

The geometric phase of a relativistically covariant four dimensional harmonic oscillator

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Abstract. We show that there is nontrivial Berry relativistically covariant phase generated by a perturbed relativistic oscillator. This phase is associated with a fractional perturbation of the azimuthal symmetry of the oscillator.

A manifestly covariant quantum mechanics was formulated by E. C. G. Stueckelberg [1] in 1941. He studied this theory for the case of a single particle in an external field. He considered the phenomenon of pair annihilation and creation as a manifestation of the development, in each case, of a single world line that curves in such a way that one part runs backwards in time, and above the turning point the line does not pass at all. This configuration was considered by Stueckelberg to represent pair annihilation. To describe such a curve, parametrization by the variable t is ineffective, since the trajectory is not single valued. He therefore introduced a parametric description, with parameter τ , along the world line. Hence one branch of the curve is generated by motion in the positive sense of t as a function of increasing τ , and the other branch by motion in the negative sense of t .

The motion, in space-time, of the point generating the world line, which we shall call an event (and has properties of space-time position and energy momentum), is governed in the classical case by the Hamiltonian equations in space-time

$$\frac{dx^\mu}{d\tau} = \frac{\partial K}{\partial p_\mu}, \quad \frac{dp^\mu}{d\tau} = -\frac{\partial K}{\partial x_\mu} \quad (1)$$

where $x^\mu = (t, \vec{x})$, $p^\mu = (E, \vec{p})$ [we take $c = 1$ and $g_{\mu\nu} = (-1, 1, 1, 1)$] and the evolution generator K is a function of the canonical variables x_μ, p_μ . For the special case of free motion,

$$K_0 = \frac{p^\mu p_\mu}{2M} \quad (2)$$

where M is an intrinsic parameter assigned to the generic event, and hence

$$\frac{dx^\mu}{d\tau} = \frac{p^\mu}{M} \quad (3)$$



It then follows that

$$\frac{d\vec{x}}{dt} = \frac{\vec{p}}{E} \quad (4)$$

consistent with standard relativistic kinematics. We note, however, that the mass squared $m^2 = -p^\mu p_\mu$ is a dynamical variable since \vec{p} and E are considered to be kinematically independent, and therefore it is not taken to be equal to a given constant. The set of values taken by m^2 in a particular dynamical context is determined by initial conditions and the dynamical equations.

In the quantum theory, \vec{x}, t (and \vec{p}, E) denote operators satisfying the commutation relations (we take $\hbar = 1$)

$$[x^\mu, p^\mu] = ig^{\mu\nu} \quad (5)$$

The state of one-event system is described by a wave function $\psi_\tau(x) \in L^2(R^4)$, a complex Hilbert space with measure $d^4x = d^3x dt$ satisfying the equation

$$i \frac{\partial \psi_\tau(x)}{\partial \tau} = K \psi_\tau(x) \quad (6)$$

This equation, designed to provide a manifestly covariant description of relativistic phenomena, is similar in form to the non-relativistic Schrodinger equation. Although free motion is determined by the operator form of K_0 of Eq. (3), i.e., the d'Alembertian, which is hyperbolic ($p_\mu p^\mu \equiv -\partial_\mu \partial^\mu$ instead of the elliptic operator $\vec{p}^2 \equiv -\nabla^2$), the same methods may be used for studying Eq. (6) as for the non-relativistic Schrodinger equation.

In 1973 Horwitz and Piron [2] generalized the Stueckelberg theory by assuming that for the treatment of systems of more than one event (generating world lines of more than one particle), one assumes the unperturbed evolution generator to be of the form ($p_i^2 = p_{i\mu} p_i^\mu$)

$$K_0 = \sum_{i=1}^N \frac{p_i^2}{2M_i} \quad (7)$$

They further assumed that there is a single universal τ which parametrizes the motion of all of the particles of the many body system (we denote this generalized theory by SHP.) There is a class of model systems, for which solutions can be achieved using straightforward methods, which involve only effective action-at-a-distance (direct action) potentials, where the evolution generator is of the form

$$K = \sum_{i=1}^N \frac{p_i^2}{2M_i} + V(x_1, x_2, \dots, x_N) \quad (8)$$

Note that in this case the potential function enters into the dynamical evolution equation as a term added to the generator of the free motion, and therefore corresponds to a space-time coordinate-dependent interaction mass.

