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A simple analytic solution for Tachyon condensation

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Abstract: In this paper we present a new and simple analytic solution for tachyon condensation in open bosonic string field theory. Unlike the $\mathcal{B}_0$ gauge solution, which requires a carefully regulated discrete sum of wedge states subtracted against a mysterious "phantom" counter term, this new solution involves a continuous integral of wedge states, and no regularization or phantom term is necessary. Moreover, we can evaluate the action and prove Sen's conjecture in a mere few lines of calculation.

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1 Introduction

The original analytic solution for tachyon condensation in open bosonic string field theory [1] (henceforth, the $\mathcal{B}_0$ gauge solution) takes the form of a regulated sum

$$\Phi = \lim_{N\to\infty}\left[\psi_N - \sum_{n=0}^{N}\frac{d}{dn}\psi_n\right], \tag{1.1}$$

where ψ_n are wedge states with certain insertions (for more details, see [1, 2]). The form of this solution has long been a puzzle. First, the limit suggests that the solution may live outside the space of well-behaved string fields — like a distribution is a limit of a sequence of functions. Second, the mysterious ψ_N term — the so-called "phantom piece" — actually *vanishes* when contracted with well-behaved states in the large N limit. But we cannot simply set $\lim_{N\to\infty}\psi_N = 0$ since, if we evaluate the action analytically [1], the ψ_N term produces a substantial portion of the energy required to prove Sen's conjecture [3]. Yet, the ψ_N term does not contribute to the energy in the ordinary level expansion [1, 4], since as a state in the Fock space it vanishes identically.

By now the regularization and phantom piece are better understood [2, 5–10], and there is little doubt that the $\mathcal{B}_0$ gauge solution is for practical purposes nonsingular. Yet, no one has found an adequate definition of the solution — or gauge equivalent alternative — which does not require the regulated sum and phantom piece.

In this note, we present an alternative solution for the tachyon vacuum which avoids the above complications. Instead of a discrete sum, the solution involves a continuous integral over wedge states, and no regularization or mysterious phantom term is necessary. Moreover, evaluation of the action and the proof of Sen's conjectures is, in contrast to the $\mathcal{B}_0$ gauge, very straightforward.

Broad classes of generalizations of the $\mathcal{B}_0$ gauge solution have been constructed in [7, 11–14]. Note in particular that our new solution is a special case of the solutions considered in [7], though our analysis will be quite different.

This paper is organized as follows. In the first half of the paper, section 2, we present the new solution for the tachyon vacuum. In section 2.1 we prove the absence of open string states around the vacuum and calculate the brane tension, giving a very simple proof of Sen's first conjecture. In section 2.2 we discuss pure gauge solutions and their relation to the mysterious phantom piece, and in section 2.3 we prove that the vacuum does not source closed strings.

In the second half of the paper, section 3, we investigate the energy of the new solution in level truncation. As a warmup exercise, in section 3.1 we consider the $\mathcal{L}_0$ level expansion. Due to the simplicity of our solution, we can solve the $\mathcal{L}_0$ expansion exactly, and we resum the expansion to confirm Sen's conjecture up to better than one part in 10 million. In section 3.2 we consider the "true" level expansion in terms of eigenstates of L_0. Surprisingly — unlike the Siegel gauge or $\mathcal{B}_0$ gauge tachyon condensates — we find that the expansion does not converge. In order to understand this, in section 3.3 we construct a toy model of our solution where the L_0 level expansion, though divergent, can be solved exactly. In the end, we are able to resum the L_0 expansion and confirm Sen's conjecture to better than 99%. We end with some discussion.

2 Solution

The new vacuum solution can be presented using the same basic algebraic setup as the original $\mathcal{B}_0$ gauge solution [2, 14]—that is, it can be built out of three "atomic" string fields K, B, c:

$$\begin{aligned} K &= \text{Grassmann even}, \ \text{gh}\# = 0, \\ B &= \text{Grassmann odd}, \ \text{gh}\# = -1, \\ c &= \text{Grassmann odd}, \ \text{gh}\# = 1, \end{aligned} \tag{2.1}$$

which satisfy the algebraic relations

$$\begin{aligned} [K,B] &= 0, \quad & Bc+cB &= 1, \\ B^2 &= 0, & c^2 &= 0, \end{aligned} \tag{2.2}$$

and have BRST variations ($Q = Q_B$)

$$QK = 0, \quad QB = K, \quad Qc = cKc. \tag{2.3}$$

All products above are open string star products. Thus, K, B, c generate a subalgebra of the open string star algebra which is closed under the action of the BRST operator. Perhaps the most useful explicit definition of K, B, c is given in terms of CFT correlation functions on the cylinder.[1] To keep the presentation self-contained, we explain how this works in appendix A. Note that the $\mathrm{SL}(2,\mathbb{R})$ vacuum can be written explicitly in terms of K [2, 14]:

$$|0\rangle \equiv \Omega = e^{-K}. \tag{2.5}$$

By extension, any power of the vacuum — that is, a wedge state [15] — can be expressed as $\Omega^t = e^{-tK}$ for $t \geq 0$.

With these preparations, the new solution for the tachyon vacuum is:

$$\Psi = \Big[c + cKBc\Big]\frac{1}{1+K}. \tag{2.6}$$

Let us be specific about the definition of $\frac{1}{1+K}$. We can invert $1+K$ using the Schwinger parameterization

$$\frac{1}{1+K} = \int_0^\infty dt\, e^{-t(1+K)} = \int_0^\infty dt\, e^{-t}\Omega^t, \tag{2.7}$$

so, if we like, we can re-express eq. (2.6) in the form

$$\Psi = \int_0^\infty dt\, e^{-t}\Big[c + cKBc\Big]\Omega^t. \tag{2.8}$$

That's all there is to it. No regularization or "phantom piece" is necessary. See figure 1 for a picture of the solution as a correlation function on the cylinder.

It is straightforward to verify the equations of motion. Note that $cKBc = Q(Bc)$ and hence

$$Q\Psi = cKc\frac{1}{1+K}. \tag{2.9}$$

To compute Ψ^2 it is convenient to write $c + cKBc$ as $c(1+K)Bc$. Then commute one of the Bs in Ψ^2 towards the other and the equations of motion are quickly established.

An important property of our solution is that it involves integration over wedge states arbitrarily close to the identity. The identity string field is a somewhat unruly object [15, 16], and indeed the solution exhibits surprising convergence properties in the level expansion. But still we have found convincing analytic and numerical evidence that the solution describes the endpoint of tachyon condensation. We explicitly construct the gauge transformation relating this solution to the $\mathcal{B}_0$ gauge vacuum in appendix B.

[1] In the operator notation these fields can be written,

$$K = \frac{\pi}{2}(K_1)_L|I\rangle, \quad B = \frac{\pi}{2}(B_1)_L|I\rangle, \quad c = \frac{1}{\pi}c(1)|I\rangle, \tag{2.4}$$

where $K_1 = L_1 + L_{-1}, B_1 = b_1 + b_{-1}$, $|I\rangle$ is the identity string field, and the subscript L denotes taking the left half of the corresponding charge — that is, integrating the current from $-i$ to i on the positive half of the unit semicircle. Note that each field K, B, c written here differs by a sign from the definitions used in [7, 14].

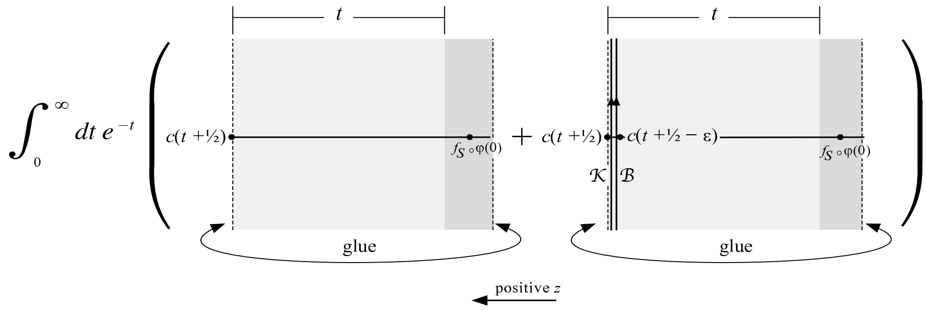

Figure 1. Overlap of the solution eq. (2.6) with a Fock space state $|\phi\rangle$, pictured as a conformal field theory correlation function on the cylinder. See appendix A for further explanation.

Eq. (2.6) is closely related to another solution which satisfies the string field reality condition:[2]

$$\hat{\Psi} = \frac{1}{\sqrt{1+K}}\Big[c + cKBc\Big]\frac{1}{\sqrt{1+K}}, \tag{2.10}$$

where the inverse square root of $1+K$ is

$$\frac{1}{\sqrt{1+K}} = \frac{1}{\sqrt{\pi}}\int_0^\infty dt\,\frac{1}{\sqrt{t}}e^{-t}\Omega^t. \tag{2.11}$$

Ψ and $\hat{\Psi}$ are related by a complex homogeneous gauge transformation

$$\hat{\Psi} = \frac{1}{\sqrt{1+K}}(Q+\Psi)\sqrt{1+K}. \tag{2.12}$$

The original Ψ is a simpler solution, but for some purposes the real $\hat{\Psi}$ is more convenient. For example, $\hat{\Psi}$ is twist even, so it lives in the same universal subspace as the $\mathcal{B}_0$ gauge vacuum and the Siegel gauge condensate. Also, the non-real Ψ has a c insertion on the boundary of the local coordinate, so Ψ could have singular contractions with states carrying insertions that collide with the c ghost.[3] For the purposes of this paper these differences

[2]In open string field theory, the string field is conventionally assumed to satisfy the following reality condition:

$$\Phi^\ddagger = \Phi,$$

where $\ddagger$ is an involution of the star algebra defined by the composition of BPZ and Hermitian conjugation [17]. K, B and c are real string fields in this sense, so in this context the reality condition simply requires that the string field read the same way from the left as from the right. The reality condition is sufficient to guarantee that the action is real and that the string field carries the correct number of perturbative degrees of freedom. However, all known observables in string field theory are invariant under "complex" gauge transformations which do not necessarily preserve the reality condition. Therefore an acceptable solution may not satisfy the reality condition, but it must be in the same (complex) gauge orbit as a solution that does.

[3]Note that this problem may also afflict $\hat{\Psi}$; though the c insertion never sits on the boundary of the local coordinate, it becomes arbitrarily close to the boundary as the integration approaches the identity string field. Hence, for example, the action of the operators $b(1)$ and $b(-1)$ on both Ψ and $\hat{\Psi}$ is divergent due to singular collisions with the c-ghost.

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will not prove to be significant. The analytic proof of Sen's conjectures is identical for either solution, and we will often use them interchangeably.

Neither Ψ nor $\hat{\Psi}$ satisfies a linear b-ghost gauge condition. However they do satisfy a linear gauge of a more general type, something we call a "dressed $\mathcal{B}_0$ gauge." We will explain this class of gauges in appendix C.

2.1 Sen's conjectures

Let us demonstrate that the solution (2.6) describes the endpoint of tachyon condensation. We need to establish two things [3]: first, no open strings are present at the vacuum, and second, that the vacuum has precisely minus the energy of an unstable D-brane.

