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HOLOGRAPHIC SUPERCONDUCTIVITY: THE PAST, THE PRESENT, AND THE FUTURE

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I review one of the recently found magic relations of "artificial" (from the CMP audience point of view) String Theory to real problems in Condensed Matter Physics.

1 Prologue

The theory of High-T superconductivity (HTS) [1] is still a theorists' battle field. Two most common approaches in the mid of 80th, partially explaining the experimental data for both, were the emergency of HTS from antiferromagnetic spin fluctuations in a doped system [2], and the interlayer coupling model of a BCS-type [3].[1] Recent experiments with cuprates [8] gave more precise data in favour of the first approach, based on the Habbard-type Hamiltonian [9]. However, the second (third, forth etc., see [4]– [7] among them) approach can not be a priori rejected for other HT superconductors.

Prospects in establishing the complete HTS theory are widely discussed in literature (see, e.g., the discussion in [10]), though the community is rather sceptical on finding the solution within the present paradigm of superconductivity – electron-electron, electron-phonon interactions, spin fluctuations etc. New ideas are welcome and really wanted to resolve this puzzle. One of the new ideas that may refresh the old-school CMP theoretical ground came recently from Strings [11] (see also the recent Lecture Notes [12] on the subject, and Refs. therein). The approach of Holographic Superconductivity (HSC) to the description of the HTS was so unexpected and "alchemical" for the CMP audience, that at the first stage of its development it was almost ignored on the traditional CMP side. Later on, some lingering criticism was addressed to the HSC approach (I will postpone the discussion around to the main body of the paper), but now ideas of the AdS/CMT correspondence are germinating into the CMP ground, mostly by efforts of distinguished persons like Subir Sachdev and Jan Zaanen.

The aim of these notes is twofold: On one hand, to make formalism of the AdS/CFT correspondence slightly convenient in use for CMP practitioners; perhaps, some of them will be interested in applying the AdS/CMT correspondence for solving specific tasks. On the other hand, the notes are aimed at explaining in brief key ideas of HTS theory to QFT or String theory practitioners; perhaps, some of them will get new excitement of doing something in another branch of physics. Limited in volume, these notes are neither comprehensive, nor pedagogical; I make no claim of originality for the content. But the hope is somebody will find the discussed subject interesting enough to start the own researches in a highly non-trivial interplay between String Theory and Condensed Matter Physics.

2 Holographic Superconductivity: the Presence

Attempts to resolve the HTS problem on the CMP side led to the important conclusion: Whatever the final theory would be, it should be a theory with a strong coupling constant. This conclusion is easy to reach, because the standard theory of superconductivity – the BCS theory – does not describe the HTS et all. Recall, the BCS theory is one of the most profound and successful theories for physical systems in the weak coupling constant regime. But the analysis of common features of different HTS samples (see, e.g., [13]) showed the strong difference between corresponding mechanisms of forming the superconducting state in conventional[2] and non-conventional[3] superconductors. Moreover, magnetism, playing the destructive role for conventional superconductors, is in the ground of new mechanism of the HTS state forming. Roughly, the BCS theory with an electric-type coupling constant has to be dual to the HTS theory with a magnetic-type coupling constant, similar to the Dirac electric-magnetic duality in QED. These reasons lead the HTS to a theory at a strong

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[1] I would like to refer to less common, if not marginal, approaches to the HTS [4]– [7], which, for my opinion, are also worth mentioning.

[2] like mercury or lead

[3] like cuprates, iron-based HT superconductors, organic compounds and heavy fermion (uranium-based) exotic compounds with superconducting properties

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coupling constant. We know, mostly from the QCD experience, that if a model with the strong coupling constant possesses non-trivial properties, they are hard to analyze with perturbative tools. Non-perturbative tools are also restrictive, that is why a real progress in formulation of the HTS theory within conventional CMP wisdom is almost stopped. But a crucial observation in Stringy description of Black Hole physics, now recognized as the AdS/CFT correspondence [15], [16], [17], gives a possibility to equate, in the strong version of this duality, non-perturbative and perturbative physics, but in framework of two different dual theories. Following the AdS/CFT, one of the dual theories, in the weak coupling constant regime, is a gravitation theory with matter on AdS background. Hence the main question in the context of High-T superconductivity is: What is the dual classical theory of gravity with matter for the strongly coupled quantum states of the HTS theory?

