



Article



Four-Dimensional Spaces of Complex Numbers and Unitary States of Two-Qubit Quantum Systems

Mars B. Gabbassov, Tolybay Z. Kuanov, Turganbay K. Yermagambetov and Berik I. Tuleuov



Article

Four-Dimensional Spaces of Complex Numbers and Unitary States of Two-Qubit Quantum Systems

Mars B. Gabbassov ^{1,2,*} , Tolybay Z. Kuanov ^{1,2}, Turganbay K. Yermagambetov ³ and Berik I. Tuleuov ^{1,*} ¹ U. A. Dzholdasbekov Institute of Mechanics and Engineering, Almaty 050010, Kazakhstan; tolybay@mail.ru² “Factor” Systems Research Company, Astana 010000, Kazakhstan³ Department of Physics and Mathematics, K. Zhubanov Aktobe Regional University, Aktobe 030000, Kazakhstan; turg@mail.ru

* Correspondence: gmarson19@gmail.com (M.B.G.); berik_t@yahoo.com (B.I.T.)

Abstract

The pure states of two-qubit quantum systems are described by a four-dimensional vector of complex numbers, and unitary operators transferring a two-qubit quantum system from one state to another have the form of a 4×4 matrix with complex elements. This fact brings to mind the idea of studying the spaces of four-dimensional numbers with complex components. Moreover, the results obtained by the authors for four-dimensional numbers with real components inspire some optimism. In this paper we construct four-dimensional spaces of complex numbers by analogy with four-dimensional spaces of real numbers. Each four-dimensional number is mapped to a matrix formed from its components and it is proved that the constructed mapping is a bijection and a homomorphism. In the space of four-dimensional numbers of the eight basis elements, half are real and half are imaginary. The presence of such symmetry distinguishes these spaces from the space of quaternions, in which one basis element is real and the rest are imaginary. The symmetry of the basis numbers makes these spaces a natural generalization of one-dimensional and two-dimensional (complex) algebra. The conditions under which the corresponding matrices are gates for two-qubit quantum systems are defined. The notion of a unitary state of a two-qubit quantum system is introduced, to which various gates from commutative groups of gates correspond. It is shown that any gate of a unitary state transforms a unitary state into a unitary state and a non-unitary state into a non-unitary state. Almost all gates used in the construction of quantum circuits, in particular H, SWAP, CX, CY, and CZ, have the same properties. The problem of searching for a gate that transfers a quantum system from one unitary state to another unitary state has been solved. Thus, with the help of four-dimensional spaces of complex numbers it was possible to construct whole classes of two-qubit gates, which opens new possibilities for the construction of quantum algorithms. The results obtained have important theoretical and practical implications for quantum computing.

Keywords: four-dimensional mathematics; four-dimensional numbers; four-dimensional complex numbers; two-qubit quantum systems; quantum states; two-qubit gates; quantum computing; abstract algebra



Academic Editors: Sergei Odintsov and Boris Malomed

Received: 6 July 2025

Revised: 11 September 2025

Accepted: 11 October 2025

Published: 22 October 2025

Citation: Gabbassov, M.B.; Kuanov, T.Z.; Yermagambetov, T.K.; Tuleuov, B.I. Four-Dimensional Spaces of Complex Numbers and Unitary States of Two-Qubit Quantum Systems. *Symmetry* **2025**, *17*, 1789. <https://doi.org/10.3390/sym17111789>

Copyright: © 2025 by the authors.

Licensee MDPI, Basel, Switzerland.

This article is an open access article distributed under the terms and conditions of the Creative Commons Attribution (CC BY) license (<https://creativecommons.org/licenses/by/4.0/>).

1. Introduction

The pure states of two-qubit quantum systems are described by a four-dimensional vector of complex numbers, and the unitary operators that transfer a two-qubit quantum

system from one state to another have the form of a 4×4 matrix with complex elements. This fact brings to mind the idea of studying the spaces of four-dimensional numbers with complex components. Moreover, the results, obtained by the authors for four-dimensional numbers with real components [1], inspire some optimism.

In this paper, the quantum states of two-qubit quantum systems and the corresponding gates, which transfer a two-qubit quantum system from one state to another, are considered. In this paper, the theory of anisotropic spaces of four-dimensional numbers with complex components is developed to describe the states of two-qubit quantum systems, and the theory of matrix spaces for four-dimensional numbers is developed to describe two-qubit gates. A total of 64 commutative groups of gates for two-qubit quantum systems are obtained and their explicit descriptions are given. The notion of a unitary state of a two-qubit quantum system is introduced, to which different gates from commutative groups of gates correspond. It is shown that any gate of a unitary state transforms a unitary state into a unitary state and a non-unitary state into a non-unitary state. Almost all gates used in the construction of quantum circuits, in particular, H, SWAP, CX, CY, and CZ, have the same properties. We also provide an algorithm for transferring a two-qubit quantum system from one unitary state to another unitary state in one step. It follows from this algorithm that quantum algorithms must contain a sequence of gates that belong to different commutative groups of gates described in this paper.

A two-qubit quantum system consists of two qubits and has four basis states in the form of the following ket vectors:

$$|00\rangle = (1, 0, 0, 0)^T, |01\rangle = (0, 1, 0, 0)^T, |10\rangle = (0, 0, 1, 0)^T, |11\rangle = (0, 0, 0, 1)^T,$$

where the index T stands for the transpose sign. Then an arbitrary state of the two-qubit quantum system can be written in the form [2,3]

$$|\varphi\rangle = x_1|00\rangle + x_2|01\rangle + x_3|10\rangle + x_4|11\rangle,$$

where $x_k = \mathbf{Re}x_k + i\mathbf{Im}x_k = x_k^{re} + ix_k^{im} \in \mathbb{C}, k = 1, 2, 3, 4$, are complex numbers satisfying the condition $\sum_{k=1}^4 |x_k|^2 = 1$. In other notations the arbitrary state of a two-qubit quantum system can be represented as

$$|\varphi\rangle = x_1 \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} + x_2 \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} + x_3 \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} + x_4 \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix},$$

that is, any state of the two-qubit quantum system is uniquely determined by complex amplitudes x_1, x_2, x_3 , and x_4 . Thus, an arbitrary state of a two-qubit quantum system is a four-dimensional number with complex components. In this paper we consider only pure states of two-qubit quantum systems.

A two-qubit quantum system is a minimal system in which quantum mechanical effects, such as entanglement, separability, coherence, quantum teleportation, etc., begin to appear. One important aspect is the visualization of two-qubit and three-qubit quantum systems on Bloch spheres [4–6]. We note the book by Nikitin et al. [7], which sets out the axiomatics of quantum mechanics. Works devoted to the development of new classes of gates for two-qubit quantum systems are virtually absent. This is not surprising, since this requires the development of a new mathematical apparatus, which is the apparatus of multidimensional mathematics, the foundations of which were laid by the famous Kazakh mathematician M. M. Abenov [8,9].

In a recent paper by the authors [1], anisotropic spaces of four-dimensional numbers with real components and the corresponding spaces of 4×4 matrices whose elements are formed from the coordinates of four-dimensional numbers were considered. All spaces of four-dimensional numbers with associative and commutative addition and multiplication were obtained. There were eight such spaces, and in six of them, a spectral norm was defined, that is, these spaces are normed spaces. A natural generalization of these results is the study of anisotropic four-dimensional spaces with complex components with associative and commutative multiplication and the corresponding spaces of matrices. Interest in this generalization is also supported by the fact that the states of a two-qubit quantum system are described by a vector of four complex numbers, and the gates for two-qubit quantum systems are a 4×4 matrix with complex elements.

Based on the developed theory of anisotropic spaces of four-dimensional numbers with complex components, all states of two-qubit quantum systems are divided into the following two categories: unitary and non-unitary states, with the structure of unitary states described explicitly. In this regard, the further study of the entanglement, separability, and coherence of unitary states is of interest. Modern research on two-qubit quantum systems is devoted specifically to the study of these quantum effects. In [10–16] and many other works, various aspects of the entanglement and separability of quantum states of two-qubit quantum systems are investigated. Of interest is the relationship between entangled and unitary states of two-qubit quantum systems. Other aspects of quantum systems, such as coherence and teleportation, presented in [17–21], can also be considered from the point of view of unitary states.

The paper has the following structure. Section 2 constructs spaces of four-dimensional numbers with complex components in general form. The operations of addition, subtraction, multiplication, and division are introduced, the operations of addition and multiplication being associative and commutative. In Section 3, the basis spaces of four-dimensional numbers with associative and commutative multiplication are distinguished. In Section 4, we construct matrix spaces corresponding to the spaces of four-dimensional numbers and construct bijective mappings between these spaces, which are homomorphisms by addition and multiplication. In Section 5 we define the conditions under which the constructed matrices are gates of two-qubit quantum systems. Section 6 introduces the concept of a unitary state of a two-qubit quantum system and studies its properties. Section 7 discusses some further development prospects of the proposed theory. The last section provides a proof of Lemma 1.

2. Spaces of Four-Dimensional Numbers with Complex Components

We will write any complex number x in the form $x = \mathbf{Re}x_k + i\mathbf{Im}x_k = x^{re} + x^{im}i$, where x^{re} and x^{im} are the real and imaginary parts of the complex number x , $i^2 = -1$. Non-standard notation for the real and imaginary parts of a complex number is adopted to avoid cumbersome expressions, to save space.

Let us denote by $X = (x_1, x_2, x_3, x_4)$, $Y = (y_1, y_2, y_3, y_4)$ four-dimensional numbers with complex components. Then the sum $X + Y$ of four-dimensional numbers X and Y is called a four-dimensional number

$$X + Y = (x_1 + y_1, x_2 + y_2, x_3 + y_3, x_4 + y_4).$$

The introduced addition operation is, as is easy to see, an associative and commutative operation. The difference of two four-dimensional numbers X and Y is called a four-dimensional number

$$X - Y = (x_1 - y_1, x_2 - y_2, x_3 - y_3, x_4 - y_4).$$

Now we will define the general form of multiplication, which will be an associative and commutative operation.

The associative and commutative multiplication of four-dimensional numbers with real components was first introduced in the work of M. Abenov [8] for one of the four-dimensional spaces. Furthermore, the general form of associative and commutative multiplication of four-dimensional numbers with real components is described in [1,9]. Particular spaces of four-dimensional numbers with real components are studied in [22,23]. The associative and commutative multiplication of four-dimensional numbers with complex components is considered for the first time.

Our further goal is to define a multiplication operation of four-dimensional numbers with complex components which will be associative and commutative. To this end, for given four-dimensional numbers with complex components X and Y , we write the general definition of the multiplication $Z = XY$ as

$$\begin{cases} z_1 = a_{11}x_1y_1 + a_{12}x_2y_2 + a_{13}x_3y_3 + a_{14}x_4y_4 \\ z_2 = a_{21}x_2y_1 + a_{22}x_1y_2 + a_{23}x_4y_3 + a_{24}x_3y_4 \\ z_3 = a_{31}x_3y_1 + a_{32}x_4y_2 + a_{33}x_1y_3 + a_{34}x_2y_4 \\ z_4 = a_{41}x_4y_1 + a_{42}x_3y_2 + a_{43}x_2y_3 + a_{44}x_1y_4 \end{cases}, \tag{1}$$

where $z_k = z_k^{re} + z_k^{im}i \in \mathbb{C}$, $k = 1, 2, 3, 4$ and the complex elements $a_{kl} = a_{kl}^{re} + a_{kl}^{im}i$, $k, l = 1, 2, 3, 4$ of the matrix $A = (a_{kl})_{k,l=1}^4$ must be defined so that the multiplication (1) becomes associative and commutative. The associativity and commutativity conditions impose certain restrictions on the elements of the matrix A .

Lemma 1. *The multiplication operation defined in (1) is associative and commutative if and only if*

$$a_{11} = a_{21} = a_{22} = a_{31} = a_{33} = a_{41} = a_{44} = \alpha, \tag{2}$$

$$a_{23} = a_{24} = \beta, a_{232} = a_{34} = \gamma, a_{42} = a_{43} = \delta, \tag{3}$$

$$a_{12} = \frac{\gamma\delta}{\alpha}, a_{13} = \frac{\beta\delta}{\alpha}, a_{14} = \frac{\beta\gamma}{\alpha}, \tag{4}$$

where $\alpha = \alpha_1 + \alpha_2i$, $\beta = \beta_1 + \beta_2i$, $\gamma = \gamma_1 + \gamma_2i$, $\delta = \delta_1 + \delta_2i$ are arbitrary non-zero complex numbers.

The proof of the lemma is given in Appendix A.

According to this lemma, the general form of associative and commutative multiplication of four-dimensional numbers with complex components has the form (A5).

Let us call the number $X^{re} = (x_1^{re} + x_1^{im}i, 0 + 0i, 0 + 0i, 0 + 0i)$ a quasi-real number. If $x_1^{im} = 0$, then the quasi-real number is called a real number. Multiply an arbitrary four-dimensional number $Y = (y_1^{re} + y_1^{im}i, y_2^{re} + y_2^{im}i, y_3^{re} + y_3^{im}i, y_4^{re} + y_4^{im}i)$ by a quasi-real number X^{re} as follows:

$$X^{re}Y = (\alpha_1 + \alpha_2i)(x_1^{re} + x_1^{im}i)Y.$$

Let us take a real number $(-1, 0, 0, 0)$ as X^{re} . Then $(-1, 0, 0, 0)Y = -(\alpha_1 + \alpha_2i)Y$. For consistency of addition and multiplication operations, we must require $X - Y = X + (-1, 0, 0, 0)Y$, whence it follows that $\alpha_1 + \alpha_2i = 1 + 0i$, or $\alpha_1 = 1, \alpha_2 = 0$. Thus, according to Formula (A5), the final form of the associative and commutative multiplication of four-dimensional numbers with complex components is written as $Z = (z_1^{re} + z_1^{im}i, z_2^{re} + z_2^{im}i, z_3^{re} + z_3^{im}i, z_4^{re} + z_4^{im}i) = (z_1, z_2, z_3, z_4) = XY$, where

$$\begin{cases} z_1 = x_1y_1 + \gamma\delta x_2y_2 + \beta\delta x_3y_3 + \beta\gamma x_4y_4 \\ z_2 = x_2y_1 + x_1y_2 + \beta x_4y_3 + \beta x_3y_4 \\ z_3 = x_3y_1 + \gamma x_4y_2 + x_1y_3 + \gamma x_2y_4 \\ z_4 = x_4y_1 + \delta x_3y_2 + \delta x_2y_3 + x_1y_4 \end{cases} \quad (5)$$

As can be seen from the course of our reasoning, other associative and commutative products of four-dimensional numbers with complex components do not exist. Thus, each point $(\beta_1, \beta_2, \gamma_1, \gamma_2, \delta_1, \delta_2) \in R^6$ of the six-dimensional space defines a particular multiplication that has the property of associativity and commutativity, as well as being consistent with the operation of addition. The obtained definition of multiplication (5) is exactly the same as the definition of the multiplication of four-dimensional numbers with real components obtained in [1,5].

The following eight numbers are called basis numbers: $J_1^1 = (1, 0, 0, 0)$, $J_1^2 = (i, 0, 0, 0)$, $J_2^1 = (0, 1, 0, 0)$, $J_2^2 = (0, i, 0, 0)$, $J_3^1 = (0, 0, 1, 0)$, $J_3^2 = (0, 0, i, 0)$, $J_4^1 = (0, 0, 0, 1)$, $J_4^2 = (0, 0, 0, i)$. Then any four-dimensional number with complex components can be represented as an expansion over the basis numbers

$$X = x_1^{re} J_1^1 + x_1^{im} J_1^2 + x_2^{re} J_2^1 + x_2^{im} J_2^2 + x_3^{re} J_3^1 + x_3^{im} J_3^2 + x_4^{re} J_4^1 + x_4^{im} J_4^2. \quad (6)$$

Let us construct the table of multiplication of basis numbers (Table 1).

