



Extend Bekenstein's theorem to Einstein–Maxwell-scalar theories with a scalar potential

Jianhui Qiu^{1,2,a} , Changjun Gao^{1,2,b}

¹ National Astronomical Observatories, Chinese Academy of Sciences, Beijing 100101, China

² School of Astronomy and Space Sciences, University of Chinese Academy of Sciences, No. 19A, Yuquan Road, Beijing 100049, China

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Abstract The Bekenstein's theorem allows us to generate an Einstein-conformal scalar solution from a single Einstein-ordinary scalar solution. In this article, we extend this theorem to the Einstein–Maxwell-scalar (EMS) theory with a non-minimal coupling between the scalar and Maxwell field when a scalar potential is also included. As applications of this extended theorem, the well-known static dilaton solution and rotating solution with a specific coupling between dilaton and Maxwell field are considered, and new conformal dilaton black hole solutions are found. The Noether charges, such as mass, electric charge, and angular momentum, are compared between the old and new black hole solutions connected by conformal transformations, and they are found conformally invariant. We speculate that the theorem may be useful in the computations of metric perturbations and spontaneous scalarization of black holes in the Einstein–Maxwell-conformal-scalar theory since they can be mapped to the corresponding EMS theories, which have been investigated in detail.

1 Introduction

As is known to all, it is very difficult to find exact solutions to Einstein's equations in General Relativity, let alone the exact solutions to Einstein's equations with a conformal scalar stress-energy. In 1974, Bekenstein [1] tried to deal with the conformally invariant scalar equation:

$$\nabla^\mu \nabla_\mu \psi - \frac{R\psi}{6} = 0, \quad (1)$$

which is the equation of motion for the scalar field in the specific Einstein–Maxwell-conformally coupled scalar (EMcS) theory:

$$\mathcal{S} = \frac{1}{16\pi} \int d^4x \sqrt{-g} \left(R - F_{\mu\nu} F^{\mu\nu} - \frac{R}{6} \psi^2 - (\nabla\psi)^2 \right). \quad (2)$$

R represents the Ricci scalar and $F_{\mu\nu}$ represents the Maxwell field by convention. We follow the convention that the Greek indices take values 0,..., 3 and Latin indices take values 1,...,3. Using conformal transformations, Bekenstein presented a theorem by means of which one can generate the solution to the theory above from a solution of the Einstein–Maxwell-ordinary scalar theory, wherein the scalar field is minimally coupled to the Maxwell invariant, i.e:

$$\mathcal{S} = \frac{1}{16\pi} \int d^4x \sqrt{-g} \left(R - F_{\mu\nu} F^{\mu\nu} - (\nabla\phi)^2 \right). \quad (3)$$

The solution to the specific EMcS theory recovered the famous Bocharova–Bronnikov–Melnikov–Bekenstein (BBMB) black hole solution [2]. The BBMB solution is the first counter-example to the no-hair theorems (see e.g. Ref. [3,4] for reviews), although it's much debated: the scalar field diverges at the horizon, even though the geometry is regular therein. Furthermore, several criticisms of the BBMB solution have been leveled at the properties of the energy-momentum tensor at the horizon [5].

From then, black solutions with non-trivial scalar fields have been constructed by dropping the assumption that the canonical scalar fields are minimally coupled to Einstein's gravity, which is called scalar-tensor theories of gravity, including Brans–Dicke theory [6], the Horndeski theory [7], the Galileon theory [8] and the generalized Galileon theory [9] (one can refer to [10–12] for explicit examples). Then over two decades ago, in the context of scalar-tensor theories, a phenomenon called spontaneous scalarization of neutron stars has aroused much interest [13]. For neutron star geometries, the non-vanishing trace of the energy-momentum tensor can source a scalar halo around the star, i.e., to scalarize. This phenomenon has now been stud-

^a e-mail: jhqi@nao.cas.cn (corresponding author)

^b e-mail: gaocj@nao.cas.cn

ied in a new form, dubbed geometric spontaneous scalarization. It is found that, in gravitational models where a real scalar field non-minimally couples to the Gauss-Bonnet invariant or its square [14–16], and under certain choices of the coupling function, both the standard (bald) vacuum solutions of General Relativity and new “hairy” black hole (BH) solutions with a scalar field profile are possible. Thus, it circumvents the black hole no-hair theorem. For some range of mass, a Schwarzschild BH becomes unstable and transfers some of its energy to a “cloud” of scalar particles around it. This spontaneous scalarization is triggered by the strong spacetime curvature, which induces non-linear curvature terms in the evolution equations. As a result, the computations are intensive and complicated [17,18].

However, the spontaneous scalarization with dynamical process could be confirmed in a cousin model – Einstein–Maxwell-scalar (EMS) theory [19–21]:

$$\mathcal{S} = \frac{1}{16\pi} \int d^4x \sqrt{-g} (R - 2\partial_\mu \phi \partial^\mu \phi - K(\phi) F_{\mu\nu} F^{\mu\nu}). \quad (4)$$

In this class of models, there is no coupling between the scalar field and the curvature, but a non-minimal coupling between the scalar and the electromagnetic field is present. For certain values of the coupling function, the conventional electrovacuum (scalar-free) Reissner–Nordström (RN) BH solves the equations of motion in EMS models, as does a novel class of BHs that allow for a non-trivial scalarfield configuration (scalar-hair). When the charge-mass ratio of the RN BH is sufficiently high, the RN BH becomes unstable to scalar perturbations, and the formation of these hairy BHs is hypothesized to represent the endpoint of the instability [22–24]. This technically simpler model facilitated the investigation of the domain of solutions’ existence, particularly outside the spherical sector, as well as the performance of completely non-linear dynamical evolutions [19].

While comparing the difference between Eqs. (3) and (4), it’s natural for one to consider whether the Bekenstein’s theorem can be extended from (3) to (4), thus considering the general non-minimal coupling between the scalar field and the Maxwell field. Since so many solutions to EMS theories with various chosen couplings have been constructed and studied as shown above, then it’s interesting to foresee that under conformal transformations, they should also circumvent the no-hair theorem as the BBMB solution did, and overcome the defects of the BBMB solution (we will apply the extended theorem to the Einstein–Maxwell-dilaton theory to illustrate this). It’s easy for one to preliminarily think that applying the conformal transformation to the metric solutions of Eq. (4) will generate solutions of EMcS theories with general non-minimal couplings between the scalar field and Maxwell field. For

example, one can refer to Zou and Myung [25,26] for the investigation of a EMcS theory with a non-minimal coupling:

$$\mathcal{S}_{\text{EMcS}} = \frac{1}{16\pi} \int d^4x \sqrt{-g} \left[R - (1 + \alpha\phi^2) F_{\mu\nu} F^{\mu\nu} - \frac{1}{6} (\phi^2 R + 6\partial_\mu \phi \partial^\mu \phi) \right] \quad (5)$$

where the coupling between the scalar field and the Maxwell field is $1 + \alpha\phi^2$. They have obtained infinite branches of scalarized charged black holes through scalarization in the EMcS theory. These are regarded as charged black holes with scalar hair because they all have a primary scalar which takes a finite value on the horizon. They also consider the radial perturbation of the scalarized black hole in this theory. Indeed, the perturbation of general EMcS theories can revert to the perturbation of corresponding EMS theories by the theorem we will introduce in this paper.

In this paper, we will not be content with extending Bekenstein’s theorem to action (4), we will also include the case of scalar potential, i.e., considering the following action:

$$\mathcal{S} = \frac{1}{16\pi} \int d^4x \sqrt{-g} (R - 2\partial_\mu \phi \partial^\mu \phi - K(\phi) F_{\mu\nu} F^{\mu\nu} - V(\phi)). \quad (6)$$

For example, One can refer to [27–29] to study spontaneous scalarization of asymptotically anti-de Sitter charged black holes in various EMS theories with different $K(\phi)$, where we can all set $V(\phi) = 2\lambda$, or refer to ([24,30]) for scalarized charged black holes with scalar mass term, where we can set $V(\phi) = 2m_\phi^2 \phi^2$. The paper is organized as follows. In Sect. 2, we exhibit two actions where the the metric is conformal to the other, and explore how the solutions of the matter fields between these two actions are connected. The strict proof of the theorem is provided in this section. In Sect. 3, we focus on the Einstein–Maxwell-dilaton theory [31] as an example to interpret the theorem. The conformal solution and corresponding thermodynamics are derived. In Sect. 4, we concentrate on the rotating dilaton solution for a specific coupling between the scalar and Maxwell fields, which is first achieved by the Kaluza–Klein black hole solution [32]. Thanks to this extended theorem, the conformal rotating dilaton solution can be achieved straightforwardly. We investigate the first law of thermodynamics and find that all the quantities are unchanged under conformal transformation. In Sect. 5, inspired by the investigation of the dilaton theory, we prove that for the general EMS theory, all the physical quantities in the first law of thermodynamics are conformally invariant provided that the spacetime is asymptotically flat and the scalar field asymptotically vanishes at infinity. In Sect. 6, we give the conclusion and discussion.