Even at this point some "mystery" has been appeared. The way to describe HTS models through AdS gravity with matter looks really strange from a CMP practitioner point of view. To resolve this puzzle, one should have in mind, the AdS/CFT prescription is just a convenient way to reformulate the problem in more tractable manner: *There is not any gravity inside a superconductor et all! It's just a recipe to compute observables on the dual field theory side.*[4]

But it is not the end of the story. When the CMP practitioner realizes that the Black Hole presence on the gravity side becomes crucial for modelling the HTS properties, his/her reaction is easy to predict: "It's a mess!" That's why only a small amount of real critics was addressed, up to now, from the CMP community side to the Holographic Superconductivity approach: People can not understand how it works.

Something interesting and important for traditional CMP practitioners is missing in this way. To realize the missing points of the Holographic Superconductivity approach, let's try to understand details.

2.1 The "alchemy" of AdS/CFT

The HSC is a part of more wide AdS/CMT correspondence – the correspondence between gravitational theories in the AdS bulk and non-gravitational Condensed Matter theories on the boundary of the AdS space.[5] Models of AdS/CMT are united by common features. Strong coupling constant, non-zero temperature, finite density of states/charges are among them. These models are featured by rejection of some CMP concepts like particles, quasi-particles, crystal lattice and their interplay, which are in the ground of any solid state theory, including the BCS theory. The AdS/CMP correspondence deals with effective ensembles/condensates instead, which come after integration over the Condensed Matter field theory degrees of freedom in the special limit, when the number of d.o.f. comes to infinity. Last but not least, all of the AdS/CMT models are effectively described in terms of gravitational theories.

To realize the gravity/field theory duality, one should follow the prescriptions proposed in early papers on the AdS/CFT [15], [16], [17]. These prescriptions were confirmed in case by case studies later on. Though there is not any explicit proof of the AdS/CFT correspondence till now, a counterexample, which falls into the requirements for such a duality, has not been found yet.

As it has been noticed, the AdS/CFT correspondence is the duality between gauge theory on the boundary of $\mathrm{AdS}_{\mathrm{d+1}}$, at the strong coupling constant regime, to gravitational theory with matter on $\mathrm{AdS}_{\mathrm{d+1}}$, at the weak coupling constant regime. Fields and their characteristics on $\mathrm{AdS}_{\mathrm{d+1}}$ are in the following correspondence with d-dimensional gauge theory observables:

- Matter fields in AdS $\rightsquigarrow$ local boundary CFT operators;
- Spin s/mass m of fields $\rightsquigarrow$ spin s/scaling dimension Δ of local operators; (1)
- Gauge fields in AdS $\rightsquigarrow$ boundary currents.

This short list can be further extended if necessary (see, e.g., [14]).

The first point of the duality (1) is the correspondence between matter fields in AdS to local Conformal Field Theory (CFT) operators. If fields, even on a constant curvature space, are easy to realize, how to define the local CFT operator corresponding to a matter field? Let's use the asymptotic expansion of an AdS field $\Theta(x,z)$ near the AdS boundary $z=\epsilon$ ($\epsilon\to 0$) to this end[6]:

$$\Theta(x,z)=\mathcal{J}(x)z^{\Delta_-}\left(1+\ldots\right)+\mathcal{B}(x)z^{\Delta_+}\left(1+\ldots\right)\,. \tag{2}$$

$\Delta_\pm$ in (2) are the scaling dimensions of the field. They can be found from the expansion of the corresponding AdS field equation of motion near the boundary $z=\epsilon$ (see, e.g., [19] for details). Clearly, the determining

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[4] The same concerns, e.g., computations with Feynman diagrams. The diagrams is a fiction, giving the recipe of computing the amplitudes.

