{(1)}{\alpha} + \frac{1}{2}(\not\in\cdot\chi_\square)\overset{(1)}{\alpha} = -2\mathcal{D}\!\!\!/_2^\star\overset{(1)}{\beta} - 3\rho\,\overset{(1)}{\hat{\chi}_\circ} - 3\sigma^\star\,\overset{(1)}{\hat{\chi}_\circ} + (4\eta+\zeta)\hat{\otimes}\overset{(1)}{\beta}\,, \tag{3.52}$$

$$\nabla\!\!\!/_4 \overset{(1)}{\underline{\alpha}} + \frac{1}{2}(\operatorname{tr}\chi_\circ)\overset{(1)}{\underline{\alpha}} + 2\,\hat{\omega}\,\overset{(1)}{\underline{\alpha}} - \frac{1}{2}(\not\in\cdot\chi_\square)\overset{(1)}{\underline{\alpha}} = 2\mathcal{D}\!\!\!/_2^\star\overset{(1)}{\underline{\beta}} - 3\rho\,\overset{(1)}{\hat{\underline{\chi}}_\circ} + 3\sigma^\star\,\overset{(1)}{\hat{\underline{\chi}}_\circ} - (4\,\underline{\eta} - \zeta)\hat{\otimes}\overset{(1)}{\underline{\beta}}\,. \tag{3.53}$$

After suitably commuting (3.52) and (3.53) with $\nabla\!\!\!/_4$ and $\nabla\!\!\!/_3$ respectively, one can show that $\overset{(1)}{\alpha}$ and $\overset{(1)}{\underline{\alpha}}$ satisfy two *decoupled spin* ± 2 *Teukolsky equations* on $(\mathcal{M}^+, g_{a,M})$. See [18].

The decoupling of the gauge invariant quantities $\overset{(1)}{\alpha}$ and $\overset{(1)}{\underline{\alpha}}$ arises as a direct consequence of the gauge condition

$$\boldsymbol{\mathcal{N}}(0) \equiv \mathcal{N}_{\text{as}}\,.$$

In the language of the introduction to this chapter, $\boldsymbol{\mathcal{N}}(0) \equiv \mathcal{N}_{as}$ *is the necessary algebraic property of the gauge to observe the decoupling and, at the same time, the property that forces the gauge to be non-integrable.* The reason why a double-null gauge does not allow for the desired decoupling is precisely the property $\mathcal{N}_{\text{as}} \not\equiv \widehat{\mathcal{N}}_{\text{ad}}$ pointed out in Section 3.2.2.

As already mentioned in the introduction, works [18, 103] ensure integrated decay for general solutions to the Teukolsky equation on $(\mathcal{M}^+, g_{a,M})$, and can be applied to establish *integrated decay for the gauge invariant quantities* of our system. Note that this resolves what we referred to as part (i) of our proof of linear stability in the *full sub-extremal range* $|a| < M$.

Part (ii) of our proof of linear stability deals with the *gauge dependent* quantities in the system. The symmetric traceless $\mathfrak{D}_{\mathcal{N}_{\text{as}}}$ two-tensor

$$\overset{(1)}{\hat{\chi}_\circ} \tag{3.54}$$

satisfies the transport equation

$$\nabla\!\!\!/_4 \overset{(1)}{\hat{\chi}_\circ} + (\operatorname{tr}\chi_\circ)\overset{(1)}{\hat{\chi}_\circ} - \hat{\omega}\,\overset{(1)}{\hat{\chi}_\circ} = -\overset{(1)}{\alpha} \tag{3.55}$$

on $(\mathcal{M}^+, g_{a,M})$. We recall that the $\mathfrak{D}_{\mathcal{N}_{\text{as}}}$ one-tensor

$$\overset{(1)}{Y}$$

vanishes indentically on $(\mathcal{M}^+, g_{a,M})$ and thus does not appear in the equation (3.55).[37] Since we can already control the right hand side of (3.55), the tensor (3.54) is the only quantity in the equation that, in order to prove stability for the full system, one needs to estimate.

An important observation is that the equation (3.55) is *blue-shifted*, namely the coefficient of the zero order term in the transport equation is *negative* in a region close to the future event horizon $\mathcal{H}^+$. Such a blue-shift term should be seen as a potential *obstruction* to proving decay for (3.54). Nonetheless, as we shall prove in forthcoming work, one can establish *integrated decay for* (3.54).[38]

The scalar functions

$$(\text{tr}\overset{(1)}{\chi_{\bigcirc}}) \, , \, (\not{\epsilon} \cdot \overset{(1)}{\chi_{\square}}) \tag{3.56}$$

satisfy the *decoupled*, *homogeneous system* of transport equations

$$\nabla\!\!\!\!/_4(\text{tr}\overset{(1)}{\chi_{\bigcirc}}) + (\text{tr}\chi_{\bigcirc})(\text{tr}\overset{(1)}{\chi_{\bigcirc}}) - \hat{\omega}(\text{tr}\overset{(1)}{\chi_{\bigcirc}}) = (\not{\epsilon} \cdot \chi_{\square})(\not{\epsilon} \cdot \overset{(1)}{\chi_{\square}}) \, , \tag{3.57}$$

$$\nabla\!\!\!\!/_4(\not{\epsilon} \cdot \overset{(1)}{\chi_{\square}}) + (\text{tr}\chi_{\bigcirc})(\not{\epsilon} \cdot \overset{(1)}{\chi_{\square}}) - \hat{\omega}(\not{\epsilon} \cdot \overset{(1)}{\chi_{\square}}) = -(\not{\epsilon} \cdot \chi_{\square})(\text{tr}\overset{(1)}{\chi_{\bigcirc}}) \tag{3.58}$$

on $(\mathcal{M}^+, g_{a,M})$. Recall that the scalar function

$$\overset{(1)}{\hat{\omega}}$$

vanishes indentically on $(\mathcal{M}^+, g_{a,M})$ and thus does not appear in the equations (3.57)-(3.58). Similarly, in view of the gauge condition (3.27), no terms of the form

$$\nabla\!\!\!\!/_{\overset{(1)}{e}_4}(\text{tr}\chi_{\bigcirc}) \, , \, \nabla\!\!\!\!/_{\overset{(1)}{e}_4}(\not{\epsilon} \cdot \chi_{\square})$$

appear in the equations. As for the equation (3.55), both the equations (3.57)-(3.58) are *blue-shifted*.[39]

[37]We note that the equation (3.55) has the same form of the correspondent equation of [19].

[38]Estimating $\overset{(1)}{\hat{\chi}_{\bigcirc}}$ from (3.55) represents the main, and in some sense most challenging, decay estimate that one has to close in our problem. In fact, such an estimate already introduces all the key technical difficulties of proving decay for the full system via our scheme.

[39]In forthcoming work, we will normalise the gauge such that any solution to the linearised system obeys

$$(\text{tr}\overset{(1)}{\chi_{\bigcirc}}) \equiv (\not{\epsilon} \cdot \overset{(1)}{\chi_{\square}}) \equiv 0 \tag{3.59}$$

along the whole initial data hypersurface. The homogeneity of the equations (3.57)-(3.58) will immediately imply that condition (3.59) *holds everywhere on* $(\mathcal{M}^+, g_{a,M})$.

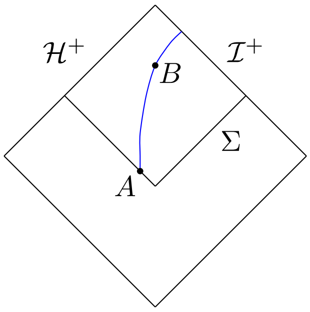

Figure 3.5: Penrose diagram of $(\mathcal{M}^+, g_{a,M})$. The initial data for the linearised system are prescribed along the hypersurface Σ. One can integrate the equation (3.61) in the τ variable from the point $A = (\tau_A, s_A, \vartheta_A, \psi_A)$ to the point $B = (\tau_B, s_A, \vartheta_A, \psi_A)$ along an integral curve of e_4. In general, via the forward integration of a transport equation from the initial data hypersurface, one can only prove *boundedness* for the solution in terms of the initial data. However, the redshift term of (3.61) crucially acts as a damping term and allows to prove *decay*.

Finally, we note that the symmetric traceless $\mathfrak{D}_{\mathcal{N}_{\text{as}}}$ two-tensor

$$\overset{(1)}{\underline{\hat{\chi}}}_{\circ} \tag{3.60}$$

satisfies the transport equation

$$\begin{aligned}\not\nabla_4 \overset{(1)}{\underline{\hat{\chi}}}_{\circ} + \frac{1}{2}(\text{tr}\chi_{\circ}) \overset{(1)}{\underline{\hat{\chi}}}_{\circ} + \hat{\omega}\, \overset{(1)}{\underline{\hat{\chi}}}_{\circ} = & -\frac{1}{2}(\text{tr}\,\underline{\chi}_{\circ}) \overset{(1)}{\hat{\chi}}_{\circ} + \frac{1}{2}(\not{\epsilon} \cdot \underline{\chi}_{\square})^{\star} \overset{(1)}{\hat{\chi}}_{\circ} - \frac{1}{2}(\not{\epsilon} \cdot \chi_{\square})^{\star} \overset{(1)}{\underline{\hat{\chi}}}_{\circ} \\ & - 2\, \widehat{\overset{(1)}{\not\Gamma}} \cdot \underline{\eta} + (\overset{(1)}{\not\Pi} \hat{\otimes} \Pi + \Pi \hat{\otimes} \overset{(1)}{\not\Pi}) \cdot (\nabla \underline{\eta}) - (\not{\text{d}}\text{iv}\, \underline{\eta} + (\underline{\eta}, \underline{\eta}))\, \widehat{\overset{(1)}{\not{g}}}\end{aligned} \tag{3.61}$$

on $(\mathcal{M}^+, g_{a,M})$. We recall that the $\mathfrak{D}_{\mathcal{N}_{\text{as}}}$ one-tensor

$$\overset{(1)}{\not\eta}$$

vanishes indentically on $(\mathcal{M}^+, g_{a,M})$ and thus does not appear in the equation (3.61).[40]

Crucially, equation (3.61) is *red-shifted*, meaning that the zero order term in the transport equation comes with a *positive* sign in a region close to the future event horizon $\mathcal{H}^+$. Relying on previous control on (3.54), and on separate estimates for the linearised metric (and its angular derivatives), the redshift property of the equation will enable us to *integrate* (3.61) *directly in the forward direction along the integral curves of* e_4 *and prove decay for* (3.60)[41] (see Figure 3.5).

The reader should regard the concomitant decoupling of the gauge invariant quantities and convenient form of equations (3.55)-(3.61) ***as the resolution, through our***

[40] This should be contrasted with the correspondent equation of [19], where the linearisation of $\underline{\eta}(\epsilon)$ does appear on the right hand side.

[41] Although discussing the particular gauge normalisation that we adopt in the problem is beyond the scope of this chapter, the reader familiar with the subject should notice that, in contrast with the approach of [19], our proof of decay for (3.60) will *not* require a *future* gauge normalisation. In fact, we will be able to prove the full linear stability result without ever renormalising the gauge to the future. In this sense, when reduced to the Schwarzschild case, our proof provides an alternative approach to that of [19].

nonlinear gauge construction, of the tension between the algebraic and geometric properties of the gauge. In fact, provided the *smallness assumption* $|a| \ll M$, equations (3.55)-(3.61) will allow to control all the quantities (3.54), (3.56), (3.60) and, in turn, to run a scheme to hierarchically control *all* the remaining quantities in the system. This resolves what we referred to as part (ii) of our proof of linear stability *in the slowly rotating regime* $|a| \ll M$.

As a concluding remark, let us note that the key feature of our hierarchy is that it allows to prove decay for the full system by ultimately *relying on transport equations in the e_4-direction only.* In fact, thanks to the remarkably simple form of the linearised transport equations in the outgoing null direction in the new gauge, *the redshift properties of the linear system can be fully exploited.*

3.3 The vacuum Einstein equations in a general gauge

The present section marks the start of the body of the chapter and represents the first of the three self-contained blocks outlined at the beginning of Section 3.2. In this section we address all the geometric preliminaries needed for our problem.

3.3.1 The geometry of non-integrable null frames

Consider a $(3+1)$-dimensional, smooth, orientable Lorentzian manifold $(\boldsymbol{\mathcal{M}}, \boldsymbol{g})$. Let

$$\boldsymbol{\mathcal{N}} = (\boldsymbol{e_1}, \boldsymbol{e_2}, \boldsymbol{e_3}, \boldsymbol{e_4})$$

be a local null frame on $(\boldsymbol{\mathcal{M}}, \boldsymbol{g})$ such that

$$\boldsymbol{g}(\boldsymbol{e_A}, \boldsymbol{e_B}) = \boldsymbol{\delta_{AB}}\,, \qquad \boldsymbol{g}(\boldsymbol{e_A}, \boldsymbol{e_3}) = \boldsymbol{g}(\boldsymbol{e_A}, \boldsymbol{e_4}) = 0\,, \tag{3.62}$$

$$\boldsymbol{g}(\boldsymbol{e_3}, \boldsymbol{e_3}) = \boldsymbol{g}(\boldsymbol{e_4}, \boldsymbol{e_4}) = 0\,, \qquad \boldsymbol{g}(\boldsymbol{e_3}, \boldsymbol{e_4}) = -2 \tag{3.63}$$

on $\boldsymbol{\mathcal{M}}$, with $\boldsymbol{A}, \boldsymbol{B} = \{\boldsymbol{1}, \boldsymbol{2}\}$. The null frame $\boldsymbol{\mathcal{N}}$ is *not* assumed to be *integrable*, meaning that at least one of the relations

$$\boldsymbol{g}([\boldsymbol{e_A}, \boldsymbol{e_B}], \boldsymbol{e_4}) \neq 0\,, \qquad \boldsymbol{g}([\boldsymbol{e_A}, \boldsymbol{e_B}], \boldsymbol{e_3}) \neq 0$$

holds on $\boldsymbol{\mathcal{M}}$.

For $k \in \mathbb{N}, k \geq 1$, we define the *horizontal tensor bundle*[42]

$$(\underline{T}^0_k \boldsymbol{\mathcal{M}})(\boldsymbol{\mathcal{N}}) := \left\{(p, \theta_p) \in T^0_k \boldsymbol{\mathcal{M}} \,|\, \theta_p(\ldots, \boldsymbol{e_3}(p), \ldots) = \theta_p(\ldots, \boldsymbol{e_4}(p), \ldots) = 0\right\}\,,$$

where $T^0_k \boldsymbol{\mathcal{M}}$ is the tensor bundle of k-covariant tensors on $\boldsymbol{\mathcal{M}}$. The tensor bundle $(\underline{T}^0_k \boldsymbol{\mathcal{M}})(\boldsymbol{\mathcal{N}})$ is a vector sub-bundle of the tensor bundle $T^0_k \boldsymbol{\mathcal{M}}$. We define $(\underline{T}^0_0 \boldsymbol{\mathcal{M}})(\boldsymbol{\mathcal{N}}) := T^0_0 \boldsymbol{\mathcal{M}} = \boldsymbol{\mathcal{M}} \times \mathbb{R}$.

[42] An analogous notion of tensor bundle is formulated in Section 2.2 of [57].

We will denote by $\Gamma((\underline{T}^0_k\boldsymbol{\mathcal{M}})(\boldsymbol{\mathcal{N}}))$ a smooth section of the tensor bundle $(\underline{T}^0_k\boldsymbol{\mathcal{M}})(\boldsymbol{\mathcal{N}})$ for any $k \geq 0$, with $\Gamma((\underline{T}^0_0\boldsymbol{\mathcal{M}})(\boldsymbol{\mathcal{N}})) = C^\infty(\boldsymbol{\mathcal{M}})$. Such a smooth section will be called *horizontal tensor field* over $\boldsymbol{\mathcal{M}}$.

We define the vector bundle

$$(\underline{T}^1_0\boldsymbol{\mathcal{M}})(\boldsymbol{\mathcal{N}}) := \left\{(p, X_p) \in T^1_0\boldsymbol{\mathcal{M}} \,|\, X_p \in \mathrm{span}_p\left\{\boldsymbol{e_1}, \boldsymbol{e_2}\right\}\right\}$$

on $\boldsymbol{\mathcal{M}}$. This is a sub-bundle of the tangent bundle $T^1_0\boldsymbol{\mathcal{M}}$, or a *distribution*, and will be denoted by

$$\mathfrak{D}_{\boldsymbol{\mathcal{N}}} := (\underline{T}^1_0\boldsymbol{\mathcal{M}})(\boldsymbol{\mathcal{N}}) \,.$$

Note that $\mathfrak{D}_{\boldsymbol{\mathcal{N}}}$ is an integrable distribution if and only if $\boldsymbol{\mathcal{N}}$ is an integrable frame. The integrability of $\boldsymbol{\mathcal{N}}$ is therefore[43] a necessary condition for $(\underline{T}^1_0\boldsymbol{\mathcal{M}})(\boldsymbol{\mathcal{N}})$ to be the tangent bundle to the leaves of a foliation.

We define $\mathfrak{D}_{\boldsymbol{\mathcal{N}}}$ *k-tensor fields* over $\boldsymbol{\mathcal{M}}$ as the restriction of tensor fields $\boldsymbol{\xi} \in \Gamma((\underline{T}^0_k\boldsymbol{\mathcal{M}})(\boldsymbol{\mathcal{N}}))$ to $\mathfrak{D}_{\boldsymbol{\mathcal{N}}}$. A $\mathfrak{D}_{\boldsymbol{\mathcal{N}}}$ *vector field* over $\boldsymbol{\mathcal{M}}$ is a vector field $\boldsymbol{X} \in \Gamma((\underline{T}^1_0\boldsymbol{\mathcal{M}})(\boldsymbol{\mathcal{N}}))$.

Consider the Levi-Civita connection $\boldsymbol{\nabla}$ on $(\boldsymbol{\mathcal{M}}, \boldsymbol{g})$. We define the connection coefficients relative to the null frame $\boldsymbol{\mathcal{N}}$ as the $\mathfrak{D}_{\boldsymbol{\mathcal{N}}}$ tensor fields

$$\begin{gathered}\hat{\boldsymbol{\omega}}, \underline{\hat{\boldsymbol{\omega}}},\\ \boldsymbol{\eta}, \underline{\boldsymbol{\eta}}, \boldsymbol{Y}, \underline{\boldsymbol{Y}}, \boldsymbol{\zeta},\\ \boldsymbol{\chi}, \underline{\boldsymbol{\chi}}\end{gathered}$$

over $\boldsymbol{\mathcal{M}}$ such that, with notation $\boldsymbol{\xi}(\boldsymbol{e_I}, \boldsymbol{e_J}, \ldots, \boldsymbol{e_K}) = \boldsymbol{\xi_{IJ\ldots K}}$, one has

$$\begin{aligned}
\boldsymbol{\chi_{AB}} &= \boldsymbol{g}(\boldsymbol{\nabla_A e_4}, \boldsymbol{e_B})\,, & \underline{\boldsymbol{\chi}}\boldsymbol{_{AB}} &= \boldsymbol{g}(\boldsymbol{\nabla_A e_3}, \boldsymbol{e_B})\,,\\
\boldsymbol{\eta_A} &= \frac{1}{2}\,\boldsymbol{g}(\boldsymbol{\nabla_3 e_4}, \boldsymbol{e_A})\,, & \underline{\boldsymbol{\eta}}\boldsymbol{_A} &= \frac{1}{2}\,\boldsymbol{g}(\boldsymbol{\nabla_4 e_3}, \boldsymbol{e_A})\,,\\
\boldsymbol{Y_A} &= \frac{1}{2}\,\boldsymbol{g}(\boldsymbol{\nabla_4 e_4}, \boldsymbol{e_A})\,, & \underline{\boldsymbol{Y}}\boldsymbol{_A} &= \frac{1}{2}\,\boldsymbol{g}(\boldsymbol{\nabla_3 e_3}, \boldsymbol{e_A})\,,\\
\hat{\boldsymbol{\omega}} &= \frac{1}{2}\,\boldsymbol{g}(\boldsymbol{\nabla_4 e_3}, \boldsymbol{e_4})\,, & \underline{\hat{\boldsymbol{\omega}}} &= \frac{1}{2}\,\boldsymbol{g}(\boldsymbol{\nabla_3 e_4}, \boldsymbol{e_3})\,,\\
\boldsymbol{\zeta_A} &= \frac{1}{2}\,\boldsymbol{g}(\boldsymbol{\nabla_A e_4}, \boldsymbol{e_3})\,.
\end{aligned}$$

These tensors naturally extend to horizontal tensors[44]

$$\begin{gathered}\hat{\boldsymbol{\omega}}, \underline{\hat{\boldsymbol{\omega}}} \in C^\infty(\boldsymbol{\mathcal{M}})\,,\\ \boldsymbol{\eta}, \underline{\boldsymbol{\eta}}, \boldsymbol{Y}, \underline{\boldsymbol{Y}}, \boldsymbol{\zeta} \in \Gamma((\underline{T}^0_1\boldsymbol{\mathcal{M}})(\boldsymbol{\mathcal{N}}))\,,\\ \boldsymbol{\chi}, \underline{\boldsymbol{\chi}} \in \Gamma((\underline{T}^0_2\boldsymbol{\mathcal{M}})(\boldsymbol{\mathcal{N}}))\end{gathered}$$

[43]We implicitly rely on the Frobenius theorem.
[44]With a slight abuse of notation, we denote the $\mathfrak{D}_{\boldsymbol{\mathcal{N}}}$ tensor and its natural extension by the same symbol.

acting on $T^1_0\boldsymbol{\mathcal{M}}$, which, by definition, satisfy

$$\begin{aligned}
\boldsymbol{\chi}(\boldsymbol{e_3},\cdot) = \boldsymbol{\chi}(\boldsymbol{e_4},\cdot) = \boldsymbol{\chi}(\cdot,\boldsymbol{e_3}) = \boldsymbol{\chi}(\cdot,\boldsymbol{e_4}) = 0\,, &\quad \underline{\boldsymbol{\chi}}(\boldsymbol{e_3},\cdot) = \underline{\boldsymbol{\chi}}(\boldsymbol{e_4},\cdot) = \underline{\boldsymbol{\chi}}(\cdot,\boldsymbol{e_3}) = \underline{\boldsymbol{\chi}}(\cdot,\boldsymbol{e_4}) = 0\,,\\
\boldsymbol{\eta_3} = \boldsymbol{\eta_4} = 0\,, &\quad \underline{\boldsymbol{\eta}}_{\boldsymbol{3}} = \underline{\boldsymbol{\eta}}_{\boldsymbol{4}} = 0\,,\\
\boldsymbol{Y_3} = \boldsymbol{Y_4} = 0\,, &\quad \underline{\boldsymbol{Y}}_{\boldsymbol{3}} = \underline{\boldsymbol{Y}}_{\boldsymbol{4}} = 0\,,\\
\boldsymbol{\zeta_3} = \boldsymbol{\zeta_4} = 0\,. &
\end{aligned}$$

Note that the non-integrability of $\boldsymbol{\mathcal{N}}$ implies that at least one of the relations

$$\boldsymbol{\chi_{AB}} \neq \boldsymbol{\chi_{BA}}\,, \qquad \underline{\boldsymbol{\chi}}_{\boldsymbol{AB}} \neq \underline{\boldsymbol{\chi}}_{\boldsymbol{BA}}$$

holds on $\boldsymbol{\mathcal{M}}$. In particular,

$$\begin{aligned}
\boldsymbol{g}([\boldsymbol{e_A},\boldsymbol{e_B}],\boldsymbol{e_4}) \neq 0 &\implies \boldsymbol{\chi_{AB}} \neq \boldsymbol{\chi_{BA}}\,,\\
\boldsymbol{g}([\boldsymbol{e_A},\boldsymbol{e_B}],\boldsymbol{e_3}) \neq 0 &\implies \underline{\boldsymbol{\chi}}_{\boldsymbol{AB}} \neq \underline{\boldsymbol{\chi}}_{\boldsymbol{BA}}\,.
\end{aligned}$$

Consider the Riemann curvature tensor $\boldsymbol{R}$ on $(\boldsymbol{\mathcal{M}},\boldsymbol{g})$. We define the curvature components relative to the null frame $\boldsymbol{\mathcal{N}}$ as the $\mathfrak{D}_{\boldsymbol{\mathcal{N}}}$ tensor fields

$$\begin{aligned}
&\boldsymbol{\rho},\boldsymbol{\sigma},\\
&\boldsymbol{\beta},\underline{\boldsymbol{\beta}},\\
&\boldsymbol{\alpha},\underline{\boldsymbol{\alpha}},
\end{aligned}$$

over $\boldsymbol{\mathcal{M}}$ such that

$$\begin{aligned}
\boldsymbol{\alpha_{AB}} &= \boldsymbol{R}(\boldsymbol{e_A},\boldsymbol{e_4},\boldsymbol{e_B},\boldsymbol{e_4})\,, & \underline{\boldsymbol{\alpha}}_{\boldsymbol{AB}} &= \boldsymbol{R}(\boldsymbol{e_A},\boldsymbol{e_3},\boldsymbol{e_B},\boldsymbol{e_3})\,,\\
\boldsymbol{\beta_A} &= \frac{1}{2}\,\boldsymbol{R}(\boldsymbol{e_A},\boldsymbol{e_4},\boldsymbol{e_3},\boldsymbol{e_4})\,, & \underline{\boldsymbol{\beta}}_{\boldsymbol{A}} &= \frac{1}{2}\,\boldsymbol{R}(\boldsymbol{e_A},\boldsymbol{e_3},\boldsymbol{e_3},\boldsymbol{e_4})\,,\\
\boldsymbol{\rho} &= \frac{1}{4}\,\boldsymbol{R}(\boldsymbol{e_4},\boldsymbol{e_3},\boldsymbol{e_4},\boldsymbol{e_3})\,, & \boldsymbol{\sigma} &= \frac{1}{4}\,{}^{\star}\boldsymbol{R}(\boldsymbol{e_4},\boldsymbol{e_3},\boldsymbol{e_4},\boldsymbol{e_3})\,,
\end{aligned}$$

where ${}^{\star}\boldsymbol{R}$ is the Hodge dual of $\boldsymbol{R}$ on $(\boldsymbol{\mathcal{M}},\boldsymbol{g})$. These tensors naturally extend to horizontal tensors[45]

$$\begin{aligned}
&\boldsymbol{\rho},\boldsymbol{\sigma} \in C^\infty(\boldsymbol{\mathcal{M}})\,,\\
&\boldsymbol{\beta},\underline{\boldsymbol{\beta}} \in \Gamma((\underline{T}^0_1\boldsymbol{\mathcal{M}})(\boldsymbol{\mathcal{N}}))\,,\\
&\boldsymbol{\alpha},\underline{\boldsymbol{\alpha}} \in \Gamma((\underline{T}^0_2\boldsymbol{\mathcal{M}})(\boldsymbol{\mathcal{N}}))
\end{aligned}$$

acting on $T^1_0\boldsymbol{\mathcal{M}}$, which, by definition, satisfy

$$\begin{aligned}
\boldsymbol{\alpha}(\boldsymbol{e_3},\cdot) = \boldsymbol{\alpha}(\boldsymbol{e_4},\cdot) = 0\,, &\qquad \underline{\boldsymbol{\alpha}}(\boldsymbol{e_3},\cdot) = \underline{\boldsymbol{\alpha}}(\boldsymbol{e_4},\cdot) = 0\,,\\
\boldsymbol{\beta_3} = \boldsymbol{\beta_4} = 0\,, &\qquad \underline{\boldsymbol{\beta}}_{\boldsymbol{3}} = \underline{\boldsymbol{\beta}}_{\boldsymbol{4}} = 0\,.
\end{aligned}$$

[45]With a slight abuse of notation, we denote the $\mathfrak{D}_{\boldsymbol{\mathcal{N}}}$ tensor and its natural extension by the same symbol.

Note that $\boldsymbol{\alpha}$ and $\underline{\boldsymbol{\alpha}}$ are *symmetric*, following from the symmetries of $\boldsymbol{R}$.

Remark 3.1. *It is important that the reader appreciates the difference between* $\mathfrak{D}_{\boldsymbol{\mathcal{N}}}$ tensor fields *and* horizontal tensor fields *over* $\boldsymbol{\mathcal{M}}$*. What follows in this section will be formulated in terms of* $\mathfrak{D}_{\boldsymbol{\mathcal{N}}}$ *tensor fields, but later in the chapter we will appeal to extensions of* $\mathfrak{D}_{\boldsymbol{\mathcal{N}}}$ *tensor fields to horizontal tensor fields.*

We define $\not{\boldsymbol{g}}$ as a symmetric $\mathfrak{D}_{\boldsymbol{\mathcal{N}}}$ two-tensor field over $\boldsymbol{\mathcal{M}}$ such that

$$\not{\boldsymbol{g}}(\boldsymbol{e_A}, \boldsymbol{e_B}) = \boldsymbol{g}(\boldsymbol{e_A}, \boldsymbol{e_B})\,.$$

By definition, $\not{\boldsymbol{g}}$ is a Riemannian metric over $\mathfrak{D}_{\boldsymbol{\mathcal{N}}}$.[46] We define $\not{\boldsymbol{g}}^{-1}$ as the inverse of the metric $\not{\boldsymbol{g}}$ and use the notation $(\not{\boldsymbol{g}}^{-1})^{\boldsymbol{AB}} = \not{\boldsymbol{g}}^{\boldsymbol{AB}}$. One can write

$$\not{\boldsymbol{g}} = \boldsymbol{e^1} \otimes \boldsymbol{e^1} + \boldsymbol{e^2} \otimes \boldsymbol{e^2}$$

and

$$\not{\boldsymbol{g}}^{-1} = \boldsymbol{e_1} \otimes \boldsymbol{e_1} + \boldsymbol{e_2} \otimes \boldsymbol{e_2}$$

over $\boldsymbol{\mathcal{M}}$, with notation $(\boldsymbol{e^1}, \boldsymbol{e^2}, \boldsymbol{e^3}, \boldsymbol{e^4})$ to denote the dual coframe to $\boldsymbol{\mathcal{N}}$ with respect to $\boldsymbol{g}$.

Given the spacetime volume form $\boldsymbol{\varepsilon}$ with respect to $\boldsymbol{g}$, we define $\not{\boldsymbol{\varepsilon}}$ as a $\mathfrak{D}_{\boldsymbol{\mathcal{N}}}$ two-form over $\boldsymbol{\mathcal{M}}$ such that

$$\not{\boldsymbol{\varepsilon}}(\boldsymbol{e_A}, \boldsymbol{e_B}) = \boldsymbol{\varepsilon}(\boldsymbol{e_A}, \boldsymbol{e_B}, \boldsymbol{e_3}, \boldsymbol{e_4})\,,$$

where the spacetime orientation is fixed by $\boldsymbol{\varepsilon}(\boldsymbol{e_1}, \boldsymbol{e_2}, \boldsymbol{e_3}, \boldsymbol{e_4}) = 1$. By definition, $\not{\boldsymbol{\varepsilon}}$ is the Riemannian volume form over $\mathfrak{D}_{\boldsymbol{\mathcal{N}}}$ associated to $\not{\boldsymbol{g}}$. One can write

$$\not{\boldsymbol{\varepsilon}} = \boldsymbol{e^1} \otimes \boldsymbol{e^2} - \boldsymbol{e^2} \otimes \boldsymbol{e^1}$$

over $\boldsymbol{\mathcal{M}}$.

One can extend $\not{\boldsymbol{g}}$ and $\not{\boldsymbol{\varepsilon}}$ to horizontal two-tensors in the natural way. We will refer to the *extension* of $\not{\boldsymbol{g}}^{-1}$ as the *projection tensor* and denote it by $\boldsymbol{\Pi}_{\boldsymbol{\mathcal{N}}}$, with

$$\boldsymbol{\Pi}_{\boldsymbol{\mathcal{N}}} = \boldsymbol{g}^{-1} + \frac{1}{2}\,\boldsymbol{e_3} \otimes \boldsymbol{e_4} + \frac{1}{2}\,\boldsymbol{e_4} \otimes \boldsymbol{e_3}\ .$$

Remark 3.2. *The notion of* S-tensors *of Christodoulou [13] relies on identifying the integral manifold of* $\mathfrak{D}_{\boldsymbol{\mathcal{N}}}$ *with the spheres of a local double-null foliation of the spacetime. If* $\mathfrak{D}_{\boldsymbol{\mathcal{N}}}$ *is non-integrable, one cannot understand our* $\mathfrak{D}_{\boldsymbol{\mathcal{N}}}$ *tensors as tensors intrinsic to some two-dimensional manifold (possibly foliating the spacetime). In particular, the reader should not regard our metric* $\not{\boldsymbol{g}}$ *as a metric induced by* $\boldsymbol{g}$ *on some smooth Riemannian sub-manifold of* $\boldsymbol{\mathcal{M}}$*.*

[46] By a metric on a vector bundle we mean an assignment $p \mapsto \boldsymbol{g}_p(\cdot,\cdot)$, $p \in \boldsymbol{\mathcal{M}}$, where $\boldsymbol{g}_p(\cdot,\cdot)$ is an inner product between elements of the fibre at p and varies smoothly with p.

Given a $\mathfrak{D}_{\mathcal{N}}$ one-tensor $\boldsymbol{\xi}$ and a $\mathfrak{D}_{\mathcal{N}}$ two-tensor $\boldsymbol{\theta}$, we introduce the notation $\boldsymbol{\xi}^{\sharp}, \boldsymbol{\theta}^{\sharp_1}, \boldsymbol{\theta}^{\sharp_2}$ such that

$$\begin{aligned}
\boldsymbol{\xi}^{\sharp A} &= \not{\boldsymbol{g}}^{AB}\boldsymbol{\xi}_B \,, \\
\boldsymbol{\theta}^{\sharp_1}{}^{B}_{A} &= \not{\boldsymbol{g}}^{BC}\boldsymbol{\theta}_{CA} \,, \\
\boldsymbol{\theta}^{\sharp_2}{}^{B}_{A} &= \not{\boldsymbol{g}}^{BC}\boldsymbol{\theta}_{AC} \,.
\end{aligned}$$

This notation allows to keep track of the position of the indices when we write expressions in tensorial form. Note that, if $\boldsymbol{\theta}$ is symmetric, then $\boldsymbol{\theta}^{\sharp_1} = \boldsymbol{\theta}^{\sharp_2}$ and the notation can be simply replaced by $\boldsymbol{\theta}^{\sharp}$. We define the contraction

$$\not{\boldsymbol{\epsilon}} \cdot \boldsymbol{\theta} = \not{\boldsymbol{\epsilon}}^{AB}\boldsymbol{\theta}_{AB}$$

and the duality relations

$$ {}^{\star}\boldsymbol{\xi}_A = \not{\boldsymbol{\epsilon}}^{\sharp_2}{}^{B}_{A}\boldsymbol{\xi}_B \,, \qquad\qquad {}^{\star}\boldsymbol{\theta}_{AB} = \not{\boldsymbol{\epsilon}}^{\sharp_2}{}^{C}_{A}\boldsymbol{\theta}_{BC} \,, $$

with, again, the indices in this precise order. By definition, we have ${}^{\star\star}\boldsymbol{\xi}_A = -\boldsymbol{\xi}_A$.

By definition of the connection coefficients, one can decompose the spacetime covariant derivatives of the frame vectors as follows

$$\begin{aligned}
\boldsymbol{\nabla}_A \boldsymbol{e}_3 &= \boldsymbol{\chi}^{\sharp_2}{}^{B}_{A}\boldsymbol{e}_B + \boldsymbol{\zeta}_A\boldsymbol{e}_3 \,, & \boldsymbol{\nabla}_A \boldsymbol{e}_4 &= \boldsymbol{\chi}^{\sharp_2}{}^{B}_{A}\boldsymbol{e}_B - \boldsymbol{\zeta}_A\boldsymbol{e}_4 \,, \\
\boldsymbol{\nabla}_3 \boldsymbol{e}_3 &= 2\underline{\boldsymbol{Y}}^A\,\boldsymbol{e}_A + \underline{\hat{\boldsymbol{\omega}}}\,\boldsymbol{e}_3 \,, & \boldsymbol{\nabla}_3 \boldsymbol{e}_4 &= 2\boldsymbol{\eta}^A\boldsymbol{e}_A - \underline{\hat{\boldsymbol{\omega}}}\,\boldsymbol{e}_4 \,, \\
\boldsymbol{\nabla}_4 \boldsymbol{e}_3 &= 2\underline{\boldsymbol{\eta}}^A\,\boldsymbol{e}_A - \hat{\boldsymbol{\omega}}\,\boldsymbol{e}_3 \,, & \boldsymbol{\nabla}_4 \boldsymbol{e}_4 &= 2\boldsymbol{Y}^A\boldsymbol{e}_A + \hat{\boldsymbol{\omega}}\,\boldsymbol{e}_4 \,,
\end{aligned}$$

$$\begin{aligned}
\boldsymbol{\nabla}_B \boldsymbol{e}_A &= \boldsymbol{\Gamma}^C_{BA}\boldsymbol{e}_C + \frac{1}{2}\underline{\boldsymbol{\chi}}_{BA}\boldsymbol{e}_3 + \frac{1}{2}\boldsymbol{\chi}_{BA}\,\boldsymbol{e}_4 \,, \\
\boldsymbol{\nabla}_3 \boldsymbol{e}_A &= \boldsymbol{\Gamma}^C_{3A}\boldsymbol{e}_C + \boldsymbol{\eta}_A\boldsymbol{e}_3 + \underline{\boldsymbol{Y}}_A\,\boldsymbol{e}_4 \,, \\
\boldsymbol{\nabla}_4 \boldsymbol{e}_A &= \boldsymbol{\Gamma}^C_{4A}\boldsymbol{e}_C + \boldsymbol{Y}_A\boldsymbol{e}_3 + \underline{\boldsymbol{\eta}}_A\,\boldsymbol{e}_4 \,,
\end{aligned}$$

where we use notation $\boldsymbol{\nabla}_I = \boldsymbol{\nabla}_{\boldsymbol{e}_I}$ and

$$\begin{aligned}
\boldsymbol{\Gamma}^C_{BA} &= \boldsymbol{g}\left(\boldsymbol{\nabla}_B\boldsymbol{e}_A, \boldsymbol{e}_D\right)\boldsymbol{g}^{DC} \,, \\
\boldsymbol{\Gamma}^C_{3A} &= \boldsymbol{g}\left(\boldsymbol{\nabla}_3\boldsymbol{e}_A, \boldsymbol{e}_B\right)\boldsymbol{g}^{BC} \,, \\
\boldsymbol{\Gamma}^C_{4A} &= \boldsymbol{g}\left(\boldsymbol{\nabla}_4\boldsymbol{e}_A, \boldsymbol{e}_B\right)\boldsymbol{g}^{BC} \,.
\end{aligned}$$

The commutators of the frame vectors read

$$[\boldsymbol{e_A}, \boldsymbol{e_B}] = \left(\boldsymbol{\Gamma}^{\boldsymbol{C}}_{\boldsymbol{AB}} - \boldsymbol{\Gamma}^{\boldsymbol{C}}_{\boldsymbol{BA}}\right) \boldsymbol{e_C} + \frac{1}{2}\left(\boldsymbol{\chi_{AB}} - \boldsymbol{\chi_{BA}}\right) \boldsymbol{e_3} + \frac{1}{2}\left(\underline{\boldsymbol{\chi}}_{\boldsymbol{AB}} - \underline{\boldsymbol{\chi}}_{\boldsymbol{BA}}\right) \boldsymbol{e_4}\,, \tag{3.64}$$

$$[\boldsymbol{e_3}, \boldsymbol{e_A}] = \left(\boldsymbol{\Gamma}^{\boldsymbol{B}}_{\boldsymbol{3A}} - \underline{\boldsymbol{\chi}}^{\sharp_2}{}^{\boldsymbol{B}}_{\boldsymbol{A}}\right) \boldsymbol{e_B} + \left(\boldsymbol{\eta_A} - \boldsymbol{\zeta_A}\right) \boldsymbol{e_3} + \underline{\boldsymbol{Y}}_{\boldsymbol{A}}\, \boldsymbol{e_4}\,, \tag{3.65}$$

$$[\boldsymbol{e_4}, \boldsymbol{e_A}] = \left(\boldsymbol{\Gamma}^{\boldsymbol{B}}_{\boldsymbol{4A}} - \boldsymbol{\chi}^{\sharp_2}{}^{\boldsymbol{B}}_{\boldsymbol{A}}\right) \boldsymbol{e_B} + \boldsymbol{Y_A} \boldsymbol{e_3} + \left(\underline{\boldsymbol{\eta}}_{\boldsymbol{A}} + \boldsymbol{\zeta_A}\right) \boldsymbol{e_4}\,, \tag{3.66}$$

$$[\boldsymbol{e_3}, \boldsymbol{e_4}] = \left(2\boldsymbol{\eta}^{\boldsymbol{A}} - 2\underline{\boldsymbol{\eta}}^{\boldsymbol{A}}\right) \boldsymbol{e_A} + \hat{\boldsymbol{\omega}}\, \boldsymbol{e_3} - \underline{\hat{\boldsymbol{\omega}}}\, \boldsymbol{e_4}\,. \tag{3.67}$$

The reader has to be careful with the order of the indices of $\boldsymbol{\chi}$ and $\underline{\boldsymbol{\chi}}$ when using the formulae above.[47] The second and third terms of the commutator $[\boldsymbol{e_A}, \boldsymbol{e_B}]$ encode the non-integrability of the frame $\boldsymbol{\mathcal{N}}$.

3.3.2 Decomposition of $\boldsymbol{\chi}$ and $\underline{\boldsymbol{\chi}}$

We define the *symmetric* $\mathfrak{D}_{\boldsymbol{\mathcal{N}}}$ two-tensors

$$\boldsymbol{\chi}_{\bigcirc}\,,\, \underline{\boldsymbol{\chi}}_{\bigcirc}$$

and the *antisymmetric* $\mathfrak{D}_{\boldsymbol{\mathcal{N}}}$ two-tensors

$$\boldsymbol{\chi}_{\square}\,,\, \underline{\boldsymbol{\chi}}_{\square}$$

such that

$$\boldsymbol{\chi}_{\bigcirc \boldsymbol{AB}} = \frac{1}{2}(\boldsymbol{\chi_{AB}} + \boldsymbol{\chi_{BA}})\,, \qquad \underline{\boldsymbol{\chi}}_{\bigcirc \boldsymbol{AB}} = \frac{1}{2}(\underline{\boldsymbol{\chi}}_{\boldsymbol{AB}} + \underline{\boldsymbol{\chi}}_{\boldsymbol{BA}})\,,$$
$$\boldsymbol{\chi}_{\square \boldsymbol{AB}} = \frac{1}{2}(\boldsymbol{\chi_{AB}} - \boldsymbol{\chi_{BA}})\,, \qquad \underline{\boldsymbol{\chi}}_{\square \boldsymbol{AB}} = \frac{1}{2}(\underline{\boldsymbol{\chi}}_{\boldsymbol{AB}} - \underline{\boldsymbol{\chi}}_{\boldsymbol{BA}})\,.$$

Tensors $\boldsymbol{\chi}_{\bigcirc}, \underline{\boldsymbol{\chi}}_{\bigcirc}$ are the *symmetric parts* of $\boldsymbol{\chi}$ and $\underline{\boldsymbol{\chi}}$ respectively, while tensors $\boldsymbol{\chi}_{\square}, \underline{\boldsymbol{\chi}}_{\square}$ are the *antisymmetric parts* of $\boldsymbol{\chi}$ and $\underline{\boldsymbol{\chi}}$ respectively. One can decompose $\boldsymbol{\chi}$ and $\underline{\boldsymbol{\chi}}$ as

$$\boldsymbol{\chi} = \boldsymbol{\chi}_{\bigcirc} + \boldsymbol{\chi}_{\square}\,, \qquad \underline{\boldsymbol{\chi}} = \underline{\boldsymbol{\chi}}_{\bigcirc} + \underline{\boldsymbol{\chi}}_{\square}\,.$$

We now define the *symmetric traceless* $\mathfrak{D}_{\boldsymbol{\mathcal{N}}}$ two-tensors

$$\hat{\boldsymbol{\chi}}_{\bigcirc}\,,\, \hat{\underline{\boldsymbol{\chi}}}_{\bigcirc}$$

[47] This is, for instance, the case of the formula for $\boldsymbol{\nabla_B e_A}$. The formula

$$\boldsymbol{\nabla_B e_A} = \boldsymbol{\Gamma}^{\boldsymbol{C}}_{\boldsymbol{BA}} \boldsymbol{e_C} + \frac{1}{2}\boldsymbol{\chi_{AB}} \boldsymbol{e_3} + \frac{1}{2}\underline{\boldsymbol{\chi}}_{\boldsymbol{AB}}\, \boldsymbol{e_4}\,,$$

which appears as formula (7.3.1d) in [14], would be incorrect here because it switches the indices of $\boldsymbol{\chi}$ and $\underline{\boldsymbol{\chi}}$.

such that

$$\hat{\boldsymbol{\chi}}_{\bigcirc} = \boldsymbol{\chi}_{\bigcirc} - \frac{1}{2}(\mathbf{tr}\boldsymbol{\chi}_{\bigcirc})\slashed{\boldsymbol{g}}\,, \qquad\qquad \hat{\underline{\boldsymbol{\chi}}}_{\bigcirc} = \underline{\boldsymbol{\chi}}_{\bigcirc} - \frac{1}{2}(\mathbf{tr}\underline{\boldsymbol{\chi}}_{\bigcirc})\slashed{\boldsymbol{g}}\,.$$

Tensors $\hat{\boldsymbol{\chi}}_{\bigcirc}$ and $\hat{\underline{\boldsymbol{\chi}}}_{\bigcirc}$ are the *symmetric traceless parts* of $\boldsymbol{\chi}$ and $\underline{\boldsymbol{\chi}}$ respectively. Note that the traces are taken relative to the inverse metric $\slashed{\boldsymbol{g}}^{-1}$, i.e.

$$\mathbf{tr}\boldsymbol{\chi}_{\bigcirc} = \slashed{\boldsymbol{g}}^{\boldsymbol{AB}}\boldsymbol{\chi}_{\bigcirc\boldsymbol{AB}} \qquad \text{and} \qquad \mathbf{tr}\underline{\boldsymbol{\chi}}_{\bigcirc} = \slashed{\boldsymbol{g}}^{\boldsymbol{AB}}\underline{\boldsymbol{\chi}}_{\bigcirc\boldsymbol{AB}}\,.$$

Since $\mathbf{tr}\boldsymbol{\chi}_{\square} = \mathbf{tr}\underline{\boldsymbol{\chi}}_{\square} = 0$, one has

$$\begin{aligned} \hat{\boldsymbol{\chi}} &= \hat{\boldsymbol{\chi}}_{\bigcirc} + \boldsymbol{\chi}_{\square}\,, & \mathbf{tr}\boldsymbol{\chi} &= \mathbf{tr}\boldsymbol{\chi}_{\bigcirc}\,, \\ \hat{\underline{\boldsymbol{\chi}}} &= \hat{\underline{\boldsymbol{\chi}}}_{\bigcirc} + \underline{\boldsymbol{\chi}}_{\square}\,, & \mathbf{tr}\underline{\boldsymbol{\chi}} &= \mathbf{tr}\underline{\boldsymbol{\chi}}_{\bigcirc}\,, \end{aligned}$$

where $\hat{\boldsymbol{\chi}}$ and $\hat{\underline{\boldsymbol{\chi}}}$ are the *traceless parts* of $\boldsymbol{\chi}$ and $\underline{\boldsymbol{\chi}}$.[48]

We denote the *antitrace* of $\boldsymbol{\chi}$ and $\underline{\boldsymbol{\chi}}$ as

$$(\slashed{\boldsymbol{\epsilon}} \cdot \boldsymbol{\chi}_{\square}) \qquad \text{and} \qquad (\slashed{\boldsymbol{\epsilon}} \cdot \underline{\boldsymbol{\chi}}_{\square})$$

respectively. In fact, since $(\slashed{\boldsymbol{\epsilon}} \cdot \boldsymbol{\chi}_{\bigcirc}) = (\slashed{\boldsymbol{\epsilon}} \cdot \underline{\boldsymbol{\chi}}_{\bigcirc}) = 0$, one has

$$(\slashed{\boldsymbol{\epsilon}} \cdot \boldsymbol{\chi}_{\square}) = (\slashed{\boldsymbol{\epsilon}} \cdot \boldsymbol{\chi})\,, \qquad\qquad (\slashed{\boldsymbol{\epsilon}} \cdot \underline{\boldsymbol{\chi}}_{\square}) = (\slashed{\boldsymbol{\epsilon}} \cdot \underline{\boldsymbol{\chi}})\,.$$

The final decomposition of $\boldsymbol{\chi}$ and $\underline{\boldsymbol{\chi}}$ that we adopt in the chapter is

$$\begin{aligned} \boldsymbol{\chi} &= \hat{\boldsymbol{\chi}}_{\bigcirc} + \frac{1}{2}(\mathbf{tr}\boldsymbol{\chi}_{\bigcirc})\slashed{\boldsymbol{g}} + \frac{1}{2}(\slashed{\boldsymbol{\epsilon}} \cdot \boldsymbol{\chi}_{\square})\slashed{\boldsymbol{\epsilon}}\,, \\ \underline{\boldsymbol{\chi}} &= \hat{\underline{\boldsymbol{\chi}}}_{\bigcirc} + \frac{1}{2}(\mathbf{tr}\underline{\boldsymbol{\chi}}_{\bigcirc})\slashed{\boldsymbol{g}} + \frac{1}{2}(\slashed{\boldsymbol{\epsilon}} \cdot \underline{\boldsymbol{\chi}}_{\square})\slashed{\boldsymbol{\epsilon}}\,. \end{aligned}$$

3.3.3 Products of $\mathfrak{D}_{\mathcal{N}}$ tensors

All tensor products and contractions of $\mathfrak{D}_{\mathcal{N}}$ tensors are defined relative to $\slashed{\boldsymbol{g}}$ and formally coincide with those defined by Christodoulou for S-tensors. However, in this chapter we consider $\mathfrak{D}_{\mathcal{N}}$ two-tensors which are *not* symmetric, in which case the order of the indices in the tensor products becomes relevant.

