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How do rotating black holes form in higher dimensions?

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Abstract Black holes are generally formed by gravitational collapse and accretion process. The necessary condition for the process to work is that overall force on collapsing/accreting matter element must be attractive. This is not so for the Myers–Perry metric describing a rotating black hole in higher dimensions. Also for accretion process to work, there should form accretion disk which requires existence of innermost stable circular orbit (ISCO). There can occur no bound orbits and consequently ISCOs in higher dimensions around a stationary black hole. Both these hurdles are overcome in pure Lovelock gravity. Rotating black holes in higher dimensions could thus form by collapse/accretion only in pure Lovelock gravity.

Mathematics Subject Classification 83

1 Introduction

Gravitational collapse and accretion onto a gravitating centre are the two main processes for formation of black holes. The former involves dynamically evolution of gravitational collapse of matter distribution with equation of state from given regular initial data. This would require fully general relativistic simulation with a dynamically evolving geometry. On the other hand, for the latter, one has the benefit of exterior metric for setting up accretion process. For accretion to set in, there should form a disk in which matter revolves around the centre in stable circular orbits (SCOs). Thus, SCOs' existence becomes a critical requirement for accretion

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process to work. In the Newtonian theory, they exist everywhere, while for general relativity (GR), there exists the minimum radius for the innermost stable circular orbit (ISCO). The ISCO angular momentum defines the lower bound on particle's angular momentum to execute an SCO. SCO in accretion disk keeps on falling inwards as it loses angular momentum by dissipative viscous forces present there. That is how matter keeps on accreting on central object positing angular momentum on to the centre. This is how rotating black hole is formed.

The primary and necessary condition for both these processes to work is that overall force involving both gravitational attraction and repulsive centrifugal component is attractive. If that is not so, collapse/accretion cannot ensue. This is what we would show in the following.

As argued above, existence of SCOs becomes the critical requirement for accretion process. It is rather well known that in GR as well as in Newtonian gravity, no bound orbits and thereby SCOs can occur in dimensions greater than the usual four. For bound orbits to occur, gravitational potential going as $1/r^{D-3}$ should be able to balance the centrifugal potential, which always goes as $1/r^2$. This would only happen when $D-3 < 2$; i.e. $D < 5$. Thus, there can occur no bound orbits in higher dimensions $D > 4$. In other words, effective potential has no minimum to form a potential well for harbouring bound orbits and SCOs.

The primary requirement for existence of bound orbits is that $\alpha < 2$ for potential due to mass falling off as $1/r^{\alpha}$. For Einstein gravity, generically $\alpha = D - 3$ which would then imply $D < 5$. However, there do exist some special cases where bound orbits could exist with specific bound on rotation parameter [23] or some specific configurations like black rings [21,22]. This happens, because in all these specific circumstances, potential due to mass tends to the four-dimensional form, i.e., $\alpha = 1$ [23].

In GR, rotation also contributes to gravitational potential as well as to the phenomenon of frame dragging. That is, space around a rotating black hole also imbibes rotation as indicated by zero angular momentum particle having non-zero angular velocity—the frame dragging. However, both these effects die out sharply with increasing r. In higher dimensions, a rotating black hole is described the Myers–Perry solution of vacuum Einstein equation [29]. As for non-rotating case, there can occur no bound orbits for higher dimensional rotating black hole, as well. This is because effect of rotation is not very significant as shown by effective potential for large r taking the same form as that due to non-rotating black hole.

Since there can occur no SCOs, there can be no accretion disk to harbour accretion process. On the other hand for gravitational collapse, one has to solve fully relativistic evolution of fluid cloud with rotation from a regular initial data. The primary and necessary condition for collapse to ensue is that the overall force on rotating fluid element must be attractive. Since collapse is to start from large r where resultant force would be $l^2/r^3 - M/r^{D-2}$, the former repulsive component would clearly override the latter for $D > 4$. Hence, gravitational collapse would not be able to ensue, and therefore, there is no question of its evolution any further.

Thus, in higher dimensions, both gravitational collapse and accretion cannot work to form rotating black hole. We would employ the Myers–Perry metric [29] for this investigation.

The distinguishing feature of rotating black holes in dimensions greater than five for a particle with angular momentum is that overall gravity is repulsive due to dominance of repulsive centrifugal component $1/r^2$ over attractive $1/r^{D-3}$ due to mass. This fact also has bearing on the phenomenon of overspinning of black hole which was initiated by Ref. [24], and following that by several others [3,4,8,19,27,30,31,35]. It turns out that it is in general possible to overspin a black hole under linear test particle accretion. However, converting a non-extremal into extremal black hole is not possible [14], and nor an extremal black hole into over-extremal state [42] through linear perturbation. The main point here is that reaching extremality state is impossible, yet it could perhaps be jumped over by a discrete non-geodetic but linear order perturbative process. It however turns out that the result is always overturned when non-linear perturbations are included [40]. Thus, weak cosmic censorship conjecture (WCCC) which may be violated at linear order is always restored at non-linear order.

The question arises, how do the Myers–Perry rotating black holes fare? Since overall force is repulsive for accreting particle with angular momentum, it would not be able to get to the hole. It may not be able to overspin even at the linear order. This is precisely what happens and has been shown in an explicit calculation in [37]. In all dimensions $D \geq 6$, a black hole cannot be overspun in [34]. In the marginal case of five dimension, it could, however, be overspun at linear order which gets overturned when second-order perturbations are included [36,38,39]. Thus, for rotating black holes, WCCC is favoured by higher dimensions.

We have also probed [33] overspinning of pure Lovelock[1] rotating black hole and have shown that it cannot be overspun in dimension $> 4N + 1$. For $N = 1$ Einstein gravity, it implies that a rotating black hole in $D > 5$ cannot be overspun.