[5] The AdS/CMT correspondence also includes holographic hydrodynamics, researches in QCD and quark-gluon plasma, (super)conductor/insulator quantum phase transitions, strange metals and more (see, e.g., a comprehensive review [14], and Refs therein).

[6] Here I use the AdS coordinate system, in which the AdS boundary is at $z=0$; the AdS metric is $ds^2_{AdS}=z^{-2}(\eta_{\mu\nu}dx^\mu dx^\nu+dz^2)$, x^μ are the coordinates of the flat d-dimensional boundary (see [18], [19] for details).

relations for Δ are different for different fields. For scalars one gets $\Delta(\Delta - d) = m^2L^2$, where m and L are, respectively, the mass of the field and the AdS space characteristic length. For a vector field in AdS one gets $\Delta(\Delta - d + 2) = m^2L^2$. Then, $\Delta_\pm$ in (2) are the highest and the lowest roots of the Δ determining relation. Other terms of (2), hidden in dots, are regular on the boundary $z = 0$ terms.

For any AdS space, masses of matter fields in AdS have to be bounded from below, to have well-defined unitary QFT in AdS. Such a restriction is known as the Breitenlohner-Freedman (BF) bound [20], which guarantees the stability of AdS space under perturbations of fields[7]

$$m^2L^2 \geq -\frac{d^2}{4}\,, \tag{3}$$

and the BF bound leads to the following inequality on the scale parameter $\Delta_- \leq 0$. Therefore, the limit $z \to 0$ is not well defined in (2) for a massive scalar field. Adding the appropriate boundary counterterm to the AdS scalar field action removes the divergency, turning the action to the following form

$$S_{AdS} \sim \lim_{\epsilon\to 0}\int_{z=\epsilon}[Dx]\ (d - 2\Delta_-)\,\mathcal{J}(x)\mathcal{B}(x) + \text{regular terms}\ . \tag{4}$$

Now, it's the GKPW celebrated rule time. Following [15], [16], [17], the main rule of the AdS/CFT correspondence is

$$\langle\exp\left(i\int d^dx\,\mathcal{J}(x)\mathcal{O}(x)\right)\rangle_{QFT} = \exp\left(-iS_{bulk}\left[\Theta(x,z)_{|z=0} \to \mathcal{J}(x)\right]\right)\ , \tag{5}$$

according to which the regularized boundary value of the AdS matter field acts like a source to the local CFT operator. Comparing (5) to (4), it is easy to identify $\mathcal{J}(x)$ with the source, and $\mathcal{B}(x)$ with the v.e.v. of the CFT operator $\mathcal{O}(x)$, since the standard QFT manipulations result in

$$\langle\mathcal{O}(x)\rangle = \frac{\delta}{\delta\mathcal{J}(x)}\left(\int d^dx\,\mathcal{J}(x)\mathcal{O}(x)\right)|_{\mathcal{J}=0}\ . \tag{6}$$

Next subtle point is to define the local CFT operators for AdS gauge fields. They are massless, hence $\Delta_- = 0$ for Maxwell and graviton fields. Their boundary values at $z = 0$ coincide with the sources $\mathcal{J}(x)$. Then the symmetry arguments can be used to identify the local CFT operator $\mathcal{O}_m$, corresponding to the gauge boson A_m, with the conserved current J_m; similarly, the local CFT operator $\mathcal{O}_{mn}$ of the graviton field g_{mn} is identified with the energy-momentum tensor T_{mn}. Looking ahead, the local CFT operator of the AdS scalar field, $\mathcal{O}(x) = \Phi(x)$, will play the role of the order parameter in the simplest model of Holographic Superconductivity.

2.2 Main ingredients to build Holographic Superconductor

Let's turn to the construction of a Holographic Supeconductor. Modelling HSC one needs [11]:

- Gravity with matter in AdS, and dual gauge CFT on the flat boundary;
- Non-extremal (charged) AdS Black Hole; (7)
- Interacting with gravity and other gauge fields charged scalar (vector/tensor) with a big enough charge.

