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Casimir densities outside a constant curvature spherical bubble

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Abstract. We investigate the Wightman function and the vacuum expectation values of the field squared and energy-momentum tensor for a massive scalar field with general curvature coupling outside a spherical bubble in Minkowski spacetime. The geometry inside the bubble corresponds to a space with constant negative curvature. The asymptotic behavior of the expectation values is studied for small values of the bubble radius and at large distances.

Keywords: vacuum polarization, energy-momentum tensor, Casimir effect.

1. Introduction

In the absence of quantum gravity, the influence of the gravitational field on quantum matter fields is studied within the framework of a semiclassical theory where the gravitational field is considered as a classical background [1]-[3]. Among the most interesting topics in this direction is the influence of the gravitational field on the properties of the quantum vacuum. Two effects, which have attracted great deal of attention, are the vacuum polarization and creation of particles. These effects may play an important role in the evolution of the early universe.

In quantum field theory, the definition of the vacuum state is based on the choice of a complete set of the solutions to the classical field equation. These solutions are sensitive to both the local and global characteristics of the background geometry and, hence, the same for the vacuum state. In particular, because of the global nature of the vacuum, the gravitational field localized in some spatial region may influence the properties of the vacuum state for a quantum field outside that region. This type of an example has been recently considered in [4], where the vacuum expectation values (VEVs) of the field squared, and the energy-momentum tensor are investigated for a scalar field in a spherically symmetric static background geometry described by two distinct metric tensors inside and outside a spherical boundary. A special case of the exterior geometry corresponding to a global monopole has been discussed in [5] and [6] for scalar and fermionic fields, respectively. Similar problems with cylindrical symmetry in background of cosmic string spacetime were studied in [7, 8]. Problems in de Sitter and anti-deSitter spacetimes were discussed in [9, 10].

In the present paper, we consider the change in the characteristics of the scalar vacuum induced by a constant negative curvature spherical region (referred below as a bubble) in the surrounding Minkowski geometry. This is a kind of gravitationally induced Casimir effect where the boundary conditions on a quantum field are imposed by the geometry inside the bubble (for a general review of the Casimir effect see, for instance, [11]-[14]). The Casimir effect for a spherical boundary in a constant negative curvature space has been recently investigated in [15] for a scalar field with Robin boundary condition.

The paper is organized as follows. In the next section, we describe the background geometry, the field content and the boundary conditions. The mode functions for a scalar field with general curvature coupling parameter are specified in Section 3. By using these functions, the VEVs of the field squared and of the energy-momentum tensor are investigated in Section 4. The main results are summarized in Section 5.

2. Problem setup

We consider a $(D+1)$- dimensional spherically symmetric spacetime described in coordinates (t,r,ϑ,ϕ), where $\vartheta=(\theta_1,...,\theta_n), n=D-2$, and $0\le\theta_k\le\pi; 0\le\phi\le 2\pi; k=1,...,n$. In the region $r>r_0$ the geometry is Minkowskian with the line element

$$ds^2 = dt^2 - dr^2 - r^2 d\Omega_{D-1}^2 \ , \tag{1}$$

where $d\Omega_{D-1}^2$ is the line element on the $(D-1)$-dimensional sphere, S^{D-1}. In the region $r>r_0$ the geometry is described by

$$ds^2 = dt^2 - dr_i^2 - a^2 \sinh^2(r_i/a) d\Omega_{D-1}^2, \tag{2}$$

with a constant a. The space corresponding to (2) has a constant negative curvature with the Ricci scalar $R=-D(D-1)/a^2$. The interior radial coordinate r_i should be chosen in a wayin order to have continuous metric tensor on the surface $r=r_0$. We take $r_i=r+r_c$ with a constant r_c. With this choice, one has the components $g_{00}=-g_{11}=1$ in both the exterior and interior regions. From the continuity of the components $g_{ll}, l=2,...,D$ one gets

$$\sinh\left(\frac{r_0}{a}+\frac{r_c}{a}\right)=\frac{r_0}{a}, \tag{3}$$

for given r_0 and a, this condition determines the constant r_c. It is negative.