Table 1. Multiplications of basis numbers.

	J_1^1	J_1^2	J_2^1	J_2^2	J_3^1	J_3^2	J_4^1	J_4^2
J_1^1	J_1^1	J_1^2	J_2^1	J_2^2	J_3^1	J_3^2	J_4^1	J_4^2
J_1^2	J_1^2	$-J_1^1$	J_2^2	$-J_2^1$	J_3^2	$-J_3^1$	J_4^2	$-J_4^1$
J_2^1	J_2^1	J_2^2	$\gamma\delta J_1^1$	$\gamma\delta J_1^2$	δJ_4^1	δJ_4^2	γJ_3^1	γJ_3^2
J_2^2	J_2^2	$-J_2^1$	$\gamma\delta J_1^2$	$-\gamma\delta J_1^1$	δJ_4^2	$-\delta J_4^1$	γJ_3^2	$-\gamma J_3^1$
J_3^1	J_3^1	J_3^2	δJ_4^1	δJ_4^2	$\beta\delta J_1^1$	$\beta\delta J_1^2$	βJ_2^1	βJ_2^2
J_3^2	J_3^2	$-J_3^1$	δJ_4^2	$-\delta J_4^1$	$\beta\delta J_1^2$	$-\beta\delta J_1^1$	βJ_2^2	$-\beta J_2^1$
J_4^1	J_4^1	J_4^2	γJ_3^1	γJ_3^2	βJ_2^1	βJ_2^2	$\beta\gamma J_1^1$	$\beta\gamma J_1^2$
J_4^2	J_4^2	$-J_4^1$	γJ_3^2	$-\gamma J_3^1$	βJ_2^2	$-\beta J_2^1$	$\beta\gamma J_1^2$	$-\beta\gamma J_1^1$

Let $x = (x_1, x_2, x_3, x_4)$ be a four-dimensional number with complex components. Consider along with it the following numbers:

$$x^{(2)} = (x_1, x_2, -x_3, -x_4) = \sum_{k=1}^4 (-1)^{\lfloor \frac{k}{3} \rfloor} (x_k^{re} J_k^1 + x_k^{im} J_k^2),$$

$$x^{(3)} = (x_1, -x_2, x_3, -x_4) = \sum_{k=1}^4 (-1)^{k-1} (x_k^{re} J_k^1 + x_k^{im} J_k^2),$$

$$x^{(4)} = (x_1, -x_2, -x_3, x_4) = \sum_{k=1}^4 (-1)^{\lfloor \frac{k}{2} \rfloor} (x_k^{re} J_k^1 + x_k^{im} J_k^2),$$

where square brackets mean the integer part of the number. Let us calculate the product of $x \cdot x^{(2)} \cdot x^{(3)} \cdot x^{(4)}$

$$\begin{aligned}
x \cdot x^{(2)} &= \left(x_1^2 + \gamma \delta x_2^2 - \beta \delta x_3^2 - \beta \gamma x_4^2, \quad 2x_1x_2 - 2\beta x_3x_4, \quad 0, \quad 0 \right), \\
x \cdot x^{(2)} \cdot x^{(3)} &= \left(x_1^3 - \gamma \delta x_1x_2^2 - \beta \delta x_1x_3^2 - \beta \gamma x_1x_4^2 + 2\beta \gamma \delta x_2x_3x_4, \right. \\
&\quad x_1^2x_2 - \gamma \delta x_2^3 + \beta \delta x_2x_3^2 + \beta \gamma x_2x_4^2 - 2\beta x_1x_3x_4, \\
&\quad x_1^2x_3 + \gamma \delta x_2^2x_3 - \beta \delta x_3^3 + \beta \gamma x_3x_4^2 - 2\gamma x_1x_2x_4, \\
&\quad \left. 2\delta x_1x_2x_3 - \beta \delta x_3^2x_4 - x_1^2x_4 - \gamma \delta x_2^2x_4 + \beta \gamma x_4^3 \right), \\
x \cdot x^{(2)} \cdot x^{(3)} \cdot x^{(4)} &= \left(x_1^4 + \gamma^2 \delta^2 x_2^4 + \beta^2 \delta^2 x_3^4 + \beta^2 \gamma^2 x_4^4 - 2\gamma \delta x_1^2 x_2^2 - 2\beta \delta x_1^2 x_3^2 \right. \\
&\quad - 2\beta \gamma x_1^2 x_4^2 - 2\beta \gamma \delta^2 x_2^2 x_3^2 - 2\beta \gamma^2 \delta x_2^2 x_4^2 - 2\beta^2 \gamma \delta x_3^2 x_4^2 + 8\beta \gamma \delta x_1x_2x_3x_4, \\
&\quad \left. 0, \quad 0, \quad 0 \right).
\end{aligned}$$

Thus, $x \cdot x^{(2)} \cdot x^{(3)} \cdot x^{(4)}$ is a quasi-real number.

Definition 1. A number $x^* = x^{(2)} \cdot x^{(3)} \cdot x^{(4)}$ is called the conjugate number to the number x .

Then $x \cdot x^* = A^{re}(X)J_1^1 + A^{im}(X)J_1^2$, where

$$\begin{aligned}
A(X) &= x_1^4 + \gamma^2 \delta^2 x_2^4 + \beta^2 \delta^2 x_3^4 + \beta^2 \gamma^2 x_4^4 - 2\gamma \delta x_1^2 x_2^2 - 2\beta \delta x_1^2 x_3^2 \\
&\quad - 2\beta \gamma x_1^2 x_4^2 - 2\beta \gamma \delta^2 x_2^2 x_3^2 - 2\beta \gamma^2 \delta x_2^2 x_4^2 - 2\beta^2 \gamma \delta x_3^2 x_4^2 + 8\beta \gamma \delta x_1x_2x_3x_4.
\end{aligned} \quad (7)$$

The real part of number A consists of 352 summands, and the imaginary part consists of 348 summands, so we do not include them here. Let us call the modulus of the number $A = A^{re} + A^{im}i$ the symplectic modulus of the number $X = (x_1, x_2, x_3, x_4)$.

By direct calculation we find the conjugate number $x^* = (x_1^*, x_2^*, x_3^*, x_4^*)$ to the four-dimensional number $x = (x_1, x_2, x_3, x_4)$

$$\begin{aligned}
x_1^* &= x_1 \left(x_1^2 - \gamma \delta x_2^2 - \beta \delta x_3^2 - \beta \gamma x_4^2 \right) + 2\beta \gamma \delta x_2x_3x_4, \\
x_2^* &= x_2 \left(-x_1^2 + \gamma \delta x_2^2 - \beta \delta x_3^2 - \beta \gamma x_4^2 \right) + 2\beta x_1x_3x_4, \\
x_3^* &= x_3 \left(-x_1^2 - \gamma \delta x_2^2 + \beta \delta x_3^2 - \beta \gamma x_4^2 \right) + 2\gamma x_1x_2x_4, \\
x_4^* &= x_4 \left(-x_1^2 - \gamma \delta x_2^2 - \beta \delta x_3^2 + \beta \gamma x_4^2 \right) + 2\delta x_1x_2x_3.
\end{aligned}$$

Accordingly, the conjugate numbers to the basis numbers are of the following form:

$$J_1^{1*} = J_1^1, \quad J_1^{2*} = -J_1^2, \quad J_2^{1*} = \gamma \delta J_2^1, \quad J_2^{2*} = -\gamma \delta J_2^2, \quad J_3^{1*} = \beta \delta J_3^1, \quad J_3^{2*} = -\beta \delta J_3^2, \quad J_4^{1*} = \beta \gamma J_4^1, \quad J_4^{2*} = -\beta \gamma J_4^2.$$

Let the symplectic modulus of the four-dimensional number $X = (x_1, x_2, x_3, x_4)$ be different from zero. Then there is a single number x^{-1} , called the inverse of x , such that $x \cdot x^{-1} = x^{-1} \cdot x = J_1^1$. Multiplying both parts of this equality by x^* yields $x^{-1} \cdot x \cdot x^* = x^*$ or $(A, 0, 0, 0)x^{-1} = x^*$. Multiplying both parts of this equality by the number $\left(\frac{\bar{A}}{|A|^2}, 0, 0, 0 \right)$, then, given that $\left(\frac{\bar{A}}{|A|^2}, 0, 0, 0 \right) \cdot (A, 0, 0, 0) = J_1^1$, we get $x^{-1} = x^* \cdot \left(\frac{\bar{A}}{|A|^2}, 0, 0, 0 \right)$, where \bar{A} is a conjugate number to A . That is,

$$X^{-1} = \left(\frac{\bar{A}}{|A|^2} x_1^*, \frac{\bar{A}}{|A|^2} x_2^*, \frac{\bar{A}}{|A|^2} x_3^*, \frac{\bar{A}}{|A|^2} x_4^* \right) \quad (8)$$

Then we define the operation of division of four-dimensional numbers as $\frac{y}{x} = y \cdot x^{-1} = \frac{\bar{A}}{|A|^2} x^* y$, if the symplectic modulus of a four-dimensional number x is different from zero.

Thus, we have defined the operations of addition, subtraction, multiplication, and division of four-dimensional numbers with complex components. Moreover, the number of multiplications is infinitely many for each value of the triple $(\beta, \gamma, \delta) \in \mathbb{C}^3$. Next, let us construct basis spaces of four-dimensional numbers in which the multiplication operation is defined concretely and all possible multiplications can be obtained from these basis multiplications. In this way, we will construct some number of basis spaces of four-dimensional numbers with specific addition and multiplication operations that have the associativity and commutativity properties.

3. Basis Spaces of Four-Dimensional Numbers

We want to obtain some finite basis on which all possible products of the form (5) are decomposed. For this purpose, similarly to the case of four-dimensional numbers with real components, we simplify this definition by getting rid of arbitrary numbers $\beta_1, \beta_2, \gamma_1, \gamma_2, \delta_1, \delta_2$. Let us make the following substitution of variables:

$$\begin{cases} \tilde{x}_1^{re} + \tilde{x}_1^{im}i = x_1^{re} + x_1^{im}i \\ \tilde{x}_2^{re} + \tilde{x}_2^{im}i = \sqrt{(\gamma_1 + \gamma_2i)(\delta_1 + \delta_2i)}(x_2^{re} + x_2^{im}i) \\ \tilde{x}_3^{re} + \tilde{x}_3^{im}i = \sqrt{(\beta_1 + \beta_2i)(\delta_1 + \delta_2i)}(x_3^{re} + x_3^{im}i) \\ \tilde{x}_4^{re} + \tilde{x}_4^{im}i = \sqrt{(\beta_1 + \beta_2i)(\gamma_1 + \gamma_2i)}(x_4^{re} + x_4^{im}i) \end{cases} \quad (9)$$

The inverse transformation has the form

$$\begin{cases} x_1^{re} + x_1^{im}i = \tilde{x}_1^{re} + \tilde{x}_1^{im}i \\ x_2^{re} + x_2^{im}i = \frac{1}{\sqrt{(\gamma_1 + \gamma_2i)(\delta_1 + \delta_2i)}}(\tilde{x}_2^{re} + \tilde{x}_2^{im}i) \\ x_3^{re} + x_3^{im}i = \frac{1}{\sqrt{(\beta_1 + \beta_2i)(\delta_1 + \delta_2i)}}(\tilde{x}_3^{re} + \tilde{x}_3^{im}i) \\ x_4^{re} + x_4^{im}i = \frac{1}{\sqrt{(\beta_1 + \beta_2i)(\gamma_1 + \gamma_2i)}}(\tilde{x}_4^{re} + \tilde{x}_4^{im}i) \end{cases} \quad .$$

Let us take the product of $z = x \cdot y$ and go to the transformed space with a wave. Then

$$\begin{aligned} Y &= \left(\tilde{y}_1^{re} + \tilde{y}_1^{im}i, \frac{1}{\sqrt{(\gamma_1 + \gamma_2i)(\delta_1 + \delta_2i)}}(\tilde{y}_2^{re} + \tilde{y}_2^{im}i), \right. \\ &\quad \left. \frac{1}{\sqrt{(\beta_1 + \beta_2i)(\delta_1 + \delta_2i)}}(\tilde{y}_3^{re} + \tilde{y}_3^{im}i), \frac{1}{\sqrt{(\beta_1 + \beta_2i)(\gamma_1 + \gamma_2i)}}(\tilde{y}_4^{re} + \tilde{y}_4^{im}i) \right), \\ Z &= \left(\tilde{z}_1^{re} + \tilde{z}_1^{im}i, \frac{1}{\sqrt{(\gamma_1 + \gamma_2i)(\delta_1 + \delta_2i)}}(\tilde{z}_2^{re} + \tilde{z}_2^{im}i), \right. \\ &\quad \left. \frac{1}{\sqrt{(\beta_1 + \beta_2i)(\delta_1 + \delta_2i)}}(\tilde{z}_3^{re} + \tilde{z}_3^{im}i), \frac{1}{\sqrt{(\beta_1 + \beta_2i)(\gamma_1 + \gamma_2i)}}(\tilde{z}_4^{re} + \tilde{z}_4^{im}i) \right). \end{aligned}$$

Substituting this into Formula (5), we have

$$\begin{cases} \tilde{z}_1^{re} + \tilde{z}_1^{im}i = (\tilde{x}_1^{re} + \tilde{x}_1^{im}i)(\tilde{y}_1^{re} + \tilde{y}_1^{im}i) + (\tilde{x}_2^{re} + \tilde{x}_2^{im}i)(\tilde{y}_2^{re} + \tilde{y}_2^{im}i) + \\ \quad + (x_3^{re} + x_3^{im}i)(\tilde{y}_3^{re} + \tilde{y}_3^{im}i) + (x_4^{re} + x_4^{im}i)(\tilde{y}_4^{re} + \tilde{y}_4^{im}i) \\ \tilde{z}_2^{re} + \tilde{z}_2^{im}i = (\tilde{x}_2^{re} + \tilde{x}_2^{im}i)(\tilde{y}_1^{re} + \tilde{y}_1^{im}i) + (\tilde{x}_1^{re} + \tilde{x}_1^{im}i)(\tilde{y}_2^{re} + \tilde{y}_2^{im}i) + \\ \quad + (x_3^{re} + x_3^{im}i)(\tilde{y}_3^{re} + \tilde{y}_3^{im}i) + (x_4^{re} + x_4^{im}i)(\tilde{y}_4^{re} + \tilde{y}_4^{im}i) \\ \tilde{z}_3^{re} + \tilde{z}_3^{im}i = (\tilde{x}_3^{re} + \tilde{x}_3^{im}i)(\tilde{y}_1^{re} + \tilde{y}_1^{im}i) + (\tilde{x}_4^{re} + \tilde{x}_4^{im}i)(\tilde{y}_2^{re} + \tilde{y}_2^{im}i) + \\ \quad + (x_1^{re} + x_1^{im}i)(\tilde{y}_3^{re} + \tilde{y}_3^{im}i) + (x_2^{re} + x_2^{im}i)(\tilde{y}_4^{re} + \tilde{y}_4^{im}i) \\ \tilde{z}_4^{re} + \tilde{z}_4^{im}i = (\tilde{x}_4^{re} + \tilde{x}_4^{im}i)(\tilde{y}_1^{re} + \tilde{y}_1^{im}i) + (\tilde{x}_3^{re} + \tilde{x}_3^{im}i)(\tilde{y}_2^{re} + \tilde{y}_2^{im}i) + \\ \quad + (x_2^{re} + x_2^{im}i)(\tilde{y}_3^{re} + \tilde{y}_3^{im}i) + (x_1^{re} + x_1^{im}i)(\tilde{y}_4^{re} + \tilde{y}_4^{im}i) \end{cases} \quad .$$

or passing to the original notations without the wave

$$\begin{cases} z_1 = x_1y_1 + x_2y_2 + x_3y_3 + x_4y_4 \\ z_2 = x_2y_1 + x_1y_2 + x_4y_3 + x_3y_4 \\ z_3 = x_3y_1 + x_4y_2 + x_1y_3 + x_2y_4 \\ z_4 = x_4y_1 + x_3y_2 + x_2y_3 + x_1y_4 \end{cases} \quad (10)$$

Equation (10) defines the canonical multiplication of four-dimensional numbers with complex components.