It is easy to show that Ψ supports no open string excitations. Following [18, 19], this follows if there exists a string field A (the homotopy operator) satisfying

$$Q_\Psi A = 1, \tag{2.13}$$

where $Q_\Psi = Q + [\Psi, \cdot]$ is the vacuum kinetic operator. If this is the case, any Q_Ψ closed state Φ can be written as $Q_\Psi(A\Phi)$ and the cohomology is trivial. The homotopy operator for our solution is easily found:

$$A = B\frac{1}{1+K}. \tag{2.14}$$

Therefore Q_Ψ has no cohomology.[4]

Let us now calculate the energy. Sen's conjecture predicts that, in the appropriate units,[5] the energy of the vacuum should be

$$E = -S(\Psi) = -\frac{1}{2\pi^2}, \tag{2.16}$$

where $S(\Psi)$ is the action. Assuming the equations of motion, we can compute the action using only the kinetic term:

$$E = \frac{1}{6}\langle\Psi, Q_B\Psi\rangle = \frac{1}{6}\,\mathrm{Tr}\left(\Big[c + cKBc\Big]\frac{1}{1+K}cKc\frac{1}{1+K}\right), \tag{2.17}$$

where we write

$$\mathrm{Tr}(\cdot) = \langle I, \cdot\rangle \tag{2.18}$$

to denote the one point vertex. Now expand the $\frac{1}{1+K}$ factors in terms of wedge states and use $cKBc = Q(Bc)$ to write the second term as a "total derivative":

$$E = \frac{1}{6}\int_0^\infty dt_1 dt_2\, e^{-t_1-t_2}\left[\mathrm{Tr}\left(c\Omega^{t_1}cKc\Omega^{t_2}\right) - \mathrm{Tr}\left(Q\Big[Bc\Omega^{t_1}cKc\Omega^{t_2}\Big]\right)\right]. \tag{2.19}$$

[4] We should mention that the existence of a homotopy operator implies the absence of cohomology at all ghost numbers, not just at the physical ghost number of 1. This appears to be in conflict with some numerical studies [20], and the paradox has yet to be resolved.

[5] We normalize the ghost correlator

$$\langle c(z_1)c(z_2)c(z_3)\rangle_{\mathrm{UHP}} = (z_1 - z_2)(z_2 - z_3)(z_1 - z_3) \tag{2.15}$$

and set the spacetime volume factor and open string coupling constant to unity. Our normalizations agree with [1, 2].

The second term is a trace of a BRST exact state, and therefore vanishes.[6] The energy reduces to:

$$E = \frac{1}{6}\int_0^\infty dt_1 dt_2\, e^{-t_1-t_2}\,\mathrm{Tr}\Big(c\Omega^{t_1}cKc\Omega^{t_2}\Big). \tag{2.20}$$

Following appendix A, we can translate the trace into a correlation function on the cylinder, which is then easy to evaluate by the usual CFT methods. (This particular correlator has already been computed e.g. in [1, 2].) The answer is,

$$\mathrm{Tr}\Big(c\Omega^{t_1}cKc\Omega^{t_2}\Big) = -\left(\frac{t_1+t_2}{\pi}\right)^2 \sin^2\frac{\pi t_1}{t_1+t_2}. \tag{2.21}$$

Therefore, we can compute the energy by evaluating the double integral,

$$E = -\frac{1}{6}\int_0^\infty dt_1 dt_2\, e^{-t_1-t_2}\left(\frac{t_1+t_2}{\pi}\right)^2 \sin^2\frac{\pi t_1}{t_1+t_2}. \tag{2.22}$$

This looks complicated, but with the substitution

$$\begin{aligned} u =&\ t_1+t_2, \quad u\in[0,\infty),\\ v =&\ \frac{t_1}{t_1+t_2}, \quad v\in[0,1],\\ dt_1dt_2 =&\ u\,dudv, \end{aligned} \tag{2.23}$$

the double integral factorizes into a product of two very simple integrals

$$E = -\frac{1}{6\pi^2}\left(\int_0^\infty du\, u^3 e^{-u}\right)\left(\int_0^1 dv\sin^2\pi v\right). \tag{2.24}$$

The first is $\Gamma(4)=6$, and the second is the integral of $\sin^2$ over a period, which produces a factor of $1/2$. Therefore

$$E = -\frac{1}{2\pi^2} \tag{2.25}$$

in agreement with Sen's conjecture.

2.2 Pure gauge solutions and the phantom piece

The absence of a phantom term in our solution comes as a surprise. To see why, let us mention a related issue: All solutions for the tachyon vacuum (constructed so far) are, in a sense, arbitrarily close to being pure gauge. In particular, for every vacuum solution Φ, there is a one parameter family of pure gauge solutions $\Phi_\lambda, \lambda\in[0,1)$ such that the Fock space component fields of Φ_λ approach those of Φ as λ approaches 1. Yet, if the tachyon vacuum is expanded in a basis of $\mathcal{L}_0$ eigenstates (see next section) the expansion coefficients never appear close to a pure gauge solution, for any λ. Therefore the tachyon vacuum and pure gauge solutions must differ by a term which vanishes in the Fock space,

[6] One should be a little careful about this. In particular, since the integration includes traces of wedge states arbitrarily close to the identity, if the insertions have net scaling dimension ≥ 2 in the sliver coordinate frame, there could be a divergence leading to an anomaly. Fortunately, the insertions in the second term have net scaling dimension -1, so such divergences are absent.

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but whose expansion in $\mathcal{L}_0$ eigenstates is nevertheless nonvanishing. This is the origin of the phantom piece.

Since the phantom piece does not explicitly appear in our solution, we need to track down where it went. Following Okawa [2],[7] we can construct the appropriate one parameter family of pure gauge solutions, Ψ_λ:

$$\Psi_\lambda = \lambda\Psi - \lambda(1-\lambda)\left(cB\frac{1+K}{1-\lambda+K}c\frac{1}{1+K}\right), \tag{2.27}$$

where Ψ is the vacuum solution eq. (2.6). Assuming the second term vanishes as λ approaches 1, the vacuum and pure gauge solutions appear to become identical. But we should be more careful. Using the Schwinger representation to expand the second term more explicitly:

$$\lim_{\lambda\to 1}(\Psi - \Psi_\lambda) = cB(1+K)\lim_{\lambda\to 1}\left[(1-\lambda)\int_0^\infty dt\, e^{-(1-\lambda)t}\Omega^t\right]c\frac{1}{1+K}. \tag{2.28}$$

In this form the subtlety of the limit is clear. Though $1-\lambda$ vanishes, as $\lambda \to 1$ there is a corresponding divergence from the integration over all wedge states (Ω^t approaches a constant — the sliver state — for large t). The product of these factors is finite, and in fact

$$\lim_{\lambda\to 1^-}(1-\lambda)\int_0^\infty dt\, e^{-(1-\lambda)t}\,\Omega^t = \Omega^\infty, \tag{2.29}$$

where Ω^∞ is the sliver state. Substituting into eq. (2.28) therefore gives[8]

$$\lim_{\lambda\to 1}(\Psi - \Psi_\lambda) = cB\Omega^\infty c\frac{1}{1+K}. \tag{2.30}$$

Since B annihilates the sliver when contracted with Fock space states [1, 7], the last term is a phantom piece. However, unlike in $\mathcal{B}_0$ gauge, the phantom term appears in the pure gauge solution (as λ approaches 1), not the tachyon vacuum.

2.3 Closed string tadpole

Since our solution describes an empty vacuum without D-branes, the field configuration should leave the closed string background undisturbed. One way to check this is to compute the closed string tadpole, which can be evaluated as a disk amplitude

$$\mathcal{A}_\Phi(\mathcal{V}) = -\langle\mathcal{V}(i\infty)c(0)\rangle_{C_1,\mathrm{BCFT}_\Phi}. \tag{2.31}$$

Here $\mathcal{V} = c\tilde{c}\mathcal{V}^m$ is an on-shell closed string vertex operator, and for convenience we have mapped the canonical unit disk to a cylinder C_1 of unit circumference; the subscript BCFT_Φ

[7] The Okawa pure gauge form for our solution is

$$\Psi_\lambda = (1-\lambda\Phi)Q\frac{1}{1-\lambda\Phi}, \quad \Phi = Bc\frac{1}{1+K}. \tag{2.26}$$

We formally obtain the vacuum solution for $\lambda = 1$.

[8] We ignore the $1+K$ factor since this would give a subleading contribution to the phantom piece, though such contributions can be important [8].

indicates that the correlator is evaluated in the boundary conformal field theory corresponding to the classical solution Φ. Ellwood [21] gave a nice prescription for computing this amplitude directly from Φ:

$$\mathcal{A}_\Phi(\mathcal{V}) = \mathcal{A}_0(\mathcal{V}) + \mathrm{Tr}(V\Phi), \tag{2.32}$$

where $\mathcal{A}_0(\mathcal{V})$ is the tadpole in the reference BCFT defining the string field theory, and $V = \mathcal{V}(i)|I\rangle$.[9] This quantity is very easy to compute. The BRST exact term in eq. (2.6) does not contribute, so we have

$$\mathrm{Tr}(V\Psi) = \mathrm{Tr}\left(Vc\frac{1}{1+K}\right) = \int_0^\infty dt\, e^{-t}\,\mathrm{Tr}(Vc\Omega^t). \tag{2.33}$$

The inner product $\mathrm{Tr}(Vc\Omega^t)$ is a correlator on a cylinder of circumference t; by a scale transformation we can reduce it to a cylinder of unit circumference, producing a factor of t for the c ghost from the conformal transformation. Thus

$$\begin{aligned}\mathrm{Tr}(V\Psi) =&\ \mathrm{Tr}(Vc\Omega)\int_0^\infty dt\, te^{-t} = \mathrm{Tr}(Vc\Omega)\\ =&\ \langle\mathcal{V}(i\infty)c(0)\rangle_{C_1} = -\mathcal{A}_0(\mathcal{V}).\end{aligned} \tag{2.34}$$

Therefore the closed string tadpole vanishes:

$$\mathcal{A}_\Psi(\mathcal{V}) = 0. \tag{2.35}$$

It is interesting to note that for our solution the contribution to the amplitude comes from the BRST nontrivial term $c\frac{1}{1+K}$, whereas in $\mathcal{B}_0$ gauge it comes exclusively from the phantom piece [21].

Before concluding, let us mention that it is possible to generalize this calculation by computing the full off-shell boundary state of our solution, following [25]. The calculation would take us too far astray to present here, but we have confirmed that the boundary state for our solution vanishes identically.

3 Level expansions

Though we have a simple analytic proof of Sen's first conjecture, it is desirable to confirm our calculation by other means. The most trusted — but also the most poorly understood — method for calculating the energy is the old L_0 level expansion, which provided the first convincing numerical evidence for Sen's conjectures in [26–29]. The level expansion of our new solution, however, brings a surprise: if we add contributions to the energy level by level, the expansion is divergent.

The situation here appears to be analogous to the "sliver frame" $\mathcal{L}_0$ level expansion, where the energy is represented as the formal sum of an asymptotic series [1, 6]. For our new solution, the $\mathcal{L}_0$ level expansion is so simple that we are able to find an exact expression for the asymptotic series and its resummation, allowing us to gain concrete insight into the nonperturbative structure of the level expansion. The L_0 case, of course, is

[9] $\mathrm{Tr}(V\Phi)$ are the gauge invariant overlaps introduced in [22–24].

more complicated, but we have found a useful toy model of our solution where, remarkably, it is possible to compute the L_0 level expansion exactly in terms of elliptic functions. In both L_0 and $\mathcal{L}_0$ expansions, we resum the divergent series to obtain good agreement with Sen's first conjecture.

3.1 Curly $\mathcal{L}_0$ level expansion

We begin by considering the $\mathcal{L}_0$ level expansion. The $\mathcal{L}_0$ level expansion is quite analogous to the ordinary L_0 level expansion, but performed in a conformal frame well-adapted to the wedge state geometry of analytic solutions. $\mathcal{L}_0$ is the dilatation generator in the sliver conformal frame [1]:

$$\begin{aligned}\mathcal{L}_0 &= f_{\mathcal{S}}^{-1} \circ L_0 \\ &= \oint_0 \frac{d\xi}{2\pi i}(1+\xi^2)\tan^{-1}\xi\, T(\xi),\end{aligned} \tag{3.1}$$

where $f_{\mathcal{S}}(z) = \frac{2}{\pi}\tan^{-1} z$ is the sliver coordinate map. We define the level L of a state to be its $\mathcal{L}_0$ eigenvalue plus one, so the tachyon is at level zero. We can write such eigenstates in the form

$$F\phi F, \tag{3.2}$$

where $F = \sqrt{\Omega}$ is the square root of the $\mathrm{SL}(2,\mathbb{R})$ vacuum, and ϕ corresponds to an insertion of an operator with scaling dimension $L-1$ in the sliver coordinate frame. K, B, c have scaling dimension $1, 1, -1$ respectively, and the dimensions are additive with the star product. Therefore, any state at level L in the KBc subalgebra can be written using states of the form

$$F\Big(K^l cBK^m cK^n\Big)F, \qquad l+m+n = L. \tag{3.3}$$

This is a different basis of eigenstates from the one used in [1], but either basis gives the same level expansion for the energy.