The continuity of the metric tensor on the bounding surface is not sufficient. The matching for the first derivatives of the metric tensor is given by the Israel condition

$$\sum_{j=i,e}\left(K_{(j)ik} - K_{(j)}h_{(j)ik}\right) = 8\pi G\tau_{ik}, \tag{4}$$

on the surface $r=r_0$. In (4), the summation goes over the interior $(j=i)$ and exterior $(j=e)$ regions, $K_{(j)ik}$ is the corresponding extrinsic curvature, $h_{(j)ik}$ is the induced metric on $r=r_0, G$ is the gravitational constant, τ_{ik} is the surface energy-momentum tensor located on the bounding surface with nonzero components τ_0^0 and $\tau_2^2=...=\tau_D^D$. For the geometry under consideration, by using (3), from (4) one finds

$$\tau_0^0 = \frac{D-1}{8\pi G r_0}\left(\sqrt{1+\frac{r_0^2}{a^2}}-1\right), \tau_2^2 = \frac{D-2}{D-1}\tau_D^D. \tag{5}$$

The corresponding energy density is positive.

Having described the geometry we pass to the field content. We consider a quantum scalar field $\varphi(x)$ with the curvature coupling parameter ξ . The field equation reads

$$\left(\nabla_l \nabla^l + m^2 + \xi R\right)\varphi(x) = 0, \tag{6}$$

where for the Ricci scalar one has $R = -D(D-1)/a^2$ in the region $r < r_0$ and $R = 0$ for $r > r_0$. In addition to the field equation in the regions $r < r_0$ and $r > r_0$, the matching conditions for the field should be specified at $r = r_0$. The field is continuous on the separating surface. In order to find the matching condition for the radial derivative of the field, note that the discontinuity in the radial derivatives of the components $g_{ll}, l = 2,...,D$, leads to the delta function term

$$-2\frac{D-1}{r_0}\left(\sqrt{1+\frac{r_0^2}{a^2}}-1\right)\delta(r-r_0) \tag{7}$$

in the Ricci scalar. By taking into account this term in the field equation (6) and integrating the corresponding radial equation near $r = r_0$, we get

$$\left(\partial_r \varphi\right)\Big|_{r=r_0-0}^{r=r_0+0} = \frac{16\pi G\xi}{D-1}\tau\varphi\Big|_{r=r_0}, \tag{8}$$

where $\tau$is the trace of the surface energy-momentum tensor. From (5) one finds

$$\frac{8\pi G}{D-1}\tau = \frac{D-1}{r_0}\left(\sqrt{1+\frac{r_0^2}{a^2}}-1\right). \tag{9}$$

For a minimally coupled field $\xi = 0$ and the radial derivative is continuous.

3. Mode functions for a scalar field

We are interested in the change of the vacuum characteristics in the region $r > r_0$ induced by the presence of the bubble. The corresponding VEVs for physical quantities bilinear in the field are expressed in terms of the mode sums over a complete set $\{\varphi_\alpha(x), \varphi_\alpha^*(x)\}$ of solutions to the classical field equation. The collective index αcorresponds to the set of quantum numbersspecifying the solutions.

In the problem under consideration, from the spherical symmetry it follows that the solutions can be presented as

$$\varphi_\alpha(x) = f_l(r) Y(m_k, \vartheta, \phi) e^{-i\omega t}, \tag{10}$$

where $Y(m_k, \vartheta, \phi)$ is the spherical harmonic of degree l. For the angular quantum numbers one has $l = 0,1,2,...; m_k = (m_0 = l), m_1, ...m_n$; where $m_1, m_2, ...m_n$ are integers obeying the relations

$$0 \le m_{n-1} \le m_{n-2} \le ... \le m_1 \le l, \tag{11}$$

and $-m_{n-1} \le m_n \le m_{n-1}$. Substituting (10) into the field equation, for the radial function in the separate regions one finds

$$f_l(r) = \begin{cases} A_{(i)} f_{(i)l}(r, \lambda), & r < r_0 \\ r^{-n/2}\left[A_{(e)1} J_{\nu_l}(\lambda r) + A_{(e)2} Y_{\nu_l}(\lambda r)\right] & r > r_0 \end{cases} \tag{12}$$

