Nonlinear superhorizon perturbations of non-canonical scalar field

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Abstract

We develop a theory of non-linear cosmological perturbations at superhorizon scales for a scalar field with a Lagrangian of the form $P(X, \phi)$, where $X = -\partial^\mu\phi\partial_\mu\phi$ and ϕ is the scalar field. We employ the ADM formalism and the spatial gradient expansion approach to obtain general solutions valid up to the second order in the gradient expansion. This formulation can be applied to, for example, DBI inflation models to investigate superhorizon evolution of non-Gaussianities. With slight modification, we also obtain general solutions valid up to the same order for a perfect fluid with a general equation of state $P = P(\rho)$.

1 Introduction

Generation of primordial fluctuations during inflation is one of the most interesting predictions of quantum field theory. Indeed, those quantum fluctuations are considered as seeds of the large scale structure of the present universe, and this picture has been accepted by many researchers as a standard scenario. The recent more accurate observation by WMAP has revealed deviation from exact scale invariance, with a slight red tilt. Moreover, there is a good possibility that deviation from Gaussianity can be detected by the future experiments such as PLANCK. With those current and future precision observations, deviation from the exact scale invariance and Gaussianity can be a powerful tool to discriminate many possible inflationary models. On the theoretical side, there are at least two known mechanisms to generate large non-Gaussianity: isocurvature perturbations and non-canonical kinetic terms. As for the former case, a typical example is the curvaton scenario is responsible for isocurvature perturbations during inflation. Now let us consider the later case, where non-canonical kinetic terms are responsible for large non-Gaussianity. Examples of this type include k-inflation [1] and DBI inflation [2]. In k-inflation and DBI inflation, large non-Gaussianity is expected when the non-linear nature of the non-canonical kinetic action becomes significant. To quantify the non-Gaussianity and clarify its observational signature, it is important to develop a theory that can deal with nonlinear cosmological perturbations. There are couple of methods to tackle this problem. One is a second-order perturbation theory. Another is based on spatial gradient expansion [3]. Closely related to the gradient expansion method, cosmological perturbations on superhorizon scales have been studied extensively in the so-called separate universe approach or δN formalism [4]. Actually, these approaches are essentially the leading order approximation to the gradient expansion. Including these, many of the previous studies were confined to the leading order approximation to the gradient expansion. However, higher order corrections to the leading order results can be important to get more detailed information about non-Gaussianity. One good example is the case studied by Leach et al [5]. They considered linear perturbations in single-field inflation models and supposed that there is a stage at which slow-roll conditions are violated. It has been then shown that, due to the decaying mode, the $O(\epsilon^2)$ corrections in spatial derivative expansion do affect the evolution of curvature perturbations on superhorizon scales. However, the linear perturbation theory is not capable for calculation of non-Gaussianity. Thus, it is necessary to develop nonlinear theory of cosmological perturbations valid up to $O(\epsilon^2)$ in the spatial gradient expansion. Gradient expansion formalism has been developed and used by many authors [3, 4, 6, 7]. Formulation valid up to $O(\epsilon^2)$ was developed, for example, by Tanaka and

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Sasaki for a universe dominated by a perfect fluid with a specific equation of state $P/\rho = const$ [6] and that dominated by a canonical scalar field [7]. However, as far as the authors know, those works have not extended to a perfect fluid with general equation of state $P = P(\rho)$ nor to a scalar field with non-canonical kinetic action, which is essential for the second type of mechanism of generating non-Gaussianity. The purpose of this paper is to fill this gap. Namely, we shall develop a theory of nonlinear superhorizon perturbations valid up to the order $O(\epsilon^2)$ for a scalar field with non-canonical kinetic action and a perfect fluid with general equation of state.