To expand the solution (2.10) in terms of $\mathcal{L}_0$ eigenstates, we multiply and divide by F,

$$\hat{\Psi} = F\left(\frac{e^{K/2}}{\sqrt{1+K}}\Big[c + cKBc\Big]\frac{e^{K/2}}{\sqrt{1+K}}\right)F, \tag{3.4}$$

and expand the factor in parentheses in powers of K. It is useful to introduce the field

$$\hat{\Psi}(z) = z^{\mathcal{L}_0}\hat{\Psi} = F\left(\frac{e^{zK/2}}{\sqrt{1+zK}}\Big[\frac{1}{z}c + cKBc\Big]\frac{e^{zK/2}}{\sqrt{1+zK}}\right)F. \tag{3.5}$$

Then the $\mathcal{L}_0$ level expansion is equivalent to a power series expansion in z.

To compute the energy we should sum the infinite series

$$E = \sum_{n=-2}^{\infty}\mathcal{E}_n, \tag{3.6}$$

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where $\mathcal{E}_n$ is the contribution to the energy (or the action) coming from fields whose $\mathcal{L}_0$ eigenvalues add up to n. Assuming the equations of motion, the $\mathcal{E}_n$s can be found from the expression

$$\mathcal{E}_n = \frac{1}{6}\oint_0 \frac{dz}{2\pi i}\frac{1}{z^{n+1}}\langle\hat{\Psi}(z), Q_B\hat{\Psi}(z)\rangle. \tag{3.7}$$

Therefore, to find the expansion we should evaluate the inner product

$$\mathcal{E}(z) = \frac{1}{6}\langle\hat{\Psi}(z), Q_B\hat{\Psi}(z)\rangle. \tag{3.8}$$

In $\mathcal{B}_0$ gauge, the computation of this quantity appears to be a nontrivial task, but for our new solution it is quite straightforward. The final answer is naturally expressed in terms of a variable Z, related to z by an $\mathrm{SL}(2,\mathbb{R})$ transformation:

$$Z = \frac{1}{2}\frac{z}{1-z}. \tag{3.9}$$

We find

$$\mathcal{E}(z) = -\frac{1}{2\pi^2}\left[1 + \frac{2}{3}\frac{1}{Z} + \frac{1}{6}\frac{1}{Z^2} + \frac{1}{6\pi}\frac{I(Z)}{Z^4}\right], \tag{3.10}$$

where $I(Z)$ is the integral

$$I(Z) = \int_0^\infty du\, e^{-u/Z}(u+1)^3 \sin\frac{\pi}{u+1}. \tag{3.11}$$

Note that as z approaches 1 (or $Z \to \infty$) the energy function approaches the expected value $\mathcal{E}(1) = -\frac{1}{2\pi^2}$.

To find the $\mathcal{E}_n$s, we need a power series expansion for this integral. To this end, expand the second factor in the integrand as a Taylor series:

$$(1+u)^3 \sin\frac{\pi}{1+u} = \sum_{n=1}^{\infty} \ell_n u^n, \tag{3.12}$$

where ℓ_ns can be expressed in terms of generalized Laguerre polynomials

$$\ell_n = (-1)^n \mathrm{Im}\big[L_n^{-4}(i\pi)\big]. \tag{3.13}$$

Integrating over u produces a factor of $n!$ in the sum, so we find the power series for $\mathcal{E}(z)$

$$\mathcal{E}(z) = -\frac{1}{2\pi^2}\left[1 + \frac{2}{3}\frac{1}{Z} + \frac{1}{6}\frac{1}{Z^2} + \frac{1}{6\pi}\sum_{n=1}^{\infty} n!\ell_n\, Z^{n-3}\right]. \tag{3.14}$$

This is a prototype for an asymptotic expansion. The $n!$ divergence of the coefficients is not helped by the ℓ_ns, which themselves diverge quite rapidly[10] due to the essential singularity in the Laguerre generating function at $u = -1$.

[10] The large n asymptotics of the Laguerre polynomials implies $\ln|\ell_n| \sim \sqrt{2\pi n}$.

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N	-2	0	2	4	6	8
New solution	-1.3333	-0.35507	-4.4137	-45.133	-269.51	22051
$\mathcal{B}_0$ gauge	-1.3333	-1.0015	-0.98539	-1.0327	-1.3054	6.7582

Table 1. Partial sum $\sum_{n=-2}^{N} \mathcal{E}_n$ up to $N = 8$ in units of $\frac{1}{2\pi^2}$, shown for the new solution eq. (2.6), eq. (2.10) and the $\mathcal{B}_0$ gauge solution, taken from [1].

From here it is a trivial extra step to expand Z in terms of z and read off the $\mathcal{E}_n$s. To the first few orders, we find explicitly:

$$\begin{aligned}\mathcal{E}(z) = \frac{1}{6}\Bigg[&-\frac{4}{\pi^2}\frac{1}{z^2} + \left(-\frac{2}{\pi^2}+\frac{1}{2}\right) - \frac{\pi^2}{8}z^2 + \frac{\pi^2}{2}z^3 + \left(-\frac{33\pi^2}{16}+\frac{\pi^4}{32}\right)z^4 \\ &+ \left(\frac{37\pi^2}{4} - \frac{3\pi^4}{8}\right)z^5 + \left(-\frac{365\pi^2}{8} + \frac{55\pi^4}{16} - \frac{\pi^6}{128}\right)z^6 \\ &+ \left(\frac{987\pi^2}{4} - \frac{235\pi^4}{8} + \frac{3\pi^6}{16}\right)z^7 + \cdots \Bigg].\end{aligned} \tag{3.15}$$

This gives an efficient method for computing $\mathcal{E}_n$s. Indeed, we were easily able to compute the $\mathcal{E}_n$s out to $n = 400$ and could have gone much further, whereas with our current understanding the calculation in $\mathcal{B}_0$ gauge becomes time consuming much beyond $n = 50$.

For illustrative purposes, we have listed the first few partial sums of the $\mathcal{E}_n$s in table 1, both for the new solution and the $\mathcal{B}_0$ gauge solution. Both reveal an "approximation" to the energy which is typical of a divergent asymptotic series. However, the partial sum for our new solution diverges much faster than in $\mathcal{B}_0$ gauge — ironically, the best approximation to the energy is the trivial one, where we truncate the solution down to the zero momentum tachyon.

To compute the energy, it is necessary to resum the asymptotic series. One way to do this is to use the method of Padé approximants [1], where we replace the asymptotic series $z^2\mathcal{E}(z)$ by a Padé approximant $P^n_m(z)$—a ratio of a degree n polynomial to a degree m polynomial chosen so that the first $m+n$ terms in the Taylor expansion of $P^n_m(z)$ match those of $z^2\mathcal{E}(z)$. The approximation to the energy is then revealed by evaluating $P^n_m(1)$. A second method[11] is to use a combination of Padé and Borel resummation. Here we replace the Borel transform of $z^2\mathcal{E}(z)$ by its Padé approximant $P^n_m(z)_{\text{Borel}}$ and evaluate the integral

$$\widetilde{P}^n_m(z) = \int_0^\infty dt\, e^{-t} P^n_m(tz)_{\text{Borel}} \tag{3.16}$$

at $z = 1$. In table 2 we list Padé and Padé-Borel approximations to the energy including fields out to level 200. Both confirm Sen's conjecture to very high accuracy. At low levels, Padé-Borel does a little better than Padé, though at very high levels Padé appears to be more accurate.[12]

[11] We thank D. Gross for suggesting this to us.

[12] Note that the convergence is slower than it is in $\mathcal{B}_0$ gauge: to get results as good as our $P^{60}_{60}(1)$, one only has to go out to $P^{18}_{18}(1)$ in $\mathcal{B}_0$ gauge.

	$P^n_n(1)$	$\widetilde{P}^n_n(1)$
$n=0$	-1.33333	-1.33333
$n=2$	-1.14334	-0.994896
$n=4$	-0.898883	-0.900412
$n=6$	-1.04241	-1.00487
$n=8$	-0.996478	-1.00029
$n=10$	-0.995773	-0.999944
$n=20$	-0.99991237	-0.99996793
$n=40$	-0.99998202	-0.99999517
$n=60$	-0.99999945	-0.99999754
$n=80$	-0.99999984	-0.99999904
$n=100$	-0.99999995	-0.99999954

Table 2. Padé and Padé-Borel approximation to the energy in units of $\frac{1}{2\pi^2}$. We have shown the approximants for $m=n$, since Padé resummation is generally most reliable when the numerator and denominator are polynomials of similar order.

It is interesting to understand why the $\mathcal{L}_0$ level expansion is asymptotic. By analogy with the old argument about the divergence of perturbation theory in QED, one suspects that something severe must happen to the energy $\mathcal{E}(z)$ as the "coupling constant" z is taken to be negative. The problem is easy to identify: for $z<0$ the string field $\hat{\Psi}(z)$ does not exist. That is, though $\hat{\Psi}(z)$ has a well-defined expansion in terms of $\mathcal{L}_0$ eigenstates, for $z<0$ the expansion does not converge to a well defined string field. The problem comes from the factor $\frac{1}{1+zK}$, which for $z<0$ would only seem to make sense as an integral over singular "inverse" wedge states. This fact should show up as some sort of pathology in the energy $z^2\mathcal{E}(z)$ for $z\le 0$. In fact, because we have a closed form expression eq. (3.10), we can plot the energy to see what happens. As can be seen from figure 2, $z^2\mathcal{E}(z)$ has a branch point at $z=0$ together with a branch cut extending to $z=\infty$. Though we can analytically continue to negative z, the continuation is not unique and moreover is *complex*, in contradiction with the fact that $\hat{\Psi}(z)$ is real to any finite level in the level expansion. Therefore $z^2\mathcal{E}(z)$ for $z<0$ cannot be interpreted as a BRST inner product of $\hat{\Psi}(z)$. Incidentally, note that there is another branch point at $z=1$. This comes from the factor $Fe^{zK/2}$, which for $z>1$ is an inverse wedge state.

We expect that this phenomenon is quite general. For any solution depending on some $f(K)$ expressed in terms of positive powers of the $SL(2,\mathbb{R})$ vacuum, $f(zK)$ for $z<0$ will be undefined. Therefore the energy function should be singular at $z=0$, rendering the $\mathcal{L}_0$ level expansion asymptotic.

3.2 Square L_0 level expansion

The traditional L_0 expansion of a string field very efficiently summarizes all possible overlaps with Fock states up to a given conformal weight. Such an information is often useful, either in explicit numerical computations, or as one possible criterion of a string field being well defined.