where $J_\nu(x)$ and $Y_\nu(x)$ are the Bessel and the Neumann functions, $\nu_l = l + D/2 - 1$ and $\lambda = \sqrt{\omega^2 - m^2}$. The function $f_{(i)l}(r, \lambda)$ is the regular solution of the radial equation for the line element (2). It is given by

$$f_{(i)l}(r, \lambda) = p_{iz-1/2}^{-\nu_l}(u), \tag{13}$$

where

$$z = \sqrt{\lambda^2 a^2 + D(D-1)(\xi - \xi_D)}, \tag{14}$$

and

$$p_\nu^{-\mu}(u) = \frac{p_\nu^{-\mu}(u)}{(u^2-1)^{(D-2)/4}}, u = \cosh(r + r_c), \tag{15}$$

with $P_\nu^{-\mu}(u)$ being the associated Legendre function of the first kind. In (14), $\xi_D = \frac{D-1}{4D}$ is the curvature coupling parameter for a conformally coupled scalar field. From the relation $P_{iz-1/2}^{-\mu}(u) = P_{-iz-1/2}^{-\mu}(u)$ it follows that the function $f_{(i)l}(r, \lambda)$ is real. The modes are specified by the set of quantum numbers $\alpha = (\lambda, m_k)$.

From the matching conditions for the field and its radial derivative on $r = r_0$ we get

$$\begin{cases} A_{(e)1} = \frac{\pi}{2} r_0^{n/2} A_{(i)} f_{(i)l}(r_0, \lambda) \bar{Y}_{\nu l}(\lambda r_0) \\ A_{(e)2} = \frac{\pi}{2} r_0^{n/2} A_{(i)} f_{(i)l}(r_0, \lambda) \bar{J}_{\nu l}(\lambda r_0) \end{cases} \tag{16}$$

where, for a given function $f(x)$ we use the notation

$$\bar{f}(\lambda r_0) = \lambda r_0 f'(\lambda r_0) - \beta(\lambda r_0) f(\lambda r_0), \tag{17}$$

and

$$\beta(r_0\lambda) = r_0\lambda \frac{f'_{(i)l}(r_0\lambda)}{f_{(i)l}(r_0\lambda)} + \frac{n}{2} + \frac{16\pi G\xi}{D-1} r_0 \tau . \tag{18}$$

Here, the prime stands for the derivative of the function with respect to the argument. From the standard Klein-Gordon normalization, condition one finds

$$A_{(i)}^2 = 2\lambda \frac{\left[\bar{J}_{\nu l}^2(\lambda r_0) + \bar{Y}_{\nu l}^2(\lambda r_0)\right]^{-1}}{\pi^2 N(m_k) \omega r_0^n f_{(i)l}^2(r_0, \lambda)} \tag{19}$$

The factor $N(m_k)$ comes from the normalization integral for $Y(m_k, \vartheta, \phi)$ and its explicit form (see, for instance, [16]) will not be required in the discussion below. As a result, for real $\lambda$the radial mode functions in the exterior region are presented in the form

$$f_l(r) = \sqrt{\frac{\lambda/\omega}{2N(m_k)}} \frac{\bar{Y}_{\nu l}(\lambda r_0) J_{\nu l}(\lambda r) - \bar{J}_{\nu l}(\lambda r_0) Y_{\nu l}(\lambda r)}{\sqrt{\bar{J}_{\nu l}^2(\lambda r_0) + \bar{Y}_{\nu l}^2(\lambda r_0)}}. \tag{20}$$

The imaginary values for λ correspond to possible bound states. For these states $\lambda = i\eta$ and $\omega = \sqrt{m^2 + \eta^2}$. The exterior radial mode functions are given by $r^{-n/2} K_{\nu l}(\eta r)$, where $K_\nu(x)$ is the Macdonald function. For $\eta > m$ the energy becomes imaginary and in order to have a stable vacuum state we will assume that $\eta < m$. For the radial functions corresponding to the boundstates one has

$$f_{bl}(r\lambda) = \begin{cases} A_{ib} f_{(i)l}(r, i\eta), & r < r_0, \\ A_{(eb)} r^{-n/2} K_{\nu l}(\eta r), & r > r_0 \end{cases} \tag{21}$$