The paper is organised as follows: In Sect. 2, we briefly describe the higher dimensional Myers–Perry rotating black hole metric. In Sect. 3, we build up the effective potential for test particles motion for ($D = 5, 6$)-dimensional black holes. In Sect. 4, we first briefly recall motivation for pure Lovelock gravity and then study pure GB rotating black hole effective potential which clearly shows existence of potential well harbouring bound orbits. We end with the discussion in the Sect. 5. Throughout we use the natural units, $G = c = 1$.

2 Myers–Perry rotating black hole

The metric describing the higher dimensional Myers–Perry rotating black hole [29] in general reads as follows:

$$\begin{aligned} ds^2 &= -dt^2 + (r^2 + a_n^2)\left(d\mu_n^2 + \mu_n^2 d\phi_n^2\right) \\ &\quad + \frac{Mr^{2n+3-D}}{\Pi F}\left(dt + a_n\mu_n^2 d\phi_n\right)^2 + \frac{\Pi F}{\Delta}dr^2 + (D-2n-1)r^2 d\alpha^2\,, \end{aligned} \tag{1}$$

where

$$\begin{aligned} \Delta &= \Pi - Mr^{2n+3-D}\,, \\ F &= 1 - \frac{a_n^2\mu_n^2}{r^2 + a_n^2}\,, \\ \Pi &= \sum_{i=1}^{n}(r^2 + a_i^2)\,, \end{aligned} \tag{2}$$

where $n = [(D-1)/2]$ is maximum number of rotation parameters a black hole can have in D dimensions (see, for example, [29]) and $\Sigma\mu_n^2 + (D-2n-1)\alpha^2 = 1$. In odd $D = 2n + 1$ dimensions, $\Delta = \Pi - Mr^2$ and $\Sigma\mu_n^2 = 1$, while for even $D = 2n + 2$, $\Delta = \Pi - Mr$ and $\Sigma\mu_n^2 + \alpha^2 = 1$ where M and a are, respectively, mass and rotation parameters. The black hole horizon is given by positive root of $\Delta = 0$.

Note that whenever one or more of rotations are switched off, $\Delta = 0$ has only one positive root indicating occurrence of only one horizon [34]. For our purpose in this investigation, it would suffice to consider only one rotation parameter. In particular, we would study the cases of $D = 5, 6$ and show that (i) effective potential, $V_{\text{eff}} > 1$ always for non-zero angular momentum, and (ii) it has only a maximum and no minimum, and hence, there can occur no bound and stable circular orbits.

We shall now specialize the above metric for $D = 5, 6$ with a single rotation and it would then read as follows: For $D = 5$:

$$\begin{aligned} ds^2 &= -dt^2 + \frac{M}{\Pi F}\left(dt - a\sin^2\theta d\phi\right)^2 + \frac{\Pi F}{\Delta}dr^2 + \Sigma' d\theta^2 \\ &\quad + \left(r^2 + a^2\right)\sin^2\theta d\phi^2 + r^2\cos^2\theta d\psi^2\,. \end{aligned} \tag{3}$$

For $D = 6$

$$\begin{aligned} ds^2 &= -dt^2 + \frac{M}{r\Pi F}\left(dt - a\sin^2\theta d\phi\right)^2 + \frac{\Pi F}{\Delta}dr^2 + \Sigma' d\theta^2 \\ &\quad + \left(r^2 + a^2\right)\sin^2\theta d\phi^2 + r^2\cos^2\theta(d\chi^2 + \sin^2\chi d\psi^2), \end{aligned} \tag{4}$$

where $\Sigma' = r^2 + a^2\cos^2\theta$. Note that $\Pi F = \Sigma'$ is always satisfied for $D = 5, 6$.

Following the standard procedure for geodesic motion of timelike particles in the equatorial plane around a rotating black hole, we would write the effective potential.

[1] Pure Lovelock means the Lovelock Lagrangian and the following equation of motion has only one Nth-order term without sum over lower orders. Here, N is degree of homogeneous Riemann curvature polynomial in action. It should, however, be noted that the metric we employ for rotating black hole is extrapolated [9] from the corresponding Myers–Perry metric, but it is not an exact solution of pure Lovelock vacuum equation. It, however, satisfies the equation in the leading order and has all the desirable and expected features.

3 Effective potential

The effective potential for timelike radial motion in equatorial plane for a rotating black hole with a single rotation is generically given by

$$V_{\text{eff}} = \Omega\mathcal{L} + \sqrt{\frac{\Delta}{g_{\phi\phi}r^{2(n-1)}}\left(\mathcal{L}^2/g_{\phi\phi}+1\right)}, \tag{5}$$

where we have defined $\mathcal{L} = L/m$ being specific angular momentum with mass of particle m and $\Omega = -g_{t\phi}/g_{\phi\phi}$ which corresponds to the frame dragging angular velocity. Note that the above expression for the effective potential stems from the geodesic equation of the Kerr geometry generalized to higher dimensions [41]. It is then written as

$$V_{\text{eff}} = \Phi\beta\frac{a\mathcal{L}}{r^2} + \frac{\sqrt{\left(\beta+\frac{\mathcal{L}^2}{r^2}\right)\left(1+\frac{a^2}{r^2}-\Phi\right)}}{\beta}, \tag{6}$$

where

$$\Phi = \frac{M}{r^{D-3}} \text{ and } \beta = 1+\frac{a^2}{r^2}+\Phi\frac{a^2}{r^2}. \tag{7}$$

On large r expansion, it takes the form $1+\mathcal{L}^2/2r^2 - M/2r^{D-3}$ which clearly shows that the repulsive centrifugal component would override the attractive gravitational one for $D > 4$.