2 Extend Bekenstein's theorem

Since we need to compare the solutions of two different actions repetitiously, we first need to list the two actions. Let us rewrite the Eq. (6),

$$\mathcal{S}_1 = \frac{1}{16\pi} \int d^4x \sqrt{-g} [R - 2\nabla_\mu \phi \nabla^\mu \phi - K_1(\phi) F_{\mu\nu} F^{\mu\nu} - V_1(\phi)]. \quad (7)$$

Variation of \mathcal{S}_1 with respect to $g_{\mu\nu}$ gives

$$G_{\mu\nu}(g_{\mu\nu}) = 2 \{ S_{\mu\nu}[g_{\mu\nu}, \phi] + \tau_{\mu\nu}[g_{\mu\nu}, K_1(\phi), V_1(\phi), F_{\mu\nu}] \}, \quad (8)$$

where

$$S_{\mu\nu}[g_{\mu\nu}, \phi] = -\frac{1}{2} g_{\mu\nu} g^{\alpha\beta} \phi_{,\alpha} \phi_{,\beta} + \phi_{,\mu} \phi_{,\nu}, \quad (9)$$

and

$$\begin{aligned} \tau_{\mu\nu}[g_{\mu\nu}, K_1(\phi), V_1(\phi), F_{\mu\nu}] &= K_1(\phi) \\ &\times \left(F_{\mu\gamma} F_{\nu}{}^{\gamma} - \frac{F^2}{4} g_{\mu\nu} \right) - \frac{1}{4} g_{\mu\nu} V_1(\phi). \end{aligned} \quad (10)$$

On the other hand, the Einstein equations for EMcS theories with general couplings can be derived from the action

$$\mathcal{S}_2 = \frac{1}{16\pi} \int d^4x \sqrt{-\tilde{g}} \left[\tilde{R} - 2\tilde{\nabla}_\mu \psi \tilde{\nabla}^\mu \psi - \frac{\tilde{R}}{3} \psi^2 - K_2(\psi) \tilde{F}_{\mu\nu} \tilde{F}^{\mu\nu} - V_2(\psi) \right], \quad (11)$$

and they read

$$\begin{aligned} G_{\mu\nu}(\tilde{g}_{\mu\nu}) &= 2 [\Theta_{\mu\nu}[\tilde{g}_{\mu\nu}, \psi] \\ &+ \tau_{\mu\nu}[\tilde{g}_{\mu\nu}, K_2(\psi), V_2(\psi), \tilde{F}_{\mu\nu}]] \\ &\times \left(1 - \frac{1}{3} \psi^2 \right)^{-1}, \end{aligned} \quad (12)$$

where

$$\begin{aligned} \Theta_{\mu\nu}[\tilde{g}_{\mu\nu}, \psi] &= S_{\mu\nu}[\tilde{g}_{\mu\nu}, \psi] - \frac{1}{6} \tilde{\nabla}_\mu \tilde{\nabla}_\nu \psi^2 \\ &+ \frac{1}{6} \tilde{g}_{\mu\nu} \tilde{\nabla}_\alpha \tilde{\nabla}^\alpha \psi^2, \end{aligned} \quad (13)$$

$$S_{\mu\nu}[\tilde{g}_{\mu\nu}, \psi] = -\frac{1}{2} \tilde{g}_{\mu\nu} \tilde{g}^{\alpha\beta} \psi_{,\alpha} \psi_{,\beta} + \psi_{,\mu} \psi_{,\nu}, \quad (14)$$

$$\begin{aligned} \tau_{\mu\nu}[\tilde{g}_{\mu\nu}, K_2(\psi), V_2(\psi), \tilde{F}_{\mu\nu}] &= K_2(\psi) \\ &\times \left(\tilde{F}_{\mu\gamma} \tilde{F}_{\nu}{}^{\gamma} - \frac{\tilde{F}^2}{4} \tilde{g}_{\mu\nu} \right) - \frac{1}{4} \tilde{g}_{\mu\nu} V_2(\psi). \end{aligned} \quad (15)$$

Here, we denote the quantities in the action \mathcal{S}_2 with a tilde except those concerning the scalar field — a symbol system similar to what was in Bekenstein's original work. The comma derivative is just a convenient notation for a partial

derivative with respect to one of the coordinates. It should be noted that the equation for the scalar field corresponding to \mathcal{S}_2 , which reads

$$4\tilde{\nabla}_\mu \tilde{\nabla}^\mu \psi - \frac{2\psi \tilde{R}}{3} - \frac{dK_2(\psi)}{d\psi} \tilde{F}^2 - \frac{dV_2(\psi)}{d\psi} = 0, \quad (16)$$

isn't conformally invariant after considering the non-minimal coupling between the Maxwell field and the scalar field. However, the name "conformally coupled" remains in the action due to the inclusion of

$$\int d^4x \sqrt{-\tilde{g}} \left(\frac{\tilde{R}}{6} \psi^2 + (\tilde{\nabla} \psi)^2 \right),$$

as it's the fundamental character of gravity admitting conformally scalar equation, and this convention can also be found in [25, 26]. We now exhibit the theorem that connects the two action \mathcal{S}_1 and \mathcal{S}_2 .

Theorem *If $g_{\mu\nu}, \phi, F_{\mu\nu}$ form a solution of Einstein's equations described by action \mathcal{S}_1 , then $\tilde{g}_{\mu\nu} = \Omega^{-2} g_{\mu\nu}$, $\psi = \sqrt{3} \tanh \frac{\phi}{\sqrt{3}}$ and $\tilde{F}_{\mu\nu} = F_{\mu\nu}$ with $\Omega^{-1} = \cosh \frac{\phi}{\sqrt{3}}$ form a solution to the EMcS theory described by action \mathcal{S}_2 provided that $K_2(\psi) = K_2\left(\sqrt{3} \tanh \frac{\phi}{\sqrt{3}}\right) \equiv K_1(\phi)$, and $\Omega^4 V_1(\phi) = \left(1 - \frac{\psi^2}{3}\right)^2 V_1\left(\sqrt{3} \tanh^{-1} \frac{\psi}{\sqrt{3}}\right) \equiv V_2(\psi)$.*

Proof Under the conformal transformation

$$g_{\mu\nu} = \Omega^2 \tilde{g}_{\mu\nu}, \quad (17)$$

the Einstein tensor transforms as

$$G_{\mu\nu}[\tilde{g}_{\mu\nu}] = G_{\mu\nu}[g_{\mu\nu}] - 6\Omega^{-2} \Theta_{\mu\nu}[\tilde{g}_{\mu\nu}, \Omega]. \quad (18)$$

According to the definition of Eq. (10), and assuming $\tilde{F}_{\mu\nu} = F_{\mu\nu}$, we have

$$\begin{aligned} \tau_{\mu\nu}[g_{\mu\nu}, K_1(\phi), V_1(\phi), F_{\mu\nu}] &= \Omega^{-2} \tau_{\mu\nu} \\ &[\tilde{g}_{\mu\nu}, K_1(\phi), \Omega^4 V_1(\phi), \tilde{F}_{\mu\nu}]. \end{aligned} \quad (19)$$

It follows then from Eq. (9)

$$S_{\mu\nu}[g_{\mu\nu}, \phi] = S_{\mu\nu}[\tilde{g}_{\mu\nu}, \phi] = \left(\frac{\partial \phi}{\partial \psi} \right)^2 S_{\mu\nu}[\tilde{g}_{\mu\nu}, \psi]. \quad (20)$$

If we let Ω satisfies

$$\Omega^2 = 1 - \frac{1}{3} \psi^2, \quad (21)$$

and regard ψ as some unspecified function of ϕ , then Eq. (18) reduces to

$$\begin{aligned} G_{\mu\nu}[\tilde{g}_{\mu\nu}] &= 2 \left\{ \left[\left(\frac{\partial \phi}{\partial \psi} \right)^2 - \left(1 - \frac{\psi^2}{3} \right)^{-2} \frac{\psi^2}{3} \right] S_{\mu\nu}[\tilde{g}_{\mu\nu}, \psi] \right. \\ &\quad \left. + \Omega^{-2} \left[-\frac{1}{6} \tilde{\nabla}_\mu \tilde{\nabla}_\nu \psi^2 + \frac{\tilde{g}_{\mu\nu}}{6} \tilde{\nabla}_\gamma \tilde{\nabla}^\gamma \psi^2 \right] \right\} \end{aligned}$$

$$+\tau_{\mu\nu}\left[\tilde{g}_{\mu\nu}, K_1(\phi), \Omega^4 V_1(\phi), \tilde{F}_{\mu\nu}\right]\Bigg\}, \quad (22)$$

in view of Eqs. (19) and (20).