The given substitution of variables (9) is not the only possible substitution. Since the square root of a complex number has two roots, we can define the following substitution of variables:

$$\begin{cases} \tilde{x}_1^{re} + \tilde{x}_1^{im}i = x_1^{re} + x_1^{im}i \\ \tilde{x}_2^{re} + \tilde{x}_2^{im}i = -\sqrt{(\gamma_1 + \gamma_2i)(\delta_1 + \delta_2i)}(x_2^{re} + x_2^{im}i) \\ \tilde{x}_3^{re} + \tilde{x}_3^{im}i = \sqrt{(\beta_1 + \beta_2i)(\delta_1 + \delta_2i)}(x_3^{re} + x_3^{im}i) \\ \tilde{x}_4^{re} + \tilde{x}_4^{im}i = \sqrt{(\beta_1 + \beta_2i)(\gamma_1 + \gamma_2i)}(x_4^{re} + x_4^{im}i) \end{cases} \quad (11)$$

Carrying out similar transformations as in (10), we obtain another canonical form of multiplication of four-dimensional numbers with complex components

$$\begin{cases} z_1 = x_1y_1 + x_2y_2 + x_3y_3 + x_4y_4 \\ z_2 = x_2y_1 + x_1y_2 - x_4y_3 - x_3y_4 \\ z_3 = x_3y_1 - x_4y_2 + x_1y_3 - x_2y_4 \\ z_4 = x_4y_1 - x_3y_2 - x_2y_3 + x_1y_4 \end{cases} \quad (12)$$

Now we make the following substitution of variables:

$$\begin{cases} \tilde{x}_1^{re} + \tilde{x}_1^{im}i = x_1^{re} + x_1^{im}i \\ \tilde{x}_2^{re} + \tilde{x}_2^{im}i = \sqrt{(\gamma_1 + \gamma_2i)(\delta_1 + \delta_2i)}(x_2^{re} + x_2^{im}i) \\ \tilde{x}_3^{re} + \tilde{x}_3^{im}i = \sqrt{(\beta_1i - \beta_2)(\delta_1 + \delta_2i)}(x_3^{re} + x_3^{im}i) \\ \tilde{x}_4^{re} + \tilde{x}_4^{im}i = \sqrt{(\beta_1i - \beta_2)(\gamma_1 + \gamma_2i)}(x_4^{re} + x_4^{im}i) \end{cases} \quad (13)$$

Then the inverse transformation has the form

$$\begin{cases} x_1^{re} + x_1^{im}i = \tilde{x}_1^{re} + \tilde{x}_1^{im}i \\ x_2^{re} + x_2^{im}i = \frac{1}{\sqrt{(\gamma_1 + \gamma_2i)(\delta_1 + \delta_2i)}}(\tilde{x}_2^{re} + \tilde{x}_2^{im}i) \\ x_3^{re} + x_3^{im}i = \frac{1}{\sqrt{(\beta_1i - \beta_2)(\delta_1 + \delta_2i)}}(\tilde{x}_3^{re} + \tilde{x}_3^{im}i) \\ x_4^{re} + x_4^{im}i = \frac{1}{\sqrt{(\beta_1i - \beta_2)(\gamma_1 + \gamma_2i)}}(\tilde{x}_4^{re} + \tilde{x}_4^{im}i) \end{cases}$$

Let us rewrite the product (5) in variables with waves $\tilde{x}_1, \tilde{x}_2, \tilde{x}_3, \tilde{x}_4$. Substituting

$$Y = \left(\tilde{y}_1, \frac{1}{\sqrt{(\gamma_1 + \gamma_2i)(\delta_1 + \delta_2i)}}\tilde{y}_2, \frac{1}{\sqrt{(\beta_1i - \beta_2)(\delta_1 + \delta_2i)}}\tilde{y}_3, \frac{1}{\sqrt{(\beta_1i - \beta_2)(\gamma_1 + \gamma_2i)}}\tilde{y}_4 \right),$$

$$Z = \left(\tilde{z}_1, \frac{1}{\sqrt{(\gamma_1 + \gamma_2i)(\delta_1 + \delta_2i)}}\tilde{z}_2, \frac{1}{\sqrt{(\beta_1i - \beta_2)(\delta_1 + \delta_2i)}}\tilde{z}_3, \frac{1}{\sqrt{(\beta_1i - \beta_2)(\gamma_1 + \gamma_2i)}}\tilde{z}_4 \right)$$

into (5), we obtain

$$\begin{cases} \tilde{z}_1 = \tilde{x}_1\tilde{y}_1 + \tilde{x}_2\tilde{y}_2 - i\tilde{x}_3\tilde{y}_3 - i\tilde{x}_4\tilde{y}_4 \\ \tilde{z}_2 = \tilde{x}_2\tilde{y}_1 + \tilde{x}_1\tilde{y}_2 - i\tilde{x}_4\tilde{y}_3 - i\tilde{x}_3\tilde{y}_4 \\ \tilde{z}_3 = \tilde{x}_3\tilde{y}_1 + \tilde{x}_4\tilde{y}_2 + \tilde{x}_1\tilde{y}_3 + \tilde{x}_2\tilde{y}_4 \\ \tilde{z}_4 = \tilde{x}_4\tilde{y}_1 + \tilde{x}_3\tilde{y}_2 + \tilde{x}_2\tilde{y}_3 + \tilde{x}_1\tilde{y}_4 \end{cases} \quad (14)$$

The above transformations suggest that the basis multiplications are the multiplications in which the coefficients of β, γ, δ take the values $[1, -1, i, -i]$. There are only 64 such multiplications, which are summarized in Table 2. Each cell in Table 2 contains a triple (a, b, c) , where a is the value of β , b is the value of γ , and c is the value of δ .

Table 2. All possible basis multiplications of four-dimensional numbers.

β	γ	$\delta = 1$	$\delta = -1$	$\delta = i$	$\delta = -i$
1	1	(1, 1, 1)	(1, 1, -1)	(1, 1, i)	(1, 1, -i)
	-1	(1-, 1, 1)	(1, -1, -1)	(1, -1, i)	(1-, 1, -i)
	i	(1, i, 1)	(1, i, -1)	(1, i, i)	(1, i, -i)
	-i	(1, -i, 1)	(1, -i, -1)	(1, -i, i)	(1, -i, -i)
-1	1	(-1, 1, 1)	(-1, 1, -1)	(-1, 1, i)	(-1, 1, -i)
	-1	(-1, -1, 1)	(-1, -1, -1)	(-1, -1, i)	(-1, -1, -i)
	i	(-1, i, 1)	(-1, i, -1)	(-1, i, i)	(-1, i, -i)
	-i	(-1, -i, 1)	(-1, -i, -1)	(-1, -i, i)	(-1, -i, -i)
i	1	(i, 1, 1)	(i, 1, -1)	(i, 1, i)	(i, 1, -i)
	-1	(i, -1, 1)	(i, -1, -1)	(i, -1, i)	(i, -1, -i)
	i	(i, i, 1)	(i, i, -1)	(i, i, i)	(i, i, -i)
	-i	(i, -i, 1)	(i, -i, -1)	(i, -i, i)	(i, -i, -i)
-i	1	(-i, 1, 1)	(-i, 1, -1)	(-i, 1, i)	(-i, 1, -i)
	-1	(-i, -1, 1)	(-i, -1, -1)	(-i, -1, i)	(-i, -1, -i)
	i	(-i, i, 1)	(-i, i, -1)	(-i, i, i)	(-i, i, -i)
	-i	(-i, -i, 1)	(-i, -i, -1)	(-i, -i, i)	(-i, -i, -i)

For each cell of Table 2, substituting the corresponding values of β, γ , and δ into Formula (5), we obtain some basis multiplication of four-dimensional numbers with complex components. Let us consider as an example the cells of the first row of the table for which $\beta = \gamma = 1$. For cell (1, 1, 1), the multiplication is defined by Formula (10); for cell (1, 1, -1), the multiplication is defined by formula

$$\begin{cases} z_1 = x_1y_1 - x_2y_2 - x_3y_3 + x_4y_4 \\ z_2 = x_2y_1 + x_1y_2 + x_4y_3 + x_3y_4 \\ z_3 = x_3y_1 + x_4y_2 + x_1y_3 + x_2y_4 \\ z_4 = x_4y_1 - x_3y_2 - x_2y_3 + x_1y_4 \end{cases}; \tag{15}$$

for cell (1, 1, i), the multiplication is defined by formula

$$\begin{cases} z_1 = x_1y_1 + ix_2y_2 + ix_3y_3 + x_4y_4 \\ z_2 = x_2y_1 + x_1y_2 + x_4y_3 + x_3y_4 \\ z_3 = x_3y_1 + x_4y_2 + x_1y_3 + x_2y_4 \\ z_4 = x_4y_1 + ix_3y_2 + ix_2y_3 + x_1y_4 \end{cases}; \tag{16}$$

and for the cell (1, 1, -i), the multiplication is defined by formula

$$\begin{cases} z_1 = x_1y_1 - ix_2y_2 - ix_3y_3 + x_4y_4 \\ z_2 = x_2y_1 + x_1y_2 + x_4y_3 + x_3y_4 \\ z_3 = x_3y_1 + x_4y_2 + x_1y_3 + x_2y_4 \\ z_4 = x_4y_1 - ix_3y_2 - ix_2y_3 + x_1y_4 \end{cases}. \tag{17}$$

The definition of multiplication (12) corresponds to the cell $(-1, -1, -1)$, and the definition (14) corresponds to the cell $(-i, 1, 1)$. The other definitions of base multiplications can be written out similarly. Obviously, any other definition of multiplication can be reduced through a substitution of variables analogous to (9), (11), or (13) to one of the basis multiplications. Thus, for four-dimensional numbers with complex components, 64 different multiplications can be defined which are associative and commutative, and this is consistent with the addition operation. As it is known [1] in the case of four-dimensional numbers with real components, there are eight different multiplication operations, for six of which a pre-norm can be defined. These definitions correspond to the cells $(1, 1, 1)$, $(1, 1, -1)$, $(1, -1, 1)$, $(1, -1, -1)$, $(-1, 1, 1)$, $(-1, 1, -1)$, $(-1, -1, 1)$, and $(-1, -1, -1)$ of Table 2.

Each basis multiplication defines some linear vector space of four-dimensional numbers with complex components over the field of complex numbers. For further convenience, we introduce the notation for these spaces. The most natural notation for them, by analogy with the case with real components, is M_j , where j varies from 1 to 64. Thus we will number the spaces in Table 2 by rows and within rows by columns. The corresponding notations are given in Table 3. It may be more convenient to use designations of the form $M(\beta, \gamma, \delta)$, where β, γ, δ takes the values $[1, -1, i, -i]$, especially for multidimensional numbers with a number of components greater than four, for example, for eight-dimensional, sixteen-dimensional, etc. With these notations, the space M_2 for the case of four-dimensional numbers with real components [1] corresponds to the space $M(-1, 1, 1)$; the space M_3 corresponds to the space $M(1, -1, 1)$, etc.; and the space M_7 corresponds to the space $M(1, -1, -1)$. These notations of spaces of four-dimensional numbers with complex components are also given in Table 3. Each cell of Table 3 contains two designations of the corresponding basis space of four-dimensional numbers with complex components.

Table 3. Denotations of basis spaces of four-dimensional numbers.

β	γ	$\delta = 1$	$\delta = -1$	$\delta = i$	$\delta = -i$
1	1	$M_1 = M(1, 1, 1)$	$M_2 = M(1, 1, -1)$	$M_3 = M(1, 1, i)$	$M_4 = M(1, 1, -i)$
	-1	$M_5 = M(1, -1, 1)$	$M_6 = M(1, -1, -1)$	$M_7 = M(1, -1, i)$	$M_8 = M(1, -1, -i)$
	i	$M_9 = M(1, i, 1)$	$M_{10} = M(1, i, -1)$	$M_{11} = M(1, i, i)$	$M_{12} = M(1, i, -i)$
	$-i$	$M_{13} = M(1, -i, 1)$	$M_{14} = M(1, -i, -1)$	$M_{15} = M(1, -i, i)$	$M_{16} = M(1, -i, -i)$
-1	1	$M_{17} = M(-1, 1, 1)$	$M_{18} = M(-1, 1, -1)$	$M_{19} = M(-1, 1, i)$	$M_{20} = M(-1, 1, -i)$
	-1	$M_{21} = M(-1, -1, 1)$	$M_{22} = M(-1, -1, -1)$	$M_{23} = M(-1, -1, i)$	$M_{24} = M(-1, -1, -i)$
	i	$M_{25} = M(-1, i, 1)$	$M_{26} = M(-1, i, -1)$	$M_{27} = M(-1, i, i)$	$M_{28} = M(-1, i, -i)$
	$-i$	$M_{29} = M(-1, -i, 1)$	$M_{30} = M(-1, -i, -1)$	$M_{31} = M(-1, -i, i)$	$M_{32} = M(-1, -i, -i)$
i	1	$M_{33} = M(i, 1, 1)$	$M_{34} = M(i, 1, -1)$	$M_{35} = M(i, 1, i)$	$M_{36} = M(i, 1, -i)$
	-1	$M_{37} = M(i, -1, 1)$	$M_{38} = M(i, -1, -1)$	$M_{39} = M(i, -1, i)$	$M_{40} = M(i, -1, -i)$
	i	$M_{41} = M(i, i, 1)$	$M_{42} = M(i, i, -1)$	$M_{43} = M(i, i, i)$	$M_{44} = M(i, i, -i)$
	$-i$	$M_{45} = M(i, -i, 1)$	$M_{46} = M(i, -i, -1)$	$M_{47} = M(i, -i, i)$	$M_{48} = M(i, -i, -i)$
-i	1	$M_{49} = M(-i, 1, 1)$	$M_{50} = M(-i, 1, -1)$	$M_{51} = M(-i, 1, i)$	$M_{52} = M(-i, 1, -i)$
	-1	$M_{53} = M(-i, -1, 1)$	$M_{54} = M(-i, -1, -1)$	$M_{55} = M(-i, -1, i)$	$M_{56} = M(-i, -1, -i)$
	i	$M_{57} = M(-i, i, 1)$	$M_{58} = M(-i, i, -1)$	$M_{59} = M(-i, i, i)$	$M_{60} = M(-i, i, -i)$
	$-i$	$M_{61} = M(-i, -i, 1)$	$M_{62} = M(-i, -i, -1)$	$M_{63} = M(-i, -i, i)$	$M_{64} = M(-i, -i, -i)$

Thus, we obtained the basis spaces of four-dimensional numbers with complex components, in which the associative and commutative operations of addition and multiplication are defined. In each space, the multiplication table of basis numbers is easily written out by substituting into Table 1 the corresponding values of (β, γ, δ) . For example, for the space $M(1, -i, -1)$ or M_{14} , the multiplication table of basis numbers is given in Table 4.

Table 4. Multiplications of basis numbers in the space M_{14} .