This would in particular for 5 and 6 dimensions read as

$$V_{\text{eff}}^{5\text{D}} = \frac{r\left[r^4+\left(r^2+M\right)a^2+r^2\mathcal{L}^2\right]^{1/2}}{r^4+\left(r^2+M\right)a^2}\left(r^2-M+a^2\right)^{1/2} + \frac{aM\mathcal{L}}{r^4+\left(r^2+M\right)a^2}, \tag{8}$$

$$V_{\text{eff}}^{6\text{D}} = \frac{r\left[r^5+\left(r^3+M\right)a^2+r^3\mathcal{L}^2\right]^{1/2}}{r^5+\left(r^3+M\right)a^2}\left(r^3-M+a^2r\right)^{1/2} + \frac{aM\mathcal{L}}{r^5+\left(r^3+M\right)a^2}, \tag{9}$$

and for $\mathcal{L} = 0$ to

$$\left(V_{\text{eff}}^{5\text{D}}\right)^2 = \left(1-\frac{M}{r^2}+\frac{a^2}{r^2}\right)\left(1+\frac{a^2}{r^2}+\frac{Ma^2}{r^4}\right)^{-1}, \tag{10}$$

$$\left(V_{\text{eff}}^{6\text{D}}\right)^2 = \left(1-\frac{M}{r^3}+\frac{a^2}{r^2}\right)\left(1+\frac{a^2}{r^2}+\frac{Ma^2}{r^5}\right)^{-1}. \tag{11}$$

Note that for zero angular momentum potential, even though repulsive component due to rotation dominates over attractive due to mass in the numerator, yet it would not turn overall repulsive because of the denominator. This becomes clear as we expand it for large r, and then, it goes as $1 - M/2r^2$, $1 - M/2r^3$, respectively, for $D = 5, 6$. It is interesting that at large r contribution due to rotation gets completely annulled out. That is why potential in the upper panel in Fig. 1 is similar to that of non rotating black hole. As shown in the upper panels of Figs. 1 and 2, accretion with zero angular momentum could proceed unhindered onto a non-rotating black hole without formation of accretion disk. However, such an object cannot acquire any rotation. It is worth noting that effective potential and circular orbits around Myers–Perry black hole have also been studied in [18,25] and it has be shown that there occur no bound orbits outside the event horizon.

When particles with angular momentum are included, the situation changes with coming in of additional centrifugal component, $\mathcal{L}^2/r^2$ in (Fig. 1, lower panel). Note that for both $D = 5, 6$ $V_{\text{eff}} = 1$ at infinity, and then, it rises as r decreases and reaches maximum before coming down at horizon. It is interesting that $V_{\text{eff}} \geq 1$ all through except very close to horizon. For a single rotation, there occurs only one horizon, and hence, there is no upper limit on rotation parameter a except for five dimension where it has to respect $a^2 \leq M$ lest it turns into a naked singularity[2]. This is an interesting case of a rotating black hole with one horizon yet having an extremal limit for its rotation parameter. Also note that in $D = 6$, as $a \to \infty$, $V_{\text{eff}} \to 1$ at all r. This is why initially maximum of curve rises with increasing a until $a \sim 1.3$, then it starts coming down.

[2] This happens only in the special case of five dimension and not in general for $D = 2n + 1$, because in this case, contribution to potential for both mass and rotation falls as $1/r^2$.

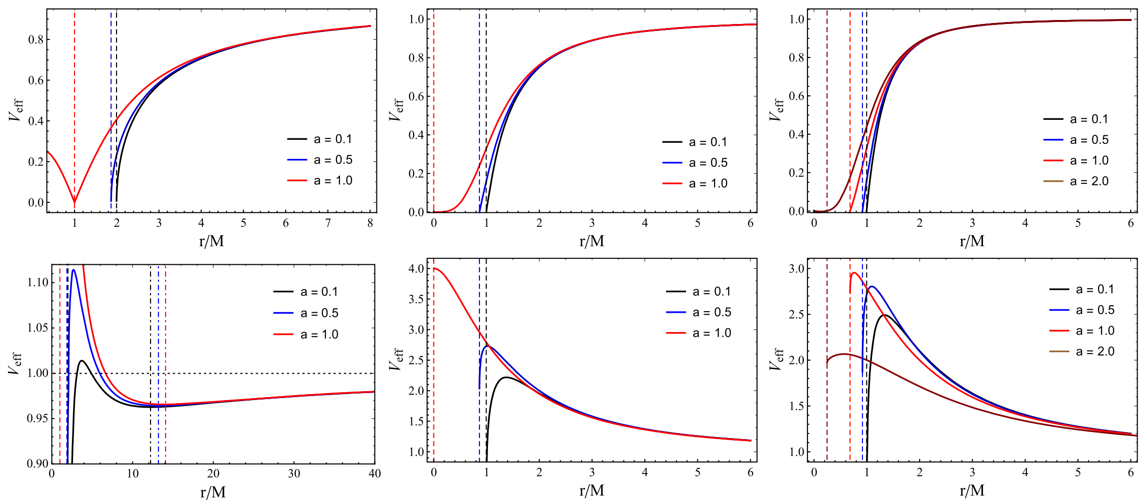

Fig. 1 Effective potential plots for $\mathcal{L} = 4$: left, middle, and right panels, respectively, refer to $D = 4, 5, 6$. The vertical dashed lines indicate location of horizon, while the vertical dot-dashed lines indicate location of minimum of V_{eff}