Let us make the following assumptions:

1. ψ satisfies

$$\left(\frac{\partial\psi}{\partial\phi}\right)^2 = \Omega^4 = \left(1 - \frac{\psi^2}{3}\right)^2, \quad (23)$$

which can be solved by

$$\psi = \sqrt{3} \tanh \frac{\phi}{\sqrt{3}} \quad \text{or} \quad \phi = \sqrt{3} \tanh^{-1} \frac{\psi}{\sqrt{3}}, \quad (24)$$

and thus the conformal factor is

$$\Omega^2 = 1 - \frac{1}{3}\psi^2 = 1 - \tanh^2 \frac{\phi}{\sqrt{3}} = \frac{1}{\cosh^2 \frac{\phi}{\sqrt{3}}}. \quad (25)$$

2. The couplings in \mathcal{S}_1 and \mathcal{S}_2 satisfy the following relations:

$$K_2(\psi) = K_2\left(\sqrt{3} \tanh \frac{\phi}{\sqrt{3}}\right) \equiv K_1(\phi), \quad (26)$$

$$\Omega^4 V_1(\phi) = \left(1 - \frac{\psi^2}{3}\right)^2 V_1\left(\sqrt{3} \tanh^{-1} \frac{\psi}{\sqrt{3}}\right) \equiv V_2(\psi).$$

Then one finds that in Eq. (22), the term $\left[\left(\frac{\partial\phi}{\partial\psi}\right)^2 - \left(1 - \frac{\psi^2}{3}\right)^{-2} \frac{\psi^2}{3}\right] = \frac{1}{\Omega^2}$, and the term $\tau_{\mu\nu}[\tilde{g}_{\mu\nu}, K_1(\phi), \Omega^4 V_1(\phi), \tilde{F}_{\mu\nu}] = \tau_{\mu\nu}[\tilde{g}_{\mu\nu}, K_2(\psi), V_2(\psi), \tilde{F}_{\mu\nu}]$, and thus the Eq. (22) will reduce to Eq. (12), which is exactly the Einstein equation for EMcS theory described by action \mathcal{S}_2 .

Now let's make an examination of whether the new solution is consistent with all the equations of motion. The equations of motion for the Maxwell field is

$$\partial_\mu [\sqrt{-g} K_1(\phi) F^{\mu\nu}] = 0. \quad (27)$$

It's not hard to verify that the quantity in the square bracket remains unchanged under conformal translations. So we have

$$\partial_\mu [\sqrt{-\tilde{g}} K_2(\psi) \tilde{F}^{\mu\nu}] = 0, \quad (28)$$

which is the equation of motion for Maxwell field in action \mathcal{S}_2 .

The last step is to check the scalar field equations in \mathcal{S}_1 and \mathcal{S}_2 , which read

$$4\nabla_\mu \nabla^\mu \phi - \frac{dK_1(\phi)}{d\phi} F^2 - \frac{dV_1(\phi)}{d\phi} = 0, \quad (29)$$

and

$$4\tilde{\nabla}_\mu \tilde{\nabla}^\mu \psi - \frac{2\psi\tilde{R}}{3} - \frac{dK_2(\psi)}{d\psi} \tilde{F}^2 - \frac{dV_2(\psi)}{d\psi} = 0. \quad (30)$$

What we are going to verify is that, basing on the equations of motion which have been previously proved together with the scalar field equation (29), the Eq. (30) is automatically

satisfied. Indeed, the trace of Einstein equation (12) gives $-\tilde{R}\Omega^2 = 2\psi\tilde{\nabla}_\mu \tilde{\nabla}^\mu \psi - 2V_2(\psi)$. Hence the left hand of side for Eq. (30) turns out to be

$$\begin{aligned} & 4\tilde{\nabla}_\mu \tilde{\nabla}^\mu \psi + \frac{2\psi}{3} \left(\frac{2\psi\tilde{\nabla}_\mu \tilde{\nabla}^\mu \psi}{\Omega^2} \right) - \frac{4\psi V_2(\psi)}{3\Omega^2} \\ & - \frac{dK_2(\psi)}{d\psi} \tilde{F}^2 - \frac{dV_2(\psi)}{d\psi} \\ & = 4\tilde{\nabla}_\mu \tilde{\nabla}^\mu \psi \left(1 + \frac{\psi^2}{3\Omega^2} \right) - \frac{4\psi V_2(\psi)}{3\Omega^2} \\ & - \frac{dK_2(\psi)}{d\psi} \tilde{F}^2 - \frac{dV_2(\psi)}{d\psi} \\ & = \frac{4\tilde{\nabla}_\mu \tilde{\nabla}^\mu \psi}{\Omega^2} - \frac{4\psi V_2(\psi)}{3\Omega^2} - \frac{dK_2(\psi)}{d\psi} \tilde{F}^2 - \frac{dV_2(\psi)}{d\psi}. \end{aligned} \quad (31)$$

On the other hand, we have

$$\begin{aligned} \nabla_\mu \nabla^\mu \phi &= \frac{1}{\sqrt{-g}} (\sqrt{-g} g^{\beta\alpha} \phi_{,\alpha})_{,\beta} \\ &= \frac{1}{\Omega^4 \sqrt{-\tilde{g}}} (\sqrt{-\tilde{g}} \tilde{g}^{\beta\alpha} \psi_{,\alpha})_{,\beta} \\ &= \frac{1}{\Omega^4} \tilde{\nabla}_\mu \tilde{\nabla}^\mu \psi \end{aligned} \quad (32)$$

and

$$\begin{aligned} \frac{dK_1(\phi)}{d\phi} F^2 &= \frac{dK_2(\psi)}{d\psi} \frac{d\psi}{d\phi} \frac{\tilde{F}^2}{\Omega^4} = \frac{dK_2(\psi)}{d\psi} \frac{\tilde{F}^2}{\Omega^2}, \\ \frac{dV_1(\phi)}{d\phi} &= \frac{d(\Omega^{-4} V_2(\psi))}{d\psi} \frac{d\psi}{d\phi} = \frac{d(\Omega^{-4} V_2(\psi))}{d\psi} \Omega^2 \\ &= \Omega^{-2} \frac{dV_2(\psi)}{d\psi} + \frac{4\psi V_2(\psi)}{3\Omega^4}. \end{aligned} \quad (33)$$

Substituting Eqs. (32) and (33) into Eq. (29), we obtain

$$\frac{4\tilde{\nabla}_\mu \tilde{\nabla}^\mu \psi}{\Omega^2} - \frac{4\psi V_2(\psi)}{3\Omega^2} - \frac{dK_2(\psi)}{d\psi} \tilde{F}^2 - \frac{dV_2(\psi)}{d\psi} = 0, \quad (34)$$

which is exactly the last line of Eq. (31). Therefore, Eq. (30) is satisfied. \square

We note that in the case of $K_1(\phi) = 1$ and $V_1(\phi) = 0$, the theorem above reduces to Bekenstein's theorem. In the next two sections, as applications of our theorem, we shall take the static dilaton black hole solution and the rotating dilaton black hole solution as seed solutions to generate two new solutions. As is known to all, it is usually very hard to obtain a rotating black hole solution in General Relativity. However, by using this theorem, one can straightforwardly get a rotating black hole solution in the conformal dilaton theory via conformal transformations.