[48]Note, again, that $\hat{\boldsymbol{\chi}}$ and $\hat{\underline{\boldsymbol{\chi}}}$ are *not* symmetric tensors.

Given $\mathfrak{D}_{\mathcal{N}}$ one-tensors $\boldsymbol{\xi}, \tilde{\boldsymbol{\xi}}$ and (possibly non-symmetric) $\mathfrak{D}_{\mathcal{N}}$ two-tensors $\boldsymbol{\theta}, \tilde{\boldsymbol{\theta}}$, we have

$$
\begin{aligned}
(\boldsymbol{\xi}, \tilde{\boldsymbol{\xi}}) &:= \not{g}^{AB} \xi_A \tilde{\xi}_B \,, \\
(\boldsymbol{\theta}, \tilde{\boldsymbol{\theta}}) &:= \not{g}^{AD} \not{g}^{CB} \theta_{AC} \tilde{\theta}_{BD} \,, \\
\boldsymbol{\xi} \wedge \tilde{\boldsymbol{\xi}} &:= \not{\epsilon}^{AB} \xi_A \tilde{\xi}_B \,, \\
\boldsymbol{\theta} \wedge \tilde{\boldsymbol{\theta}} &:= \not{\epsilon}^{AD} \not{g}^{CB} \theta_{AC} \tilde{\theta}_{BD} \,, \\
(\boldsymbol{\theta} \times \tilde{\boldsymbol{\theta}})_{AB} &:= \theta^{\sharp_2}{}_A^{\ C} \tilde{\theta}_{CB} \,,
\end{aligned}
$$

with the indices in this precise order. We also have

$$
\begin{aligned}
(\boldsymbol{\xi} \otimes \tilde{\boldsymbol{\xi}})_{AB} &:= \xi_A \tilde{\xi}_B \,, \\
(\boldsymbol{\xi} \widehat{\otimes} \tilde{\boldsymbol{\xi}})_{AB} &:= (\boldsymbol{\xi} \otimes \tilde{\boldsymbol{\xi}})_{AB} + (\tilde{\boldsymbol{\xi}} \otimes \boldsymbol{\xi})_{AB} - (\boldsymbol{\xi}, \tilde{\boldsymbol{\xi}}) \not{g}_{AB} \,.
\end{aligned}
$$

The product $(\boldsymbol{\xi} \widehat{\otimes} \tilde{\boldsymbol{\xi}})$ is symmetric and traceless relative to $\not{g}^{-1}$.

3.3.4 Differential operators on $\mathfrak{D}_{\mathcal{N}}$ tensors

In this section we define first and second order differential operators from $\mathfrak{D}_{\mathcal{N}}$ (q, k)-tensors to $\mathfrak{D}_{\mathcal{N}}$ (q, k)-tensors, with $q, k \geq 0$.

Given a vector field $\boldsymbol{X}$ over $\mathcal{M}$, we define the differential operator

$$\not{\nabla}_{\boldsymbol{X}}$$

such that

- $\not{\nabla}_{\boldsymbol{X}} \boldsymbol{f} := \boldsymbol{X}(\boldsymbol{f})$ for any $\boldsymbol{f} \in C^\infty(\mathcal{M})$;
- $\not{\nabla}_{\boldsymbol{X}} \boldsymbol{Y} := \boldsymbol{g}^{BA} \boldsymbol{g}(\nabla_{\boldsymbol{X}} \boldsymbol{Y}, \boldsymbol{e_B}) \boldsymbol{e_A}$ for any $\mathfrak{D}_{\mathcal{N}}$ vector field $\boldsymbol{Y}$, where on the right hand side we have, with a slight abuse of notation, the natural extension[49] of $\boldsymbol{Y}$ to a smooth section of $T\mathcal{M}$;
- $(\not{\nabla}_{\boldsymbol{X}} \boldsymbol{\xi})(\boldsymbol{Y_1}, \ldots, \boldsymbol{Y_k}) := \boldsymbol{X}(\boldsymbol{\xi}(\boldsymbol{Y_1}, \ldots, \boldsymbol{Y_k})) - \boldsymbol{\xi}(\not{\nabla}_{\boldsymbol{X}} \boldsymbol{Y_1}, \ldots, \boldsymbol{Y_k}) - \cdots - \boldsymbol{\xi}(\boldsymbol{Y_1}, \ldots, \not{\nabla}_{\boldsymbol{X}} \boldsymbol{Y_k})$ for any $\mathfrak{D}_{\mathcal{N}}$ k-tensor $\boldsymbol{\xi}$ and $\mathfrak{D}_{\mathcal{N}}$ vector fields $\boldsymbol{Y_1}, \ldots, \boldsymbol{Y_k}$.

We define the linear map

$$\not{\nabla}$$

from $\mathfrak{D}_{\mathcal{N}}$ (q, k)-tensors to $\mathfrak{D}_{\mathcal{N}}$ $(q, k+1)$-tensors such that

- $(\not{\nabla} \boldsymbol{f})(\boldsymbol{X}) := \not{\nabla}_{\boldsymbol{X}} \boldsymbol{f}$ for any $\boldsymbol{f} \in C^\infty(\mathcal{M})$ and $\mathfrak{D}_{\mathcal{N}}$ vector field $\boldsymbol{X}$;
- $(\not{\nabla} \boldsymbol{Y})(\boldsymbol{X}) := \not{\nabla}_{\boldsymbol{X}} \boldsymbol{Y}$ for any $\mathfrak{D}_{\mathcal{N}}$ vector fields $\boldsymbol{X}, \boldsymbol{Y}$;

[49] We extend by imposing $\boldsymbol{g}(\boldsymbol{Y}, \boldsymbol{e_3}) = \boldsymbol{g}(\boldsymbol{Y}, \boldsymbol{e_4}) = 0$ on $\mathcal{M}$.

- $(\nabla\!\!\!/\, \boldsymbol{\xi})(\boldsymbol{X}, \boldsymbol{Y_1}, \ldots, \boldsymbol{Y_k}) := (\nabla\!\!\!/_{\boldsymbol{X}}\, \boldsymbol{\xi})(\boldsymbol{Y_1}, \ldots, \boldsymbol{Y_k})$ for any $\mathfrak{D}_{\boldsymbol{\mathcal{N}}}$ k-tensor $\boldsymbol{\xi}$ and $\mathfrak{D}_{\boldsymbol{\mathcal{N}}}$ vector fields $\boldsymbol{X}, \boldsymbol{Y_1}, \ldots, \boldsymbol{Y_k}$.

It is easy to check that the linear map $\nabla\!\!\!/$ defines a (linear) connection of $\mathfrak{D}_{\boldsymbol{\mathcal{N}}}$ over $\boldsymbol{\mathcal{M}}$.

One can now decompose the spacetime covariant derivatives of the frame vectors as

$$\begin{aligned}
\boldsymbol{\nabla_B e_A} &= \nabla\!\!\!/_{\boldsymbol{B}}\, \boldsymbol{e_A} + \frac{1}{2}\boldsymbol{\chi_{BA} e_3} + \frac{1}{2}\boldsymbol{\underline{\chi}_{BA} e_4}\,, \\
\boldsymbol{\nabla_3 e_A} &= \nabla\!\!\!/_{\boldsymbol{3}}\, \boldsymbol{e_A} + \boldsymbol{\eta_A e_3} + \boldsymbol{\underline{Y}_A}\, \boldsymbol{e_4}\,, \\
\boldsymbol{\nabla_4 e_A} &= \nabla\!\!\!/_{\boldsymbol{4}}\, \boldsymbol{e_A} + \boldsymbol{Y_A e_3} + \boldsymbol{\underline{\eta}_A}\, \boldsymbol{e_4}\,.
\end{aligned}$$

It is easy to check that the connection $\nabla\!\!\!/$ as defined above is compatible with the metric $g\!\!\!/$, i.e. given any vector field $\boldsymbol{X}$, one has

$$\begin{aligned}
\nabla\!\!\!/_{\boldsymbol{X}} g\!\!\!/ &= 0\,, \\
\nabla\!\!\!/_{\boldsymbol{X}} \epsilon\!\!/ &= 0\,.
\end{aligned}$$

For $\mathfrak{D}_{\boldsymbol{\mathcal{N}}}$ one-tensors $\boldsymbol{\xi}$ and $\mathfrak{D}_{\boldsymbol{\mathcal{N}}}$ two-tensors $\boldsymbol{\theta}$, we define the divergence operator as[50]

$$\begin{aligned}
\mathrm{d}\!\!/\mathrm{iv}\, \boldsymbol{\xi} &= g\!\!\!/^{\boldsymbol{AB}} \nabla\!\!\!/_{\boldsymbol{A}} \boldsymbol{\xi_B}\,, \\
(\mathrm{d}\!\!/\mathrm{iv}\, \boldsymbol{\theta})_{\boldsymbol{A}} &= g\!\!\!/^{\boldsymbol{CB}} (\nabla\!\!\!/_{\boldsymbol{C}} \boldsymbol{\theta})_{\boldsymbol{AB}}\,,
\end{aligned}$$

the curl operator as

$$\mathrm{c}\!\!/\mathrm{url}\, \boldsymbol{\xi} = \epsilon\!\!/^{\boldsymbol{AB}} \nabla\!\!\!/_{\boldsymbol{A}} \boldsymbol{\xi_B}$$

and the operator

$$(\mathcal{D}\!\!\!/^{\star}_{\boldsymbol{2}} \boldsymbol{\xi})_{\boldsymbol{AB}} = -\frac{1}{2}\Big((\nabla\!\!\!/\, \boldsymbol{\xi})_{\boldsymbol{AB}} + (\nabla\!\!\!/\, \boldsymbol{\xi})_{\boldsymbol{BA}} - (\mathrm{d}\!\!/\mathrm{iv}\, \boldsymbol{\xi}) g\!\!\!/_{\boldsymbol{AB}} \Big)\,,$$

with the indices in this precise order. The $\mathfrak{D}_{\boldsymbol{\mathcal{N}}}$ two-tensor $\mathcal{D}\!\!\!/^{\star}_{\boldsymbol{2}} \boldsymbol{\xi}$ is symmetric and traceless relative to $g\!\!\!/^{\boldsymbol{-1}}$.

Given $\mathfrak{D}_{\boldsymbol{\mathcal{N}}}$ vector fields $\boldsymbol{X}, \boldsymbol{Y}$, we define the differential operator

$$\nabla\!\!\!/^2_{\boldsymbol{X},\boldsymbol{Y}}$$

such that, for any $\mathfrak{D}_{\boldsymbol{\mathcal{N}}}$ k-tensor $\boldsymbol{\xi}$ and $\mathfrak{D}_{\boldsymbol{\mathcal{N}}}$ vector field $\boldsymbol{Z}$, we have

$$\begin{aligned}
\nabla\!\!\!/^2_{\boldsymbol{X},\boldsymbol{Y}} \boldsymbol{\xi} &:= \nabla\!\!\!/_{\boldsymbol{X}} (\nabla\!\!\!/_{\boldsymbol{Y}} \boldsymbol{\xi}) - \nabla\!\!\!/_{\nabla\!\!\!/_{\boldsymbol{X}} \boldsymbol{Y}} \boldsymbol{\xi}\,, \\
\nabla\!\!\!/^2_{\boldsymbol{X},\boldsymbol{Y}} \boldsymbol{Z} &:= \nabla\!\!\!/_{\boldsymbol{X}} (\nabla\!\!\!/_{\boldsymbol{Y}} \boldsymbol{Z}) - \nabla\!\!\!/_{\nabla\!\!\!/_{\boldsymbol{X}} \boldsymbol{Y}} \boldsymbol{Z}\,.
\end{aligned}$$

[50]The reader familiar with [19] should note that our definition of $\mathrm{d}\!\!/\mathrm{iv}\boldsymbol{\theta}$ contracts the indices in a different order.

We define the curvature tensor

$$\not{\boldsymbol{R}}$$

as the $\mathfrak{D}_{\boldsymbol{\mathcal{N}}}$ four-tensor such that

$$\not{\boldsymbol{R}}(\boldsymbol{W}, \boldsymbol{Z}, \boldsymbol{X}, \boldsymbol{Y}) := \not{\boldsymbol{g}}(\not{\boldsymbol{\nabla}}^2_{\boldsymbol{X},\boldsymbol{Y}} \boldsymbol{Z} - \not{\boldsymbol{\nabla}}^2_{\boldsymbol{Y},\boldsymbol{X}} \boldsymbol{Z}, \boldsymbol{W})$$

for any $\mathfrak{D}_{\boldsymbol{\mathcal{N}}}$ vector fields $\boldsymbol{X}, \boldsymbol{Y}, \boldsymbol{W}, \boldsymbol{Z}$. We define the smooth scalar function

$$\not{\boldsymbol{K}}$$

such that

$$\not{\boldsymbol{K}} := \not{\boldsymbol{g}}^{\boldsymbol{AC}} \not{\boldsymbol{g}}^{\boldsymbol{BD}} \not{\boldsymbol{R}}_{\boldsymbol{ADCB}} \,.$$

Note that, if $\mathfrak{D}_{\boldsymbol{\mathcal{N}}}$ was integrable, then $\not{\boldsymbol{K}}$ would correspond to the Gaussian curvature of the integral manifold of $\mathfrak{D}_{\boldsymbol{\mathcal{N}}}$.

Given a vector field $\boldsymbol{X}$, we define the differential operator

$$\not{\boldsymbol{\mathcal{L}}}_{\boldsymbol{X}}$$

such that, for any $\mathfrak{D}_{\boldsymbol{\mathcal{N}}}$ k-tensor $\boldsymbol{\xi}$, $\mathfrak{D}_{\boldsymbol{\mathcal{N}}}$ vector field $\boldsymbol{Y}$ and scalar function $\boldsymbol{f}$, we have

$$\begin{aligned} \not{\boldsymbol{\mathcal{L}}}_{\boldsymbol{X}} \boldsymbol{f} &= \not{\boldsymbol{\nabla}}_{\boldsymbol{X}} \, \boldsymbol{f} \,, \\ (\not{\boldsymbol{\mathcal{L}}}_{\boldsymbol{X}} \boldsymbol{\xi})(\boldsymbol{Y}) &= (\boldsymbol{\mathcal{L}}_{\boldsymbol{X}} \boldsymbol{\xi})(\boldsymbol{Y}) \,, \end{aligned} \tag{3.68}$$

with $\boldsymbol{\mathcal{L}}_{\boldsymbol{X}} \boldsymbol{\xi}$ the spacetime Lie derivatives of $\boldsymbol{\xi}$ with respect to $\boldsymbol{X}$.[51] For $\mathfrak{D}_{\boldsymbol{\mathcal{N}}}$ one-tensors $\boldsymbol{\xi}$ and $\mathfrak{D}_{\boldsymbol{\mathcal{N}}}$ two-tensors $\boldsymbol{\theta}$, we have relations

$$\not{\boldsymbol{\mathcal{L}}}_{\boldsymbol{4}} \boldsymbol{\xi} = \not{\boldsymbol{\nabla}}_{\boldsymbol{4}} \, \boldsymbol{\xi} + \boldsymbol{\chi}^{\sharp_2} \cdot \boldsymbol{\xi} \,, \tag{3.69}$$

$$\not{\boldsymbol{\mathcal{L}}}_{\boldsymbol{3}} \boldsymbol{\xi} = \not{\boldsymbol{\nabla}}_{\boldsymbol{3}} \, \boldsymbol{\xi} + \underline{\boldsymbol{\chi}}^{\sharp_2} \cdot \boldsymbol{\xi} \,, \tag{3.70}$$

$$\not{\boldsymbol{\mathcal{L}}}_{\boldsymbol{4}} \boldsymbol{\theta} = \not{\boldsymbol{\nabla}}_{\boldsymbol{4}} \, \boldsymbol{\theta} + \boldsymbol{\chi}_{\circ} \times \boldsymbol{\theta} + \boldsymbol{\theta} \times \boldsymbol{\chi}_{\circ} + \boldsymbol{\chi}_{\square} \times \boldsymbol{\theta} - \boldsymbol{\theta} \times \boldsymbol{\chi}_{\square} \,, \tag{3.71}$$

$$\not{\boldsymbol{\mathcal{L}}}_{\boldsymbol{3}} \boldsymbol{\theta} = \not{\boldsymbol{\nabla}}_{\boldsymbol{3}} \, \boldsymbol{\theta} + \underline{\boldsymbol{\chi}}_{\circ} \times \boldsymbol{\theta} + \boldsymbol{\theta} \times \underline{\boldsymbol{\chi}}_{\circ} + \underline{\boldsymbol{\chi}}_{\square} \times \boldsymbol{\theta} - \boldsymbol{\theta} \times \underline{\boldsymbol{\chi}}_{\square} \,. \tag{3.72}$$

[51] As noted for the definition of $\not{\boldsymbol{\nabla}}_{\boldsymbol{X}} \, \boldsymbol{Y}$, the right hand side of (3.68) involves, with a slight abuse of notation, the natural extension of $\boldsymbol{Y}$ to a smooth section of $T\boldsymbol{\mathcal{M}}$.

The commutation formulae

Let $\boldsymbol{\xi}$ be a $\mathfrak{D}_{\mathcal{N}}$ one-tensor and $\boldsymbol{\theta}$ a $\mathfrak{D}_{\mathcal{N}}$ two-tensor. We have the following commutation formulae

$$\begin{aligned}[\boldsymbol{\nabla}\!\!\!/_4, \boldsymbol{\nabla}\!\!\!/]\boldsymbol{\xi} = & -\boldsymbol{\chi} \times (\boldsymbol{\nabla}\!\!\!/\,\boldsymbol{\xi}) + \boldsymbol{Y} \otimes (\boldsymbol{\nabla}\!\!\!/_3\,\boldsymbol{\xi}) + (\underline{\boldsymbol{\eta}} + \boldsymbol{\zeta}) \otimes (\boldsymbol{\nabla}\!\!\!/_4\,\boldsymbol{\xi}) \\ & + (\boldsymbol{Y}, \boldsymbol{\xi})\boldsymbol{\chi} - (\boldsymbol{\chi}^{\sharp_2} \cdot \boldsymbol{\xi}) \otimes \boldsymbol{Y} + (\underline{\boldsymbol{\eta}}, \boldsymbol{\xi})\boldsymbol{\chi} - (\boldsymbol{\chi}^{\sharp_2} \cdot \boldsymbol{\xi}) \otimes \underline{\boldsymbol{\eta}} + ({}^{\star}\boldsymbol{\beta}) \otimes ({}^{\star}\boldsymbol{\xi})\,,\end{aligned} \tag{3.73}$$

$$\begin{aligned}[\boldsymbol{\nabla}\!\!\!/_3, \boldsymbol{\nabla}\!\!\!/]\boldsymbol{\xi} = & -\underline{\boldsymbol{\chi}} \times (\boldsymbol{\nabla}\!\!\!/\,\boldsymbol{\xi}) + \underline{\boldsymbol{Y}} \otimes (\boldsymbol{\nabla}\!\!\!/_4\,\boldsymbol{\xi}) + (\boldsymbol{\eta} - \boldsymbol{\zeta}) \otimes (\boldsymbol{\nabla}\!\!\!/_3\,\boldsymbol{\xi}) \\ & + (\underline{\boldsymbol{Y}}, \boldsymbol{\xi})\underline{\boldsymbol{\chi}} - (\underline{\boldsymbol{\chi}}^{\sharp_2} \cdot \boldsymbol{\xi}) \otimes \underline{\boldsymbol{Y}} + (\boldsymbol{\eta}, \boldsymbol{\xi})\underline{\boldsymbol{\chi}} - (\underline{\boldsymbol{\chi}}^{\sharp_2} \cdot \boldsymbol{\xi}) \otimes \boldsymbol{\eta} - ({}^{\star}\,\underline{\boldsymbol{\beta}}) \otimes ({}^{\star}\boldsymbol{\xi})\,,\end{aligned} \tag{3.74}$$

$$\begin{aligned}[\boldsymbol{\nabla}\!\!\!/_3, \boldsymbol{\nabla}\!\!\!/_4]\boldsymbol{\xi} = & \,\hat{\boldsymbol{\omega}}(\boldsymbol{\nabla}\!\!\!/_3\boldsymbol{\xi}) - \underline{\hat{\boldsymbol{\omega}}}(\boldsymbol{\nabla}\!\!\!/_4\boldsymbol{\xi}) + 2(\boldsymbol{\nabla}\!\!\!/\,\boldsymbol{\xi})^{\sharp_1} \cdot (\boldsymbol{\eta} - \underline{\boldsymbol{\eta}}) \\ & + 2(\boldsymbol{\eta}, \boldsymbol{\xi})\underline{\boldsymbol{\eta}} - 2(\underline{\boldsymbol{\eta}}, \boldsymbol{\xi})\boldsymbol{\eta} + 2(\underline{\boldsymbol{Y}}, \boldsymbol{\xi})\boldsymbol{Y} - 2(\boldsymbol{Y}, \boldsymbol{\xi})\underline{\boldsymbol{Y}} + 2\boldsymbol{\sigma}({}^{\star}\boldsymbol{\xi})\end{aligned} \tag{3.75}$$

and

$$\begin{aligned}([\boldsymbol{\nabla}\!\!\!/_4, \boldsymbol{\nabla}\!\!\!/]\boldsymbol{\theta})_{ABC} = & -\boldsymbol{\chi}_A^{\sharp_2 D}(\boldsymbol{\nabla}\!\!\!/_D\,\boldsymbol{\theta})_{BC} + \boldsymbol{Y}_A(\boldsymbol{\nabla}\!\!\!/_3\,\boldsymbol{\theta})_{BC} + (\underline{\boldsymbol{\eta}}_A + \boldsymbol{\zeta}_A)(\boldsymbol{\nabla}\!\!\!/_4\,\boldsymbol{\theta})_{BC} \\ & + \boldsymbol{\chi}_{AB}\,\boldsymbol{Y}^D\boldsymbol{\theta}_{DC} - \boldsymbol{Y}_B\boldsymbol{\chi}_A^{\sharp_2 D}\boldsymbol{\theta}_{DC} + \boldsymbol{\chi}_{AB}\,\underline{\boldsymbol{\eta}}^D\,\boldsymbol{\theta}_{DC} - \underline{\boldsymbol{\eta}}_B\boldsymbol{\chi}_A^{\sharp_2 D}\boldsymbol{\theta}_{DC} + \boldsymbol{\not\epsilon}_B^{\sharp_2 D}({}^{\star}\boldsymbol{\beta})_A\boldsymbol{\theta}_{DC} \\ & + \boldsymbol{\chi}_{AC}\,\boldsymbol{Y}^D\boldsymbol{\theta}_{BD} - \boldsymbol{Y}_C\boldsymbol{\chi}_A^{\sharp_2 D}\boldsymbol{\theta}_{BD} + \boldsymbol{\chi}_{AC}\,\underline{\boldsymbol{\eta}}^D\,\boldsymbol{\theta}_{BD} - \underline{\boldsymbol{\eta}}_C\boldsymbol{\chi}_A^{\sharp_2 D}\boldsymbol{\theta}_{BD} + \boldsymbol{\not\epsilon}_C^{\sharp_2 D}({}^{\star}\boldsymbol{\beta})_A\boldsymbol{\theta}_{BD}\,,\end{aligned} \tag{3.76}$$

$$\begin{aligned}([\boldsymbol{\nabla}\!\!\!/_3, \boldsymbol{\nabla}\!\!\!/]\boldsymbol{\theta})_{ABC} = & -\underline{\boldsymbol{\chi}}_A^{\sharp_2 D}(\boldsymbol{\nabla}\!\!\!/_D\,\boldsymbol{\theta})_{BC} + \underline{\boldsymbol{Y}}_A(\boldsymbol{\nabla}\!\!\!/_4\,\boldsymbol{\theta})_{BC} + (\boldsymbol{\eta}_A - \boldsymbol{\zeta}_A)(\boldsymbol{\nabla}\!\!\!/_3\,\boldsymbol{\theta})_{BC} \\ & + \underline{\boldsymbol{\chi}}_{AB}\,\underline{\boldsymbol{Y}}^D\,\boldsymbol{\theta}_{DC} - \underline{\boldsymbol{Y}}_B\underline{\boldsymbol{\chi}}_A^{\sharp_2 D}\boldsymbol{\theta}_{DC} + \underline{\boldsymbol{\chi}}_{AB}\,\boldsymbol{\eta}^D\boldsymbol{\theta}_{DC} - \boldsymbol{\eta}_B\underline{\boldsymbol{\chi}}_A^{\sharp_2 D}\boldsymbol{\theta}_{DC} - \boldsymbol{\not\epsilon}_B^{\sharp_2 D}({}^{\star}\,\underline{\boldsymbol{\beta}})_A\boldsymbol{\theta}_{DC} \\ & + \underline{\boldsymbol{\chi}}_{AC}\,\underline{\boldsymbol{Y}}^D\,\boldsymbol{\theta}_{BD} - \underline{\boldsymbol{Y}}_C\underline{\boldsymbol{\chi}}_A^{\sharp_2 D}\boldsymbol{\theta}_{BD} + \underline{\boldsymbol{\chi}}_{AC}\,\boldsymbol{\eta}^D\boldsymbol{\theta}_{BD} - \boldsymbol{\eta}_C\underline{\boldsymbol{\chi}}_A^{\sharp_2 D}\boldsymbol{\theta}_{BD} - \boldsymbol{\not\epsilon}_C^{\sharp_2 D}({}^{\star}\,\underline{\boldsymbol{\beta}})_A\boldsymbol{\theta}_{BD}\,,\end{aligned} \tag{3.77}$$

$$\begin{aligned}([\boldsymbol{\nabla}\!\!\!/_3, \boldsymbol{\nabla}\!\!\!/_4]\boldsymbol{\theta})_{AB} = & \,\hat{\boldsymbol{\omega}}(\boldsymbol{\nabla}\!\!\!/_3\boldsymbol{\theta})_{AB} - \underline{\hat{\boldsymbol{\omega}}}(\boldsymbol{\nabla}\!\!\!/_4\boldsymbol{\theta})_{AB} + 2(\boldsymbol{\nabla}\!\!\!/\,\boldsymbol{\theta})^{\sharp_1}{}^{C}{}_{AB}(\boldsymbol{\eta} - \underline{\boldsymbol{\eta}})_C \\ & + 2\boldsymbol{\eta}^C\boldsymbol{\theta}_{CB}\underline{\boldsymbol{\eta}}_A - 2\underline{\boldsymbol{\eta}}^C\boldsymbol{\theta}_{CB}\boldsymbol{\eta}_A + 2\underline{\boldsymbol{Y}}^C\boldsymbol{\theta}_{CB}\boldsymbol{Y}_A - 2\boldsymbol{Y}^C\boldsymbol{\theta}_{CB}\underline{\boldsymbol{Y}}_A + 2\boldsymbol{\sigma}\boldsymbol{\not\epsilon}_A^{\sharp_2 C}\boldsymbol{\theta}_{CB} \\ & + 2\boldsymbol{\eta}^C\boldsymbol{\theta}_{AC}\underline{\boldsymbol{\eta}}_B - 2\underline{\boldsymbol{\eta}}^C\boldsymbol{\theta}_{AC}\boldsymbol{\eta}_B + 2\underline{\boldsymbol{Y}}^C\boldsymbol{\theta}_{AC}\boldsymbol{Y}_B - 2\boldsymbol{Y}^C\boldsymbol{\theta}_{AC}\underline{\boldsymbol{Y}}_B + 2\boldsymbol{\sigma}\boldsymbol{\not\epsilon}_B^{\sharp_2 C}\boldsymbol{\theta}_{AC}\,.\end{aligned} \tag{3.78}$$

Proof. The commutation formulae are analogous to the ones derived by [110]. Note that, in our case, one has to be careful with the order of the indices of $\boldsymbol{\chi}$ and $\underline{\boldsymbol{\chi}}$ in (3.73), (3.74), (3.76) and (3.77). □

3.3.5 The vacuum Einstein equations

Consider a $(3+1)$-dimensional, smooth Lorentzian manifold $(\boldsymbol{\mathcal{M}}, \boldsymbol{g})$ which is a solution to the vacuum Einstein equations

$$\mathbf{Ric}(\boldsymbol{g}) = 0\,.$$

Then, the connection coefficients and curvature components of $(\boldsymbol{\mathcal{M}}, \boldsymbol{g})$ relative to a null frame $\boldsymbol{\mathcal{N}}$ satisfy a nonlinear system of equations that we present in this section and is derived in Appendix A.[52] Recall that $\boldsymbol{\mathcal{N}}$ is *not* assumed to be integrable.

[52] The system of vacuum Einstein equations relative to a non-integrable null frame has recently appeared, in a similar form, in [46].

The full list of unknowns in the system of equations is

$$\boldsymbol{\not g}\,,\, \boldsymbol{e_4}\,,\, \boldsymbol{e_3}\,,\, \boldsymbol{e_A}\,,$$
$$\hat{\boldsymbol{\omega}}\,,\, \underline{\hat{\boldsymbol{\omega}}}\,,\, \boldsymbol{\eta}\,,\, \underline{\boldsymbol{\eta}}\,,\, \boldsymbol{Y}\,,\, \underline{\boldsymbol{Y}}\,,\, \boldsymbol{\zeta}\,,\, \hat{\boldsymbol{\chi}}_\bigcirc\,,\, \underline{\hat{\boldsymbol{\chi}}}_\bigcirc\,,\, (\mathbf{tr}\boldsymbol{\chi}_\bigcirc)\,,\, (\mathbf{tr}\,\underline{\boldsymbol{\chi}}_\bigcirc)\,,\, (\not\epsilon\cdot\boldsymbol{\chi}_\square)\,,\, (\not\epsilon\cdot\underline{\boldsymbol{\chi}}_\square)\,, \boldsymbol{\Gamma}^{\boldsymbol{B}}_{\boldsymbol{4A}}\,,\, \boldsymbol{\Gamma}^{\boldsymbol{B}}_{\boldsymbol{3A}}\,,\, \boldsymbol{\Gamma}^{\boldsymbol{C}}_{\boldsymbol{AB}}\,,$$
$$\boldsymbol{\rho}\,,\, \boldsymbol{\sigma}\,,\, \boldsymbol{\beta}\,,\, \underline{\boldsymbol{\beta}}\,,\, \boldsymbol{\alpha}\,,\, \underline{\boldsymbol{\alpha}}\,.$$

We define the scalars

$$(\boldsymbol{\Gamma_3}, \boldsymbol{\chi}_\square) := \boldsymbol{\not g}^{\boldsymbol{BD}}\boldsymbol{\chi}_{\square\boldsymbol{CB}}\boldsymbol{\Gamma}^{\boldsymbol{C}}_{\boldsymbol{3D}}\,, \qquad (\boldsymbol{\Gamma_4}, \underline{\boldsymbol{\chi}}_\square) := \boldsymbol{\not g}^{\boldsymbol{BD}}\underline{\boldsymbol{\chi}}_{\square\boldsymbol{CB}}\boldsymbol{\Gamma}^{\boldsymbol{C}}_{\boldsymbol{4D}}\,.$$

Null frame equations

The frame vector fields satisfy the commutators (3.64)-(3.67), which read

$$\boldsymbol{\nabla_A}\,\boldsymbol{e_B} - \boldsymbol{\nabla_B}\,\boldsymbol{e_A} = \left(\boldsymbol{\Gamma}^{\boldsymbol{C}}_{\boldsymbol{AB}} - \boldsymbol{\Gamma}^{\boldsymbol{C}}_{\boldsymbol{BA}}\right)\boldsymbol{e_C} + \boldsymbol{\chi}_{\square\boldsymbol{AB}}\boldsymbol{e_3} + \underline{\boldsymbol{\chi}}_{\square\boldsymbol{AB}}\boldsymbol{e_4}\,, \tag{3.79}$$
$$\boldsymbol{\nabla_3}\,\boldsymbol{e_A} - \boldsymbol{\nabla_A}\,\boldsymbol{e_3} = \left(\boldsymbol{\Gamma}^{\boldsymbol{B}}_{\boldsymbol{3A}} - \underline{\boldsymbol{\chi}}^{\sharp_2}{}^{\boldsymbol{B}}_{\boldsymbol{A}}\right)\boldsymbol{e_B} + (\boldsymbol{\eta_A} - \boldsymbol{\zeta_A})\,\boldsymbol{e_3} + \underline{\boldsymbol{Y}}_{\boldsymbol{A}}\,\boldsymbol{e_4}\,, \tag{3.80}$$
$$\boldsymbol{\nabla_4}\,\boldsymbol{e_A} - \boldsymbol{\nabla_A}\,\boldsymbol{e_4} = \left(\boldsymbol{\Gamma}^{\boldsymbol{B}}_{\boldsymbol{4A}} - \boldsymbol{\chi}^{\sharp_2}{}^{\boldsymbol{B}}_{\boldsymbol{A}}\right)\boldsymbol{e_B} + \boldsymbol{Y_A}\boldsymbol{e_3} + (\underline{\boldsymbol{\eta}}_{\boldsymbol{A}} + \boldsymbol{\zeta_A})\,\boldsymbol{e_4}\,, \tag{3.81}$$
$$\boldsymbol{\nabla_3}\,\boldsymbol{e_4} - \boldsymbol{\nabla_4}\,\boldsymbol{e_3} = \left(2\boldsymbol{\eta}^{\boldsymbol{A}} - 2\underline{\boldsymbol{\eta}}^{\boldsymbol{A}}\right)\boldsymbol{e_A} + \hat{\boldsymbol{\omega}}\,\boldsymbol{e_3} - \underline{\hat{\boldsymbol{\omega}}}\,\boldsymbol{e_4}\,. \tag{3.82}$$

Null structure equations

We have the first variational formulae

$$\boldsymbol{\not\mathcal{L}}_{\boldsymbol{e_4}}\boldsymbol{\not g} = 2\hat{\boldsymbol{\chi}}_\bigcirc + (\mathbf{tr}\boldsymbol{\chi}_\bigcirc)\boldsymbol{\not g}\,, \tag{3.83}$$
$$\boldsymbol{\not\mathcal{L}}_{\boldsymbol{e_3}}\boldsymbol{\not g} = 2\underline{\hat{\boldsymbol{\chi}}}_\bigcirc + (\mathbf{tr}\underline{\boldsymbol{\chi}}_\bigcirc)\boldsymbol{\not g} \tag{3.84}$$

and the second variational formulae

$$\boldsymbol{\not\nabla_4}\,\hat{\boldsymbol{\chi}}_\bigcirc + (\mathbf{tr}\boldsymbol{\chi}_\bigcirc)\hat{\boldsymbol{\chi}}_\bigcirc - \hat{\boldsymbol{\omega}}\,\hat{\boldsymbol{\chi}}_\bigcirc = -2\boldsymbol{\not{\mathcal{D}}}^\star_{\boldsymbol{2}}\boldsymbol{Y} + (\boldsymbol{\eta} + \underline{\boldsymbol{\eta}} + 2\boldsymbol{\zeta})\hat{\otimes}\boldsymbol{Y} - \boldsymbol{\alpha}\,, \tag{3.85}$$
$$\boldsymbol{\not\nabla_3}\,\underline{\hat{\boldsymbol{\chi}}}_\bigcirc + (\mathbf{tr}\,\underline{\boldsymbol{\chi}}_\bigcirc)\underline{\hat{\boldsymbol{\chi}}}_\bigcirc - \underline{\hat{\boldsymbol{\omega}}}\,\underline{\hat{\boldsymbol{\chi}}}_\bigcirc = -2\boldsymbol{\not{\mathcal{D}}}^\star_{\boldsymbol{2}}\,\underline{\boldsymbol{Y}} + (\boldsymbol{\eta} + \underline{\boldsymbol{\eta}} - 2\boldsymbol{\zeta})\hat{\otimes}\,\underline{\boldsymbol{Y}} - \underline{\boldsymbol{\alpha}}\,. \tag{3.86}$$

The Raychaudhuri equations read

$$\boldsymbol{\not\nabla_4}(\mathbf{tr}\boldsymbol{\chi}_\bigcirc) + \frac{1}{2}(\mathbf{tr}\boldsymbol{\chi}_\bigcirc)^2 - \hat{\boldsymbol{\omega}}\,\mathbf{tr}\boldsymbol{\chi}_\bigcirc = -(\hat{\boldsymbol{\chi}}_\bigcirc, \hat{\boldsymbol{\chi}}_\bigcirc) + \frac{1}{2}(\not\epsilon\cdot\boldsymbol{\chi}_\square)^2 + 2\,\mathbf{d\not{i}v}\boldsymbol{Y} + 2(\boldsymbol{\eta} + \underline{\boldsymbol{\eta}} + 2\boldsymbol{\zeta}, \boldsymbol{Y})\,, \tag{3.87}$$

$$\boldsymbol{\not\nabla_3}(\mathbf{tr}\,\underline{\boldsymbol{\chi}}_\bigcirc) + \frac{1}{2}(\mathbf{tr}\,\underline{\boldsymbol{\chi}}_\bigcirc)^2 - \underline{\hat{\boldsymbol{\omega}}}\,\mathbf{tr}\,\underline{\boldsymbol{\chi}}_\bigcirc = -(\underline{\hat{\boldsymbol{\chi}}}_\bigcirc, \underline{\hat{\boldsymbol{\chi}}}_\bigcirc) + \frac{1}{2}(\not\epsilon\cdot\underline{\boldsymbol{\chi}}_\square)^2 + 2\,\mathbf{d\not{i}v}\,\underline{\boldsymbol{Y}} + 2(\boldsymbol{\eta} + \underline{\boldsymbol{\eta}} - 2\boldsymbol{\zeta}, \underline{\boldsymbol{Y}})\,. \tag{3.88}$$

We have the mixed transport equations

$$\nabla\!\!\!/_4\, \hat{\chi}_\circ + \frac{1}{2}(\mathbf{tr}\chi_\circ)\hat{\chi}_\circ + \hat{\omega}\,\hat{\chi}_\circ = -2\mathcal{D}\!\!\!/_2^\star\,\underline{\eta} - \frac{1}{2}(\mathbf{tr}\,\chi_\circ)\hat{\chi}_\circ + \frac{1}{2}(\not\in\cdot\chi_\Box)({}^\star\hat{\chi}_\circ) - \frac{1}{2}(\not\in\cdot\chi_\Box)({}^\star\hat{\chi}_\circ) \tag{3.89}$$

$$+\,\underline{\eta}\,\hat{\otimes}\,\underline{\eta} + Y\,\hat{\otimes}\,\underline{Y}\,,$$

$$\nabla\!\!\!/_3\, \hat{\chi}_\circ + \frac{1}{2}(\mathbf{tr}\,\chi_\circ)\hat{\chi}_\circ + \underline{\hat{\omega}}\,\hat{\chi}_\circ = -2\mathcal{D}\!\!\!/_2^\star\eta - \frac{1}{2}(\mathbf{tr}\chi_\circ)\hat{\chi}_\circ + \frac{1}{2}(\not\in\cdot\chi_\Box)({}^\star\hat{\chi}_\circ) - \frac{1}{2}(\not\in\cdot\chi_\Box)({}^\star\hat{\chi}_\circ) \tag{3.90}$$

$$+\,\eta\,\hat{\otimes}\,\eta + \underline{Y}\,\hat{\otimes}\,Y\,,$$

$$\nabla\!\!\!/_4(\mathbf{tr}\,\chi_\circ) + \frac{1}{2}(\mathbf{tr}\chi_\circ)(\mathbf{tr}\,\chi_\circ) + \hat{\omega}\,\mathbf{tr}\,\chi_\circ = -(\hat{\chi}_\circ, \hat{\chi}_\circ) + \frac{1}{2}(\not\in\cdot\chi_\Box)(\not\in\cdot\chi_\Box) + 2(\underline{\eta}, \underline{\eta}) + 2\rho \tag{3.91}$$

$$+\,2\,\mathbf{d}\!\!\!/\mathbf{iv}\,\underline{\eta} + 2(Y, \underline{Y})\,,$$

$$\nabla\!\!\!/_3(\mathbf{tr}\chi_\circ) + \frac{1}{2}(\mathbf{tr}\,\chi_\circ)(\mathbf{tr}\chi_\circ) + \underline{\hat{\omega}}\,\mathbf{tr}\chi_\circ = -(\hat{\chi}_\circ, \hat{\chi}_\circ) + \frac{1}{2}(\not\in\cdot\chi_\Box)(\not\in\cdot\chi_\Box) + 2(\eta, \eta) + 2\rho \tag{3.92}$$

$$+\,2\,\mathbf{d}\!\!\!/\mathbf{iv}\eta + 2(\underline{Y}, Y)\,,$$

and the transport equations

$$\nabla\!\!\!/_4(\not\in\cdot\chi_\Box) + (\mathbf{tr}\chi_\circ)(\not\in\cdot\chi_\Box) - \hat{\omega}(\not\in\cdot\chi_\Box) = 2(\underline{\eta} + 2\zeta)\wedge Y + 2Y\wedge\eta + 2\,\mathbf{c}\!\!\!/\mathbf{url}\,Y\,, \tag{3.93}$$

$$\nabla\!\!\!/_3(\not\in\cdot\chi_\Box) + (\mathbf{tr}\,\chi_\circ)(\not\in\cdot\chi_\Box) - \underline{\hat{\omega}}(\not\in\cdot\chi_\Box) = 2(\eta - 2\zeta)\wedge\underline{Y} + 2\underline{Y}\wedge\underline{\eta} + 2\,\mathbf{c}\!\!\!/\mathbf{url}\,\underline{Y}\,, \tag{3.94}$$

$$\nabla\!\!\!/_4(\not\in\cdot\chi_\Box) + \frac{1}{2}(\mathbf{tr}\chi_\circ)(\not\in\cdot\chi_\Box) + \hat{\omega}(\not\in\cdot\chi_\Box) = -\hat{\chi}_\circ\wedge\hat{\chi}_\circ - \frac{1}{2}(\mathbf{tr}\,\chi_\circ)(\not\in\cdot\chi_\Box) + 2Y\wedge\underline{Y} \tag{3.95}$$

$$+\,2\sigma + 2\,\mathbf{c}\!\!\!/\mathbf{url}\,\underline{\eta}\,,$$

$$\nabla\!\!\!/_3(\not\in\cdot\chi_\Box) + \frac{1}{2}(\mathbf{tr}\,\chi_\circ)(\not\in\cdot\chi_\Box) + \underline{\hat{\omega}}(\not\in\cdot\chi_\Box) = -\hat{\chi}_\circ\wedge\hat{\chi}_\circ - \frac{1}{2}(\mathbf{tr}\chi_\circ)(\not\in\cdot\chi_\Box) + 2\underline{Y}\wedge Y \tag{3.96}$$

$$-\,2\sigma + 2\,\mathbf{c}\!\!\!/\mathbf{url}\,\eta\,.$$

We have equations

$$\nabla\!\!\!/_4\,\eta = \nabla\!\!\!/_3\,Y - \chi^{\sharp_1}\cdot(\eta - \underline{\eta}) + 2\underline{\hat{\omega}}\,Y - \beta\,, \tag{3.97}$$

$$\nabla\!\!\!/_3\,\underline{\eta} = \nabla\!\!\!/_4\,\underline{Y} + \chi^{\sharp_1}\cdot(\eta - \underline{\eta}) + 2\hat{\omega}\,\underline{Y} + \underline{\beta}\,, \tag{3.98}$$

$$\nabla\!\!\!/_4\,\underline{\hat{\omega}} + \nabla\!\!\!/_3\,\hat{\omega} = 2(\eta, \underline{\eta}) - 2(Y, \underline{Y}) - 2\hat{\omega}\,\underline{\hat{\omega}} - 2(\eta - \underline{\eta}, \zeta) - 2\rho \tag{3.99}$$

and the equations for the torsion

$$\nabla\!\!\!\!/_{\mathbf{4}}\,\boldsymbol{\zeta} = -\nabla\!\!\!\!/\,\hat{\omega} + \chi^{\sharp_2}\cdot(\underline{\boldsymbol{\eta}} - \zeta) - \underline{\chi}^{\sharp_2}\cdot \boldsymbol{Y} - \hat{\omega}(\underline{\boldsymbol{\eta}} + \zeta) + \underline{\hat{\omega}}\,\boldsymbol{Y} - \boldsymbol{\beta}\,, \tag{3.100}$$

$$\nabla\!\!\!\!/_{\mathbf{3}}\,\boldsymbol{\zeta} = \nabla\!\!\!\!/\,\underline{\hat{\omega}} - \underline{\chi}^{\sharp_2}\cdot(\boldsymbol{\eta} + \boldsymbol{\zeta}) + \chi^{\sharp_2}\cdot \underline{\boldsymbol{Y}} + \underline{\hat{\omega}}(\boldsymbol{\eta} - \boldsymbol{\zeta}) - \hat{\omega}\,\underline{\boldsymbol{Y}} - \underline{\boldsymbol{\beta}}\,, \tag{3.101}$$

$$\mathbf{cu\!\!\!/rl}\,\boldsymbol{\zeta} = -\frac{1}{2}\,\hat{\boldsymbol{\chi}}_{\circ}\wedge\hat{\underline{\boldsymbol{\chi}}}_{\circ} + \frac{1}{4}\,(\mathbf{tr}\,\boldsymbol{\chi}_{\circ})(\not\in\cdot\,\underline{\boldsymbol{\chi}}_{\square}) - \frac{1}{4}\,(\mathbf{tr}\;\underline{\boldsymbol{\chi}}_{\circ})(\not\in\cdot\,\boldsymbol{\chi}_{\square}) - \frac{1}{2}\,(\not\in\cdot\,\underline{\boldsymbol{\chi}}_{\square})\,\hat{\omega} + \frac{1}{2}\,(\not\in\cdot\,\boldsymbol{\chi}_{\square})\,\underline{\hat{\omega}} + \boldsymbol{\sigma}\,. \tag{3.102}$$