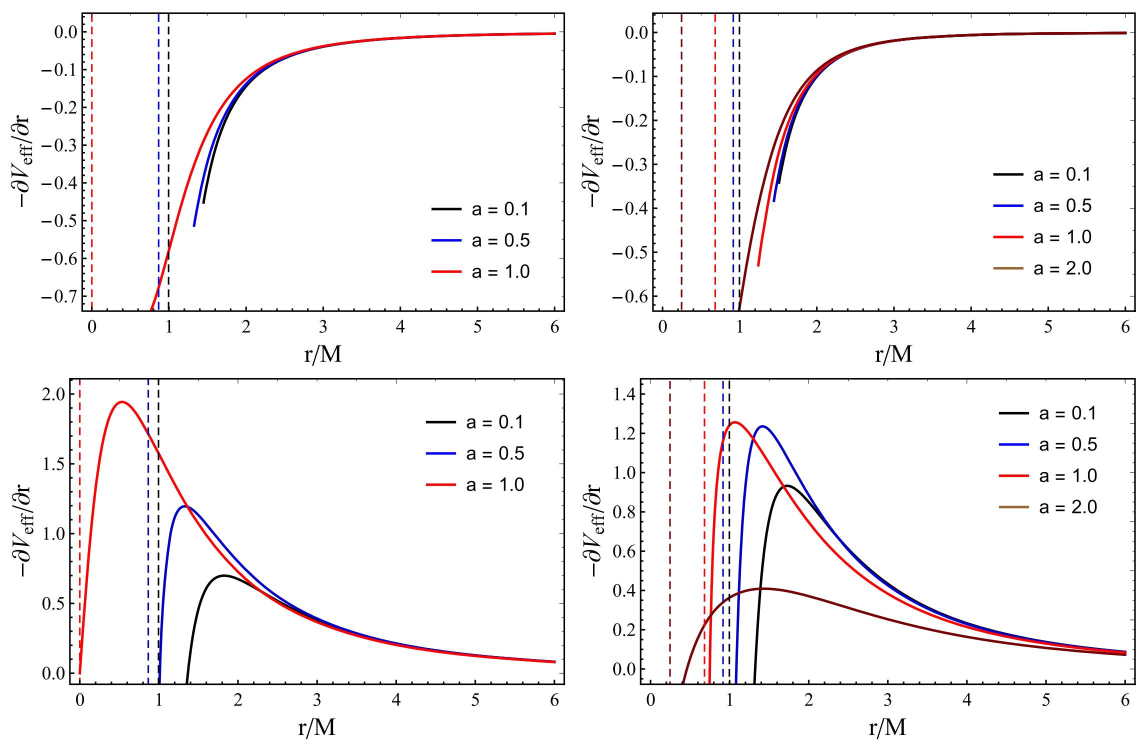

Fig. 2 $-\partial V_{\text{eff}}/\partial r$ plots for $\mathcal{L} = 5$: left and right panels, respectively, refer to $D = 5, 6$. The vertical lines indicate location of horizon

4 Pure Lovelock rotating black hole and orbits

One of the most fascinating and remarkable generalizations of Einstein gravity in higher dimensions is Lovelock gravity [28]. Its action is a homogeneous polynomial of degree N in Riemann tensor including Hilbert–Einstein, Gauss–Bonnet, and so on, respectively, for $N = 1, 2, \ldots$ It is the dimensionally extended Euler densities summed over all N, and each N comes with a dimensionful coupling constant. It is most remarkable that in spite of non-linearity in Riemann in the action, the resulting equation still remains second order. This is the unique feature of Lovelock gravity that singles it out from all other generalizations. Physically, it is most desirable and attractive as it wards off unpleasant and uncomfortable entities like ghosts.

Higher derivatives and higher dimensions are the natural arena for string theory, and Gauss–Bonnet term [32,46] turns up at the one loop correction. This indicates that higher derivative GB/Lovelock terms could be string theory correction to Einstein theory. The first vacuum solution of Einstein–Gauss–Bonnet equation was obtained in [5] describing a static back hole. For higher N, the equation would be quite formidable, yet it is interesting that it could be solved for static black holes [43–45].

In here we would take pure GB/Lovelock theory as a proper gravitational theory on its own right in higher dimensions, and not as string inspired correction to the Einstein theory. In particular, we would like to study pure GB rotating black hole spacetime. Unlike the Myers–Perry solution, there exists no exact solution for a rotating black hole of the pure GB vacuum equation which would indeed be formidably hard to solve. There was a novel and innovative method adopted in [12] in which the Kerr metric was obtained without solving the field equations. It appeals to the two physically motivated properties: one, a photon falling along the axis of rotation experiences no three acceleration and, two, a timelike particle experiences the inverse square Newtonian acceleration. One has however to begin with an appropriate 3-geometry, spherical for non-rotating and ellipsoidal for rotating black hole. It turns out that the same method has recently been generalized to higher dimensions [2] to write the Myers–Perry rotating black hole metric. What really happens is that one begins with an appropriate ellipsoidal flat metric and then introduces in the Newtonian potential, M/r^{D-3}, and thus results the rotating black hole metric. This was the method employed in [9] to write the pure GB rotating black hole metric in six dimension, where Newtonian potential was now replaced by the corresponding pure GB potential, $M/\sqrt{r}$ (pure Lovelock potential goes as $1/r^{\alpha}$ where $\alpha = (D-2N-1)/N$). The resulting metric has all the desirable and expected features, but it fails to satisfy the pure GB vacuum equation. However, it does satisfy the equation in leading orders. We shall however work with this metric for six-dimensional pure GB rotating black hole.

Unfortunately, this straightforward and physically appealing method does not quite work for pure Lovelock gravity. This may be due to equation being non-linear in Riemann curvature. It is nevertheless physically well motivated where pure Lovelock potential is incorporated into the corresponding ellipsoidal rotating geometry. It should therefore represent a valid rotating black hole metric which may not though be an exact solution of pure Lovelock equation. Yet, it should be taken as a good rotating black hole spacetime that incorporates pure Lovelock potential. Its energetics and optical properties have also been studied in [1]. Recently, the metric for $4D$-Einstein-GB rotating black hole has similarly been obtained [26] which is also not an exact solution of the equation. We shall take this metric to study six-dimensional pure GB rotating black hole.