3 Generating new static dilaton black hole solution

The action of the Einstein–Maxwell–dilaton theory is given by

$$S = \frac{1}{16\pi} \int d^4x \sqrt{-g} \left[R - 2(\nabla\phi)^2 - e^{-2\alpha\phi} F^2 \right]. \quad (35)$$

Now we apply our theorem to this theory which has $K_1(\phi) = e^{-2\alpha\phi}$ and $V_1(\phi) = 0$. The corresponding spherically symmetric black hole solution takes the form [31]

$$ds^2 = -\lambda^2 dt^2 + \frac{dr^2}{\lambda^2} + R^2 d\Omega_2^2, \quad (36)$$

$$e^{2\phi} = \left(1 - \frac{r_-}{r}\right)^{\frac{2\alpha}{1+\alpha^2}}, \quad F_{tr} = \frac{Q}{r^2}, \quad (37)$$

with

$$\begin{aligned} \lambda^2 &= \left(1 - \frac{r_+}{r}\right) \left(1 - \frac{r_-}{r}\right)^{\frac{1-\alpha^2}{1+\alpha^2}}, \\ R &= r \left(1 - \frac{r_-}{r}\right)^{\frac{\alpha^2}{1+\alpha^2}}, \\ d\Omega_2^2 &= d\theta^2 + \sin^2(\theta) d\phi^2. \end{aligned} \quad (38)$$

The physical mass M , charge Q are related to r_- and r_+ as follows

$$M = \frac{r_+}{2} + \left(\frac{1-\alpha^2}{1+\alpha^2}\right) \frac{r_-}{2}, \quad Q = \left(\frac{r_+ r_-}{1+\alpha^2}\right)^{\frac{1}{2}}. \quad (39)$$

According to the theorem, the scalar field ψ in corresponding S_2 is

$$\psi = \sqrt{3} \tanh \frac{\phi}{\sqrt{3}} = \sqrt{3} \frac{\left(1 - \frac{r_-}{r}\right)^{\frac{2\alpha}{\sqrt{3}(1+\alpha^2)}} - 1}{\left(1 - \frac{r_-}{r}\right)^{\frac{2\alpha}{\sqrt{3}(1+\alpha^2)}} + 1}, \quad (40)$$

while the Maxwell field remains unchanged. It is simple to recognize that ψ takes a limit value at the horizon $r = r_+$ from the Eq. (40). A more general conclusion could be drawn from Eq. (24), which shows that ψ takes a limit value as long as ϕ does. And for the dilaton solution, Eq. (37) shows that the scalar field ϕ does take a limit value at the horizon $r = r_+$.

We have the corresponding coupling

$$\begin{aligned} K_2(\psi) &= e^{-2\alpha\sqrt{3}\tanh^{-1}(\psi/\sqrt{3})} = \left(\frac{3 + \sqrt{3}\psi}{3 - \sqrt{3}\psi}\right)^{-\sqrt{3}\alpha}, \\ V_2(\psi) &= 0, \end{aligned} \quad (41)$$

and the conformal factor

$$\begin{aligned} \Omega^2 &= 1 - \frac{\psi^2}{3} \\ &= \left[\frac{2}{\left(1 - \frac{r_-}{r}\right)^{\frac{\alpha}{\sqrt{3}(1+\alpha^2)}} + \left(1 - \frac{r_-}{r}\right)^{\frac{-\alpha}{\sqrt{3}(1+\alpha^2)}}} \right]^2. \end{aligned} \quad (42)$$

It is straightforward to prove that Ω approaches unit one when r approaches $+\infty$. The black hole solution corresponding to S_2 is then

$$d\tilde{s}^2 = \frac{1}{\Omega^2} \left(-\lambda^2 dt^2 + \frac{dr^2}{\lambda^2} + R^2 d\Omega_2^2 \right). \quad (43)$$

Since we have $\Omega^2 \rightarrow 1$ in the spacial infinity, the spacetime is also asymptotically flat.

Now we turn to the thermodynamics of the old (Eq. (36)) and new black hole (Eq. (43)) solutions, which are connected by conformal transformation. The energy can be calculated for the new black hole by the Komar Mass. If one denotes the timelike vector by \mathbf{k} , which is normalized at spatial infinity, the Komar mass is then given by

$$M_K := -\frac{1}{8\pi} \oint_{\mathcal{S}_t} \nabla^\mu k^\nu dS_{\mu\nu}. \quad (44)$$

The integration is performed on a closed 2-surface \mathcal{S}_t in Σ_t . Here Σ_t provides a 3 + 1 foliation $(\Sigma_t)_{t \in \mathbb{R}}$ of the spacetime. The line element of the surface \mathcal{S}_t is

$$dS_{\mu\nu} = (s_\mu n_\nu - n_\mu s_\nu) \sqrt{q} d^2y, \quad (45)$$

where \mathbf{n} is the unit timelike vector normal to Σ_t , \mathbf{s} is the unit vector normal to \mathcal{S}_t within Σ_t which is oriented towards the exterior of \mathcal{S}_t . $y^a = (y^1, y^2)$ are coordinates spanning \mathcal{S}_t . q is defined as $q := \det(q_{ab})$ and q_{ab} are the components of the metric induced by g on \mathcal{S}_t . In order to calculate the Komar mass (and the angular momentum in the next sections), we closely follow the detailed procedures given by Gourgoulhon [33]. The timelike vector $\partial_t = \mathbf{k}$ of the spacetime can be decomposed using the lapse vector $N\mathbf{n}$ and shift vector $\boldsymbol{\beta}$

$$\mathbf{k} = N\mathbf{n} + \boldsymbol{\beta}. \quad (46)$$

Then the Komar mass (44) takes the form

$$M_K = \frac{1}{4\pi} \oint_{\mathcal{S}_t} \left(s^i D_i N - K_{ij} s^i \beta^j \right) \sqrt{q} d^2y. \quad (47)$$

K_{ij} the extrinsic curvature tensor of Σ_t , and D_i is the covariant derivative operator which is compatible with the induced metric on Σ_t . The components in the integrand can be easily expressed with the metric $g_{\mu\nu}$ according to [33]. The shift vector $\beta^i = -g^{0i}/g^{00}$ vanishes for a static spacetime. Thus, only the first term in the integrand makes a contribution to the integration. The unit vector s^i normal to \mathcal{S}_t is

$$s^i = \left(\sqrt{1/g_{11}}, 0, 0 \right), \quad (48)$$

and $N = \sqrt{-1/g^{00}}$, $\sqrt{q} = \sqrt{g_{22}g_{33}}$. To calculate the Komar mass \tilde{M} for the corresponding S_2 , we should replace the metric in the integrand with $\tilde{g}_{\mu\nu}$. Therefore, we obtain

$$\tilde{s}^1 = \sqrt{\Omega^2 \lambda^2} \sim 1 + O(1/r), \quad (49)$$

$$\partial_r \tilde{N} = \partial_r \sqrt{\frac{\lambda^2}{\Omega^2}} \sim \left[\frac{r_+}{2} + \left(\frac{1-\alpha^2}{1+\alpha^2} \right) \frac{r_-}{2} \right] / r^2 + O(1/r^3), \quad (50)$$

$$\sqrt{q} = R^2 / \Omega^2 \sin \theta \sim r^2 \sin \theta + O(r). \quad (51)$$

The Komar mass is

$$\begin{aligned} \tilde{M} &= \frac{1}{4\pi} \oint_{\mathcal{S}_t} \left[\frac{r_+}{2} + \left(\frac{1-\alpha^2}{1+\alpha^2} \right) \frac{r_-}{2} \right] \sin \theta d\theta d\varphi \\ &= \frac{r_+}{2} + \left(\frac{1-\alpha^2}{1+\alpha^2} \right) \frac{r_-}{2} = M, \end{aligned} \quad (52)$$

namely the mass is invariant.