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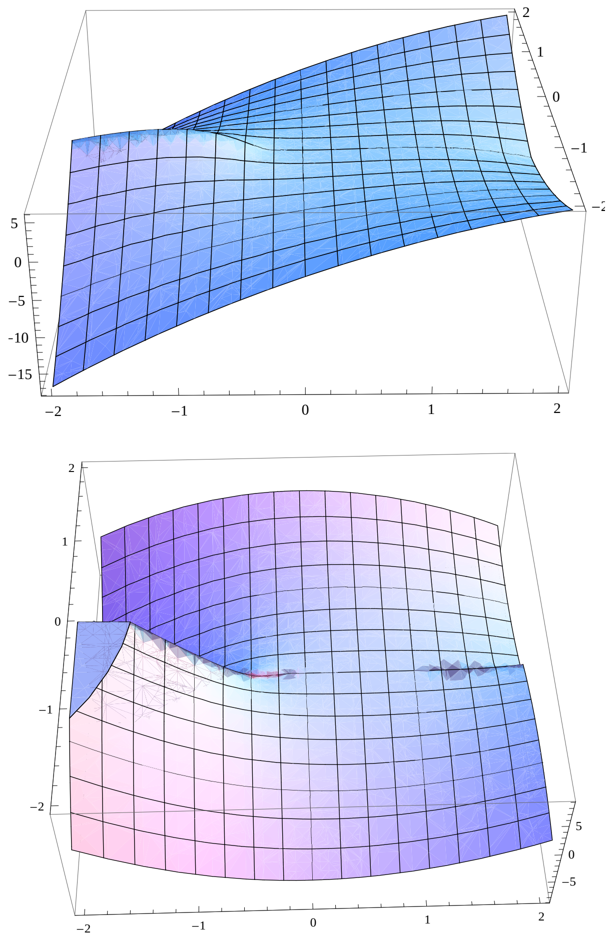

Figure 2. Real and imaginary parts of $z^2\mathcal{E}(z)$ for $-2 < \mathrm{Re}(z) < 2$ and $-2 < \mathrm{Im}(z) < 2$, shown left and right, respectively. Note that the function is very smooth at $z = 0$ and 1, but they are nevertheless branch points.

To expand our solution in the eigenstates of L_0 it is convenient to use the techniques and formalism of [1]. The twist even (real) solution can be written as

$$\hat{\Psi} = \frac{1}{\pi}\int_0^\infty\!\!\int_0^\infty dt\, ds\, \frac{e^{-t-s}}{\sqrt{ts}}\widehat{U}_{t+s+1}\left[\frac{2}{\pi}\tilde{c}\left(\frac{\pi}{4}(s-t)\right) + \frac{1}{\pi}Q_B\widehat{\mathcal{B}}\tilde{c}\left(\frac{\pi}{4}(s-t)\right)\right]|0\rangle, \qquad (3.17)$$

where $\widehat{U}_r = U_r U_r^\star$ and the star denotes the BPZ conjugate. The rest of the notation follows [1], in particular $U_r = (2/r)^{\mathcal{L}_0}$. The tilde is used to translate the c insertions in the cylinder frame to the canonical upper half plane, explicitly $\tilde{c}(x) = \cos^2 x\, c(\tan x)$.

The string field can be readily expanded and the individual coefficients can be numer-

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ically integrated. We find

$$\begin{aligned}\hat{\Psi} = &\ 0.509038\, c_1|0\rangle + 0.13231\, c_{-1}|0\rangle - 0.00157618\, L_{-2}\, c_1|0\rangle + \\ &-0.0135795\, L_{-4}\, c_1|0\rangle + 0.0231579\, L_{-2}L_{-2}\, c_1|0\rangle + 0.0893356\, c_{-3}|0\rangle \\ &-0.00694698\, L_{-2}\, c_{-1}|0\rangle + \cdots + (Q_B\text{-exact}).\end{aligned} \tag{3.18}$$

For example the first coefficient is given by

$$\begin{aligned} t &= \frac{1}{2\pi^2}\int_0^\infty du \int_{-1}^1 dw\, e^{-u}\frac{(u+1)^2}{\sqrt{1-w^2}}\cos^2\left(\frac{\pi}{2}\frac{u}{u+1}w\right) \\ &= \frac{1}{4\pi}\int_1^\infty du\, e^{1-u}u^2\left(1+J_0\left(\pi\frac{u-1}{u}\right)\right) \\ &= 0.509038, \end{aligned} \tag{3.19}$$

where J_0 is a Bessel function of the first kind. To obtain eq. (3.19) from eq. (3.17) we have made a change of variables $u = t+s$ and $w = (t-s)/(t+s)$. In more generality all the coefficients are given by an integral of the form

$$\int_0^\infty du(u+1)^2 P\left(\frac{1}{u+1}\right)e^{-u}\int_{-1}^1 dw\frac{1}{\sqrt{1-w^2}}\cos^2\left(\frac{\pi}{2}\frac{u}{u+1}w\right)\tan^n\left(\frac{\pi}{2}\frac{u}{u+1}w\right), \tag{3.20}$$

where P is a polynomial whose detailed form depends on the coefficient in question. These integrals are absolutely convergent, but to evaluate them numerically with enough precision we found necessary to make a further change of variables $w = \sin\phi$ upon which the integrable singularity at $w = \pm 1$ disappears.

The apparently rapid decay of the coefficients suggests that the energy of the solution computed in level truncation should converge quite well. Let us compute the regularized energy, the analogue of eq. (3.10):

$$E(z) = \frac{1}{6}\langle z^{L_0}\hat{\Psi}, Q_B z^{L_0}\hat{\Psi}\rangle. \tag{3.21}$$

For $z = 1$ we recover the exact expression, and because the kinetic term is diagonal in L_0 eigenstates, the coefficients of the energy at order z^{2L-2} are exactly the contributions from fields at level L. With the help of the computer[13] we have computed the energy up to level 30 which in our basis includes contributions from 2455 fields. The resulting (normalized) energy takes the form

$$\begin{aligned} 2\pi^2 E(z) = &\ -\frac{0.85247}{z^2} - 0.0616762z^2 - 0.120529z^6 + 0.104037z^{10} - 0.132712z^{14} \\ &+0.158365z^{18} - 0.204746z^{22} + 0.268088z^{26} - 0.363999z^{30} + 0.496009z^{34} \\ &-0.682054z^{38} + 0.942044z^{42} - 1.30865z^{46} + 1.81739z^{50} - 2.52216z^{54} \\ &+3.49649z^{58} + \cdots . \end{aligned} \tag{3.22}$$

[13] Part of our computer code was written by Ian Ellwood while working on an unpublished project with the second author [30]. We thank him for kindly letting us use his code.

	$P^n_n(1)$	$\widetilde{P}^n_n(1)$
$n = 0$	-0.852470	-0.852470
$n = 4$	-0.787834	-0.871988
$n = 8$	-0.992052	-0.983243
$n = 12$	-0.992013	-0.984516
$n = 16$	-0.996081	-0.993936
$n = 20$	-0.999595	-0.993687
$n = 24$	-0.997322	-0.995001
$n = 28$	-0.997690	-0.993253

Table 3. Padé and Padé-Borel approximation to the energy in units of $\frac{1}{2\pi^2}$. We have shown only the diagonal approximants P^n_n and $\tilde{P}^n_n$ for $z^2E(z)$ at $z = 1$. Note that they depend on the contributions of fields up to level n.

The result for the lowest levels is encouraging: at lowest truncation level we find 85% of the expected energy, at level 2 we get 91% and at level 4 103%. But that is as close as we get to the correct answer; in fact it is obvious from eq. (3.22) that the contributions of higher levels are increasing in magnitude and therefore the series cannot converge.

As we've seen, a similar divergence occurs in the $\mathcal{L}_0$ level expansion, but this is the first time such behavior has appeared in the canonical L_0 level truncation scheme. We can evaluate the energy using either Padé or Padé-Borel resummation; as shown in table 3, both types of resummation confirm Sen's conjecture to better than 99% at level 28. It is of great interest to understand why the expansion of our solution is divergent. We explore the answer to this question using an explicitly soluble toy model in section 3.3.

Let us give the expansion of our solution in the original matter Virasoro+ghost oscillator basis used by Sen and Zwiebach [27], out to level 4:

$$\begin{aligned}\hat{\Psi} = &\, tc_1|0\rangle + uc_{-1}|0\rangle + vL^m_{-2}\,c_1|0\rangle + wb_{-2}c_0c_1|0\rangle + \\ &+AL^m_{-4}\,c_1|0\rangle + BL^m_{-2}L^m_{-2}\,c_1|0\rangle + Cc_{-3}|0\rangle + Db_{-3}c_{-1}c_1|0\rangle + \\ &+Eb_{-2}c_{-2}c_1|0\rangle + FL^m_{-2}c_{-1}|0\rangle + w_1L^m_{-3}c_0|0\rangle + w_2b_{-2}c_{-1}c_0|0\rangle + \\ &+w_3b_{-4}c_0c_1|0\rangle + w_4L^m_{-2}b_{-2}c_0c_1|0\rangle + \cdots .\end{aligned} \tag{3.23}$$

The coefficients above are given by

$$\begin{array}{llll} t = 0.509038 & A = -0.10674 & E = 0.242131 & w_1 = 0 \\ u = 0.772988 & B = 0.106714 & F = 0.673728 & w_2 = 1.13718 \\ v = 0.213559 & C = 1.11009 & & w_3 = 0.3338 \\ w = -0.211983 & D = 0.887287 & & w_4 = -0.343299. \end{array}$$

Surprisingly, the expectation values do not appear to be getting smaller at higher levels, at least out to level 4. Apparently this is an artifact of the choice of basis, since in the simpler basis eq. (3.18) the coefficients appear to decay quite rapidly. Of course, the level approximation to the energy is the same in either case.

	$P_n^n(1)$	$\widetilde{P}_n^n(1)$
$n=0$	-0.266085	-0.266085
$n=4$	-0.679355	-0.679026
$n=8$	-0.935655	-0.883524
$n=12$	-0.940574	-0.920585
$n=16$	-0.971911	-0.950665
$n=20$	$+0.452292$	-0.946722
$n=24$	-0.974222	-0.955226
$n=28$	-0.974103	-0.954514

Table 4. Padé and Padé-Borel approximation to the energy for the asymmetric solution in units of $\frac{1}{2\pi^2}$. We have shown the approximants for $m = n$. The value P_{20}^{20} is anomalously large due to an accidental position of a zero and a pole of the Padé approximant very near the value $z = 1$.

It is of interest to consider the level expansion of the non-real solution eq. (2.6) as well. Focusing on the BRST nontrivial part of the string field we find by numerical integration

$$\begin{aligned}\Psi \;=\; & 0.284394\, c_1|0\rangle + 0.249034\, c_0|0\rangle + 0.244516\, c_{-1}|0\rangle + 0.0359031\, L_{-2}\, c_1|0\rangle + \\ & +0.252567\, c_{-2}|0\rangle + 0.00302175\, L_{-2}\, c_0|0\rangle - 0.0177251\, L_{-4}\, c_1|0\rangle + \\ & +0.0175741\, L_{-2}L_{-2}\, c_1|0\rangle + 0.268936\, c_{-3}|0\rangle - 0.010923\, L_{-2}\, c_{-1}|0\rangle + \cdots \\ & +(Q_B\text{-exact}).\end{aligned} \tag{3.24}$$

We have computed the components of the string field up to level 30. The resulting z-dependent energy is given by

$$\begin{aligned}2\pi^2 E_{\text{asym}}(z) = & -\frac{0.266085}{z^2} - 0.408062 - 0.00644403z^2 + 0.0200865z^4 - 0.292541z^6 \\ & -0.108361z^8 + 0.23035z^{10} + 0.0672657z^{12} - 0.275233z^{14} - 0.074523z^{16} \\ & +0.299372z^{18} + 0.0574889z^{20} - 0.362862z^{22} - 0.0592361z^{24} + 0.440743z^{26} \\ & +0.0513536z^{28} - 0.563397z^{30} - 0.0524896z^{32} + 0.721687z^{34} + 0.0471252z^{36} \\ & -0.944548z^{38} - 0.0474732z^{40} + 1.24749z^{42} + 0.0439229z^{44} - 1.67218z^{46} \\ & -0.0442855z^{48} + 2.25055z^{50} + 0.0415004z^{52} - 3.04491z^{54} - 0.0416184z^{56} \\ & +4.13094z^{58}.\end{aligned} \tag{3.25}$$

There are twice as many terms here because the solution is not twist even, so odd levels contribute to the action as well. Again the expansion is divergent and we can resum the series using Padé or Padé-Borel resummation. The results in table 4 nicely confirm Sen's conjecture, though we do not get quite as close to the expected answer as with the real solution.