In pure Lovelock theory, gravity is kinematic in all critical odd $D = 2N+1$ dimensions [6,15], and hence, for black hole solutions, $D \geq 2N+2$. The Myers–Perry analogue for Lovelock rotating black hole would then be given by Eqs. (1) and (2) with Δ being replaced by

$$\Delta = \Pi - 2Mr^{2n-\alpha}\,, \tag{12}$$

which for the six-dimensional pure GB rotating black hole with single rotation where $\alpha = 1/2$ would be

$$\Delta = \Pi - 2Mr^{3/2}\,. \tag{13}$$

Black hole horizons would be given by $\Delta = 0$ which for single rotation reduces to

$$r^4 - 4M^2r^3 + 2a^2r^2 + a^4 = 0\,. \tag{14}$$

It solves to give

$$r_{\pm} = M^2 + \frac{X(M,a)}{\sqrt{6}} \pm \left(8M^4 - \frac{8a^4 - 2Y^2(M,a)}{3Y(M,a)} + \sqrt{3}\frac{64M^6 - 32a^2M^2)}{4\sqrt{2}X(M,a)} - \frac{8a^2}{3}\right)^{1/2}, \tag{15}$$

with

$$X^2(M,a) = 6M^4 - 2a^2 + \frac{4a^4}{Y(M,a)} + Y(M,a)\,,$$
$$Y^3(M,a) = 27M^4a^4 - 8a^6 + 3\sqrt{3}\,M^2\,a^4\sqrt{27M^4 - 16a^2}\,.$$

For pure GB rotating black hole having single rotation, the effective potential in Eq. (5) is then written by

$$V_{\text{eff}}^{\text{6D}}(r) = \frac{2aM\mathcal{L}}{r^{5/2}+\left(r^{1/2}+2M\right)a^2} + \frac{\left(r^3+\left(r+2Mr^{1/2}\right)a^2+r\mathcal{L}^2\right)^{1/2}}{r^{5/2}+\left(r^{1/2}+2M\right)a^2} \times\left(r\left(r-2Mr^{1/2}\right)+a^2\right)^{1/2}, \tag{16}$$

and

$$V_{\text{eff}}^{\text{8D}}(r) = \frac{2aM\mathcal{L}}{r^{7/2}+\left(r^{3/2}+2M\right)a^2} + \frac{\left(r^5+\left(r^3+2Mr^{3/2}\right)a^2+r^3\mathcal{L}^2\right)^{1/2}}{r^{7/2}+\left(r^{3/2}+2M\right)a^2} \times\left(r^2-2Mr^{1/2}+a^2\right)^{1/2}, \tag{17}$$

$$V_{\text{eff}}^{\text{9D}}(r) = \frac{2aM\mathcal{L}}{r^4+\left(r^2+2M\right)a^2} + \frac{\left(r^6+\left(r^4+2Mr^2\right)a^2+r^4\mathcal{L}^2\right)^{1/2}}{r^4+\left(r^2+2M\right)a^2} \times\left(r^2-2M+a^2\right)^{1/2}, \tag{18}$$

respectively, for $D = 8, 9$. In the limit of large r, these expressions reduce to

$$V_{\text{eff}}^{\text{6D}}(r\to r_\infty) \sim 1-\frac{M}{r^{1/2}}-\frac{M^2}{2r}-\frac{M^3}{r^{3/2}}+\left(\frac{\mathcal{L}^2}{2r^2}-\frac{5M^4}{8r^2}\right)+\mathcal{O}(r^{5/2}), \tag{19}$$

$$V_{\text{eff}}^{\text{8D}}(r\to r_\infty) \sim 1-\frac{M}{r^{3/2}}-\frac{M^2}{2r^3}+\frac{\mathcal{L}^2}{2r^2}+\mathcal{O}(r^{7/2}), \tag{20}$$

$$V_{\text{eff}}^{\text{9D}}(r\to r_\infty) \sim 1+\frac{\mathcal{L}^2}{2r^2}-\frac{M}{r^2}-\frac{a^2\mathcal{L}^2}{2r^4}+\mathcal{O}(r^{9/2}). \tag{21}$$

Note that for the former two (as well as for $D = 7$ for which potential falls off as $1/r$), V_{eff} tends to unity at infinity from the below while for the latter from the above. This means that bound orbits can exist only in $D = 6, 7, 8$. For bound orbits to exist, there must occur a potential well with a minimum, $V_{\text{eff}} < 1$ there. This cannot happen for $D > 4N$ in general and $D > 8$ in particular for pure GB.

Here, we consider the particular case of $D = 6$. The following required condition should be satisfied for circular orbits:

$$\frac{\partial V_{\text{eff}}(r)}{\partial r} = 0, \tag{22}$$

giving the equation

$$\frac{\left(r^{5/2}+a^2\left(2M+\sqrt{r}\right)\right)\left(r^2-2Mr^{3/2}+a^2\right)^{-1/2}}{\left(r^3+r\mathcal{L}^2+a^2\left(2M\sqrt{r}+r\right)\right)^{1/2}} \times\left[a^4\left(M+\sqrt{r}\right)+a^2\sqrt{r}\left(6r^2-8M^2r+\mathcal{L}^2\right)+3r^{5/2}\mathcal{L}^2 +5r^{9/2}-9Mr^4-5Mr^2\mathcal{L}^2\right] -\left(a^2+5r^2\right)\left[2aM\mathcal{L}+\left(a^2-2Mr^{3/2}+r^2\right)^{1/2} \times\left(r^3+r\mathcal{L}^2+a^2\left(2M\sqrt{r}+r\right)\right)^{1/2}\right]=0. \tag{23}$$

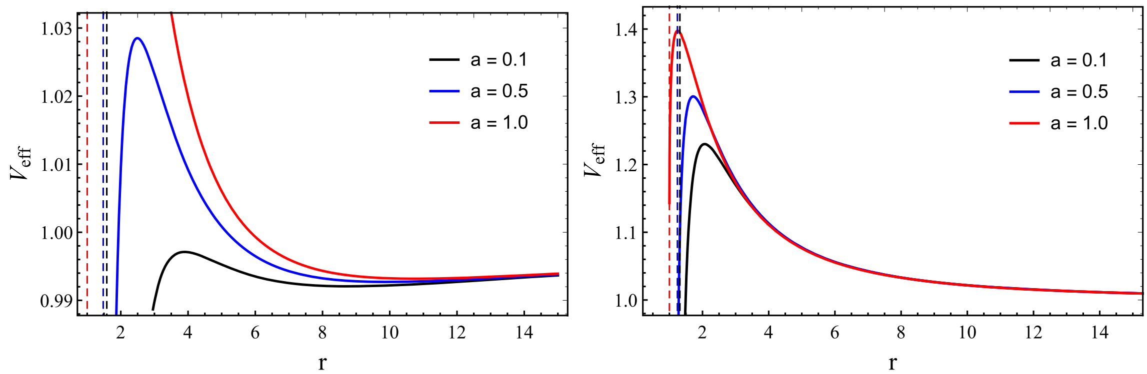

Fig. 3 V_{eff} for $D = 6$, prograde (left) and retrograde (right) orbits for $\mathcal{L} = \pm 8.5$. Vertical lines indicate location of horizon