	J_1^1	J_1^2	J_2^1	J_2^2	J_3^1	J_3^2	J_4^1	J_4^2
J_1^1	J_1^1	J_1^2	J_2^1	J_2^2	J_3^1	J_3^2	J_4^1	J_4^2
J_1^2	J_1^2	$-J_1^1$	J_2^2	$-J_2^1$	J_3^2	$-J_3^1$	J_4^2	$-J_4^1$
J_2^1	J_2^1	J_2^2	J_1^1	$-J_1^2$	$-J_4^1$	$-J_4^2$	$-J_3^1$	J_3^2
J_2^2	J_2^2	$-J_2^1$	$-J_1^2$	$-J_1^1$	$-J_4^2$	J_4^1	J_3^2	$-J_3^1$
J_3^1	J_3^1	J_3^2	$-J_4^1$	$-J_4^2$	$-J_1^1$	$-J_1^2$	J_2^1	J_2^2
J_3^2	J_3^2	$-J_3^1$	$-J_4^2$	J_4^1	$-J_1^2$	J_1^1	J_2^2	$-J_2^1$
J_4^1	J_4^1	J_4^2	$-J_3^1$	J_3^2	J_2^1	J_2^2	$-J_1^1$	J_1^2
J_4^2	J_4^2	$-J_4^1$	J_3^2	$-J_3^1$	J_2^2	$-J_2^1$	J_1^2	$-J_1^1$

The constructed basis spaces have a remarkable symmetry. Indeed, as can be seen from the diagonal elements of Table 4, in the space M_{14} , of the eight basis numbers, four are real and four numbers are imaginary. It is easy to realize that a similar property is possessed by the basis numbers of any space M_j , where $j \in [1, 64]$. The presence of such symmetry distinguishes these spaces from the space of quaternions, in which one basis element is real and the rest are imaginary. The symmetry of the basis numbers makes these spaces a natural generalization of one-dimensional and two-dimensional (complex) algebra.

4. Matrix Spaces for Four-Dimensional Numbers

Let us introduce the operator $F : X \rightarrow F(X)$, which maps a matrix $X = (x_1, x_2, x_3, x_4) \in \mathbb{C}^4$ to each four-dimensional number.

$$F(X) = \begin{pmatrix} x_1 & \gamma\delta x_2 & \beta\delta x_3 & \beta\gamma x_4 \\ x_2 & x_1 & \beta x_4 & \beta x_3 \\ x_3 & \gamma x_4 & x_1 & \gamma x_2 \\ x_4 & \delta x_3 & \delta x_2 & x_1 \end{pmatrix}, \tag{18}$$

where $\beta = \beta_1 + \beta_2i$, $\gamma = \gamma_1 + \gamma_2i$, $\delta = \delta_1 + \delta_2i$ are the coefficients in the definition of multiplication (5). All elements of the matrix $F(X)$ are formed from the coordinates of number X .

The mapping $F : X \rightarrow F(X)$ is one-to-one and on. Indeed, two different numbers, x and y , correspond to different matrices, and for any matrix of the above kind, we can find the corresponding four-dimensional number with complex components.

Theorem 1. *The set of all matrices of the form (18) is closed with respect to the operations of addition, subtraction, and multiplication of matrices, as well as multiplication of a matrix by a scalar. The inverse matrix to a nondegenerate matrix has the same form (18).*

Proof. It is proved by direct inspection. \square

The multiplication of two four-dimensional numbers $X = (x_1, x_2, x_3, x_4)$ and $Y = (y_1, y_2, y_3, y_4)$ with complex components can be represented as $X \cdot Y = F(X) \cdot Y$, where the

sign of the multiplication in the left-hand side is understood in the sense of (5), and the sign of the multiplication in the right-hand side is understood as the multiplication of a matrix by a vector. Thus, we have defined an alternative definition of multiplication of four-dimensional numbers using matrix (18).

Considering as (β, γ, δ) their possible values for the basis spaces of four-dimensional numbers, we obtain 64 different mappings $F : X \rightarrow F(X)$. Let us denote these mappings by $F_j, j = 1, 2, \dots, 64$, or $F[\beta, \gamma, \delta]$, where β, γ , and δ take the values $[1, -1, i, -i]$. For example, for the space $M(1, -i, -1)$ or M_{14} , the mapping F is as follows:

$$F_{14}(X) = \begin{pmatrix} x_1 & ix_2 & -x_3 & -ix_4 \\ x_2 & x_1 & x_4 & x_3 \\ x_3 & -ix_4 & x_1 & -ix_2 \\ x_4 & -x_3 & -x_2 & x_1 \end{pmatrix}, \tag{19}$$

Similarly for four-dimensional number spaces with real components, the mapping F has unique properties, as given in the following theorem.

Theorem 2. Every mapping $F_j (j = 1, 2, \dots, 64)$ for arbitrary four-dimensional numbers $X, Y \in \mathbb{C}$ has the following properties:

1. $F_j(X \pm Y) = F_j(X) \pm F_j(Y)$;
2. $F_j(cX) = cF_j(X)$ for any $c \in \mathbb{C}$;
3. $F_j(XY) = F_j(X)F_j(Y)$;
4. $F_j(X^{-1}) = F_j^{-1}(X)$;
5. $\det(F_j(X)) = A(X)$;
6. $\det(F_j(X) \pm F(Y)) = A(X \pm Y)$;
7. $\det(F_j(X)F_j(Y)) = A(XY)$;
8. $\det(F_j^{-1}(X)) = A(X^{-1})$, if the symplectic module of X is nonzero,

where $A(X), A(X \pm Y)$, and $A(XY)$ are defined in (7).

Proof. Let us prove the theorem for the space of numbers M_{14} ; for the other spaces, it is proved analogously. Properties (1) and (2) are obvious. Let us prove property (3). In the basis space M_{14} , the multiplication operation has the form

$$\begin{cases} z_1 = x_1y_1 + ix_2y_2 - x_3y_3 - ix_4y_4 \\ z_2 = x_2y_1 + x_1y_2 + x_4y_3 + x_3y_4 \\ z_3 = x_3y_1 - ix_4y_2 + x_1y_3 - ix_2y_4 \\ z_4 = x_4y_1 - x_3y_2 - x_2y_3 + x_1y_4 \end{cases}, \tag{20}$$

in which $(z_1, z_2, z_3, z_4) = XY$.

$$F_{14}(X)F_{14}(Y) = \begin{pmatrix} x_1 & ix_2 & -x_3 & -ix_4 \\ x_2 & x_1 & x_4 & x_3 \\ x_3 & -ix_4 & x_1 & -ix_2 \\ x_4 & -x_3 & -x_2 & x_1 \end{pmatrix} \begin{pmatrix} y_1 & iy_2 & -y_3 & -iy_4 \\ y_2 & y_1 & y_4 & y_3 \\ y_3 & -iy_4 & y_1 & -iy_2 \\ y_4 & -y_3 & -y_2 & y_1 \end{pmatrix} = B,$$

where B is the resulting matrix. Let us calculate the elements of matrix B .

$$\begin{aligned} b_{11} &= x_1y_1 + ix_2y_2 - x_3y_3 - ix_4y_4 = z_1, \\ b_{12} &= ix_1y_2 + ix_2y_1 + ix_3y_4 + ix_4y_3 = iz_2, \\ b_{13} &= -x_1y_3 + ix_2y_4 - x_3y_1 + ix_4y_2 = -z_3, \end{aligned}$$

$$\begin{aligned}
 b_{14} &= -ix_1y_4 + ix_2y_3 + ix_3y_2 - ix_4y_1 = -iz_4, \\
 b_{21} &= x_2y_1 + x_1y_2 + x_4y_3 + x_3y_4 = z_2, \\
 b_{22} &= ix_2y_2 + x_1y_1 - ix_4y_4 - x_3y_3 = z_1, \\
 b_{23} &= -x_2y_3 + x_1y_4 + x_4y_1 - x_3y_2 = z_4, \\
 b_{24} &= -ix_2y_4 + x_1y_3 - ix_4y_2 + x_3y_1 = z_3, \\
 b_{31} &= x_3y_1 - ix_4y_2 + x_1y_3 - ix_2y_4 = z_3, \\
 b_{32} &= ix_3y_2 - ix_4y_1 - ix_1y_4 + ix_2y_3 = -iz_4, \\
 b_{33} &= -x_3y_3 - ix_4y_4 + x_1y_1 + ix_2y_2 = z_1, \\
 b_{34} &= -ix_3y_4 - ix_4y_3 - ix_1y_2 - ix_2y_1 = -iz_2, \\
 b_{41} &= x_4y_1 - x_3y_2 - x_2y_3 + x_1y_4 = z_4, \\
 b_{42} &= ix_4y_2 - x_3y_1 + ix_2y_4 - x_1y_3 = -z_3, \\
 b_{43} &= -x_4y_3 - x_3y_4 - x_2y_1 - x_1y_2 = -z_2, \\
 b_{44} &= -ix_4y_4 - x_3y_3 + ix_2y_2 + x_1y_1 = z_1.
 \end{aligned}$$

Thus, matrix B coincides with matrix (19), in which x is replaced by z . Let us prove property

(4). By Formula (8), $X^{-1} = \left(\frac{\bar{A}}{|A|^2}x_1^*, \frac{\bar{A}}{|A|^2}x_2^*, \frac{\bar{A}}{|A|^2}x_3^*, \frac{\bar{A}}{|A|^2}x_4^* \right)$, and by Formula (7),

$$A(X) = x_1^4 - x_2^4 + x_3^4 - x_4^4 - 2ix_1^2x_2^2 + 2x_1^2x_3^2 + 2ix_1^2x_4^2 + 2ix_2^2x_3^2 - 2x_2^2x_4^2 - 2ix_3^2x_4^2 + 8ix_1x_2x_3x_4; \tag{21}$$

hence,

$$\begin{aligned}
 (X^{-1})_1 &= \frac{1}{A(X)}x_1(x_1^2 - ix_2^2 + x_3^2 + ix_4^2) + \frac{1}{A(X)}2ix_2x_3x_4, \\
 (X^{-1})_2 &= \frac{1}{A(X)}x_2(-x_1^2 + ix_2^2 + x_3^2 + ix_4^2) + \frac{1}{A(X)}2x_1x_3x_4, \\
 (X^{-1})_3 &= \frac{1}{A(X)}x_3(-x_1^2 - ix_2^2 - x_3^2 + ix_4^2) - \frac{1}{A(X)}2ix_1x_2x_4, \\
 (X^{-1})_4 &= \frac{1}{A(X)}x_4(-x_1^2 - ix_2^2 + x_3^2 - ix_4^2) - \frac{1}{A(X)}2x_1x_2x_3.
 \end{aligned}$$

Then we have the matrix

$$F_{14}(X^{-1}) = \begin{pmatrix} (X^{-1})_1 & i(X^{-1})_2 & -(X^{-1})_3 & -i(X^{-1})_4 \\ (X^{-1})_2 & (X^{-1})_1 & (X^{-1})_4 & (X^{-1})_3 \\ (X^{-1})_3 & -i(X^{-1})_4 & (X^{-1})_1 & -i(X^{-1})_2 \\ (X^{-1})_4 & -(X^{-1})_3 & -(X^{-1})_2 & (X^{-1})_1 \end{pmatrix}.$$

Multiplying the matrices $F_{14}(X^{-1})$ and $F_{14}(X)$ by each other, we obtain $F_{14}(X^{-1})F_{14}(X) = E$, where E is a unit matrix, which proves the required.

Let us prove property (5). Let us calculate the determinant of the matrix $F_{14}(X)$.

$$\begin{aligned}
 \det F_{14}(X) &= x_1 \begin{vmatrix} x_1 & x_4 & x_3 \\ -ix_4 & x_1 & -ix_2 \\ -x_3 & -x_2 & x_1 \end{vmatrix} - ix_2 \begin{vmatrix} x_2 & x_4 & x_3 \\ x_3 & x_1 & -ix_2 \\ x_4 & -x_2 & x_1 \end{vmatrix} - x_3 \begin{vmatrix} x_2 & x_1 & x_3 \\ x_3 & -ix_4 & -ix_2 \\ x_4 & -x_3 & x_1 \end{vmatrix} \\
 &+ ix_4 \begin{vmatrix} x_2 & x_1 & x_4 \\ x_3 & -ix_4 & x_1 \\ x_4 & -x_3 & -x_2 \end{vmatrix} \\
 &= x_1^2(x_1^2 - ix_2^2 + x_3^2 + ix_4^2) + 2ix_1x_2x_3x_4 - ix_2^2(x_1^2 - ix_2^2 - x_3^2 - ix_4^2) + 2ix_1x_2x_3x_4 \\
 &- x_3^2(-x_1^2 - ix_2^2 - x_3^2 + ix_4^2) + 2ix_1x_2x_3x_4 + ix_4^2(x_1^2 + ix_2^2 - x_3^2 + ix_4^2) + 2ix_1x_2x_3x_4 \\
 &= A(X),
 \end{aligned}$$

according to (21).

Property (6) follows automatically from properties (1) and (5). Property (7) follows from properties (3) and (5), and property (8) follows from (4) and (5). □

Thus, between the space of four-dimensional numbers with complex components and the space of (4×4) -matrices of the form (18), there exist bijections F_j ($j = 1, 2, \dots, 64$), which preserve arithmetic operations; that is, the existing bijections are homomorphisms by both addition and multiplication. That is, we have constructed 64 matrix spaces, each of which is isomorphic and homomorphic to the space of four-dimensional numbers with complex components. The elements of each matrix space form an abelian group by multiplication and by addition. In [1] it is shown that to each four-dimensional number with real components correspond eight matrix spaces. To illustrate the results of Theorems 1 and 2, we give the following example.

Example 1. Take two four-dimensional numbers $X = (1, i, 0 - 1)$ and $Y = (-1, 0, 0, i)$ in the space $M_{14} = M(1, -i, -1)$. The product of these numbers, according to Formula (5), is equal to $Z = XY = (-2, -i, i, 1 + i)$. The corresponding matrices $F_{14}(X)$, $F_{14}(Y)$, and $F_{14}(Z)$, according to Formula (19), respectively, have the forms,

$$\begin{pmatrix} 1 & -1 & 0 & i \\ i & 1 & -1 & 0 \\ 0 & i & 1 & 1 \\ -1 & 0 & -i & 1 \end{pmatrix}, \begin{pmatrix} -1 & 0 & 0 & 1 \\ 0 & -1 & i & 0 \\ 0 & 1 & -1 & 0 \\ i & 0 & 0 & -1 \end{pmatrix}, \begin{pmatrix} -2 & 1 & -i & 1-i \\ -i & -2 & 1+i & i \\ i & 1-i & -2 & -1 \\ 1+i & -i & i & -2 \end{pmatrix}.$$

Hence, it is easy to verify that $F_{14}(X)F_{14}(Y) = F_{14}(Z)$. The matrix inverse of $F_{14}(Y)$ has the form

$$F_{14}^{-1}(Y) = -\frac{1+i}{2} \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & i & 0 \\ 0 & 1 & 1 & 0 \\ i & 0 & 0 & 1 \end{pmatrix}.$$

Comparing this matrix with the matrix (19), we notice that in the space M_{14} , it is the matrix of the four-dimensional number $\tilde{Y} = (-\frac{1+i}{2}, 0, 0, \frac{1-i}{2})$. Multiplying the numbers Y and \tilde{Y} by each other, we can see that $\tilde{Y} = Y^{-1}$. Hence, $F_{14}^{-1}(Y) = F_{14}(Y^{-1})$.