3.3 Exactly soluble model for the L_0 level expansion

Let us now try to understand why the L_0 expansion of our solution is divergent. Following the logic of section 3.1, the divergence should be related to the analytic structure of the

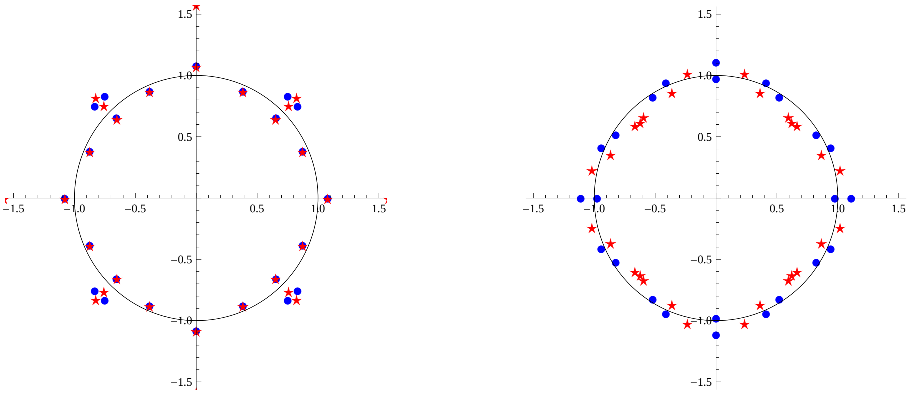

Figure 3. a) Location of the poles and zeros of the Padé approximant P^{30}_{30} of $z^2 E(z)$ in eq. (3.21). Red asterisks indicate position of poles; blue dots indicate location of zeros. b) The analogous picture for the identity correlator (3.29). Note that for the true solution the poles and zeros almost coincide, which suggests milder singularities along the unit circle than for the identity correlator.

energy as a function of the parameter z. Given the slow non-exponential growth of the coefficients in eq. (3.22) we expect the function $z^2E(z)$ to be holomorphic inside the unit disk but with some singularities on its boundary. Plotting the distribution of poles and zeros of Padé approximants (see figure 3) suggests that $z^2E(z)$ cannot be analytically continued beyond the unit disk, just like elliptic functions in the q variable.

We can gain an important insight into this problem by looking at a certain class of coefficients in eq. (3.18). For example the family of states $(L_{-2})^n c_1|0\rangle$ comes with coefficients given by

$$v_n = \frac{(-3)^{-n}}{\pi(n-1)!}\int_0^\infty du\, e^{-u}\left(1+J_0\left(\pi\frac{u}{u+1}\right)\right)\left(\frac{(u+3)(u-1)}{4n}-\frac{2}{u+1}\right)\left(1-\frac{4}{(u+1)^2}\right)^{n-1}. \tag{3.26}$$

For large n, these behave as

$$v_n = \frac{1}{2\pi n!}\left(1+O\left(\frac{1}{n}\right)\right). \tag{3.27}$$

This looks exactly as if the coefficients were coming from the identity string field. This identity-like behavior is not surprising. The dominant contribution to our solution comes from wedge states close to the identity, since larger wedges are exponentially suppressed.

This suggests that we consider the level expansion of the field $c = \frac{1}{\pi}U_1^* c_1|0\rangle$ as a toy model for the level expansion of our solution $\hat{\Psi}$. The level expansion of c will not yield the brane tension (c is not a solution), but it is of interest in its own right in relation to certain other energy computations, as we will describe shortly. The analogue of the z-dependent energy for c is:

$$F(z) = \left\langle z^{L_0}c, z^{L_0}Q_B c\right\rangle = \frac{1}{\pi^2}\langle\, 0|c_{-1}U_1 z^{2L_0}U_1^* c_1 c_0|0\,\rangle. \tag{3.28}$$

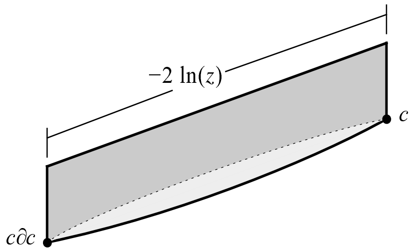

Figure 4. Worldsheet picture of our toy correlator eq. (3.28).

To our great surprise, we found that the contribution to $F(z)$ from each level is exactly an integer:

$$F(z) = -\frac{1}{4\pi^2}\left[\frac{1}{z^2} - 4z^2 + 10z^6 - 24z^{10} + 55z^{14} - 116z^{18} + 230z^{22} - \right.$$
$$\left. -440z^{26} + 819z^{30} - 1480z^{34} + 2602z^{38} + \cdots\right]. \tag{3.29}$$

Such a nice expansion is sure to have an analytic explanation, but before we derive it, let us note that the question about the analytic behavior of $F(z)$ is essentially answered at this point. By the Polya-Carlson theorem a function with integer coefficients in its Taylor expansion cannot be extended beyond the unit disk unless it is rational (which, as we will show, it is not). Therefore $F(z)$ must have an essential singularity at every point on the unit circle. This agrees well with the analytic structure $z^2E(z)$ in eq. (3.21), as suggested by position of the Padé poles and zeros.

Let us now see how to evaluate $F(z)$ analytically. Geometrically, eq. (3.28) can be represented as a correlator of ghost operators on a paper-bag-shaped surface obtained by taking a rectangular strip, folding it in half and gluing together adjacent edges of the folded boundary (see figure 4). To evaluate the correlator directly one would have to conformally map the geometry to the upper half plane where we know all the correlation functions. Undoubtedly such a map can be constructed (along the lines of [31]),[14] but there is a simple shortcut.

Algebraically, our task is to "normal order" $U_1 z^{2L_0} U_1^*$, that is, to find a conformal map $\psi(\xi)$, holomorphic in the vicinity of $\xi = 0$ such that

$$U_1 z^{2L_0} U_1^* = U_\psi^* U_\psi, \tag{3.30}$$

where U_ψ is the action of a finite conformal transformation $\psi(\xi)$ (note that ψ implicitly depends on z). If we can find such a ψ, then we can easily compute $F(z)$:

$$F(z) = -\frac{1}{\pi^2}\psi'(0)^{-2}. \tag{3.31}$$

[14] Upon completion of this paper we were informed by Ian Ellwood that such a map has been constructed in [32, 33].

In terms of conformal transformations the problem can be stated equivalently as finding $\psi(\xi)$ holomorphic around the origin, such that

$$f \circ I \circ f^{-1} \circ I = I \circ \psi^{-1} \circ I \circ \psi, \tag{3.32}$$

where I stands for the inversion $I : \xi \to -1/\xi$, and f is the map entering the definition of the star algebra identity composed with rescaling by z, $f(\xi) = \frac{2\xi}{1-z\xi^2}$. To make sense of the equation eq. (3.32) we have to assume that f is holomorphic and univalent in some domain which includes the unit disk. Both sides of the equation have to match in some annular region around the unit circle where both are simultaneously meaningful. Alternatively, one can demand that both sides agree as formal power series in the scaling parameter z, not to be confused with the coordinate ξ. This is a well known problem in mathematics related to uniformization and the existence of the Neretin semigroup [34, 35].

Although in general it is more convenient to carry out computations in a CFT-independent way, for this particular problem it is useful to pick the simplest CFT corresponding to strings propagating freely in flat space. The identity string field has a very simple expression and its correlators can be easily evaluated by oscillator methods, see e.g. [36–38]. Consider the following correlator

$$\left(i\sqrt{2/\alpha'}\right)^2 \langle\, I \circ \partial X(x) U_1 z^{2L_0} U_1^* \partial X(y)\,\rangle. \tag{3.33}$$

Here we assume the total central charge is zero, so an insertion of a weight zero operator like $c\partial c\partial^2 c$ is implicit. We can compute the correlator in two different ways: Either using formula eq. (3.30), upon which we find the correlator is equal to

$$\frac{\psi'(x)\psi'(y)}{(1+\psi(x)\psi(y))^2}, \tag{3.34}$$

or we can compute it with the oscillator formalism. Let us commute ∂X towards the center of the correlator and write it in its mode expansion

$$i\sqrt{2/\alpha'}\,\partial X(w) = \sum_{n=-\infty}^{\infty} \alpha_n w^{-n-1}. \tag{3.35}$$

Next let us introduce normalized oscillators $a_n = \alpha_n/\sqrt{n}$ for $n > 0$ and rewrite

$$U_1^*|0\rangle = e^{-\frac{1}{2}\sum_{n=1}^{\infty}(-1)^n a_n^\dagger a_n^\dagger}|0\rangle. \tag{3.36}$$

Using the formula

$$\langle\, 0|e^{\frac{1}{2}a.S.a} a_n a_m^\dagger e^{\frac{1}{2}a^\dagger.V.a^\dagger}|0\,\rangle = \det(1-S.V)^{-1/2}(1-V.S)^{-1}_{\rm nm}, \tag{3.37}$$

we find

$$\frac{\psi'(x)\psi'(y)}{(1+\psi(x)\psi(y))^2} = \sum_{n=1}^{\infty} n z^{2n}\left(\tilde{x}^{-n} + (-)^{n+1}\tilde{x}^n\right)\left(\tilde{y}^{-n} + (-)^{n+1}\tilde{y}^n\right)\frac{1}{1-z^{4n}}\frac{1}{\tilde{x}\tilde{y}}\frac{d\tilde{x}}{dx}\frac{d\tilde{y}}{dy}, \tag{3.38}$$

where

$$\tilde{x} = x - \sqrt{1+x^2},$$
$$\tilde{y} = y + \sqrt{1+y^2}. \tag{3.39}$$

Note that thanks to the vanishing total central charge the determinant factor from eq. (3.37) cancels against normalization constants from the other sectors.

Imposing $\psi(0) = 0$ the equation can be easily integrated. Expanding $1/(1 - z^{4n})$ into a geometric series the two infinite sums can be interchanged and one finds

$$1 + \psi(x)\psi(y) = \prod_{k=0}^{\infty} \frac{(1 - \frac{\tilde{y}}{\tilde{x}} z^{4k+2})(1 - \frac{\tilde{x}}{\tilde{y}} z^{4k+2})(1 + \frac{1}{\tilde{x}\tilde{y}} z^{4k+2})(1 + \tilde{x}\tilde{y} z^{4k+2})}{(1 - \frac{1}{\tilde{x}^2} z^{4k+4})(1 - \tilde{x}^2 z^{4k+4})(1 - \frac{1}{\tilde{y}^2} z^{4k+4})(1 - \tilde{y}^2 z^{4k+4})} (1 - z^{8k+4})^2. \tag{3.40}$$

This equation at first sight seems rather unlikely to be self-consistent, the right hand side does not look anything like one plus something factorizable. Fortunately, the infinite product can be expressed in terms of Jacobi theta functions:[15]

$$1 + \psi(x)\psi(y) = \theta_4(1)\theta_3(1) \frac{\theta_4\left(\frac{\tilde{x}}{\tilde{y}}\right)\theta_3(\tilde{x}\tilde{y})}{\theta_4(\tilde{x})\theta_3(\tilde{x})\theta_4(\tilde{y})\theta_3(\tilde{y})}. \tag{3.41}$$

The theta functions all depend on common *nome* $q = e^{2\pi i\tau}$ which we suppressed and which is related to our previous scaling parameter z by $q = z^4$. Explicitly the theta functions are given by

$$\theta_3(x) = \sum_{n=-\infty}^{\infty} q^{n^2/2} x^n = \prod_{m=1}^{\infty} (1 - q^m)(1 + x q^{m-1/2})(1 + x^{-1} q^{m-1/2}), \tag{3.42}$$

$$\theta_4(x) = \sum_{n=-\infty}^{\infty} (-1)^n q^{n^2/2} x^n = \prod_{m=1}^{\infty} (1 - q^m)(1 - x q^{m-1/2})(1 - x^{-1} q^{m-1/2}), \tag{3.43}$$

$$\begin{aligned} \theta_2(x) &= \sum_{n=-\infty}^{\infty} q^{(n-1/2)^2/2} x^{n-1/2} \\ &= q^{1/8}(x^{1/2} + x^{-1/2}) \prod_{m=1}^{\infty} (1 - q^m)(1 + x q^m)(1 + x^{-1} q^m), \end{aligned} \tag{3.44}$$

$$\begin{aligned} \theta_1(x) &= i \sum_{n=-\infty}^{\infty} (-1)^n q^{(n-1/2)^2/2} x^{n-1/2} \\ &= -i q^{1/8}(x^{1/2} - x^{-1/2}) \prod_{m=1}^{\infty} (1 - q^m)(1 - x q^m)(1 - x^{-1} q^m). \end{aligned} \tag{3.45}$$

From the representation in terms of infinite sums, one can easily derive an identity

$$\theta_4\left(\frac{\tilde{x}}{\tilde{y}}\right)\theta_3(\tilde{x}\tilde{y}) = \frac{\theta_4(\tilde{x})\theta_3(\tilde{x})\theta_4(\tilde{y})\theta_3(\tilde{y})}{\theta_4(1)\theta_3(1)} - \frac{\theta_1(\tilde{x})\theta_2(\tilde{x})\theta_1(\tilde{y})\theta_2(\tilde{y})}{\theta_4(1)\theta_3(1)}. \tag{3.46}$$

[15] We use the notation of Polchinski, String Theory, Vol I.