Let us turn to the other physical quantities corresponding to \mathcal{S}_1 in this case. The electric potential Φ conjugated to Q is given by $\Phi = A_0(r_+) = \frac{Q}{r_+}$. The temperature T is

$$\begin{aligned} T &= \frac{1}{4\pi} \frac{-\partial_r g_{00}}{\sqrt{-g_{00}g_{11}}} \Big|_{r_+} = \frac{1}{4\pi} (\lambda^2)' \Big|_{r_+} \\ &= \frac{(1-r_-/r_+)^{\frac{1-\alpha^2}{1+\alpha^2}}}{4\pi r_+}. \end{aligned} \quad (53)$$

Temperature's conjugate variable can be regraded as the Wald entropy [34,35]. Given the Lagrangian density \mathcal{L} , the Wald entropy is

$$S = 2\pi \int_{\mathcal{S}_t} \frac{\delta \mathcal{L}}{\delta R_{\mu\nu\alpha\beta}} \epsilon_{\mu\alpha} \epsilon_{\nu\beta} \sqrt{q} d\Omega_2^2, \quad (54)$$

where $\epsilon^{\mu\nu}$ is the binormal vector to the bifurcation surface \mathcal{S}_t , q is the determinant of induced metric on \mathcal{S}_t . The variation of the action with respect to $R_{\mu\nu\alpha\beta}$ is to be carried out by regarding the Riemann tensor $R_{\mu\nu\alpha\beta}$ as formally independent on the metric $g_{\mu\nu}$. In the static spacetime, \mathcal{S}_t has two normal directions along r and t . We can construct an antisymmetric 2-tensor $\epsilon_{\mu\nu}$ along these directions so that $\epsilon_{rt} = \epsilon_{tr} = -1$. For the action \mathcal{S}_1 we only need to concentrate on the scalar curvature

$$\begin{aligned} \mathcal{L} &= \frac{1}{16\pi} R_{\mu\nu\alpha\beta} g^{\nu\alpha} g^{\mu\beta}, \\ \frac{\partial \mathcal{L}}{\partial R_{\mu\nu\alpha\beta}} &= \frac{1}{16\pi} \frac{1}{2} (g^{\mu\alpha} g^{\nu\beta} - g^{\nu\alpha} g^{\mu\beta}). \end{aligned} \quad (55)$$

Then the Wald entropy yields

$$\begin{aligned} S &= \frac{1}{8} \int_{\mathcal{S}_t} \frac{1}{2} (g^{\mu\alpha} g^{\nu\beta} - g^{\nu\alpha} g^{\mu\beta}) (\epsilon_{\mu\nu} \epsilon_{\alpha\beta}) \sqrt{q} d\Omega_2^2 \\ &= \frac{1}{4} \int_{\mathcal{S}_t} \sqrt{q} d\Omega_2^2 = \frac{A_H}{4} = r_+^2 \left(1 - \frac{r_-}{r_+} \right)^{\frac{2\alpha^2}{1+\alpha^2}}, \end{aligned} \quad (56)$$

which is equal to the Bekenstein–Hawking entropy [36] as expected. Armed with these expressions, one can construct the first law of thermodynamics and the Smarr formula

$$dM = TdS + \Phi dQ, \quad M = 2TS + \Phi Q. \quad (57)$$

Now let us discuss the thermodynamic quantities of the conformal version \mathcal{S}_2 . The temperature of the black hole (43) is

$$\begin{aligned} \tilde{T} &= \frac{1}{4\pi} \frac{\left(\frac{\lambda^2}{\Omega^2} \right)'}{\sqrt{\frac{\lambda^2}{\Omega^2} \cdot \frac{1}{\Omega^2 \lambda^2}}} \Big|_{r_+} = \frac{1}{4\pi} \frac{\left(\frac{\lambda^2}{\Omega^2} \right)' + \left(\frac{1}{\Omega^2} \right)' \lambda^2}{\frac{1}{\Omega^2}} \Big|_{r_+} \\ &= \frac{1}{4\pi} (\lambda^2)' \Big|_{r_+} = T. \end{aligned} \quad (58)$$

The penultimate equation holds because $\lambda^2|_{r_+} = 0$. Thus, the temperature remains unchanged after the conformal transformation. In the calculation of the corresponding Wald entropy in \mathcal{S}_2 , one need only consider the term $\frac{1-\psi^2/3}{16\pi} R_{\mu\nu\alpha\beta} g^{\nu\alpha} g^{\mu\beta}$. Compared to Eq. (55), there is only one more factor: $1-\psi^2/3$. Therefore, the Wald entropy \tilde{S} in the corresponding \mathcal{S}_2 is

$$\begin{aligned} \tilde{S} &= \frac{1 - \frac{1}{3}\psi^2}{4} \int_{\mathcal{S}_t} \sqrt{\tilde{h}} d\Omega_2^2 = \frac{\Omega^2 \tilde{A}_H}{4} \\ &= r_+^2 \left(1 - \frac{r_-}{r_+} \right)^{\frac{2\alpha^2}{1+\alpha^2}} = S. \end{aligned} \quad (59)$$

Namely, the Wald entropy remains unchanged. We should emphasize that the entropy here is not the Bekenstein–Hawking entropy $\tilde{A}_H/4$, but the Wald entropy $\Omega^2 \tilde{A}_H/4$ (or equivalently, $(1 - \frac{\psi^2}{3}) \tilde{A}_H/4$). It is only the Wald entropy that plays the role of the Noether charge that can be applied in the first law of thermodynamics. For more instances of theories involving a conformally coupled scalar field, one can refer to [37,38].

Since the Faraday tensor is invariant as requested by the theorem $\tilde{F}_{\mu\nu} = F_{\mu\nu}$, we have $\tilde{A}_\mu = A_\mu$. Therefore, the conjugate variable to charge \tilde{Q} , i.e., $\tilde{\Phi} = \tilde{A}_0(r_+)$ is equal to $\Phi = A_0(r_+)$. The electric charge \tilde{Q} is determined by

$$\tilde{Q} \equiv -\frac{1}{4\pi} \int_{\mathcal{S}_t} K_2(\psi)^* F d\Omega_2^2 \quad (60)$$

where $*F$ denotes the dual form of Faraday tensor. Note that $Q \equiv -\frac{1}{4\pi} \int_{\mathcal{S}_t} K_1(\phi)^* F d\Omega_2^2$ and $K_1(\phi) = K_2(\psi)$, thus, we have $Q = \tilde{Q}$.

In conclusion, both the old and new black holes have identical thermodynamics. Concretely, they have the same Komar masses $\tilde{M} = M$, electric charges $\tilde{Q} = Q$, electric potentials $\tilde{\Phi} = \Phi$, temperature $\tilde{T} = T$ and Wald entropy $\tilde{S} = S$ (but different Bekenstein–Hawking entropy). We underline that, while having identical physical quantities in the first rule of thermodynamics, $dM = TdS + \Phi dQ$, they are fundamentally distinct spacetimes.

4 Generating new rotating dilaton black hole

A rotating dilaton black hole solution with a special coupling constant $\alpha = \sqrt{3}$ is found by Frolov et al. [32]. For this value of α , the action is simply the Kaluza–Klein action, which is obtained by dimensionally reduction of five dimensional vacuum Einstein action. Applying our theorem, we can derive a new rotating black hole solution.

The metric for a rotating dilaton black hole with $\alpha = \sqrt{3}$ is given by [31].

$$ds^2 = -\frac{1-Z}{B}dt^2 - \frac{2aZ\sin^2\theta}{B\sqrt{1-v^2}}dtd\varphi + B\frac{\Sigma}{\Delta_0}dr^2 + B\Sigma d\theta^2 + \left[B(r^2 + a^2) + a^2\sin^2\theta\frac{Z}{B} \right] \sin^2\theta d\varphi^2, \quad (61)$$

where

$$B = \sqrt{1 + \frac{v^2 Z}{1-v^2}}, \quad Z = \frac{2mr}{\Sigma}, \quad (62)$$

$$\Delta_0 = r^2 + a^2 - 2mr,$$

$$\Sigma = r^2 + a^2 \cos^2\theta$$

Here, m and a are related to the mass M , angular momentum J and charge Q of the black hole by

$$M = m \left[1 + \frac{v^2}{2(1-v^2)} \right], \quad Q = \frac{mv}{1-v^2},$$

$$J = \frac{ma}{\sqrt{1-v^2}}. \quad (63)$$

The constant v is the velocity of the boost. The vector potential and the dilaton field are

$$A_t = \frac{v}{2(1-v^2)} \frac{Z}{B^2},$$

$$A_\varphi = -a\sin^2\theta \frac{v}{2\sqrt{1-v^2}} \frac{Z}{B^2}, \quad (64)$$

$$\phi = -\frac{\sqrt{3}}{2} \log B$$

The theorem tells us $K_2(\psi) = \left(\frac{3+\sqrt{3}\psi}{3-\sqrt{3}\psi} \right)^{-3}$ and the conformal factor is

$$\Omega^2 = \left(\frac{2}{e^{\phi/\sqrt{3}} + e^{-\phi/\sqrt{3}}} \right)^2 = \left(\frac{2}{\frac{1}{\sqrt{B}} + \sqrt{B}} \right)^2. \quad (65)$$