From the matching conditions of the solutions on $r = r_0$ it follows that the eigenvalues for η are solutions of the equation

$$\widetilde{K}_{\nu l}\left(\eta r_0\right)=0\,. \tag{22}$$

Here and in what follows, for a function $f\left(x\right)$ we use the notation

$$\widetilde{f}\left(x\right)=xf'\left(x\right)-\beta\left(r_0,ix/r_0\right)f\left(x\right). \tag{23}$$

For the normalization coefficient of the bound states one obtains

$$A_{(eb)}^2=-\frac{\eta\tilde{I}_{\nu l}\left(\eta r_0\right)}{N\left(m_k\right)\omega\partial_\eta\widetilde{K}_{\nu l}\left(\eta r_0\right)}, \tag{24}$$

where $I_\nu\left(x\right)$ is the modified Bessel function of the first kind. The coefficient $A_{(ib)}$ is found from

$$A_{(ib)}=\frac{A_{(eb)}K_{\nu l}\left(\eta r_0\right)}{r_0^{n/2}f_{(i)l}^{(1)}\left(r_0,i\eta\right)}. \tag{25}$$

For small values of r_0 one has

$$\widetilde{K}_{\nu l}\left(\eta r_0\right)\approx-\left(2l+3n/2\right)K_{\nu l}\left(\eta r_0\right), \tag{26}$$

and the function $\widetilde{K}_{\nu l}\left(\eta r_0\right)$ is negative. For large r_0/a, by using the asymptotic expression

$$P_{z-1/2}^{-\mu}\left(u\right)\approx\frac{\Gamma\left(z\right)\left(2r_0/a\right)^{z-1/2}}{\sqrt{\pi}\Gamma\left(1/2+z+\mu\right)}, \tag{27}$$

we see that

$$\frac{\widetilde{K}_{\nu l}\left(\eta r_0\right)}{K_{\nu l}\left(\eta r_0\right)}\approx-\left(\frac{r_0}{a}\right)\left[\eta a+\sqrt{\eta^2a^2+D\left(D-1\right)\left(\xi_D-\xi\right)}-1/2+2\xi\left(D-1\right)\right]. \tag{28}$$

We have numerically checked that in $D=3$ there are no bound states for both minimally and conformally coupled scalar fields.

4. Wightman function and the VEVs

The scalar modes we have presented above are obtained from the modes outside a spherical shell with the radius r_0 on which the field obeys the Robin boundary condition

$$\left(\beta_R + \partial_r\right)\varphi = 0, \quad r = r_0, \tag{29}$$

with the replacement

$$r_0\beta_R \to -\beta\left(r_0, \lambda\right) + \frac{n}{2}. \tag{30}$$

As a result, the corresponding Wightman function and the VEVs of the field squared and of the energy-momentum tensor are obtained in a way similar to that described in [17] for the case of the Robin sphere. Here we omit the details and present the final results.

The Wightman function is presented in the form

$$\begin{aligned} W\left(x, x'\right) = W_M\left(x, x'\right) - \frac{2\left(rr'\right)^{-n/2}}{n\pi S_D} \sum_{l=0}^{\infty} v_l C_l^{n/2}\left(\cos\theta\right) \int_m^{\infty} dy y \times \\ \times \frac{\tilde{I}_{vl}\left(yr_0\right)}{\widetilde{K}_{vl}\left(yr_0\right)} \frac{K_{vl}\left(yr\right) K_{vl}\left(yr'\right)}{\sqrt{y^2 - m^2}} \cosh\left(\left(t' - t\right)\sqrt{y^2 - m^2}\right), \end{aligned} \tag{31}$$

where $W_M\left(x, x'\right)$ is the Wightman function in the Minkowski spacetime in the absence of the bubble. In (31), $S_D = 2\pi^{D/2}/\Gamma\left(D/2\right)$ is the surface area of the sphere with unit radius in D - dimensional space, θ is the angle between directions determined by the sets of angles $\left(\vartheta, \phi\right)$ and $\left(\vartheta', \phi'\right), C_p^q\left(x\right)$ is the Gegenbauer polynomial of degree p and order q. The second term in the right-hand side of (31) is induced by the bubble.