Using this identity the expression for $1+\psi(x)\psi(y)$ simplifies and we find

$$1+\psi(x)\psi(y) = 1-\frac{\theta_1(\tilde{x})\theta_2(\tilde{x})}{\theta_3(\tilde{x})\theta_4(\tilde{x})}\frac{\theta_1(\tilde{y})\theta_2(\tilde{y})}{\theta_3(\tilde{y})\theta_4(\tilde{y})}, \tag{3.47}$$

and hence

$$\psi(x) = i\frac{\theta_1(\tilde{x})\theta_2(\tilde{x})}{\theta_3(\tilde{x})\theta_4(\tilde{x})} = q^{\frac{1}{4}}(\tilde{x}-\tilde{x}^{-1})\prod_{m=1}^{\infty}\frac{1-\tilde{x}^2q^{2m}}{1-\tilde{x}^2q^{2m-1}}\frac{1-\tilde{x}^{-2}q^{2m}}{1-\tilde{x}^{-2}q^{2m-1}}. \tag{3.48}$$

We see that indeed $\psi(0)=0$ and

$$\psi'(0) = 2q^{1/4}\prod_{m=1}^{\infty}\left(\frac{1-q^{2m}}{1-q^{2m-1}}\right)^2 = \frac{\eta(2\tau)^4}{\eta(\tau)^2}. \tag{3.49}$$

Now we can very easily compute the correlator eq. (3.28):

$$F(z) = -\frac{1}{\pi^2}\frac{\eta(\tau)^4}{\eta(2\tau)^8}, \qquad z = e^{i\pi\tau/2}. \tag{3.50}$$

This function is holomorphic inside the unit circle $|z|<1$, but every point on the unit circle is an essential singularity and the function cannot be analytically continued beyond the unit disk (see figure 3b for the distribution of poles and zeros of its Padé approximant).

We can gain some intuition into the origin of these singularities by looking at figure 4. For $z=1$, the c insertions sit right on top of each other, but for $z>1$ the picture does not appear to make sense — formally, the cs should be separated by a worldsheet of "negative" length. This is quite analogous to the worldsheet interpretation of inverse wedge states, which are responsible for the divergence of the $\mathcal{L}_0$ level expansion. Therefore it is not surprising that $F(z)$ is undefined for $|z|>1$. Note also that the $F(z)$ occurs in the lower limit of integration when we evaluate $E(z)$. Therefore figure 4 for $z>0$ gives a nice intuitive picture for why the L_0 level expansion of our solution is divergent.

Now that we have a closed form solution for the level expansion, we can evaluate $F(1)=\mathrm{Tr}[cQc]$ and see what we get:[16]

$$\mathrm{Tr}(cQc) = -\frac{1}{\pi^2}\lim_{z\to 1^-}\frac{\eta(\tau)^4}{\eta(2\tau)^8} = 0. \tag{3.51}$$

We have checked that this result agrees with the Padé resummation of the series eq. (3.29). In fact, we get the same answer when computing in the $\mathcal{L}_0$ level expansion:

$$\langle z^{\mathcal{L}_0}c, z^{\mathcal{L}_0}Q_Bc\rangle = -\frac{4}{\pi^2}\left(\frac{1-z}{z}\right)^2. \tag{3.52}$$

Again this vanishes at $z=1$. Given that cQc is an identity-like string field, it may be surprising that $\mathrm{Tr}(cQc)$ appears to vanish regardless of the regularization, and even holds

[16]To prove this limit we use the formula $\eta(-1/\tau)=\sqrt{-i\tau}\,\eta(\tau)$ and $\eta(\tau)\sim e^{i\pi\tau/12}$ for large and positive $\mathrm{Im}(\tau)$. Note that because $F(z)$ has essential singularities on the unit circle, in taking the limit $z\to 1$ we should be careful to follow a contour that intersects the real axis at an angle of less than 90°.

in the L_0 level expansion. There is actually a good formal argument for believing this result. Consider the energy of a vacuum solution Φ computed in the $\frac{1}{2}\mathcal{L}_0^-$ expansion. The energy function

$$\mathcal{E}^-(z) = \frac{1}{6}\langle z^{\frac{1}{2}\mathcal{L}_0^-}\Phi, z^{\frac{1}{2}\mathcal{L}_0^-}Q_B\Phi\rangle \tag{3.53}$$

is *independent* of z because $\frac{1}{2}\mathcal{L}_0^-$ is a reparameterization generator. Expanding Φ in a basis of $\frac{1}{2}\mathcal{L}_0^-$ eigenstates

$$\Phi \propto c + \text{higher levels}\ldots. \tag{3.54}$$

we can formally rewrite eq. (3.53) in the form,

$$\mathcal{E}^-(z) = \sum_{n=-2}^{\infty} z^n \mathcal{E}_n^-, \tag{3.55}$$

where $\mathcal{E}_n^-$ is the contribution to the action of fields whose total $\frac{1}{2}\mathcal{L}_0^-$ eigenvalues adds up to n. But since the energy is independent of z, only the contribution $\mathcal{E}_0^-$ can be nonvanishing, and in particular

$$\mathcal{E}_{-2}^- \propto \mathrm{Tr}(cQc) = 0, \tag{3.56}$$

consistent with the prediction of the L_0 and $\mathcal{L}_0$ level expansions. It would be interesting to test this formal argument by extending the above computations to the other $\mathcal{E}_n^-$.

4 Discussion

In this paper we have given a simple analytic solution for tachyon condensation in open bosonic string field theory. The absence of a regulator and phantom term makes the solution easier to work with than in $\mathcal{B}_0$ gauge. Moreover, the physics is much easier to see, as it is almost exclusively contained in the term:

$$c\frac{1}{1+K}, \tag{4.1}$$

which is nothing more than the zero momentum tachyon, albeit expressed in an unusual gauge (see appendix C). The second term

$$cKBc\frac{1}{1+K} \tag{4.2}$$

is BRST exact, and its only purpose is to make the tachyon eq. (4.1) satisfy the equation of motion. Of course, this fits nicely with the intuition that the condensation of the tachyon field is really what's responsible for the physics of tachyon condensation.

A novel feature of our solution is that it involves a continuous superposition of wedge states arbitrarily close to the identity. The fact that it is a continuous superposition, and not, say, an isolated identity-like piece, is crucial for the consistency of our solution. Indeed, many identity-based solutions have been proposed in the past, but such solutions provide no unambiguous calculation of the action.[17] Still, there are certain types of calculations

[17] Though identity based solutions are singular, some still correctly capture some nontrivial open string physics. See especially [39].

that would be problematic for our solution. For example, $b(1)|\Psi\rangle$ and $b(1)|\hat{\Psi}\rangle$ are divergent because the b ghost gets "too close" to the c insertion inside $\Psi, \hat{\Psi}$. We hope that such issues will not limit the utility of our solution.

Since the beginning, one of the great mysteries of string field theory has been the remarkable success of the level expansion. One byproduct of our analysis has been a more detailed picture of why the level expansion works, and in particular how it may fail to converge. It is quite remarkable that we were able to solve the L_0 level expansion exactly for the field c — it would be very interesting to find analogous solutions for other states. Ideas along these lines could be important for constructing a solution for the tachyon vacuum in Siegel gauge.

There are many questions related to the tachyon vacuum that have yet to be understood. For example, finding an analytic construction of the tachyon potential, understanding vacuum string field theory and multiple D-branes [24, 40, 41], recovering closed string physics around the tachyon vacuum, and finding an analytic tachyon vacuum in superstring field theory [42–47]. Perhaps this solution could inspire new approaches to marginal deformations [48–55], or help in the construction of lump solutions [56]. We hope that our work will be useful for studying these important issues.

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A Star products and cylinder correlators

In this appendix we explain how to translate expressions given in the text into conformal field theory correlation functions on the cylinder. The basic starting point are string fields Φ which can be represented as a correlation function on a semi-infinite vertical strip of worldsheet in the complex plane, with some operator insertions placed inside. The bottom edge of the strip lies on the real axis, and corresponds to the boundary of the open string; the "top" of the strip is at $+i\infty$, and corresponds to the open string midpoint. On the positive and negative vertical edges of the strip we impose boundary conditions corresponding to the left and right halves of the open string,[18] respectively. Evaluating the

[18]Fixing these boundary conditions requires a choice of parameterization of the string along the vertical edges. Different parameterizations correspond to different choices of projector conformal frames [12]. In this paper we have been using the sliver conformal frame, where the standard parameterization of the half

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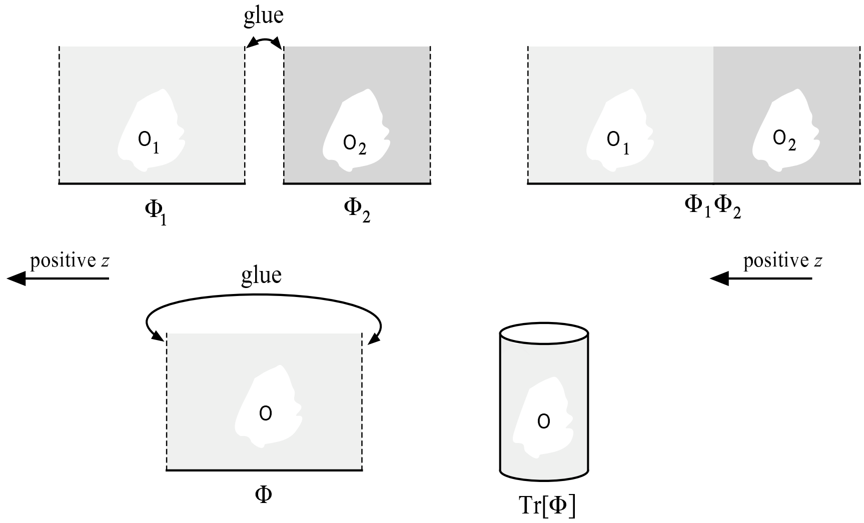

Figure 5. Star product and trace of open string functionals, represented as correlation functions on a semi-infinite strip with possible operator insertions. Note that if we visualize the real axis as increasing towards the left, the order of the multiplication matches the geometrical order of the gluing.

resulting correlator gives a representation of Φ as a Schroedinger functional of a classical open string configuration $\Phi[x(\sigma), ghosts]$.

Perhaps there is a possibility for geometrical confusion here, since the *left* half of the string lies on the *right* (positive) edge of the strip in the complex plane. This is an artifact of our star product convention, which adheres to [1, 14, 27, 57]. To solve this problem, [2] introduced a different convention for the star product with the opposite identification of left and right. We keep the old convention, but to avoid confusing pictures it is helpful to visualize the complex plane so that the positive real axis increases towards the left — that is, our complex plane is related to the old one by $z \to -z^*$. Then the left half of the string lies on the left (positive) boundary of the strip. Note that closed contours in our visualization move clockwise — so our convention might be called the *left handed* picture for the star product, whereas that of [2] is the *right handed* picture.