So the line element for the black hole gives

$$d\tilde{s}^2 = \frac{1}{\Omega^2} \left\{ -\frac{1-Z}{B}dt^2 - \frac{2aZ\sin^2\theta}{B\sqrt{1-v^2}}dtd\varphi + B\frac{\Sigma}{\Delta_0}dr^2 + B\Sigma d\theta^2 + \left[B(r^2 + a^2) + a^2\sin^2\theta\frac{Z}{B} \right] \sin^2\theta d\varphi^2 \right\}$$

$$+ \left\{ B(r^2 + a^2) + a^2\sin^2\theta\frac{Z}{B} \right\} \sin^2\theta d\varphi^2 + B\frac{\Sigma}{\Delta_0}dr^2 + B\Sigma d\theta^2 \quad (66)$$

The scalar field is

$$\psi = \sqrt{3} \tanh \frac{\phi}{\sqrt{3}} = \sqrt{3} \frac{e^{\frac{2\phi}{\sqrt{3}}} - 1}{e^{\frac{2\phi}{\sqrt{3}}} + 1} = \sqrt{3} \frac{1-B}{1+B}. \quad (67)$$

To compute the Komar mass \tilde{M} of the rotating black hole corresponding to S_2 , it is necessary to account for the extrinsic curvature term in (47). It is obvious that the components of shift vector $\beta^i = -g^{0i}/g^{00}$ are conformal invariant. Taking into consideration the “3 + 1” foliation associated with the standard Boyer Lindquist coordinates (t, r, θ, φ) , the non-vanishing component of β^i is β^φ (or β^3). Therefore, we only need to concentrate on $K_{r\varphi} s^r \beta^\varphi$ considering that $s^i = (s^r, 0, 0)$ (see Eq. (48)). The extrinsic curvature $K_{r\varphi}$ is evaluated via $2N K_{ij} = \mathcal{L}_\beta \gamma_{ij}$ in the case of $\partial \gamma_{ij} / \partial t = 0$, with γ_{ij} the induced metric on the space. Then we have

$$K_{r\varphi} = \frac{1}{2N} \mathcal{L}_\beta \gamma_{r\varphi}$$

$$= \frac{1}{2N} \left(\beta^\varphi \underbrace{\frac{\partial \gamma_{r\varphi}}{\partial \varphi}}_0 + \gamma_{\varphi\varphi} \frac{\partial \beta^\varphi}{\partial r} + \gamma_{r\varphi} \underbrace{\frac{\partial \beta^\varphi}{\partial \varphi}}_0 \right) = \frac{1}{2N} \gamma_{\varphi\varphi} \frac{\partial \beta^\varphi}{\partial r}. \quad (68)$$

Therefore, and recalling that the components in the integrand can be express with the metric components: $s^r = \sqrt{1/g_{11}}$, $N = \sqrt{-1/g^{00}}$, $\gamma_{\varphi\varphi} = g_{33}$, $\beta^\varphi = -g^{03}/g^{00}$, $\sqrt{q} = \sqrt{g_{22}g_{33}}$; we have

$$K_{r\varphi} s^r \beta^\varphi \sqrt{q} = \frac{s^r}{2N} \gamma_{\varphi\varphi} \frac{\partial \beta^\varphi}{\partial r} \beta^\varphi \sqrt{q}$$

$$= \frac{1}{4} \sqrt{\frac{-g^{00}}{g_{11}}} g_{33} \sqrt{g_{22}g_{33}} \frac{\partial (g_{00}^{03})^2}{\partial r}. \quad (69)$$

In order to calculate the mass \tilde{M} of corresponding S_2 in this situation, we must insert the metric $\tilde{g}_{\mu\nu}$ into the aforementioned equation. In the limit of $r \rightarrow \infty$, we get $\sqrt{\frac{-\tilde{g}^{00}}{\tilde{g}_{11}}} \sim 1$, $\tilde{g}_{33} \sim r^2 \sin \theta$, $\sqrt{\tilde{g}_{22}\tilde{g}_{33}} \sim r^2 \sin \theta$, $\frac{\partial (\frac{\tilde{g}^{03}}{\tilde{g}^{00}})^2}{\partial r} \sim \frac{24a^2 m^2}{(-1+v^2)r^7}$. Therefore, we have $\tilde{K}_{r\varphi} \tilde{s}^r \tilde{\beta}^\varphi \sqrt{\tilde{q}} \sim O(1/r^3)$ which does not contribute to the integral in the definition of Komar mass. Then, considering the term $\tilde{s}^i \tilde{D}_i \tilde{N} \sqrt{\tilde{q}}$ again, we have

$$\tilde{s}^r \sim 1, \quad \partial_r \tilde{N} \sim \frac{m(2-v^2)}{2(1-v^2)r^2}, \quad \sqrt{\tilde{q}} \sim r^2 \sin \theta. \quad (70)$$

As a result, the Komar mass for the rotating black hole (66) according to Eq. (47) is

$$\tilde{M} = \frac{1}{4\pi} \oint_{\mathcal{S}_t} \frac{m(2-v^2)}{2(1-v^2)} \sin\theta d\theta d\varphi = \frac{m(2-v^2)}{2(1-v^2)} = M. \quad (71)$$

As for the electric charge \tilde{Q} and the Wald entropy \tilde{S} , they also remain unchanged under the conformal transformation, i.e. $\tilde{Q} = Q$, $\tilde{S} = S$. In order to calculate the temperature of this rotating black hole, one can refer to [39,40] where it shows the temperature is given by

$$T_H = \lim_{r \rightarrow r_H} \frac{\partial_r \sqrt{-g_{tt} - 2g_{t\varphi}\Omega_H - g_{\phi\phi}\Omega_H^2}}{2\pi \sqrt{g_{rr}}}, \quad (72)$$

for the four-dimensional rotating black hole. Here $\Omega_H = -\frac{g_{t\varphi}}{g_{\phi\phi}}|_{r \rightarrow r_H}$ is the conjugate variable to angular momentum of the black hole — angular velocity of the black hole. It is very obvious that the angular velocity of the black hole is not changed under conformal transformation $g_{\mu\nu} \rightarrow \frac{1}{\Omega^2} g_{\mu\nu}$. For the metric, there exists the the Killing field $\chi = \partial_t + \Omega_H \partial_\varphi$, which is normal to the horizon. If we denote $-\chi^\mu \chi_\mu = -g_{tt} - 2g_{t\varphi}\Omega_H - g_{\phi\phi}\Omega_H^2$ by X , then we get $X = 0$ on the horizon since the horizon is a null surface. Then, under conformal transformation $g_{\mu\nu} \rightarrow \frac{1}{\Omega^2} g_{\mu\nu}$, we arrive at

$$\begin{aligned} \frac{\partial_r \sqrt{X}}{2\pi \sqrt{g_{rr}}} &\rightarrow \frac{\partial_r \sqrt{\frac{X}{\Omega^2}}}{2\pi \sqrt{\frac{g_{rr}}{\Omega^2}}} \\ &= \frac{\partial_r \sqrt{X}}{2\pi \sqrt{g_{rr}}} + \frac{\partial_r \left(\frac{1}{\Omega^2}\right) X}{2\pi \sqrt{\frac{g_{rr}}{\Omega^2}}} = \frac{\partial_r \sqrt{X}}{2\pi \sqrt{g_{rr}}}. \end{aligned} \quad (73)$$

Then it follows that $T = \tilde{T}$ from Eqs. (72) and (73). The electric potential is defined by

$$\Phi = A_\mu \chi^\mu|_{r=r_H} - A_\mu \chi^\mu|_{r \rightarrow \infty}, \quad (74)$$

where A_μ denotes the vector potential. Since $\Omega_H = \tilde{\Omega}_H$ we achieve $\chi_\mu = \tilde{\chi}_\mu$. Taking $A_\mu = \tilde{A}_\mu$ into account, we obtain $\Phi = \tilde{\Phi}$.