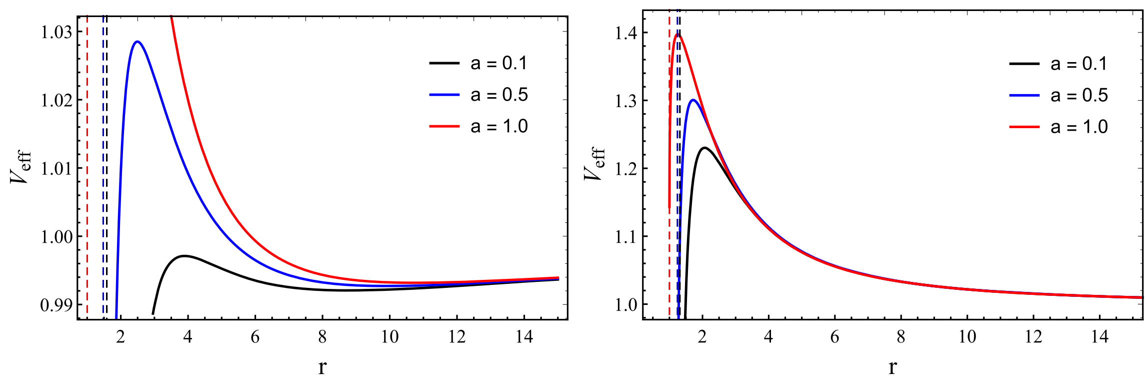

Fig. 4 Left and right panels show V_{eff} for $\mathcal{L} = 2.25$ in $D = 8, 9$, respectively. Vertical lines indicate location of horizon

The above equation solves to give two positive roots for which V_{eff} reaches maximum and minimum (see Fig. 3, left panel). Since for bound orbits for pure GB, the allowed range is $6 \leq D \leq 8$, the behavior of V_{eff} would be so for $D = 7, 8$, as well. In particular, for $D = 7$ (in general for $D = 3N + 1$ [7]), potential has the same fall off, $1/r$ as for the Kerr black hole, and hence, the effective potential would be the same as for the Kerr metric in four dimension. However, for $D \geq 9$, there would occur only a maximum indicating absence of bound orbits and consequently also of stable circular orbits (see Fig. 4, right panel).

For the angular momentum, Eq. (23) solves to give

$$\mathcal{L}^2_{\pm\,GB}(r) = \frac{MrAB \mp 2\sqrt{2}\left(a^2 - 2Mr^{3/2} + r^2\right)C}{r^{3/2}\left(r^{3/2}\left(2\sqrt{r} - 5M\right)^2 - 8a^2M\right)A}\,, \tag{24}$$

where

$$\begin{aligned} A &= \left(r^{5/2} + a^2\left(\sqrt{r} + 2M\right)\right)^2\,, \\ B &= 2\sqrt{r}\left(r^2 + a^2\right)^2 + M\left(22a^2r^2 + 11a^4 - 5r^4\right) - 40M^2a^2r^{3/2}\,, \\ C &= \left(M^3\sqrt{r}\left(a^3 + 5\,ar^2\right)^2\,A^2\right)^{1/2}\,. \end{aligned} \tag{25}$$

In Fig. 5, we demonstrate the radial profile of $\mathcal{L}$ (see, left panel), where minimum shows the ISCO radius and describes the lower bound on angular momentum for particles to be at stable circular orbits, i.e., particle would fall in carrying angular momentum to black hole if its angular momentum is less than this critical value. Therefore, existence of this bound on angular momentum, i.e., ISCO is of primary critical importance for formation of rotating black holes [11].

To understand more deeply the behavior of stable orbits, we further analyse the threshold value of angular momentum numerically, i.e., we explore the ISCO (see Table 1). As seen in Table 1, ISCO radius and corresponding angular momentum decrease as rotation parameter a increase.

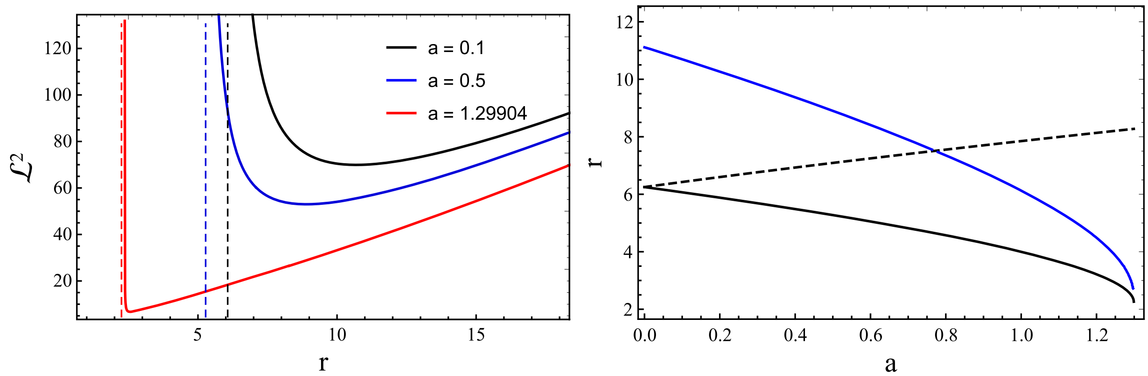

Fig. 5 Left panel: radial profile of the specific angular momentum $\mathcal{L}^2$ as a function of spatial distance r, where vertical lines indicates location of r_{ph}. Right panel: r_{ph} (black and dashed lines for co and counter-rotating) and ISCO radius (blue line) plotted against rotation parameter a. Here, $D = 6$, and $r_{\text{ph}} = 6.25$ corresponds to $a = 0$