We have the two Codazzi-like equations

$$\mathbf{di\!\!\!/v}\;\hat{\boldsymbol{\chi}}_{\circ} = -\,\frac{1}{2}\not\in^{\sharp_2}\cdot\,\nabla\!\!\!\!/(\not\in\cdot\,\boldsymbol{\chi}_{\square}) - \hat{\boldsymbol{\chi}}^{\sharp}_{\circ}\cdot\boldsymbol{\zeta} - \frac{1}{2}(\not\in\cdot\,\boldsymbol{\chi}_{\square})^{\star}\boldsymbol{\zeta} + \frac{1}{2}(\mathbf{tr}\,\boldsymbol{\chi}_{\circ})\boldsymbol{\zeta} + \frac{1}{2}\nabla\!\!\!\!/(\mathbf{tr}\,\boldsymbol{\chi}_{\circ}) - (\not\in\cdot\,\boldsymbol{\chi}_{\square})^{\star}\boldsymbol{Y} \tag{3.103}$$

$$-\,(\not\in\cdot\,\boldsymbol{\chi}_{\square})^{\star}\boldsymbol{\eta} - \boldsymbol{\beta}\,,$$

$$\mathbf{di\!\!\!/v}\;\hat{\underline{\boldsymbol{\chi}}}_{\circ} = -\,\frac{1}{2}\not\in^{\sharp_2}\cdot\,\nabla\!\!\!\!/(\not\in\cdot\,\underline{\boldsymbol{\chi}}_{\square}) + \hat{\underline{\boldsymbol{\chi}}}^{\sharp}_{\circ}\cdot\boldsymbol{\zeta} + \frac{1}{2}(\not\in\cdot\,\underline{\boldsymbol{\chi}}_{\square})^{\star}\boldsymbol{\zeta} - \frac{1}{2}(\mathbf{tr}\,\underline{\boldsymbol{\chi}}_{\circ})\boldsymbol{\zeta} + \frac{1}{2}\nabla\!\!\!\!/(\mathbf{tr}\,\underline{\boldsymbol{\chi}}_{\circ}) - (\not\in\cdot\,\underline{\boldsymbol{\chi}}_{\square})^{\star}\,\underline{\boldsymbol{Y}} \tag{3.104}$$

$$-\,(\not\in\cdot\,\underline{\boldsymbol{\chi}}_{\square})^{\star}\,\underline{\boldsymbol{\eta}} + \underline{\boldsymbol{\beta}}\,,$$

and the Gauss-like equation

$$\boldsymbol{K\!\!\!/} = \frac{1}{2}(\hat{\boldsymbol{\chi}}_{\circ},\hat{\underline{\boldsymbol{\chi}}}_{\circ}) - \frac{1}{4}(\mathbf{tr}\boldsymbol{\chi}_{\circ})(\mathbf{tr}\,\underline{\boldsymbol{\chi}}_{\circ}) - \frac{1}{4}(\not\in\cdot\,\boldsymbol{\chi}_{\square})(\not\in\cdot\,\underline{\boldsymbol{\chi}}_{\square}) + (\boldsymbol{\Gamma_3},\boldsymbol{\chi}_{\square}) + (\boldsymbol{\Gamma_4},\underline{\boldsymbol{\chi}}_{\square}) - \boldsymbol{\rho}\,. \tag{3.105}$$

Bianchi equations

The Bianchi equations read

$$\nabla\!\!\!\!/_{\mathbf{3}}\,\boldsymbol{\alpha} + \frac{1}{2}(\mathbf{tr}\,\underline{\boldsymbol{\chi}}_{\circ})\boldsymbol{\alpha} + 2\underline{\hat{\omega}}\,\boldsymbol{\alpha} + \frac{1}{2}(\not\in\cdot\,\underline{\boldsymbol{\chi}}_{\square})^{\star}\boldsymbol{\alpha} = -2\mathcal{D\!\!\!/}^{\star}_{\mathbf{2}}\boldsymbol{\beta} - 3\boldsymbol{\rho}\,\hat{\boldsymbol{\chi}}_{\circ} - 3\boldsymbol{\sigma}^{\star}\,\hat{\boldsymbol{\chi}}_{\circ} + (4\boldsymbol{\eta} + \boldsymbol{\zeta})\hat{\otimes}\,\boldsymbol{\beta}\,, \tag{3.106}$$

$$\nabla\!\!\!\!/_{\mathbf{4}}\,\boldsymbol{\beta} + 2(\mathbf{tr}\boldsymbol{\chi}_{\circ})\boldsymbol{\beta} - \hat{\omega}\,\boldsymbol{\beta} - 2(\not\in\cdot\,\boldsymbol{\chi}_{\square})^{\star}\boldsymbol{\beta} = \mathbf{di\!\!\!/v}\boldsymbol{\alpha} + (\underline{\boldsymbol{\eta}}^{\sharp} + 2\zeta^{\sharp})\cdot\boldsymbol{\alpha} + 3\boldsymbol{\rho}\boldsymbol{Y} + 3\boldsymbol{\sigma}^{\star}\boldsymbol{Y}\,, \tag{3.107}$$

$$\nabla\!\!\!\!/_{\mathbf{3}}\,\boldsymbol{\beta} + (\mathbf{tr}\,\underline{\boldsymbol{\chi}}_{\circ})\boldsymbol{\beta} + \underline{\hat{\omega}}\,\boldsymbol{\beta} + (\not\in\cdot\,\underline{\boldsymbol{\chi}}_{\square})^{\star}\boldsymbol{\beta} = \mathcal{D\!\!\!/}^{*}_{\mathbf{1}}(-\boldsymbol{\rho},\boldsymbol{\sigma}) + 3\boldsymbol{\rho}\boldsymbol{\eta} + 3\boldsymbol{\sigma}^{\star}\boldsymbol{\eta} + 2\hat{\chi}^{\sharp}_{\circ}\cdot\underline{\boldsymbol{\beta}} + \underline{\boldsymbol{Y}}^{\sharp}\cdot\boldsymbol{\alpha}\,, \tag{3.108}$$

$$\nabla\!\!\!\!/_{\mathbf{4}}\,\boldsymbol{\rho} + \frac{3}{2}(\mathbf{tr}\boldsymbol{\chi}_{\circ})\boldsymbol{\rho} = \mathbf{di\!\!\!/v}\boldsymbol{\beta} + (2\underline{\boldsymbol{\eta}} + \boldsymbol{\zeta},\boldsymbol{\beta}) - \frac{1}{2}(\hat{\underline{\boldsymbol{\chi}}}_{\circ},\boldsymbol{\alpha}) - 2(\boldsymbol{Y},\underline{\boldsymbol{\beta}}) - \frac{3}{2}(\not\in\cdot\,\underline{\boldsymbol{\chi}}_{\square})\boldsymbol{\sigma}\,, \tag{3.109}$$

$$\nabla\!\!\!\!/_{\mathbf{4}}\,\boldsymbol{\sigma} + \frac{3}{2}(\mathbf{tr}\boldsymbol{\chi}_{\circ})\boldsymbol{\sigma} = -\mathbf{cu\!\!\!/rl}\boldsymbol{\beta} - (2\underline{\boldsymbol{\eta}} + \boldsymbol{\zeta})\wedge\boldsymbol{\beta} + \frac{1}{2}\hat{\underline{\boldsymbol{\chi}}}_{\circ}\wedge\boldsymbol{\alpha} - 2\boldsymbol{Y}\wedge\underline{\boldsymbol{\beta}} + \frac{3}{2}(\not\in\cdot\,\underline{\boldsymbol{\chi}}_{\square})\boldsymbol{\rho}\,, \tag{3.110}$$

$$\nabla\!\!\!\!/_{\mathbf{3}}\,\boldsymbol{\rho} + \frac{3}{2}(\mathbf{tr}\,\underline{\boldsymbol{\chi}}_{\circ})\boldsymbol{\rho} = -\mathbf{di\!\!\!/v}\,\underline{\boldsymbol{\beta}} - (2\boldsymbol{\eta} - \boldsymbol{\zeta},\underline{\boldsymbol{\beta}}) - \frac{1}{2}(\hat{\chi}_{\circ},\underline{\boldsymbol{\alpha}}) + 2(\underline{\boldsymbol{Y}},\boldsymbol{\beta}) + \frac{3}{2}(\not\in\cdot\,\boldsymbol{\chi}_{\square})\boldsymbol{\sigma}\,, \tag{3.111}$$

$$\nabla\!\!\!\!/_{\mathbf{3}}\,\boldsymbol{\sigma} + \frac{3}{2}(\mathbf{tr}\,\underline{\boldsymbol{\chi}}_{\circ})\boldsymbol{\sigma} = -\mathbf{cu\!\!\!/rl}\,\underline{\boldsymbol{\beta}} - (2\boldsymbol{\eta} - \boldsymbol{\zeta})\wedge\underline{\boldsymbol{\beta}} - \frac{1}{2}\hat{\chi}_{\circ}\wedge\underline{\boldsymbol{\alpha}} - 2\underline{\boldsymbol{Y}}\wedge\boldsymbol{\beta} - \frac{3}{2}(\not\in\cdot\,\boldsymbol{\chi}_{\square})\boldsymbol{\rho}\,, \tag{3.112}$$

$$\nabla\!\!\!/_4\,\underline{\beta}+(\mathrm{tr}\chi_{\bigcirc})\underline{\beta}+\hat{\omega}\,\underline{\beta}+(\not{e}\cdot\chi_{\square})^{\star}\,\underline{\beta}=\mathcal{D}\!\!\!/_1^{\star}(\rho,\sigma)-3\rho\,\underline{\eta}+3\sigma^{\star}\,\underline{\eta}+2\hat{\chi}_{\bigcirc}^{\sharp}\cdot\beta-Y^{\sharp}\cdot\underline{\alpha}\,, \quad (3.113)$$

$$\nabla\!\!\!/_3\,\underline{\beta}+2(\mathrm{tr}\,\underline{\chi}_{\bigcirc})\underline{\beta}-\underline{\hat{\omega}}\,\underline{\beta}-2(\not{e}\cdot\underline{\chi}_{\square})^{\star}\,\underline{\beta}=-\mathrm{d}\!\!\!/\mathrm{iv}\,\underline{\alpha}-(\eta^{\sharp}-2\zeta^{\sharp})\cdot\underline{\alpha}-3\rho\,\underline{Y}+3\sigma^{\star}\,\underline{Y}\,, \quad (3.114)$$

$$\nabla\!\!\!/_4\,\underline{\alpha}+\frac{1}{2}(\mathrm{tr}\chi_{\bigcirc})\underline{\alpha}+2\hat{\omega}\,\underline{\alpha}-\frac{1}{2}(\not{e}\cdot\chi_{\square})^{\star}\,\underline{\alpha}=2\mathcal{D}\!\!\!/_2^{\star}\,\underline{\beta}-3\rho\,\underline{\hat{\chi}}_{\bigcirc}+3\sigma^{\star}\,\underline{\hat{\chi}}_{\bigcirc}-(4\underline{\eta}-\zeta)\hat{\otimes}\,\underline{\beta}\,. \quad (3.115)$$

Some remarks on the system of equations

We collect here some remarks about the system of vacuum Einstein equations:

- The null frame equations (3.79)-(3.82) should be seen as equations for the frame components e_I^{μ}, which are scalar quantities at the level of the metric components.
- If one compares our system of equations to the systems of [14, 13], both the outgoing and ingoing shears are here replaced by $\hat{\chi}_{\bigcirc}$ and $\underline{\hat{\chi}}_{\bigcirc}$. The elliptic equations for $Y, \underline{Y}, \eta$ and $\underline{\eta}$ are now seen, in our system, as transport equations for $(\not{e}\cdot\chi_{\square})$ and $(\not{e}\cdot\underline{\chi}_{\square})$.
- If one sets
$$(\not{e}\cdot\chi_{\square})=(\not{e}\cdot\underline{\chi}_{\square})=0\,,$$
then our system reduces to the system of [14], where indeed the frame adopted *is* integrable. If one sets
$$(\not{e}\cdot\chi_{\square})=(\not{e}\cdot\underline{\chi}_{\square})=0\,, \qquad Y=\underline{Y}=0\,,$$
then our system reduces to the system of [13], where the system is derived in a double-null gauge.

Remark 3.3. *Crucially, the algebraically special frame of the Kerr exterior manifold is* non-integrable. *The connection coefficients and curvature components for the Kerr metric relative to the algebraically special frame solve our system of equations, but they do* not *solve the systems of [14, 13].*

3.4 The Kerr exterior manifold

This section opens the second of the three self-contained blocks outlined at the beginning of Section 3.2 and concerns the main object of the chapter, namely the Kerr exterior manifold.

We consider the *Kerr exterior manifold* $(\mathcal{M}, g_{a,M})$, with real parameters a and $M>0$, $0<|a|<M$, and a system of double-null coordinates (u,v,θ^A) that *globally* covers $(\mathcal{M}, g_{a,M})$.[53,54]

[53] For the existence of such a global coordinate system, the reader should refer to [93].

[54] We note that the double-null coordinates (u,v,θ^A) are not defined along the Kerr event horizon. Nonetheless, one can make formal sense of the event horizon as an asymptotic null hypersurface corresponding to the limits $\{u\to\infty\}\cup\{v\to-\infty\}$. See later in this section.

We introduce the Boyer–Lindquist coordinates

$$(t_{bl}, r_{bl}, \theta_{bl}, \phi_{bl})$$

on $(\mathcal{M}, g_{a,M})$, with coordinate functions

$$\begin{aligned} t_{bl} &= t_{bl}(u,v)\,, & r_{bl} &= r_{bl}(u,v,\theta^A)\,, \\ \theta_{bl} &= \theta_{bl}(u,v,\theta^A)\,, & \phi_{bl} &= \phi_{bl}(u,v,\theta^A) \end{aligned}$$

as defined in [93]. We define the scalar functions

$$\begin{aligned} \Sigma(r_{bl},\theta_{bl}) &:= r_{bl}^2 + a^2\cos^2\theta_{bl}\,, \\ R^2(r_{bl},\theta_{bl}) &:= r_{bl}^2 + a^2 + \frac{2Ma^2 r_{bl}\sin^2\theta_{bl}}{\Sigma}\,, \\ \Delta(r_{bl}) &:= r_{bl}^2 - 2Mr_{bl} + a^2\,. \end{aligned}$$

on $(\mathcal{M}, g_{a,M})$.

The Kerr metric $g_{a,M}$ in coordinates (u, v, θ^A) takes the double-null form

$$g_{a,M} = -4\,\Omega^2(u,v,\theta^A)du\,dv + \not{g}_{\theta^A\theta^B}(u,v,\theta^A)(d\theta^A - b^{\theta^A}(u,v,\theta^A)dv)(d\theta^B - b^{\theta^B}(u,v,\theta^A)dv)\,,$$

on $(\mathcal{M}, g_{a,M})$, with $\Omega^2 \in C^\infty(\mathcal{M})$, $\not{g}$ a symmetric $S^2_{u,v}$ two-tensor[55] and b a vector field tangent to the $S^2_{u,v}$-spheres such that

$$\Omega^2 = \frac{\Delta}{R^2}\,, \qquad b^{\theta^1} = 0\,, \qquad b^{\theta^2} = \frac{4Mar_{bl}}{\Sigma R^2}\,, \qquad \not{g}_{\theta^2\theta^2} = R^2\sin^2\theta_{bl} \tag{3.116}$$

and satisfying the implicit relations (218) of [17].[56] The future and past event horizons of $(\mathcal{M}, g_{a,M})$ correspond to the (asymptotic) null hypersurfaces

$$\mathcal{H}^+ \equiv \{u = \infty\}\,, \qquad \mathcal{H}^- \equiv \{v = -\infty\}\,,$$

with $\mathcal{H} = \mathcal{H}^+ \cup \mathcal{H}^+$. We note that

$$\Omega^2|_{\mathcal{H}} = 0\,.$$

[55] From now on, we will refer to $\mathfrak{D}_{\widehat{\mathcal{N}}_{\rm ad}}$ tensors as $S^2_{u,v}$ tensors, with $S^2_{u,v} \equiv \{u,v\}\times S^2$. See later in this section for the definition of the frame $\widehat{\mathcal{N}}_{\rm ad}$ on $(\mathcal{M}, g_{a,M})$.

[56] *Note on notation*: Given a one-tensor ξ, the reader should note the difference between

$$\xi_{\theta^A} := \xi(\partial_{\theta^A})$$

and

$$\xi_A = \xi(e_A)\,.$$

We refer to the former indices as *coordinate* indices and the latter indices as *horizontal* indices. The two notations become equivalent when the frame vector fields e_A are *coordinate* vector fields, which will *not* be the case for the main frame adopted in this chapter (cf. [19]).

Future and past null infinity correspond to the (asymptotic) null hypersurfaces

$$\mathcal{I}^+ \equiv \{v = \infty\} \ , \qquad\qquad \mathcal{I}^- \equiv \{u = -\infty\} \ .$$

We define the null frame

$$\widehat{\mathcal{N}}_{\text{ad}} = (\widehat{e}_1, \widehat{e}_2, \widehat{e}_3, \widehat{e}_4)$$

adapted to the double-null foliation of $(\mathcal{M}, g_{a,M})$, with frame vectors

$$\widehat{e}_4 = \partial_v + b^{\theta^A} \partial_{\theta^A} \, , \qquad \widehat{e}_3 = \frac{1}{\Omega^2} \partial_u \, , \qquad \widehat{e}_A = h_A^{\theta^B} \partial_{\theta^B} \, ,$$

where the scalar functions $h_A^{\theta^B}$ are such that $\widehat{\mathcal{N}}_{\text{ad}}$ satisfies the frame conditions (3.63). For our problem, three important properties of the frame $\widehat{\mathcal{N}}_{\text{ad}}$ are the following:

- The frame $\widehat{\mathcal{N}}_{\text{ad}}$ is an *integrable* frame on $(\mathcal{M}, g_{a,M})$,
- We have

$$\begin{aligned} \widehat{\underline{\omega}} &= 0 \, , \\ \widehat{\zeta} &= -\widehat{\underline{\eta}} \, , \\ \widehat{(\not{\epsilon} \cdot \chi_{\square})} &= \widehat{(\not{\epsilon} \cdot \underline{\chi}_{\square})} = 0 \, , \\ \widehat{Y} &= \widehat{\underline{Y}} = 0 \, , \\ \widehat{\Gamma}_{4A}^B &= 0 \end{aligned}$$

  on $(\mathcal{M}, g_{a,M})$, where the connection coefficients denoted by a hat are defined relative to the frame $\widehat{\mathcal{N}}_{\text{ad}}$,
- The frame $\widehat{\mathcal{N}}_{\text{ad}}$ extends to a *regular* frame along the event horizon $\mathcal{H}^+$. See Section 5.1.1 of [19] for a precise definition of regularity of the frame along $\mathcal{H}^+$.

We now consider the *algebraically special frame* of $(\mathcal{M}, g_{a,M})$

$$\mathcal{N}_{\text{as}} = (e_1^{\text{as}}, e_2^{\text{as}}, e_3^{\text{as}}, e_4^{\text{as}}) \ ,$$

with frame vectors

$$\begin{aligned} e_4^{\text{as}} &= \partial_{t_{bl}} + \frac{\Delta}{r_{bl}^2 + a^2} \partial_{r_{bl}} + \frac{a}{r_{bl}^2 + a^2} \partial_{\phi_{bl}} \, , \\ e_3^{\text{as}} &= \frac{(r_{bl}^2 + a^2)^2}{\Sigma\Delta} \partial_{t_{bl}} - \frac{r_{bl}^2 + a^2}{\Sigma} \partial_{r_{bl}} + \frac{a(r_{bl}^2 + a^2)}{\Sigma\Delta} \partial_{\phi_{bl}} \, , \\ e_1^{\text{as}} &= \frac{a^2 \sin\theta_{bl} \cos\theta_{bl}}{\Sigma} \partial_{t_{bl}} + \frac{r_{bl}}{\Sigma} \partial_{\theta_{bl}} + \frac{a \cot\theta_{bl}}{\Sigma} \partial_{\phi_{bl}} \, , \\ e_2^{\text{as}} &= \frac{a r_{bl} \sin\theta_{bl}}{\Sigma} \partial_{t_{bl}} - \frac{a \cos\theta_{bl}}{\Sigma} \partial_{\theta_{bl}} + \frac{r_{bl} \csc\theta_{bl}}{\Sigma} \partial_{\phi_{bl}} \, . \end{aligned}$$

The relevant properties of $\mathcal{N}_{\rm as}$ for our problem are the following:

- We have
$$\mathcal{N}_{\rm as} \not\equiv \widehat{\mathcal{N}}_{\rm ad}$$
on $(\mathcal{M}, g_{a,M})$,
- The frame $\mathcal{N}_{\rm as}$ is a *non-integrable* frame on $(\mathcal{M}, g_{a,M})$,
- The frame $\mathcal{N}_{\rm as}$ extends to a *regular* frame along the event horizon $\mathcal{H}^+$,
- We have
$$e_4^{\rm as} \equiv \widehat{e}_4 \qquad \text{along } \mathcal{H}^+. \tag{3.117}$$
This geometric property[57] will be crucially exploited in forthcoming work by the author. To check (3.117), we note that, given regular coordinates
$$(t^*, r, \theta, \phi^*)$$
on $(\mathcal{M}, g_{a,M})$ such that $r = r_{bl}$, $\theta = \theta_{bl}$ and
$$dt^* = dt_{bl} + \frac{r_{bl}^2 + a^2}{\Delta}\, dr_{bl}\,, \qquad\qquad d\phi^* = d\phi_{bl} + \frac{a}{\Delta}\, dr_{bl}\,,$$
one has
$$e_4^{\rm as}|_{r=r_{bl_+}} = 2\,\partial_{t^*} + 2\frac{a}{r_{bl_+}^2 + a^2}\,\partial_{\phi^*}\,, \qquad\qquad e_3^{\rm as}|_{r_{bl}=r_{bl_+}} = -\frac{r_{bl_+}^2 + a^2}{\Sigma(r_{bl_+}, \theta)}\,\partial_r\,,$$
with $\mathcal{H} \equiv \left\{ r = r_{bl_+} \right\}$. One can conclude $e_4^{\rm as}{}_\flat|_{r=r_{bl_+}} \propto dr$.
- The frame vector $e_A^{\rm as}$ is *tangent* to $\mathcal{H}^+$. In particular, $e_A^{\rm as}(u) = 0$ along $\mathcal{H}^+$.
To check this, one can decompose $e_A^{\rm as} = c_1\widehat{e}_4 + c_2\widehat{e}_3 + c_3\widehat{e}_A$ for some scalar functions c_1, c_2, c_3. The condition $g(e_4^{\rm as}, e_A^{\rm as}) = 0$ combined to (3.117) implies that $e_A^{\rm as}$ has no $\widehat{e}_3$-component along $\mathcal{H}^+$, i.e. $e_A^{\rm as} = c_1\widehat{e}_4 + c_2\widehat{e}_A$ along $\mathcal{H}^+$.
- The frame has the asymptotic property
$$\mathcal{N}_{\rm as} \to \widehat{\mathcal{N}}_{\rm ad}$$
as $r_{bl} \to \infty$, meaning that the limit of scalar functions $e_I^{\rm as\,\mu} \to \widehat{e}_I^{\,\mu}$ holds for all $I = \{1, 2, 3, 4\}$ as $r_{bl} \to \infty$,
- The integral curves of $e_4^{\rm as}$ generate *timelike* hypersurfaces which intersect future null infinity. As $v \to \infty$ along each one of these hypersrufaces, the hypersurface asymptotes an outgoing null cone of the Kerr exterior manifold. In the limit $u \to \infty$, the hypersurface generated by $e_4^{\rm as}$ coincides with the future event horizon $\mathcal{H}^+$, and thus becomes null. The

[57]This property is a manifestation of a more general one. In fact, for general stationary black hole solutions, the null generators of the event horizon are principal null directions of the Weyl tensor.

integral curves of e_3^{as} generate *timelike* hypersurfaces which intersect $\mathcal{H}^+$. In the limit $v \to \infty$, the hypersurface generated by e_3^{as} coincides with future null infinity, and thus becomes null.

Note that all the properties listed are *geometric* properties of the algebraically special frame of $(\mathcal{M}, g_{a,M})$.

The connection coefficients of the Kerr metric $g_{a,M}$ relative to $\mathcal{N}_{\mathrm{as}}$ are

$$\chi_{11} = \chi_{22} = \frac{r_{bl}\Delta}{\left(r_{bl}^2 + a^2\right)\Sigma}\,, \qquad \chi_{12} = \frac{a\Delta\cos\theta_{bl}}{\left(r_{bl}^2 + a^2\right)\Sigma}\,, \qquad \chi_{21} = -\chi_{12}\,,$$

$$\underline{\chi}_{11} = \underline{\chi}_{22} = -\frac{r_{bl}\left(r_{bl}^2 + a^2\right)}{\Sigma^2}\,, \qquad \underline{\chi}_{12} = \frac{a\left(r_{bl}^2 + a^2\right)\cos\theta_{bl}}{\Sigma^2}\,, \qquad \underline{\chi}_{21} = -\underline{\chi}_{12}\,,$$

$$\hat{\omega} = \frac{2M\left(r_{bl}^2 - a^2\right)}{\left(r_{bl}^2 + a^2\right)^2}\,, \qquad \underline{\hat{\omega}} = \frac{2a^2 r_{bl}\sin^2\theta_{bl}}{\Sigma^2}\,,$$

$$\eta_1 = 0\,, \qquad \eta_2 = \frac{2a\sin\theta_{bl}}{\Sigma}\,,$$

$$\underline{\eta}_1 = -\frac{a^2 r_{bl}\sin(2\theta_{bl})}{\Sigma^2}\,, \qquad \underline{\eta}_2 = \frac{a\sin\theta_{bl}\left(a^2\cos^2\theta_{bl} - r_{bl}^2\right)}{\Sigma^2}\,,$$

$$Y_A = 0\,, \qquad \underline{Y}_A = 0\,,$$

$$\zeta_1 = \frac{a^2 r_{bl}\sin(2\theta_{bl})}{\Sigma^2}\,, \qquad \zeta_2 = -\frac{a\sin\theta_{bl}\left(a^2\cos^2\theta_{bl} - r_{bl}^2\right)}{\Sigma^2}\,,$$

with

$$\hat{\chi}_{\circ AB} = 0\,, \qquad \underline{\hat{\chi}}_{\circ AB} = 0\,,$$

$$(\not\in \cdot \chi_{\square}) = \frac{2a\Delta\cos\theta_{bl}}{\left(r_{bl}^2 + a^2\right)\Sigma}\,, \qquad (\not\in \cdot \underline{\chi}_{\square}) = \frac{2a\left(r_{bl}^2 + a^2\right)\cos\theta_{bl}}{\Sigma^2}\,,$$

$$\mathrm{tr}\chi_{\circ} = \frac{2r_{bl}\Delta}{\left(r_{bl}^2 + a^2\right)\Sigma}\,, \qquad \mathrm{tr}\,\underline{\chi}_{\circ} = -\frac{2r_{bl}\left(r_{bl}^2 + a^2\right)}{\Sigma^2}\,.$$

Note that

$$\underline{\eta} + \zeta = 0 \tag{3.118}$$

and

$$\Gamma^B_{4A} = 0\,, \qquad \Gamma^C_{3A}\, g_{CB} = 2\,\underline{\chi}_{\square AB}\,.$$

The curvature components read

$$\alpha_{AB} = 0\,, \qquad \underline{\alpha}_{AB} = 0\,,$$

$$\beta_A = 0\,, \qquad \underline{\beta}_A = 0\,,$$

$$\rho = \frac{2Mr_{bl}\left(3a^2\cos^2\theta_{bl} - r_{bl}^2\right)}{\Sigma^3}\,, \qquad \sigma = \frac{aM\cos\theta_{bl}\left(3r_{bl}^2 - a^2\cos^2\theta_{bl}\right)}{\Sigma^3}\,.$$

All the connection coefficients and curvature components are *regular* quantities along $\mathcal{H}^+$.

Note the identities

$$\nabla\!\!\!/\, \hat{\omega} = 0\,,$$
$$-2 \mathcal{D}\!\!\!/_2^* \eta + \eta \hat{\otimes} \eta = 0\,,$$
$$-2 \mathcal{D}\!\!\!/_2^* \underline{\eta} + \underline{\eta} \hat{\otimes} \underline{\eta} = 0$$

on $(\mathcal{M}, g_{a,M})$ and the identities

$$\nabla\!\!\!/_4\, \eta|_{\mathcal{H}^+} = 0\,,$$
$$\nabla\!\!\!/_4\, \underline{\eta}|_{\mathcal{H}^+} = 0\,,$$
$$\nabla\!\!\!/_4\, \zeta|_{\mathcal{H}^+} = 0$$

along $\mathcal{H}^+$.

We have the frame commutators

$$[e_A^{\mathrm{as}}, e_B^{\mathrm{as}}] = \left(\Gamma^C_{AB} - \Gamma^C_{BA}\right) e_C^{\mathrm{as}} + \chi_{\square AB} e_3^{\mathrm{as}} + \underline{\chi}_{\square AB} e_4^{\mathrm{as}}\,,$$
$$[e_3^{\mathrm{as}}, e_A^{\mathrm{as}}] = \left(\Gamma^B_{3A} - \underline{\chi}^{\sharp_2}{}_A^B\right) e_B^{\mathrm{as}} + (\eta_A - \zeta_A)\, e_3^{\mathrm{as}}\,,$$
$$[e_4^{\mathrm{as}}, e_A^{\mathrm{as}}] = -\chi^{\sharp_2}{}_A^B e_B^{\mathrm{as}}\,,$$
$$[e_3^{\mathrm{as}}, e_4^{\mathrm{as}}] = \left(2\eta^A - 2\,\underline{\eta}^A\right) e_A^{\mathrm{as}} + \hat{\omega}\, e_3^{\mathrm{as}} - \underline{\hat{\omega}}\, e_4^{\mathrm{as}}\,.$$

Note that

$$[e_4^{\mathrm{as}}, e_A^{\mathrm{as}}]|_{\mathcal{H}^+} = 0\,.$$

The commutation formulae for a $\mathfrak{D}_{\mathcal{N}_{\mathrm{as}}}$ one-tensor ξ on $(\mathcal{M}, g_{a,M})$ read

$$[\nabla\!\!\!/_4, \nabla\!\!\!/]\xi = -\chi \times (\nabla\!\!\!/\, \xi) + \sum_{i=1}^k \left((\underline{\eta}^\sharp \cdot \xi_{\hat{i}})\chi - (\chi^{\sharp_2} \cdot \xi_{\hat{i}}) \otimes \underline{\eta} \right)\,,$$
$$[\nabla\!\!\!/_3, \nabla\!\!\!/]\xi = -\,\underline{\chi} \times (\nabla\!\!\!/\, \xi) + (\eta - \zeta) \otimes (\nabla\!\!\!/_3\, \xi) + \sum_{i=1}^k \left((\eta^\sharp \cdot \xi_{\hat{i}})\, \underline{\chi} - (\underline{\chi}^{\sharp_2} \cdot \xi_{\hat{i}}) \otimes \eta \right)\,,$$
$$[\nabla\!\!\!/_3, \nabla\!\!\!/_4]\xi = \hat{\omega}(\nabla\!\!\!/_3\, \xi) - \underline{\hat{\omega}}(\nabla\!\!\!/_4\, \xi) + 2(\nabla\!\!\!/\, \xi)^{\sharp_1} \cdot (\eta - \underline{\eta}) + \sum_{i=1}^k \left(2(\eta^\sharp \cdot \xi_{\hat{i}})\, \underline{\eta} - 2(\underline{\eta}^\sharp \cdot \xi_{\hat{i}})\eta \right) + 2\sigma({}^\star\xi)\,.$$

Note that

$$[\nabla\!\!\!/_4, \nabla\!\!\!/]\xi\,|_{\mathcal{H}^+} = 0\,.$$

Crucially, the commutation of $\nabla\!\!\!/_3$ and $\nabla\!\!\!/_4$ produces a term $2(\nabla\!\!\!/\, \xi)^{\sharp_1} \cdot (\eta - \underline{\eta})$ that does *not* vanish along $\mathcal{H}^+$.[58] The commutation formulae for a $\mathfrak{D}_{\mathcal{N}_{\mathrm{as}}}$ two-tensor on $(\mathcal{M}, g_{a,M})$ can be easily deduced from the formulae (3.76)-(3.78) and exhibit analogous properties.

Remark 3.4. *Looking at the frame commutators, it is easy to check that the distribution gen-*

[58] Note that this term does not appear when one considers the analogous Schwarzschild commutator and it has to be seen as a potentially problematic term in the context of a redshift estimate.

erated by the frame vector fields

$$(e_4^{\rm as}, e_1^{\rm as}, e_2^{\rm as})$$

is *integrable* *along* $\mathcal{H}^+$. *Since, as previously observed, all the three frame vector fields are tangent to* $\mathcal{H}^+$*, the distribution coincides with the tangent bundle of* $\mathcal{H}^+$*.*

We now define a new differentiable structure on a Lorentzian sub-manifold of $(\mathcal{M}, g_{a,M})$. We first introduce the Kruskal coordinates

$$(U, V, \theta^A)$$

on $(\mathcal{M}, g_{a,M})$. We then consider the union of two null hypersurfaces

$$C_{U_0, V\geq V_0} \cup \underline{C}_{U\geq U_0, V_0} \tag{3.119}$$

on $(\mathcal{M}, g_{a,M})$, with $U_0 \in (-\infty, 0)$ and $V_0 \in (0, \infty)$, and *define* the manifold with boundary

$$\mathcal{M}^+ := [U_0, 0] \times [V_0, \infty) \times S^2 . \tag{3.120}$$

We define the new differentiable structure

$$(\tau, s, \vartheta, \psi) ,$$

with

$$\tau \in [0, \infty) , \qquad s \in (-\infty, \infty) , \qquad (\vartheta, \psi) \in S^2 ,$$

on $\mathcal{M}^+$ such that

$$(\tau, s, \vartheta, \psi) = \begin{cases} (0, v_0 - u, \theta^1, \theta^2) & \text{along } \underline{C}_{U\geq U_0, V_0} \\ (0, v - u_0, \theta^1, \theta^2) & \text{along } C_{U_0, V\geq V_0} \end{cases}$$

and

$$e_4^{\rm as}(\tau) = 1$$
$$e_4^{\rm as}(s) = e_4^{\rm as}(\vartheta) = e_4^{\rm as}(\psi) = 0$$

on $(\mathcal{M}^+, g_{a,M})$, with $v_0 = v(V_0)$ and $u_0 = u(U_0)$. The new system of coordinates is well-defined globally on $(\mathcal{M}^+, g_{a,M})$, with the caveat that it breaks down along the future event horizon $\mathcal{H}^+$. However, the future event horizon can still be formally parametrised as

$$\mathcal{H}^+ = (\tau_{\geq 0}, -\infty, \vartheta, \psi) .$$

Coordinates $(\tau, s, \vartheta, \psi)$ induce a *global* foliation of the Lorentzian manifold $(\mathcal{M}^+, g_{a,M})$ already discussed in Section 3.2.2, where the hypersurfaces of constant τ and s were depicted in Figure

3.2. We recall that the foliation of the null hypersurface $\{\tau = 0\}$ by the two-spheres

$$S^2_{\tau,s} = \{\tau, s\} \times S^2 \tag{3.121}$$

coincides with the foliation by double-null spheres $S^2_{u,v}$.

The Kerr metric $g_{a,M}$ in coordinates $(\tau, s, \vartheta, \psi)$ and associated algebraically special frame $\mathcal{N}_{\text{as}}$ on $\mathcal{M}^+$ satisfies

$$g_{a,M\,\tau\tau} = 0\,, \tag{3.122}$$
$$\Gamma^s_{\tau\tau} = \Gamma^\vartheta_{\tau\tau} = \Gamma^\psi_{\tau\tau} = 0 \tag{3.123}$$

on $(\mathcal{M}^+, g_{a,M})$ and will be referred to as the Kerr metric in an *outgoing frame-calibrated gauge.*

3.5 A new gauge for perturbations of the Kerr solution

The present section is the core of the chapter. We consider a one-parameter family of metric perturbations $\boldsymbol{g}(\epsilon)$ around Kerr in a new gauge and develop the formalism necessary to its correct formulation and to discuss its properties. The question of whether the gauge considered is a well-posed gauge is also carefully addressed.

Before starting, we clarify that, for us, a *gauge* for the family of metrics $\boldsymbol{g}(\epsilon)$ on a (to be specified) manifold $\mathcal{M}^+$ is the identification of the fixed (for all $\epsilon \geq 0$) differentiable structure on $\mathcal{M}^+$ relative to which the family $\boldsymbol{g}(\epsilon)$ is prescribed *together with* the one-parameter family of null frames on $\mathcal{M}^+$ associated to $\boldsymbol{g}(\epsilon)$.

We start our discussion by *fixing* the manifold $\mathcal{M}^+$ as in (3.120) and its differentiable structure $(\tau, s, \vartheta, \psi)$. We consider a one-parameter family of smooth Lorentzian metrics $\boldsymbol{g}(\epsilon)$ on $\mathcal{M}^+$

$$\boldsymbol{g}(\epsilon) = \boldsymbol{g}_{x^\mu x^\nu}(\epsilon)\, dx^\mu\, dx^\nu\,,$$

with $x^\mu = (\tau, s, \vartheta, \psi)$, and the associated one-parameter family of null frames

$$\boldsymbol{\mathcal{N}}(\epsilon) = (\boldsymbol{e_1}(\epsilon), \boldsymbol{e_2}(\epsilon), \boldsymbol{e_3}(\epsilon), \boldsymbol{e_4}(\epsilon))$$

on $\mathcal{M}^+$ such that

(i) For any $\epsilon \geq 0$, the metric identity

$$\Pi\phi^*_\epsilon(\boldsymbol{g}_{\tau\tau}(\epsilon)) = 0 \tag{3.124}$$

and the identities for the Christoffel symbols

$$\Pi\phi_\epsilon^*(\boldsymbol{\Gamma}^s_{\tau\tau}(\epsilon)) = 0\,, \tag{3.125}$$
$$\Pi\phi_\epsilon^*(\boldsymbol{\Gamma}^\vartheta_{\tau\tau}(\epsilon)) = 0\,, \tag{3.126}$$
$$\Pi\phi_\epsilon^*(\boldsymbol{\Gamma}^\psi_{\tau\tau}(\epsilon)) = 0 \tag{3.127}$$

hold on $\mathcal{M}^+$,

(ii) The connection coefficients

$$\hat{\boldsymbol{\omega}}(\epsilon)\,,\ \boldsymbol{Y}(\epsilon)\,,\ \underline{\boldsymbol{\eta}}(\epsilon)\,,\ \boldsymbol{\Gamma}^{\boldsymbol{B}}_{\boldsymbol{4A}}(\epsilon)$$

are fixed on $\mathcal{M}^+$ such that

$$\Pi\phi_\epsilon^*(\hat{\boldsymbol{\omega}}(\epsilon)) = \hat{\omega}^{\text{ as}}_{\text{ Kerr}}\,, \tag{3.128}$$
$$\Pi\phi_\epsilon^*(\boldsymbol{Y}(\epsilon)) = 0\,, \tag{3.129}$$
$$\Pi\phi_\epsilon^*(\underline{\boldsymbol{\eta}}(\epsilon)) = \underline{\eta}^{\text{ as}}_{\text{ Kerr}}\,, \tag{3.130}$$
$$\Pi\phi_\epsilon^*(\boldsymbol{\Gamma}^{\boldsymbol{B}}_{\boldsymbol{4A}}(\epsilon)) = 0 \tag{3.131}$$

for all $\epsilon \geq 0$, where the tensors $\hat{\omega}^{\text{as}}_{\text{Kerr}}$ and $\underline{\eta}^{\text{as}}_{\text{Kerr}}$ are known quantities relative to the fixed differentiable structure on $\mathcal{M}^+$.

(iii) The frame vector field $\boldsymbol{e_4}(\epsilon)$ is fixed on $\mathcal{M}^+$ such that

$$\phi_{-\epsilon *}(\boldsymbol{e_4}(\epsilon)) = \partial_\tau \tag{3.132}$$

for all $\epsilon \geq 0$, where the vector field ∂_τ is a known quantity relative to the fixed differentiable structure on $\mathcal{M}^+$.

(iv) For any $\epsilon \geq 0$, the frame $\boldsymbol{\mathcal{N}}(\epsilon)$ extends to a regular frame for $s \to -\infty$,

and such that the metric $\boldsymbol{g}(0) \equiv g_{a,M}$ in the new gauge, with associated frame $\boldsymbol{\mathcal{N}}(0) \equiv \mathcal{N}_{\text{as}}$ on $\mathcal{M}^+$.[59]

We refer to the family of metrics $\boldsymbol{g}(\epsilon)$ as a one-parameter family of metric perturbations around the Kerr solution in an *outgoing frame-calibrated gauge*.

Remark 3.5. *As we shall discuss later in the chapter, there is a residual gauge freedom underlying the definition of the fixed differentiable structure on $\mathcal{M}^+$ and the family of frames $\boldsymbol{\mathcal{N}}(\epsilon)$ associated to $\boldsymbol{g}(\epsilon)$.*

[59] The formal meaning of the projected pullback $\Pi\phi_\epsilon^*$ and of the gauge conditions is explained in Section 3.5.2. The reader should think that the tensorial identity (3.130) is now correctly comparing one-tensors living in the same tensor bundle, as opposed to the informal identity (3.25) in Section 3.2.3 of the overview.

The connection coefficients and curvature components relative to $\boldsymbol{\mathcal{N}}(\epsilon)$ are one-parameter families of $\mathfrak{D}_{\boldsymbol{\mathcal{N}}(\epsilon)}$ tensor fields on $\mathcal{M}^+$. They will be denoted by

$$\begin{gathered}\hat{\boldsymbol{\omega}}(\epsilon)\,,\ \hat{\underline{\boldsymbol{\omega}}}(\epsilon)\,,\ \boldsymbol{\eta}(\epsilon)\,,\ \underline{\boldsymbol{\eta}}(\epsilon)\,,\ \boldsymbol{Y}(\epsilon)\,,\ \underline{\boldsymbol{Y}}(\epsilon)\,,\ \boldsymbol{\zeta}(\epsilon)\,,\\ \hat{\boldsymbol{\chi}}_{\circ}(\epsilon)\,,\ \hat{\underline{\boldsymbol{\chi}}}_{\circ}(\epsilon)\,,\ (\mathbf{tr}\boldsymbol{\chi}_{\circ})(\epsilon)\,,\ (\mathbf{tr}\,\underline{\boldsymbol{\chi}}_{\circ})(\epsilon)\,,\ (\not\in\cdot\,\boldsymbol{\chi}_{\square})(\epsilon)\,,\ (\not\in\cdot\,\underline{\boldsymbol{\chi}}_{\square})(\epsilon)\,,\\ \boldsymbol{\rho}(\epsilon)\,,\ \boldsymbol{\sigma}(\epsilon)\,,\ \boldsymbol{\beta}(\epsilon)\,,\ \underline{\boldsymbol{\beta}}(\epsilon)\,,\ \boldsymbol{\alpha}(\epsilon)\,,\ \underline{\boldsymbol{\alpha}}(\epsilon)\,.\end{gathered}$$

For $\epsilon = 0$, we will adopt the notation

$$\begin{gathered}\boldsymbol{\mathcal{N}}(0) \equiv \mathcal{N}\,,\\ (\boldsymbol{e_4}(0), \boldsymbol{e_3}(0), \boldsymbol{e_A}(0)) \equiv (e_4, e_3, e_A)\,.\end{gathered}$$

The connection coefficients and curvature components relative to $\mathcal{N}$ are $\mathfrak{D}_{\mathcal{N}}$ tensor fields denoted by[60]

$$\begin{gathered}\hat{\omega}\,,\ \hat{\underline{\omega}}\,,\ \eta\,,\ \underline{\eta}\,,\ Y\,,\ \underline{Y}\,,\ \zeta\,,\ \hat{\chi}_{\circ}\,,\ \hat{\underline{\chi}}_{\circ}\,,\ (\mathrm{tr}\chi_{\circ})\,,\ (\mathrm{tr}\,\underline{\chi}_{\circ})\,,\ (\not\in\cdot\,\chi_{\square})\,,\ (\not\in\cdot\,\underline{\chi}_{\square})\,,\\ \rho\,,\ \sigma\,,\ \beta\,,\ \underline{\beta}\,,\ \alpha\,,\ \underline{\alpha}\,.\end{gathered}$$

3.5.1 The construction of the new gauge

In this section we state, and prove, the well-posedness of the new gauge. Some motivation and a preliminary discussion of the result can be found in Section 3.2.3 of the overview. For completeness, we will recall the full geometric setting needed to formulate the theorem.

Let $(\boldsymbol{\mathcal{M}}, \boldsymbol{g})$ be a $(3+1)$-dimensional, smooth, orientable Lorentzian manifold with topology

$$\boldsymbol{\mathcal{M}} \cong \mathbb{R}^2 \times \boldsymbol{S}^2\,.$$

For any such Lorentzian manifold $(\boldsymbol{\mathcal{M}}, \boldsymbol{g})$, there exist local coordinates

$$(\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{\theta^1}, \boldsymbol{\theta^2})$$

such that $\boldsymbol{g}$ takes the double-null form

$$\boldsymbol{g} = -4\,\boldsymbol{\Omega}^2 \boldsymbol{du}\,\boldsymbol{dv} + \boldsymbol{\gamma\!\!\!/}_{\boldsymbol{\theta^A\theta^B}}(\boldsymbol{d\theta^A} - \boldsymbol{b^{\theta^A}}\boldsymbol{dv})(\boldsymbol{d\theta^B} - \boldsymbol{b^{\theta^B}}\boldsymbol{dv}) \tag{3.133}$$

on a *sufficiently small* neighbourhood $\widetilde{\boldsymbol{\mathcal{B}}} \subseteq \boldsymbol{\mathcal{M}}$, with $\boldsymbol{\Omega}^2 \in C^\infty(\boldsymbol{\mathcal{M}})$, $\boldsymbol{\gamma\!\!\!/}$ a symmetric two-tensor on the two-spheres

$$S^2_{\boldsymbol{u},\boldsymbol{v}} \equiv \{\boldsymbol{u}, \boldsymbol{v}\} \times \boldsymbol{S}^2$$

and $\boldsymbol{b}$ a vector field tangent to the $S^2_{\boldsymbol{u},\boldsymbol{v}}$ spheres.