Equations (2) become

$$\frac{dx_i^\mu}{d\tau} = \frac{\partial K}{\partial p_{i\mu}}, \quad \frac{dp_i^\mu}{d\tau} = -\frac{\partial K}{\partial x_{i\mu}} \quad (9)$$

In 1989 Horwitz and Arshansky [3] demonstrated the existence of bound state solutions for the quantum case by solving the dynamical equation (6) associated with the dynamical evolution

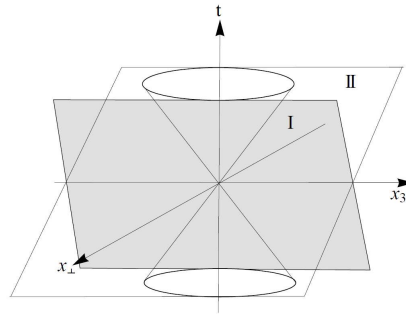


Figure 1. RMS

operator (8). The two-body potential function, which they chose for Poincaré invariance was of the form $V(\rho)$, where $x_1^\mu - x_2^\mu$ is spacelike,

$$\rho = \sqrt{(x_1^\mu - x_2^\mu)(x_{1\mu} - x_{2\mu})} \equiv \sqrt{(x_1 - x_2)^2} \quad (10)$$

Instead of the more widely used decomposition of Minkowski space into the timelike and full spacelike regions, they used the decomposition of the spacelike region into two subregions [invariant under an $O(2,1)$ subgroup of $O(3,1)$]. (see Fig.1) One of these (called the RMS - Reduced Minkowski Space) sectors (I) consists of the space-time points external (in spacelike directions) to two hyperplanes tangent to the light cone that are oriented along the z axis (the direction must be chosen to define this space). The second (II) consists of the space-time points in the sector interior (timelike direction) to these hyperplanes, but excluding the light cone. In the figure 1, this composition is shown schematically by folding the two space axes x, y together (defining the coordinate x_\perp); in the resulting three-dimensional space, the two hyperplanes become planes and intersect along the z axis.

where

$$x^0 = \rho \sin \theta \sinh \beta \quad x^1 = \rho \sin \theta \cos \phi \cosh \beta \quad x^2 = \rho \sin \theta \sin \phi \sinh \beta \quad x^3 = \rho \cos \theta \quad (11)$$

This procedure resulted in the correct (Schrodinger) spectrum for the bound state. When they applied the method of treating the relativistic quantum two body problem to the case of the four dimensional Harmonic Oscillator so that the reduced Hamiltonian is

$$\begin{aligned} K &= \frac{p_\mu p^\mu}{2m} + \frac{1}{2} k x_\mu x^\mu = -\frac{1}{2m} \partial_\mu \partial^\mu + \frac{1}{2} m w^2 \rho^2 \\ &= \frac{1}{2m} \left[-\frac{\partial^2}{\partial \rho^2} - \frac{3}{\rho} \frac{\partial}{\partial \rho} + \frac{\Lambda}{\rho^2} \right] + \frac{1}{2} m w^2 \rho^2 \end{aligned} \quad (12)$$

They obtained the complete set of eigenvectors which span the RMS, represented by the τ independent wave functions¹

¹ Cook [4] solved this problem with full spacelike support for the wavefunction and obtained an incorrect spectrum. Note than Zmuidzinas [5] first showed that there is no complete set of orthogonal functions in the full spacelike region, and constructed the RMS

$$\begin{aligned} \psi_{mnl n_a}(\phi, \beta, \theta, \rho) = & \frac{1}{2\pi} e^{i(m+1/2)\phi} \cdot \sqrt{n} \sqrt{\Gamma(1+m+n)/\Gamma(1+m-n)} \cdot \\ & \cdot (1 - \tanh^2 \beta)^{\frac{1}{4}} P_m^{-n}(\tanh \beta) \cdot (1 - \cos^2 \theta)^{-\frac{1}{4}} \cdot \\ & \cdot P_l^n(\cos \theta) \frac{1}{\sqrt{\rho}} \left(\frac{m w \rho^2}{\hbar}\right)^{l/2} e^{-\frac{m w \rho^2}{2\hbar}} L_{n_a}^{l+1/2}\left(\frac{m w \rho^2}{\hbar}\right) \end{aligned} \quad (13)$$