Table 1 Numerical values of $r_{\text{min}}(r_{\text{ISCO}})$, $\mathcal{L}_{\text{min}}$ and r_{mb} for co- and counter-rotating orbits are tabulated for different values of rotation parameter a

a	r_{min}	$\mathcal{L}_{\text{min}}$	$r_{\text{mb}}(co)$	r_{mb}(counter)
0.0	11.1111	8.60663	7.11111	7.11111
0.1	10.6931	8.36242	6.87725	7.46032
0.5	8.89855	7.27935	5.86838	8.89124
1.0	6.12307	5.46058	4.29261	10.6881
1.29904	2.2500	5.39805	–	11.7449

Here we note that an extremal black hole corresponds to $a_{\text{extremal}} = 1.29904$

For pure GB rotating black hole, ISCOs will only occur in $D = 6, 7, 8$ and not in any dimension > 8.

We now consider another limit on existence of the circular geodesics. ISCO defines the stability threshold and below which would occur unstable orbits which would naturally be bounded from below by the photon circular orbit. That is defined by $\mathcal{L}^2_{\pm} \to \infty$, and that gives

$$r^{3/2}\left(2\sqrt{r} - 5M\right)^2 - 8a^2M = 0\,. \tag{26}$$

For $a = 0$, $r_{\text{ph}}/M^2 = 25/4$, while for $a \neq 0$, its location is shown in Fig. 5 (right panel).

We note that $r > r_{\text{ph}}$ therefore defines the existence threshold for circular orbit, while $r \geq r_{ISCO}$ defines the stability threshold. The marginally (energetically) bound threshold defined by $\mathcal{E}(r_{\text{mb}}) = 1$ exists in between these two above-mentioned bounds. Here, we have defined $\mathcal{E} = E/m$. Note that $\mathcal{E} \leq 1$ and $\mathcal{E} > 1$, respectively, refer to energetically bound and unbound orbits.

All unstable orbits occurs between $r_{\text{ph}} < r < r_{\text{ISCO}}$, and their ultimate fate is decided by the marginally bound condition, i.e. $V_{\text{eff}} = 1$. We now analyse $V_{\text{eff}}(\text{mb}) = V_{\text{eff}} - 1$. On perturbation, particle can escape to infinity in case it is positive, while it climbs up to $r > r_{\text{ISCO}}$ to a stable circular orbit in case it is negative. The radius threshold for marginally bound orbit, r_{mb} is given by the solution of $V_{\text{eff}}(\text{mb}) = 0$. Therefore, for $r_{\text{ph}} < r < r_{\text{mb}}$ orbit is unbounded, while for $r > r_{\text{mb}}$, it is bounded. For co and counter-rotating orbits, we explore the radius threshold for marginally bound orbit in Table 1.

For $D = 6$, on substituting Eq. (24) in Eq. (16), we obtain

$$\begin{aligned} V_{mb}^{\pm GB}(r) \;&=\; \frac{2aM}{r^{5/2} + \left(r^{1/2} + 2M\right)a^2}\left(\frac{MrAB \mp 2\sqrt{2}\left(a^2 - 2Mr^{3/2} + r^2\right)C}{r^{3/2}\left(r^{3/2}\left(2\sqrt{r} - 5M\right)^2 - 8a^2M\right)A}\right)^{1/2} \\ &\quad -1 + \frac{\left(r\left(r - 2r^{1/2}M\right) + a^2\right)^{1/2}}{r^{5/2} + \left(r^{1/2} + 2M\right)a^2} \end{aligned}$$

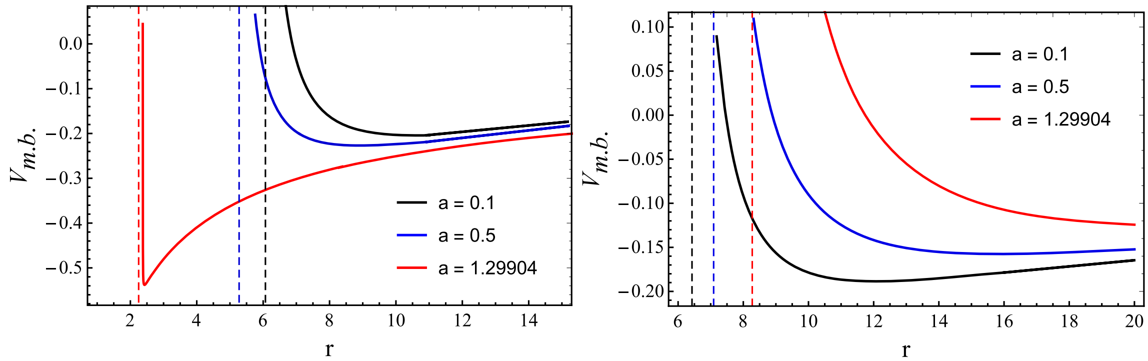

Fig. 6 Marginally bound circular orbits in $D = 6$: left and right panels, respectively, refer to co and counter-rotating orbits. Vertical lines indicate location of r_{ph}

$$\times\left[r^3 + \left(r + 2Mr^{1/2}\right)a^2 + \frac{MrAB \mp 2\sqrt{2}\left(a^2 - 2Mr^{3/2} + r^2\right)C}{r^{1/2}\left(r^{3/2}\left(2\sqrt{r} - 5M\right)^2 - 8a^2M\right)A}\right]^{1/2}, \tag{27}$$

with A, B and C given in Eq. (25). In Fig. 6, we show the radial profile of $V^{\pm}_{mb}$ for co- and counter-rotating orbits.