[60] With a slight abuse of notation, we use the same notation that we adopted to define the connection coefficients and curvature components relative to the algebraically special frame of Kerr (see Section 3.4). Note that the $\mathcal{N}$ and $\mathcal{N}_{\mathrm{as}}$ are identified by the new gauge.

We consider the union of two null hypersurfaces

$$\widetilde{\mathcal{B}} \cap (\boldsymbol{C}_{\boldsymbol{u_0},\boldsymbol{v}\geq \boldsymbol{v_0}} \cup \underline{\boldsymbol{C}}_{\boldsymbol{u}\geq \boldsymbol{u_0},\boldsymbol{v_0}}) \tag{3.134}$$

on $\widetilde{\mathcal{B}}$, with $\boldsymbol{S}^2_{\boldsymbol{u_0},\boldsymbol{v_0}} \subset \widetilde{\mathcal{B}}$. We define the set

$$\mathcal{B}_{\boldsymbol{\delta}} := \bigcup_{(\boldsymbol{u}-\boldsymbol{u_0})^2+(\boldsymbol{v}-\boldsymbol{v_0})^2<\boldsymbol{\delta}^2} \boldsymbol{S}^2_{\boldsymbol{u},\boldsymbol{v}} \,,$$

with $\boldsymbol{\delta} > 0$ *sufficiently small* and such that $\mathcal{B}_{\boldsymbol{\delta}} \subset \widetilde{\mathcal{B}}$, and the set

$$\mathcal{B}^+_{\boldsymbol{\delta}} := \mathcal{B}_{\boldsymbol{\delta}} \cap \{\boldsymbol{v} \geq \boldsymbol{v_0}\} \cap \{\boldsymbol{u} \geq \boldsymbol{u_0}\} \ .$$

See Figure 3.3 in the overview for a pictorial representation of the set $\mathcal{B}^+_{\boldsymbol{\delta}}$.

We have the following theorem.

Theorem 3.2 (Outgoing frame-calibrated gauge)**.** *Consider the Lorentzian manifold* $(\mathcal{M}, \boldsymbol{g})$, *its local differentiable structure* $(\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{\theta^A})$ *and the sufficiently small set* $\mathcal{B}^+_{\boldsymbol{\delta}} \subset \mathcal{M}$ *as above. Then, for any* $\widetilde{\boldsymbol{\omega}}, \mathfrak{f}_{\boldsymbol{1}}, \mathfrak{f}_{\boldsymbol{2}}$ *smooth scalar functions of the coordinates* $(\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{\theta^A})$ *on* $\mathcal{B}^+_{\boldsymbol{\delta}}$*, there exist a system of coordinates*

$$(\boldsymbol{\tau}, \boldsymbol{s}, \boldsymbol{\vartheta}, \boldsymbol{\psi}) \tag{3.135}$$

on $\mathcal{B}^+_{\boldsymbol{\delta}}$*, with* $(\boldsymbol{\vartheta}, \boldsymbol{\psi}) \in \boldsymbol{S}^2$*, and a null frame*

$$\mathcal{N} = (\boldsymbol{e_1}, \boldsymbol{e_2}, \boldsymbol{e_3}, \boldsymbol{e_4}) \tag{3.136}$$

on $\mathcal{B}^+_{\boldsymbol{\delta}}$ *such that*

(i) ***Anchoring of the gauge****: The restriction of the coordinates* (3.135) *to*

$$\mathcal{B}^+_{\boldsymbol{\delta}} \cap (\boldsymbol{C}_{\boldsymbol{u_0},\boldsymbol{v}\geq \boldsymbol{v_0}} \cup \underline{\boldsymbol{C}}_{\boldsymbol{u}\geq \boldsymbol{u_0},\boldsymbol{v_0}}) \tag{3.137}$$

satisfies

$$(\boldsymbol{\tau}, \boldsymbol{s}, \boldsymbol{\vartheta}, \boldsymbol{\psi}) = \begin{cases} (0, \boldsymbol{v_0} - \boldsymbol{u}, \boldsymbol{\theta^1}, \boldsymbol{\theta^2}) & \text{along } \mathcal{B}^+_{\boldsymbol{\delta}} \cap \underline{\boldsymbol{C}}_{\boldsymbol{u}\geq \boldsymbol{u_0},\boldsymbol{v_0}} \\ (0, \boldsymbol{v} - \boldsymbol{u_0}, \boldsymbol{\theta^1}, \boldsymbol{\theta^2}) & \text{along } \mathcal{B}^+_{\boldsymbol{\delta}} \cap \boldsymbol{C}_{\boldsymbol{u_0},\boldsymbol{v}\geq \boldsymbol{v_0}} \end{cases} .$$

In particular, each of the two-spheres

$$\boldsymbol{S}^2_{0,\boldsymbol{s}} \equiv \{0, \boldsymbol{s}\} \times \boldsymbol{S}^2$$

on the null hypersurface (3.137) *coincides with a double-null sphere* $\boldsymbol{S}^2_{\boldsymbol{u},\boldsymbol{v}}$ *for some* $(\boldsymbol{u}, \boldsymbol{v})$*.*

(ii) ***Metric conditions**: The metric identity*

$$\boldsymbol{g}_{\boldsymbol{\tau}\boldsymbol{\tau}} = 0$$

and the identities for the Christoffel symbols

$$\begin{aligned}\boldsymbol{\Gamma}^{\boldsymbol{s}}_{\boldsymbol{\tau}\boldsymbol{\tau}} &= 0\,,\\ \boldsymbol{\Gamma}^{\boldsymbol{\vartheta}}_{\boldsymbol{\tau}\boldsymbol{\tau}} &= 0\,,\\ \boldsymbol{\Gamma}^{\boldsymbol{\psi}}_{\boldsymbol{\tau}\boldsymbol{\tau}} &= 0\end{aligned}$$

hold on $\boldsymbol{\mathcal{B}}^{+}_{\boldsymbol{\delta}}$*.*

(iii) ***Connection conditions**: The connection coefficients*

$$\hat{\boldsymbol{\omega}}\,,\ \boldsymbol{Y}\,,\ \underline{\boldsymbol{\eta}}\,,\ \boldsymbol{\Gamma}^{\boldsymbol{B}}_{\boldsymbol{4A}}$$

are such that

$$\begin{aligned}\hat{\boldsymbol{\omega}} &= \widetilde{\hat{\boldsymbol{\omega}}}\,,\\ \boldsymbol{Y} &= 0\,,\\ \underline{\boldsymbol{\eta}} &= \widetilde{\underline{\boldsymbol{\eta}}}\,,\\ \boldsymbol{\Gamma}^{\boldsymbol{B}}_{\boldsymbol{4A}} &= 0\end{aligned}$$

on $\boldsymbol{\mathcal{B}}^{+}_{\boldsymbol{\delta}}$*, where* $\widetilde{\underline{\boldsymbol{\eta}}}$ *is a* $\mathfrak{D}_{\boldsymbol{\mathcal{N}}}$ *one-tensor such that* $\widetilde{\underline{\boldsymbol{\eta}}}(\boldsymbol{e_1}) = \mathfrak{f}_{\boldsymbol{1}}$ *and* $\widetilde{\underline{\boldsymbol{\eta}}}(\boldsymbol{e_2}) = \mathfrak{f}_{\boldsymbol{2}}$ *on* $\boldsymbol{\mathcal{B}}^{+}_{\boldsymbol{\delta}}$*.*

(iv) ***Frame condition**: The coordinates* $(\boldsymbol{s}, \boldsymbol{\vartheta}, \boldsymbol{\psi})$ *are transported along the integral curves of* $\boldsymbol{e_4}$*, meaning that the identity*

$$\boldsymbol{e_4} = \partial_{\boldsymbol{\tau}} \tag{3.138}$$

holds on $\boldsymbol{\mathcal{B}}^{+}_{\boldsymbol{\delta}}$*.*

(v) ***Regularity**: The frame* $\boldsymbol{\mathcal{N}}$ *is a regular frame on* $\boldsymbol{\mathcal{B}}^{+}_{\boldsymbol{\delta}}$*.*

Proof. We consider the null hypersurface (3.134) and prescribe a *regular* null frame

$$\boldsymbol{\mathcal{N}}_0 = (\boldsymbol{e}^0_{\boldsymbol{1}}, \boldsymbol{e}^0_{\boldsymbol{2}}, \boldsymbol{e}^0_{\boldsymbol{3}}, \boldsymbol{e}^0_{\boldsymbol{4}}) \tag{3.139}$$

along it. We then define the frame

$$\boldsymbol{\mathcal{N}} = (\boldsymbol{e_1}, \boldsymbol{e_2}, \boldsymbol{e_3}, \boldsymbol{e_4}) \tag{3.140}$$

as the unique, *local* solution to the system of nonlinear ODEs for the frame along the integral curves of $\boldsymbol{e_4}$

$$\boldsymbol{\nabla}_{\boldsymbol{e_4}}\boldsymbol{e_4} = \widetilde{\boldsymbol{\omega}}\,\boldsymbol{e_4}\,, \tag{3.141}$$

$$\boldsymbol{\nabla}_{\boldsymbol{e_4}}\boldsymbol{e_3} = -\widetilde{\boldsymbol{\omega}}\,\boldsymbol{e_3} + 2\,\mathfrak{f}_{\boldsymbol{1}}\,\boldsymbol{e_1} + 2\,\mathfrak{f}_{\boldsymbol{2}}\,\boldsymbol{e_2}\,, \tag{3.142}$$

$$\boldsymbol{\nabla}_{\boldsymbol{e_4}}\boldsymbol{e_1} = \mathfrak{f}_{\boldsymbol{1}}\,\boldsymbol{e_4}\,, \tag{3.143}$$

$$\boldsymbol{\nabla}_{\boldsymbol{e_4}}\boldsymbol{e_2} = \mathfrak{f}_{\boldsymbol{2}}\,\boldsymbol{e_4} \tag{3.144}$$

on $\widetilde{\boldsymbol{\mathcal{B}}}$, with initial frame prescribed on (3.134) and coinciding with the frame $\boldsymbol{\mathcal{N}}_0$ along (3.134).

We note that the system (3.141)-(3.144) can also be written as a system for the frame components

$$(\boldsymbol{e_1}^{\boldsymbol{\mu}}, \boldsymbol{e_2}^{\boldsymbol{\mu}}, \boldsymbol{e_3}^{\boldsymbol{\mu}}, \boldsymbol{e_4}^{\boldsymbol{\mu}})$$

as scalar functions of the double-null coordinates on $\widetilde{\boldsymbol{\mathcal{B}}}$

$$\begin{aligned}
&\boldsymbol{e}_{\boldsymbol{4}}^{\boldsymbol{\nu}}\partial_{\boldsymbol{\nu}}(\boldsymbol{e}_{\boldsymbol{4}}^{\boldsymbol{\mu}}) + \boldsymbol{e_4}^{\boldsymbol{\sigma}}\,\boldsymbol{\Gamma}^{\boldsymbol{\mu}}_{\boldsymbol{\sigma\nu}}\,\boldsymbol{e_4}^{\boldsymbol{\nu}} = \widetilde{\boldsymbol{\omega}}\,\boldsymbol{e}_{\boldsymbol{4}}^{\boldsymbol{\mu}}\,,\\
&\boldsymbol{e}_{\boldsymbol{4}}^{\boldsymbol{\nu}}\partial_{\boldsymbol{\nu}}(\boldsymbol{e}_{\boldsymbol{3}}^{\boldsymbol{\mu}}) + \boldsymbol{e_4}^{\boldsymbol{\sigma}}\,\boldsymbol{\Gamma}^{\boldsymbol{\mu}}_{\boldsymbol{\sigma\nu}}\,\boldsymbol{e_3}^{\boldsymbol{\nu}} = -\widetilde{\boldsymbol{\omega}}\,\boldsymbol{e}_{\boldsymbol{3}}^{\boldsymbol{\mu}} + 2\,\mathfrak{f}_{\boldsymbol{1}}\,\boldsymbol{e_1}^{\boldsymbol{\mu}} + 2\,\mathfrak{f}_{\boldsymbol{2}}\,\boldsymbol{e_2}^{\boldsymbol{\mu}}\,,\\
&\boldsymbol{e}_{\boldsymbol{4}}^{\boldsymbol{\nu}}\partial_{\boldsymbol{\nu}}(\boldsymbol{e}_{\boldsymbol{1}}^{\boldsymbol{\mu}}) + \boldsymbol{e_4}^{\boldsymbol{\sigma}}\,\boldsymbol{\Gamma}^{\boldsymbol{\mu}}_{\boldsymbol{\sigma\nu}}\,\boldsymbol{e_1}^{\boldsymbol{\nu}} = \mathfrak{f}_{\boldsymbol{1}}\,\boldsymbol{e}_{\boldsymbol{4}}^{\boldsymbol{\mu}}\,,\\
&\boldsymbol{e}_{\boldsymbol{4}}^{\boldsymbol{\nu}}\partial_{\boldsymbol{\nu}}(\boldsymbol{e}_{\boldsymbol{2}}^{\boldsymbol{\mu}}) + \boldsymbol{e_4}^{\boldsymbol{\sigma}}\,\boldsymbol{\Gamma}^{\boldsymbol{\mu}}_{\boldsymbol{\sigma\nu}}\,\boldsymbol{e_2}^{\boldsymbol{\nu}} = \mathfrak{f}_{\boldsymbol{2}}\,\boldsymbol{e}_{\boldsymbol{4}}^{\boldsymbol{\mu}}\,,
\end{aligned}$$

with $\boldsymbol{x}^{\boldsymbol{\mu}} = (\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{\theta}^{\boldsymbol{1}}, \boldsymbol{\theta}^{\boldsymbol{2}})$ and $\boldsymbol{\Gamma}$ the known Christoffel symbols of $\boldsymbol{g}$ with respect to the double-null coordinates. The initial data now correspond to the frame components

$$(\boldsymbol{e}_{\boldsymbol{1}}^{0\,\boldsymbol{\mu}}, \boldsymbol{e}_{\boldsymbol{2}}^{0\,\boldsymbol{\mu}}, \boldsymbol{e}_{\boldsymbol{3}}^{0\,\boldsymbol{\mu}}, \boldsymbol{e}_{\boldsymbol{4}}^{0\,\boldsymbol{\mu}})$$

along (3.134).

The system of transport equations (3.141)-(3.144) for the frame ensures that $\boldsymbol{\mathcal{N}}$ is a *null* frame once $\boldsymbol{\mathcal{N}}_0$ is prescribed to be a *null* frame along (3.134). One can indeed derive the *homogeneous* system of linear ODEs

$$\begin{aligned}
&\boldsymbol{\nabla}_{\boldsymbol{e_4}}\left(\boldsymbol{g}(\boldsymbol{e_1},\boldsymbol{e_1})\right) = 2\mathfrak{f}_{\boldsymbol{1}}\boldsymbol{g}(\boldsymbol{e_4},\boldsymbol{e_1})\\
&\boldsymbol{\nabla}_{\boldsymbol{e_4}}\left(\boldsymbol{g}(\boldsymbol{e_1},\boldsymbol{e_2})\right) = \mathfrak{f}_{\boldsymbol{1}}\boldsymbol{g}(\boldsymbol{e_4},\boldsymbol{e_2}) + \mathfrak{f}_{\boldsymbol{2}}\boldsymbol{g}(\boldsymbol{e_1},\boldsymbol{e_4})\\
&\boldsymbol{\nabla}_{\boldsymbol{e_4}}\left(\boldsymbol{g}(\boldsymbol{e_2},\boldsymbol{e_2})\right) = 2\mathfrak{f}_{\boldsymbol{2}}\boldsymbol{g}(\boldsymbol{e_4},\boldsymbol{e_2})
\end{aligned}$$

$$\begin{aligned}
&\boldsymbol{\nabla}_{\boldsymbol{e_4}}\left(\boldsymbol{g}(\boldsymbol{e_4},\boldsymbol{e_4})\right) = 2\widetilde{\boldsymbol{\omega}}\,\boldsymbol{g}(\boldsymbol{e_4},\boldsymbol{e_4})\\
&\boldsymbol{\nabla}_{\boldsymbol{e_4}}\left(\boldsymbol{g}(\boldsymbol{e_3},\boldsymbol{e_3})\right) = 2\left(2\mathfrak{f}_{\boldsymbol{1}}\,\boldsymbol{g}(\boldsymbol{e_1},\boldsymbol{e_3}) + 2\mathfrak{f}_{\boldsymbol{2}}\,\boldsymbol{g}(\boldsymbol{e_2},\boldsymbol{e_3}) - \widetilde{\boldsymbol{\omega}}\,\boldsymbol{g}(\boldsymbol{e_3},\boldsymbol{e_3})\right)\\
&\boldsymbol{\nabla}_{\boldsymbol{e_4}}\left(\boldsymbol{g}(\boldsymbol{e_4},\boldsymbol{e_3})\right) = 2\mathfrak{f}_{\boldsymbol{1}}\,\boldsymbol{g}(\boldsymbol{e_1},\boldsymbol{e_4}) + 2\mathfrak{f}_{\boldsymbol{2}}\,\boldsymbol{g}(\boldsymbol{e_2},\boldsymbol{e_4})
\end{aligned}$$

$$
\begin{aligned}
\nabla_{e_4}\left(g(e_4,e_1)\right) &= \widetilde{\omega}\, g(e_4,e_1) + \mathfrak{f}_1 g(e_4,e_4)\\
\nabla_{e_4}\left(g(e_3,e_1)\right) &= 2\mathfrak{f}_1\, g(e_1,e_1) + 2\mathfrak{f}_2\, g(e_2,e_1) - \widetilde{\omega}\, g(e_3,e_1) + \mathfrak{f}_1 g(e_4,e_3)\\
\nabla_{e_4}\left(g(e_4,e_2)\right) &= \widetilde{\omega}\, g(e_4,e_2) + \mathfrak{f}_2 g(e_4,e_4)\\
\nabla_{e_4}\left(g(e_3,e_2)\right) &= 2\mathfrak{f}_1\, g(e_1,e_2) + 2\mathfrak{f}_2\, g(e_2,e_2) - \widetilde{\omega}\, g(e_3,e_2) + \mathfrak{f}_2 g(e_4,e_3)
\end{aligned}
$$

on $\widetilde{\mathcal{B}}$ for the unknowns

$$
g(e_A,e_B)\,,\; g(e_A,e_4)\,,\; g(e_A,e_3)\,,\; g(e_4,e_4)\,,\; g(e_3,e_3)\,,\; g(e_4,e_3)\,,
$$

$A, B = \{1, 2\}$, with initial conditions

$$
\begin{aligned}
g(e_A,e_B) &= g({e_A}^0,{e_B}^0) = \delta_{AB}\\
g(e_A,e_4) &= g({e_A}^0,{e_4}^0) = 0\\
g(e_A,e_3) &= g({e_A}^0,{e_3}^0) = 0\\
g(e_4,e_4) &= g({e_4}^0,{e_4}^0) = 0\\
g(e_3,e_3) &= g({e_3}^0,{e_3}^0) = 0\\
g(e_4,e_3) &= g({e_4}^0,{e_3}^0) = -2
\end{aligned}
$$

along (3.134). The system admits

$$
\begin{aligned}
g(e_A,e_B) &= \delta_{AB}\,, & g(e_A,e_4) = g(e_A,e_3) &= 0\,,\\
g(e_4,e_4) = g(e_3,e_3) &= 0\,, & g(e_4,e_3) &= -2
\end{aligned}
$$

as the unique solution on $\widetilde{\mathcal{B}}$, verifying the null frame conditions for the frame $\mathcal{N}$.

We now prescribe coordinates

$$
(\tau_0, s_0, \vartheta_0, \psi_0) \tag{3.145}
$$

along (3.134) such that

$$
(\tau_0, s_0, \vartheta_0, \psi_0) = \begin{cases} (0, v_0 - u, \theta^1, \theta^2) & \text{along } \widetilde{\mathcal{B}} \cap \underline{C}_{u\geq u_0, v_0} \\ (0, v - u_0, \theta^1, \theta^2) & \text{along } \widetilde{\mathcal{B}} \cap C_{u_0, v\geq v_0} \end{cases} .
$$

We define the coordinates

$$
(\tau, s, \vartheta, \psi)
$$

as the unique solution to the system of linear ODEs along the integral curves of e_4

$$
\begin{aligned}
e_4(\tau) &= 1 && (3.146)\\
e_4(s) &= 0 && (3.147)\\
e_4(\vartheta) &= 0 && (3.148)\\
e_4(\psi) &= 0 && (3.149)
\end{aligned}
$$

on $\widetilde{\mathcal{B}}$, with initial coordinates prescribed on (3.134) and coinciding with the coordinates $(\tau_0, s_0, \vartheta_0, \psi_0)$

along (3.134).

It is easy to check that the resulting local gauge on $\boldsymbol{\mathcal{M}}$ satisfies all the gauge conditions of Theorem 3.2 on the sufficiently small neighbourhood $\boldsymbol{\mathcal{B}}^{+}_{\boldsymbol{\delta}}$. The anchoring of the gauge (i) holds by construction. In general, the covariant derivative $\boldsymbol{\nabla_4}$ of the frame vectors read

$$\begin{aligned}
\boldsymbol{\nabla_4 e_4} &= 2\boldsymbol{Y^A e_A} + \hat{\boldsymbol{\omega}}\,\boldsymbol{e_4}\,,\\
\boldsymbol{\nabla_4 e_3} &= 2\boldsymbol{\underline{\eta}^A}\,\boldsymbol{e_A} - \hat{\boldsymbol{\omega}}\,\boldsymbol{e_3}\,,\\
\boldsymbol{\nabla_4 e_A} &= \boldsymbol{\Gamma^B_{4A} e_B} + \boldsymbol{Y_A e_3} + \boldsymbol{\underline{\eta}_A}\,\boldsymbol{e_4}\,,
\end{aligned}$$

which, compared with the system of equations (3.141)-(3.144), yield the gauge conditions for the connection coefficients. The system of equations (3.146)-(3.149) implies that $\boldsymbol{e_4} = \partial_{\boldsymbol{\tau}}$. Since $\boldsymbol{e_4}$ is null, we have $\boldsymbol{g_{\tau\tau}} = 0$. Moreover, we can write

$$\boldsymbol{\nabla_\tau}\partial_{\boldsymbol{\tau}} = \widetilde{\hat{\boldsymbol{\omega}}}\,\partial_{\boldsymbol{\tau}}\,,$$

which implies the gauge conditions for the Christoffel symbols. □

We end the section with a remark.

Remark 3.6. *Note that, in Theorem 3.2, we do* not *assume any special algebraic property for* $(\boldsymbol{\mathcal{M}}, \boldsymbol{g})$*. We also do* not *assume that* $(\boldsymbol{\mathcal{M}}, \boldsymbol{g})$ *is a solution to the vacuum Einstein equations. These two facts may be contrasted to the* linear *gauge adopted by [1].*

The residual gauge freedom

Our outgoing frame-calibrated gauge of Theorem 3.2 comes with a residual freedom in constructing both the coordinates $(\boldsymbol{\tau}, \boldsymbol{s}, \boldsymbol{\vartheta}, \boldsymbol{\psi})$ and the frame $\boldsymbol{\mathcal{N}}$.

The residual gauge freedom *for the frame* $\boldsymbol{\mathcal{N}}$ can be understood as the freedom to prescribe the initial frame $\boldsymbol{\mathcal{N}}_0$ along the hypersurface (3.134) in the proof of the theorem.

The residual gauge freedom *for the coordinates* $(\boldsymbol{\tau}, \boldsymbol{s}, \boldsymbol{\vartheta}, \boldsymbol{\psi})$ coincides, by construction, with the residual gauge freedom to redefine the double-null foliation of the hypersurface (3.134) (and, in turn, with redefining the initial coordinates $(\boldsymbol{\tau}_0, \boldsymbol{s}_0, \boldsymbol{\vartheta}_0, \boldsymbol{\psi}_0)$ along (3.134) in the proof of the theorem).

The residual freedom of the new gauge induces some residual freedom in the definition of the one-parameter family of metrics $\boldsymbol{g}(\epsilon)$ in the new gauge. Such a residual freedom will be crucially exploited in forthcoming work by the author.

3.5.2 Tensor perturbations

In this section of the chapter, we present a rigorous definition of tensor perturbations. The precise notion of tensor perturbations is important to understand the formal meaning of our gauge conditions for $\boldsymbol{g}(\epsilon)$ [61], to correctly formulate the nonlinear vacuum Einstein equations for $\boldsymbol{g}(\epsilon)$ in our outgoing frame-calibrated gauge and for the linearisation procedure of Section 3.7.1.

A rigorous definition of covariant tensor perturbations

We consider the one-parameter family of metrics $\boldsymbol{g}(\epsilon)$ on $\mathcal{M}^+$ in the new gauge. To formulate tensor perturbations, we will need to compare $\mathfrak{D}_{\boldsymbol{\mathcal{N}}(\epsilon)}$ k-tensors $\boldsymbol{\mathcal{T}}(\epsilon)$ to *non-vanishing*[62] $\mathfrak{D}_{\boldsymbol{\mathcal{N}}(0)}$ k-tensors $\boldsymbol{\mathcal{T}}(0) = \mathcal{T}$ on $\mathcal{M}^+$. Since the frame vectors $\boldsymbol{e_A}(\epsilon)$ are *not* fixed on $\mathcal{M}^+$ for all $\epsilon > 0$, we have

$$\mathfrak{D}_{\boldsymbol{\mathcal{N}}(\epsilon)} \not\equiv \mathfrak{D}_{\boldsymbol{\mathcal{N}}(0)} \tag{3.150}$$

on $\mathcal{M}^+$ for all $\epsilon > 0$. The tensors $\boldsymbol{\mathcal{T}}(\epsilon)$ and $\boldsymbol{\mathcal{T}}(0)$ thus live in different tensor bundles on $\mathcal{M}^+$.

Remark 3.7. *Relation* (3.150) *does not hold for a double-null gauge, for which* $\mathfrak{D}_{\boldsymbol{\mathcal{N}}(\epsilon)} \equiv \mathfrak{D}_{\boldsymbol{\mathcal{N}}(0)}$ *for all* $\epsilon \geq 0$*.*

To understand the comparison between such tensors, we first *extend* the $\mathfrak{D}_{\boldsymbol{\mathcal{N}}(\epsilon)}$ k-tensors to *horizontal* k-tensors on $\mathcal{M}^+$. We then introduce a five-dimensional manifold $\mathcal{V}$, that we think as foliated by submanifolds diffeomorphic to $\mathcal{M}^+$ such that

$$\mathcal{V} \cong \mathcal{M}^+ \times \mathbb{R}_\epsilon \,. \tag{3.151}$$

We denote the slices of the foliation of $\mathcal{V}$ by $\mathcal{M}^+_\epsilon$. Each $\mathcal{M}^+_\epsilon$ corresponds to the Lorentzian manifold $(\mathcal{M}^+, \boldsymbol{g}(\epsilon))$ for a fixed value of $\epsilon \geq 0$.

We choose the differentiable structure of $\mathcal{V}$ to be the one induced by the fixed differentiable structure of $\mathcal{M}^+$, namely we have coordinates

$$(\tau, s, \vartheta, \psi, \epsilon) \tag{3.152}$$

on $\mathcal{V}$, where each of the leaves $\mathcal{M}^+_\epsilon$ corresponds to a level set of the coordinate ϵ.

The one-parameter families of metric perturbations $\boldsymbol{g}(\epsilon)$ and horizontal k-tensors $\boldsymbol{\mathcal{T}}(\epsilon)$ can now be understood as genuine *tensors* (not a *one-parameter family* any more) on $\mathcal{V}$ such that

$$\boldsymbol{g}(p,\epsilon) := \boldsymbol{g}_p(\epsilon)\,, \qquad \boldsymbol{\mathcal{T}}(p,\epsilon) := \boldsymbol{\mathcal{T}}_p(\epsilon)\,,$$

for any $p \in \mathcal{M}^+_\epsilon$ and

$$\boldsymbol{g}(p,\epsilon)(\partial_\epsilon) := 0\,, \qquad \boldsymbol{\mathcal{T}}(p,\epsilon)(\partial_\epsilon) := 0\,.$$

The one-parameter family of null frames $\boldsymbol{\mathcal{N}}(\epsilon)$ naturally induces a *frame* (again, not a *one-*

[61] See, for instance, the identity (3.130).
[62] In [19], the only non-vanishing tensor for $\epsilon = 0$ is the metric $\not{g}(0)$.

parameter family any more) on $\mathcal{V}$ with frame vectors

$$\begin{aligned} \boldsymbol{e_4}(p,\epsilon) &:= \boldsymbol{e_4}_p(\epsilon)\,,\\ \boldsymbol{e_3}(p,\epsilon) &:= \boldsymbol{e_3}_p(\epsilon)\,,\\ \boldsymbol{e_A}(p,\epsilon) &:= \boldsymbol{e_A}_p(\epsilon) \end{aligned}$$

and

$$\boldsymbol{e_5}(p,\epsilon) := \partial_\epsilon|_p\,,$$

where the coordinate form of the frame vector $\boldsymbol{e_5}$ is relative to the differentiable structure (3.152) on $\mathcal{V}$.

Comparing $\boldsymbol{\mathcal{T}}(\epsilon)$ and $\mathcal{T}$ can now be seen as comparing the *same* tensor at two different points (lying on two different slices $\mathcal{M}^+_\epsilon$) of $\mathcal{V}$. To identify points on different slices, we define the diffeomorphism

$$\begin{aligned} \phi_\epsilon &: \mathcal{V} \to \mathcal{V}\\ \phi_\epsilon|_{\mathcal{M}^+_0} &: \mathcal{M}^+_0 \to \mathcal{M}^+_\epsilon \end{aligned}$$

generated by the frame vector field $\boldsymbol{e_5}$. We consider the pull-back tensor

$$\phi^*_\epsilon \boldsymbol{\mathcal{T}} \tag{3.153}$$

on $\mathcal{M}^+_0$, where, in view of the definition of ϕ_ϵ, we have

$$(\phi^*_\epsilon \boldsymbol{\mathcal{T}})_{\mu_1\cdots\mu_k} = \phi^*_\epsilon\left(\boldsymbol{\mathcal{T}}_{\mu_1\cdots\mu_k}\right) \tag{3.154}$$

on $\mathcal{M}^+_0$.

We observe that the tensor (3.153) is *not*, in general, horizontal on $\mathcal{M}^+_0$ (relative to $\boldsymbol{\mathcal{N}}(0)$). In fact, pulling back the distribution $\mathfrak{D}_{\boldsymbol{\mathcal{N}}(\epsilon)}$, one has

$$\phi^*_\epsilon(\mathfrak{D}_{\boldsymbol{\mathcal{N}}(\epsilon)}) \not\equiv \mathfrak{D}_{\boldsymbol{\mathcal{N}}(0)}$$

on $\mathcal{M}^+_0$.[63]

Given the projection tensor Π on $\mathcal{M}^+_0$ (relative to $\boldsymbol{\mathcal{N}}(0)$), we have the following definition:

Definition 3.1. *Given a $\mathfrak{D}_{\boldsymbol{\mathcal{N}}(0)}$ k-tensor $\mathcal{T}$ on $\mathcal{M}^+_0$, we define the* <u>*k-tensor perturbation of $\mathcal{T}$*</u> *as the restriction to a $\mathfrak{D}_{\boldsymbol{\mathcal{N}}(0)}$ k-tensor of the horizontal k-tensor*

$$\Pi\phi^*_\epsilon \boldsymbol{\mathcal{T}} - \mathcal{T} \tag{3.155}$$

on $\mathcal{M}^+_0$, where

$$(\Pi\phi^*_\epsilon \boldsymbol{\mathcal{T}})_{\mu_1\cdots\mu_k} := \Pi^{\nu_1}_{\mu_1}\cdots\Pi^{\nu_k}_{\mu_k}\,(\phi^*_\epsilon \boldsymbol{\mathcal{T}})_{\nu_1\cdots\nu_k}$$

[63] Note that tensor (3.153) is horizontal on $\mathcal{M}^+_0$ relative to $\phi^*_\epsilon(\mathfrak{D}_{\boldsymbol{\mathcal{N}}(\epsilon)})$.

on $\mathcal{M}_0^+$.

Note that tensor perturbations preserve the symmetries of tensors. We conclude with two caveats. First, note that

$$(\Pi\phi_\epsilon^*\boldsymbol{\mathcal{T}})_{\mu_1\cdots\mu_k} \neq \Pi\phi_\epsilon^*(\boldsymbol{\mathcal{T}}_{\mu_1\cdots\mu_k})$$

on $\mathcal{M}_0^+$, which should be contrasted to the identity (3.154). Second, note that

$$\Pi\phi_\epsilon^*(\boldsymbol{\mathcal{T}}^{\boldsymbol{\sharp}}) \neq (\Pi\phi_\epsilon^*\boldsymbol{\mathcal{T}})^\sharp$$

on $\mathcal{M}_0^+$, where the sharp symbol on the right hand side is unbolded.

Remark 3.8. *The choice of diffeomorphism ϕ_ϵ encodes the fact that, although we formulate tensor perturbations* geometrically*, we still rely on the fixed differentiable structure of $\mathcal{M}^+$.*

A rigorous definition of frame perturbations

We give the following definition, using notation $\boldsymbol{e_I}(0) = e_I$ for frame vectors of $\boldsymbol{\mathcal{N}}(0)$.

Definition 3.2. *Given a frame vector field e_I of the frame $\boldsymbol{\mathcal{N}}(0)$ on $\mathcal{M}_0^+$, $I = 1, \ldots, 4$, we define the frame perturbation of e_I as the vector field*

$$\phi_{-\epsilon *}\boldsymbol{e_I} - e_I$$

on $\mathcal{M}_0^+$, where

$$(\phi_{-\epsilon *}\boldsymbol{e_I})^\mu := \phi_\epsilon^*\left(\boldsymbol{e_I}^\mu\right)$$

on $\mathcal{M}_0^+$.

Remark 3.9. *The reader should note that the frame perturbation of e_A* can *fail to be in $\mathfrak{D}_{\boldsymbol{\mathcal{N}}(0)}$.*

A rigorous definition of scalar perturbations

Consider a smooth scalar function $\mathfrak{f}(\epsilon)$ on $\mathcal{V}$, with $\mathfrak{f}(0) = \mathfrak{f}$ on $\mathcal{M}_0^+$. We have the following definition, which concludes the rigorous definitions of tensor perturbations.

Definition 3.3. *Given a scalar function $\mathfrak{f} \in C^\infty(\mathcal{M}_0^+)$, we define the scalar perturbation of $\mathfrak{f}$ as the smooth scalar function*

$$\Pi\phi_\epsilon^*\mathfrak{f} - \mathfrak{f}$$

on $\mathcal{M}_0^+$.

Note that

$$\Pi\phi_\epsilon^*\mathfrak{f} = \phi_\epsilon^*\mathfrak{f} \tag{3.156}$$

on $\mathcal{M}_0^+$.

Perturbing products and covariant derivatives of tensors

In this section we clarify how to treat perturbations of products and covariant derivatives of $\mathfrak{D}_{\boldsymbol{\mathcal{N}}(\epsilon)}$ covariant tensors on $\mathcal{M}^+$ in view of the definitions of the previous sections. The products and covariant derivatives of $\mathfrak{D}_{\boldsymbol{\mathcal{N}}(\epsilon)}$ covariant tensors are $\mathfrak{D}_{\boldsymbol{\mathcal{N}}(\epsilon)}$ covariant tensors. Their perturbations are therefore defined as in Definition 3.1. For future convenience,[64] we derive formulae for the projected pullback $\Pi\phi_\epsilon^*$ when applied to products and covariant derivatives.

Consider the $\mathfrak{D}_{\boldsymbol{\mathcal{N}}(\epsilon)}$ k-tensors $\boldsymbol{\mathcal{T}}(\epsilon), \widetilde{\boldsymbol{\mathcal{T}}}(\epsilon)$ on $\mathcal{V}$, with $\mathcal{V}$ as in (3.151). We have

$$\Pi\phi_\epsilon^*(\boldsymbol{\mathcal{T}} \otimes \widetilde{\boldsymbol{\mathcal{T}}}) = (\Pi\phi_\epsilon^*\, \boldsymbol{\mathcal{T}}) \otimes (\Pi\phi_\epsilon^*\, \widetilde{\boldsymbol{\mathcal{T}}}) \tag{3.157}$$

and

$$(\Pi\phi_\epsilon^*(\boldsymbol{\mathcal{T}}^{\sharp_1}))^\nu_{\mu_2\cdots\mu_k} = \Pi^{\alpha_2}_{\mu_2}\cdots\Pi^{\alpha_k}_{\mu_k}\left(\phi_\epsilon^*\, \not{\boldsymbol{g}}^{-1}\right)^{\nu\alpha_1}(\phi_\epsilon^*\, \boldsymbol{\mathcal{T}})_{\alpha_1\alpha_2\cdots\alpha_k} \tag{3.158}$$

on $\mathcal{M}_0^+$. Given $\mathfrak{D}_{\boldsymbol{\mathcal{N}}(\epsilon)}$ one-tensors $\boldsymbol{\xi}(\epsilon), \tilde{\boldsymbol{\xi}}(\epsilon)$ and $\mathfrak{D}_{\boldsymbol{\mathcal{N}}(\epsilon)}$ two-tensors $\boldsymbol{\theta}(\epsilon), \tilde{\boldsymbol{\theta}}(\epsilon)$ on $\mathcal{V}$, we have

$$\begin{aligned}
\Pi\phi_\epsilon^*((\boldsymbol{\xi}, \tilde{\boldsymbol{\xi}})) &= \phi_\epsilon^*((\boldsymbol{\xi}, \tilde{\boldsymbol{\xi}}))\,,\\
\Pi\phi_\epsilon^*((\boldsymbol{\theta}, \tilde{\boldsymbol{\theta}})) &= \phi_\epsilon^*((\boldsymbol{\theta}, \tilde{\boldsymbol{\theta}}))\,,\\
\Pi\phi_\epsilon^*(\boldsymbol{\xi} \wedge \tilde{\boldsymbol{\xi}}) &= \phi_\epsilon^*(\boldsymbol{\xi} \wedge \tilde{\boldsymbol{\xi}})\,,\\
\Pi\phi_\epsilon^*(\boldsymbol{\theta} \wedge \tilde{\boldsymbol{\theta}}) &= \phi_\epsilon^*(\boldsymbol{\theta} \wedge \tilde{\boldsymbol{\theta}})\,,\\
(\Pi\phi_\epsilon^*(\boldsymbol{\theta} \times \tilde{\boldsymbol{\theta}}))_{\mu\nu} &= \Pi^\alpha_\nu(\Pi\phi_\epsilon^*(\boldsymbol{\theta}^{\sharp_2}))^\sigma_\mu(\phi_\epsilon^*\, \tilde{\boldsymbol{\theta}})_{\sigma\alpha}
\end{aligned}$$

and

$$\Pi\phi_\epsilon^*(\boldsymbol{\xi}\,\widehat{\otimes}\,\tilde{\boldsymbol{\xi}}) = \Pi\phi_\epsilon^*(\boldsymbol{\xi} \otimes \tilde{\boldsymbol{\xi}}) + \Pi\phi_\epsilon^*(\tilde{\boldsymbol{\xi}} \otimes \boldsymbol{\xi}) - (\Pi\phi_\epsilon^*((\boldsymbol{\xi}, \tilde{\boldsymbol{\xi}})))(\Pi\phi_\epsilon^*\, \not{\boldsymbol{g}}) \tag{3.159}$$

on $\mathcal{M}_0^+$, where we used (3.156), (3.157) and (3.158). Note that, in general, the product (3.159) is *not* traceless relative to $\not{\boldsymbol{g}}^{-1}$.

Given the $\mathfrak{D}_{\boldsymbol{\mathcal{N}}(\epsilon)}$ $(1,1)$-projection tensor $\boldsymbol{\Pi}(\epsilon)$ on $\mathcal{V}$, we have

$$(\Pi\phi_\epsilon^*\, \boldsymbol{\Pi})^\nu_\mu = \Pi^\sigma_\mu\,(\phi_\epsilon^*\boldsymbol{\delta})^\nu_\sigma + \frac{1}{2}\,\Pi^\sigma_\mu\,(\phi_{-\epsilon *}\boldsymbol{e_3})^\gamma\,(\phi_{-\epsilon *}\boldsymbol{e_4})^\nu\,(\phi_\epsilon^*\,\boldsymbol{g})_{\sigma\gamma} + \frac{1}{2}\,\Pi^\sigma_\mu\,(\phi_{-\epsilon *}\boldsymbol{e_4})^\gamma\,(\phi_{-\epsilon *}\boldsymbol{e_3})^\nu\,(\phi_\epsilon^*\,\boldsymbol{g})_{\sigma\gamma} \tag{3.160}$$

and

$$(\Pi\phi_\epsilon^*(\not{\boldsymbol{\nabla}}\, \mathfrak{f}))_\mu = (\Pi\phi_\epsilon^*\boldsymbol{\Pi})^\sigma_\mu(\phi_\epsilon^*\boldsymbol{\nabla})_\sigma(\phi_\epsilon^*\, \mathfrak{f})\,, \tag{3.161}$$

$$\Pi\phi_\epsilon^*(\not{\boldsymbol{\nabla}}_{\boldsymbol{e_4}}\, \mathfrak{f}) = (\phi_{-\epsilon *}\, \boldsymbol{e_4})^\mu\partial_\mu(\phi_\epsilon^*\, \mathfrak{f})\,, \tag{3.162}$$

$$\Pi\phi_\epsilon^*(\not{\boldsymbol{\nabla}}_{\boldsymbol{e_3}}\, \mathfrak{f}) = (\phi_{-\epsilon *}\, \boldsymbol{e_3})^\mu\partial_\mu(\phi_\epsilon^*\, \mathfrak{f})\,, \tag{3.163}$$

[64] See the linearisation procedure of Section 3.7.1.

$$(\Pi\phi_\epsilon^*(\nabla\!\!\!/\,\boldsymbol{\mathcal{T}}))_{\nu\mu_1\cdots\mu_k} = (\Pi\phi_\epsilon^*\mathbf{\Pi})_\nu^\sigma(\Pi\phi_\epsilon^*\mathbf{\Pi})_{\mu_1}^{\alpha_1}\cdots(\Pi\phi_\epsilon^*\mathbf{\Pi})_{\mu_k}^{\alpha_k}((\phi_\epsilon^*\boldsymbol{\nabla})_\sigma(\phi_\epsilon^*\,\boldsymbol{\mathcal{T}}))_{\alpha_1\cdots\alpha_k}\,, \tag{3.164}$$

$$(\Pi\phi_\epsilon^*(\nabla\!\!\!/_{\boldsymbol{e_4}}\,\boldsymbol{\mathcal{T}}))_{\mu_1\cdots\mu_k} = (\Pi\phi_\epsilon^*\mathbf{\Pi})_{\mu_1}^{\alpha_1}\cdots(\Pi\phi_\epsilon^*\mathbf{\Pi})_{\mu_k}^{\alpha_k}(\phi_{-\epsilon*}\boldsymbol{e_4})^\nu((\phi_\epsilon^*\boldsymbol{\nabla})_\nu(\phi_\epsilon^*\,\boldsymbol{\mathcal{T}}))_{\alpha_1\cdots\alpha_k}\,, \tag{3.165}$$

$$(\Pi\phi_\epsilon^*(\nabla\!\!\!/_{\boldsymbol{e_3}}\,\boldsymbol{\mathcal{T}}))_{\mu_1\cdots\mu_k} = (\Pi\phi_\epsilon^*\mathbf{\Pi})_{\mu_1}^{\alpha_1}\cdots(\Pi\phi_\epsilon^*\mathbf{\Pi})_{\mu_k}^{\alpha_k}(\phi_{-\epsilon*}\boldsymbol{e_3})^\nu((\phi_\epsilon^*\boldsymbol{\nabla})_\nu(\phi_\epsilon^*\,\boldsymbol{\mathcal{T}}))_{\alpha_1\cdots\alpha_k} \tag{3.166}$$

on $\mathcal{M}_0^+$.