For $r > r_0$ the bubble-induced contribution is finite in the coincidence limit. For the renormalized VEV of the field squared from (31) we directly get

$$\left\langle \varphi^2 \right\rangle = -\frac{r^{2-D}}{\pi S_D} \sum_{l=0}^{\infty} D_l \int_m^{\infty} dy y \frac{\tilde{I}_{vl}\left(yr_0\right)}{\widetilde{K}_{vl}\left(yr_0\right)} \frac{K_{vl}^2\left(yr\right)}{\sqrt{y^2 - m^2}}, \tag{32}$$

with the notation

$$D_l = \left(2l + D - 2\right) \frac{\Gamma\left(l + D - 2\right)}{\Gamma\left(D - 1\right) l!}. \tag{33}$$

Note that the renormalization is reduced to the omission of the part corresponding to the Minkowski spacetime.

The vacuum energy-momentum tensor is diagonal with the components (no summation over μ)

$$\left\langle T_{\mu}^{\mu}\right\rangle = -\frac{r^{2-D}}{2\pi S_D}\sum_{l=0}^{\infty} D_l \int_m^{\infty} dy y^3 \frac{\tilde{I}_{\nu l}\left(yr_0\right)}{\widetilde{K}_{\nu l}\left(yr_0\right)} \frac{F\left[K_{\nu l}\left(yr\right)\right]}{\sqrt{y^2-m^2}}, \tag{34}$$

where for a given function $f(y)$ we have introduced the following notations

$$\begin{aligned}
F^{(0)}\left[f(y)\right] &= (4\xi-1)\left[f'^2(y) - \frac{n}{y} f(y) f'(y) + \left(\frac{v_l^2}{y^2} - \frac{1+4\xi-2(mr/y)^2}{1-4\xi}\right)\right], \\
F^{(1)}\left[f(y)\right] &= f'^2(y) + \frac{\xi_1}{y} f(y) f'(y) - \left(1 + \frac{v_l^2 + \xi_1 n/2}{y^2}\right) f^2(y), \\
F^{(i)}\left[f(y)\right] &= 4\xi f'^2(y) + \left[4\xi - 2 + \frac{v_l^2 + (\xi_1 - n) - \xi_1 n^2/2}{(n+1) y^2}\right] f^2(y) - F^{(1)}\left[f(y)\right],
\end{aligned} \tag{35}$$

with $i = 2,3,\ldots,D$ and $\xi_1 = 4(D-1)\xi - D + 2$.

When the bubble radius goes to zero, $r_0 \to 0$, one has

$$\frac{\tilde{I}_{\nu_l}\left(yr_0\right)}{\widetilde{K}_{\nu_l}\left(yr_0\right)} \approx \frac{n\left(yr_0/2\right)^{2\nu_l}}{(2l+3n/2)\nu_l \Gamma^2\left(\nu_l\right)}. \tag{36}$$

This shows that the dominant contribution to the VEVs comes from the $l = 0$ term. To the leading, order one gets

$$\begin{aligned}
\left\langle \varphi^2 \right\rangle &\approx -\frac{2^{-n}\left(r_0/r\right)^n}{3\pi^{D/2+1}\Gamma(n/2)} \int_m^{\infty} dy y^{n+1} \frac{K_{n/2}^2(yr)}{\sqrt{y^2-m^2}}, \\
\left\langle T_{\mu}^{\mu} \right\rangle &\approx \frac{2^{-n-1}\left(r_0/r\right)^n}{3\pi^{D/2+1}\Gamma(n/2)} \int_m^{\infty} dy y^{n+3} \frac{F^{(\mu)}\left[K_{n/2}^2(yr)\right]}{\sqrt{y^2-m^2}},
\end{aligned} \tag{37}$$

and for a fixed r the VEVs vanish as r_0^n.