5. Commutative Gate Groups for Two-Qubit Quantum Systems

Quantum computation consists of successively applying unitary operators U to the quantum state of a quantum system, which are called gates. The unitary operators or gates applied to a n -qubit quantum system are represented as a matrix of size $2^n \times 2^n$. For

example, unitary operators for a two-qubit system take the form of a matrix of size 4×4 as follows:

$$U = \begin{pmatrix} x_{11}^{re} + x_{11}^{im}i & x_{12}^{re} + x_{12}^{im}i & x_{13}^{re} + x_{13}^{im}i & x_{14}^{re} + x_{14}^{im}i \\ x_{21}^{re} + x_{21}^{im}i & x_{22}^{re} + x_{22}^{im}i & x_{23}^{re} + x_{23}^{im}i & x_{24}^{re} + x_{24}^{im}i \\ x_{31}^{re} + x_{31}^{im}i & x_{32}^{re} + x_{32}^{im}i & x_{33}^{re} + x_{33}^{im}i & x_{34}^{re} + x_{34}^{im}i \\ x_{41}^{re} + x_{41}^{im}i & x_{42}^{re} + x_{42}^{im}i & x_{43}^{re} + x_{43}^{im}i & x_{44}^{re} + x_{44}^{im}i \end{pmatrix}, \quad (22)$$

where x_{ij}^{re} and x_{ij}^{im} are real and imaginary parts of the elements of the matrix U . It is known that if the matrix (22) is a gate, then the Hermite conjugate matrix U^* is also a gate, i.e., the matrix

$$U^* = \begin{pmatrix} x_{11}^{re} - x_{11}^{im}i & x_{21}^{re} - x_{21}^{im}i & x_{31}^{re} - x_{31}^{im}i & x_{41}^{re} - x_{41}^{im}i \\ x_{12}^{re} - x_{12}^{im}i & x_{22}^{re} - x_{22}^{im}i & x_{32}^{re} - x_{32}^{im}i & x_{42}^{re} - x_{42}^{im}i \\ x_{13}^{re} - x_{13}^{im}i & x_{23}^{re} - x_{23}^{im}i & x_{33}^{re} - x_{33}^{im}i & x_{43}^{re} - x_{43}^{im}i \\ x_{14}^{re} - x_{14}^{im}i & x_{24}^{re} - x_{24}^{im}i & x_{34}^{re} - x_{34}^{im}i & x_{44}^{re} - x_{44}^{im}i \end{pmatrix} \quad (23)$$

is a gate for a two-qubit quantum system. By definition $U \cdot U^* = E$, where E is a unit matrix of size 4×4 .

The main two-qubit gates used in the construction of quantum circuits are the Hadamard gate H , with unitary matrix

$$H = \frac{1}{2} \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 \end{pmatrix},$$

SWAP gate with matrix

$$SWAP = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix},$$

CNOT or CX gate with matrix

$$CX = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix},$$

as well as gates CY, CZ, with matrices

$$CY = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & -i \\ 0 & 0 & i & 0 \end{pmatrix}, CZ = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}.$$

In practice, quantum circuits for quantum computations are built from existing gates. A quantum circuit is a sequence of gates applied to one or more qubits in a quantum register. A quantum register is a collection of qubits that we use for computation [18]. At the hardware level, experimental physicists and engineers are working on optimizing the main gates. In addition, other physicists and computer scientists are trying to create the most efficient high-level gates [24]. Our goal is to create new groups of two-qubit gates. In the previous section, we constructed 64 abelian groups of matrices, with elements formed

from the components of a four-dimensional number. Under some additional conditions, the constructed matrices turn into unitary operators.

Let the four-dimensional number $X = (x_1, x_2, x_3, x_4) \in \mathbb{C}^4$ be the quantum state $|\varphi\rangle$ of a two-qubit quantum circuit, i.e.,

$$|\varphi\rangle = x_1|00\rangle + x_2|01\rangle + x_3|10\rangle + x_4|11\rangle. \tag{24}$$

Then to each such state we can map 64 matrices $F_j(X)$, $j = 1, 2, \dots, 64$, of size 4×4 to each such state, as follows:

$$F_j(X) = \begin{pmatrix} x_1^{re} + x_1^{im}i & \gamma_j\delta_j(x_2^{re} + x_2^{im}i) & \beta_j\delta_j(x_3^{re} + x_3^{im}i) & \beta_j\gamma_j(x_4^{re} + x_4^{im}i) \\ x_2^{re} + x_2^{im}i & x_1^{re} + x_1^{im}i & \beta_j(x_4^{re} + x_4^{im}i) & \beta_j(x_3^{re} + x_3^{im}i) \\ x_3^{re} + x_3^{im}i & \gamma_j(x_4^{re} + x_4^{im}i) & x_1^{re} + x_1^{im}i & \gamma_j(x_2^{re} + x_2^{im}i) \\ x_4^{re} + x_4^{im}i & \delta_j(x_3^{re} + x_3^{im}i) & \delta_j(x_2^{re} + x_2^{im}i) & x_1^{re} + x_1^{im}i \end{pmatrix}, \tag{25}$$

where $\beta_j, \gamma_j, \delta_j$ takes one of the values of $\{+1, -1, +i, -i\}$ depending on the value of j according to Table 3. Note that since $X = (x_1, x_2, x_3, x_4) \in \mathbb{C}^4$ is a state of a two-qubit quantum system, it is true that

$$\sum_{j=1}^4 |x_j|^2 = \sum_{j=1}^4 \left((x_j^{re})^2 + (x_j^{im})^2 \right) = 1. \tag{26}$$

Theorem 3. Let $X = (x_1, x_2, x_3, x_4) \in \mathbb{C}^4$ be the state of the two-qubit quantum system and $F_j(X)$ be the matrix (25) corresponding to X , for some j . Let the elements of the matrix $F_j(X)$ satisfy the following conditions:

$$\begin{cases} \rho_1(X) \equiv x_1\bar{x}_2 + \gamma\delta\bar{x}_1x_2 + \delta x_3\bar{x}_4 + \gamma\bar{x}_3x_4 = 0 \\ \rho_2(X) \equiv x_1\bar{x}_3 + \delta x_2\bar{x}_4 + \beta\delta\bar{x}_1x_3 + \beta\bar{x}_2x_4 = 0 \\ \rho_3(X) \equiv x_1\bar{x}_4 + \gamma x_2\bar{x}_3 + \beta\bar{x}_2x_3 + \beta\gamma\bar{x}_1x_4 = 0 \end{cases}, \tag{27}$$

where \bar{x}_j means complex conjugate to x_j . Then the matrix $F_j(X)$ is a two-qubit gate.

Proof. Consider the Hermite conjugate matrix $F_j^*(X)$ to matrix $F_j(X)$

$$F_j^*(X) = \begin{pmatrix} x_1^{re} - x_1^{im}i & x_2^{re} - x_2^{im}i & x_3^{re} - x_3^{im}i & x_4^{re} - x_4^{im}i \\ \bar{\gamma}_j\bar{\delta}_j(x_2^{re} - x_2^{im}i) & x_1^{re} - x_1^{im}i & \gamma_j(x_4^{re} - x_4^{im}i) & \delta_j(x_3^{re} - x_3^{im}i) \\ \beta_j\delta_j(x_3^{re} - x_3^{im}i) & \beta_j(x_4^{re} - x_4^{im}i) & x_1^{re} - x_1^{im}i & \delta_j(x_2^{re} - x_2^{im}i) \\ \beta_j\gamma_j(x_4^{re} - x_4^{im}i) & \beta_j(x_3^{re} - x_3^{im}i) & \gamma_j(x_2^{re} - x_2^{im}i) & x_1^{re} - x_1^{im}i \end{pmatrix}.$$

Multiply the matrices $F_j^*(X)$ and $F_j(X)$ by each other as follows:

$$F_j(X)F_j^*(X) = \begin{pmatrix} \rho_0 & \rho_1 & \rho_2 & \rho_3 \\ \bar{\gamma}\bar{\delta}\rho_1 & \rho_0 & \bar{\gamma}\rho_3 & \bar{\delta}\rho_2 \\ \bar{\beta}\bar{\delta}\rho_2 & \bar{\beta}\rho_3 & \rho_0 & \bar{\delta}\rho_2 \\ \bar{\beta}\bar{\gamma}\rho_3 & \bar{\beta}\rho_2 & \bar{\gamma}\rho_2 & \rho_0 \end{pmatrix},$$

where $\rho_0 = \sum_{j=1}^4 |x_j|^2$. Here, we took advantage of the fact that $\beta^2 = \gamma^2 = \delta^2 = 1$. Using relations (26) and (27), we have $F_j(X)F_j^*(X) = E$, where E is a unit matrix, which proves the theorem. \square

Remark 1. Conditions (27) in the notation of matrix (25) can be written in the form of the following six equations:

$$\begin{aligned} & x_1^{re} x_2^{re} + x_1^{im} x_2^{im} + (\gamma^{re} \delta^{re} - \gamma^{im} \delta^{im}) (x_1^{re} x_2^{re} + x_1^{im} x_2^{im}) \\ & + (\gamma^{im} \delta^{re} + \gamma^{re} \delta^{im}) (x_1^{im} x_2^{re} - x_1^{re} x_2^{im}) + \delta^{re} (x_3^{re} x_4^{re} + x_3^{im} x_4^{im}) - \delta^{im} (x_3^{im} x_4^{re} - x_3^{re} x_4^{im}) \\ & + \gamma^{re} (x_3^{re} x_4^{re} + x_3^{im} x_4^{im}) + \gamma^{im} (x_3^{im} x_4^{re} - x_3^{re} x_4^{im}) = 0. \end{aligned}$$

$$\begin{aligned} & x_1^{im} x_2^{re} - x_1^{re} x_2^{im} + (\gamma^{im} \delta^{re} + \gamma^{re} \delta^{im}) (x_1^{re} x_2^{re} + x_1^{im} x_2^{im}) \\ & - (\gamma^{re} \delta^{re} - \gamma^{im} \delta^{im}) (x_1^{im} x_2^{re} - x_1^{re} x_2^{im}) + \delta^{re} (x_3^{im} x_4^{re} - x_3^{re} x_4^{im}) + \delta^{im} (x_3^{re} x_4^{re} + x_3^{im} x_4^{im}) \\ & - \gamma^{re} (x_3^{im} x_4^{re} - x_3^{re} x_4^{im}) + \gamma^{im} (x_3^{re} x_4^{re} + x_3^{im} x_4^{im}) = 0. \end{aligned}$$

$$\begin{aligned} & x_1^{re} x_3^{re} + x_1^{im} x_3^{im} + \delta^{re} (x_2^{re} x_4^{re} + x_2^{im} x_4^{im}) - \delta^{im} (x_2^{im} x_4^{re} - x_2^{re} x_4^{im}) \\ & + (\beta^{re} \delta^{re} - \beta^{im} \delta^{im}) (x_1^{re} x_3^{re} + x_1^{im} x_3^{im}) + (\beta^{im} \delta^{re} + \beta^{re} \delta^{im}) (x_1^{im} x_3^{re} - x_1^{re} x_3^{im}) \\ & + \beta^{re} (x_2^{re} x_4^{re} + x_2^{im} x_4^{im}) + \beta^{im} (x_2^{im} x_4^{re} - x_2^{re} x_4^{im}) = 0. \end{aligned}$$

$$\begin{aligned} & x_1^{im} x_3^{re} - x_1^{re} x_3^{im} + \delta^{im} (x_2^{re} x_4^{re} + x_2^{im} x_4^{im}) + \delta^{re} (x_2^{im} x_4^{re} - x_2^{re} x_4^{im}) \\ & + (\beta^{im} \delta^{re} + \beta^{re} \delta^{im}) (x_1^{re} x_3^{re} + x_1^{im} x_3^{im}) - (\beta^{re} \delta^{re} - \beta^{im} \delta^{im}) (x_1^{im} x_3^{re} - x_1^{re} x_3^{im}) \\ & + \beta^{im} (x_2^{re} x_4^{re} + x_2^{im} x_4^{im}) - \beta^{re} (x_2^{im} x_4^{re} - x_2^{re} x_4^{im}) = 0. \end{aligned}$$

$$\begin{aligned} & x_1^{re} x_4^{re} + x_1^{im} x_4^{im} + \gamma^{re} (x_2^{re} x_3^{re} + x_2^{im} x_3^{im}) - \gamma^{im} (x_2^{im} x_3^{re} - x_2^{re} x_3^{im}) + \beta^{re} (x_2^{re} x_3^{re} + x_2^{im} x_3^{im}) \\ & + \beta^{im} (x_2^{im} x_3^{re} - x_2^{re} x_3^{im}) + (\beta^{re} \gamma^{re} - \beta^{im} \gamma^{im}) (x_1^{re} x_4^{re} + x_1^{im} x_4^{im}) \\ & + (\beta^{im} \gamma^{re} + \beta^{re} \gamma^{im}) (x_1^{im} x_4^{re} - x_1^{re} x_4^{im}) = 0. \end{aligned}$$

$$\begin{aligned} & x_1^{im} x_4^{re} - x_1^{re} x_4^{im} + \gamma^{re} (x_2^{im} x_3^{re} - x_2^{re} x_3^{im}) + \gamma^{im} (x_2^{re} x_3^{re} + x_2^{im} x_3^{im}) - \beta^{re} (x_2^{im} x_3^{re} - x_2^{re} x_3^{im}) \\ & + \beta^{im} (x_2^{re} x_3^{re} + x_2^{im} x_3^{im}) + (\beta^{im} \gamma^{re} + \beta^{re} \gamma^{im}) (x_1^{re} x_4^{re} + x_1^{im} x_4^{im}) \\ & - (\beta^{re} \gamma^{re} - \beta^{im} \gamma^{im}) (x_1^{im} x_4^{re} - x_1^{re} x_4^{im}) = 0. \end{aligned}$$

Thus, we have found 64 commutative groups of gates, when conditions (26) and (27) are satisfied. For example, for the space of matrices $F_{14}(X)$, conditions (27) have the form

$$\begin{cases} \rho_1 \equiv x_1 \bar{x}_2 + i \bar{x}_1 x_2 - x_3 \bar{x}_4 - i \bar{x}_3 x_4 = 0 \\ \rho_2 \equiv x_1 \bar{x}_3 - x_2 \bar{x}_4 - \bar{x}_1 x_3 + \bar{x}_2 x_4 = 0 \\ \rho_3 \equiv x_1 \bar{x}_4 - i x_2 \bar{x}_3 + \bar{x}_2 x_3 - i \bar{x}_1 x_4 = 0 \end{cases}, \quad (28)$$

and in the matrix notation (25), the above six equations turn into the following three equations:

$$x_1^{im} x_2^{re} - x_1^{re} x_2^{im} - x_1^{re} x_2^{re} - x_1^{im} x_2^{im} + x_3^{re} x_4^{re} + x_3^{im} x_4^{im} + x_3^{im} x_4^{re} - x_3^{re} x_4^{im} = 0.$$

$$x_1^{im} x_3^{re} - x_1^{re} x_3^{im} + x_2^{re} x_4^{re} + x_2^{im} x_4^{im} + x_1^{re} x_3^{re} + x_1^{im} x_3^{im} - x_2^{im} x_4^{re} + x_2^{re} x_4^{im} = 0.$$

$$x_1^{im} x_4^{re} - x_1^{re} x_4^{im} - x_2^{im} x_3^{re} + x_2^{re} x_3^{im} = 0.$$

Similarly, it is possible to write conditions (27) for all other groups, substituting instead of β, γ, δ the corresponding values.