Given a string field defined as a correlator on the strip, we can compute star products and traces as follows: To compute the product $\Phi_1\Phi_2[x(\sigma)]$, we glue Φ_1's negative vertical edge to Φ_2's positive vertical edge, and evaluate the resulting correlator. To compute the trace, we glue the positive and negative edges of the strip together to form a correlation function on the cylinder. See figure 5. The gluing of edges is analogous to the contraction of matrix indices — this is the essential intuition behind the split string formalism [14, 58, 59]. Note that with our picture of the complex plane, Φ_1's strip appears to the left of Φ_2's in the product $\Phi_1\Phi_2[x(\sigma)]$, as would seem natural.

string with $\sigma \in [0, \frac{\pi}{2}]$ maps to the vertical height $y = \frac{1}{\pi}\tanh^{-1}\sin\sigma \in [0, \infty]$ on the strip edge. If we had used the butterfly frame, the edges would be parameterized as $y = \frac{1}{4}\tan\sigma \in [0, \infty]$.

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Let us demonstrate how this works for fields in the KBc subalgebra. We use the doubling trick to extend holomorphically to the lower half plane, so the semi-infinite vertical strip becomes an infinite vertical strip extending from $-i\infty$ to $+i\infty$. The wedge state Ω^t is then represented as an infinite vertical strip of worldsheet of width t, without any operator insertions. A Fock space state $|\phi\rangle = \phi(0)|0\rangle$ is a vertical strip of width 1, with an insertion $f_\mathcal{S} \circ \phi(0)$ placed halfway between the edges of the strip, on the real axis. Here

$$f_\mathcal{S}(z) = \frac{2}{\pi} \tan^{-1} z \tag{A.1}$$

is called the sliver conformal map, and maps the unit disk to an infinite vertical strip of width 1. Finally, consider the string fields K, B, c. We take them to be *infinitely thin* vertical strips of worldsheet carrying operator insertions

$$\begin{aligned} K &\quad\longrightarrow\quad \mathfrak{K} \equiv \int_{-i\infty}^{i\infty} \frac{dz}{2\pi i} T(z), \\ B &\quad\longrightarrow\quad \mathfrak{B} \equiv \int_{-i\infty}^{i\infty} \frac{dz}{2\pi i} b(z), \\ c &\quad\longrightarrow\quad c(z), \end{aligned} \tag{A.2}$$

where $c(z)$ is inserted exactly on the strip, on the real axis. We can now compute star products and traces of fields in the KBc subalgebra by gluing strip edges, as described above. The procedure is illustrated for an example $\mathrm{Tr}(cKBc\Omega^t\phi)$ in figure 6.

Using this basic procedure, we can calculate the overlap of our solution eq. (2.6) with any Fock space state:

$$\mathrm{Tr}(\Psi\phi) = \int_0^\infty dt\, e^{-t} \left\langle \left[c(t+\tfrac{1}{2}) + c(t+\tfrac{1}{2})\, \mathfrak{K}\mathfrak{B} \lim_{\epsilon\to 0} c(t+\tfrac{1}{2}-\epsilon) \right] f_\mathcal{S} \circ \phi(0) \right\rangle_{C_{t+1}}, \tag{A.3}$$

where $\langle\cdot\rangle_{C_{t+1}}$ is the correlation function on the cylinder of circumference $t+1$ and the $\mathfrak{B}$ and $\mathfrak{K}$ contour insertions must be integrated between the c ghosts on either side. It is often convenient to represent the $\mathfrak{K}$ insertion as a derivative of a wedge state $K = \frac{d}{ds}\Omega^s\big|_{s=0}$. Therefore we can also write

$$\begin{aligned} \mathrm{Tr}(\Psi\phi) = \int_0^\infty dt\, e^{-t} \Bigg[&\left\langle c(t+\tfrac{1}{2}) f_\mathcal{S} \circ \phi(0) \right\rangle_{C_{t+1}} \\ &+ \frac{d}{ds} \left\langle c(t+s+\tfrac{1}{2}) \mathfrak{B} c(t+\tfrac{1}{2}) f_\mathcal{S} \circ \phi(0) \right\rangle_{C_{t+s+1}} \Bigg|_{s=0} \Bigg]. \end{aligned} \tag{A.4}$$

Note that the gluing prescription does not determine the absolute location of the operator insertions in the complex plane — it only determines their relative positions, modulo the circumference of the cylinder. Here we have made some convenient choice for the coordinates of the insertions.

Since both left and right handed star products have become common in the literature, let us explain how to relate theories which use these conventions. The right handed star product is related to the left handed one by

$$[AB]_R = (-1)^{AB} BA, \tag{A.5}$$

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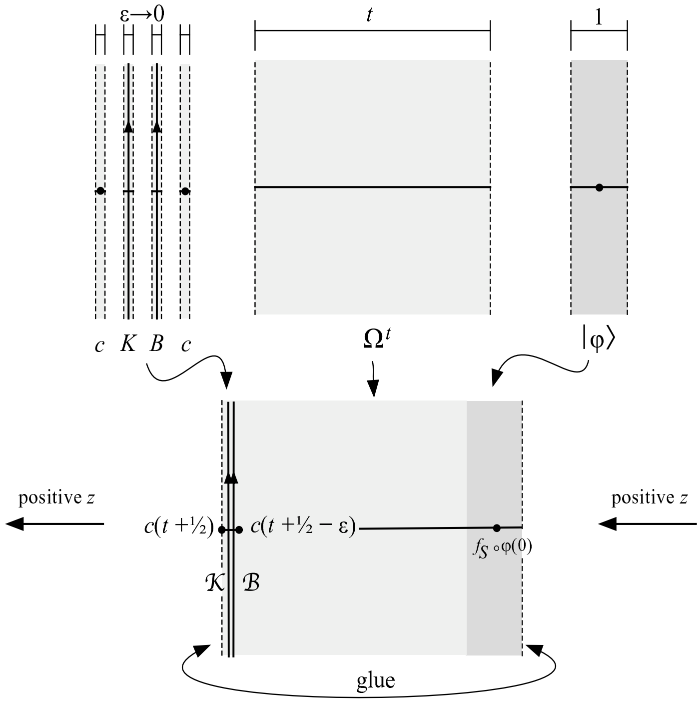

Figure 6. Representation of the inner product $\mathrm{Tr}(cKBc\Omega^t\phi)$ as a correlation function on the cylinder. The parameter ϵ above is introduced for visual purposes, and should be taken to zero. Note that positive z increases from right to left in this picture.

where the bracket $[\cdot]_R$ indicates that all star products inside are right handed. We define a string field A in our theory to be *equivalent* to a string field A' in the right handed theory if they are related by:

$$A' = A^{\S}, \tag{A.6}$$

where $A^{\S} = (-1)^{L_0}A$ denotes twist conjugation, a graded involution of the star product corresponding to a reversal of the parameterization of the open string[19] [17, 57]. This involution satisfies

$$(QA)^{\S} = Q(A^{\S}), \qquad (AB)^{\S} = (-1)^{AB}B^{\S}A^{\S}, \qquad \mathrm{Tr}\left(A^{\S}\right) = \mathrm{Tr}(A). \tag{A.7}$$

For fields in the KBc subalgebra

$$c^{\S} = -c, \qquad K^{\S} = K, \qquad B^{\S} = B. \tag{A.8}$$

If string fields in the left and right handed theory are related by this twist, one can show:

$$[f(A', B', \ldots)]_R = f(A, B, \ldots)^{\S}, \tag{A.9}$$

[19] $A^{\S}$ is related to the twist conjugation introduced in [17, 57] by a minus sign. Thus a twist even solution acquires a minus sign under conjugation with §.

where f is any function of a list of string fields. This has two consequences: First, if we have a relation between string fields of the form

$$f(A, B, \ldots) = 0, \tag{A.10}$$

then the corresponding relation holds in the right handed theory:

$$[f(A', B', \ldots)]_R = 0. \tag{A.11}$$

Second, traces between the two theories agree:

$$\mathrm{Tr}\Big([f(A', B', \ldots)]_R\Big) = \mathrm{Tr}\Big(f(A, B, \ldots)\Big). \tag{A.12}$$

Therefore we know how to translate any statement about string fields in our left handed convention to a statement about string fields in the right handed convention. One can check that the $\mathcal{B}_0$ gauge vacuum picks up an extra sign under twist conjugation, which accounts for the sign discrepancy between the solutions presented in [1] and [2]. Our solution Ψ maps to

$$\Psi' = \frac{1}{1+K}(-c + cKBc) = -\left[(c + cKBc)\frac{1}{1+K}\right]_R. \tag{A.13}$$

Note that in the right handed convention, the sign in front of c insertion is negative. This is because in the right handed picture the tachyon condenses towards the left of the perturbative vacuum in the tachyon potential.

B Equivalence to the $\mathcal{B}_0$ gauge solution

In this appendix we explicitly construct the gauge parameter relating our solution to the $\mathcal{B}_0$ gauge solution. Consider two dressed $\mathcal{B}_0$ gauge solutions[20]

$$\Phi = fc\frac{KB}{1-fg}cg, \qquad \Phi' = f'c\frac{KB}{1-f'g'}cg', \tag{B.1}$$

where f, f', g, g' are functions of K. If these solutions are gauge equivalent, they can be related by the transformation

$$\Phi' = U^{-1}(Q + \Phi)U, \tag{B.2}$$

where

$$\begin{aligned} U =&\ 1 - fBcg + \left(\frac{1-fg}{1-f'g'}\right) f'Bcg', \\ U^{-1} =&\ 1 - f'Bcg' + \left(\frac{1-f'g'}{1-fg}\right) fBcg. \end{aligned} \tag{B.3}$$

If they are not gauge equivalent, than either U or U^{-1} must be singular. The only part of the above expressions which could potentially cause problems are the factors in parentheses. Therefore, Φ and Φ' are gauge equivalent if and only if the string field

$$M = \frac{1-fg}{1-f'g'} \tag{B.4}$$

[20] We discuss dressed $\mathcal{B}_0$ gauges in appendix C. Note that not all solutions within the KBc subalgebra can be written in this form.

and its inverse are well defined. In practice, the easiest way to see this is to check that both M and M^{-1} are analytic functions of K at $K = 0$.[21] Since fg and $f'g'$ must also be analytic, this amounts to the requirement that the first nonvanishing powers in a Taylor series expansion of $1 - fg$ and $1 - f'g'$ must be the same:

$$1 - fg \sim K^n + \text{higher powers}\ldots, \qquad 1 - f'g' \sim K^n + \text{higher powers}\ldots. \tag{B.5}$$

The integer n plays the role of an index labeling physically inequivalent solutions in the KBc subalgebra. $n = 0$ describes the perturbative vacuum and $n = 1$ describes the closed string vacuum. Other possible values of n are mysterious since the corresponding solutions do not appear to be well-defined. They have been conjectured to be related to multiple brane solutions [30].

For the $\mathcal{B}_0$ gauge vacuum and our new solution, we have

$$\begin{aligned} 1 - fg &= \frac{K}{1+K} = K + \text{higher powers}\ldots, \\ 1 - f'g' &= 1 - \Omega = K + \text{higher powers}\ldots. \end{aligned} \tag{B.6}$$

Therefore the solutions are gauge equivalent and describe the closed string vacuum. Explicitly, M and M^{-1} are,

$$\begin{aligned} M &= \lim_{N\to\infty} \int_0^\infty dt e^{-t}\left[\Omega^{N+t} - \sum_{n=0}^{N} \frac{d}{dt}\Omega^{n+t}\right], \\ M^{-1} &= 1 - \Omega + \int_0^1 dt\, \Omega^t. \end{aligned} \tag{B.7}$$

Note the presence of a limit and sliver-like term in the expression for M. This is the origin of the regulator and phantom piece in the $\mathcal{B}_0$ gauge solution.