2 Formalism

Throughout this paper we consider a minimally-coupled scalar field described by an action of the form $I = \int d^4x\sqrt{-g}P(X,\phi)$ where $X = -g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi$, and suppose that $-g^{\mu\nu}\partial_\nu\phi$ is timelike and future-directed. The following relation among first-order variations of P, ρ and ϕ will be useful in the analysis below. $\delta P = c_s^2\delta\rho + \rho\Gamma\delta\phi$, where

$$c_s^2 = \frac{P_X}{2P_{XX}X + P_X}, \quad \Gamma = \frac{1}{\rho}\left(P_\phi - c_s^2\rho_\phi\right), \tag{1}$$

where the subscripts X and ϕ represent derivative with respect to X and ϕ, respectively. Note that c_s is the speed of sound for the gauge invariant scalar perturbation in the linear theory.

We shall develop a theory of nonlinear cosmological perturbations on superhorizon scales. For this purpose we employ the ADM formalism and the gradient expansion in the uniform Hubble slicing. In the (3+1)-decomposition, the metric is expressed as $ds^2 = g_{\mu\nu}dx^\mu dx^\nu = -\alpha^2dt^2 + \gamma_{ij}(dx^i+\beta^idt)(dx^j+\beta^jdt)$ where α is the lapse function, β^i is the shift vector and Latin indices run over $1,2,3$. Since α and β^i represent gauge degrees of freedom for diffeomorphism and appear as Lagrange multipliers in the action, the corresponding equations of motion leads to constraint equations. Contrary to α and β, components of the spatial metric γ_{ij} are dynamical variables (subject to the constraint equations) and the corresponding equations of motion are called dynamical equations. In what follows we shall express the dynamical equations as a set of first-order differential equations with respect to the time t. For this purpose we introduce the extrinsic curvature K_{ij} defined by $K_{ij} = -\frac{1}{2\alpha}\left(\partial_t\gamma_{ij} - D_i\beta_j - D_j\beta_i\right)$ where D is the covariant derivative compatible with the spatial metric γ_{ij}. For the stress-energy tensor in the perfect fluid form, we define the 3-vector v^i as $v^i \equiv u^i/u^0$. Hereafter, we shall use γ_{ij} and its inverse γ^{ij} to raise and lower indices of K, D, v, β. In addition to the standard ADM decomposition briefly reviewed above, we further decompose the spatial metric and the extrinsic curvature as

$$\gamma_{ij} = a^2\psi^4\tilde{\gamma}_{ij}, \quad K_{ij} = a^2\psi^4\left(\frac{1}{3}K\tilde{\gamma}_{ij} + \tilde{A}_{ij}\right), \tag{2}$$

where $a(t)$ is the scale factor of a fiducial Friedmann background (specified later) and the determinant of $\tilde{\gamma}_{ij}$ is constrained to be unity: $\det\tilde{\gamma}_{ij} = 1$. Throughout this paper we adopt the uniform Hubble slicing

$$K = -3H(t), \quad H(t) \equiv \frac{\partial_t a}{a}. \tag{3}$$

In the gradient expansion approach we introduce a flat FRW universe $(a(t),\ \phi_0(t))$ as a background and suppose that the characteristic length scale L of perturbations is longer than the Hubble length scale $1/H$ of the background, i.e. $HL \gg 1$. Therefore, we consider $\epsilon \equiv 1/(HL)$ as a small parameter and systematically expand our equations by ϵ, considering a spatial derivative acted on perturbations is of order $O(\epsilon)$. The background flat FRW universe $(a(t),\ \phi_0(t))$ satisfies the Friedmann equation and the equation of motion. Since the FRW background is recovered in the limit $\epsilon \to 0$, we naturally have the assumptions $v^i = O(\epsilon), \beta^i = O(\epsilon)$ and $\partial_t\tilde{\gamma}_{ij} = O(\epsilon)$. Actually, following the arguments in refs. [6, 7], we assume a stronger condition $\partial_t\tilde{\gamma}_{ij} = O(\epsilon^2)$. This assumption significantly simplifies our analysis and, we believe, still allows many useful applications of the formalism. On the other hand, we consider ψ and $\tilde{\gamma}_{ij}$ (without derivatives acted on them) as quantities of order $O(1)$. We can estimate orders of magnitude of various quantities by using the above assumption and the basic equations. In summary, we have the following estimates (including assumptions):