The idea of the geometric phase proposed by Berry [6] in 1984 asserts that under adiabatic processes the wave function of a system picks up a phase factor that can be found in the nonrelativistic case by the integral

$$\gamma_n(t) \equiv i \int_0^t \langle \psi_n(t') | \frac{\partial \psi_n(t')}{\partial t} \rangle dt' \quad (14)$$

where ψ_n is the n th eigenvector of the evolving Hamiltonian and t is the time between final and initial state of the time dependent parameters of the Hamiltonian (i.e. the evolution time). To construct a manifestly covariant form of (14), we have to replace the variable t by the variable τ which is the evolution time according to SHP quantum mechanics. Suppose there are N τ dependent parameters: $R_1(\tau), R_2(\tau), \dots, R_N(\tau)$ in the Hamiltonian of a given problem then

$$\frac{\partial \psi_n}{\partial \tau} = \frac{\partial \psi_n}{\partial R_1} \frac{dR_1}{d\tau} + \frac{\partial \psi_n}{\partial R_2} \frac{dR_2}{d\tau} + \dots + \frac{\partial \psi_n}{\partial R_N} \frac{dR_N}{d\tau} = (\nabla_{\vec{R}} \psi_n) \cdot \frac{d\vec{R}}{d\tau} \quad (15)$$

where

$$\vec{R} = (R_1, R_2, \dots, R_N) \quad (16)$$

is the gradient with respect to these parameters. We then have

$$\gamma_n(\tau) = i \int_{\vec{R}_i}^{\vec{R}_f} \langle \psi_n | \nabla_{\vec{R}} \psi_n \rangle \cdot d\vec{R} \quad (17)$$

Now, if the Hamiltonian returns to its original form after a time $\tau = T$, then the geometric phase is

$$\gamma_n(T) = i \oint \langle \psi_n | \nabla_{\vec{R}} \psi_n \rangle \cdot d\vec{R} \quad (18)$$

We wish to demonstrate the realization of the manifestly covariant Berry phase in the example of the four dimensional oscillator. For that propose we add a perturbation to the Hamiltonian (12) which breaks the hyperangular symmetry of this Hamiltonian. Since the complex valued matrix elements necessary to develop a dynamical phase arise in the example from the ϕ dependance, one must perturb the azimuthal symmetry with a fractional coefficient, as we shall see below.

$$\begin{aligned} K = & \frac{1}{2m} \left[-\frac{\partial^2}{\partial \rho^2} - \frac{3}{\rho} \frac{\partial}{\partial \rho} + \frac{\Lambda}{\rho^2} \right] + \frac{1}{2} m w^2 \rho^2 + \\ & + 2 \left(\varepsilon_1 \rho^2 \sin^2 \theta \cos^2((2/3)\phi) \cosh^2 \beta + \right. \\ & + \varepsilon_2 \rho^2 \sin^2 \theta \sin^2((2/3)\phi) \cosh^2 \beta + \\ & \left. + \varepsilon_3 \rho^2 \cos^2 \theta - \varepsilon_0 \rho^2 \sin^2 \theta \sinh^2 \beta \right) \end{aligned} \quad (19)$$

where $\varepsilon_0, \varepsilon_1, \varepsilon_2, \varepsilon_3$ are small parameters of the perturbation.

Now using degenerate time independent perturbation theory, we calculate the first order correction to the wave function (13).

The new wave function produced by the Hamiltonian (19) is equal to a linear combination of the wave function (13) and the first order correction to this wave function which is given by

$$\psi_{mnl n_a}^{(1)} = \sum_{m'n'l'n'_a \neq mnl n_a} \frac{\langle \Phi_{m'n'l'n'_a} | V | \Phi_{mnl n_a} \rangle}{K_a - K'_a} \Phi_{m'n'l'n'_a} \quad (20)$$

where V is the perturbation given in (19), K_a is the eigenvalue of the unperturbed Hamiltonian (12), and $\Phi_{m'n'l'n'_a}$ is a linear combination of the $\psi_{m'n'l'n'_a}$ due to the degeneracy of the energy eigenvalues

$$K_a = \hbar\omega(l + 2n_a + \frac{3}{2}) \quad (21)$$

Now suppose ε_0 and ε_3 are equal to zero so Eq.(20) will be

$$\begin{aligned} \psi_{mnl n_a}^{(1)} = & \varepsilon_1 \sum_{m'n'l'n'_a \neq mnl n_a} \frac{\langle \Phi_{m'n'l'n'_a} | \rho^2 \sin^2 \theta \cos^2((2/3)\phi) \cosh^2 \beta | \Phi_{mnl n_a} \rangle}{l - l' + 2(n_a - n'_a)} \Phi_{m'n'l'n'_a} \\ & + \varepsilon_2 \sum_{m'n'l'n'_a \neq mnl n_a} \frac{\langle \Phi_{m'n'l'n'_a} | \rho^2 \sin^2 \theta \sin^2((2/3)\phi) \cosh^2 \beta | \Phi_{mnl n_a} \rangle}{l - l' + 2(n_a - n'_a)} \Phi_{m'n'l'n'_a} \end{aligned} \quad (22)$$