If the quantum state consists of real components, conditions (27) take the form

$$\begin{cases} \rho_1 \equiv (1 + \gamma\delta)x_1x_2 + (\delta + \gamma)x_3x_4 = 0 \\ \rho_2 \equiv (1 + \beta\delta)x_1x_3 + (\delta + \beta)x_2x_4 = 0 \\ \rho_3 \equiv (1 + \beta\gamma)x_1x_4 + (\gamma + \beta)x_2x_3 = 0 \end{cases}, \tag{29}$$

As can be seen from these equalities, if $\beta = \gamma = 1, \delta = -1$, then the first two equalities are fulfilled automatically and the conditions (29) turn into one condition $x_1x_4 + x_2x_3 = 0$. Indeed, this case corresponds to the matrix

$$F_2(X) = F(1, 1, -1) = \begin{pmatrix} x_1 & -x_2 & -x_3 & x_4 \\ x_2 & x_1 & x_4 & x_3 \\ x_3 & x_4 & x_1 & x_2 \\ x_4 & -x_3 & -x_2 & x_1 \end{pmatrix}.$$

Multiplying this matrix by its transpose, we verify that it is a two-qubit gate if conditions (26) and $x_1x_4 + x_2x_3 = 0$ are satisfied. In other cases, for example, when $\beta = \gamma = \delta = -1$, which corresponds to the matrix $F_{22}(X)$, condition (29) consists of three equalities. Similarly, we can investigate different variants of quantum gates for quantum states with purely imaginary components.

So, we have constructed an infinite number of different two-qubit gates that lie in 64 commutative groups of gates. Furthermore, the multiplication of two gates from two different commutative groups is also a gate. It is easy to see that a gate obtained by multiplying two gates from two different groups is, in general, not an element of any commutative group $F_j(X)$. As for the gates H, SWAP, CX, CY, and CZ, which are often used in practice, they are also not elements of $F_j(X)$. But note that

$$H \cdot SWAP = F_{22}(X_{\frac{1}{2}}), \quad \text{or} \quad F_{22}(X_{\frac{1}{2}})SWAP = H, \tag{30}$$

where $X_{\frac{1}{2}} = \left(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}\right)^T$.

The quantum algorithm assumes the sequential application of different gates to qubits in the quantum circuit. It follows from the obtained results that if in a quantum algorithm there are gates belonging to one of 64 groups, this sequence can be replaced by one gate from the same group, since the sequential application of gates from one group does not lead out of this group. Hence, an optimal quantum algorithm should consist of gates which belong to different groups $F_j(X)$, where $j \in [1, 64]$.

6. Unitary States of Two-Qubit Quantum Systems

The ultimate meaning of any quantum algorithm is to translate the state of a quantum system into the desired state for solving the problem. Therefore, finding a fan that translates a quantum system from one given state to another given state is an important practical problem. Let us consider this problem for a two-qubit quantum system. The proposed approach can be generalized to an n -qubit system.

Definition 2. The state of a two-qubit quantum system $X = (x_1, x_2, x_3, x_4) \in \mathbb{C}^4$ is called a unitary state if there exists a triple $(\beta, \gamma, \delta) \in \mathbb{C}^3$, such that $\beta, \gamma, \delta \in \{+1, -1, +i, -i\}$ and the components X satisfy (27).

In other words, the state of a two-qubit quantum system is unitary if it is defined by space M_j , where $j \in [1, 64]$, in which the matrix $F_j(X)$ corresponding to this state is a two-qubit gate.

Remark 2. The conditions $\beta, \gamma, \delta \in \{+1, -1, +i, -i\}$ in the definition of a unitary state are imposed only to consider states only from basis spaces. They are not mandatory.

Note that all basis states of a quantum system are unitary states in all 64 basis spaces.

Definition 3. Quantum states having one of the following $X_1 = (x_1, 0, 0, 0)^T$, $X_2 = (0, x_2, 0, 0)^T$, $X_3 = (0, 0, x_3, 0)^T$, $X_4 = (0, 0, 0, x_4)^T$, where $x_j \in \mathbb{C}$, $|x_j| = 1$, $j = 1, 2, 3, 4$, are called quasi-basic states.

Obviously, all quasi-basic states satisfy (27) for any $j \in [1, 64]$, that is, they are unitary states.

Lemma 2. A quasi-basic state X_1 corresponds to a single gate in all spaces M_j , $j = 1, 2, \dots, 64$. Each of the quasi-basic states X_2, X_3, X_4 corresponds to 16 different gates in all spaces M_j , $j = 1, 2, \dots, 64$.

Proof. The first statement concerning X_1 is obvious. Consider the quasi-basic state X_2 . The corresponding matrix has the form

$$F_j(X_2) = \begin{pmatrix} 0 & \gamma\delta x_2 & 0 & 0 \\ x_2 & 0 & 0 & 0 \\ 0 & 0 & 0 & \gamma x_2 \\ 0 & 0 & \delta x_2 & 0 \end{pmatrix}.$$

Since γ and δ take values from the set $\{1, -1, i, -i\}$, there are only 16 different choices. Similarly, consider the states X_3 and X_4 .

Each unitary state X from M_j defines a unitary operator (gate) from $F_j(X)$, which we will call a gate of unitary state X . Let Y be an arbitrary state of a two-qubit quantum system. We apply the gate $F_j(X)$ to the state Y : $F_j(X)Y = X \cdot Y$; that is, applying the gate $F_j(X)$ is equivalent to a four-dimensional multiplication of the unitary state X by the state Y in the space M_j . If Y is also a unitary state from the same space, then we will get a unitary state from the same space. To what state will the state Y go if it is not unitary? The following theorem gives the answer to this question. \square

Theorem 4. A unitary state gate from space $M_j(\beta, \gamma, \delta)$ translates any unitary (non-unitary) state into a unitary (non-unitary) state.

Proof. For simplicity, we prove the theorem for the space M_{14} , and for other spaces, it is proved analogously. Let $X = (x_1, x_2, x_3, x_4) \in \mathbb{C}^4$ be a unitary state from space M_{14} and $F_{14}(X)$, the corresponding gate of a unitary state X , which has the form (19), and the components of the vector X satisfy (28). Let Y be an arbitrary state of a two-qubit quantum system. Then the gate $F_{14}(X)$ moves the quantum system from state Y to the state of

$$F_{14}(X)Y = \begin{pmatrix} x_1 & ix_2 & -x_3 & -ix_4 \\ x_2 & x_1 & x_4 & x_3 \\ x_3 & -ix_4 & x_1 & -ix_2 \\ x_4 & -x_3 & -x_2 & x_1 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \end{pmatrix} = \begin{pmatrix} x_1y_1 + ix_2y_2 - x_3y_3 - ix_4y_4 \\ x_2y_1 + x_1y_2 + x_4y_3 + x_3y_4 \\ x_3y_1 - ix_4y_2 + x_1y_3 - ix_2y_4 \\ x_4y_1 - x_3y_2 - x_2y_3 + x_1y_4 \end{pmatrix}.$$

Let us compute $\rho_1(F_{14}(X)Y)$, where ρ_1 is defined in (28).

$$\begin{aligned} \rho_1(F_{14}(X)Y) &= (x_1y_1 + ix_2y_2 - x_3y_3 - ix_4y_4)(\overline{x_2y_1} + \overline{x_1y_2} + \overline{x_4y_3} + \overline{x_3y_4}) \\ &\quad + (ix_1y_1 + \overline{x_2y_2} - ix_3y_3 - \overline{x_4y_4})(x_2y_1 + x_1y_2 + x_4y_3 + x_3y_4) \\ &\quad - (x_3y_1 - ix_4y_2 + x_1y_3 - ix_2y_4)(\overline{x_4y_1} - \overline{x_3y_2} - \overline{x_2y_3} + \overline{x_1y_4}) \\ &\quad - (ix_3y_1 - \overline{x_4y_2} + ix_1y_3 - \overline{x_2y_4})(x_4y_1 - x_3y_2 - x_2y_3 + x_1y_4) \\ &= (x_1\overline{x_2} + ix_1x_2 - x_3\overline{x_4} - ix_3x_4)(|y_1|^2 + |y_2|^2 + |y_3|^2 + |y_4|^2) \\ &\quad + (y_1\overline{y_2} + iy_1y_2 - y_3\overline{y_4} - iy_3y_4)(|x_1|^2 + |x_2|^2 + |x_3|^2 + |x_4|^2) \\ &\quad + (x_1\overline{x_4} - ix_2\overline{x_3} + \overline{x_2}x_3 - ix_1x_4)(y_1\overline{y_3} - y_2\overline{y_4} - \overline{y_1}y_3 + \overline{y_2}y_4) \\ &\quad + (x_1\overline{x_3} - x_2\overline{x_4} - \overline{x_1}x_3 + \overline{x_2}x_4)(y_1\overline{y_4} - iy_2\overline{y_3} + \overline{y_2}y_3 - iy_1y_4). \end{aligned}$$

Passing to the notation ρ_k from (28), we obtain

$$\rho_1(F_{14}(X)Y) = \rho_1(X)\rho_0(Y) + \rho_1(Y)\rho_0(X) + \rho_3(X)\rho_2(Y) + \rho_3(Y)\rho_2(X),$$

where $\rho_0(Y) = |y_1|^2 + |y_2|^2 + |y_3|^2 + |y_4|^2 = 1$, $\rho_0(X) = |x_1|^2 + |x_2|^2 + |x_3|^2 + |x_4|^2 = 1$. Since X is a unitary state in space M_{14} , by virtue of (28), $\rho_1(X) = \rho_2(X) = \rho_3(X) = 0$. Hence, $\rho_1(F_{14}(X)Y) = \rho_1(Y)$. Performing similar calculations, we obtain $\rho_2(F_{14}(X)Y) = \rho_2(Y)$ and $\rho_3(F_{14}(X)Y) = \rho_3(Y)$. It follows from these equalities that if Y is a unitary state in M_{14} , then $F_{14}(X)Y$ is also a unitary state in M_{14} , and conversely, if Y is a non-unitary state in M_{14} , then $F_{14}(X)Y$ is also a non-unitary state in M_{14} .

Thus, the unitary state gate translates unitary states into unitary states and non-unitary states into non-unitary states. All the above gates, H, SWAP, CX, CY, and CZ have similar properties. □

Lemma 3. *The gates H, SWAP, CX, CY, and CZ convert a unitary (non-unitary) state of a two-qubit quantum system into a unitary (non-unitary) state.*

Proof. Let us first prove the lemma for the SWAP fan. Let $X = (x_1, x_2, x_3, x_4) \in \mathbb{C}^4$ be a unitary state of a two-qubit quantum system from the space $M(\beta, \gamma, \delta)$. Let us apply the SWAP gate to state X as follows: $Y = \text{SWAP} \cdot X = (x_1, x_3, x_2, x_4)$. Since X is a unitary state, its components satisfy (27). Consequently, the components of state Y satisfy equations

$$\begin{cases} x_1\overline{x_3} + \gamma\delta\overline{x_1}x_3 + \delta x_2\overline{x_4} + \gamma\overline{x_2}x_4 = 0 \\ x_1\overline{x_2} + \delta x_3\overline{x_4} + \beta\delta\overline{x_1}x_2 + \beta\overline{x_3}x_4 = 0 \\ x_1\overline{x_4} + \gamma x_3\overline{x_2} + \beta\overline{x_3}x_2 + \beta\gamma\overline{x_1}x_4 = 0 \end{cases} \quad (31)$$

But these equations represent Equation (27) for the space $M(\gamma, \beta, \delta)$. Thus, the SWAP gate translates a state from the space $M(\beta, \gamma, \delta)$ to a state from the space $M(\gamma, \beta, \delta)$. Suppose now that X is a non-unitary state of a two-qubit quantum system. Suppose that $Y = \text{SWAP} \cdot X$ is a unitary state. But then, as just proven, state $X = \text{SWAP}^1 \cdot Y$ is a unitary state, which contradicts the original assumption.

For the gates CX, CY, and CZ the proof is completely analogous. Let us prove Hadamard gate H . For this purpose, let us use the equality (30). Since the statements of the lemma are true for the SWAP and $F_{22}\left(X_{\frac{1}{2}}\right)$ gates, they are true for H as well.

Unitary states play an important role in the construction of quantum algorithms, since for them we can explicitly specify a gate, which transfers a quantum system from one given state to another. \square

Theorem 5. *Let $X \in M_j$ be a unitary state of a two-qubit quantum system, where $1 \leq j \leq 64$. Then for any quasi-basic state $X_k, k = 1, 2, 3, 4$, there exists a gate $F_j(Y)$, such that $F_j(Y)X = X_k$, for some state Y .*

Proof. Take $Y = X_k X^{-1}$. Since $X \in M_j$, then $X^{-1} \in M_j$. X_k also belongs to M_j ; hence, $Y \in M_j$. Let us multiply both parts of the equation by X : $YX = X_k$. As we saw above, $YX = F_j(Y)X$. Whence, $F_j(Y)X = X_k$. The gate we are looking for is $F_j(X_k X^{-1})$.

This theorem states that, from any unitary state, one can go from any unitary state to any quasi-basic state, including any basis state, in one step. \square

Corollary 1. *Let X, Y be two unitary states lying in the same space M_j , where $1 \leq j \leq 64$. Then there exists a gate $F_j(Z)$, which transfers a two-qubit quantum system from state X to state Y .*

Proof. According to Theorem 4, there exist gates $F_j(\tilde{X})$ and $F_j(\tilde{Y})$, translating X and Y to X_k for some k . Then the gate $F_j^{-1}(\tilde{Y})F_j(\tilde{X})$ translates state X into state Y . \square

Theorem 6. *Let X, Y be two unitary states of a two-qubit quantum system. Then there exists a gate translating the two-qubit quantum system from state X to state Y .*

Proof. Let state X lie in space M_{j_1} and state Y lie in space M_{j_2} . The base state $X_1 = (1, 0, 0, 0)^T$, according to Lemma 2, lies in both spaces. Hence, by Theorem 5, there exists a gate $F_{j_1}(Z_1)$, which translates X into X_1 , and a gate $F_{j_2}(Z_2)$, which translates Y into X_1 . Then the gate $F_{j_2}^{-1}(Z_2)F_{j_1}(Z_1)$ is the desired gate translating the quantum system from state X to state Y .

Thus, we have divided all possible states of a two-qubit quantum system into the following two classes: unitary and non-unitary. Unitary states include all quasi-basic states (hence, all basis states) and play an important role in the construction of quantum algorithms. Theorem 6 makes it possible to go from any unitary state to any other unitary state in one step, and the gate of this transition can be explicitly described and there are infinitely many such gates (since there are infinitely many quasi-basic states). \square

7. Discussions

In this paper, we constructed four-dimensional spaces of complex numbers by analogy with four-dimensional spaces of real numbers [1]. Each four-dimensional number is mapped to a matrix formed from its components, and it is proved that the constructed mapping is bijection and homomorphism. The conditions under which the corresponding matrices are gates for two-qubit quantum systems are defined. The notion of a unitary state of a two-qubit quantum system is introduced. It is shown that any gate of a unitary state transforms a unitary state into a unitary state and a non-unitary state into a non-unitary state. Almost all gates used in the construction of quantum circuits, in particular, H, SWAP, CX, CY, and CZ, have the same properties. The question of the existence of a gate for a two-qubit quantum system, which translates a unitary state into a non-unitary state or vice versa, is of interest.

The problem of finding a gate that transfers a quantum system from one unitary state to another unitary state is solved. Thus, with the help of the four-dimensional spaces of complex numbers, it was possible to construct whole classes of two-qubit gates, which opens new possibilities for the construction of quantum algorithms. The results obtained have important theoretical and practical implications for quantum computing.

In this paper, we have only concerned ourselves with the four-dimensional algebra of complex numbers. It is possible to develop a four-dimensional mathematical analysis of complex numbers by analogy with the four-dimensional mathematical analysis of real numbers [8,9,22,23]. In the works [8,9], the applications of the four-dimensional analysis of real numbers to the solution of some problems of mathematical physics are given.