$$\begin{aligned}
&\psi = O(1), \quad \tilde{\gamma}_{ij} = O(1), \quad v^i = O(\epsilon), \quad \beta^i = O(\epsilon), \quad \chi \equiv \alpha - 1 = O(\epsilon^2), \\
&\tilde{A}_{ij} = O(\epsilon^2), \quad \delta \equiv \frac{\rho - \rho_0}{\rho_0} = O(\epsilon^2), \quad \pi \equiv \phi - \phi_0 = O(\epsilon^2), \quad p \equiv P - P_0 = O(\epsilon^2), \\
&\partial_t \tilde{\gamma}_{ij} = O(\epsilon^2), \quad \partial_t \psi = O(\epsilon^2), \quad v^i + \beta^i = O(\epsilon^3).
\end{aligned} \tag{4}$$

We substitute the order of magnitude shown in (4) into the conservation equations $\nabla_\mu T^\mu_{\ \nu} = 0$, the Hamiltonian and momentum constraint equations, and the evolution equations for the spatial metric and for the extrinsic curvature. By using these equations and the background conservation equation $\partial_t \rho_0 + 3H(\rho_0 + P_0) = 0$, a single equation for δ is easily obtained, $\partial_t(a^2 \rho_0 \delta) = O(\epsilon^4)$. It is intriguing to note that we have not yet specified the form of p. The form of p for the scalar field system is specified by the relation as

$$p = \rho_0 (c_{s0}^2 \delta + \Gamma_0 \pi) + O(\epsilon^4), \tag{5}$$

where $c_{s0}^2 = P_{0X}/(2P_{0XX}X_0 + P_{0X})$ and $\Gamma_0 = (P_{0\phi} - c_{s0}^2 \rho_{0\phi})/\rho_0$. We can obtain another equation relating p and π, by expanding p as $p = P_{0X}(X - X_0) + P_{0\phi}\pi + O(\epsilon^4)$, where $X - X_0 = 2(\partial_t \phi_0 \partial_t \pi - \chi X_0) + O(\epsilon^4)$. Actually, this equation can be interpreted as a first-order equation for π.

3 General solution

Having written down all relevant equations up to the order $O(\epsilon^2)$ in the gradient expansion, we now seek a general solution. First, $\psi = O(1)$ and $\partial_t \psi = O(\epsilon^2)$ imply that $\psi = L^{(0)}(x^k) + O(\epsilon^2)$ where $L^{(0)}(x^k)$ is an arbitrary function of the spatial coordinates $\{x^k\}$ $(k = 1, 2, 3)$. Hereafter, the superscript (n) indicates that the corresponding quantity is of order $O(\epsilon^n)$. Similarly, $\tilde{\gamma}_{ij} = O(1)$ and $\partial_t \tilde{\gamma}_{ij} = O(\epsilon^2)$ imply that $\tilde{\gamma}_{ij} = f_{ij}^{(0)}(x^k) + O(\epsilon^2)$ where $f_{ij}^{(0)}(x^k)$ is a (3×3)-matrix with unit determinant whose components depend only on the spatial coordinates. By using these equations and background conservation equation, equations for δ and u_i are easily obtained. The traceless part of the extrinsic curvature $\tilde{A}_{ij}$ is solved by using the leading part of ψ and $\tilde{\gamma}_{ij}$. The 'constants' of integration are not independent but are related to each other by the two constraint equations. Indeed, by solving the Hamiltonian and momentum constraints, the 'constants' are expressed in terms of other integration 'constants'. Until now, we have not used (5). Therefore, the general solutions presented above are valid not only for the scalar field system but also for radiation, dust or any other sources, provided that the stress-energy tensor is of the perfect fluid form and that $p = O(\epsilon^2)$. We now use (5) to proceed further. It is easy to integrate the corresponding eqs to give the solutions of π and χ.