Implementation of (7) results in modeling the phase transition, at some critical temperature $\mathrm{T_c}$, with forming the energy gap. The charged scalar (possibly vector/tensor) field is the BH "hair", which condenses at the boundary. This condensate models the Cooper pairs in HSC.

This is a general scheme. But what are the things behind? How does it work?

Now, the first point in the HSC building instruction (7) does not seem so magic: We are trying to describe a theory at the strong coupling constant by use of the AdS/CFT machinery. But the second point of (7) contains something new. So, we need to understand the role of Black Holes (BHs) in the HSC picture.

It is well known, after Hawking, fact that Black Holes are not black. They can radiate, and its radiation can be measured in thermal units. In other words, Black Holes are warm (or even hot), until they will be evaporated, or will reach a stable extremal limit with zero Hawking temperature. This is the fate of all BHs, unless they live in a space with a compact boundary. The latter is realized only for one case of constant curvature manifolds: For Anti-de-Sitter space. The fate of AdS BHs is more optimistic; even some of them may be transformed into "eternal" BHs. Such transformation becomes possible due to condensation and evaporation of the emitted particles with further absorption by the BH, establishing the thermodynamic equilibrium in the end, see Fig. 1.[8] But it means, that the boundary has the same temperature as the AdS bulk, and this temperature is the

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[7] Roughly, one may neglect the back-reaction of fields on the AdS geometry when the BF bound is satisfied. But such a backreaction has to be taken into account once the BF bound is violated.

[8] This picture is very close to the Earth's water cycle: The amount of water on the Earth did not change since the dinosaurs epoch. The Earth's atmosphere plays the role of the compact boundary.

Hawking temperature of the AdS-BH. Therefore, the first role of BHs in forming the HSC is to set non-zero temperature in the boundary field theory.

Another role of the AdS-BHs in the story is more subtle: Some of the emitted particles may condense on the boundary. They will not evaporate, and will form the condensate of the corresponding BHs "hairs". To realize a simple scenario with non-trivial scalar hair, which will be responsible for the phase transition to the superconducting state, one should find a black hole that has the scalar hair at low temperatures, but has not the hair at high temperatures. This is a non-trivial task. However, it was shown [21] that, for a real scalar field Ψ with arbitrary potential $V(\Psi)$, neutral AdS black holes have the scalar hair iff AdS is unstable. Gubser [22] argued that a charged scalar field around a charged black hole would have the desired property. It's because of the scalar effective potential in the background of the AdS charged BH (the Maxwell field is chosen to be $A_m = (A_t, 0, 0, \dots, 0)$)

$$V_{eff.}(\Psi) = \sqrt{-g}\left[m^2 + A_t g^{tt} A_t\right]\Psi^2\,. \tag{8}$$

Since $g^{tt} < 0$ for AdS space, the effective mass of the scalar field decreases; there is a chance to form a non-trivial scalar hair (see below). Once formed, the scalar hair condensate plays the role of the "Cooper pairs" condensate, if, of course, it remains stable.

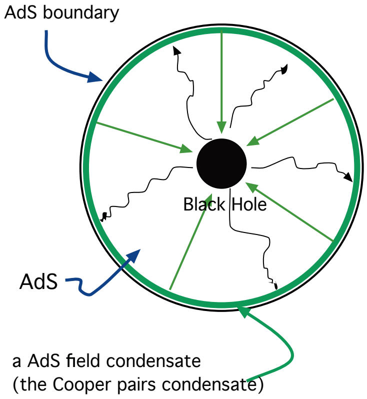

Figure 1. Thermal equilibrium in AdS-Black-Hole space.

Finally, the realization of the third point of (7), on a big enough charge of the BH hair, makes possible to focus on the much simple probe limit, neglecting the back-reaction of the AdS matter fields on the AdS-Black Hole background.