As can be seen from Fig. 6, one can observe that $V^{\pm}_{\text{mb}} < 0$ always for $r > r_{\text{mb}}$ for co- and counter-rotating orbits, thereby indicating that orbits remain bounded. In the case of $r < r_{\text{mb}}$ orbits remain unbounded on perturbation.

5 Discussion

We shall first discuss rotating black holes in Einstein gravity (Figs. 1, 2 refer to Einstein gravity while from Fig. 3 to pure GB). In Fig. 1, effective potential is plotted for $\mathcal{L} = 4$ and $D = 4, 5, 6$ (from left to right). For large r, $V_{\text{eff}} \sim 1 - M/2r^{D-3} \leq 1$ which shows asymptotically it is the Newtonian contribution that survives. It is this property that distinguishes four from higher dimensions. In contrast for $D > 4$, $V_{\text{eff}} \geq 1$, it comes down to unity from the above and therefore clearly indicates absence of any potential well to harbour bound orbit and SCOs. First, it was shown for non-rotating black holes in higher dimensions [16]. It is however carried forward to rotating black holes simply because rotation contributes repulsively. Thus, no bound orbits could exist around Myer–Perry rotating black holes in higher dimensions, save for some special cases [21–23].

This raises the question about accretion process in higher dimensions. Accretion is mediated through accretion disk and occurrence of bound and SCOs is a necessary requirement. Accretion disk provides avenue for other interactions involving viscosity and collisions between particles from which particles can lose angular momentum and keep on falling inward and spiral into the hole. This is how angular momentum gets carried onto the hole through the accretion process. Since stable circular orbits cannot exist in higher dimensions for accretion disk to form, hence an accretion process cannot ensue. It can therefore play no role in formation of a rotating black hole in higher dimensions.

Particles with angular momentum could impinge black hole only if they are energised energy exceeding the maximum of potential barrier Ref. [4,34]. Even then it was shown that overspinning was not possible, because particles with angular momentum appropriate for overspinning cannot be able to reach the horizon. Since $V_{\text{eff}} > 1$ always, hence they would not be able to reach rotating black hole horizon unless they were somehow energised to a value overriding maximum of the potential barrier. The only possibility of such an energising process could perhaps be collision with other compact objects, like neutron stars or black holes. It would be a very complex and involved process which would require detailed simulation of collision process.

How about taking the question to generalized theories of gravity which could provide the possibility of occurrence of bound orbits? The only theory for which bound orbits exist in higher dimensions is uniquely the pure Lovelock gravity [13]. It turns out that bound orbits occur in the dimension range $2N + 2 \leq D \leq 4N$ [16]. For the linear order Einstein gravity, bound orbits could exist only in $D = 4$, while for the quadratic pure Gauss–Bonnet (GB) gravity, they do exist for $D = 6, 7, 8$ (Fig. 5). It may also be noted that non-rotating black hole [17] is stable [20] only in dimensions $D \geq 3N + 1$; i.e., for pure GB in $D = 7, 8$. Thus, for pure GB rotating black holes would have bound orbits and SCOs around them. And so would be formed accretion disk to facilitate accretion process leading to formation of rotating black holes [3]. Now, the situation is exactly similar to the Kerr black hole in GR, rotating black holes in pure GB gravity could be formed by the usual accretion process with an accretion disk in $D = 7, 8$ (leaving out $D = 6$ for which black hole would be unstable). Thus, we have the dimension window, $2N + 2 \leq D \leq 4N$, that allows for the usual accretion process with accretion disk and that could very well work leading to formation of pure Lovelock rotating black holes. However, one has not yet been able to find an exact solution of pure Gauss–Bonnet vacuum equation describing a rotating black hole [9].

Now, we come to gravitational collapse. It involves a matter configuration with rotation collapsing under its own gravity from a regular initial data, and the geometry is not fixed but dynamically evolving. This is a very complex and involved problem which could properly be addressed through sophisticated numerical simulations of fully relativistic hydrodynamical flow. However, the primary requirement for gravitational collapse to ensue is that overall gravity is attractive. As we see in Fig. 2 that gravity is indeed repulsive for an exterior metric of a higher dimensional rotating black hole. It is true that this is not the metric in the collapsing interior configuration; however, overall sign of gravitational force is determined by what happens in the exterior vacuum metric. Note that for four-dimensional Kerr geometry, the overall gravity is indeed attractive, and hence, it satisfies the primary condition for collapse. That is not the case in higher dimensions > 4.

Thus, a detailed analysis would be pertinent only if the necessary key condition for collapse to ensue is satisfied. As we have shown that it cannot be satisfied for Einstein gravity in higher dimensions.

As for accretion, in pure Lovelock gravity, the necessary condition for gravitational collapse would also be satisfied. Note that attractive component due to mass in this case goes as $1/r^\alpha$ where $\alpha = (D - 2N - 1)/N$, while the repulsive centrifugal one always falls off as $1/r^2$. The former would ride over the latter so long as $\alpha < 2$ which implies $D < 4N + 1$. That is in the dimension range $2N + 2 \leq D \leq 4N$, overall gravitational force would be attractive and thereby satisfying the necessary condition for gravitational collapse.

Thus, in Einstein gravity, higher dimensional rotating black holes cannot be formed by gravitational accretion and collapse processes. They could however be formed in pure GB/Lovelock gravity where both processes could very well work in the dimension window, $2N + 2 \leq D \leq 4N$. In particular, for pure GB, rotating black hole could in principle form by accretion and collapse in $D = 6, 7, 8$. Besides kinematicity of gravity in all critical odd dimensions $D = 2N + 1$ and existence of bound orbits [13], the formation of rotating black holes in higher dimensions is yet another property that singles out pure Lovelock gravity.

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Declaration

Conflict of interest The authors declare no conflict of interest.

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[3] In a separate paper, we would be studying particle motion for pure GB rotating black hole and, in particular, obtain the threshold value of angular momentum given by ISCO [10].

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