Turning to angular momentum for (66), we have the definition of Komar angular momentum [33,41]

$$J_K := \frac{1}{16\pi} \oint_{\mathcal{S}_t} \nabla^\mu \phi^\nu dS_{\mu\nu} \quad (75)$$

where ϕ^ν is the Killing vector, and \mathcal{S}_t share the same meaning as in the definition of Komar mass in Sect. 3. It is natural to choose a foliation adapted to the axisymmetric in the sense that the Killing vector ϕ is tangent to the hypersurface \mathcal{S}_t . Then we have $\mathbf{n} \cdot \phi = 0$ and the integrand in (75) is

$$\begin{aligned} \nabla^\mu \phi^\nu dS_{\mu\nu} &= \nabla_\mu \phi_\nu (s^\mu n^\nu - n^\mu s^\nu) \sqrt{q} d^2y \\ &= 2\nabla_\mu \phi_\nu s^\mu n^\nu \sqrt{q} d^2y \end{aligned}$$

$$= -2s^\mu \phi_\nu \nabla_\mu n^\nu \sqrt{q} d^2y = 2K_{ij} s^i \phi^j \sqrt{q} d^2y. \quad (76)$$

So Eq. (75) becomes

$$J_K = \frac{1}{8\pi} \oint_{\mathcal{S}_t} K_{ij} s^i \phi^j \sqrt{q} d^2y. \quad (77)$$

Let us use the “3 + 1” foliation associated with the standard Boyer-Lindquist coordinates (t, r, θ, φ) and evaluate the integral (77) by choosing for \mathcal{S}_t as a sphere $r = \text{const}$. Then, we have $y^a = (\theta, \varphi)$. The Boyer-Lindquist components of ϕ are $\phi^i = (0, 0, 1)$ and those of s are $s^i = (s^r, 0, 0)$ because γ_{ij} is diagonal in this coordinate system. Thus Eq. (77) reduces to

$$J_K = \frac{1}{8\pi} \oint_{r=\text{const}} K_{r\varphi} s^r \sqrt{q} d\theta d\varphi. \quad (78)$$

Recalling the Eq. (68) for extrinsic curvature $K_{r\varphi}$, one needs to read off the components of the shift vector for the rotating conformal dilaton black hole

$$(\tilde{\beta}^r, \tilde{\beta}^\theta, \tilde{\beta}^\varphi) = \left(0, 0, \frac{-az}{\sqrt{1-v^2} [B^2(r^2+a^2) + za^2 \sin^2 \theta]}\right). \quad (79)$$

Keeping in mind the expression for extrinsic curvature (68), we find the Komar angular momentum is

$$J_K = \frac{1}{16\pi} \oint_{r=\text{const}} \frac{s^r}{N} \gamma_{\varphi\varphi} \frac{\partial \beta^\varphi}{\partial r} \sqrt{q} d\theta d\varphi. \quad (80)$$

For the new rotating dilaton black hole, we must replace the metric $\tilde{g}_{\mu\nu}$ into the integrand, which results in

$$\begin{aligned} \frac{\tilde{s}^r}{\tilde{N}} \tilde{\gamma}_{\varphi\varphi} \sqrt{\tilde{q}} &= \sqrt{\frac{-\tilde{g}^{00}}{\tilde{g}_{11}}} \tilde{g}_{33} \sqrt{\tilde{g}_{22} \tilde{g}_{33}} = \frac{1}{\Omega^2} \sqrt{\frac{-g^{00}}{g_{11}}} g_{33} \sqrt{g_{22} g_{33}} \\ &= \frac{1}{\Omega^2} \frac{s^r}{N} \gamma_{\varphi\varphi} \sqrt{q}, \quad \tilde{\beta}^i = -\tilde{g}^{0i} / \tilde{g}^{00} = \beta^i. \end{aligned} \quad (81)$$

In the limit of $r \rightarrow \infty$, the following relationships hold: $s^r \sim 1 + O(1/r)$, $\frac{1}{N} \sim 1 + O(1/r)$, $\gamma_{\varphi\varphi} \sim r^2 \sin^2 \theta + O(r)$, $\sqrt{q} \sim r^2 \sin \theta + O(r)$, $\frac{\partial \beta^\varphi}{\partial r} \sim \frac{6am}{\sqrt{1-v^2} r^4} + O(1/r^5)$. Therefore, we arrive at $\frac{s^r}{N} \gamma_{\varphi\varphi} \frac{\partial \beta^\varphi}{\partial r} \sqrt{q} \sim \frac{6am}{\sqrt{1-v^2}} \sin^3 \theta$. Replace it with the metric $\tilde{g}_{\mu\nu}$, and we get that $\frac{\tilde{s}^r}{\tilde{N}} \tilde{\gamma}_{\varphi\varphi} \frac{\partial \tilde{\beta}^\varphi}{\partial r} \sqrt{\tilde{q}} \sim \frac{6am}{\sqrt{1-v^2}} \sin^3 \theta$ because of $\frac{1}{\Omega^2} \sim 1 + O(1/r^2)$. Thus, the Komar angular momentum for (61) is

$$\tilde{J} = J = \frac{1}{16\pi} \int \frac{6am}{\sqrt{1-v^2}} \sin^3 \theta d\theta d\varphi = \frac{ma}{\sqrt{1-v^2}}. \quad (82)$$

All physical quantities of the rotating black holes in the first law of thermodynamics $dM = TdS + \Omega_H dJ + \Phi dQ$ are not changed after conformal transformation, just as they are in the case of static black holes. In this sense, they are twin solutions, yet they are fundamentally distinct.

5 Conformal invariance of physical quantities when the scalar potential vanishes

So far, we have found that physical quantities in the first law of thermodynamics remain unchanged under the conformal transformation for dilaton black hole solutions. By examining the aforementioned procedures for calculating quantities, one can observe that, according to the definition, quantities such as temperature T , entropy S , angular velocity of black hole Ω_H , electric potential Φ and electric charge Q do not change under the conformal transformations $\Omega^{-1} = \cosh \frac{\phi}{\sqrt{3}}$ regardless of choices of $K_1(\phi)$. The remaining quantities are mass M and angular momentum J , which raises the question of whether they are also invariant under general conformal transformations $\Omega^{-1} = \cosh \frac{\phi}{\sqrt{3}}$ corresponding to EMS theories beyond the dilaton theory? We will investigate the problem for the case of $V(\phi) = 0$ to address the case of asymptotical black holes, so that Komar mass and angular momentum can be well defined.

Indeed, one may deduce from the definition of angular momentum (80), that the term $\frac{s^r}{N} \gamma_{\phi\phi} \frac{\partial \beta^\phi}{\partial r} \sqrt{q}$ is multiplied by $\frac{1}{\Omega^2}$ under conformal transformations. Therefore, to guarantee the convergence of the integration, $\frac{s^r}{N} \gamma_{\phi\phi} \frac{\partial \beta^\phi}{\partial r} \sqrt{q}$ should converge to a constant a_1 multiplied by $\sin^3 \theta$, i.e. $\frac{s^r}{N} \gamma_{\phi\phi} \frac{\partial \beta^\phi}{\partial r} \sqrt{q} \sim a_1 \sin^3 \theta + O(1/r)$. Then the associated angular momentum is

$$J = \frac{1}{16\pi} \int a_1 \sin^3 \theta d\theta d\varphi = \frac{a_1}{6}. \quad (83)$$

If the scalar field ϕ vanishes at infinity, we obtain $\Omega^{-1} = \cosh(0) = 1$ at space infinity. Therefore, according to (81), we obtain $\frac{s^r}{N} \gamma_{\phi\phi} \frac{\partial \beta^\phi}{\partial r} \sqrt{q} \sim a_1 \sin^3 \theta + O(1/r)$, indicating that the black hole's angular momentum is conformally invariant $\tilde{J} = J$. By returning to the definition of Komar mass (47) and concentrating first on the second term, i.e. (69), it is trivial to demonstrate that under the conformal transformation, we have

$$\tilde{K}_{r\phi} \tilde{s}^r \tilde{\beta}^\phi \sqrt{\tilde{q}} = \frac{1}{\Omega^2} K_{r\phi} s^r \beta^\phi \sqrt{q}. \quad (84)$$