Solutions obtained so far are correct up to leading order in the gradient expansion. Among them, the spatial metric ψ and $\tilde{\gamma}_{ij}$ have been obtained only up to $O(1)$ while all other variables are correct at least up to $O(\epsilon^2)$. In this subsection we seek $O(\epsilon^2)$ corrections to ψ and $\tilde{\gamma}_{ij}$. For this purpose it is convenient to specify the shift vector β^i more accurately than indicated by (4) as $\beta^i = O(\epsilon^3)$. With this gauge choice, in summary we have obtained the following solutions in the gradient expansion for the scalar field system.

$$
\begin{aligned}
\delta &= \frac{R^{(0)}}{2\kappa^2\rho_0 a^2} + O(\epsilon^4),\\
u_i &= \frac{1}{6\kappa^2(\rho_0+P_0)a^3}\partial_i\left(R^{(0)}\int_{t_0}^{t} a(t')dt' + C^{(2)}\right) + O(\epsilon^5),\\
\pi &= -\frac{\partial_t\phi_0}{6\kappa^2(\rho_0+P_0)a^3}\left(R^{(0)}\int_{t_0}^{t} a(t')dt' + C^{(2)}\right) + O(\epsilon^4),\\
\chi &= -\frac{1}{6\kappa^2(\rho_0+P_0)a^2}\left[\left(1+3c_{s0}^2 - \frac{\rho_0\Gamma_0\partial_t\phi_0}{(\rho_0+P_0)a}\int_{t_0}^{t} a(t')dt'\right)R^{(0)} - \frac{\rho_0\Gamma_0\partial_t\phi_0}{(\rho_0+P_0)a}C^{(2)}\right]\\
&\quad +O(\epsilon^4),\\
\psi &= L^{(0)}\left(1+\frac{1}{2}\int_{t_0}^{t} H(t')\chi(t')dt'\right) + O(\epsilon^4),\\
\tilde{\gamma}_{ij} &= f_{ij}^{(0)} - 2\left(F_{ij}^{(2)}\int_{t_0}^{t}\frac{dt'}{a^3(t')}\int_{t_0}^{t'} a(t'')dt'' + C_{ij}^{(2)}\int_{t_0}^{t}\frac{dt'}{a^3(t')}\right) + O(\epsilon^4),\\
\tilde{A}_{ij} &= \frac{1}{a^3}\left(F_{ij}^{(2)}\int_{t_0}^{t} a(t')dt' + C_{ij}^{(2)}\right) + O(\epsilon^4),
\end{aligned}
\tag{6}
$$

where $R^{(0)} = R\left[(L^{(0)})^4 f^{(0)}\right]$, $C^{(2)}$ in this expression is related to $\Pi^{(2)}$, and ‘constants‘ of integration $L^{(0)}$, $f_{ij}^{(0)}$, $C^{(2)}$ and $C_{ij}^{(2)}$ depend only on the spatial coordinates $\{x^k\}$ $(k = 1, 2, 3)$.

4 Summary and discussion

We have developed a theory of nonlinear cosmological perturbations on superhorizon scales for a scalar field described by a Lagrangian of the form $P(X, \phi)$, where $X = -\partial^\mu\phi\partial_\mu\phi$ and ϕ is the scalar field, and also for a perfect fluid with a general equation of state $P = P(\rho)$. The general solutions valid up to the order $O(\epsilon^2)$ in the spatial gradient expansion have been presented for the scalar field system and in Appendix of [8] for the perfect fluid. This formalism can be applied to many interesting circumstances. Some particular examples including a scalar with shift symmetry, a canonical scalar and a DBI scalar. Thus, the formalism can be used to investigate superhorizon evolution of nonlinear cosmological perturbations in k-inflation and DBI inflation. If matching subhorizon perturbation with superhorizon perturbation occurs in enough large scale (or enough late time), the $O(\epsilon^0)$ effect becomes dominant. However, near crossing the horizon, there is the case where the $O(\epsilon^2)$ corrections need to estimate. In order to quantify the non-Gaussianity, it needs to translate curvature perturbation ψ in our solutions which has been obtained on the uniform Hubble hypersurface, into ones on the uniform density ζ, which have been discussed in the first section. Calculating three point correlation function of ζ including the $O(\epsilon^0)$ corrections will be addressed in future publication.

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