Another direction for the development of the proposed theory consists of the development of eight-dimensional algebra of real and complex numbers. As we have seen, the four-dimensional algebra of complex numbers is an ideal model for two-qubit quantum systems. Correspondingly, the eight-dimensional algebra of complex numbers will describe the mathematical model of three-qubit quantum systems, the sixteen-dimensional algebra of complex numbers of four-qubit quantum systems, etc.

And of course, the potential of four-dimensional algebra of complex numbers for studying two-qubit quantum systems is not fully revealed. It would be interesting to study the properties of gates formed by the multiplication of several gates from different commutative groups of matrices constructed in this paper. As follows from the results obtained, optimal quantum algorithms consist of a sequence of gates from different commutative groups. As shown above, gates from different commutative groups do not commute with each other; in fact, a new gate formation mechanism for two-qubit quantum systems is proposed. The new gate formation mechanism is of great importance for quantum computing.

Author Contributions: Conceptualization, M.B.G.; Methodology, M.B.G. and B.I.T.; Software, M.B.G., T.Z.K. and B.I.T.; Validation, M.B.G., T.Z.K., T.K.Y. and B.I.T.; Formal analysis, M.B.G. and B.I.T.; Investigation, M.B.G.; Resources, T.K.Y.; Writing—original draft, M.B.G.; Writing—review & editing, M.B.G. and B.I.T. All authors have read and agreed to the published version of the manuscript

Funding: The authors express their sincere gratitude for the financial support of the Ministry of Science and Higher Education of the Republic of Kazakhstan (grant number AR23489825).

Data Availability Statement: The original contributions presented in this study are included in the article. Further inquiries can be directed to the corresponding authors.

Conflicts of Interest: Authors Mars B. Gabbassov and Tolybay Z. Kuanov were employed by the “Factor” Systems Research Company. The remaining authors declare that the research was conducted in the absence of any commercial or financial relationships that could be construed as a potential conflict of interest.

Appendix A. Proof of Lemma 1

For clarity, let us rewrite equalities (1) in a more expanded form

$$\left\{ \begin{array}{l} z_1 = (a_{11}^{re} + a_{11}^{im}i)(x_1^{re} + x_1^{im}i)(y_1^{re} + y_1^{im}i) + (a_{12}^{re} + a_{12}^{im}i)(x_2^{re} + x_2^{im}i)(y_2^{re} + y_2^{im}i) \\ \quad + (a_{13}^{re} + a_{13}^{im}i)(x_3^{re} + x_3^{im}i)(y_3^{re} + y_3^{im}i) + (a_{14}^{re} + a_{14}^{im}i)(x_4^{re} + x_4^{im}i)(y_4^{re} + y_4^{im}i) \\ z_2 = (a_{21}^{re} + a_{21}^{im}i)(x_2^{re} + x_2^{im}i)(y_1^{re} + y_1^{im}i) + (a_{22}^{re} + a_{22}^{im}i)(x_1^{re} + x_1^{im}i)(y_2^{re} + y_2^{im}i) \\ \quad + (a_{23}^{re} + a_{23}^{im}i)(x_4^{re} + x_4^{im}i)(y_3^{re} + y_3^{im}i) + (a_{24}^{re} + a_{24}^{im}i)(x_3^{re} + x_3^{im}i)(y_4^{re} + y_4^{im}i) \\ z_3 = (a_{31}^{re} + a_{31}^{im}i)(x_3^{re} + x_3^{im}i)(y_1^{re} + y_1^{im}i) + (a_{32}^{re} + a_{32}^{im}i)(x_4^{re} + x_4^{im}i)(y_2^{re} + y_2^{im}i) \\ \quad + (a_{33}^{re} + a_{33}^{im}i)(x_1^{re} + x_1^{im}i)(y_3^{re} + y_3^{im}i) + (a_{34}^{re} + a_{34}^{im}i)(x_2^{re} + x_2^{im}i)(y_4^{re} + y_4^{im}i) \\ z_4 = (a_{41}^{re} + a_{41}^{im}i)(x_4^{re} + x_4^{im}i)(y_1^{re} + y_1^{im}i) + (a_{42}^{re} + a_{42}^{im}i)(x_3^{re} + x_3^{im}i)(y_2^{re} + y_2^{im}i) \\ \quad + (a_{43}^{re} + a_{43}^{im}i)(x_2^{re} + x_2^{im}i)(y_3^{re} + y_3^{im}i) + (a_{44}^{re} + a_{44}^{im}i)(x_1^{re} + x_1^{im}i)(y_4^{re} + y_4^{im}i) \end{array} \right. \quad (A1)$$

Let us first satisfy the commutativity condition, i.e., write down the equation $XY = YX$; give similar terms of the form $x_i^{re}y_j^{re}$, $x_i^{re}y_j^{im}$, $x_i^{im}y_j^{re}$, and $x_i^{im}y_j^{im}$; and equate the coefficients at them to zero. Removing the same equations, we obtain 12 equations with respect to the elements a_{kl}^{re}, a_{kl}^{im} , $k, l = 1, 2, 3, 4$, solving which we have

$$\begin{aligned} a_{21}^{re} &= a_{22}^{re}, & a_{21}^{im} &= a_{22}^{im}, & a_{23}^{re} &= a_{24}^{re}, & a_{23}^{im} &= a_{24}^{im}, & a_{31}^{re} &= a_{33}^{re}, & a_{31}^{im} &= a_{33}^{im}, \\ a_{32}^{re} &= a_{34}^{re}, & a_{32}^{im} &= a_{34}^{im}, & a_{41}^{re} &= a_{44}^{re}, & a_{41}^{im} &= a_{44}^{im}, & a_{42}^{re} &= a_{43}^{re}, & a_{42}^{im} &= a_{43}^{im}. \end{aligned}$$

Substituting these values into (A1), we obtain a general form of a commutative multiplication of four-dimensional numbers with complex components as follows:

$$\left\{ \begin{aligned} z_1 &= (a_{11}^{re} + a_{11}^{im}i)(x_1^{re} + x_1^{im}i)(y_1^{re} + y_1^{im}i) + (a_{12}^{re} + a_{12}^{im}i)(x_2^{re} + x_2^{im}i)(y_2^{re} + y_2^{im}i) \\ &\quad + (a_{13}^{re} + a_{13}^{im}i)(x_3^{re} + x_3^{im}i)(y_3^{re} + y_3^{im}i) + (a_{14}^{re} + a_{14}^{im}i)(x_4^{re} + x_4^{im}i)(y_4^{re} + y_4^{im}i) \\ z_2 &= (a_{22}^{re} + a_{22}^{im}i)(x_2^{re} + x_2^{im}i)(y_1^{re} + y_1^{im}i) + (a_{22}^{re} + a_{22}^{im}i)(x_1^{re} + x_1^{im}i)(y_2^{re} + y_2^{im}i) \\ &\quad + (a_{23}^{re} + a_{23}^{im}i)(x_4^{re} + x_4^{im}i)(y_3^{re} + y_3^{im}i) + (a_{23}^{re} + a_{23}^{im}i)(x_3^{re} + x_3^{im}i)(y_4^{re} + y_4^{im}i) \\ z_3 &= (a_{33}^{re} + a_{33}^{im}i)(x_3^{re} + x_3^{im}i)(y_1^{re} + y_1^{im}i) + (a_{32}^{re} + a_{32}^{im}i)(x_4^{re} + x_4^{im}i)(y_2^{re} + y_2^{im}i) \\ &\quad + (a_{33}^{re} + a_{33}^{im}i)(x_1^{re} + x_1^{im}i)(y_3^{re} + y_3^{im}i) + (a_{32}^{re} + a_{32}^{im}i)(x_2^{re} + x_2^{im}i)(y_4^{re} + y_4^{im}i) \\ z_4 &= (a_{44}^{re} + a_{44}^{im}i)(x_4^{re} + x_4^{im}i)(y_1^{re} + y_1^{im}i) + (a_{42}^{re} + a_{42}^{im}i)(x_3^{re} + x_3^{im}i)(y_2^{re} + y_2^{im}i) \\ &\quad + (a_{42}^{re} + a_{42}^{im}i)(x_2^{re} + x_2^{im}i)(y_3^{re} + y_3^{im}i) + (a_{44}^{re} + a_{44}^{im}i)(x_1^{re} + x_1^{im}i)(y_4^{re} + y_4^{im}i) \end{aligned} \right. \quad (A2)$$

Thus, the number of independent coefficients a_{kl}^{re}, a_{kl}^{im} , $k, l = 1, 2, 3, 4$, is reduced from 32 to 20. Let us now write the equations $(XY)Z = X(YZ)$, where $Z = (z_1, z_2, z_3, z_4)$, $z_k \in \mathbb{C}$, $k = 1, 2, 3, 4$, using (2), and equating the coefficients at the same products $x_l^i y_j^m z_k^n$, $l, m, n = re, im$, we obtain 48 different equations with respect to a_{kl}^{re}, a_{kl}^{im} . Consider the equations obtained by equating to zero the coefficients at products $z_1^{re} x_2^{re} y_2^{re}$ and $z_1^{re} x_2^{im} y_2^{re}$

$$\begin{aligned} a_{11}^{re} a_{12}^{re} - a_{11}^{im} a_{12}^{im} - a_{12}^{re} a_{22}^{re} + a_{12}^{im} a_{22}^{im} &= 0, \\ -a_{11}^{re} a_{12}^{im} - a_{12}^{re} a_{11}^{im} + a_{22}^{re} a_{12}^{im} + a_{12}^{re} a_{22}^{im} &= 0. \end{aligned}$$

Let us transform these equations as follows:

$$\begin{aligned} a_{12}^{re} (a_{11}^{re} - a_{22}^{re}) - a_{12}^{im} (a_{11}^{im} - a_{22}^{im}) &= 0, \\ -a_{12}^{im} (a_{11}^{re} - a_{22}^{re}) - a_{12}^{re} (a_{11}^{im} - a_{22}^{im}) &= 0. \end{aligned}$$

Multiply the first equation by a_{12}^{re} , multiply the second equation by a_{12}^{im} , and subtract these from each other as follows:

$$|a_{12}|^2 (a_{11}^{re} - a_{22}^{re}) = 0.$$

Now multiply the first equation by a_{12}^{im} , multiply the second equation by a_{12}^{re} , and add the resulting equations as follows:

$$|a_{12}|^2 (a_{11}^{im} - a_{22}^{im}) = 0.$$

Hence, $a_{11}^{re} = a_{22}^{re}$, $a_{11}^{im} = a_{22}^{im}$.

Let us equate to zero the coefficients at the products $z_1^{re} x_3^{re} y_3^{re}$ and $z_1^{re} x_3^{im} y_3^{re}$

$$\begin{aligned} a_{11}^{re} a_{13}^{re} - a_{11}^{im} a_{13}^{im} - a_{13}^{re} a_{33}^{re} + a_{13}^{im} a_{33}^{im} &= 0, \\ -a_{11}^{re} a_{13}^{im} - a_{13}^{re} a_{11}^{im} + a_{33}^{re} a_{13}^{im} + a_{13}^{re} a_{33}^{im} &= 0. \end{aligned}$$

Performing similar transformations, we obtain $a_{11}^{re} = a_{33}^{re}$, $a_{11}^{im} = a_{33}^{im}$.

Further equating to zero the coefficients of $z_1^r x_4^r y_4^r$ and $z_1^i x_4^i y_4^i$, we obtain $a_{11}^r = a_{44}^r$, $a_{11}^i = a_{44}^i$.

Substituting the found values into (A2) and getting rid of $a_{22}^r, a_{33}^r, a_{44}^r, a_{22}^i, a_{33}^i, a_{44}^i$, we obtain a new definition of multiplication of four-dimensional numbers with complex components.

$$\left\{ \begin{aligned} z_1 &= (a_{11}^r + a_{11}^i i)(x_1^r + x_1^i i)(y_1^r + y_1^i i) + (a_{12}^r + a_{12}^i i)(x_2^r + x_2^i i)(y_2^r + y_2^i i) \\ &\quad + (a_{13}^r + a_{13}^i i)(x_3^r + x_3^i i)(y_3^r + y_3^i i) + (a_{14}^r + a_{14}^i i)(x_4^r + x_4^i i)(y_4^r + y_4^i i) \\ z_2 &= (a_{11}^r + a_{11}^i i)[(x_2^r + x_2^i i)(y_1^r + y_1^i i) + (x_1^r + x_1^i i)(y_2^r + y_2^i i)] \\ &\quad + (a_{23}^r + a_{23}^i i)[(x_4^r + x_4^i i)(y_3^r + y_3^i i) + (x_3^r + x_3^i i)(y_4^r + y_4^i i)] \\ z_3 &= (a_{11}^r + a_{11}^i i)[(x_3^r + x_3^i i)(y_1^r + y_1^i i) + (x_1^r + x_1^i i)(y_3^r + y_3^i i)] \\ &\quad + (a_{32}^r + a_{32}^i i)[(x_2^r + x_2^i i)(y_4^r + y_4^i i) + (x_4^r + x_4^i i)(y_2^r + y_2^i i)] \\ z_4 &= (a_{11}^r + a_{11}^i i)[(x_4^r + x_4^i i)(y_1^r + y_1^i i) + (x_1^r + x_1^i i)(y_4^r + y_4^i i)] \\ &\quad + (a_{42}^r + a_{42}^i i)[(x_2^r + x_2^i i)(y_3^r + y_3^i i) + (x_3^r + x_3^i i)(y_2^r + y_2^i i)] \end{aligned} \right. \quad (A3)$$

In definition (3), the number of independent coefficients is 14. Let us again write the equations $(XY)Z = X(YZ)$ on the basis of definition (3), and by using similar terms, we obtain 12 different equations. Let us equate to zero the coefficients at the products $z_4^r x_3^i y_3^r, z_4^i x_3^r y_3^i, z_3^r x_4^r y_4^r, z_3^i x_4^i y_4^i, z_3^r x_4^i y_4^r, z_3^i x_4^r y_4^i, z_2^r x_2^i y_2^r, z_2^i x_2^r y_2^i$

$$\begin{aligned} a_{11}^r a_{13}^r - a_{11}^i a_{13}^i - a_{23}^r a_{42}^r + a_{23}^i a_{42}^i &= 0, \\ -a_{11}^r a_{13}^i - a_{11}^i a_{13}^r + a_{23}^i a_{42}^r + a_{23}^r a_{42}^i &= 0, \\ a_{11}^r a_{14}^r - a_{11}^i a_{14}^i - a_{23}^r a_{32}^r + a_{23}^i a_{32}^i &= 0, \\ -a_{11}^r a_{14}^i - a_{11}^i a_{14}^r + a_{23}^i a_{32}^r + a_{23}^r a_{32}^i &= 0, \\ a_{32}^r a_{42}^r - a_{32}^i a_{42}^i - a_{11}^r a_{12}^r + a_{11}^i a_{12}^i &= 0, \\ -a_{32}^r a_{42}^i - a_{32}^i a_{42}^r + a_{11}^i a_{12}^r + a_{11}^r a_{12}^i &= 0. \end{aligned}$$

Multiply the first equation by a_{11}^r ; multiply the second equation by a_{11}^i ; and from the first equation, subtract the second equation, as follows:

$$\left((a_{11}^r)^2 + (a_{11}^i)^2 \right) a_{13}^r = a_{11}^r \left(a_{23}^r a_{42}^r - a_{23}^i a_{42}^i \right) + a_{11}^i \left(a_{23}^r a_{42}^i + a_{23}^i a_{42}^r \right),$$

from where we have

$$a_{13}^r = \frac{a_{11}^r (a_{23}^r a_{42}^r - a_{23}^i a_{42}^i) + a_{11}^i (a_{23}^r a_{42}^i + a_{23}^i a_{42}^r)}{(a_{11}^r)^2 + (a_{11}^i)^2}.$$