or

$$\psi_{mnl n_a}^{(1)} = \varepsilon_1 \psi' + \varepsilon_2 \psi'' \quad (23)$$

The calculation of ψ' and ψ'' will done numerically by letting each of the indices ($m, n, l, n_a, m', n', l', n'_a$) run from 2 to 5 and computing the ratios in (22) for each permutation of these indices. In this way the matrix of the perturbation will have an order 256x256 and each of the eigenvectors, $\psi_{m'n'l'n'_a}$, and the first order correction will be 256x1 column vector. The degeneracy is 16 fold beside the fact that we have additional degeneracy due to the linearity of the energy values in n and l (so the difference in eigenvalues can be zero even if $l \neq l'$ and $n_a \neq n'_a$ which requires a more complete treatment, now in preparation [7]) and the corrected wave function will be

$$\psi = \psi_{mnl n_a} + \psi_{mnl n_a}^{(1)} = \psi_{mnl n_a} + \varepsilon_1 \psi' + \varepsilon_2 \psi'' \quad (24)$$

Eq. (24) can be written as

$$\psi = \psi_{mnl n_a} + \varepsilon_1 \sum_{m'n'l'n'_a} c_{mm'} A_{mm'nn'll'n_a n'_a} \Phi_{m'n'l'n'_a} + \varepsilon_2 \sum_{m'n'l'n'_a} d_{mm'} B_{mm'nn'll'n_a n'_a} \Phi_{m'n'l'n'_a} \quad (25)$$

where $A_{mm'nn'll'n_a n'_a}$ and $B_{mm'nn'll'n_a n'_a}$ are the factors of the perturbation matrix elements which results from integration on β, ρ and θ divided by the energy differences (and hence are real) and the factors $c_{mm'}$ and $d_{mm'}$ are given by

$$c_{mm'} = \int_0^{2\pi} e^{i(m'-m)\phi} \cos^2((2/3)\phi) d\phi = \frac{\frac{3}{2}i(m-m') - \frac{1}{\sqrt{3}}}{\frac{4}{9} - (m'-m)^2} \quad (26)$$

and

$$d_{mm'} = \int_0^{2\pi} e^{i(m'-m)\phi} \sin^2((2/3)\phi) d\phi = \frac{\frac{\sqrt{3}}{2}i(m-m' + \frac{2}{3\sqrt{3}}) + \frac{2}{3}}{\frac{4}{9} - (m'-m)^2} \quad (27)$$

substitute (25) in Eq. (18) where $\vec{R} = (\varepsilon_1, \varepsilon_2)$ and expressing ε_1 and ε_2 in polar coordinates, then taking the line integral over the unit circle, we get

$$\begin{aligned} \gamma_{mnl n_a}(T) &= i\pi(\langle \psi' | \psi'' \rangle - \langle \psi'' | \psi' \rangle) = i\pi \cdot 2i \text{Im}[\langle \psi' | \psi'' \rangle] = \\ &= -2\pi \text{Im} \left[\sum_{m'n'l'n'_a} c_{mm'} A_{mm'nn'll'n_a n'_a} d_{mm'}^* B_{mm'nn'll'n_a n'_a} \right] \end{aligned} \quad (28)$$

We now substitute (26) and (27) into (28) and obtain

$$\gamma_{mnl n_a}(T) = 3\pi \sum_{m'n'l'n'_a} A'_{mm'nn'll'n_a n'_a} B'_{mm'nn'll'n_a n'_a} (m' - m - \frac{2}{9\sqrt{3}}) \quad (29)$$

Which is the geometric phase due to the perturbation on the eigenvector (13) where

$$A'_{mm'nn'll'n_a n'_a} = \frac{A_{mm'nn'll'n_a n'_a}}{\frac{4}{9} - (m' - m)^2}; \quad B'_{mm'nn'll'n_a n'_a} = \frac{B_{mm'nn'll'n_a n'_a}}{\frac{4}{9} - (m' - m)^2} \quad (30)$$

We have shown in this note that a nontrivial Berry phase can be associated with a perturbed relativistically covariant oscillator. The fractional breaking of the azimuthal symmetry was necessary to achieve this, and makes explicit, in this example, the association of the Berry phase with topological properties of the system (as in the Aharonov-Bohm effect [8], [9]). We remark that the Berry phase computed on this way is covariant, since the RMS coordinates in every Lorentz frame are isomorphic [10].

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