Proof. We compute

$$\begin{aligned}
(\Pi\phi_\epsilon^*(\nabla\!\!\!/\,\mathfrak{f}))_\mu &= \Pi_\mu^\nu(\phi_\epsilon^*(\nabla\!\!\!/\,\mathfrak{f}))_\nu \\
&= \Pi_\mu^\nu(\phi_\epsilon^*(\mathbf{\Pi}_\nu^\sigma(\boldsymbol{\nabla}\mathfrak{f})_\sigma)) \\
&= (\Pi\phi_\epsilon^*\mathbf{\Pi})_\mu^\sigma(\phi_\epsilon^*(\boldsymbol{\nabla}\mathfrak{f}))_\sigma \\
&= (\Pi\phi_\epsilon^*\mathbf{\Pi})_\mu^\sigma((\phi_\epsilon^*\boldsymbol{\nabla})_\sigma(\phi_\epsilon^*\mathfrak{f}))\,, \\
\Pi\phi_\epsilon^*(\nabla\!\!\!/_{\boldsymbol{e_4}}\,\mathfrak{f}) &= \phi_\epsilon^*(\nabla\!\!\!/_{\boldsymbol{e_4}}\,\mathfrak{f}) \\
&= \phi_\epsilon^*(\boldsymbol{e_4}(\mathfrak{f})) \\
&= (\phi_{-\epsilon*}\,\boldsymbol{e_4})^\mu\partial_\mu(\phi_\epsilon^*\mathfrak{f})\,, \\
\Pi\phi_\epsilon^*(\nabla\!\!\!/_{\boldsymbol{e_3}}\,\mathfrak{f}) &= (\phi_{-\epsilon*}\,\boldsymbol{e_3})^\mu\partial_\mu(\phi_\epsilon^*\mathfrak{f})
\end{aligned}$$

and

$$\begin{aligned}
(\Pi\phi_\epsilon^*(\nabla\!\!\!/\,\boldsymbol{\xi}))_{\mu\nu} &= \Pi_\mu^\alpha\Pi_\nu^\beta(\phi_\epsilon^*(\nabla\!\!\!/\,\boldsymbol{\xi}))_{\alpha\beta} \\
&= \Pi_\mu^\alpha\Pi_\nu^\beta(\phi_\epsilon^*(\mathbf{\Pi}_\alpha^\sigma\mathbf{\Pi}_\beta^\gamma\boldsymbol{\nabla}\boldsymbol{\xi}))_{\sigma\gamma} \\
&= (\Pi\phi_\epsilon^*\mathbf{\Pi})_\mu^\sigma(\Pi\phi_\epsilon^*\mathbf{\Pi})_\nu^\gamma(\phi_\epsilon^*(\boldsymbol{\nabla}\boldsymbol{\xi}))_{\sigma\gamma} \\
&= (\Pi\phi_\epsilon^*\mathbf{\Pi})_\mu^\sigma(\Pi\phi_\epsilon^*\mathbf{\Pi})_\nu^\gamma((\phi_\epsilon^*\boldsymbol{\nabla})_\sigma(\phi_\epsilon^*\boldsymbol{\xi}))_\gamma\,, \\
(\Pi\phi_\epsilon^*(\nabla\!\!\!/_{\boldsymbol{e_4}}\,\boldsymbol{\xi}))_\mu &= \Pi_\mu^\nu(\phi_\epsilon^*(\nabla\!\!\!/_{\boldsymbol{e_4}}\,\boldsymbol{\xi}))_\nu \\
&= \Pi_\mu^\nu(\phi_\epsilon^*(\mathbf{\Pi}_\nu^\alpha\boldsymbol{\nabla}_{\boldsymbol{e_4}}\boldsymbol{\xi}))_\alpha \\
&= (\Pi\phi_\epsilon^*\mathbf{\Pi})_\mu^\alpha(\phi_\epsilon^*(\boldsymbol{\nabla}_{\boldsymbol{e_4}}\boldsymbol{\xi}))_\alpha \\
&= (\Pi\phi_\epsilon^*\mathbf{\Pi})_\mu^\alpha(\phi_{-\epsilon*}\boldsymbol{e_4})^\nu((\phi_\epsilon^*\boldsymbol{\nabla})_\nu(\phi_\epsilon^*\boldsymbol{\xi}))_\alpha\,, \\
(\Pi\phi_\epsilon^*(\nabla\!\!\!/_{\boldsymbol{e_3}}\,\boldsymbol{\xi}))_\mu &= (\Pi\phi_\epsilon^*\mathbf{\Pi})_\mu^\alpha(\phi_{-\epsilon*}\boldsymbol{e_3})^\nu((\phi_\epsilon^*\boldsymbol{\nabla})_\nu(\phi_\epsilon^*\boldsymbol{\xi}))_\alpha
\end{aligned}$$

for $\boldsymbol{\xi}(\epsilon)$ a $\mathfrak{D}_{\boldsymbol{\mathcal{N}}(\epsilon)}$ one-tensor on $\mathcal{V}$. The higher rank formulae can be derived analogously. □

Consider now the $\mathfrak{D}_{\boldsymbol{\mathcal{N}}(\epsilon)}$ one-tensor $\boldsymbol{\xi}(\epsilon)$ and the $\mathfrak{D}_{\boldsymbol{\mathcal{N}}(\epsilon)}$ two-tensor $\boldsymbol{\theta}(\epsilon)$ on $\mathcal{V}$. We have

$$\begin{aligned}
\Pi\phi_\epsilon^*(\mathbf{d\!\!\!/iv}\,\boldsymbol{\xi}) &= \phi_\epsilon^*(\mathbf{d\!\!\!/iv}\,\boldsymbol{\xi})\,, \\
(\Pi\phi_\epsilon^*(\mathbf{d\!\!\!/iv}\,\boldsymbol{\theta}))_\mu &= \Pi_\mu^\alpha(\phi_\epsilon^*\,\boldsymbol{g\!\!\!/}^{-1})^{\sigma\gamma}(\phi_\epsilon^*(\nabla\!\!\!/\boldsymbol{\theta}))_{\sigma\alpha\gamma}
\end{aligned}$$

and

$$\Pi\phi_\epsilon^*(\mathbf{cu\!\!\!/rl}\,\boldsymbol{\xi}) = \phi_\epsilon^*(\mathbf{cu\!\!\!/rl}\,\boldsymbol{\xi})$$

on $\mathcal{M}_0^+$. We also have

$$(\Pi\phi_\epsilon^*(\not{\mathcal{D}}_{\mathbf{2}}^\star\,\boldsymbol{\xi}))_{\mu\nu} = -\frac{1}{2}\left((\Pi\phi_\epsilon^*(\not{\nabla}\,\boldsymbol{\xi}))_{\mu\nu} + (\Pi\phi_\epsilon^*(\not{\nabla}\,\boldsymbol{\xi}))_{\nu\mu} - (\Pi\phi_\epsilon^*(\mathbf{d}\not{\mathbf{i}}\mathbf{v}\,\boldsymbol{\xi}))(\Pi\phi_\epsilon^*\,\not{\boldsymbol{g}})_{\mu\nu}\right)$$

on $\mathcal{M}_0^+$.

In view of the pullback formulae (3.165), (3.166) and the pullback formulae for the products of covariant tensors, one can use relations (3.69)-(3.72) to derive the pullback formulae for

$$\Pi\phi_\epsilon^*(\not{\mathcal{L}}_{\mathbf{4}}\boldsymbol{\xi})\,,\ \Pi\phi_\epsilon^*(\not{\mathcal{L}}_{\mathbf{3}}\boldsymbol{\xi})\,,\ \Pi\phi_\epsilon^*(\not{\mathcal{L}}_{\mathbf{4}}\boldsymbol{\theta})\,,\ \Pi\phi_\epsilon^*(\not{\mathcal{L}}_{\mathbf{3}}\boldsymbol{\theta})\,.$$

Alternatively, one can simply write

$$\begin{aligned}
(\Pi\phi_\epsilon^*(\not{\mathcal{L}}_{\mathbf{4}}\boldsymbol{\xi}))_\mu &= (\Pi\phi_\epsilon^*\mathbf{\Pi})_\mu^\sigma(\phi_\epsilon^*(\boldsymbol{\mathcal{L}}_{\mathbf{4}}\boldsymbol{\xi}))_\sigma\,,\\
(\Pi\phi_\epsilon^*(\not{\mathcal{L}}_{\mathbf{3}}\boldsymbol{\xi}))_\mu &= (\Pi\phi_\epsilon^*\mathbf{\Pi})_\mu^\sigma(\phi_\epsilon^*(\boldsymbol{\mathcal{L}}_{\mathbf{3}}\boldsymbol{\xi}))_\sigma\,,\\
(\Pi\phi_\epsilon^*(\not{\mathcal{L}}_{\mathbf{4}}\boldsymbol{\theta}))_{\mu\nu} &= (\Pi\phi_\epsilon^*\mathbf{\Pi})_\mu^\sigma(\Pi\phi_\epsilon^*\mathbf{\Pi})_\nu^\gamma(\phi_\epsilon^*(\boldsymbol{\mathcal{L}}_{\mathbf{4}}\boldsymbol{\theta}))_{\sigma\gamma}\,,\\
(\Pi\phi_\epsilon^*(\not{\mathcal{L}}_{\mathbf{3}}\boldsymbol{\theta}))_{\mu\nu} &= (\Pi\phi_\epsilon^*\mathbf{\Pi})_\mu^\sigma(\Pi\phi_\epsilon^*\mathbf{\Pi})_\nu^\gamma(\phi_\epsilon^*(\boldsymbol{\mathcal{L}}_{\mathbf{3}}\boldsymbol{\theta}))_{\sigma\gamma}
\end{aligned}$$

on $\mathcal{M}_0^+$.

3.5.3 Properties of the new gauge

In this section we collect some properties of the family of metrics $\boldsymbol{g}(\epsilon)$ on $\mathcal{M}^+$ in our outgoing frame-calibrated gauge:

- The family $\boldsymbol{\mathcal{N}}(\epsilon)$ is a one-parameter family of *non-integrable* null frames on $\mathcal{M}^+$. This motivates, a posteriori, the formalism developed in Section 3.3.1.
- We have the one-parameter family of systems of transport equations for the frame in the $\boldsymbol{e_4}(\epsilon)$-direction

$$\begin{aligned}
\boldsymbol{\nabla}_{\boldsymbol{e_4}(\epsilon)}\boldsymbol{e_4}(\epsilon) &= \hat{\boldsymbol{\omega}}(\epsilon)\,\boldsymbol{e_4}(\epsilon) + 2\,\boldsymbol{Y}^{\boldsymbol{A}}(\epsilon)\,\boldsymbol{e_A}(\epsilon)\,,\\
\boldsymbol{\nabla}_{\boldsymbol{e_4}(\epsilon)}\boldsymbol{e_3}(\epsilon) &= -\hat{\boldsymbol{\omega}}(\epsilon)\,\boldsymbol{e_3}(\epsilon) + 2\,\underline{\boldsymbol{\eta}}^{\boldsymbol{A}}(\epsilon)\,\boldsymbol{e_A}(\epsilon)\,,\\
\boldsymbol{\nabla}_{\boldsymbol{e_4}(\epsilon)}\boldsymbol{e_A}(\epsilon) &= \boldsymbol{\Gamma}^{\boldsymbol{B}}_{\boldsymbol{4A}}(\epsilon)\,\boldsymbol{e_B}(\epsilon) + \boldsymbol{Y_A}(\epsilon)\,\boldsymbol{e_3}(\epsilon) + \underline{\boldsymbol{\eta}}_{\boldsymbol{A}}(\epsilon)\,\boldsymbol{e_4}(\epsilon)\,,
\end{aligned}$$

  on $\mathcal{M}^+$, with

$$\Pi\phi_\epsilon^*(\hat{\boldsymbol{\omega}}(\epsilon)) - \hat{\omega}^{\text{as}}_{\text{Kerr}} = 0\,, \tag{3.167}$$

$$\Pi\phi_\epsilon^*(\boldsymbol{Y}(\epsilon)) = 0\,,$$

$$\Pi\phi_\epsilon^*(\underline{\boldsymbol{\eta}}(\epsilon)) - \underline{\eta}^{\text{as}}_{\text{Kerr}} = 0\,,$$

$$\Pi\phi_\epsilon^*(\boldsymbol{\Gamma}^{\boldsymbol{B}}_{\boldsymbol{4A}}(\epsilon)) = 0\,. \tag{3.168}$$

  for all $\epsilon \geq 0$ on $\mathcal{M}^+$.

- Coordinates transported along the integral curves of a frame vector field, such as the coordinates $(\tau, s, \vartheta, \psi)$ on $\mathcal{M}^+$, are sometimes called *Lagrangian coordinates.* Note however that this terminology is more frequent in the context of orthonormal frames, where $\boldsymbol{e_4}(\epsilon)$ is typically a *timelike* frame vector.

Remark 3.10. *We note that one could consider a one-parameter family of metric perturbations* $\boldsymbol{g}(\epsilon)$ *on* $\mathcal{M}^+$*, with an associated one-parameter family of null frames* $\boldsymbol{\mathcal{N}}(\epsilon)$*, satisfying the gauge conditions*

$$\begin{aligned} \Pi\phi_\epsilon^*(\underline{\hat{\boldsymbol{\omega}}}(\epsilon)) - \underline{\hat{\omega}}\,^{\text{as}}_{\text{Kerr}} &= 0\,, \\ \Pi\phi_\epsilon^*(\underline{\boldsymbol{Y}}(\epsilon)) &= 0\,, \\ \Pi\phi_\epsilon^*(\boldsymbol{\eta}(\epsilon)) - \eta^{\text{as}}_{\text{Kerr}} &= 0\,, \\ \Pi\phi_\epsilon^*(\boldsymbol{\Gamma^B_{3A}}(\epsilon)) - \Gamma^{B\ \text{as}}_{3A\text{Kerr}} &= 0 \end{aligned}$$

and

$$\boldsymbol{e_3}(\epsilon) = \partial_\tau$$

on $\mathcal{M}^+$ *for all* $\epsilon \geq 0$*. Our choice of gauge is motivated by the form of the vacuum Einstein equations for* $\boldsymbol{g}(\epsilon)$*. In fact, our gauge conditions allow to handle some potentially problematic terms appearing in the transport equations in the* $\boldsymbol{e_4}(\epsilon)$*-direction. This will be crucial in our analysis of the linearised system.*

3.6 The vacuum Einstein equations around Kerr in the new gauge

This section concludes the second of the three self-contained blocks composing the body of the chapter. See the beginning of Section 3.2 for the general structure of the chapter.

In this section we address the nonlinear vacuum Einstein equations for the family of metrics $\boldsymbol{g}(\epsilon)$ in our outgoing frame-calibrated gauge. In principle, one could re-write the full system of equations presented in Section 3.3.5 (and derived in Appendix A) specialised to our gauge. However, since our ultimate goal is the analysis of the *linearised* system of vacuum Einstein equations, we find more instructive to present only some selected nonlinear equations that will play an important role (once linearised) in our problem.

To achieve the formally correct form of the nonlinear equations in the new gauge, we crucially employ the formalism to treat nonlinear tensor perturbations discussed in Section 3.5.2. The reader should contrast the equations of this section with their informal version of Section 3.2.3 of the overview.

Consider the one-parameter family of metrics $\boldsymbol{g}(\epsilon)$ on $\mathcal{M}^+$ introduced in Section 3.5. If the

Lorentzian manifold $(\mathcal{M}^+, \boldsymbol{g}(\epsilon))$ satisfies the vacuum Einstein equations

$$\mathbf{Ric}(\boldsymbol{g}(\epsilon)) = 0\,,$$

for all $\epsilon \geq 0$, then we have[65]

- the set of transport equations

$$\Pi\phi_\epsilon^*((\boldsymbol{\nabla\!\!\!/}_\tau \hat{\boldsymbol{\chi}}_\bigcirc)(\epsilon) + (\mathbf{tr}\boldsymbol{\chi}_\bigcirc)\,\hat{\boldsymbol{\chi}}_\bigcirc(\epsilon)) - \hat{\omega}\,(\Pi\phi_\epsilon^*\,\hat{\boldsymbol{\chi}}_\bigcirc(\epsilon)) = -(\Pi\phi_\epsilon^*\,\boldsymbol{\alpha}(\epsilon))\,,$$
$$\Pi\phi_\epsilon^*(\partial_\tau(\mathbf{tr}\boldsymbol{\chi}_\bigcirc)(\epsilon) + \frac{1}{2}\,(\mathbf{tr}\boldsymbol{\chi}_\bigcirc)^2(\epsilon)) - \hat{\omega}\,(\Pi\phi_\epsilon^*(\mathbf{tr}\boldsymbol{\chi}_\bigcirc)(\epsilon)) = \Pi\phi_\epsilon^*(-(\hat{\boldsymbol{\chi}}_\bigcirc, \hat{\boldsymbol{\chi}}_\bigcirc)(\epsilon) + \frac{1}{2}(\boldsymbol{\in\!\!\!/} \cdot \boldsymbol{\chi}_\square)^2(\epsilon))\,,$$
$$\Pi\phi_\epsilon^*\,(\partial_\tau(\boldsymbol{\in\!\!\!/} \cdot \boldsymbol{\chi}_\square)(\epsilon) + (\mathbf{tr}\boldsymbol{\chi}_\bigcirc)(\boldsymbol{\in\!\!\!/} \cdot \boldsymbol{\chi}_\square)(\epsilon)) - \hat{\omega}\,(\Pi\phi_\epsilon^*\,(\boldsymbol{\in\!\!\!/} \cdot \boldsymbol{\chi}_\square)(\epsilon)) = 0$$

  on $\mathcal{M}^+$, where we exploit the gauge conditions (3.128), (3.129) and (3.132),

- the transport equation

$$\begin{aligned}\Pi\phi_\epsilon^*((\boldsymbol{\nabla\!\!\!/}_\tau \hat{\boldsymbol{\chi}}_\bigcirc)(\epsilon) + \frac{1}{2}\,(\mathbf{tr}\boldsymbol{\chi}_\bigcirc)\,\hat{\boldsymbol{\chi}}_\bigcirc(\epsilon)\; + \frac{1}{2}\,(\boldsymbol{\in\!\!\!/} \cdot \boldsymbol{\chi}_\square)\,{}^\star\hat{\boldsymbol{\chi}}_\bigcirc(\epsilon)) + \hat{\omega}\,(\Pi\phi_\epsilon^*\,\hat{\boldsymbol{\chi}}_\bigcirc(\epsilon)) \\ = \Pi\phi_\epsilon^*((-2\,\boldsymbol{\mathcal{D}\!\!\!/}_{\mathbf{2}}^\star\,\boldsymbol{\underline{\eta}} + \boldsymbol{\underline{\eta}}\,\hat{\otimes}\,\boldsymbol{\underline{\eta}})(\epsilon)) \\ + \Pi\phi_\epsilon^*(-\frac{1}{2}\,(\mathbf{tr}\,\boldsymbol{\chi}_\bigcirc)\,\hat{\boldsymbol{\chi}}_\bigcirc(\epsilon) + \frac{1}{2}\,(\boldsymbol{\in\!\!\!/} \cdot \boldsymbol{\chi}_\square)\,{}^\star\hat{\boldsymbol{\chi}}_\bigcirc(\epsilon))\end{aligned}$$

  on $\mathcal{M}^+$, where, again, we exploit the gauge conditions (3.128), (3.129) and (3.132). *Crucially*, the gauge condition (3.130) implies that we have the formal Taylor expansion

$$\begin{aligned}\Pi\phi_\epsilon^*((-2\boldsymbol{\mathcal{D}\!\!\!/}_{\mathbf{2}}^*\;\boldsymbol{\underline{\eta}} + \boldsymbol{\underline{\eta}}\;\hat{\otimes}\;\boldsymbol{\underline{\eta}})(\epsilon)) = \epsilon\cdot(-2\,\widehat{\overset{(1)}{\Gamma\!\!\!/}}\cdot\underline{\eta} - (\mathrm{d\!\!\!/iv}\;\underline{\eta} + (\underline{\eta},\underline{\eta}))\,\widehat{\overset{(1)}{g\!\!\!/}} \\ + (\overset{(1)}{\Pi\!\!\!/}\,\widehat{\otimes}\,\Pi + \Pi\,\hat{\otimes}\,\overset{(1)}{\Pi\!\!\!/})\cdot(\nabla\,\underline{\eta})) + \mathcal{O}(\epsilon^2)\end{aligned} \tag{3.169}$$

  around $\epsilon = 0$ on $\mathcal{M}^+$. At this point, the reader should not worry about the meaning of the terms appearing on the right hand side of the expression (3.169), which will be illustrated in Section 3.7.1. The fundamental fact about (3.169) that has to be noticed is the absence of terms depending on the *vanishing* (for all $\epsilon \geq 0$) tensor perturbation

$$\Pi\phi_\epsilon^*(\boldsymbol{\underline{\eta}}(\epsilon)) - \underline{\eta}$$

  on $\mathcal{M}^+$.

- the identities

$$[\boldsymbol{e_4}, \boldsymbol{e_3}]^\mu = \partial_\tau\,(\boldsymbol{e}_{\boldsymbol{3}}^\mu)\,, \qquad\qquad [\boldsymbol{e_4}, \boldsymbol{e_A}]^\mu = \partial_\tau\,(\boldsymbol{e}_{\boldsymbol{A}}^\mu)$$

[65] Note that, in all the following equations, the connection coefficient $\hat{\omega}$ appears de-bolded.

and the set of equations

$$\begin{aligned}
\partial_\tau \left(\Pi\phi_\epsilon^*(\boldsymbol{e}_{\boldsymbol{3}}^\mu(\epsilon))\right) =& -2\left(\Pi\phi_\epsilon^*(\boldsymbol{\eta}^\mu - \underline{\boldsymbol{\eta}}^\mu)(\epsilon)\right) - \hat{\omega}\left(\Pi\phi_\epsilon^*(\boldsymbol{e}_{\boldsymbol{3}}^\mu(\epsilon))\right) + (\Pi\phi_\epsilon^*\,\underline{\hat{\boldsymbol{\omega}}}(\epsilon))(\Pi\phi_\epsilon^*(\boldsymbol{e}_{\boldsymbol{4}}^\mu(\epsilon)))\,,\\
\partial_\tau \left(\Pi\phi_\epsilon^*(\boldsymbol{e}_{\boldsymbol{A}}^\mu(\epsilon))\right) =& -\Pi\phi_\epsilon^*({\boldsymbol{\chi}^{\sharp_2}}^\mu{}_{\boldsymbol{A}}(\epsilon)) + (\Pi\phi_\epsilon^*(\boldsymbol{Y}_{\boldsymbol{A}}(\epsilon)))(\Pi\phi_\epsilon^*(\boldsymbol{e}_{\boldsymbol{3}}^\mu(\epsilon)))\\
&+ \left(\Pi\phi_\epsilon^*\left(\underline{\boldsymbol{\eta}}_{\boldsymbol{A}}(\epsilon) + \boldsymbol{\zeta}_{\boldsymbol{A}}(\epsilon)\right)\right)(\Pi\phi_\epsilon^*(\boldsymbol{e}_{\boldsymbol{4}}^\mu(\epsilon)))\,,
\end{aligned}$$

on $\mathcal{M}^+$, where we exploit the gauge conditions (3.128), (3.131) and (3.132).

As a final general observation, the equations of the system of Section 3.3.5 that get more heavily simplified in our gauge are the *transport equations in the* $\boldsymbol{e}_{\boldsymbol{4}}(\epsilon)$*-direction.* This should be seen as the underlying motivation for the construction of the outgoing frame-calibrated gauge of Theorem 3.2. See Section 3.2.4 of the overview for a preliminary discussion of the crucial role of these simplified equations in the problem.

3.7 The linearised vacuum Einstein equations around Kerr

In this section of the chapter we present the system of nonlinear vacuum Einstein equations *linearised* around the Kerr solution $(\mathcal{M}^+, g_{a,M})$ in our outgoing frame-calibrated gauge. This section constitutes the third of the three self-contained blocks into which the body of the chapter is divided, as outlined at the beginning of Section 3.2.

Most of the section is devoted to a careful presentation of the linearisation procedure. The main new technical difficulty lies in the presence of connection coefficients of the Kerr metric $g_{a,M}$ which are *non-vanishing* $\mathfrak{D}_{\mathcal{N}}$ *covariant* tensors on $(\mathcal{M}^+, g_{a,M})$.

We linearise the vacuum Einstein equations in their *tensorial* form. To do that, we first pull-back the entire system of nonlinear equations via $\Pi\phi_\epsilon^*$ (or, for what concerns the null frame equations, via ϕ_ϵ^*). See the definition of the projected pull-back $\Pi\phi_\epsilon^*$ in Section 3.5.2. We then subtract to each of the pulled-back nonlinear equations on $(\mathcal{M}^+, g_{a,M})$ its correspondent Kerr equation and understand the linearisation of each of the terms in the resulting equation as the *linearisation of a nonlinear tensor perturbation* on $(\mathcal{M}^+, g_{a,M})$. Note that most of the conceptual difficulties that arise when one considers tensor perturbations have already been addressed, at the *nonlinear* level, in Section 3.5.2. To fully appreciate the linearisation procedure, the reader should first review the definitions therein.

The reader only interested in the final linearised system of equations can refer to Section 3.7.2.

3.7.1 The linearisation procedure

In this section we linearise tensor perturbations building on the definitions and formalism of Section 3.5.2.

Note that the linearisation of a $\mathfrak{D}_{\mathcal{N}}$ covariant tensor perturbation will be defined to be a $\mathfrak{D}_{\mathcal{N}}$

covariant tensor on $(\mathcal{M}^+, g_{a,M})$, while the linearisation of a frame perturbation will *not* be required to be a vector field in $\mathfrak{D}_{\mathcal{N}}$ (see related Remark 3.9). Let us also note that, with the exception of the tensor perturbation of the projection tensor Π, *we do not linearise tensor perturbations of mixed type.* We will linearise perturbations of smooth scalar functions, frame vector fields, $\mathfrak{D}_{\mathcal{N}}$ covariant tensors, of the projection tensor Π and of the contravariant two-tensors $\not{g}^{-1}$ and $\not{\epsilon}^{-1}$ on $(\mathcal{M}^+, g_{a,M})$.[66]

Consider the one-parameter family of $\mathfrak{D}_{\boldsymbol{\mathcal{N}}(\epsilon)}$ k-tensors $\boldsymbol{\mathcal{T}}(\epsilon)$, the one-parameter family of frames $\boldsymbol{\mathcal{N}}(\epsilon)$ and the one-parameter family of smooth scalar functions $\boldsymbol{\mathfrak{f}}(\epsilon)$ on $\mathcal{M}^+$. We have the following four definitions, where we denote by $\mathcal{O}(\epsilon^2)$ tensors whose components are $\mathcal{O}(\epsilon^2)$ smooth scalar functions on $\mathcal{M}^+$. We recall the notation $\boldsymbol{\mathcal{T}}(0) = \mathcal{T}$, $\boldsymbol{e_I}(0) = e_I$ and $\boldsymbol{\mathfrak{f}}(0) = \mathfrak{f}$.

Definition 3.4. *Given the $\mathfrak{D}_{\mathcal{N}}$ k-tensor $\mathcal{T}$ on $(\mathcal{M}^+, g_{a,M})$, we define the k-tensor*

$$\overset{(1)}{\mathcal{T}}$$

on $(\mathcal{M}^+, g_{a,M})$ such that

$$\phi_\epsilon^* \left(\boldsymbol{\mathcal{T}}(\epsilon)\right) - \mathcal{T} = \epsilon \cdot \overset{(1)}{\mathcal{T}} + \mathcal{O}(\epsilon^2) \tag{3.170}$$

on $(\mathcal{M}^+, g_{a,M})$.[67]

Definition 3.5. *Given the $\mathfrak{D}_{\mathcal{N}}$ k-tensor $\mathcal{T}$ on $(\mathcal{M}^+, g_{a,M})$, we define the* <u>*linearised k-tensor perturbation of $\mathcal{T}$*</u> *as the linearisation of the (nonlinear) k-tensor perturbation of $\mathcal{T}$ defined in Definition 3.1, namely as the $\mathfrak{D}_{\mathcal{N}}$ k-tensor*[68]

$$\overset{(1)}{\not{\mathcal{T}}}$$

on $(\mathcal{M}^+, g_{a,M})$ such that

$$\Pi\phi_\epsilon^* \left(\boldsymbol{\mathcal{T}}(\epsilon)\right) - \mathcal{T} = \epsilon \cdot \overset{(1)}{\not{\mathcal{T}}} + \mathcal{O}(\epsilon^2)$$

on $(\mathcal{M}^+, g_{a,M})$.

Definition 3.6. *Given the frame vector field e_I of the frame $\mathcal{N}$ on $(\mathcal{M}^+, g_{a,M})$, $I = 1, \ldots, 4$, we define the* <u>*linearised frame perturbation of e_I*</u> *as the linearisation of the (nonlinear) frame perturbation of e_I defined in Definition 3.2, namely as the vector field*

$$\overset{(1)}{e}_I \tag{3.171}$$

[66] It is already worth noticing that, for instance, the tensor $\boldsymbol{\eta}^{\boldsymbol{\sharp}}(\epsilon)$ (with the sharp symbol in bold) will be linearised by considering it as the contraction of the tensor $\boldsymbol{\eta}(\epsilon)$ with the inverse metric $\boldsymbol{\not{g}}^{-1}(\epsilon)$. One therefore gets two non-trivial terms in the linearisation, one coming from linearising $\boldsymbol{\eta}(\epsilon)$ and one from linearising $\boldsymbol{\not{g}}^{-1}(\epsilon)$. The reader should realise that linearising $\boldsymbol{\eta}(\epsilon)$ and then raising the index with the inverse metric $\not{g}^{-1}$ is different from linearising $\boldsymbol{\eta}^{\boldsymbol{\sharp}}(\epsilon)$.

[67] The tensors $\boldsymbol{\mathcal{T}}(\epsilon)$ and $\mathcal{T}$ in (3.170) are, with a slight abuse of notation, the natural extensions of $\boldsymbol{\mathcal{T}}(\epsilon)$ and $\mathcal{T}$ to horizontal tensors on $\mathcal{M}^+$. Note that, in view of the fact that the pullback $\phi_\epsilon^* \left(\boldsymbol{\mathcal{T}}(\epsilon)\right)$ is not projected, the k-tensor $\overset{(1)}{\mathcal{T}}$ is *not*, in general, horizontal.

[68] The reader should note that the linearised tensor is now *slashed*.

on $(\mathcal{M}^+, g_{a,M})$ *such that*

$$\phi_{-\epsilon *}(\boldsymbol{e_I}(\epsilon)) - e_I = \epsilon \cdot \overset{(1)}{e}_I + \mathcal{O}(\epsilon^2)$$

on $(\mathcal{M}^+, g_{a,M})$.

Definition 3.7. *Given the scalar function* $\mathfrak{f} \in C^\infty(\mathcal{M}^+)$*, we define the* *linearised scalar perturbation of* $\mathfrak{f}$ *as the linearisation of the (nonlinear) scalar perturbation of* $\mathfrak{f}$ *defined in Definition 3.3, namely as the smooth scalar function*

$$\overset{(1)}{\mathfrak{f}} \tag{3.172}$$

on $(\mathcal{M}^+, g_{a,M})$ *such that*

$$\phi^*_\epsilon(\mathfrak{f}(\epsilon)) - \mathfrak{f} = \epsilon \cdot \overset{(1)}{\mathfrak{f}} + \mathcal{O}(\epsilon^2)$$

on $(\mathcal{M}^+, g_{a,M})$.

Remark 3.11. *The reader should keep in mind that* $\overset{(1)}{\mathcal{T}}$ *is a k-tensor on* $(\mathcal{M}^+, g_{a,M})$*, while* $\overset{(1)}{\mathcal{T}}$ *is a* $\mathfrak{D}_\mathcal{N}$ *k-tensor on* $(\mathcal{M}^+, g_{a,M})$*. By naturally extending* $\overset{(1)}{\mathcal{T}}$ *to an horizontal k-tensor on* $(\mathcal{M}^+, g_{a,M})$*,*[69] *one can compare* $\overset{(1)}{\mathcal{T}}$ *and* $\overset{(1)}{\mathcal{T}}$ *and note that the identity*

$$\overset{(1)}{\mathcal{T}}_{\mu_1\cdots\mu_k} = \Pi^{\nu_1}_{\mu_1}\cdots\Pi^{\nu_k}_{\mu_k}\overset{(1)}{\mathcal{T}}_{\nu_1\cdots\nu_k}$$

holds on $(\mathcal{M}^+, g_{a,M})$*. In what follows, whether we are dealing with* $\overset{(1)}{\mathcal{T}}$ *or its natural extension will be clear by the context.*

We have the two following useful facts.

Claim 3.1. *If* $\mathcal{T} \equiv 0$ *on* $(\mathcal{M}^+, g_{a,M})$*, then* $\overset{(1)}{\mathcal{T}} \equiv \overset{(1)}{\mathcal{T}}$ *on* $(\mathcal{M}^+, g_{a,M})$.

Proof. We compute

$$\begin{aligned} 0 &= \phi^*_\epsilon(\boldsymbol{\mathcal{T}}(\boldsymbol{e_{A_1}}, \ldots, \boldsymbol{e_3}, \ldots, \boldsymbol{e_{A_k}})) \\ &= (\phi^*_\epsilon\boldsymbol{\mathcal{T}})_{\mu_1\cdots\mu_k}(\phi_{\epsilon *}\boldsymbol{e_{A_1}})^{\mu_1}\cdots(\phi_{\epsilon *}\boldsymbol{e_3})^{\mu_i}\cdots(\phi_{\epsilon *}\boldsymbol{e_{A_k}})^{\mu_k} \\ &= \epsilon \cdot \overset{(1)}{\mathcal{T}}(e_{A_1}, \ldots, e_3, \ldots, e_{A_k}) + \mathcal{O}(\epsilon^2) \end{aligned}$$

for any $\epsilon \geq 0$, which implies

$$\overset{(1)}{\mathcal{T}}(e_{A_1}, \ldots, e_3, \ldots, e_{A_k}) = 0\,.$$

[69] Which, again, with an abuse of notation, will be denoted by the same letter.

Similarly,

$$\overset{(1)}{\mathcal{T}}(e_{A_1}, \ldots, e_4, \ldots, e_{A_k}) = 0\,.$$

□

Claim 3.2. *In our gauge, we have*

$$\overset{(1)}{\mathcal{T}}(e_{A_1}, \ldots, e_4, \ldots, e_{A_k}) = 0$$

on $(\mathcal{M}^+, g_{a,M})$.

Proof. We compute

$$\begin{aligned} 0 &= \phi_\epsilon^*(\boldsymbol{\mathcal{T}}(\boldsymbol{e_{A_1}}, \ldots, \boldsymbol{e_4}, \ldots, \boldsymbol{e_{A_k}})) \\ &= (\phi_\epsilon^* \boldsymbol{\mathcal{T}})_{\mu_1 \cdots \mu_k} (\phi_{\epsilon *} \boldsymbol{e_{A_1}})^{\mu_1} \cdots (\phi_{\epsilon *}\, \boldsymbol{e_4})^{\mu_i} \cdots (\phi_{\epsilon *} \boldsymbol{e_{A_k}})^{\mu_k} \\ &= \epsilon \cdot \overset{(1)}{\mathcal{T}}(e_{A_1}, \ldots, e_4, \ldots, e_{A_k}) + \mathcal{O}(\epsilon^2) \end{aligned}$$

for any $\epsilon \geq 0$, which implies

$$\overset{(1)}{\mathcal{T}}(e_{A_1}, \ldots, e_4, \ldots, e_{A_k}) = 0\,.$$

Note that we used the gauge condition (3.132) to set $\mathcal{T}(e_{A_1}, \ldots, \overset{(1)}{e}_4, \ldots, e_{A_k}) = 0$. □

For all the linearised perturbations of frame vectors, connection coefficients and curvature components, we will adopt the same notation of Definition 3.5, e.g. we write

$$\overset{(1)}{e}_3\,,\ \overset{(1)}{\not\eta}\,,\ \overset{(1)}{\not\zeta}\,,\ \overset{(1)}{\rho}\ \ldots$$

In view of Claim 3.1, we will *not* slash the linearised connection coefficients and curvature components when they vanish on $(\mathcal{M}^+, g_{a,M})$, e.g. we write

$$\overset{(1)}{\hat{\chi}}_{\bigcirc}\,, \overset{(1)}{\underline{Y}}\,, \overset{(1)}{\alpha}\ \ldots$$

In our outgoing frame-calibrated gauge, we have

$$\begin{gathered} \overset{(1)}{\Gamma}{}^{s}_{\tau\tau} = \overset{(1)}{\Gamma}{}^{\vartheta}_{\tau\tau} = \overset{(1)}{\Gamma}{}^{\psi}_{\tau\tau} = 0\,, \\ \overset{(1)}{\hat{\omega}} = \overset{(1)}{Y} = \overset{(1)}{\not\underline{\eta}} = \overset{(1)}{\Gamma}{}^{B}_{4A} = 0\,, \\ \overset{(1)}{e}_4 = 0 \end{gathered}$$

on $(\mathcal{M}^+, g_{a,M})$. This follows immediately from the gauge conditions satisfied by the family of metrics $\boldsymbol{g}(\epsilon)$ on $\mathcal{M}^+$.

Note that $\boldsymbol{\eta}(\epsilon)$ and $\boldsymbol{\zeta}(\epsilon)$ are the only one-parameter families of $\mathfrak{D}_{\boldsymbol{\mathcal{N}}(\epsilon)}$ covariant tensors which do *not* vanish on the Kerr background and produce *non-zero* linearised tensor perturbations.[70]

Remark 3.12. *Claim 3.1 shows that the linearisation of* $\mathfrak{D}_{\boldsymbol{\mathcal{N}}(\epsilon)}$ *covariant tensors which vanish on* $(\mathcal{M}^+, g_{a,M})$ *(i.e. for* $\epsilon = 0$*) does* not *need any projection. Claim 3.2 shows that, in our gauge, the projection procedure introduced for nonlinear perturbations in Section 3.5.2 is,* to linear order, *only killing the* e_3*-component of the pulled-back covariant tensors. Their* e_4*-component automatically vanishes on* $(\mathcal{M}^+, g_{a,M})$*.*

Remark 3.13. *The reader only interested in the* linear *stability problem can neglect the Definitions 3.1, 3.2 and 3.3 of Section 3.5.2. In fact, to achieve our linearisation, one can simply pull-back the covariant tensor* $\boldsymbol{\mathcal{T}}(\epsilon)$*, linearise the pulled-back tensor* $\phi^*_\epsilon(\boldsymbol{\mathcal{T}}(\epsilon))$ *on* $(\mathcal{M}^+, g_{a,M})$ *and* then *project the linearised tensor relative to* $\mathcal{N}$*. This is, to linear order, equivalent to first projecting the pulled-back tensor* $\phi^*_\epsilon(\boldsymbol{\mathcal{T}}(\epsilon))$ *(i.e. considering* $\Pi\phi^*_\epsilon(\boldsymbol{\mathcal{T}}(\epsilon))$*) and* then *linearising, as we do. On the other hand, our procedure provides the right formulation of covariant tensor perturbations at the* nonlinear *level.*

Linearisation of the metric and volume form

Using Definition 3.5, we define the *linearised metric* as the $\mathfrak{D}_{\mathcal{N}}$ symmetric two-tensor[71]

$$\overset{(1)}{\not{\not{g}}}$$

on $(\mathcal{M}^+, g_{a,M})$ such that

$$\Pi\phi^*_\epsilon\left(\not{\boldsymbol{g}}(\epsilon)\right) - \not{g} = \epsilon\cdot\overset{(1)}{\not{\not{g}}} + \mathcal{O}(\epsilon^2)$$

on $(\mathcal{M}^+, g_{a,M})$. We define the symmetric two-tensors

$$\overset{(1)}{g}\,,\ \overset{(1)}{\not{g}}$$

on $(\mathcal{M}^+, g_{a,M})$ such that

$$\phi^*_\epsilon\left(\boldsymbol{g}(\epsilon)\right) - g = \epsilon\cdot\overset{(1)}{g} + \mathcal{O}(\epsilon^2)\,,$$

$$\phi^*_\epsilon\left(\not{\boldsymbol{g}}(\epsilon)\right) - \not{g} = \epsilon\cdot\overset{(1)}{\not{g}} + \mathcal{O}(\epsilon^2)$$

on $(\mathcal{M}^+, g_{a,M})$. Note that

$$\overset{(1)}{\not{\not{g}}}_{\mu\nu} = \Pi^\alpha_\mu\,\Pi^\beta_\nu\,\overset{(1)}{\not{g}}_{\alpha\beta}\,,$$

but

$$\overset{(1)}{\not{g}}_{\mu\nu} \neq \Pi^\alpha_\mu\,\Pi^\beta_\nu\,\overset{(1)}{g}_{\alpha\beta}\,.$$

[70]For this reason, combined with Claim 3.1, $\overset{(1)}{\not{\eta}}$ and $\overset{(1)}{\not{\zeta}}$ are, in fact, the only *slashed* $\mathfrak{D}_{\mathcal{N}}$ covariant tensors (together with the linearised metric, see next section) appearing in the system of linearised Einstein equations of Section 3.7.2.

[71]The reader should note that the linearised metric tensor is *double-slashed*.

In particular, $\overset{(1)}{\not{g}}$ is *not*, in general, an horizontal two-tensor.

We define the *linearised inverse metric* as the symmetric contravariant two-tensor

$$\overset{(1)}{\not{g}}{}^{-1}$$

on $(\mathcal{M}^+, g_{a,M})$ such that

$$\phi_\epsilon^* \left(\boldsymbol{\not{g}}^{-1}(\epsilon)\right) - \not{g}^{-1} = \epsilon \cdot \overset{(1)}{\not{g}}{}^{-1} + \mathcal{O}(\epsilon^2)$$

on $(\mathcal{M}^+, g_{a,M})$.

From the gauge conditions for the family of metrics $\boldsymbol{g}(\epsilon)$ on $\mathcal{M}^+$, we have

$$\overset{(1)}{\not{\not{g}}}{}_{\tau\tau} = 0 \tag{3.173}$$

on $(\mathcal{M}^+, g_{a,M})$. One can adopt the decomposition

$$\overset{(1)}{\not{\not{g}}} = \widehat{\overset{(1)}{\not{\not{g}}}} + \frac{1}{2}(\mathrm{tr}\,\overset{(1)}{\not{\not{g}}})\not{g}\,,$$

where the trace is taken with respect to $\not{g}^{-1}$.[72] Note the following contraction identity

$$\overset{(1)}{\not{g}}{}^{\mu\nu}\not{g}_{\mu\nu} = -(\mathrm{tr}\,\overset{(1)}{\not{\not{g}}})\,,$$

obtained by linearising $\phi_\epsilon^*(\boldsymbol{\not{g}}^{\mu\nu}\boldsymbol{\not{g}}_{\mu\nu}) = 2$ and using $(\mathrm{tr}\,\overset{(1)}{\not{g}}) = (\mathrm{tr}\,\overset{(1)}{\not{\not{g}}})$.

Using Definition 3.5, we define the *linearised volume form* as the $\mathfrak{D}_\mathcal{N}$ antisymmetric two-tensor[73]

$$\overset{(1)}{\not{\not{\epsilon}}}$$

on $(\mathcal{M}^+, g_{a,M})$ such that

$$\Pi\phi_\epsilon^* \left(\boldsymbol{\not{\epsilon}}(\epsilon)\right) - \not{\epsilon} = \epsilon \cdot \overset{(1)}{\not{\not{\epsilon}}} + \mathcal{O}(\epsilon^2)\,.$$

on $(\mathcal{M}^+, g_{a,M})$. We define the antisymmetric two-tensor

$$\overset{(1)}{\not{\epsilon}}$$

on $(\mathcal{M}^+, g_{a,M})$ such that

$$\phi_\epsilon^* \left(\boldsymbol{\not{\epsilon}}(\epsilon)\right) - \not{\epsilon} = \epsilon \cdot \overset{(1)}{\not{\epsilon}} + \mathcal{O}(\epsilon^2)\,.$$

[72] Note that, in our notation, $\widehat{\overset{(1)}{\not{\not{g}}}} \neq \overset{(1)}{\widehat{\not{g}}}$. In fact, the right hand side is identically zero.
[73] The reader should note that the linearised volume form is *double-slashed*.

on $(\mathcal{M}^+, g_{a,M})$. Note that

$$\overset{(1)}{\not\epsilon}_{\mu\nu} = \Pi^\alpha_\mu \, \Pi^\beta_\nu \, \overset{(1)}{\not\epsilon}_{\alpha\beta} \, .$$

We define the *linearised inverse volume form* as the antisymmetric contravariant two-tensor

$$\overset{(1)}{\not\epsilon}{}^{-1}$$

on $(\mathcal{M}^+, g_{a,M})$ such that

$$\phi^*_\epsilon \left(\boldsymbol{\not\epsilon}^{-1}(\epsilon)\right) - \not\epsilon^{-1} = \epsilon \cdot \overset{(1)}{\not\epsilon}{}^{-1} + \mathcal{O}(\epsilon^2)$$

on $(\mathcal{M}^+, g_{a,M})$.

We have the formula

$$\overset{(1)}{\not\epsilon} = \frac{1}{2}(\mathrm{tr}\,\overset{(1)}{\not g})\not\epsilon \, .$$

Note also the following contraction identity

$$\begin{aligned}\overset{(1)}{\not\epsilon}{}^{\mu\nu}\not\epsilon_{\mu\nu} &= -\not\epsilon^{\mu\nu}\,\overset{(1)}{\not\epsilon}_{\mu\nu} \\ &= -\not\epsilon^{\mu\nu}\,\overset{(1)}{\not\epsilon}_{\mu\nu} \, .\end{aligned}$$

To conclude the section, we note that the linearisation of a *(anti-)symmetric* $\mathfrak{D}_{\mathcal{N}(\epsilon)}$ two-tensor is a *(anti-)symmetric* $\mathfrak{D}_{\mathcal{N}}$ two-tensor on $(\mathcal{M}^+, g_{a,M})$. However, the linearisation of a *traceless*[74] $\mathfrak{D}_{\mathcal{N}(\epsilon)}$ two-tensor is guaranteed to be *traceless*[75] only when the two-tensor *vanishes* on $(\mathcal{M}^+, g_{a,M})$ (i.e. for $\epsilon = 0$).

The *traceless* $\mathfrak{D}_{\mathcal{N}(\epsilon)}$ two-tensors that we linearise in the Einstein equations include, for instance, the volume form $\boldsymbol{\not\epsilon}(\epsilon)$ and the following symmetric two-tensors

$$\begin{gathered}\hat{\boldsymbol{\chi}}_\circ(\epsilon)\,,\ \hat{\underline{\boldsymbol{\chi}}}_\circ(\epsilon)\,,\ (-2\boldsymbol{\not{\mathcal{D}}}^*_2\boldsymbol{\eta} + \boldsymbol{\eta}\hat{\otimes}\boldsymbol{\eta})(\epsilon)\,,\ (-2\boldsymbol{\not{\mathcal{D}}}^*_2\,\underline{\boldsymbol{\eta}} + \underline{\boldsymbol{\eta}}\,\hat{\otimes}\,\underline{\boldsymbol{\eta}})(\epsilon)\,,\ (\boldsymbol{\not{\mathcal{D}}}^*_2\,\underline{\boldsymbol{Y}})(\epsilon)\,, \\ \boldsymbol{\alpha}(\epsilon)\,,\ \underline{\boldsymbol{\alpha}}(\epsilon)\,,\ (\boldsymbol{\not{\mathcal{D}}}^*_2\boldsymbol{\beta})(\epsilon)\,,\ (\boldsymbol{\not{\mathcal{D}}}^*_2\,\underline{\boldsymbol{\beta}})(\epsilon)\ \ldots\end{gathered}$$

Since $\boldsymbol{\not\epsilon}(\epsilon)$ is antisymmetric, the linearised volume form $\overset{(1)}{\not\epsilon}$ is antisymmetric (and therefore *traceless*) on $(\mathcal{M}^+, g_{a,M})$. All the symmetric traceless $\mathfrak{D}_{\mathcal{N}(\epsilon)}$ two-tensors listed above *do* vanish on $(\mathcal{M}^+, g_{a,M})$ *relative to the algebraically special frame* $\mathcal{N}$ of our gauge. Their linearisation is therefore a symmetric *traceless* $\mathfrak{D}_{\mathcal{N}}$ two-tensor on $(\mathcal{M}^+, g_{a,M})$. Note that this fact would not hold true relative to a general frame on $(\mathcal{M}^+, g_{a,M})$, in particular relative to the frame $\widehat{\mathcal{N}}_{\mathrm{ad}}$.

[74] With respect to $\boldsymbol{\not g}^{-1}(\epsilon)$.
[75] With respect to $\not g^{-1}$.