Next, multiply the first equation by a_{11}^i , multiply the second equation by a_{11}^r and add the resulting equations as follows:

$$\left((a_{11}^r)^2 + (a_{11}^i)^2 \right) a_{13}^i = a_{11}^r \left(a_{23}^r a_{42}^i + a_{23}^i a_{42}^r \right) + a_{11}^i \left(-a_{23}^r a_{42}^r + a_{23}^i a_{42}^i \right),$$

whence

$$a_{13}^i = \frac{a_{11}^r (a_{23}^r a_{42}^i + a_{23}^i a_{42}^r) - a_{11}^i (a_{23}^r a_{42}^r - a_{23}^i a_{42}^i)}{(a_{11}^r)^2 + (a_{11}^i)^2}.$$

Now notice that

$$a_{13}^r + a_{13}^i i = \frac{(a_{23}^r + a_{23}^i i)(a_{42}^r + a_{42}^i i)}{a_{11}^r + a_{11}^i i}.$$

Similarly, from the third and fourth equations, we easily obtain

$$a_{14}^{re} = \frac{a_{11}^{re} (a_{23}^{re} a_{32}^{re} - a_{23}^{im} a_{32}^{im}) + a_{11}^{im} (a_{23}^{re} a_{32}^{im} + a_{23}^{im} a_{32}^{re})}{(a_{11}^{re})^2 + (a_{11}^{im})^2},$$

$$a_{14}^{im} = \frac{a_{11}^{re} (a_{23}^{re} a_{32}^{im} + a_{23}^{im} a_{32}^{re}) - a_{11}^{im} (a_{23}^{re} a_{32}^{re} - a_{23}^{im} a_{32}^{im})}{(a_{11}^{re})^2 + (a_{11}^{im})^2},$$

and, respectively,

$$a_{14}^{re} + a_{14}^{im} i = \frac{(a_{23}^{re} + a_{23}^{im} i)(a_{32}^{re} + a_{32}^{im} i)}{a_{11}^{re} + a_{11}^{im} i},$$

And from the fifth and sixth equations we have

$$a_{12}^{re} = \frac{a_{11}^{re} (a_{32}^{re} a_{42}^{re} - a_{32}^{im} a_{42}^{im}) + a_{11}^{im} (a_{32}^{re} a_{42}^{im} + a_{32}^{im} a_{42}^{re})}{(a_{11}^{re})^2 + (a_{11}^{im})^2},$$

$$a_{12}^{im} = \frac{a_{11}^{re} (a_{32}^{re} a_{42}^{im} + a_{32}^{im} a_{42}^{re}) - a_{11}^{im} (a_{32}^{re} a_{42}^{re} - a_{32}^{im} a_{42}^{im})}{(a_{11}^{re})^2 + (a_{11}^{im})^2},$$

$$a_{12}^{re} + a_{12}^{im} i = \frac{(a_{32}^{re} + a_{32}^{im} i)(a_{42}^{re} + a_{42}^{im} i)}{a_{11}^{re} + a_{11}^{im} i}.$$

Substituting the found expressions into the other equations obtained from the equality $(XY)Z = X(YZ)$, we make sure that all 12 equations are satisfied. Thus, the general form of associative and commutative anisotropic multiplication of four-dimensional numbers with complex components is as follows

$$\left\{ \begin{aligned} & z_1 = (a_{11}^{re} + a_{11}^{im} i)(x_1^{re} + x_1^{im} i)(y_1^{re} + y_1^{im} i) + \\ & + \frac{(a_{32}^{re} + a_{32}^{im} i)(a_{42}^{re} + a_{42}^{im} i)}{a_{11}^{re} + a_{11}^{im} i} (x_2^{re} + x_2^{im} i)(y_2^{re} + y_2^{im} i) + \\ & + \frac{(a_{23}^{re} + a_{23}^{im} i)(a_{42}^{re} + a_{42}^{im} i)}{a_{11}^{re} + a_{11}^{im} i} (x_3^{re} + x_3^{im} i)(y_3^{re} + y_3^{im} i) + \\ & + \frac{(a_{23}^{re} + a_{23}^{im} i)(a_{32}^{re} + a_{32}^{im} i)}{a_{11}^{re} + a_{11}^{im} i} (x_4^{re} + x_4^{im} i)(y_4^{re} + y_4^{im} i) \\ & z_2 = (a_{11}^{re} + a_{11}^{im} i) [(x_2^{re} + x_2^{im} i)(y_1^{re} + y_1^{im} i) + (x_1^{re} + x_1^{im} i)(y_2^{re} + y_2^{im} i)] \\ & + (a_{23}^{re} + a_{23}^{im} i) [(x_4^{re} + x_4^{im} i)(y_3^{re} + y_3^{im} i) + (x_3^{re} + x_3^{im} i)(y_4^{re} + y_4^{im} i)] \\ & z_3 = (a_{11}^{re} + a_{11}^{im} i) [(x_3^{re} + x_3^{im} i)(y_1^{re} + y_1^{im} i) + (x_1^{re} + x_1^{im} i)(y_3^{re} + y_3^{im} i)] \\ & + (a_{32}^{re} + a_{32}^{im} i) [(x_2^{re} + x_2^{im} i)(y_4^{re} + y_4^{im} i) + (x_4^{re} + x_4^{im} i)(y_2^{re} + y_2^{im} i)] \\ & z_4 = (a_{11}^{re} + a_{11}^{im} i) [(x_4^{re} + x_4^{im} i)(y_1^{re} + y_1^{im} i) + (x_1^{re} + x_1^{im} i)(y_4^{re} + y_4^{im} i)] \\ & + (a_{42}^{re} + a_{42}^{im} i) [(x_2^{re} + x_2^{im} i)(y_3^{re} + y_3^{im} i) + (x_3^{re} + x_3^{im} i)(y_2^{re} + y_2^{im} i)] \end{aligned} \right. \cdot \quad (A4)$$

The number of independent coefficients is eight. Let us introduce the following notations:

$$a_{11}^{re} + a_{11}^{im} i = \alpha_1 + \alpha_2 i, a_{23}^{re} + a_{23}^{im} i = \beta_1 + \beta_2 i, a_{32}^{re} + a_{32}^{im} i = \gamma_1 + \gamma_2 i, a_{42}^{re} + a_{42}^{im} i = \delta_1 + \delta_2 i.$$

Then the general formula of associative and commutative multiplication is written in the following form

$$\left\{ \begin{array}{l} z_1 = (\alpha_1 + \alpha_2 i)(x_1^{re} + x_1^{im} i)(y_1^{re} + y_1^{im} i) + \\ + \frac{(\gamma_1 + \gamma_2 i)(\delta_1 + \delta_2 i)}{\alpha_1 + \alpha_2 i}(x_2^{re} + x_2^{im} i)(y_2^{re} + y_2^{im} i) + \\ + \frac{(\beta_1 + \beta_2 i)(\delta_1 + \delta_2 i)}{\alpha_1 + \alpha_2 i}(x_3^{re} + x_3^{im} i)(y_3^{re} + y_3^{im} i) + \\ + \frac{(\beta_1 + \beta_2 i)(\gamma_1 + \gamma_2 i)}{\alpha_1 + \alpha_2 i}(x_4^{re} + x_4^{im} i)(y_4^{re} + y_4^{im} i) \\ z_2 = (\alpha_1 + \alpha_2 i)(x_2^{re} + x_2^{im} i)(y_1^{re} + y_1^{im} i) + (\alpha_1 + \alpha_2 i)(x_1^{re} + x_1^{im} i)(y_2^{re} + y_2^{im} i) \\ + (\beta_1 + \beta_2 i)(x_4^{re} + x_4^{im} i)(y_3^{re} + y_3^{im} i) + (\beta_1 + \beta_2 i)(x_3^{re} + x_3^{im} i)(y_4^{re} + y_4^{im} i) \\ z_3 = (\alpha_1 + \alpha_2 i)(x_3^{re} + x_3^{im} i)(y_1^{re} + y_1^{im} i) + (\gamma_1 + \gamma_2 i)(x_4^{re} + x_4^{im} i)(y_2^{re} + y_2^{im} i) \\ + (\alpha_1 + \alpha_2 i)(x_1^{re} + x_1^{im} i)(y_3^{re} + y_3^{im} i) + (\gamma_1 + \gamma_2 i)(x_2^{re} + x_2^{im} i)(y_4^{re} + y_4^{im} i) \\ z_4 = (\alpha_1 + \alpha_2 i)(x_4^{re} + x_4^{im} i)(y_1^{re} + y_1^{im} i) + (\delta_1 + \delta_2 i)(x_3^{re} + x_3^{im} i)(y_2^{re} + y_2^{im} i) \\ + (\delta_1 + \delta_2 i)(x_2^{re} + x_2^{im} i)(y_3^{re} + y_3^{im} i) + (\alpha_1 + \alpha_2 i)(x_1^{re} + x_1^{im} i)(y_4^{re} + y_4^{im} i) \end{array} \right. ,$$

with eight independent coefficients $\alpha_j, \beta_j, \gamma_j, \delta_j \in R, j = 1, 2$. It is easy to verify by direct inspection that the multiplication operation thus defined is associative and commutative for any values of the coefficients $\alpha_j, \beta_j, \gamma_j, \delta_j \in R, j = 1, 2$.

The last equalities are expressed in a more compact form as follows:

$$\left\{ \begin{array}{l} z_1 = \alpha x_1 y_1 + \frac{\gamma \delta}{\alpha} x_2 y_2 + \frac{\beta \delta}{\alpha} x_3 y_3 + \frac{\beta \gamma}{\alpha} x_4 y_4 \\ z_2 = \alpha x_2 y_1 + \alpha x_1 y_2 + \beta x_4 y_3 + \beta x_3 y_4 \\ z_3 = \alpha x_3 y_1 + \gamma x_4 y_2 + \alpha x_1 y_3 + \gamma x_2 y_4 \\ z_4 = \alpha x_4 y_1 + \delta x_3 y_2 + \delta x_2 y_3 + \alpha x_1 y_4 \end{array} \right. \quad (A5)$$

Lemma 1 is proven.

References

- Abenov, M.M.; Gabbassov, M.B.; Kuanov, T.Z.; Tuleuov, B.I. Anisotropic Four-Dimensional Spaces of Real Numbers. *Symmetry* **2025**, *17*, 795. [\[CrossRef\]](#)
- Sysoev, S.S. *Introduction to Quantum Computing. Quantum Algorithms. Study Guide*; Publishers of St. Petersburg University: St. Petersburg, Russia, 2019; p. 144.
- Hidari, J.D. *Quantum Computing: An Applied Approach*; DMK Press: Moscow, Russia, 2021; p. 370.
- Wie, C.-R. Two-Qubit Bloch Sphere. *Physics* **2020**, *2*, 383–396. [\[CrossRef\]](#)
- Filatov, S.; Auzinsh, M. Towards Two Bloch Sphere Representation of Pure Two-Qubit States and Unitaries. *Entropy* **2024**, *26*, 280. [\[CrossRef\]](#) [\[PubMed\]](#)
- Alsing, P.M.; Cafaro, C.; Felice, D.; Luongo, O. Geometric Aspects of Mixed Quantum States inside the Bloch Sphere. *Quantum Rep.* **2024**, *6*, 90–109. [\[CrossRef\]](#)
- Nikitin, N.V.; Toms, K.S.; Fotina, O.V. *Axioms of Quantum Mechanics*; Universitetskaya Kniga: Moscow, Russia, 2015; p. 126.
- Abenov, M.M. *Four-Dimensional Mathematics: Methods and Applications*; KazNU Publishing: Almaty, Kazakhstan, 2019; p. 175.
- Abenov, M.M.; Gabbassov, M.B. *Anisotropic Four-Dimensional Spaces*; Preprint: Astana, Kazakhstan, 2020; p. 65.
- Lopez-Saldivar, J.A.; Castanos, O.; Nahmad-Achar, E.; Lopez-Pena, R.; Man'ko, M.A.; Man'ko, V.I. Geometry and Entanglement of Two-Qubit States in the Quantum Probabilistic Representation. *Entropy* **2018**, *20*, 630. [\[CrossRef\]](#) [\[PubMed\]](#)
- Abdel-Khalek, S.; Berrada, K.; Khalil, E.M.; Almalki, F. Entanglement of General Two-Qubit States in a Realistic Framework. *Symmetry* **2021**, *13*, 386. [\[CrossRef\]](#)
- Gnonfin, H.; Gouba, L. The Separability Problem in Two-Qubits Revisited. *Symmetry* **2023**, *15*, 2089. [\[CrossRef\]](#)
- Man'ko, M.A.; Man'ko, V.I. Probability Distributions Describing Qubit-State Superpositions. *Entropy* **2023**, *25*, 1366. [\[CrossRef\]](#) [\[PubMed\]](#)
- Mohamed, A.-B.A.; Khalil, E.M.; Yassen, M.F.; Eleuch, H. Two-Qubit Local Fisher Information Correlation beyond Entanglement in a Nonlinear Generalized Cavity with an Intrinsic Decoherence. *Entropy* **2021**, *23*, 311. [\[CrossRef\]](#) [\[PubMed\]](#)
- Yu, C.-S.; Song, H.-S. Global Entanglement for Multipartite Quantum States. *Phys. Rev. A* **2006**, *73*, 022325. [\[CrossRef\]](#)
- Neven, A.; Bastin, T. The Quantum Separability Problem is a Simultaneous Hollowisation Matrix Analysis Problem. *J. Phys. A Math. Theor.* **2018**, *51*, 315305. [\[CrossRef\]](#)

17. De Zela, F. Generalizing Wave-Particle Duality: Two-Qubit Extension of the Polarization Coherence Theorem. *Quantum Rep.* **2020**, *2*, 501–513. [[CrossRef](#)]
18. Wang, K.; Yu, X.-T.; Cai, X.-F.; Zhang, Z.-C. Probabilistic Teleportation of Arbitrary Two-Qubit Quantum State via Non-Symmetric Quantum Channel. *Entropy* **2018**, *20*, 238. [[CrossRef](#)] [[PubMed](#)]
19. Kalaga, J.K.; Leonski, W.; Szczesniak, R.; Perina, J., Jr. Mixedness, Coherence and Entanglement in a Family of Three-Qubit States. *Entropy* **2022**, *24*, 324. [[CrossRef](#)] [[PubMed](#)]
20. Perina, J. *Coherence of Light*; Kluwer: Dordrecht, The Netherlands, 1985.
21. Perina, J. *Quantum Statistics of Linear and Nonlinear Optical Phenomena*; Kluwer: Dordrecht, The Netherlands, 1991.
22. Rakhimova, A.T.; Gabbassov, M.B.; Shapen, K.M. About One Space of Four-Dimensional Numbers. *Vestnik KazNU. Ser. Math. Mech. Inform.* **2020**, *108*, 81–98.
23. Rakhimova, A.T.; Gabbassov, M.B.; Shapen, K.M. Functions in One Space of Four-Dimensional Numbers. *Vestnik KazNU. Ser. Math. Mech. Comp. Sci.* **2021**, *110*, 139–154. [[CrossRef](#)]
24. Sutor, R.S. *Dancing with Qubits: How Quantum Computing Works and How It Can Change the World*; Packt Publishing: Birmingham, UK, 2019; p. 510.

Disclaimer/Publisher’s Note: The statements, opinions and data contained in all publications are solely those of the individual author(s) and contributor(s) and not of MDPI and/or the editor(s). MDPI and/or the editor(s) disclaim responsibility for any injury to people or property resulting from any ideas, methods, instructions or products referred to in the content.