Now, how it looks in details. The proposed in [11] Lagrangian is

$$\frac{\mathcal{L}}{\sqrt{-g}} = \frac{1}{2k^2}\left(R + \frac{6}{L^2}\right) - \frac{1}{4e^2}F_{mn}F^{mn} - \frac{1}{e^2}\left(|\nabla\Psi - iA\Psi|^2 + m^2\Psi^2 + V(|\Psi|)\right), \tag{9}$$

and it describes the interacting system of AdS$_4$ gravity with Maxwell and charged scalar fields. In the probe limit $e^2 \gg k^2$, and with $V(|\Psi|) \sim |\Psi|^4$, the system is reduced to that of the Ginzburg-Landau (GL) phenomenological theory. But it turns out that for our purposes it is enough to consider trivial potential $V(|\Psi|) = 0$; the phase transition occurs even with this choice.

The gravity part of (9) supports the Reissner-Nordstrom (RN) charged BH solution, geometry of which is, in general, defined by[9]

$$ds^2 = \left[-f(r)dt^2 + g(r)dr^2 + r^2(dx^2 + dy^2)\right], \qquad (r = L/z), \tag{10}$$
$$A = \gamma h(r)dt \quad \rightsquigarrow \quad A_m = (\gamma h(r), 0, 0, 0).$$

The unitarity requirement restricts the effective mass of the scalar field $(m^2 - 2\gamma^2)/6$ to the BF bound value (recall, $m^2_{BF}L^2 = -d^2/4$ for AdS$_{d+1}$), but near-horizon the AdS-RN BH has the geometry of AdS$_2\times$R$_2$, so the effective mass of the scalar field, in $\Psi = \Psi(r)$ ansatz, can be chosen below the AdS$_2$ BF bound

$$(m^2 - 2\gamma^2) < -\frac{3}{2}\,. \tag{11}$$

This choice provides the instability in deep inside of the AdS space, with forming the scalar hair of the AdS-RN Black Hole. At the same time, the effective mass may satisfy the AdS$_4$ BF bound

$$\frac{(m^2 - 2\gamma^2)}{6} \geq -\frac{9}{4}\,. \tag{12}$$

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[9] Here I use other AdS coordinates, related to the previous ones via $x^\mu \to x^\mu$, $r = L/z$.

Thus, the stable scalar hair condensate is formed on the boundary. Rising the BH temperature results in increasing the effective mass of the scalar field, so at some critical value T_c the AdS_2 geometry becomes stable near the horizon, and the boundary condensate of the BH scalar hair collapses. It corresponds to the phase transition from superconducting to normal state.

2.3 Phase transition

The above mentioned scenario does work even in the background of the neutral planar AdS Black Hole [11], with the geometry

$$ds^2 = -f(r)dt^2 + \frac{dr^2}{f(r)} + r^2(dx^2+dy^2), \qquad f(r) = \frac{r^2}{L^2} - \frac{M}{r}, \quad (r = L/z) \tag{13}$$

and the Hawking temperature

$$T = (3M^{1/3})/(4\pi L^{4/3})\,. \tag{14}$$

In the probe limit, the Lagrangian (9) is reduced to

$$\frac{\mathcal{L}}{\sqrt{-g}} = -\frac{1}{4e^2}F_{mn}F^{mn} - \frac{1}{e^2}\left(|\nabla\Psi - iA\Psi|^2 + m^2\Psi^2 + V(|\Psi|)\right), \tag{15}$$

the interacting fields propagate on the background (13). Equations of motion, coming from (15),

$$(\nabla_m - iA_m)(\nabla^m - iA^m)\Psi - \frac{1}{2}\frac{\Psi}{|\Psi|}(2m^2|\Psi| + V'(|\Psi|)) = 0,$$

$$\nabla^m F_{mn} = i\left[\Psi^*(\nabla_n - iA_n)\Psi - \Psi(\nabla_n + iA_n)\Psi^*\right], \tag{16}$$

set up the system of coupled partial differential equations, which has to be solved. The ansatz of [11] is

$$A_m = (\phi(r), 0, 0, 0), \qquad \Psi = \psi(r), \qquad V(|\Psi|) = 0\,. \tag{17}$$

The resulting system of nonlinear equations (the phase of the scalar field can be fixed to be a constant, so one may consider the real scalar ψ)

$$\psi'' + \left(\frac{f'}{f} + \frac{2}{r}\right)\psi' + \frac{\phi^2}{f^2}\psi - \frac{m^2}{f}\psi = 0,$$

$$\psi'' + \frac{2}{r}\psi' - \frac{2\psi^2}{f}\phi = 0\,, \tag{18}$$

will be numerically solved. But having the non-trivial boundary, we need to supply the system (18) with the correct Boundary Conditions (BCs).