The bubble induced VEVs diverge on the bounding surface. These divergences are well known in the problems of quantum field theory on manifolds with boundaries. In the problem under consideration, these divergences are weaker, compared with the case of a sphere with Dirichlet, Neumann or, more general, Robin boundary conditions. This is a consequence of the dependence of the effective Robin coefficient, given by the right-hand side of (30), on λ. At large distances from the bubble and for a massive field, assuming that $mr \gg 1$, the dominant contribution to the integrals come from the region near the lower limit of the integration. By taking into account that over all the integration region the argument of the Macdonald function is large, for the leading order term in the VEV of the field squared we find

$$\left\langle \varphi^2 \right\rangle \approx -\frac{\sqrt{\pi} m^{D-1} e^{-2mr}}{4S_D (mr)^{D-1/2}} \sum_{l=0}^{\infty} D_l \frac{\tilde{I}_{\nu l}\left(mr_0\right)}{\widetilde{K}_{\nu l}\left(mr_0\right)}. \tag{38}$$

In a similar way, for the components of the energy-momentum tensor one obtains

$$\langle T_0^0 \rangle \approx 4\xi m^2 \langle \varphi^2 \rangle, \langle T_2^2 \rangle \approx \frac{\xi - 1/8}{\xi} \langle T_0^0 \rangle, \langle T_1^1 \rangle \approx \frac{1 - D/2}{mr} \langle T_2^2 \rangle. \tag{39}$$

For a minimally coupled field the leading term for the energy density vanishes and it is required to keep the next to the leading order terms. The leading term also vanishes for the radial stress. In order to find the corresponding asymptotic it is most easily to use the relation $r\partial_r \langle T_\mu^\nu \rangle = 0$ which is a consequence of the covariant conservation equation $\nabla_\nu \langle T_\mu^\nu \rangle = 0$.

For a mass less field and at large distances from the bubble surface, $r \gg r_0, a,$ we introduce in (32) a new integration variable $x = yr$ and expand the integrand over r_0/r and a/r. The leading contribution comes from the $l = 0$ term and, after the evaluation of the remaining integral over x, for the VEV of the field squared one finds

$$\langle \varphi^2 \rangle \approx \frac{2^{1-D}}{\pi^{D/2}} \frac{n - 2\beta(r_0 \cdot 0)}{n + 2\beta(r_0 \cdot 0)} \frac{\Gamma(n+1/2) r_0^n}{n\Gamma^2(n/2) r^{2n+1}} \Gamma\left(\frac{n+1}{2}\right). \tag{40}$$

For the components of the vacuum energy-momentum tensor, in a similar way, we find (no summation over μ)

$$\langle T_\mu^\mu \rangle \approx -\frac{2^{-1-n}}{\pi^{D/2+1}} \frac{n - 2\beta(r_0 \cdot 0)}{n + 2\beta(r_0 \cdot 0)} \frac{r_0^n r^{1-2D}}{\Gamma(n/2)} \int_0^\infty dy y^{n+2} F^{(\mu)} \left[K_{n/2}(y) \right]. \tag{41}$$

Similar to the case of the field squared, the integrals are expressed in terms of the gamma function.

5. Conclusion

We have investigated the change in the local characteristics of the quantum vacuum for a scalar field induced by the presence of a spherical bubble of a constant curvature space. The interior geometry is described by the line element (2) and outside the bubble, one has Minkowski space-time. The exterior and interior radial coordinates are shifted by r_C that is determined from (3). From the Israel matching condition one obtains the corresponding surface energy-momentum tensor with the components (5) located on the bubble surface. The scalar field is continuous and the jump of the corresponding radial derivative is given by (8).

For the evaluation of the VEVs, a complete set of mode functions are required for a scalar field. These functions are presented as (10) where inside the bubble the radial function is expressed in terms of the associated Legendre function of the first kind, whereas the solution in the exterior region is given by a linear combination of the Bessel and Neumann functions. The corresponding coefficients are determined by the matching conditions and the exterior radial function is presented as (20). The corresponding Wightman function and the VEVs of the field squared and energy-momentum tensor are obtained in a way similar to that for a spherical boundary in Minkowski spacetime with Robin boundary condition for a scalar field. In particular, the VEVs of the field squared and energy-momentum tensor are given by the expressions (32) and (34). At large distances from the bubble, the VEVs decay as power law for mass less field and exponentially in the massive case. The divergences on the bubble surface here are weaker compared to the case of a sphere with Robin boundary condition.

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