Linearisation of the projection tensor

We define the *linearised projection tensor* as the $(1,1)$ tensor

$$\overset{(1)}{\not{\Pi}}$$

on $(\mathcal{M}^+, g_{a,M})$ such that

$$\Pi\phi^*_\epsilon\left(\mathbf{\Pi}(\epsilon)\right) - \Pi = \epsilon \cdot \overset{(1)}{\not{\Pi}} + \mathcal{O}(\epsilon^2)$$

on $(\mathcal{M}^+, g_{a,M})$. In components, we have

$$\overset{(1)}{\not{\Pi}}{}_\mu^\nu = \Pi_\mu^\alpha \, \overset{(1)}{\Pi}{}_\alpha^\nu \,,$$

with $\overset{(1)}{\Pi}$ a $(1,1)$ tensor on $(\mathcal{M}^+, g_{a,M})$ such that

$$\phi^*_\epsilon\left(\mathbf{\Pi}(\epsilon)\right) - \Pi = \epsilon \cdot \overset{(1)}{\Pi} + \mathcal{O}(\epsilon^2)$$

on $(\mathcal{M}^+, g_{a,M})$. We have

$$\overset{(1)}{\not{\Pi}}{}_A^\nu = \overset{(1)}{\Pi}{}_A^\nu \,, \qquad\qquad \overset{(1)}{\not{\Pi}}{}_3^\nu = \overset{(1)}{\not{\Pi}}{}_4^\nu = 0 \,.$$

It is useful to note that

$$\overset{(1)}{\Pi}{}_\mu^\nu = \frac{1}{2}\, \overset{(1)}{e}{}_3^\sigma \, e_4^\nu \, g_{\mu\sigma} + \frac{1}{2}\, e_4^\sigma \overset{(1)}{e}{}_3^\nu \, g_{\mu\sigma} + \frac{1}{2}\left(e_4^\sigma \, e_3^\nu + e_3^\sigma \, e_4^\nu\right) \overset{(1)}{g}{}_{\mu\sigma} \,,$$

where we used the gauge relation $\overset{(1)}{e}{}_4 \equiv 0$ on $(\mathcal{M}^+, g_{a,M})$. In particular, we have

$$\begin{aligned} \overset{(1)}{\Pi}{}_A^\nu &= \frac{1}{2}\, \overset{(1)}{e}{}_3^\sigma \, e_4^\nu \, g_{A\sigma} + \frac{1}{2}\left(e_4^\sigma \, e_3^\nu + e_3^\sigma \, e_4^\nu\right) \overset{(1)}{g}{}_{A\sigma} \\ &= -\frac{1}{2}\, g(e_3, \overset{(1)}{e}{}_A)\, e_4^\nu - \frac{1}{2}\, g(e_4, \overset{(1)}{e}{}_A)\, e_3^\nu \,, \end{aligned} \tag{3.174}$$

which implies

$$\overset{(1)}{\Pi}{}_A^B = 0 \,. \tag{3.175}$$

One can conclude

$$\overset{(1)}{\not{\Pi}}{}_A^B = 0 \,. \tag{3.176}$$

Remark 3.14. *If*

$$g(e_3, \overset{(1)}{e}{}_A) \equiv g(e_4, \overset{(1)}{e}{}_A) \equiv 0 \tag{3.177}$$

on $(\mathcal{M}^+, g_{a,M})$, then

$$\overset{(1)}{\not{\Pi}}{}_\mu^{\nu} \equiv 0$$

on $(\mathcal{M}^+, g_{a,M})$. Indeed, the linearised projection tensor accounts for the fact that the horizontal structure of our gauge is variable, which, to linear order, is quantified by the vector field $\overset{(1)}{e}_A$ gaining components in e_3 and e_4. If one adopts our formalism to linearise the system of Einstein equations in a double-null gauge, then the identity (3.177) *holds and the linearised projection tensor vanishes identically on $(\mathcal{M}^+, g_{a,M})$.*

As a last remark, note the following non-trivial identity

$$\Pi_\mu^\alpha\, \Pi_\nu^\beta\, \overset{(1)}{g}{}_{\alpha\beta} = \overset{(1)}{\not{g}}{}_{\mu\nu}\,. \tag{3.178}$$

To see that (3.178) holds, we compute

$$\begin{aligned}
\overset{(1)}{\not{g}}{}_{\mu\nu} &= \Pi_\mu^\alpha\, \Pi_\nu^\beta\, \overset{(1)}{\not{g}}{}_{\alpha\beta} \\
&= \Pi_\mu^\alpha\, \Pi_\nu^\beta\, (\overset{(1)}{\Pi}{}_\alpha^\gamma\, \Pi_\beta^\sigma\, g_{\gamma\sigma} + \Pi_\alpha^\gamma\, \overset{(1)}{\Pi}{}_\beta^\sigma\, g_{\gamma\sigma} + \Pi_\alpha^\gamma\, \Pi_\beta^\sigma\, \overset{(1)}{g}{}_{\gamma\sigma}) \\
&= \overset{(1)}{\not{\Pi}}{}_\mu^\alpha\, \Pi_\nu^\beta\, g_{\alpha\beta} + \Pi_\mu^\alpha\, \overset{(1)}{\not{\Pi}}{}_\nu^\beta\, g_{\alpha\beta} + \Pi_\mu^\alpha\, \Pi_\nu^\beta\, \overset{(1)}{g}{}_{\alpha\beta} \\
&= \Pi_\mu^\alpha\, \Pi_\nu^\beta\, \overset{(1)}{g}{}_{\alpha\beta}\,,
\end{aligned}$$

where, in the last equality, we used (3.176) and $g(e_3, e_A) = g(e_4, e_A) = 0$. Identity (3.178) will be employed in some of our computations.

Since

$$(\Pi\phi_\epsilon^*\,(\boldsymbol{g}(\epsilon)))_{\mu\nu} - \not{g}_{\mu\nu} = \epsilon \cdot \Pi_\mu^\alpha\, \Pi_\nu^\beta\, \overset{(1)}{g}{}_{\alpha\beta} + \mathcal{O}(\epsilon^2)\,,$$

identity (3.178) implies that, *to linear order*, we have

$$\Pi\phi_\epsilon^*\,\boldsymbol{g} = \Pi\phi_\epsilon^*\,\boldsymbol{\not{g}}$$

on $(\mathcal{M}^+, g_{a,M})$.

Linearisation of products and covariant derivatives of tensors

In view of what discussed in the first part of the present Section 3.7.1 and the formulae for nonlinear perturbation of tensor products derived in Section 3.5.2, the linearisation of the tensor products appearing in the Einstein equations does not introduce any new technical difficulty. On the other hand, the linearisation of the covariant derivatives of tensors is somewhat more involved. Before presenting the relevant linearisation formulae, we shall give some definitions and useful identities. The reader only interested in the final formulae should refer to the formulae (3.182)-(3.190).

We define the $(1,2)$-*tensor*

$$\overset{(1)}{\Gamma} \tag{3.179}$$

on $(\mathcal{M}^+, g_{a,M})$ such that

$$(\phi_\epsilon^* \boldsymbol{\nabla}(\epsilon) - \nabla)_\mu \, \partial_\nu = \epsilon \cdot \overset{(1)}{\Gamma}{}^\sigma_{\mu\nu} \, \partial_\sigma + \mathcal{O}(\epsilon^2)$$

on $(\mathcal{M}^+, g_{a,M})$. Note that $\overset{(1)}{\Gamma}$ is *symmetric* in the lower indices and can be written as

$$\overset{(1)}{\Gamma}{}^\sigma_{\mu\nu} = \frac{1}{2} \, g^{\sigma\alpha}((\nabla_\mu \overset{(1)}{g})_{\nu\alpha} + (\nabla_\nu \overset{(1)}{g})_{\alpha\mu} - (\nabla_\alpha \overset{(1)}{g})_{\mu\nu}) \, .$$

We define the $(1,2)$-tensor

$$\overset{(1)}{\not{\Gamma}} \tag{3.180}$$

on $(\mathcal{M}^+, g_{a,M})$ such that

$$\overset{(1)}{\not{\Gamma}}{}^\sigma_{\mu\nu} := \Pi^\alpha_\mu \, \Pi^\beta_\nu \, \overset{(1)}{\Gamma}{}^\sigma_{\alpha\beta} \, .$$

Claim 3.3. *Let ξ be a $\mathfrak{D}_\mathcal{N}$ one-tensor on $(\mathcal{M}^+, g_{a,M})$. We have*

$$\begin{aligned} \overset{(1)}{\not{\Gamma}}{}^\sigma_{\mu\nu} \, \xi_\sigma = & \frac{1}{2} \, \not{g}^{\sigma\alpha} \, ((\not{\nabla}_\mu \overset{(1)}{\not{g}})_{\nu\alpha} + (\not{\nabla}_\nu \overset{(1)}{\not{g}})_{\alpha\mu} - (\not{\nabla}_\alpha \overset{(1)}{\not{g}})_{\mu\nu}) \, \xi_\sigma \\ & - \frac{1}{2} \, \not{g}^{\sigma\alpha} \, (\chi_{\bigcirc\mu\nu} \overset{(1)}{g}_{3\alpha} + \chi_{\square\mu\alpha} \overset{(1)}{g}_{3\nu} + \chi_{\square\nu\alpha} \overset{(1)}{g}_{3\mu}) \, \xi_\sigma \\ & - \frac{1}{2} \, \not{g}^{\sigma\alpha} \, (\underline{\chi}_{\bigcirc\mu\nu} \overset{(1)}{g}_{4\alpha} + \underline{\chi}_{\square\mu\alpha} \overset{(1)}{g}_{4\nu} + \underline{\chi}_{\square\nu\alpha} \overset{(1)}{g}_{4\mu}) \, \xi_\sigma \end{aligned}$$

and

$$\begin{aligned} \not{g}^{\mu\nu} \, \overset{(1)}{\not{\Gamma}}{}^\sigma_{\mu\nu} \, \xi_\sigma = & - \frac{1}{2} \, \not{g}^{\sigma\alpha} \, ((\mathrm{tr}\chi_\bigcirc) \overset{(1)}{g}_{3\alpha} + 2 \, \not{g}^{\mu\nu} \chi_{\square\mu\alpha} \overset{(1)}{g}_{3\nu}) \, \xi_\sigma \\ & - \frac{1}{2} \, \not{g}^{\sigma\alpha}((\mathrm{tr} \, \underline{\chi}_\bigcirc) \overset{(1)}{g}_{4\alpha} + 2 \, \not{g}^{\mu\nu} \underline{\chi}_{\square\mu\alpha} \overset{(1)}{g}_{4\nu}) \, \xi_\sigma \\ & + \not{g}^{\sigma\alpha}(\not{\mathrm{d}}\mathrm{iv} \, \widehat{\overset{(1)}{\not{g}}})_\alpha \, \xi_\sigma \, . \end{aligned}$$

Proof. The first identity follows from an easy (but lengthy) computation. It essentially relies on noting that

$$\Pi^\gamma_\mu \Pi^\delta_\nu g^{\sigma\alpha} (\nabla_\gamma \overset{(1)}{g})_{\delta\alpha} \xi_\sigma = \not{g}^{\sigma\alpha} \xi_\sigma (\nabla_{\Pi^\gamma_\mu \partial_\gamma} \overset{(1)}{g})(\Pi^\delta_\nu \partial_\delta, \partial_\alpha) \, ,$$

applying the Leibniz rule and using identity (3.178) to obtain the first line of the formula. To prove the second formula, we note that

$$\begin{aligned}\frac{1}{2}\not{g}^{\mu\nu}\not{g}^{\sigma\alpha}((\not\nabla_\mu \overset{(1)}{\not{g}})_{\nu\alpha} + (\not\nabla_\nu \overset{(1)}{\not{g}})_{\alpha\mu} - (\not\nabla_\alpha \overset{(1)}{\not{g}})_{\mu\nu})\xi_\sigma &= \not{g}^{\mu\nu}\not{g}^{\sigma\alpha}(\not\nabla_\mu \overset{(1)}{\not{g}})_{\nu\alpha}\xi_\sigma - \frac{1}{2}\not{g}^{\sigma\alpha}\not\nabla_\alpha(\mathrm{tr}\overset{(1)}{\not{g}})\xi_\sigma \\ &= \not{g}^{\mu\nu}\not{g}^{\sigma\alpha}(\not\nabla_\mu \widehat{\overset{(1)}{\not{g}}})_{\nu\alpha}\xi_\sigma + \frac{1}{2}\not{g}^{\mu\nu}\not{g}^{\sigma\alpha}\not\nabla_\mu(\mathrm{tr}\overset{(1)}{\not{g}})\not{g}_{\nu\alpha}\xi_\sigma \\ &\quad - \frac{1}{2}\not{g}^{\sigma\alpha}\not\nabla_\alpha(\mathrm{tr}\overset{(1)}{\not{g}})\xi_\sigma \\ &= \not{g}^{\sigma\alpha}(\mathrm{d}\not{\mathrm{i}}\mathrm{v}\,\widehat{\overset{(1)}{\not{g}}})_\alpha\xi_\sigma\,.\end{aligned}$$

□

We define the $(1,1)$-tensors

$$\overset{(1)}{\not\Gamma}_4\,,\ \overset{(1)}{\not\Gamma}_3 \tag{3.181}$$

on $(\mathcal{M}^+, g_{a,M})$ such that

$$\overset{(1)}{\not\Gamma}{}^\sigma_{4\nu} := e_4^\mu\,\Pi^\beta_\nu\,\overset{(1)}{\Gamma}{}^\sigma_{\mu\beta}\,, \qquad \overset{(1)}{\not\Gamma}{}^\sigma_{3\nu} := e_3^\mu\,\Pi^\beta_\nu\,\overset{(1)}{\Gamma}{}^\sigma_{\mu\beta}\,.$$

Claim 3.4. *Let ξ be a $\mathfrak{D}_{\mathcal{N}}$ one-tensor on $(\mathcal{M}^+, g_{a,M})$. We have*

$$\overset{(1)}{\not\Gamma}{}^\sigma_{4A}\xi_\sigma = \overset{(1)}{\hat{\chi}_\circ}{}^\sigma_A\xi_\sigma + \frac{1}{2}(tr\overset{(1)}{\chi_\circ})\xi_A + \frac{1}{2}(\overset{(1)}{\not\in}\cdot\chi_\square)^\star\xi_A + \chi^{\sharp_2}{}^B_A\,\xi(\overset{(1)}{e}_B) + g(\overset{(1)}{e}_A, e_4)\,\eta\cdot\xi - g(\overset{(1)}{e}_A, e_C)\not{g}^{CD}\chi^{\sharp_2}{}^B_D\xi_B\,.$$

Along $\mathcal{H}^+$, we have

$$\overset{(1)}{\not\Gamma}{}^\sigma_{4A}\xi_\sigma = \overset{(1)}{\hat{\chi}_\circ}{}^\sigma_A\xi_\sigma + \frac{1}{2}(tr\overset{(1)}{\chi_\circ})\xi_A + \frac{1}{2}(\overset{(1)}{\not\in}\cdot\chi_\square)^\star\xi_A + g(\overset{(1)}{e}_A, e_4)\,\eta\cdot\xi\,.$$

Proof. We first note that

$$(\overset{(1)}{\nabla_4 e_A})^\sigma = e_4(\overset{(1)}{e^\sigma_A}) + e_4^\mu\overset{(1)}{e^\nu_A}\Gamma^\sigma_{\mu\nu} + e_4^\mu e_A^\nu\overset{(1)}{\Gamma}{}^\sigma_{\mu\nu}\,.$$

and, using the transport equations for the frame,

$$(\overset{(1)}{\nabla_4 e_A})^\sigma = \underline{\eta}(\overset{(1)}{e}_A)e_4^\sigma\,.$$

We conclude

$$\overset{(1)}{\not\Gamma}{}^\sigma_{4A} = -e_4(\overset{(1)}{e^\sigma_A}) - e_4^\mu\overset{(1)}{e^\nu_A}\Gamma^\sigma_{\mu\nu} + \underline{\eta}(\overset{(1)}{e}_A)e_4^\sigma\,.$$

Note that

$$\begin{aligned} e_4^\mu \overset{(1)}{e}{}_A^\nu \Gamma^\sigma_{\mu\nu} &= e_4^\mu \overset{(1)}{e}{}_A^\nu \Gamma^\sigma_{\nu\mu} \\ &= g(\nabla_{\overset{(1)}{e}_A} e_4, \partial_\gamma) g^{\gamma\sigma} , \end{aligned}$$

where we use the symmetry of Γ and the fact that $e_4^\mu = \delta^\mu_\tau$. When we contract with ξ, we obtain

$$\begin{aligned} \overset{(1)}{\not{\Gamma}}{}^\sigma_{4A}\xi_\sigma &= \overset{(1)}{\hat{\chi}}{}_\circ{}^\sigma{}_A \xi_\sigma + \frac{1}{2}(\mathrm{tr}\overset{(1)}{\chi}_\circ)\xi_A + \frac{1}{2}(\not{\epsilon}\cdot\overset{(1)}{\chi}_\square)^\star \xi_A + \chi^{\sharp_2}{}^B_A\,\xi(\overset{(1)}{e}_B) - g(\nabla_{\overset{(1)}{e}_A} e_4, \partial_\mu) g^{\mu\sigma}\xi_\sigma \\ &= \overset{(1)}{\hat{\chi}}{}_\circ{}^\sigma{}_A \xi_\sigma + \frac{1}{2}(\mathrm{tr}\overset{(1)}{\chi}_\circ)\xi_A + \frac{1}{2}(\not{\epsilon}\cdot\overset{(1)}{\chi}_\square)^\star \xi_A + \chi^{\sharp_2}{}^B_A\,\xi(\overset{(1)}{e}_B) + g(\overset{(1)}{e}_A, e_4)\eta\cdot\xi - g(\overset{(1)}{e}_A, e_C)\not{g}^{CD}\chi^{\sharp_2}{}^B_D \xi_B , \end{aligned}$$

where we used the linearisation of the commutator $[\boldsymbol{e_4}, \boldsymbol{e_A}]$ (see Section 3.7.2). □

Claim 3.5. *We have*

$$\overset{(1)}{\not{\Gamma}}{}^C_{4A}\not{g}_{CB} + \overset{(1)}{\not{\Gamma}}{}^C_{4B}\not{g}_{AC} = 2\,\overset{(1)}{\hat{\chi}}{}_{\circ AB} + (tr\overset{(1)}{\chi}_\circ)\not{g}_{AB} - (\not{\epsilon}\cdot\chi_\square)^\star(\widehat{\overset{(1)}{\not{g}}})_{AB} + g(\overset{(1)}{e}_A, e_4)\,\eta_B + g(\overset{(1)}{e}_B, e_4)\,\eta_A .$$

Proof. Using Claim 3.4, we compute

$$\begin{aligned} \overset{(1)}{\not{\Gamma}}{}^C_{4A}\not{g}_{CB} + \overset{(1)}{\not{\Gamma}}{}^C_{4B}\not{g}_{AC} =& 2\,\overset{(1)}{\hat{\chi}}{}_{\circ AB} + (\mathrm{tr}\overset{(1)}{\chi}_\circ)\not{g}_{AB} + \chi^{\sharp_2}{}^C_A\,\not{g}(\overset{(1)}{e}_C, e_B) + g(\overset{(1)}{e}_A, e_4)\eta^C\not{g}_{CB} - g(\overset{(1)}{e}_A, e_C)g^{CD}\chi^{\sharp_2}{}^F_D\not{g}_{FB} \\ &+ \chi^{\sharp_2}{}^C_B\,\not{g}(\overset{(1)}{e}_C, e_A) + g(\overset{(1)}{e}_B, e_4)\eta^C\not{g}_{AC} - g(\overset{(1)}{e}_B, e_C)g^{CD}\chi^{\sharp_2}{}^F_D\not{g}_{FA} \\ =& 2\,\overset{(1)}{\hat{\chi}}{}_{\circ AB} + (\mathrm{tr}\overset{(1)}{\chi}_\circ)\not{g}_{AB} - \frac{1}{2}(\not{\epsilon}\cdot\chi_\square)(\not{\epsilon}^{\sharp_2}{}^C_A\,\overset{(1)}{\not{g}}(e_C, e_B) + \not{\epsilon}^{\sharp_2}{}^C_B\,\overset{(1)}{\not{g}}(e_A, e_C)) \\ &+ g(\overset{(1)}{e}_A, e_4)\eta_B + g(\overset{(1)}{e}_B, e_4)\eta_A \\ =& 2\,\overset{(1)}{\hat{\chi}}{}_{\circ AB} + (\mathrm{tr}\overset{(1)}{\chi}_\circ)\not{g}_{AB} - (\not{\epsilon}\cdot\chi_\square)^\star(\widehat{\overset{(1)}{\not{g}}})_{AB} + g(\overset{(1)}{e}_A, e_4)\eta_B + g(\overset{(1)}{e}_B, e_4)\eta_A . \end{aligned}$$

□

We are now ready to introduce the linearisation of the covariant derivatives of tensors. The reader should refer back to Section 3.5.2 to see how the projected pullback $\Pi\phi^*_\epsilon$ of the covariant derivatives is performed.

Consider the one-parameter family of $\mathfrak{D}_{\boldsymbol{\mathcal{N}}(\epsilon)}$ two-tensors $\boldsymbol{\theta}(\epsilon)$, the one-parameter family of $\mathfrak{D}_{\boldsymbol{\mathcal{N}}(\epsilon)}$ one-tensors $\boldsymbol{\xi}(\epsilon)$ and the one-parameter family of smooth scalar functions $\mathfrak{f}(\epsilon)$ on $\mathcal{M}^+$. We have

$$\Pi\phi^*_\epsilon((\boldsymbol{\nabla\!\!\!/}\,\mathfrak{f})(\epsilon)) - \nabla\!\!\!/\,\mathfrak{f} = \epsilon\cdot(\nabla\!\!\!/\,\overset{(1)}{\mathfrak{f}} + \overset{(1)}{\not{\Pi}}\cdot(\nabla\mathfrak{f})) + \mathcal{O}(\epsilon^2) , \tag{3.182}$$

$$\Pi\phi^*_\epsilon((\boldsymbol{\nabla\!\!\!/}_{\boldsymbol{3}}\,\mathfrak{f})(\epsilon)) - \nabla\!\!\!/_3\,\mathfrak{f} = \epsilon\cdot(\nabla\!\!\!/_{\overset{(1)}{e}_3}\mathfrak{f} + \nabla\!\!\!/_3\,\overset{(1)}{\mathfrak{f}}) + \mathcal{O}(\epsilon^2) , \tag{3.183}$$

$$\Pi\phi^*_\epsilon((\boldsymbol{\nabla\!\!\!/}_{\boldsymbol{4}}\,\mathfrak{f})(\epsilon)) - \nabla\!\!\!/_4\,\mathfrak{f} = \epsilon\cdot(\nabla\!\!\!/_4\,\overset{(1)}{\mathfrak{f}}) + \mathcal{O}(\epsilon^2) , \tag{3.184}$$

$$\Pi\phi_\epsilon^*((\slashed{\boldsymbol{\nabla}}\,\boldsymbol{\xi})(\epsilon)) - \slashed{\nabla}\,\xi = \epsilon\cdot(\slashed{\nabla}\,\overset{(1)}{\slashed{\xi}} - \overset{(1)}{\slashed{\Gamma}}\cdot\xi - \frac{1}{2}(\overset{(1)}{\xi}(e_3))\chi + (\overset{(1)}{\slashed{\Pi}}\otimes\Pi + \Pi\otimes\overset{(1)}{\slashed{\Pi}})\cdot(\nabla\xi)) + \mathcal{O}(\epsilon^2)\,, \tag{3.185}$$

$$\Pi\phi_\epsilon^*((\slashed{\boldsymbol{\nabla}}_{\mathbf{3}}\,\boldsymbol{\xi})(\epsilon)) - \slashed{\nabla}_3\,\xi = \epsilon\cdot(\slashed{\nabla}_{\overset{(1)}{e_3}}\xi + \slashed{\nabla}_3\,\overset{(1)}{\slashed{\xi}} - \overset{(1)}{\slashed{\Gamma}}{}_3\cdot\xi - (\overset{(1)}{\xi}(e_3))\eta + \overset{(1)}{\slashed{\Pi}}\cdot(\nabla_3\xi)) + \mathcal{O}(\epsilon^2)\,, \tag{3.186}$$

$$\Pi\phi_\epsilon^*((\slashed{\boldsymbol{\nabla}}_{\mathbf{4}}\,\boldsymbol{\xi})(\epsilon)) - \slashed{\nabla}_4\,\xi = \epsilon\cdot(\slashed{\nabla}_4\,\overset{(1)}{\slashed{\xi}} - \overset{(1)}{\slashed{\Gamma}}{}_4\cdot\xi + \overset{(1)}{\slashed{\Pi}}\cdot(\nabla_4\xi)) + \mathcal{O}(\epsilon^2) \tag{3.187}$$

on $(\mathcal{M}^+, g_{a,M})$. In coordinate components, we have

$$\begin{aligned}(\Pi\phi_\epsilon^*((\slashed{\boldsymbol{\nabla}}_{\mathbf{3}}\,\boldsymbol{\theta})(\epsilon)))_{\mu\nu} - (\slashed{\nabla}_3\,\theta)_{\mu\nu} = \epsilon\cdot((\slashed{\nabla}_{\overset{(1)}{e_3}}\theta)_{\mu\nu} + (\slashed{\nabla}_3\,\overset{(1)}{\slashed{\theta}})_{\mu\nu} - \overset{(1)}{\slashed{\Gamma}}{}^{\sigma}{}_{3\mu}\theta_{\sigma\nu} - \overset{(1)}{\slashed{\Gamma}}{}^{\sigma}{}_{3\nu}\theta_{\mu\sigma} - \eta_\mu\overset{(1)}{\theta}{}_{3\nu} - \eta_\nu\overset{(1)}{\theta}{}_{\mu3} \\ + ((\overset{(1)}{\slashed{\Pi}}\otimes\Pi + \Pi\otimes\overset{(1)}{\slashed{\Pi}})\cdot(\nabla_3\theta))_{\mu\nu}) + \mathcal{O}(\epsilon^2)\,,\end{aligned} \tag{3.188}$$

$$(\Pi\phi_\epsilon^*((\slashed{\boldsymbol{\nabla}}_{\mathbf{4}}\,\boldsymbol{\theta})(\epsilon)))_{\mu\nu} - (\slashed{\nabla}_4\,\theta)_{\mu\nu} = \epsilon\cdot((\slashed{\nabla}_4\,\overset{(1)}{\slashed{\theta}})_{\mu\nu} - \overset{(1)}{\slashed{\Gamma}}{}^{\sigma}{}_{4\mu}\theta_{\sigma\nu} - \overset{(1)}{\slashed{\Gamma}}{}^{\sigma}{}_{4\nu}\theta_{\mu\sigma} \tag{3.189}$$

$$+ ((\overset{(1)}{\slashed{\Pi}}\otimes\Pi + \Pi\otimes\overset{(1)}{\slashed{\Pi}})\cdot(\nabla_4\theta))_{\mu\nu}) + \mathcal{O}(\epsilon^2) \tag{3.190}$$

on $(\mathcal{M}^+, g_{a,M})$.

Remark 3.15. *By setting $\overset{(1)}{\slashed{\Pi}}\equiv 0$, $\xi\equiv 0$ and $\theta\equiv 0$, one can recover the linearisation formulae of [19] (see Remark 3.14, where the identity $\overset{(1)}{\slashed{\Pi}}\equiv 0$ is discussed). As noticed in Claim 3.1, the linearisation of $\mathfrak{D}_{\boldsymbol{\mathcal{N}}(\epsilon)}$ covariant tensors which vanish on the Kerr (or, in the context of [19], Schwarzschild) background does not need any projection. The reproduce the linearisation formulae of [19], one can therefore replace $\overset{(1)}{\slashed{\xi}}$ by $\overset{(1)}{\xi}$, $\overset{(1)}{\slashed{\theta}}$ by $\overset{(1)}{\theta}$ and set $\overset{(1)}{\xi}(e_3)=0$ in the formulae.*

Proof. To prove the linearisation formulae (3.182)-(3.190), the reader should first refer back to the formulae (3.161)-(3.166) of Section 3.5.2 and realise that all we need to prove is

$$\Pi\phi_\epsilon^*(\boldsymbol{\nabla}\mathfrak{f}) - \slashed{\nabla}\,\mathfrak{f} = \epsilon\cdot(\slashed{\nabla}\,\overset{(1)}{\mathfrak{f}}) + \mathcal{O}(\epsilon^2)\,, \tag{3.191}$$

$$\Pi\phi_\epsilon^*(\boldsymbol{\nabla}\boldsymbol{\xi}) - \slashed{\nabla}\,\xi = \epsilon\cdot(\slashed{\nabla}\,\overset{(1)}{\slashed{\xi}} - \overset{(1)}{\slashed{\Gamma}}\cdot\xi - \frac{1}{2}(\overset{(1)}{\xi}(e_3))\chi) + \mathcal{O}(\epsilon^2)\,, \tag{3.192}$$

$$\Pi\phi_\epsilon^*(\boldsymbol{\nabla}_{\mathbf{3}}\boldsymbol{\xi}) - \slashed{\nabla}_3\,\xi = \epsilon\cdot(\slashed{\nabla}_{\overset{(1)}{e_3}}\xi + \slashed{\nabla}_3\,\overset{(1)}{\slashed{\xi}} - \overset{(1)}{\slashed{\Gamma}}{}_3\cdot\xi - (\overset{(1)}{\xi}(e_3))\eta) + \mathcal{O}(\epsilon^2)\,, \tag{3.193}$$

$$\Pi\phi_\epsilon^*(\boldsymbol{\nabla}_{\mathbf{4}}\boldsymbol{\xi}) - \slashed{\nabla}_4\,\xi = \epsilon\cdot(\slashed{\nabla}_4\,\overset{(1)}{\slashed{\xi}} - \overset{(1)}{\slashed{\Gamma}}{}_4\cdot\xi) + \mathcal{O}(\epsilon^2) \tag{3.194}$$

on $(\mathcal{M}^+, g_{a,M})$. The contribution in the linearisation formulae coming from the projection tensor can be easily checked. Similarly, one can easily derive the formulae for $\mathfrak{D}_{\boldsymbol{\mathcal{N}}(\epsilon)}$ two-tensors from the ones that we are about to prove for $\mathfrak{D}_{\boldsymbol{\mathcal{N}}(\epsilon)}$ one-tensors. Note also that the derivation of the linearisation formulae (3.183) and (3.184) is trivial.

We compute

$$
\begin{aligned}
(\Pi\phi_\epsilon^*(\boldsymbol{\nabla}\mathfrak{f}))_\mu &= \Pi_\mu^\nu(\phi_\epsilon^*\boldsymbol{\nabla})_\nu(\phi_\epsilon^*\mathfrak{f}) \\
&= \nabla\!\!\!/_\mu(\phi_\epsilon^*\mathfrak{f})\,,
\end{aligned}
$$

where we used the fact that all connections on $(\mathcal{M}^+, g_{a,M})$ agree on scalar functions, i.e. $(\phi_\epsilon^*\boldsymbol{\nabla})_\mu(\phi_\epsilon^*\mathfrak{f}) = \nabla_\mu(\phi_\epsilon^*\mathfrak{f})$. One linearises and obtains (3.191).

It is easy to check that

$$
\begin{aligned}
\Pi\phi_\epsilon^*(\boldsymbol{\nabla}\boldsymbol{\xi}) - \nabla\!\!\!/\,\xi &= \epsilon\cdot((\Pi\otimes\Pi)\cdot(\nabla\overset{(1)}{\xi}) - \overset{(1)}{\Gamma\!\!\!/}\cdot\xi + \mathcal{O}(\epsilon^2)\,,\\
\Pi\phi_\epsilon^*(\boldsymbol{\nabla_3}\boldsymbol{\xi}) - \nabla\!\!\!/_3\,\xi &= \epsilon\cdot(\nabla\!\!\!/_{\overset{(1)}{e_3}}\xi + \Pi\cdot(\nabla_3\overset{(1)}{\xi}) - \overset{(1)}{\Gamma\!\!\!/}_3\cdot\xi + \mathcal{O}(\epsilon^2)\,,\\
\Pi\phi_\epsilon^*(\boldsymbol{\nabla_4}\boldsymbol{\xi}) - \nabla\!\!\!/_4\,\xi &= \epsilon\cdot(\Pi\cdot(\nabla_4\overset{(1)}{\xi}) - \overset{(1)}{\Gamma\!\!\!/}_4\cdot\xi) + \mathcal{O}(\epsilon^2)\,.
\end{aligned}
$$

We now note that

$$
\begin{aligned}
(\Pi\otimes\Pi)\cdot(\nabla\overset{(1)}{\xi}) &= (\Pi\otimes\Pi)\cdot(\nabla\overset{(1)}{\xi\!\!\!/}) - \frac{1}{2}(\overset{(1)}{\xi}(e_3))\chi\,,\\
\Pi\cdot(\nabla_3\overset{(1)}{\xi}) &= \Pi\cdot(\nabla_3\overset{(1)}{\xi\!\!\!/}) - (\overset{(1)}{\xi}(e_3))\eta\,,\\
\Pi\cdot(\nabla_4\overset{(1)}{\xi}) &= \Pi\cdot(\nabla_4\overset{(1)}{\xi\!\!\!/})\,,
\end{aligned}
$$

where we crucially use Claim 3.2. One can then conclude the proof of the formulae (3.192)-(3.194). □

Remark 3.16. *As already observed, $\boldsymbol{\eta}(\epsilon)$ and $\boldsymbol{\zeta}(\epsilon)$ are, together with the metric $\boldsymbol{g\!\!\!/}(\epsilon)$, the only one-parameter families of $\mathfrak{D}_{\boldsymbol{\mathcal{N}}(\epsilon)}$ covariant tensors which do not vanish on $(\mathcal{M}^+, g_{a,M})$ and have* non-zero *linearisation. For this reason, the formulae* (3.185)-(3.187) *applied to $\boldsymbol{\eta}(\epsilon)$ and $\boldsymbol{\zeta}(\epsilon)$ have each single term that is non-vanishing. When applied to all the other tensors, some of the terms vanish.*

We now define the notation

$$
\begin{aligned}
(\mathrm{tr}(\overset{(1)}{\Pi\!\!\!/}\otimes\Pi))^{\alpha\beta} &:= g\!\!\!/^{\mu\nu}\overset{(1)}{\Pi\!\!\!/}{}^\alpha_\mu\Pi^\beta_\nu\,, & (\mathrm{tr}(\Pi\otimes\overset{(1)}{\Pi\!\!\!/}))^{\alpha\beta} &:= g\!\!\!/^{\mu\nu}\Pi^\alpha_\mu\overset{(1)}{\Pi\!\!\!/}{}^\beta_\nu\,,\\
(\in\!\!\!\!/\cdot(\overset{(1)}{\Pi\!\!\!/}\otimes\Pi))^{\alpha\beta} &:= \in\!\!\!\!/^{\mu\nu}\overset{(1)}{\Pi\!\!\!/}{}^\alpha_\mu\Pi^\beta_\nu\,, & (\in\!\!\!\!/\cdot(\Pi\otimes\overset{(1)}{\Pi\!\!\!/}))^{\alpha\beta} &:= \in\!\!\!\!/^{\mu\nu}\Pi^\alpha_\mu\overset{(1)}{\Pi\!\!\!/}{}^\beta_\nu
\end{aligned}
$$

and, using formula (3.185), we derive

$$\Pi\phi_\epsilon^*((\mathbf{d\!\!/iv}\,\boldsymbol{\xi})(\epsilon)) - \mathrm{d\!\!/iv}\,\xi = \epsilon\cdot(\mathrm{d\!\!/iv}\,\overset{(1)}{\not\xi} - \mathrm{tr}(\overset{(1)}{\not\Gamma}\cdot\xi) - \frac{1}{2}(\overset{(1)}{\xi}(e_3))(\mathrm{tr}\chi) + \mathrm{tr}(\overset{(1)}{\not\Pi}\otimes\Pi + \Pi\otimes\overset{(1)}{\not\Pi})\cdot(\nabla\xi)$$
$$+\overset{(1)}{\not g}{}^{\mu\nu}(\not\nabla\xi)_{\mu\nu}) + \mathcal{O}(\epsilon^2)\,, \tag{3.195}$$

$$\Pi\phi_\epsilon^*((\mathbf{cu\!\!/rl}\,\boldsymbol{\xi})(\epsilon)) - \mathrm{cu\!\!/rl}\,\xi = \epsilon\cdot(\mathrm{cu\!\!/rl}\,\overset{(1)}{\not\xi} - \frac{1}{2}(\overset{(1)}{\xi}(e_3))(\not\in\cdot\chi) + (\not\in\cdot(\overset{(1)}{\not\Pi}\otimes\Pi + \Pi\otimes\overset{(1)}{\not\Pi}))\cdot(\nabla\xi)$$
$$-\frac{1}{2}(\mathrm{cu\!\!/rl}\,\xi)(\mathrm{tr}\,\overset{(1)}{\not g})) + \mathcal{O}(\epsilon^2) \tag{3.196}$$

on $(\mathcal{M}^+, g_{a,M})$. The linearisation of the divergence of a $\mathfrak{D}_{\boldsymbol{\mathcal{N}}(\epsilon)}$ *two*-tensor that we will need to perform will be trivial, since the $\mathfrak{D}_{\boldsymbol{\mathcal{N}}(\epsilon)}$ two-tensor will vanish on the Kerr background, e.g.

$$(\mathbf{d\!\!/iv}\,\hat{\boldsymbol{\chi}}_\circ)(\epsilon)\,,\ (\mathbf{d\!\!/iv}\,\underline{\boldsymbol{\alpha}})(\epsilon)\ \ldots$$

For this reason, we do not derive a complete linearisation formula for the divergence of $\mathfrak{D}_{\boldsymbol{\mathcal{N}}(\epsilon)}$ two-tensors.

We note that <u>if</u>

$$-2\not{\mathcal{D}}_2^*\,\xi + \xi\,\hat{\otimes}\,\xi = 0$$

on $(\mathcal{M}^+, g_{a,M})$, then we have

$$\Pi\phi_\epsilon^*((\mathbf{d\!\!/iv}\,\boldsymbol{\xi} + (\boldsymbol{\xi},\boldsymbol{\xi})(\epsilon))) - (\mathrm{d\!\!/iv}\,\xi + (\xi,\xi)) = \epsilon\cdot(\mathrm{d\!\!/iv}\,\overset{(1)}{\not\xi} + 2(\xi,\overset{(1)}{\not\xi}) - \frac{1}{2}(\mathrm{d\!\!/iv}\,\xi + (\xi,\xi))(\mathrm{tr}\,\overset{(1)}{\not g}) \tag{3.197}$$
$$-\mathrm{tr}(\overset{(1)}{\not\Gamma}\cdot\xi) - \frac{1}{2}(\overset{(1)}{\xi}(e_3))(\mathrm{tr}\chi) + \mathrm{tr}(\overset{(1)}{\not\Pi}\otimes\Pi + \Pi\otimes\overset{(1)}{\not\Pi})\cdot(\nabla\xi))$$
$$+\mathcal{O}(\epsilon^2)$$

and

$$\Pi\phi_\epsilon^*((-2\not{\boldsymbol{\mathcal{D}}}_{\mathbf{2}}^*\,\boldsymbol{\xi} + \boldsymbol{\xi}\,\hat{\otimes}\,\boldsymbol{\xi})(\epsilon)) = \epsilon\cdot(-2\not{\mathcal{D}}_2^*\,\overset{(1)}{\not\xi} + 2(\xi\,\hat{\otimes}\,\overset{(1)}{\not\xi}) - 2\,\widehat{\overset{(1)}{\not\Gamma}}\cdot\xi - (\mathrm{d\!\!/iv}\,\xi + (\xi,\xi))\,\widehat{\overset{(1)}{\not g}} \tag{3.198}$$
$$+(\overset{(1)}{\not\Pi}\,\widehat{\otimes}\,\Pi + \Pi\,\hat{\otimes}\,\overset{(1)}{\not\Pi})\cdot(\nabla\xi)) + \mathcal{O}(\epsilon^2)$$

on $(\mathcal{M}^+, g_{a,M})$, where we used

$$(\overset{(1)}{\xi}(e_3))\,\hat{\chi}_\circ = 0$$

and the linearisation of the product

$$\Pi\phi_\epsilon^*((\boldsymbol{\xi},\boldsymbol{\xi})(\epsilon)) - (\xi,\xi) = \epsilon\cdot(2\,(\xi,\overset{(1)}{\not\xi}) + \overset{(1)}{\not g}{}^{\mu\nu}\,\xi_\mu\,\xi_\nu) + \mathcal{O}(\epsilon^2)$$

on $(\mathcal{M}^+, g_{a,M})$.

3.7.2 The equations of linearised gravity

In this section we present the system of linearised vacuum Einstein equations around the Kerr solution $(\mathcal{M}^+, g_{a,M})$ in our outgoing frame-calibrated gauge.

With the formulae derived in Section 3.7.1, the reader should be able to linearise all the equations in the system of Section 3.3.5. To give some non-trivial examples, we linearise the null frame equation

$$(\boldsymbol{\nabla}_{\boldsymbol{4}}\, \boldsymbol{e_A})(\epsilon) - (\boldsymbol{\nabla}_{\boldsymbol{A}}\, \boldsymbol{e_4})(\epsilon) = \left(\boldsymbol{\Gamma}^{\boldsymbol{B}}_{\boldsymbol{4A}}(\epsilon) - \boldsymbol{\chi}^{\sharp_2}{}^{\boldsymbol{B}}{}_{\boldsymbol{A}}(\epsilon)\right) \boldsymbol{e_B}(\epsilon) + \boldsymbol{Y_A}(\epsilon)\boldsymbol{e_3}(\epsilon) + (\underline{\boldsymbol{\eta}}_{\boldsymbol{A}}(\epsilon) + \boldsymbol{\zeta_A}(\epsilon))\, \boldsymbol{e_4}(\epsilon)\,, \tag{3.199}$$

the first variational formula

$$(\boldsymbol{\mathcal{L}\!\!\!/}_{\boldsymbol{e_4}}\boldsymbol{g\!\!\!/})(\epsilon) = 2\,\hat{\boldsymbol{\chi}}_{\circ}(\epsilon) + (\mathbf{tr}\boldsymbol{\chi}_{\circ})(\epsilon)\boldsymbol{g\!\!\!/}(\epsilon) \tag{3.200}$$

and the Bianchi equation

$$\begin{aligned}(\boldsymbol{\nabla\!\!\!\!/}_{\boldsymbol{3}}\, \boldsymbol{\beta})(\epsilon) + (\mathbf{tr}\, \boldsymbol{\chi}_{\circ})(\epsilon)\boldsymbol{\beta}(\epsilon) + \underline{\hat{\boldsymbol{\omega}}}(\epsilon)\boldsymbol{\beta}(\epsilon) + (\boldsymbol{\epsilon\!\!/} \cdot \boldsymbol{\chi}_{\square})(\epsilon)^{\star}\boldsymbol{\beta}(\epsilon) = {} & \boldsymbol{\mathcal{D}\!\!\!\!/}^{*}_{\boldsymbol{1}}(-\boldsymbol{\rho}, \boldsymbol{\sigma})(\epsilon) + 3\boldsymbol{\rho}(\epsilon)\boldsymbol{\eta}(\epsilon) \\ & + 3\boldsymbol{\sigma}(\epsilon)^{\star}\boldsymbol{\eta}(\epsilon) + 2\hat{\boldsymbol{\chi}}_{\circ}{}^{\sharp}(\epsilon) \cdot \underline{\boldsymbol{\beta}}(\epsilon) \\ & + \underline{\boldsymbol{Y}}^{\sharp}(\epsilon) \cdot \boldsymbol{\alpha}(\epsilon)\,. \end{aligned} \tag{3.201}$$

Starting with the linearisation of equation (3.199) for the frame vector fields, we consider, according to the Definition 3.6, the pushed-forward equation

$$\begin{aligned}\phi_{-\epsilon*}(\boldsymbol{\nabla}_{\boldsymbol{4}}\, \boldsymbol{e_A}) - \phi_{-\epsilon*}(\boldsymbol{\nabla}_{\boldsymbol{A}}\, \boldsymbol{e_4}) = {} & \left(\phi^*_\epsilon(\boldsymbol{\Gamma}^{\boldsymbol{B}}_{\boldsymbol{4A}}) - \phi^*_\epsilon(\boldsymbol{\chi}^{\sharp_2}{}^{\boldsymbol{B}}{}_{\boldsymbol{A}})\right)(\phi_{-\epsilon*}\boldsymbol{e_B}) + (\phi^*_\epsilon(\boldsymbol{Y_A}))(\phi_{-\epsilon*}\boldsymbol{e_3}) \\ & + (\phi^*_\epsilon(\underline{\boldsymbol{\eta}}_{\boldsymbol{A}}) + \phi^*_\epsilon(\boldsymbol{\zeta_A}))\,(\phi_{-\epsilon*}\boldsymbol{e_4})\end{aligned}$$

on $(\mathcal{M}^+, g_{a,M})$ and linearise each of the terms following the Definitions 3.6 and 3.7. Note that, for instance,

$$\phi^*_\epsilon(\boldsymbol{Y_A}) = (\phi^*_\epsilon\boldsymbol{Y})(\phi_{-\epsilon*}\boldsymbol{e_A})\,,$$

where

$$(\phi^*_\epsilon\boldsymbol{Y})(e_A) = (\Pi\phi^*_\epsilon\boldsymbol{Y})(e_A) = 0\,, \qquad\qquad Y(\phi_{-\epsilon*}\boldsymbol{e_A}) = 0\,.$$

We obtain the linearised null frame equation

$$e_4(\overset{(1)}{e}{}^{\mu}_{A}) = -\overset{(1)}{\hat{\chi}}_{\circ}{}^{\mu}{}_{A} - \frac{1}{2}\,(\overset{(1)}{\mathrm{tr}\chi_{\circ}})\,\delta^{\mu}_{A} - \frac{1}{2}\,(\overset{(1)}{\epsilon\!\!/ \cdot \chi_{\square}})\,\epsilon\!\!/^{\sharp_2}{}^{\mu}{}_{A} - \chi^{\sharp_2}{}^{B}{}_{A}\,\overset{(1)}{e}{}^{\mu}_{B} + \overset{(1)}{\zeta}{}_{A}\,e^{\mu}_{4}\,.$$

To linearise equation (3.200), we first note that the equation is equivalent to

$$(\boldsymbol{\nabla\!\!\!\!/}_{\boldsymbol{4}}\, \boldsymbol{g\!\!\!/})(\epsilon) = 0\,.$$

We linearise equation (3.200) in this latter form using the linearisation formula (3.190), i.e. we linearise the tensorial equation

$$\Pi\phi^*_\epsilon((\boldsymbol{\nabla\!\!\!/}_{\mathbf{4}}\,\boldsymbol{g\!\!\!/})(\epsilon))=0 \tag{3.202}$$

on $(\mathcal{M}^+,g_{a,M})$. We define the $\mathfrak{D}_{\mathcal{N}}$ one-tensor

$$\overset{(1)}{\mathfrak{h}}$$

on $(\mathcal{M}^+,g_{a,M})$ such that

$$\overset{(1)}{\mathfrak{h}}{}_A:=g(\overset{(1)}{e}_A,e_4)$$

on $(\mathcal{M}^+,g_{a,M})$ and we note that

$$((\overset{(1)}{\Pi\!\!\!/}\otimes\Pi+\Pi\otimes\overset{(1)}{\Pi\!\!\!/})\cdot(\nabla_4 g\!\!\!/))_{AB}=g(\overset{(1)}{e}_A,e_4)\,\underline{\eta}_B+g(\overset{(1)}{e}_B,e_4)\,\underline{\eta}_A\,.$$

By applying Claim 3.5 and taking the *symmetric traceless* and *trace* parts of the linearisation of (3.202), we obtain the pair of linearised equations

$$\nabla\!\!\!/_4\,\widehat{\overset{(1)}{g\!\!\!/}}+(\not{\epsilon}\cdot\chi_{\square})^\star(\widehat{\overset{(1)}{g\!\!\!/}})=2\,\overset{(1)}{\hat{\chi}}_{\bigcirc}+\overset{(1)}{\mathfrak{h}}\,\hat{\otimes}\,(\eta-\underline{\eta})\,, \tag{3.203}$$

$$\nabla\!\!\!/_4(\mathrm{tr}\,\overset{(1)}{g\!\!\!/})=(\overset{(1)}{\mathrm{tr}\chi_{\bigcirc}})+(\overset{(1)}{\mathfrak{h}},\eta-\underline{\eta})\,. \tag{3.204}$$

Equation (3.203) is an equation for symmetric traceless $\mathfrak{D}_{\mathcal{N}}$ two-tensors on $(\mathcal{M}^+,g_{a,M})$. Since $\Gamma^B_{4A}=0$ on $(\mathcal{M}^+,g_{a,M})$, one can equivalently see it as an equation for the linearised metric *components*, namely

$$\nabla\!\!\!/_4(\widehat{\overset{(1)}{g\!\!\!/}}_{AB})+(\not{\epsilon}\cdot\chi_{\square})^\star(\widehat{\overset{(1)}{g\!\!\!/}})_{AB}=2\,\overset{(1)}{\hat{\chi}}_{\bigcirc AB}+(\overset{(1)}{\mathfrak{h}}\,\hat{\otimes}\,(\eta-\underline{\eta}))_{AB}\,.$$

The linearisation of equation (3.201) is conceptually analogous to that of equation (3.200). However, we want to briefly elaborate on the linearisation of the terms

$$\Pi\phi^*_\epsilon((\boldsymbol{\nabla\!\!\!/}_{\mathbf{3}}\,\boldsymbol{\beta})(\epsilon)) \tag{3.205}$$

and

$$\Pi\phi^*_\epsilon((^\star\boldsymbol{\eta})(\epsilon)) \tag{3.206}$$

on $(\mathcal{M}^+,g_{a,M})$. To linearise (3.205), we apply formula (3.186). We have

$$\Pi\phi^*_\epsilon((\boldsymbol{\nabla\!\!\!/}_{\mathbf{3}}\,\boldsymbol{\beta})(\epsilon))-\nabla\!\!\!/_3\,\beta=\epsilon\cdot(\nabla\!\!\!/_3\,\overset{(1)}{\beta})+\mathcal{O}(\epsilon^2)$$

on $(\mathcal{M}^+,g_{a,M})$, where we used $\beta\equiv 0$ and, in view of Claim 3.1, the identity

$$(\overset{(1)}{\beta}(e_3))\eta=0\,.$$

To linearise (3.206), we note that

$$(\Pi\phi_\epsilon^*((^\star\boldsymbol{\eta})(\epsilon)))_\mu = \Pi_\mu^\nu\,(\phi_\epsilon^*\,\boldsymbol{\not g}^{-1}(\epsilon))^{\sigma\gamma}(\phi_\epsilon^*\,\boldsymbol{\not\epsilon}(\epsilon))_{\nu\sigma}(\phi_\epsilon^*\,\boldsymbol{\eta}(\epsilon))_\gamma$$

on $(\mathcal{M}^+, g_{a,M})$ and linearise

$$(\Pi\phi_\epsilon^*(^\star\boldsymbol{\eta}(\epsilon)))_\mu - (^\star\eta)_\mu = \epsilon\cdot(\overset{(1)}{\not g}{}^{\sigma\gamma}\not\epsilon_{\mu\sigma}\eta_\gamma + (\overset{(1)}{\not\epsilon}{}^{\sharp_2}\cdot\eta)_\mu + (^\star\overset{(1)}{\not\eta})_\mu) + \mathcal{O}(\epsilon^2)\,.$$

The final linearisation of (3.201) reads

$$\begin{aligned}\not\nabla_3\overset{(1)}{\beta} + (\mathrm{tr}\,\underline{\chi}_\circ)\overset{(1)}{\beta} + \underline{\hat\omega}\,\overset{(1)}{\beta} + (\not\epsilon\cdot\chi_\square)^\star\overset{(1)}{\beta} = {}& \not{\mathcal{D}}_1^*(-\overset{(1)}{\rho},\overset{(1)}{\sigma}) + 3\rho\overset{(1)}{\not\eta} + 3\overset{(1)}{\rho}\,\eta + 3\overset{(1)}{\sigma}{}^\star\eta \\ & + 3\sigma^\star\overset{(1)}{\not\eta} + 3\sigma\overset{(1)}{\not g}{}^{\sigma\gamma}\not\epsilon_{\mu\sigma}\eta_\gamma + 3\sigma\overset{(1)}{\not\epsilon}{}^{\sharp_2}\cdot\eta \\ & + \overset{(1)}{\not g}{}^{\sigma\gamma}\not\epsilon_{\mu\sigma}\not\nabla_\gamma\sigma + \overset{(1)}{\not\epsilon}{}^{\sharp_2}\cdot(\not\nabla\sigma) + \overset{(1)}{\not\Pi}\cdot(\nabla\rho + \nabla\sigma)\,.\end{aligned}$$

We are now ready to present the full system of linearised vacuum Einstein equations.