Recalling the computation in Sect. 4, the first term of the integrand in Eq. (47) yields

$$s^i D_i N \sqrt{q} = \sqrt{\frac{g_{22} g_{33}}{g_{11}}} \partial_r \sqrt{-\frac{1}{g^{00}}}, \quad (85)$$

and it is straightforward to demonstrate that under conformal transformations, we have

$$\tilde{s}^i \tilde{D}_i \tilde{N} \sqrt{\tilde{q}} = \frac{1}{\Omega^2} s^i D_i N \sqrt{q} + \partial_r \left(\frac{1}{2\Omega^2} \right) s^r N \sqrt{q}. \quad (86)$$

If it is shown that the second term $\partial_r \left(\frac{1}{2\Omega^2} \right) s^r N \sqrt{q}$ makes no contribution to the integral used to define Komar mass, then

combining Eqs. (84) and (86) yields

$$\left(\tilde{s}^i \tilde{D}_i \tilde{N} - \tilde{K}_{ij} s^i \tilde{\beta}^j \right) \sqrt{\tilde{q}} = \frac{1}{\Omega^2} \left(s^i D_i N - K_{ij} s^i \beta^j \right) \sqrt{q}. \quad (87)$$

In this case, considering that we have $\frac{1}{\Omega^2} \sim 1$ at space infinity, the Komar mass (see Eq. (47)) stays unaltered $M = \tilde{M}$ for the same reason as for conformal invariance of angular momentum (see discussion concerning the angular momentum above). Therefore, the last challenge is to demonstrate that $\partial_r \left(\frac{1}{2\Omega^2} \right) s^r N \sqrt{q}$ genuinely does not make contributions in the integral. Assuming that at space infinity, the scalar field ϕ vanishes and can be expanded into the series in the following form:

$$\phi = c_1/r + c_2/r^2 + \dots, \quad (88)$$

then the conformal transformations can be expanded into

$$\frac{1}{\Omega^2} = \cosh^2 \left(\frac{\phi}{\sqrt{3}} \right) = 1 + \frac{c_1^2}{3r^2} + O(1/r^3), \quad (89)$$

resulting in $\partial_r \left(\frac{1}{2\Omega^2} \right) \sim -\frac{c_1^2}{3r^3} + O(1/r^4)$. If the solution of the black hole for \mathcal{S}_1 is also asymptotically flat, we obtain at space infinity $s^r = \sqrt{1/g_{11}} \sim 1$, $N = \sqrt{-1/g^{00}} \sim 1$, $\sqrt{q} \sim r^2$. Thus, when all of the criteria above are satisfied, we get $\partial_r \left(\frac{1}{2\Omega^2} \right) s^r N \sqrt{q} \sim O(1/r)$, which doesn't contribute to the integral used to define Komar mass. As a result, all the physical quantities in the first law of thermodynamics $dM = TdS + \Phi dQ + \Omega_H dJ$ are conformally invariant considering the Komar mass is conformally invariant.

Why is it necessary that ϕ should be expanded into the Taylor series at space infinity? The following counterexample can illustrate this point. If ϕ equals $1/\sqrt{r}$, it cannot be expanded to Taylor series. In this instance, the conformal transformation may be expanded into

$$\frac{1}{\Omega^2} = 1 + \frac{1}{3r} + O(1/r^2). \quad (90)$$

This yields $\partial_r \left(\frac{1}{2\Omega^2} \right) s^r N \sqrt{q} \sim -\frac{1}{6}$, which contributes to the integral in the definition of Komar mass. Consequently, $M = \tilde{M}$ cannot be established in this circumstance.

To summarize, In the case of $V(\phi) \equiv 0$, the following criteria must be fulfilled in order for all physical quantities in the first law of thermodynamics to remain conformally invariant regardless of the choices of $K_1(\phi)$

1. The spacetime is asymptotically flat so that quantities such as Komar mass and angular momentum can be well defined;
2. At spatial infinity, the scalar field should vanish and be expanded into the Taylor series: $\phi = c_1/r + c_2/r^2 + \dots$

One may check that both the static dilaton black hole solution and the precise rotating dilaton black hole solution satisfy the two criteria above, ensuring that the prior analysis of these two solutions is self-consistent.

6 Conclusion and discussion

Originally, Bekenstein attempted to obtain exact solutions of conformally invariant scalar equations in Einstein's frame. He established the theorem by which the Einstein-conformal-scalar solution can be deduced from the initial ordinary solution. We take it a step further by taking into account the non-minimal coupling between the scalar and Maxwell fields when a scalar potential is also included, which has generated considerable attention as a consequence of studies on the scalarization of black holes.

The paper starts with the extension of Bekenstein's theorem to EMS theories. Then, using the dilaton theory as an illustration of the extended theorem, the conformal dilaton solution is found straightforwardly. Concerning the exact spinning dilaton black hole, its conformal counterpart is obtained and studied. We examine the thermodynamics of the two solutions. We find that the Bekenstein–Hawking entropy formula no longer applies in the conformal theory and should be replaced with the Wald entropy formulas, since the geometry quantity is $(1 - \frac{\psi^2}{3})\tilde{R}$ instead of the Ricci scalar \tilde{R} . A thorough study shows that the solutions in the dilaton theory and their conformal counterparts analogues possess the same thermodynamic quantities, and therefore they are cousin models.

After examining these two specific theories, we find that if one can establish that the mass M and angular momentum J are conformally invariant for any conformal factor $\Omega^{-1} = \cosh \frac{\phi}{\sqrt{3}}$ regardless of choices of $K_1(\phi)$, then all the quantities in the first law of thermodynamics $dM = TdS + \Omega_H dJ + \Phi dQ$ are conformally invariant. It is therefore found that when $V(\phi) = 0$, if the following conditions are satisfied: (i) the spacetime is asymptotically flat; (ii) the solution for scalar field in \mathcal{S}_1 vanishes at spatial infinity and can be extended into the Taylor series: $\phi = c_1/r + c_2/r^2 + \dots$, then the physical quantities in the first law of thermodynamics are always conformally invariant.

In terms of potential applications of the extended theorem, one might start with the spontaneous scalarization of black holes. The following three instances has been found to allow for spontaneous scalarization of charged black holes. [19–21]: (i). exponential coupling, $K_1(\phi) = e^{-\alpha\phi^2}$, $V_1(\phi) = 0$, (ii). power-law coupling, $K_1(\phi) = 1 - \alpha\phi^2$, $V_1(\phi) = 0$; (iii). fractional coupling, $K_1(\phi) = \frac{1}{1+\alpha\phi^2}$, $V_1(\phi) = 0$; Therefore, it is intriguing to examine the scalarization of charged black holes' conformal counterparts. Another possi-

ble application of the extended theorem is to study black hole perturbations in EMcS theories. The theorem establishes that the solutions of \mathcal{S}_1 and \mathcal{S}_2 are connected via conformal translation regardless of whether they are static, stationary, or even non-stationary. Considering that the perturbations of black holes are always added to background spacetime and the backreactions are neglected, thus the perturbation in generic EMcS theories can be converted into the investigation of the perturbation in EMS theories. Zou et al. [25,26] examined the stability problem of scalar hairy black holes in the EMcS theory with the quadratic coupling $K_2(\psi) = 1 + \alpha\psi^2$, $V_2(\psi) = 0$. Indeed, one may perform the stability analysis by using the conformal transformation on the initial solutions in the EMS theory first, and then the perturbation of the EMcS is accomplished by multiplying the perturbation in the EMS theory by the conformal factor, which is determined by the unperturbed solution of the EMS theory. Furthermore, Blázquez-Salcedo et al. [42] find that, the EMS theory with the higher power-law coupling $K_1(\phi) = 1 - \alpha\phi^4$, $V_1(\phi) = 0$ exhibits an intriguing two-branch space of scalar hair solutions that coexists with the conventional Reissner–Nordstrom black hole. We think this character may then be readily extended to its conformal counterpart. For the case of $V(\phi) \neq 0$, one can take the dilaton black holes in de Sitter or Anti-de Sitter universe as an example [43], where $K(\phi) = e^{-2\phi}$ and $V(\phi) = \frac{4}{3}\lambda + \frac{1}{3}\lambda(e^{2\phi} + e^{-2\phi})$. It's interesting to research the corresponding EMcS theory. Finally, one can also apply the theorem to EMS theories with $V(\phi) = 2\lambda$ [27–29] or $V(\phi) = 2m_\phi^2\phi^2$ [24,30] and investigate the spontaneous scalarization in the corresponding EMcS theories.

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