C Gauge fixing

In this appendix we give a setup for understanding the gauge fixing of the new solution (2.6), (2.8) and related solutions appearing in [7]. To this end, we define the operator

$$\mathcal{B}_{f,g}\Phi = \frac{1}{2} f[\mathcal{B}_0^-(f^{-1}\Phi g^{-1})]g, \tag{C.1}$$

where f, g are functions of K and $\mathcal{B}_0^- = \mathcal{B}_0 - \mathcal{B}_0^*$. Also define

$$\mathcal{L}_{f,g}\Phi = \frac{1}{2} f[\mathcal{L}_0^-(f^{-1}\Phi g^{-1})]g. \tag{C.2}$$

[21] For the sake of discussion we presume that elements of the wedge algebra are analytic functions of K at $K = 0$, though this may be a stronger regularity requirement than is necessary for some purposes. For example, in a general proof of Sen's conjectures [7] it only appears necessary to assume $fg(K)$ is once differentiable at $K = 0$. However in the general case such solutions fail to have a well-defined $\mathcal{L}_0$ level expansion.

These operators are easy to evaluate on wedge states with insertions since $\mathcal{B}_0^-, \mathcal{L}_0^-$ are derivations and

$$\begin{aligned}
\frac{1}{2}\mathcal{B}_0^- K &= B, & \frac{1}{2}\mathcal{L}_0^- K &= K,\\
\frac{1}{2}\mathcal{B}_0^- B &= 0, & \frac{1}{2}\mathcal{L}_0^- B &= B,\\
\frac{1}{2}\mathcal{B}_0^- c &= 0, & \frac{1}{2}\mathcal{L}_0^- c &= -c.
\end{aligned} \tag{C.3}$$

We should think of $\mathcal{B}_{f,g}, \mathcal{L}_{f,g}$ as generalizations of $\mathcal{B}_0, \mathcal{L}_0$. In fact

$$\mathcal{L}_{F,F} = \mathcal{L}_0, \qquad \mathcal{B}_{F,F} = \mathcal{B}_0, \tag{C.4}$$

where $F = \sqrt{\Omega}$ is the square root of the SL$(2,\mathbb{R})$ vacuum. In particular, $\mathcal{B}_0$ gauge is just an example of a large family of gauges

$$\mathcal{B}_{f,g}\Phi = 0. \tag{C.5}$$

Note that the string field must be "dressed" by factors of f^{-1}, g^{-1} before it is annihilated by $\mathcal{B}_0^-$. For this reason, we call these *dressed* $\mathcal{B}_0$ gauges. The new solutions Ψ and the real $\hat{\Psi}$ satisfy gauge conditions of this type:

$$\mathcal{B}_{1,\frac{1}{1+K}}\Psi = \; 0, \tag{C.6}$$

$$\mathcal{B}_{\frac{1}{\sqrt{1+K}},\frac{1}{\sqrt{1+K}}}\hat{\Psi} = \; 0. \tag{C.7}$$

Equation (C.6) can be reexpressed in a particularly simple form:

$$\mathcal{B}_0^-\left(1 - \frac{\pi}{2}(K_1)_R\right)\Phi = 0. \tag{C.8}$$

It could be interesting to explore the consequences of these gauges in perturbation theory.

Of all these gauges, $\mathcal{B}_0$ gauge certainly appears to be the most natural one. It is reasonable to wonder, then, in what sense our new gauge $\mathcal{B}_{\frac{1}{\sqrt{1+K}},\frac{1}{\sqrt{1+K}}}\Phi = 0$ is special or unique. One answer to this question is given by the level expansion. Given any solution satisfying a linear gauge condition $\mathcal{O}\Phi = 0$, one can define a "natural" level expansion in terms of eigenstates of the operator $[Q_B, \mathcal{O}]$. For Siegel gauge, this leads to the ordinary L_0 level expansion; for $\mathcal{B}_0$ gauge, this gives the $\mathcal{L}_0$ level expansion. For the new solution $\hat{\Psi}$, the natural expansion is in terms of eigenstates of $\mathcal{L}_{\frac{1}{\sqrt{1+K}},\frac{1}{\sqrt{1+K}}}$. Remarkably, this expansion of eq. (2.8) terminates after just two levels:

$$\text{Level}\,0: \quad \frac{1}{\sqrt{1+K}}c\frac{1}{\sqrt{1+K}}, \qquad \text{Level}\,1: \quad \frac{1}{\sqrt{1+K}}cKBc\frac{1}{\sqrt{1+K}}. \tag{C.9}$$

Indeed this is remarkable — certainly we do not find the tachyon condensate in Siegel gauge after expanding out to level 2. In fact, this can be taken as the defining property of our solution, according to the following claim:

Claim. Eq. (2.6) is the unique, regular dressed $\mathcal{B}_0$ gauge solution in the KBc subalgebra that terminates at finite level in its own level expansion, up to homogeneous gauge transformations.

We can establish this as follows. For a solution to terminate at level $n-1$ in its own level expansion, the function of K sandwiched between the c insertions must be an nth degree polynomial, call it P_n. The non-real form of the solution is then

$$\Phi = cBP_nc\left(1-\frac{K}{P_n}\right), \qquad \mathcal{B}_{1,1-\frac{K}{P_n}}\Phi = 0. \tag{C.10}$$

It is helpful to cancel the K in the numerator. Assuming $n \geq 1$, P_n has at least one root, which we can call $-\frac{1}{\gamma}$. Then write $P_n = \left(K+\frac{1}{\gamma}\right)\pi_{n-1}$ with π_{n-1} some polynomial of order $n-1$, and the solution becomes

$$\Phi = cBP_nc\left(1-\frac{1}{\pi_{n-1}}+\frac{1}{\gamma}\frac{1}{P_n}\right). \tag{C.11}$$

The first term is the identity string field with some insertions. Unless the identity piece cancels, the action evaluated on the solution will be undefined.[22] For $n \geq 2$, the inverses of P_n and π_{n-1} can be found by making a partial fraction decomposition and expressing the resulting terms as integrals over wedge states via the Schwinger parameterization. None of this produces a piece which would cancel the identity string field, so for $n \geq 2$ the solutions are ill-defined. However, for $n = 1$, $\pi_{n-1} = \pi_0$ is a constant; if we choose $\pi_0 = 1$ the identity is exactly canceled, leaving $P_n = \frac{1}{\gamma}+K$ and

$$\Phi = \left(\frac{1}{\gamma}c + cKBc\right)\frac{1}{1+\gamma K}. \tag{C.12}$$

This is our original solution eq. (2.6), up to a reparameterization $\gamma^{\mathcal{L}_0^-/2}$. This leaves the case $n = 0$; the solution there is

$$\Phi = \frac{1}{\gamma}c(1-\gamma K). \tag{C.13}$$

This is a singular identity-based solution. Therefore only $n = 1$ admits a regular solution to the equations of motion, as claimed.

Let us list a few useful properties of dressed $\mathcal{B}_0$ operators. Dressed $\mathcal{B}_0$ operators have the following symmetries under conjugation:

$$\mathcal{B}_{f,g}^* = -\mathcal{B}_{f^{-1},g^{-1}}, \tag{C.14}$$
$$\mathcal{B}_{f,g}^\dagger = -\mathcal{B}_{\bar{g}^{-1},\bar{f}^{-1}}, \tag{C.15}$$
$$\mathcal{B}_{f,g}^\ddagger = \mathcal{B}_{\bar{g},\bar{f}}, \tag{C.16}$$
$$\mathcal{B}_{f,g}^\S = \mathcal{B}_{g,f}. \tag{C.17}$$

[22]Note also that the trace of an identity-like string field is undefined if the field carries insertions with total zero or positive scaling dimension in the sliver coordinate frame. This is certainly true of eq. (C.11).

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Here $*$ denotes BPZ conjugation, $\dagger$ denotes Hermitian conjugation, $\ddagger$ is reality conjugation, $\S$ is twist conjugation, and $\bar{f}, \bar{g}$ are the complex conjugates of f, g. The same properties also hold for $\mathcal{L}_{f,g}$. Note that equations (C.16), (C.17) imply that a dressed $\mathcal{B}_0$ gauge solution is consistent with the reality condition only when $f = \bar{g}$, and it is twist even only when $f = g$.

To give some other formulas, it is helpful to introduce the string fields,

$$B_f = Bf\frac{d}{dK}f^{-1}, \qquad K_f = Kf\frac{d}{dK}f^{-1}. \tag{C.18}$$

We have for example,

$$B_1 = 0, \qquad B_\Omega = B, \qquad B_{\frac{1}{1+K}} = \frac{B}{1+K}. \tag{C.19}$$

B_f and K_f characterize the failure of $\mathcal{B}_{f,g}, \mathcal{L}_{f,g}$ to be derivations of the star product:

$$\mathcal{B}_{f,g}(\Phi\Lambda) = \ (\mathcal{B}_{f,v}\Phi)\,\Lambda + (-1)^\Phi\Phi\,(\mathcal{B}_{u,g}\Lambda) - (-1)^\Phi\Phi B_{uv}\Lambda, \tag{C.20}$$

$$\mathcal{L}_{f,g}(\Phi\Lambda) = \ (\mathcal{L}_{f,v}\Phi)\,\Lambda + \Phi\,(\mathcal{L}_{u,g}\Lambda) - \Phi K_{uv}\Lambda. \tag{C.21}$$

To give a slightly more general formula we have introduced arbitrary u, v on the right hand side. Note that this implies that $\mathcal{B}_{f,f^{-1}}, \mathcal{L}_{f,f^{-1}}$ are derivations of the star product. Also note

$$\mathcal{B}_{f,g}|I\rangle = B_{fg}, \qquad \mathcal{L}_{f,g}|I\rangle = K_{fg}. \tag{C.22}$$

Two dressed $\mathcal{B}_0$ operators can be related by left/right multiplication with B_f:

$$\mathcal{B}_{f,g}\Phi = \mathcal{B}_{u,v}\Phi + B_{f/u}\Phi + (-1)^\Phi\Phi B_{g/v} \tag{C.23}$$

with a similar formula for $\mathcal{L}_{f,g}$. B_f and K_f satisfy a logarithmic sum/product rule:

$$aB_f + bB_g = B_{f^a g^b}, \qquad a, b \in \mathbb{C} \tag{C.24}$$

which implies a similar rule for $\mathcal{B}_{f,g}, \mathcal{L}_{f,g}$:

$$a\mathcal{B}_{f,g} + b\mathcal{B}_{h,j} = \mathcal{B}_{f^a h^b, g^a j^b}, \qquad a, b \in \mathbb{C}, \quad a + b = 1. \tag{C.25}$$

The restriction $a+b = 1$ gives a simpler formula, but the general case follows by multiplying this equation by a constant. Thus dressed $\mathcal{B}_0, \mathcal{L}_0$ operators form a closed linear space; in particular, we cannot make new gauges by taking linear combinations of $\mathcal{B}_{f,g}$s.

The special projector algebra [1, 11] $[\mathcal{L}_0, \mathcal{L}_0^*] = \mathcal{L}_0 + \mathcal{L}_0^*$ plays an important role in the algebraic structure of analytic solutions. There is an analogue of this algebra for dressed $\mathcal{L}_0$ operators. To display this algebra is is useful to introduce a "dressed" analogue of a wedge state:

$$\Omega(f) = e^{-K_f}, \tag{C.26}$$

and,

$$\Omega(f^a g^b) = \Omega(f)^a\Omega(g)^b \quad a, b \in \mathbb{C}. \tag{C.27}$$

The generalization of the special projector algebra is then,

$$[\mathcal{L}_{f,g}, \mathcal{L}_{u,v}^*] = \mathcal{L}_{\Omega(f),\Omega(g)} + \mathcal{L}_{\Omega(u),\Omega(v)}^*. \tag{C.28}$$

Note that $\Omega(\cdot)$ acts as the identity on wedge states, so we recover the usual formula when $f = u = F$ and $g = v = F$.

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