Linearised null frame equations

The linearised null frame equations read

$$\begin{aligned}\overset{(1)}{e}_A(e^\mu_B) + e_A(\overset{(1)}{e^\mu_B}) - \overset{(1)}{e}_B(e^\mu_A) - e_B(\overset{(1)}{e^\mu_A}) = {}& (\overset{(1)}{\Gamma^C_{AB}} - \overset{(1)}{\Gamma^C_{BA}})e^\mu_C + (\Gamma^C_{AB} - \Gamma^C_{BA})\overset{(1)}{e^\mu_C} \\ & + \frac{1}{2}(\not\epsilon\cdot\overset{(1)}{\chi_\square})\not\epsilon_{AB}e^\mu_3 + \frac{1}{2}(\not\epsilon\cdot\chi_\square)\not\epsilon_{AB}\overset{(1)}{e^\mu_3} \\ & + \frac{1}{2}(\not\epsilon\cdot\overset{(1)}{\underline{\chi}_\square})\not\epsilon_{AB}e^\mu_4\,,\end{aligned} \tag{3.207}$$

$$\begin{aligned}\overset{(1)}{e}_3(e^\mu_A) + e_3(\overset{(1)}{e^\mu_A}) - \overset{(1)}{e}_A(e^\mu_3) - e_A(\overset{(1)}{e^\mu_3}) = {}& \overset{(1)}{\Gamma^B_{3A}}e^\mu_B - \overset{(1)}{\hat{\underline{\chi}}}_\circ{}^\mu{}_A - \frac{1}{2}\,(\mathrm{tr}\,\overset{(1)}{\underline{\chi}_\circ})\,\delta^\mu_A - \frac{1}{2}\,(\not\epsilon\cdot\overset{(1)}{\underline{\chi}_\square})\,\not\epsilon^{\sharp_2\mu}{}_A \\ & + (\Gamma^B_{3A} - \underline{\chi}^{\sharp_2 B}{}_A)\,\overset{(1)}{e^\mu_B} + (\overset{(1)}{\eta} - \overset{(1)}{\zeta})_A e^\mu_3 + (\eta - \zeta)(\overset{(1)}{e}_A)e^\mu_3 \\ & + (\eta - \zeta)_A\overset{(1)}{e^\mu_3} + \overset{(1)}{\underline{Y}}_A\,e^\mu_4\end{aligned} \tag{3.208}$$

and

$$e_4(\overset{(1)}{e^\mu_A}) = -\overset{(1)}{\hat\chi}_\circ{}^\mu{}_A - \frac{1}{2}\,(\mathrm{tr}\overset{(1)}{\chi_\circ})\,\delta^\mu_A - \frac{1}{2}\,(\not\epsilon\cdot\overset{(1)}{\chi_\square})\,\not\epsilon^{\sharp_2\mu}{}_A - \chi^{\sharp_2 B}{}_A\,\overset{(1)}{e^\mu_B} + \overset{(1)}{\zeta}_A\,e^\mu_4\,, \tag{3.209}$$

$$\overset{(1)}{e}_3(e^\mu_4) - e_4(\overset{(1)}{e^\mu_3}) = 2\overset{(1)}{\not g}{}^{\mu\nu}(\eta_\nu - \underline{\eta}_\nu) + 2\not g^{\mu\nu}(\overset{(1)}{\eta}_\nu - \overset{(1)}{\underline{\eta}}_\nu) + \hat\omega\,\overset{(1)}{e^\mu_3} - \underline{\hat\omega}\,\overset{(1)}{e^\mu_4}\,. \tag{3.210}$$

Linearised null structure equations

We have the linearised first variational formulae

$$\not\nabla_4 \widehat{\overset{(1)}{\not g}} + (\not e \cdot \chi_\square)^\star(\widehat{\overset{(1)}{\not g}}) = 2\,\overset{(1)}{\hat{\chi}_\bigcirc} + \overset{(1)}{\mathfrak{h}}\,\hat{\otimes}\,(\eta - \underline{\eta})\,, \tag{3.211}$$

$$\not\nabla_4(\mathrm{tr}\,\overset{(1)}{\not g}) = (\mathrm{tr}\overset{(1)}{\chi_\bigcirc}) + (\overset{(1)}{\mathfrak{h}}, \eta - \underline{\eta}) \tag{3.212}$$

and

$$\begin{aligned}
&\not\nabla_3(\overset{(1)}{\not g}_{\mu\nu}) - \not\nabla_3(\Pi^\sigma_\mu \Pi^\gamma_\nu)\overset{(1)}{\not g}_{\sigma\gamma} + (\overset{(1)}{\not\Pi}{}^\sigma_\mu \Pi^\gamma_\nu + \Pi^\sigma_\mu \overset{(1)}{\not\Pi}{}^\gamma_\nu)(\mathcal{L}_3 \not g)_{\sigma\gamma} \\
&\quad + \Pi^\sigma_\mu \Pi^\gamma_\nu (\overset{(1)}{e_3}(\not g_{\sigma\gamma}) + (\partial_\sigma \overset{(1)}{e^\rho_3})\not g_{\gamma\rho} + (\partial_\gamma \overset{(1)}{e^\rho_3})\not g_{\sigma\rho} + (\partial_\sigma e^\rho_3)\overset{(1)}{\not g}_{\gamma\rho} + (\partial_\gamma e^\rho_3)\overset{(1)}{\not g}_{\sigma\rho}) \\
&\quad = 2\,\overset{(1)}{\hat{\underline{\chi}}}_{\bigcirc\mu\nu} + (\mathrm{tr}\,\overset{(1)}{\underline{\chi}_\bigcirc})\not g_{\mu\nu} + (\mathrm{tr}\,\underline{\chi}_\bigcirc)\overset{(1)}{\not g}_{\mu\nu}\,.
\end{aligned} \tag{3.213}$$

We have the linearised second variational formulae

$$\not\nabla_4 \overset{(1)}{\hat{\chi}_\bigcirc} + (\mathrm{tr}\chi_\bigcirc)\overset{(1)}{\hat{\chi}_\bigcirc} - \hat{\omega}\,\overset{(1)}{\hat{\chi}_\bigcirc} = -\overset{(1)}{\alpha}\,, \tag{3.214}$$

$$\not\nabla_3 \overset{(1)}{\hat{\underline{\chi}}_\bigcirc} + (\mathrm{tr}\,\underline{\chi}_\bigcirc)\overset{(1)}{\hat{\underline{\chi}}_\bigcirc} - \hat{\underline{\omega}}\,\overset{(1)}{\hat{\underline{\chi}}_\bigcirc} = -2\not{\mathcal{D}}^\star_2 \overset{(1)}{\underline{Y}} + (\eta + \underline{\eta} - 2\zeta)\hat{\otimes}\overset{(1)}{\underline{Y}} - \overset{(1)}{\underline{\alpha}} \tag{3.215}$$

and the linearised Raychaudhuri equations

$$\not\nabla_4(\mathrm{tr}\overset{(1)}{\chi_\bigcirc}) + (\mathrm{tr}\chi_\bigcirc)(\mathrm{tr}\overset{(1)}{\chi_\bigcirc}) - \hat{\omega}(\mathrm{tr}\overset{(1)}{\chi_\bigcirc}) = (\not e \cdot \chi_\square)(\overset{(1)}{\not e} \cdot \chi_\square)\,, \tag{3.216}$$

$$\begin{aligned}
\not\nabla_3(\mathrm{tr}\,\overset{(1)}{\underline{\chi}_\bigcirc}) + (\mathrm{tr}\,\underline{\chi}_\bigcirc)(\mathrm{tr}\,\overset{(1)}{\underline{\chi}_\bigcirc}) - \hat{\underline{\omega}}(\mathrm{tr}\,\overset{(1)}{\underline{\chi}_\bigcirc}) &= (\not e \cdot \underline{\chi}_\square)(\overset{(1)}{\not e} \cdot \underline{\chi}_\square) + (\mathrm{tr}\,\underline{\chi}_\bigcirc)\,\overset{(1)}{\hat{\underline{\omega}}} - \overset{(1)}{e_3}(\mathrm{tr}\,\underline{\chi}_\bigcirc) \\
&\quad + 2\,\mathrm{d}\!\!\!/\mathrm{iv}\,\overset{(1)}{\underline{Y}} + 2(\eta + \underline{\eta} - 2\zeta, \overset{(1)}{\underline{Y}})\,.
\end{aligned} \tag{3.217}$$

We have the linearised mixed transport equations

$$\begin{aligned}
\not\nabla_4 \overset{(1)}{\hat{\underline{\chi}}_\bigcirc} + \frac{1}{2}(\mathrm{tr}\chi_\bigcirc)\overset{(1)}{\hat{\underline{\chi}}_\bigcirc} + \hat{\omega}\,\overset{(1)}{\hat{\underline{\chi}}_\bigcirc} &= -\frac{1}{2}(\mathrm{tr}\,\underline{\chi}_\bigcirc)\overset{(1)}{\hat{\chi}_\bigcirc} + \frac{1}{2}(\not e \cdot \chi_\square)^\star\overset{(1)}{\hat{\underline{\chi}}_\bigcirc} - \frac{1}{2}(\not e \cdot \underline{\chi}_\square)^\star\overset{(1)}{\hat{\chi}_\bigcirc} \\
&\quad - 2\,\widehat{\overset{(1)}{\not\Gamma}} \cdot \underline{\eta} + (\overset{(1)}{\not\Pi}\hat{\otimes}\Pi + \Pi\hat{\otimes}\overset{(1)}{\not\Pi}) \cdot (\nabla\underline{\eta}) - (\mathrm{d}\!\!\!/\mathrm{iv}\,\underline{\eta} + (\underline{\eta}, \underline{\eta}))\widehat{\overset{(1)}{\not g}}\,,
\end{aligned} \tag{3.218}$$

$$\begin{aligned}
\not\nabla_3 \overset{(1)}{\hat{\chi}_\bigcirc} + \frac{1}{2}(\mathrm{tr}\,\underline{\chi}_\bigcirc)\overset{(1)}{\hat{\chi}_\bigcirc} + \hat{\underline{\omega}}\,\overset{(1)}{\hat{\chi}_\bigcirc} &= -\frac{1}{2}(\mathrm{tr}\chi_\bigcirc)\overset{(1)}{\hat{\underline{\chi}}_\bigcirc} + \frac{1}{2}(\not e \cdot \chi_\square)^\star\overset{(1)}{\hat{\underline{\chi}}_\bigcirc} - \frac{1}{2}(\not e \cdot \underline{\chi}_\square)^\star\overset{(1)}{\hat{\chi}_\bigcirc} \\
&\quad - 2\not{\mathcal{D}}^*_2 \overset{(1)}{\not\eta} + 2(\eta\hat{\otimes}\overset{(1)}{\not\eta}) - 2\,\widehat{\overset{(1)}{\not\Gamma}} \cdot \eta + (\overset{(1)}{\not\Pi}\hat{\otimes}\Pi + \Pi\hat{\otimes}\overset{(1)}{\not\Pi}) \cdot (\nabla\eta) \\
&\quad - (\mathrm{d}\!\!\!/\mathrm{iv}\eta + (\eta, \eta))\widehat{\overset{(1)}{\not g}}
\end{aligned} \tag{3.219}$$

$$\begin{aligned}\nabla\!\!\!/_4(\mathrm{tr}\overset{(1)}{\underline{\chi}_\circ}) + \frac{1}{2}(\mathrm{tr}\chi_\circ)(\mathrm{tr}\overset{(1)}{\underline{\chi}_\circ}) + \hat{\omega}(\mathrm{tr}\overset{(1)}{\underline{\chi}_\circ}) = &-\frac{1}{2}(\mathrm{tr}\,\underline{\chi}_\circ)(\mathrm{tr}\overset{(1)}{\chi_\circ}) + \frac{1}{2}(\not\in\cdot\underline{\chi}_\square)(\not\in\cdot\overset{(1)}{\chi_\square}) + \frac{1}{2}(\not\in\cdot\chi_\square)(\not\in\cdot\overset{(1)}{\underline{\chi}_\square}) \\ &-(\mathrm{d\!\!\!/iv}\,\underline{\eta}+(\underline{\eta},\underline{\eta}))(\mathrm{tr}\overset{(1)}{\not g}) + 2\,\mathrm{tr}(\overset{(1)}{\Pi\!\!\!/}\otimes\Pi+\Pi\otimes\overset{(1)}{\Pi\!\!\!/})\cdot(\nabla\,\underline{\eta}) \\ &-2\,\mathrm{tr}(\overset{(1)}{\Gamma\!\!\!/}\cdot\underline{\eta}) + 2\overset{(1)}{\rho}\,,\end{aligned} \tag{3.220}$$

$$\begin{aligned}\nabla\!\!\!/_3(\mathrm{tr}\overset{(1)}{\chi_\circ}) + \frac{1}{2}(\mathrm{tr}\,\underline{\chi}_\circ)(\mathrm{tr}\overset{(1)}{\chi_\circ}) + \underline{\hat{\omega}}(\mathrm{tr}\overset{(1)}{\chi_\circ}) = &-\frac{1}{2}(\mathrm{tr}\chi_\circ)(\mathrm{tr}\overset{(1)}{\underline{\chi}_\circ}) + \frac{1}{2}(\not\in\cdot\chi_\square)(\not\in\cdot\overset{(1)}{\underline{\chi}_\square}) + \frac{1}{2}(\not\in\cdot\underline{\chi}_\square)(\not\in\cdot\overset{(1)}{\chi_\square}) \\ &-(\mathrm{d\!\!\!/iv}\eta+(\eta,\eta))(\mathrm{tr}\overset{(1)}{\not g}) + 2\,\mathrm{tr}(\overset{(1)}{\Pi\!\!\!/}\otimes\Pi+\Pi\otimes\overset{(1)}{\Pi\!\!\!/})\cdot(\nabla\eta) \\ &-2\,\mathrm{tr}(\overset{(1)}{\Gamma\!\!\!/}\cdot\eta) - \overset{(1)}{e_3}(\mathrm{tr}\chi_\circ) - (\mathrm{tr}\chi_\circ)\overset{(1)}{\underline{\hat{\omega}}} + 2\,\mathrm{d\!\!\!/iv}\overset{(1)}{\not\eta} + 4(\eta,\overset{(1)}{\not\eta}) \\ &-(\overset{(1)}{\eta}(e_3))(\mathrm{tr}\chi) + 2\overset{(1)}{\rho}\end{aligned} \tag{3.221}$$

and the transport equations

$$\nabla\!\!\!/_4(\not\in\cdot\overset{(1)}{\chi_\square}) + (\mathrm{tr}\chi_\circ)(\not\in\cdot\overset{(1)}{\chi_\square}) - \hat{\omega}(\not\in\cdot\overset{(1)}{\chi_\square}) = -(\not\in\cdot\chi_\square)(\mathrm{tr}\overset{(1)}{\chi_\circ})\,, \tag{3.222}$$

$$\begin{aligned}\nabla\!\!\!/_3(\not\in\cdot\overset{(1)}{\underline{\chi}_\square}) + (\mathrm{tr}\,\underline{\chi}_\circ)(\not\in\cdot\overset{(1)}{\underline{\chi}_\square}) - \underline{\hat{\omega}}(\not\in\cdot\overset{(1)}{\underline{\chi}_\square}) = &-(\not\in\cdot\underline{\chi}_\square)(\mathrm{tr}\overset{(1)}{\underline{\chi}_\circ}) + (\not\in\cdot\underline{\chi}_\square)\overset{(1)}{\underline{\hat{\omega}}} - \overset{(1)}{e_3}(\not\in\cdot\underline{\chi}_\square) \\ &+2(\eta-2\zeta)\wedge\overset{(1)}{\underline{Y}} + 2\overset{(1)}{\underline{Y}}\wedge\underline{\eta} + 2\,\mathrm{cu\!\!\!/rl}\overset{(1)}{\underline{Y}}\,,\end{aligned} \tag{3.223}$$

$$\begin{aligned}\nabla\!\!\!/_4(\not\in\cdot\overset{(1)}{\underline{\chi}_\square}) + \frac{1}{2}(\mathrm{tr}\chi_\circ)(\not\in\cdot\overset{(1)}{\underline{\chi}_\square}) + \hat{\omega}(\not\in\cdot\overset{(1)}{\underline{\chi}_\square}) = &-\frac{1}{2}(\not\in\cdot\underline{\chi}_\square)(\mathrm{tr}\overset{(1)}{\chi_\circ}) - \frac{1}{2}(\mathrm{tr}\,\underline{\chi}_\circ)(\not\in\cdot\overset{(1)}{\chi_\square}) - \frac{1}{2}(\not\in\cdot\chi_\square)(\mathrm{tr}\overset{(1)}{\underline{\chi}_\circ}) \\ &-(\mathrm{cu\!\!\!/rl}\,\underline{\eta})(\mathrm{tr}\overset{(1)}{\not g}) + 2(\not\in\cdot(\overset{(1)}{\Pi\!\!\!/}\otimes\Pi+\Pi\otimes\overset{(1)}{\Pi\!\!\!/}))\cdot(\nabla\,\underline{\eta}) + 2\overset{(1)}{\sigma}\,,\end{aligned} \tag{3.224}$$

$$\begin{aligned}\nabla\!\!\!/_3(\not\in\cdot\overset{(1)}{\chi_\square}) + \frac{1}{2}(\mathrm{tr}\,\underline{\chi}_\circ)(\not\in\cdot\overset{(1)}{\chi_\square}) + \underline{\hat{\omega}}(\not\in\cdot\overset{(1)}{\chi_\square}) = &-\frac{1}{2}(\not\in\cdot\chi_\square)(\mathrm{tr}\overset{(1)}{\underline{\chi}_\circ}) - \frac{1}{2}(\mathrm{tr}\chi_\circ)(\not\in\cdot\overset{(1)}{\underline{\chi}_\square}) - \frac{1}{2}(\not\in\cdot\underline{\chi}_\square)(\mathrm{tr}\overset{(1)}{\chi_\circ}) \\ &-\overset{(1)}{e_3}(\not\in\cdot\chi_\square) - (\mathrm{cu\!\!\!/rl}\,\eta)(\mathrm{tr}\overset{(1)}{\not g}) + 2(\not\in\cdot(\overset{(1)}{\Pi\!\!\!/}\otimes\Pi+\Pi\otimes\overset{(1)}{\Pi\!\!\!/}))\cdot(\nabla\eta) \\ &-(\not\in\cdot\chi_\square)\overset{(1)}{\underline{\hat{\omega}}} + 2\,\mathrm{cu\!\!\!/rl}\overset{(1)}{\not\eta} - (\overset{(1)}{\eta}(e_3))(\not\in\cdot\chi_\square) - 2\overset{(1)}{\sigma}\,.\end{aligned} \tag{3.225}$$

We have equations

$$\begin{aligned}\not\nabla_4 \overset{(1)}{\not\eta} + \chi^{\sharp_1}\cdot \overset{(1)}{\not\eta} &= \overset{(1)}{\not\Gamma}_4\cdot\eta - \overset{(1)}{\not\Pi}\cdot(\nabla_4\eta) - \overset{(1)}{\hat\chi_\circ}{}^{\sharp}\cdot(\eta-\underline\eta) - \frac{1}{2}(\mathrm{tr}\overset{(1)}{\chi_\circ})(\eta-\underline\eta)\\ &+ \frac{1}{2}(\not\epsilon\cdot\overset{(1)}{\chi_\square})({}^\star\eta - {}^\star\underline\eta) + \frac{1}{2}(\not\epsilon\cdot\chi_\square)\overset{(1)}{\not g}{}^{\mu\nu}\not\epsilon_{\cdot\mu}(\eta_\nu-\underline\eta_\nu)\\ &+ \frac{1}{4}(\not\epsilon\cdot\chi_\square)(\mathrm{tr}\overset{(1)}{\not g})({}^\star\eta - {}^\star\underline\eta) - \overset{(1)}{\beta}\,,\end{aligned} \tag{3.226}$$

$$\begin{aligned}\not\nabla_4 \overset{(1)}{\underline Y} + 2\,\hat\omega\,\overset{(1)}{\underline Y} &= \not\nabla_{\overset{(1)}{e}_3}\underline\eta - \overset{(1)}{\not\Gamma}_3\cdot\underline\eta - (\overset{(1)}{\underline\eta}(e_3))\eta + \overset{(1)}{\not\Pi}\cdot(\nabla_3\,\underline\eta) - \underline\chi^{\sharp_1}\cdot\overset{(1)}{\not\eta} - \overset{(1)}{\hat{\underline\chi}_\circ}{}^{\sharp}\cdot(\eta-\underline\eta)\\ &- \frac{1}{2}(\mathrm{tr}\,\overset{(1)}{\underline\chi_\circ})(\eta-\underline\eta) + \frac{1}{2}(\not\epsilon\cdot\overset{(1)}{\underline\chi_\square})({}^\star\eta - {}^\star\underline\eta) + \frac{1}{2}(\not\epsilon\cdot\underline\chi_\square)\overset{(1)}{\not g}{}^{\mu\nu}\not\epsilon_{\cdot\mu}(\eta_\nu-\underline\eta_\nu)\\ &+ \frac{1}{4}(\not\epsilon\cdot\underline\chi_\square)(\mathrm{tr}\overset{(1)}{\not g})({}^\star\eta - {}^\star\underline\eta) - \overset{(1)}{\underline\beta}\,,\end{aligned} \tag{3.227}$$

$$\begin{aligned}\not\nabla_4 \overset{(1)}{\hat{\underline\omega}} + 2\,\hat\omega\,\overset{(1)}{\hat{\underline\omega}} &= 2(\underline\eta - \zeta, \overset{(1)}{\not\eta}) - 2(\eta-\underline\eta, \overset{(1)}{\not\zeta}) - 2\overset{(1)}{\rho}\\ &- \overset{(1)}{e}_3(\hat\omega) + 2\overset{(1)}{\not g}{}^{\mu\nu}\eta_\mu\,\underline\eta_\nu - 2\overset{(1)}{\not g}{}^{\mu\nu}(\eta-\underline\eta)_\mu\zeta_\nu\end{aligned} \tag{3.228}$$

and the equations for the torsion

$$\begin{aligned}\not\nabla_4 \overset{(1)}{\not\zeta} + \chi^{\sharp_2}\cdot\overset{(1)}{\not\zeta} + \hat\omega\,\overset{(1)}{\not\zeta} &= \overset{(1)}{\not\Gamma}_4\cdot\zeta - \overset{(1)}{\not\Pi}\cdot(\nabla_4\zeta) + \frac{1}{4}(\not\epsilon\cdot\chi_\square)(\mathrm{tr}\overset{(1)}{\not g})({}^\star\underline\eta - {}^\star\zeta)\\ &- \overset{(1)}{\not\Pi}\cdot(\nabla\,\hat\omega) + \frac{1}{2}(\mathrm{tr}\overset{(1)}{\chi_\circ})(\underline\eta - \zeta) + \frac{1}{2}(\not\epsilon\cdot\chi_\square)\overset{(1)}{\not g}{}^{\mu\nu}\not\epsilon_{\cdot\mu}(\underline\eta_\nu - \zeta_\nu)\\ &+ \overset{(1)}{\hat\chi_\circ}{}^{\sharp}\cdot(\underline\eta - \zeta) + \frac{1}{2}(\not\epsilon\cdot\overset{(1)}{\chi_\square})({}^\star\underline\eta - {}^\star\zeta) - \overset{(1)}{\beta}\,,\end{aligned} \tag{3.229}$$

$$\begin{aligned}\not\nabla_3 \overset{(1)}{\not\zeta} + \underline\chi^{\sharp_2}\cdot\overset{(1)}{\not\zeta} + \hat{\underline\omega}\,\overset{(1)}{\not\zeta} &= -\not\nabla_{\overset{(1)}{e}_3}\zeta + \overset{(1)}{\not\Gamma}_3\cdot\zeta + (\overset{(1)}{\zeta}(e_3))\eta - \overset{(1)}{\not\Pi}\cdot(\nabla_3\zeta)\\ &+ \not\nabla\,\overset{(1)}{\hat{\underline\omega}} + \overset{(1)}{\not\Pi}\cdot(\nabla\,\underline\omega) - \underline\chi^{\sharp_2}\cdot\overset{(1)}{\not\eta} - \overset{(1)}{\hat{\underline\chi}_\circ}{}^{\sharp}\cdot(\eta+\zeta) - \frac{1}{2}(\mathrm{tr}\,\overset{(1)}{\underline\chi_\circ})(\eta+\zeta)\\ &- \frac{1}{2}(\not\epsilon\cdot\overset{(1)}{\underline\chi_\square})({}^\star\eta + {}^\star\zeta) - \frac{1}{2}(\not\epsilon\cdot\underline\chi_\square)\overset{(1)}{\not g}{}^{\mu\nu}\not\epsilon_{\cdot\mu}(\eta_\nu + \zeta_\nu) - \frac{1}{4}(\not\epsilon\cdot\underline\chi_\square)(\mathrm{tr}\overset{(1)}{\not g})({}^\star\eta + {}^\star\zeta)\\ &+ \chi^{\sharp_2}\cdot\overset{(1)}{\underline Y} + \hat{\underline\omega}\,\overset{(1)}{\not\eta} + \overset{(1)}{\hat{\underline\omega}}(\eta-\zeta) - \hat\omega\,\overset{(1)}{\underline Y} - \overset{(1)}{\underline\beta}\,,\end{aligned} \tag{3.230}$$

$$\begin{aligned}\mathrm{c\not{u}rl}\,\overset{(1)}{\not\zeta} &= \frac{1}{2}(\overset{(1)}{\zeta}(e_3))(\not\epsilon\cdot\chi) - (\not\epsilon\cdot(\overset{(1)}{\not\Pi}\otimes\Pi + \Pi\otimes\overset{(1)}{\not\Pi}))\cdot(\nabla\zeta)\\ &+ \frac{1}{4}(\mathrm{tr}\overset{(1)}{\chi_\circ})(\not\epsilon\cdot\underline\chi_\square) + \frac{1}{4}(\mathrm{tr}\chi_\circ)(\not\epsilon\cdot\overset{(1)}{\underline\chi_\square}) - \frac{1}{4}(\mathrm{tr}\,\overset{(1)}{\underline\chi_\circ})(\not\epsilon\cdot\chi_\square)\\ &- \frac{1}{4}(\mathrm{tr}\,\underline\chi_\circ)(\not\epsilon\cdot\overset{(1)}{\chi_\square}) - \frac{1}{2}\hat\omega(\not\epsilon\cdot\overset{(1)}{\underline\chi_\square}) + \frac{1}{2}\hat{\underline\omega}(\not\epsilon\cdot\overset{(1)}{\chi_\square}) + \frac{1}{2}(\not\epsilon\cdot\chi_\square)\,\overset{(1)}{\hat{\underline\omega}}\\ &+ \frac{1}{2}(\mathrm{c\not{u}rl}\,\zeta)(\mathrm{tr}\overset{(1)}{\not g}) + \overset{(1)}{\sigma}\,.\end{aligned} \tag{3.231}$$

We have the linearised Codazzi-like equations

$$\operatorname{d\!\!/iv} \overset{(1)}{\hat{\chi}_\circ} + \overset{(1)}{\hat{\chi}_\circ}{}^\sharp \cdot \zeta = -\frac{1}{2}\overset{(1)}{\not g}{}^{\mu\nu}\not e_{\cdot\nu}\,\not\nabla_\mu(\not e\cdot\chi_\square) - \frac{1}{2}\overset{(1)}{\not\!\#}{}^{\sharp_2}\cdot(\not\nabla(\not e\cdot\chi_\square)) - \frac{1}{2}\not e^{\sharp_2}\cdot(\not\nabla(\not e\cdot\overset{(1)}{\chi_\square})) \tag{3.232}$$
$$-\frac{1}{2}\overset{(1)}{\not\Pi}\cdot(\nabla(\not e\cdot\chi_\square)) - \frac{1}{2}(\not e\cdot\overset{(1)}{\chi_\square})({}^\star\zeta + 2{}^\star\eta) - \frac{1}{2}(\not e\cdot\chi_\square)\overset{(1)}{\not g}{}^{\mu\nu}\not e_{\cdot\nu}(\zeta+2\eta)_\mu$$
$$-\frac{1}{2}(\not e\cdot\chi_\square)\overset{(1)}{\not\!\#}{}^{\sharp_2}\cdot(\zeta+2\eta) - \frac{1}{2}(\not e\cdot\chi_\square)({}^\star\overset{(1)}{\not\zeta} + 2{}^\star\overset{(1)}{\not\eta}) + \frac{1}{2}(\mathrm{tr}\overset{(1)}{\chi_\circ})\zeta$$
$$+\frac{1}{2}(\mathrm{tr}\chi_\circ)\overset{(1)}{\not\zeta} + \frac{1}{2}\not\nabla(\mathrm{tr}\overset{(1)}{\chi_\circ}) + \frac{1}{2}\overset{(1)}{\not\Pi}\cdot(\nabla(\mathrm{tr}\chi_\circ)) - \overset{(1)}{\beta},$$

$$\operatorname{d\!\!/iv} \overset{(1)}{\hat{\underline{\chi}}_\circ} - \overset{(1)}{\hat{\underline{\chi}}_\circ}{}^\sharp \cdot \zeta = -\frac{1}{2}\overset{(1)}{\not g}{}^{\mu\nu}\not e_{\cdot\nu}\,\not\nabla_\mu(\not e\cdot\underline{\chi}_\square) - \frac{1}{2}\overset{(1)}{\not\!\#}{}^{\sharp_2}\cdot(\not\nabla(\not e\cdot\underline{\chi}_\square)) - \frac{1}{2}\not e^{\sharp_2}\cdot(\not\nabla(\not e\cdot\overset{(1)}{\underline{\chi}_\square})) \tag{3.233}$$
$$-\frac{1}{2}\overset{(1)}{\not\Pi}\cdot(\nabla(\not e\cdot\underline{\chi}_\square)) + \frac{1}{2}(\not e\cdot\overset{(1)}{\chi_\square})({}^\star\zeta - 2{}^\star\underline{\eta}) + \frac{1}{2}(\not e\cdot\underline{\chi}_\square)\overset{(1)}{\not g}{}^{\mu\nu}\not e_{\cdot\nu}(\zeta-2\underline{\eta})_\mu$$
$$+\frac{1}{2}(\not e\cdot\underline{\chi}_\square)\overset{(1)}{\not\!\#}{}^{\sharp_2}\cdot(\zeta-2\underline{\eta}) + \frac{1}{2}(\not e\cdot\underline{\chi}_\square){}^\star\overset{(1)}{\not\zeta} - \frac{1}{2}(\mathrm{tr}\overset{(1)}{\underline{\chi}_\circ})\zeta$$
$$-\frac{1}{2}(\mathrm{tr}\,\underline{\chi}_\circ)\overset{(1)}{\not\zeta} + \frac{1}{2}\not\nabla(\mathrm{tr}\overset{(1)}{\underline{\chi}_\circ}) + \frac{1}{2}\overset{(1)}{\not\Pi}\cdot(\nabla(\mathrm{tr}\,\underline{\chi}_\circ)) - (\not e\cdot\chi_\square)^\star\overset{(1)}{\underline{Y}} + \overset{(1)}{\underline{\beta}}$$

and the linearised Gauss-like equation

$$\overset{(1)}{\not K} = -\frac{1}{4}(\mathrm{tr}\,\underline{\chi}_\circ)(\mathrm{tr}\overset{(1)}{\chi_\circ}) - \frac{1}{4}(\mathrm{tr}\chi_\circ)(\mathrm{tr}\overset{(1)}{\underline{\chi}_\circ}) - \frac{1}{4}(\not e\cdot\underline{\chi}_\square)(\not e\cdot\overset{(1)}{\chi_\square}) - \frac{1}{4}(\not e\cdot\chi_\square)(\not e\cdot\overset{(1)}{\underline{\chi}_\square}) \tag{3.234}$$
$$+\frac{1}{2}(\not e\cdot\overset{(1)}{\chi_\square})\not g^{BD}\not e_{CB}\Gamma^C_{3D} + \frac{1}{2}(\not e\cdot\chi_\square)\not g^{BD}\not e_{CB}\overset{(1)}{\Gamma^C_{3D}} - \overset{(1)}{\rho}\,.$$

Linearised Bianchi equations

The linearised Bianchi equations read

$$\nabla\!\!\!\!/_3 \overset{(1)}{\alpha} + \frac{1}{2}(\mathrm{tr}\,\underline{\chi}_\circ)\overset{(1)}{\alpha} + 2\,\underline{\hat{\omega}}\,\overset{(1)}{\alpha} + \frac{1}{2}(\not\epsilon\cdot\underline{\chi}_\square)\overset{(1)}{\alpha} = -2\mathcal{D}\!\!\!\!/_2^\star\overset{(1)}{\beta} - 3\rho\overset{(1)}{\hat{\chi}}_\circ - 3\sigma^\star\overset{(1)}{\hat{\chi}}_\circ + (4\eta+\zeta)\hat{\otimes}\overset{(1)}{\beta}\,, \quad (3.235)$$

$$\nabla\!\!\!\!/_4 \overset{(1)}{\beta} + 2(\mathrm{tr}\chi_\circ)\overset{(1)}{\beta} - \hat{\omega}\,\overset{(1)}{\beta} - 2(\not\epsilon\cdot\chi_\square)^\star\overset{(1)}{\beta} = \mathrm{d}\!\!\!/\mathrm{iv}\,\overset{(1)}{\alpha} + (\underline{\eta}^\sharp + 2\zeta^\sharp)\cdot\overset{(1)}{\alpha}\,, \quad (3.236)$$

$$\begin{aligned}\nabla\!\!\!\!/_3 \overset{(1)}{\beta} + (\mathrm{tr}\,\underline{\chi}_\circ)\overset{(1)}{\beta} + \underline{\hat{\omega}}\,\overset{(1)}{\beta} + (\not\epsilon\cdot\underline{\chi}_\square)^\star\overset{(1)}{\beta} = {}& \mathcal{D}\!\!\!\!/_1^*(-\overset{(1)}{\rho},\overset{(1)}{\sigma}) + 3\rho\overset{(1)}{\eta\!\!\!/} + 3\overset{(1)}{\rho}\,\eta \qquad (3.237)\\ & + 3\sigma^\star\overset{(1)}{\eta\!\!\!/} + 3\overset{(1)}{\sigma}{}^\star\eta + 3\sigma\overset{(1)}{g\!\!\!/^{\mu\nu}}\not\epsilon_{\cdot\nu}\eta_\mu + 3\sigma\overset{(1)}{g\!\!\!/}{}^{\sharp_2}\cdot\eta\\ & + \overset{(1)}{I\!\!\!/}\cdot(\nabla\rho) + \overset{(1)}{g\!\!\!/^{\mu\nu}}\not\epsilon_{\cdot\nu}\nabla\!\!\!\!/_\mu\sigma + \overset{(1)}{g\!\!\!/}{}^{\sharp_2}\cdot(\nabla\!\!\!\!/\,\sigma) + \overset{(1)}{I\!\!\!/}\cdot(\nabla\sigma)\,,\end{aligned}$$

$$\begin{aligned}\nabla\!\!\!\!/_4 \overset{(1)}{\rho} + \frac{3}{2}(\mathrm{tr}\chi_\circ)\overset{(1)}{\rho} = {}& \mathrm{d}\!\!\!/\mathrm{iv}\,\overset{(1)}{\beta} + (2\,\underline{\eta}+\zeta,\overset{(1)}{\beta}) - \frac{3}{2}\rho(\mathrm{tr}\overset{(1)}{\chi}_\circ) - \frac{3}{2}\sigma(\epsilon\cdot\overset{(1)}{\chi}_\square) \qquad (3.238)\\ & - \frac{3}{2}(\epsilon\cdot\chi_\square)\overset{(1)}{\sigma}\,,\end{aligned}$$

$$\nabla\!\!\!\!/_4 \overset{(1)}{\sigma} + \frac{3}{2}(\mathrm{tr}\chi_\circ)\overset{(1)}{\sigma} + \frac{3}{2}\sigma(\mathrm{tr}\overset{(1)}{\chi}_\circ) = -\,\mathrm{c}\!\!\!/\mathrm{url}\,\overset{(1)}{\beta} - (2\,\underline{\eta}+\zeta)\wedge\overset{(1)}{\beta} + \frac{3}{2}\rho(\not\epsilon\cdot\overset{(1)}{\chi}_\square) + \frac{3}{2}(\not\epsilon\cdot\chi_\square)\overset{(1)}{\rho}\,, \quad (3.239)$$

$$\begin{aligned}\nabla\!\!\!\!/_3 \overset{(1)}{\rho} + \overset{(1)}{e}{}_3(\rho) + \frac{3}{2}(\mathrm{tr}\,\underline{\chi}_\circ)\overset{(1)}{\rho} = {}& -\,\mathrm{d}\!\!\!/\mathrm{iv}\,\overset{(1)}{\underline{\beta}} - (2\eta-\zeta,\overset{(1)}{\underline{\beta}}) - \frac{3}{2}\rho(\mathrm{tr}\,\overset{(1)}{\underline{\chi}}_\circ) + \frac{3}{2}\sigma(\epsilon\cdot\overset{(1)}{\underline{\chi}}_\square) \qquad (3.240)\\ & + \frac{3}{2}(\epsilon\cdot\chi_\square)\overset{(1)}{\sigma}\,,\end{aligned}$$

$$\nabla\!\!\!\!/_3 \overset{(1)}{\sigma} + \overset{(1)}{e}{}_3(\sigma) + \frac{3}{2}(\mathrm{tr}\,\underline{\chi}_\circ)\overset{(1)}{\sigma} + \frac{3}{2}\sigma(\mathrm{tr}\,\overset{(1)}{\underline{\chi}}_\circ) = -\,\mathrm{c}\!\!\!/\mathrm{url}\,\overset{(1)}{\underline{\beta}} - (2\eta-\zeta)\wedge\overset{(1)}{\underline{\beta}} - \frac{3}{2}\rho(\not\epsilon\cdot\overset{(1)}{\underline{\chi}}_\square) - \frac{3}{2}(\not\epsilon\cdot\underline{\chi}_\square)\overset{(1)}{\rho}\,, \quad (3.241)$$

$$\begin{aligned}\nabla\!\!\!\!/_4 \overset{(1)}{\underline{\beta}} + (\mathrm{tr}\chi_\circ)\overset{(1)}{\underline{\beta}} + \hat{\omega}\,\overset{(1)}{\underline{\beta}} + (\not\epsilon\cdot\chi_\square)^\star\overset{(1)}{\underline{\beta}} = {}& \mathcal{D}\!\!\!\!/_1^*(\overset{(1)}{\rho},\overset{(1)}{\sigma}) - 3\overset{(1)}{\rho}\,\underline{\eta} \qquad (3.242)\\ & + 3\overset{(1)}{\sigma}{}^\star\underline{\eta} + 3\sigma\overset{(1)}{g\!\!\!/^{\mu\nu}}\not\epsilon_{\cdot\nu}\underline{\eta}_\mu + 3\sigma\overset{(1)}{g\!\!\!/}{}^{\sharp_2}\cdot\underline{\eta}\\ & - \overset{(1)}{I\!\!\!/}\cdot(\nabla\rho) + \overset{(1)}{g\!\!\!/^{\mu\nu}}\not\epsilon_{\cdot\nu}\nabla\!\!\!\!/_\mu\sigma + \overset{(1)}{g\!\!\!/}{}^{\sharp_2}\cdot(\nabla\!\!\!\!/\,\sigma) + \overset{(1)}{I\!\!\!/}\cdot(\nabla\sigma)\,,\end{aligned}$$

$$\nabla\!\!\!\!/_3 \overset{(1)}{\underline{\beta}} + 2(\mathrm{tr}\,\underline{\chi}_\circ)\overset{(1)}{\underline{\beta}} - \underline{\hat{\omega}}\,\overset{(1)}{\underline{\beta}} - 2(\not\epsilon\cdot\underline{\chi}_\square)^\star\overset{(1)}{\underline{\beta}} = -\,\mathrm{d}\!\!\!/\mathrm{iv}\,\overset{(1)}{\underline{\alpha}} - (\eta^\sharp - 2\zeta^\sharp)\cdot\overset{(1)}{\underline{\alpha}} - 3\rho\overset{(1)}{\underline{Y}} + 3\sigma^\star\overset{(1)}{\underline{Y}}\,, \quad (3.243)$$

$$\nabla\!\!\!\!/_4 \overset{(1)}{\underline{\alpha}} + \frac{1}{2}(\mathrm{tr}\chi_\circ)\overset{(1)}{\underline{\alpha}} + 2\,\hat{\omega}\,\overset{(1)}{\underline{\alpha}} - \frac{1}{2}(\not\epsilon\cdot\chi_\square)\overset{(1)}{\underline{\alpha}} = 2\mathcal{D}\!\!\!\!/_2^\star\overset{(1)}{\underline{\beta}} - 3\rho\overset{(1)}{\hat{\underline{\chi}}}_\circ + 3\sigma^\star\overset{(1)}{\hat{\underline{\chi}}}_\circ - (4\,\underline{\eta}-\zeta)\hat{\otimes}\overset{(1)}{\underline{\beta}}\,. \quad (3.244)$$

Appendix A: The formal derivation of the equations

We derive the system of nonlinear vacuum Einstein equations of Section 3.3.5. In this appendix we abandon the bold notation adopted throughout the chapter. Instead, we write in bold those terms that arise in our derivation of the equations but do not appear in the derivation of the equations of [14, 13].

The *null frame equations* of Section 3.3.5 are simply the commutation formulae for the frame $\mathcal{N}$.

To derive the *null structure equations*, one applies the formula

$$g(\nabla_{e_I}\nabla_{e_J}e_K, e_L) - g(\nabla_{e_J}\nabla_{e_I}e_K, e_L) - g(\nabla_{[e_I,e_J]}e_K, e_L) = R(e_L, e_K, e_I, e_J)\,, \tag{3.245}$$

with $I, J, K, L = \{1,2,3,4\}$. The reader should already note that, when $I, J = \{1,2\}$, *the commutator in the last term on the left hand side of* (3.245) *captures the non-integrability of the frame* $\mathcal{N}$.

We will derive a subset of the null structure equations in that we omit the derivation of the conjugate equations. We will also omit the straightforward derivation of the first variational formulae. Before starting, let us note that, crucially, in our computations one has to keep track of the *order of the indices of* χ *and* $\underline{\chi}$.

We compute

$$\begin{aligned}
g(\nabla_3\nabla_B e_3, e_A) &= (\nabla_3\,\underline{\chi})_{BA} + \Gamma^C_{3B}\,\underline{\chi}_{CA} + 2\zeta_B\,\underline{Y}_A\,,\\
g(\nabla_B\nabla_3 e_3, e_A) &= 2(\nabla\,\underline{Y})_{BA} + \hat{\underline{\omega}}\,\underline{\chi}_{BA}\,,\\
g(\nabla_{[e_3,e_B]}e_3, e_A) &= (\Gamma^C_{3B} - \underline{\chi}^{\sharp_2}{}_B^C)\,\underline{\chi}_{CA} + 2(\eta_B - \zeta_B)\,\underline{Y}_A + 2\,\underline{Y}_B\,\underline{\eta}_A
\end{aligned}$$

and obtain the equation

$$(\nabla_3\,\underline{\chi})_{BA} = 2(\nabla_B\,\underline{Y})_A + \hat{\underline{\omega}}\,\underline{\chi}_{BA} - \underline{\chi}^{\sharp_2}{}_B^C\,\underline{\chi}_{CA} + 2(\eta_B - 2\zeta_B)\,\underline{Y}_A + 2\,\underline{Y}_B\,\underline{\eta}_A - \underline{\alpha}_{BA}\,. \tag{3.246}$$

We compute

$$\begin{aligned}
g(\nabla_4\nabla_B e_3, e_A) &= (\nabla_4\,\underline{\chi})_{BA} + \Gamma^C_{4B}\,\underline{\chi}_{CA} + 2\zeta_B\,\underline{\eta}_A\,,\\
g(\nabla_B\nabla_4 e_3, e_A) &= 2(\nabla\,\underline{\eta})_{BA} - \hat{\omega}\,\underline{\chi}_{BA}\,,\\
g(\nabla_{[e_4,e_B]}e_3, e_A) &= (\Gamma^C_{4B} - \chi^{\sharp_2}{}_B^C)\,\underline{\chi}_{CA} + 2Y_B\,\underline{Y}_A + 2(\underline{\eta}_B + \zeta_B)\,\underline{\eta}_A
\end{aligned}$$

and obtain the equation

$$(\nabla_4\,\underline{\chi})_{BA} = -2\zeta_B\,\underline{\eta}_A + 2(\nabla\,\underline{\eta})_{BA} - \hat{\omega}\,\underline{\chi}_{BA} - \chi^{\sharp_2}{}_B^C\,\underline{\chi}_{CA} + 2Y_B\,\underline{Y}_A + 2(\underline{\eta}_B + \zeta_B)\,\underline{\eta}_A + R(e_A, e_3, e_4, e_B)\,. \tag{3.247}$$

We compute

$$g(\nabla_C \nabla_B e_3, e_A) = (\nabla \chi)_{CBA} + \Gamma^D_{CB}\, \chi_{DA} + \zeta_B\, \chi_{CA}\,,$$
$$g(\nabla_B \nabla_C e_3, e_A) = (\nabla \chi)_{BCA} + \Gamma^D_{BC}\, \chi_{DA} + \zeta_C\, \chi_{BA}\,,$$
$$g(\nabla_{[e_C, e_B]} e_3, e_A) = (\Gamma^D_{CB} - \Gamma^D_{BC})\, \chi_{DA} + \boldsymbol{(\chi_{CB} - \chi_{BC})\, \underline{Y}_A} + \boldsymbol{(\chi_{CB} - \chi_{BC})\, \underline{\eta}_A}$$

and obtain the equation

$$(\nabla_C\, \chi)_{BA} + \zeta_B\, \chi_{CA} = (\nabla_B\, \chi)_{CA} + \zeta_C\, \chi_{BA} + \boldsymbol{(\chi_{CB} - \chi_{BC})\, \underline{Y}_A} + \boldsymbol{(\chi_{CB} - \chi_{BC})\, \underline{\eta}_A} \quad (3.248)$$
$$+ R(e_A, e_3, e_C, e_B)\,.$$

We compute

$$g((\nabla_C(\nabla e_D))(e_B), e_A) = \not{g}((\not{\nabla}_C(\not{\nabla}\, e_D))(e_B), e_A) + \frac{1}{2}\chi_{BD}\, \chi_{CA} + \frac{1}{2}\chi_{CA}\, \chi_{BD} - \frac{1}{2}\chi_{CB} \not{g}(\not{\nabla}_3\, e_D, e_A)$$
$$- \frac{1}{2}\, \chi_{CB}\, \not{g}(\not{\nabla}_4\, e_D, e_A)\,,$$
$$g((\nabla_B(\nabla e_D))(e_C), e_A) = \not{g}((\not{\nabla}_B(\not{\nabla}\, e_D))(e_C), e_A) + \frac{1}{2}\chi_{CD}\, \chi_{BA} + \frac{1}{2}\chi_{BA}\, \chi_{CD} - \frac{1}{2}\chi_{BC} \not{g}(\not{\nabla}_3\, e_D, e_A)$$
$$- \frac{1}{2}\, \chi_{BC}\, \not{g}(\not{\nabla}_4\, e_D, e_A)$$

and obtain the equation

$$\not{R}(e_A, e_D, e_C, e_B) + \frac{1}{2}(\chi_{BD}\, \chi_{CA} + \chi_{CA}\, \chi_{BD}) - \frac{1}{2}(\chi_{CD}\, \chi_{BA} + \chi_{BA}\, \chi_{CD}) \quad (3.249)$$
$$\boldsymbol{- \frac{1}{2}(\chi_{CB} - \chi_{BC}) \not{g}(\not{\nabla}_3\, e_D, e_A) - \frac{1}{2}(\chi_{CB} - \chi_{BC}) \not{g}(\not{\nabla}_4\, e_D, e_A)}$$
$$= R(e_A, e_D, e_C, e_B)\,.$$

We compute

$$g(\nabla_3 \nabla_A e_3, e_4) = -2(\nabla_3 \zeta)_A - 2\Gamma^B_{3A} \zeta_B - 2{\chi^{\sharp_2}}^B_{\ A} \eta_B - 2\, \underline{\hat{\omega}}\, \zeta_A\,,$$
$$g(\nabla_A \nabla_3 e_3, e_4) = -2(\nabla\, \underline{\hat{\omega}})_A - 2{\chi^{\sharp_2}}^B_{\ A}\, \underline{Y}_B - 2\, \underline{\hat{\omega}}\, \zeta_A\,,$$
$$g(\nabla_{[e_3, e_A]} e_3, e_4) = -2(\Gamma^B_{3A} - {\chi^{\sharp_2}}^B_{\ A}) \zeta_B - 2(\eta_A - \zeta_A)\, \underline{\hat{\omega}} + 2\, \hat{\omega}\, \underline{Y}_A$$

and obtain the equation

$$(\nabla_3 \zeta)_A = (\nabla\, \underline{\hat{\omega}})_A - {\chi^{\sharp_2}}^B_{\ A} \eta_B + {\chi^{\sharp_2}}^B_{\ A}\, \underline{Y}_B - {\chi^{\sharp_2}}^B_{\ A} \zeta_B + \underline{\hat{\omega}}(\eta_A - \zeta_A) - \hat{\omega}\, \underline{Y}_A - \underline{\beta}_A\,. \quad (3.250)$$

We compute

$$
\begin{aligned}
g(\nabla_3 \nabla_4 e_3, e_A) &= 2(\nabla_3 \underline{\eta})_A - 2\,\hat{\omega}\,\underline{Y}_A\,,\\
g(\nabla_4 \nabla_3 e_3, e_A) &= 2(\nabla_4 \underline{Y})_A + 2\,\underline{\hat{\omega}}\,\underline{\eta}_A\,,\\
g(\nabla_{[e_3,e_4]} e_3, e_A) &= 2\chi^{\sharp_1}{}_A^B(\eta_B - \underline{\eta}_B) + 2\,\hat{\omega}\,\underline{Y}_A - 2\,\underline{\hat{\omega}}\,\underline{\eta}_A
\end{aligned}
$$

and obtain the equation

$$
(\nabla_3 \underline{\eta})_A - (\nabla_4 \underline{Y})_A = \chi^{\sharp_1}{}_A^B(\eta_B - \underline{\eta}_B) + 2\,\hat{\omega}\,\underline{Y}_A + \underline{\beta}_A\,. \tag{3.251}
$$

We compute

$$
\begin{aligned}
g(\nabla_4 \nabla_3 e_3, e_4) &= -2\nabla_4\,\underline{\hat{\omega}} - 4Y^A\,\underline{Y}_A + 2\,\hat{\omega}\,\underline{\hat{\omega}}\,,\\
g(\nabla_3 \nabla_4 e_3, e_4) &= 2\nabla_3\,\hat{\omega} - 4\eta^A\,\underline{\eta}_A + 2\,\hat{\omega}\,\underline{\hat{\omega}}\,,\\
g(\nabla_{[e_4,e_3]} e_3, e_4) &= 4(\eta^A - \underline{\eta}^A)\zeta_A + 4\,\hat{\omega}\,\underline{\hat{\omega}}
\end{aligned}
$$

and obtain the equation

$$
\nabla_4\,\underline{\hat{\omega}} + \nabla_3\,\hat{\omega} = 2\eta^A\,\underline{\eta}_A - 2Y^A\,\underline{Y}_A - 2\,\hat{\omega}\,\underline{\hat{\omega}} - 2(\eta^A - \underline{\eta}^A)\zeta_A - 2\rho\,. \tag{3.252}
$$

We compute

$$
\begin{aligned}
g(\nabla_A \nabla_B e_4, e_3) &= 2(\nabla\zeta)_{AB} - \chi_{BD}\chi^{\sharp_2}{}_A^D - 2\zeta_A\zeta_B\,,\\
g(\nabla_B \nabla_A e_4, e_3) &= 2(\nabla\zeta)_{BA} - \chi_{AD}\chi^{\sharp_2}{}_B^D - 2\zeta_A\zeta_B\,,\\
g(\nabla_{[e_A,e_B]} e_4, e_3) &= 2(\Gamma^C_{AB} - \Gamma^C_{BA})\zeta_C \boldsymbol{- \hat{\omega}(\chi_{AB} - \chi_{BA}) + \underline{\hat{\omega}}(\chi_{AB} - \chi_{BA})}
\end{aligned}
$$

and obtain the equation

$$
\begin{aligned}
2(\nabla\zeta)_{AB} - 2(\nabla\zeta)_{BA} - \chi_{BD}\chi^{\sharp_2}{}_A^D + \chi_{AD}\chi^{\sharp_2}{}_B^D - 2(\Gamma^C_{AB} - \Gamma^C_{BA})\zeta_C&\\
\boldsymbol{+ \hat{\omega}(\chi_{AB} - \chi_{BA}) - \underline{\hat{\omega}}(\chi_{AB} - \chi_{BA})} = R(e_3, e_4, e_A, e_B)\,.&
\end{aligned} \tag{3.253}
$$

Let $(\mathcal{M}, g)$ be a solution to the vacuum Einstein equations

$$
\mathrm{Ric}(g) = 0\,.
$$

Then

$$
\mathrm{tr}\,\alpha = \mathrm{tr}\,\underline{\alpha} = 0\,,
$$

where the trace is taken with respect to $\not{g}^{-1}$, and

$$R_{A34B} = \rho \not{g}_{AB} - \sigma \not{\in}_{AB} \,, \tag{3.254}$$

$$R_{A3CB} = -\not{\in}_{CB} ({}^{\star}\underline{\beta})_A \,, \tag{3.255}$$

$$\not{g}^{AC} \not{g}^{BD} R_{ADCB} = -2\rho \,. \tag{3.256}$$

The identities (3.254)-(3.256) rely on the symmetries of the Riemann curvature tensor R, the Bianchi identities for R and the fact that any antisymmetric horizontal two-tensor is proportional to the (extension to an horizontal tensor of the) induced volume form $\not{\in}$ on $\mathfrak{D}_{\mathcal{N}}$. The proof of the identities (3.254)-(3.256) is analogous to that of [14, 13].

Note that all the spacetime covariant derivatives ∇ appearing in the equations (3.246)-(3.253) are applied to horizontal tensors on $(\mathcal{M}, g)$ and evaluated on the horizontal frame vectors. All the other terms in the equations are horizontal components of horizontal tensors on $(\mathcal{M}, g)$. One can therefore replace ∇ by the projected covariant derivative $\not{\nabla}$ and understand the equations as equations for $\mathfrak{D}_{\mathcal{N}}$ tensors.

Taking the *trace* (with respect to $\not{g}^{-1}$) part, the *antitrace* (with respect to $\not{\in}^{-1}$) part and the *symmetric traceless* part of equation (3.246), one obtains

$$\not{\nabla}_3 (\operatorname{tr} \underline{\chi}_{\bigcirc}) + \frac{1}{2} (\operatorname{tr} \underline{\chi}_{\bigcirc})^2 - \underline{\hat{\omega}} \operatorname{tr} \underline{\chi}_{\bigcirc} = -(\hat{\underline{\chi}}_{\bigcirc}, \hat{\underline{\chi}}_{\bigcirc}) - (\underline{\chi}_{\square}, \underline{\chi}_{\square}) + 2 \operatorname{d\!\!/iv} \underline{Y} + 2(\eta + \underline{\eta} - 2\zeta, \underline{Y}) \,,$$

$$\not{\nabla}_3 (\not{\in} \cdot \underline{\chi}_{\square}) - \underline{\hat{\omega}} (\not{\in} \cdot \underline{\chi}_{\square}) = -\underline{\chi} \wedge \underline{\chi} + 2(\eta - 2\zeta) \wedge \underline{Y} + 2 \underline{Y} \wedge \underline{\eta} + 2 \operatorname{c\!\!/url} \underline{Y} \,,$$

$$\not{\nabla}_3 \hat{\underline{\chi}}_{\bigcirc} + (\operatorname{tr} \underline{\chi}_{\bigcirc}) \hat{\underline{\chi}}_{\bigcirc} - \underline{\hat{\omega}} \hat{\underline{\chi}}_{\bigcirc} = -2 \not{\mathcal{D}}_2^{\star} \underline{Y} + (\eta + \underline{\eta} - 2\zeta) \hat{\otimes} \underline{Y} - \underline{\alpha}$$

respectively. Taking the *trace* part, the *antitrace* part and the *symmetric traceless* part of equation (3.247), one obtains

$$\not{\nabla}_4 (\operatorname{tr} \underline{\chi}_{\bigcirc}) + \frac{1}{2} (\operatorname{tr} \chi_{\bigcirc})(\operatorname{tr} \underline{\chi}_{\bigcirc}) + \hat{\omega} \operatorname{tr} \underline{\chi}_{\bigcirc} = -(\hat{\chi}_{\bigcirc}, \hat{\underline{\chi}}_{\bigcirc}) - (\chi_{\square}, \underline{\chi}_{\square}) + 2(\underline{\eta}, \underline{\eta}) + 2\rho + 2 \operatorname{d\!\!/iv} \underline{\eta} + 2(Y, \underline{Y}) \,,$$

$$\not{\nabla}_4 (\not{\in} \cdot \underline{\chi}_{\square}) + \hat{\omega} (\not{\in} \cdot \underline{\chi}_{\square}) = -\chi \wedge \underline{\chi} + 2Y \wedge \underline{Y} + 2\sigma + 2 \operatorname{c\!\!/url} \underline{\eta} \,,$$

$$\not{\nabla}_4 \hat{\underline{\chi}}_{\bigcirc} + \frac{1}{2} (\operatorname{tr} \chi_{\bigcirc}) \hat{\underline{\chi}}_{\bigcirc} + \hat{\omega} \hat{\underline{\chi}}_{\bigcirc} = -2 \not{\mathcal{D}}_2^{\star} \underline{\eta} - \frac{1}{2} (\operatorname{tr} \underline{\chi}_{\bigcirc}) \hat{\chi}_{\bigcirc} + \underline{\eta} \hat{\otimes} \underline{\eta} + Y \hat{\otimes} \underline{Y}$$

respectively. By taking the $\not{g}^{CA}$-*trace* of the equation (3.248),[76] one obtains the Codazzi-like equation

$$\operatorname{d\!\!/iv} \hat{\underline{\chi}}_{\bigcirc} = -\operatorname{d\!\!/iv} \underline{\chi}_{\square} + \hat{\underline{\chi}}_{\bigcirc}^{\sharp} \cdot \zeta + \underline{\chi}_{\square}^{\sharp_2} \cdot \zeta - \frac{1}{2} (\operatorname{tr} \underline{\chi}_{\bigcirc}) \zeta + \frac{1}{2} \not{\nabla} (\operatorname{tr} \underline{\chi}_{\bigcirc}) - 2 \chi_{\square}^{\sharp_2} \cdot \underline{Y} - 2 \underline{\chi}_{\square}^{\sharp_2} \cdot \underline{\eta} + \underline{\beta} \,.$$

By taking the $\not{g}^{AC} \not{g}^{BD}$-*trace* of the equation (3.249), one obtains the Gauss-like equation

$$\not{K} = \frac{1}{2} (\hat{\chi}_{\bigcirc}, \hat{\underline{\chi}}_{\bigcirc}) - \frac{1}{4} (\operatorname{tr} \chi_{\bigcirc})(\operatorname{tr} \underline{\chi}_{\bigcirc}) - \frac{1}{4} (\not{\in} \cdot \chi_{\square})(\not{\in} \cdot \underline{\chi}_{\square}) + (\Gamma_3, \chi_{\square}) + (\Gamma_4, \underline{\chi}_{\square}) - \rho \,.$$

[76]Note that one can equivalently trace by $\not{g}^{BA}$. The $\not{g}^{CB}$-trace vanishes because the equation is antisymmetric in the horizontal indices B, C.

By taking the *antitrace* of equation (3.253), one obtains

$$\not{\mathrm{curl}}\,\zeta = -\frac{1}{2}\,\hat{\chi}_{\circ} \wedge \hat{\underline{\chi}}_{\circ} + \frac{1}{4}\,(\mathrm{tr}\,\chi_{\circ})(\not{\epsilon}\cdot\underline{\chi}_{\square}) - \frac{1}{4}\,(\mathrm{tr}\,\underline{\chi}_{\circ})(\not{\epsilon}\cdot\chi_{\square}) - \frac{1}{2}\,(\not{\epsilon}\cdot\underline{\chi}_{\square})\,\hat{\omega} + \frac{1}{2}\,(\not{\epsilon}\cdot\chi_{\square})\,\hat{\underline{\omega}} + \sigma\,.$$

This completes the derivation of the null structure equations.

The *Bianchi equations* are derived via the *contracted Bianchi identities*[77]

$$\nabla^I R_{IJKL} = 0\,,$$

with $I, J, K, L = \{1, 2, 3, 4\}$.

[77]These are equivalent to the Bianchi identities

$$\nabla_{[I} R_{JK]LM} = 0$$

when $(\mathcal{M}, g)$ solves the vacuum Einstein equations.

We first compute

$$
\begin{aligned}
(\nabla_3 R)_{A4B4} &= (\nabla_3 \alpha)_{AB} + 2\,\underline{\hat{\omega}}\,\alpha_{AB} - 4(\eta \hat{\otimes} \beta)_{AB}\,, \\
(\nabla_4 R)_{A3B3} &= (\nabla_4\,\underline{\alpha})_{AB} + 2\,\hat{\omega}\,\underline{\alpha}_{AB} + 4(\underline{\eta}\,\hat{\otimes}\,\underline{\beta})_{AB}\,, \\
(\nabla_4 R)_{A434} &= 2(\nabla_4 \beta)_A - 2\,\hat{\omega}\,\beta_A - 2\,\underline{\eta}^B\,\alpha_{AB} - 6Y_A\rho - 6{}^\star Y_A\sigma\,, \\
(\nabla_3 R)_{A434} &= 2(\nabla_3 \beta)_A + 2\,\underline{\hat{\omega}}\,\beta_A - 2\,\underline{Y}^B\,\alpha_{AB} - 6\eta_A\rho - 6{}^\star \eta_A\sigma\,, \\
(\nabla_A R)_{B4C4} &= (\nabla_A \alpha)_{BC} + 2\zeta_A \alpha_{BC} - 2(\chi_{AB}\beta_C + \chi_{AC}\beta_B - \chi_{AD}\beta^D g_{BC})\,, \\
(\nabla^B R)_{A4B4} &= (g^{CB}\nabla_C R)_{A4B4} \\
&= g^{CB}((\nabla_C \alpha)_{AB} + 2\zeta_C\alpha_{AB} - 2(\chi_{CA}\beta_B + \chi_{CB}\beta_A - \chi_{CD}\beta^D g_{AB})) \\
&= (\nabla^B \alpha)_{AB} + 2\zeta^B\alpha_{AB} - 2(\mathrm{tr}\,\chi)\beta_A \boldsymbol{+2(\chi_{AB} - \chi_{BA})\beta^B}\,, \\
(\nabla^B R)_{A3B3} &= (\nabla^B\,\underline{\alpha})_{AB} - 2\zeta^B\,\underline{\alpha}_{AB} + 2(\mathrm{tr}\,\underline{\chi})\,\underline{\beta}_A \boldsymbol{-2(\underline{\chi}_{AB} - \underline{\chi}_{BA})\,\underline{\beta}^B}\,, \\
(\nabla_A R)_{B434} &= 2(\nabla_A \beta)_B + 2\zeta_A\beta_B - \underline{\chi}^{\sharp_2}{}_A{}^C\alpha_{BC} - 3\rho\chi_{AB} - 3\sigma\epsilon^{\sharp_2}{}_B{}^C\chi_{AC}\,, \\
\frac{1}{2}((\nabla_B R)_{A434} + (\nabla_A R)_{B434}) &= \frac{1}{2}\Big(2(\nabla_B \beta)_A + 2\zeta_B\beta_A - \underline{\chi}^{\sharp_2}{}_B{}^C\alpha_{AC} - 3\rho\chi_{BA} - 3\sigma\epsilon^{\sharp_2}{}_A{}^C\chi_{BC} \\
&\quad +2(\nabla_A \beta)_B + 2\zeta_A\beta_B - \underline{\chi}^{\sharp_2}{}_A{}^C\alpha_{BC} - 3\rho\chi_{AB} - 3\sigma\epsilon^{\sharp_2}{}_B{}^C\chi_{AC}\Big)\,, \\
\frac{1}{2}((\nabla_B R)_{A334} + (\nabla_A R)_{B334}) &= \frac{1}{2}\Big(2(\nabla_B\,\underline{\beta})_A - 2\zeta_B\,\underline{\beta}_A + \chi^{\sharp_2}{}_B{}^C\,\underline{\alpha}_{AC} + 3\rho\,\underline{\chi}_{BA} - 3\sigma\epsilon^{\sharp_2}{}_A{}^C\,\underline{\chi}_{BC} \\
&\quad +2(\nabla_A\,\underline{\beta})_B - 2\zeta_A\,\underline{\beta}_B + \chi^{\sharp_2}{}_A{}^C\,\underline{\alpha}_{BC} + 3\rho\,\underline{\chi}_{AB} - 3\sigma\epsilon^{\sharp_2}{}_B{}^C\,\underline{\chi}_{AC}\Big)\,, \\
(\nabla_C R)_{A3B4} &= \epsilon_{AB}\nabla_C\sigma - g_{AB}\nabla_C\rho + \underline{\chi}_{CA}\,\beta_B + g_{AB}\,\underline{\chi}_{CD}\,\beta^D - \underline{\chi}_{CB}\,\beta_A \\
&\quad - \chi_{CB}\,\underline{\beta}_A - g_{AB}\chi_{CD}\,\underline{\beta}^D + \chi_{CA}\,\underline{\beta}_B\,, \\
(g^{CB}\nabla_C R)_{A3B4} &= \epsilon^{\sharp_2}{}_A{}^C\nabla_C\sigma - \nabla_A\rho - (\mathrm{tr}\,\chi)\,\underline{\beta}_A \boldsymbol{+(\chi_{BA} - \chi_{AB})\,\underline{\beta}^B} \\
&\quad + (\underline{\chi}_{BA} + \underline{\chi}_{AB})\beta^B - (\mathrm{tr}\,\underline{\chi})\beta_A\,, \\
(g^{CB}\nabla_C R)_{B3A4} &= -\epsilon^{\sharp_2}{}_A{}^C\nabla_C\sigma - \nabla_A\rho + (\mathrm{tr}\,\underline{\chi})\beta_A \boldsymbol{-(\underline{\chi}_{BA} - \underline{\chi}_{AB})\beta^B} \\
&\quad - (\chi_{BA} + \chi_{AB})\,\underline{\beta}^B + (\mathrm{tr}\,\chi)\,\underline{\beta}_A\,, \\
(g^{AB}\nabla_B R)_{A434} &= 2\,\mathrm{div}\,\beta + 2(\zeta, \beta) - (\underline{\chi}, \alpha) - 3\rho(\mathrm{tr}\,\chi) \boldsymbol{-3\sigma\epsilon^{AC}\chi_{AC}}\,, \\
(g^{AB}\nabla_B R)_{A334} &= 2\,\mathrm{div}\,\underline{\beta} - 2(\zeta, \underline{\beta}) + (\chi, \underline{\alpha}) + 3\rho(\mathrm{tr}\,\underline{\chi}) \boldsymbol{-3\sigma\epsilon^{AC}\,\underline{\chi}_{AC}}\,.
\end{aligned}
$$

We now apply the contracted Bianchi identities. We compute

$$
\begin{aligned}
0 &= (\nabla^I R)_{I434} \\
&= (\nabla^A R)_{A434} + (\nabla^3 R)_{3434} + (\nabla^4 R)_{4434} \\
&= (g^{AB}\nabla_B R)_{A434} + (g^{34}\nabla_4 R)_{3434} + (g^{34}\nabla_3 R)_{4434} \\
&= 2\,\mathrm{div}\,\beta + 2(\zeta, \beta) - (\underline{\chi}, \alpha) - 3\rho(\mathrm{tr}\,\chi) - 3\sigma\epsilon^{AC}\chi_{AC} - 2\nabla_4\rho + 4(\underline{\eta}, \beta) - 4(Y, \underline{\beta})\,,
\end{aligned}
$$

which corresponds to the Bianchi equation for $\nabla_4 \rho$. We compute

$$
\begin{aligned}
0 &= (\nabla^I R)_{I4A4} \\
&= (g^{BC} \nabla_C R)_{B4A4} - \frac{1}{2} (\nabla_4 R)_{34A4} \\
&= (\nabla^B \alpha)_{AB} + 2\zeta^B \alpha_{AB} - 2(\operatorname{tr} \chi)\beta_A + 2(\chi_{AB} - \chi_{BA})\beta^B - (\nabla_4 \beta)_A + \hat{\omega}\, \beta_A + \underline{\eta}^B\, \alpha_{AB} \\
&\quad + 3Y_A \rho + 3^\star Y_A \sigma \,,
\end{aligned}
$$

which corresponds to the Bianchi equation for $(\nabla_4 \beta)_A$. We compute

$$
\begin{aligned}
0 &= (\nabla^I R)_{I3A4} \\
&= (g^{BC} \nabla_C R)_{B3A4} - \frac{1}{2} (\nabla_3 R)_{43A4} \\
&= -\epsilon^{\sharp_2}{}_A^C \nabla_C \sigma - \nabla_A \rho + (\operatorname{tr} \underline{\chi})\beta_A - (\chi_{BA} - \chi_{AB})\beta^B - (\chi_{BA} + \chi_{AB})\, \underline{\beta}^B + (\operatorname{tr} \chi)\, \underline{\beta}_A \\
&\quad + (\nabla_3 \beta)_A + \underline{\hat{\omega}}\, \beta_A - \underline{Y}^B\, \alpha_{AB} - 3\eta_A \rho - 3^\star \eta_A \sigma \,,
\end{aligned}
$$

which corresponds to the Bianchi equation for $(\nabla_3 \beta)_A$. We compute

$$
(\nabla_3 R)_{A4B4} = \frac{1}{2} \left((\nabla_B R)_{A434} + (\nabla_A R)_{B434} \right) + \frac{1}{2} \left((\nabla_4 R)_{A3B4} + (\nabla_4 R)_{B3A4} \right)
$$

and, by substitution, obtain the equation

$$
\begin{aligned}
(\nabla_3 \alpha)_{AB} + 2\, \underline{\hat{\omega}}\, \alpha_{AB} - 4(\eta \hat{\otimes} \beta)_{AB} = \frac{1}{2} \Big(& 2(\nabla_B \beta)_A + 2\zeta_B \beta_A - \chi^{\sharp_2}{}_B^C \alpha_{AC} - 3\rho \chi_{BA} - 3\sigma \epsilon^{\sharp_2}{}_A^C \chi_{BC} \\
& + 2(\nabla_A \beta)_B + 2\zeta_A \beta_B - \chi^{\sharp_2}{}_A^C \alpha_{BC} - 3\rho \chi_{AB} - 3\sigma \epsilon^{\sharp_2}{}_B^C \chi_{AC} \Big) \\
& + (-\nabla_4 \rho - 2(Y, \underline{\beta}) + 2(\underline{\eta}, \beta)) g_{AB} \,.
\end{aligned}
$$

Using the Bianchi equation for $\nabla_4 \rho$, this corresponds to the Bianchi equation for $(\nabla_3 \alpha)_{AB}$.

As for the null structure equations, we omit the derivation of the conjugate equations. The Bianchi equations for σ are derived by taking the Hodge dual of the Bianchi equations for ρ. Note that, as observed before, one can replace ∇ by the projected covariant derivative $\nabla\!\!\!\!/$ and understand the equations as equations for $\mathfrak{D}_{\mathcal{N}}$ tensors.

This concludes the derivation of the Bianchi equations and, in fact, of the whole system of vacuum Einstein equations of Section 3.3.5.

References

[1] L. Andersson, T. Bäckdahl, P. Blue, and S. Ma, *Stability for linearized gravity on the Kerr spacetime*, arXiv e-prints, (2019), 1903.03859.

[2] L. Andersson and P. Blue, *Hidden symmetries and decay for the wave equation on the Kerr spacetime*, Ann. of Math. (2), 182 (2015), pp. 787–853.

[3] S. Aretakis, *Horizon Instability of Extremal Black Holes*, Adv. Theor. Math. Phys., 19 (2015), pp. 507–530, 1206.6598.

[4] P. Blue and J. Sterbenz, *Uniform decay of local energy and the semi-linear wave equation on Schwarzschild space*, Comm. Math. Phys., 268 (2006), pp. 481–504.

[5] N. Burq, *Décroissance de l'énergie locale de l'équation des ondes pour le problème extérieur et absence de résonance au voisinage du réel*, Acta Math., 180 (1998), pp. 1–29.

[6] V. Cardoso, O. J. C. Dias, and S. Yoshida, *Perturbations and absorption cross section of infinite-radius black rings*, Phys. Rev. D, 72 (2005), p. 024025.

[7] V. Cardoso and J. P. Lemos, *New instability for rotating black branes and strings*, Physics Letters B, 621 (2005), pp. 219 – 223.

[8] V. Cardoso and S. Yoshida, *Superradiant instabilities of rotating black branes and strings*, Journal of High Energy Physics, 2005 (2005), p. 009.

[9] B. Carter, *Black hole equilibrium states*, Black holes/Les astres occlus (École d'Été Phys. Théor., Les Houches, 1972), (1973), pp. 57–214.

[10] S.-S. Chern, *An elementary proof of the existence of isothermal parameters on a surface*, Proceedings of the American Mathematical Society, 6 (1955), pp. 771–782.

[11] Y. Choquèt-Bruhat and R. Geroch, *Global aspects of the Cauchy problem in general relativity*, Comm. Math. Phys., 14 (1969), pp. 329–335.

[12] D. Christodoulou, *Global solutions of nonlinear hyperbolic equations for small initial data*, Communications on Pure and Applied Mathematics, 39, pp. 267–282, https://onlinelibrary.wiley.com/doi/pdf/10.1002/cpa.3160390205.

[13] D. Christodoulou, *The formation of black holes in general relativity*, EMS Monographs in Mathematics, European Mathematical Society (EMS), Zürich, 2009.

[14] D. Christodoulou and S. Klainerman, *The global nonlinear stability of the Minkowski space*, vol. 41 of Princeton Mathematical Series, Princeton University Press, Princeton, NJ, 1993.

[15] S. C. Collingbourne, *The Gregory-Laflamme Instability of the Schwarzschild Black String Exterior*, (2020), 2007.08441.

[16] M. Dafermos and G. Holzegel, *Dynamic instability of solitons in 4 + 1-dimensional gravity with negative cosmological constant*, Unpublished, (2006).

[17] M. Dafermos, G. Holzegel, and I. Rodnianski, *A scattering theory construction of dynamical vacuum black holes*, ArXiv e-prints, (2013), 1306.5364.

[18] M. Dafermos, G. Holzegel, and I. Rodnianski, *Boundedness and Decay for the Teukolsky Equation on Kerr Spacetimes I: The Case $|a| \ll M$*, Annals of PDE, 5 (2019), p. 2.

[19] ——, *The linear stability of the Schwarzschild solution to gravitational perturbations*, Acta Math., 222 (2019), pp. 1–214.

[20] M. Dafermos, G. Holzegel, I. Rodnianski, and M. Taylor, *The nonlinear stability of the Schwarzschild family of black holes*, To appear.

[21] M. Dafermos and J. Luk, *The interior of dynamical vacuum black holes I: The C^0-stability of the Kerr Cauchy horizon*, ArXiv e-prints, (2017), 1710.01722.

[22] M. Dafermos and I. Rodnianski, *A new physical-space approach to decay for the wave equation with applications to black hole spacetimes*, XVIth International Congress on Mathematical Physics, P. Exner (ed.), World Scientific, (2009), pp. 421 – 433.

[23] M. Dafermos and I. Rodnianski, *The red-shift effect and radiation decay on black hole spacetimes*, Comm. Pure Appl. Math., 62 (2009), pp. 859–919.

[24] M. Dafermos and I. Rodnianski, *Decay for solutions of the wave equation on Kerr exterior spacetimes I-II: The cases $|a| \ll M$ or axisymmetry*, ArXiv e-prints, (2010), 1010.5132.

[25] M. Dafermos and I. Rodnianski, *A proof of the uniform boundedness of solutions to the wave equation on slowly rotating Kerr backgrounds*, Invent. Math., 185 (2011), pp. 467–559.

[26] ——, *Lectures on black holes and linear waves*, in Evolution equations, vol. 17 of Clay Math. Proc., Amer. Math. Soc., Providence, RI, 2013, pp. 97–205.

[27] M. Dafermos, I. Rodnianski, and Y. Shlapentokh-Rothman, *Decay for solutions of the wave equation on Kerr exterior spacetimes III: The full subextremal case $|a| < M$*, Ann. of Math. (2), 183 (2016), pp. 787–913.

[28] O. J. C. Dias, *Superradiant instability of large radius doubly spinning black rings*, Phys. Rev. D, 73 (2006), p. 124035.

[29] D. Dold, *Unstable mode solutions to the Klein-Gordon equation in Kerr–anti–de Sitter spacetimes*, Comm. Math. Phys., 350 (2017), pp. 639–697.

[30] M. Durkee, *Geodesics and Symmetries of Doubly-Spinning Black Rings*, Class. Quant. Grav., 26 (2009), p. 085016, 0812.0235.

[31] H. Elvang, *Black saturn*, Journal of High Energy Physics, 2007 (2007), pp. 050–050.

[32] H. Elvang and R. Emparan, *Black rings, supertubes, and a stringy resolution of black hole non-uniqueness*, Journal of High Energy Physics, 2003 (2003), p. 035.

[33] R. Emparan and H. S. Reall, *Generalized Weyl solutions*, Phys. Rev., D65 (2002), p. 084025, hep-th/0110258.

[34] ——, *A rotating black ring solution in five dimensions*, Phys. Rev. Lett., 88 (2002), pp. 101101, 4.

[35] ——, *Black rings*, Classical Quantum Gravity, 23 (2006), pp. R169–R197.

[36] ——, *Black holes in higher dimensions*, Living Reviews in Relativity, 11 (2008), p. 6.

[37] F. C. Eperon, *Wave Equations on Curved Spacetimes*, doctoral thesis, 2019.

[38] F. C. Eperon, H. S. Reall, and J. E. Santos, *Instability of supersymmetric microstate geometries*, JHEP, 10 (2016), p. 031, 1607.06828.

[39] P. Figueras, *A black ring with a rotating 2-sphere*, JHEP, 07 (2005), p. 039, hep-th/0505244.

[40] P. Figueras, M. Kunesch, and S. Tunyasuvunakool, *End point of black ring instabilities and the weak cosmic censorship conjecture*, Physical review letters, 116 (2016).

[41] P. Figueras, K. Murata, and H. S. Reall, *Black hole instabilities and local Penrose inequalities*, Classical Quantum Gravity, 28 (2011), pp. 225030, 30.

[42] G. Fournodavlos and J. Smulevici, *On the initial boundary value problem for the Einstein vacuum equations in the maximal gauge*, (2019), 1912.07338.

[43] H. Friedrich, *On the existence of n-geodesically complete or future complete solutions of Einstein's field equations with smooth asymptotic structure*, Comm. Math. Phys., 107 (1986), pp. 587–609.

[44] O. Gannot, *Quasinormal modes for Schwarzschild-AdS black holes: exponential convergence to the real axis*, Comm. Math. Phys., 330 (2014), pp. 771–799.

[45] E. Giorgi, *The linear stability of Reissner-Nordström spacetime: the full subextremal range*, (2019), 1910.05630.

[46] E. Giorgi, S. Klainerman, and J. Szeftel, *A general formalism for the stability of Kerr*, (2020), 2002.02740.

[47] R. Gregory and R. Laflamme, *Hypercylindrical Black Holes*, Phys. Rev., D37 (1988), p. 305.

[48] R. Gregory and R. Laflamme, *Black strings and p-branes are unstable*, Physical Review Letters, 70 (1993), pp. 2837–2840.

[49] S. Grunau and B. Khamesra, *Geodesic motion in the (rotating) black string spacetime*, Phys. Rev., D87 (2013), p. 124019, 1303.6863.

[50] S. Gunasekaran and H. K. Kunduri, *Slow decay of waves in gravitational solitons*, (2020), 2007.04283.

[51] E. Hackmann, V. Kagramanova, J. Kunz, and C. Lammerzahl, *Analytic solutions of the geodesic equation in higher dimensional static spherically symmetric space-times*, Phys. Rev., D78 (2008), p. 124018, 0812.2428. [Addendum: Phys. Rev.D79,no.2,029901(2009)].

[52] D. Häfner, P. Hintz, and A. Vasy, *Linear stability of slowly rotating Kerr black holes*, arXiv e-prints, (2019), 1906.00860.

[53] B. Helffer, *Introduction to semi-classical methods for the Schrödinger operator with magnetic field*, https://www.math.u-psud.fr/helffer/syrieoctobre2005.pdf.

[54] P. Hintz and A. Vasy, *The global non-linear stability of the kerrde sitter family of black holes*, Acta Math., 220 (2018), pp. 1–206.

[55] G. Holzegel, *On the massive wave equation on slowly rotating Kerr-AdS spacetimes*, Comm. Math. Phys., 294 (2010), pp. 169–197.

[56] ——, *Conservation laws and flux bounds for gravitational perturbations of the Schwarzschild metric*, Class. Quant. Grav., 33 (2016), p. 205004, 1602.04524.

[57] G. Holzegel and A. Shao, *Unique continuation from infinity in asymptotically anti-de sitter spacetimes ii: Non-static boundaries*, Communications in Partial Differential Equations, 42 (2017), pp. 1871–1922, https://doi.org/10.1080/03605302.2017.1390677.

[58] G. Holzegel and J. Smulevici, *Decay properties of Klein-Gordon fields on Kerr-AdS spacetimes*, Comm. Pure Appl. Math., 66 (2013), pp. 1751–1802.

[59] ——, *Quasimodes and a lower bound on the uniform energy decay rate for Kerr-AdS spacetimes*, Anal. PDE, 7 (2014), pp. 1057–1090.

[60] G. T. Horowitz, ed., *Black holes in higher dimensions*, Cambridge Univ. Pr., Cambridge, UK, 2012.

[61] J. Hoskisson, *Particle Motion in the Rotating Black Ring Metric*, Phys. Rev., D78 (2008), p. 064039, 0705.0117.

[62] J. L. Hovdebo and R. C. Myers, *Black rings, boosted strings and Gregory-Laflamme*, Phys. Rev., D73 (2006), p. 084013, hep-th/0601079.

[63] P.-K. Hung, J. Keller, and M.-T. Wang, *Linear Stability of Schwarzschild Spacetime: Decay of Metric Coefficients*, (2017), 1702.02843.

[64] G. Idelon-Riton, *Quasimodes and a lower bound for the local energy decay of the Dirac equation in Schwarzschild-Anti-de Sitter spacetime*, ArXiv e-prints, (2016), 1610.05542.

[65] T. Igata, *Stable bound orbits in six-dimensional Myers-Perry black holes*, Phys. Rev. D, 92 (2015), pp. 024002, 8.

[66] T. Igata, H. Ishihara, and Y. Takamori, *Stable bound orbits of massless particles around a black ring*, Phys. Rev. D, 87 (2013), p. 104005.

[67] T. W. Johnson, *The linear stability of the Schwarzschild solution to gravitational perturbations in the generalised wave gauge*, Ann. PDE, 5 (2019), pp. Paper No. 13, 92.

[68] B. S. Kay and R. M. Wald, *Linear stability of Schwarzschild under perturbations which are non-vanishing on the bifurcation 2-sphere*, Classical and Quantum Gravity, 4 (1987), p. 893.

[69] J. Keir, *Slowly decaying waves on spherically symmetric spacetimes and ultracompact neutron stars*, Classical Quantum Gravity, 33 (2016), pp. 135009, 42.

[70] J. Keir, *Wave propagation on microstate geometries*, Annales Henri Poincare, 21 (2019), pp. 705–760, 1609.01733.

[71] R. P. Kerr, *Gravitational field of a spinning mass as an example of algebraically special metrics*, Phys. Rev. Lett., 11 (1963), pp. 237–238.

[72] S. Klainerman, *The null condition and global existence to nonlinear wave equations*, in Nonlinear systems of partial differential equations in applied mathematics, Part 1 (Santa Fe, N.M., 1984), vol. 23 of Lectures in Appl. Math., Amer. Math. Soc., Providence, RI, 1986, pp. 293–326.

[73] S. Klainerman, *Mathematical challenges of general relativity*, Rend. Mat. Appl. (7), 27(2) (2007), pp. 105–122.

[74] S. Klainerman and J. Szeftel, *Global Nonlinear Stability of Schwarzschild Spacetime under Polarized Perturbations*, arXiv e-prints, (2017), 1711.07597.

[75] S. Klainerman and J. Szeftel, *Constructions of GCM spheres in perturbations of Kerr*, (2019), 1911.00697.

[76] ——, *Effective results on uniformization and intrinsic GCM spheres in perturbations of Kerr*, (2019), 1912.12195.

[77] H. K. Kunduri and J. Lucietti, *Supersymmetric Black Holes with Lens-Space Topology*, Phys. Rev. Lett., 113 (2014), p. 211101, 1408.6083.

[78] P. Laul and J. Metcalfe, *Localized energy estimates for wave equations on high-dimensional Schwarzschild space-times*, Proc. Amer. Math. Soc., 140 (2012), pp. 3247–3262.

[79] P. Laul, J. Metcalfe, S. Tikare, and M. Tohaneanu, *Localized energy estimates for wave equations on* (1+4)*-dimensional Myers-Perry space-times*, SIAM J. Math. Anal., 47 (2015), pp. 1933–1957.

[80] L. Lehner and F. Pretorius, *Black Strings, Low Viscosity Fluids, and Violation of Cosmic Censorship*, Phys. Rev. Lett., 105 (2010), p. 101102, 1006.5960.

[81] H. Lindblad and I. Rodnianski, *The global stability of Minkowski space-time in harmonic gauge*, Ann. of Math. (2), 171 (2010), pp. 1401–1477.

[82] H. Lindblad and M. Taylor, *Global stability of Minkowski space for the Einstein-Vlasov system in the harmonic gauge*, Arch. Ration. Mech. Anal., 235 (2020), pp. 517–633.

[83] H. Lindblad and M. Tohaneanu, *A local energy estimate for wave equations on metrics asymptotically close to Kerr*, arXiv e-prints, (2020), 2004.05664.

[84] J. Luk, *Improved decay for solutions to the linear wave equation on a Schwarzschild black hole*, Ann. Henri Poincaré, 11 (2010), pp. 805–880.

[85] ——, *A vector field method approach to improved decay for solutions to the wave equation on a slowly rotating Kerr black hole*, Anal. PDE, 5 (2012), pp. 553–625.

[86] ——, *The null condition and global existence for nonlinear wave equations on slowly rotating Kerr spacetimes*, J. Eur. Math. Soc. (JEMS), 15 (2013), pp. 1629–1700.

[87] H. Masaood, *A Scattering Theory for Linearised Gravity on the Exterior of the Schwarzschild Black Hole I: The Teukolsky Equations*, (2020), 2007.13658.

[88] T. Mishima and H. Iguchi, *New axisymmetric stationary solutions of five-dimensional vacuum Einstein equations with asymptotic flatness*, Phys. Rev., D73 (2006), p. 044030, hep-th/0504018.

[89] G. Moschidis, *Logarithmic local energy decay for scalar waves on a general class of asymptotically flat spacetimes*, Ann. PDE, 2 (2016), pp. Art. 5, 124.

[90] ——, *A proof of the instability of AdS for the Einstein–massless Vlasov system*, ArXiv e-prints, (2018), 1812.04268.

[91] R. Myers and M. Perry, *Black holes in higher dimensional space-times*, Annals of Physics, 172 (1986), pp. 304 – 347.

[92] A. A. Pomeransky and R. A. Sen'kov, *Black ring with two angular momenta*, ArXiv High Energy Physics - Theory e-prints, (2006), hep-th/0612005.

[93] F. Pretorius and W. Israel, *Quasispherical light cones of the Kerr geometry*, Class. Quant. Grav., 15 (1998), pp. 2289–2301, gr-qc/9803080.

[94] J. V. Ralston, *Solutions of the wave equation with localized energy*, Communications on Pure and Applied Mathematics, 22, pp. 807–823, https://onlinelibrary.wiley.com/doi/pdf/10.1002/cpa.3160220605.

[95] I. Rodnianski and Y. Shlapentokh-Rothman, *The Asymptotically Self-Similar Regime for the Einstein Vacuum Equations*, Geometric and Functional Analysis, 28 (2018), pp. 755–878.

[96] ——, *Naked Singularities for the Einstein Vacuum Equations: The Exterior Solution*, (2019), 1912.08478.

[97] I. Rodnianski and J. Speck, *A regime of linear stability for the einstein-scalar field system with applications to nonlinear big bang formation*, Annals of Mathematics, 187 (2018), pp. 65–156.

[98] I. Rodnianski and T. Tao, *Effective limiting absorption principles, and applications*, Communications in Mathematical Physics, 333 (2015), pp. 1–95.

[99] J. G. Rosa, *Boosted black string bombs*, JHEP, 02 (2013), p. 014, 1209.4211.

[100] J. E. Santos and B. Way, *Neutral black rings in five dimensions are unstable*, Phys. Rev. Lett., 114 (2015), p. 221101.

[101] J. Sbierski, *Characterisation of the energy of Gaussian beams on Lorentzian manifolds: with applications to black hole spacetimes*, Anal. PDE, 8 (2015), pp. 1379–1420.

[102] V. Schlue, *Decay of linear waves on higher-dimensional Schwarzschild black holes*, Anal. PDE, 6 (2013), pp. 515–600.

[103] Y. Shlapentokh-Rothman and R. Teixeira da Costa, *Boundedness and decay for the Teukolsky equation on Kerr in the full subextremal range $|a| < M$: frequency space analysis*, (2020), 2007.07211.

[104] J. Smulevici, *Between modes and quasinormal modes: quasimodes of Kerr-AdS spacetimes*, https://www.newton.ac.uk/seminar/20131007103011301, (October 2013).

[105] M. Stevenson, *Weyl's Law*, http://www-personal.umich.edu/stevmatt/weyl-law.pdf.

[106] W. Strauss, *Partial Differential Equations: An Introduction, 2nd Edition*, Wiley Global Education, 2007.

[107] F. R. Tangherlini, *Schwarzschild field in n dimensions and the dimensionality of space problem*, Nuovo Cim., 27 (1963), pp. 636–651.

[108] D. Tataru, *Local decay of waves on asymptotically flat stationary space-times*, Amer. J. Math., 135 (2013), pp. 361–401.

[109] D. Tataru and M. Tohaneanu, *A local energy estimate on Kerr black hole backgrounds*, Int. Math. Res. Not. IMRN, (2011), pp. 248–292.

[110] M. Taylor, *The global nonlinear stability of Minkowski space for the massless Einstein-Vlasov system*, Ann. PDE, 3 (2017), pp. Paper No. 9, 177.