Near the boundary, according to the general expansion (2), the massless vector field is expanded as

$$A_t(r \to \infty) = A_t^{(0)} + \frac{A_t^{(1)}}{r} + \ldots \tag{19}$$

The finite part of (19) has to be associated with the source of the local boundary operator. Another part of (19) corresponds to the conserved current. Having it in mind, the $A_t^{(1)}$ part of the near boundary expansion is equated with the temporal component of the current J_m, i.e. with the electric charge density ρ. The finite part of the expansion (19) corresponds to the chemical potential of the electric charge density μ. Therefore, near the boundary, the Maxwell field BC is

$$\phi(r \to \infty) = \mu - \frac{\rho}{r} + \ldots \tag{20}$$

The choice of the sign in (20) is fixed from the requirement of smooth behaviour of the Maxwell field near the horizon, $\phi(r_H) = 0$. As usual, the horizon position r_H is determined from vanishing the BH red shift factor $f(r)$ (cf. (13)) on the horizon, $f(r_H) = 0$. Clearly, $\rho = \mu r_H$.

Concerning the BC for the charged scalar field, to get the simple falloff in the asymptote of the scalar field at $r \to \infty$, the authors of [11] fixed its mass to the nearest to the AdS_4 BF bound integer value, $m^2L^2 = -2$. Then, the near boundary expansion of ψ becomes

$$\psi(r \to \infty) = \frac{\psi^{(1)}}{r} + \frac{\psi^{(2)}}{r^2} + \ldots\,, \tag{21}$$

so if one chooses $\psi^{(1)} = 0$, then $\langle\mathcal{O}_2\rangle_\psi = \sqrt{2}\psi^{(2)}$ and viceversa. In what follows we will focus on $\psi^{(1)} = 0$ case, hence

$$\psi(r \to \infty) = \frac{\psi^{(2)}}{r^2} + \ldots\,. \tag{22}$$

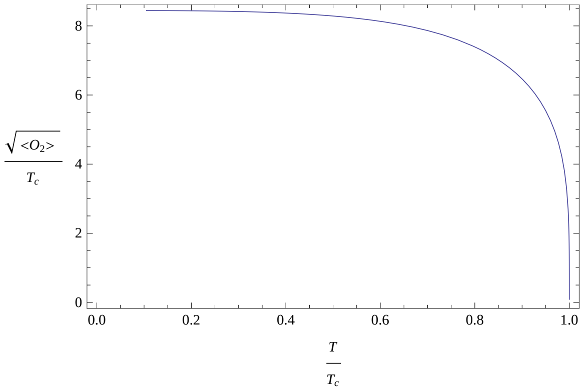

Figure 2. The second order phase transition in the s-wave 3D HSC.

Numerical solution of (16) at the fixed ansatz for AdS fields (17), and at the fixed BCs (20), (22) leads to a phase transition at a critical temperature T_c, see Fig. 2. Note that the critical exponent of the phase transition coincides with that of the GL theory: The results of numerical calculations[10] on the plot Fig. 2 is well approximated with

$$\langle \mathcal{O}_2 \rangle_\psi / T_c^2 \sim (1 - T/T_c)^{1/2}, \quad T \to T_c \,. \tag{23}$$

Also note that $\langle \mathcal{O}_2 \rangle_\psi / T_c \approx 8.3$ at $T \to 0$, and the condensation occurs at $T_c \approx 0.118\sqrt{\rho}$.

2.4 Conductivity

Having established the behavior of the order parameter $\langle \mathcal{O}_2 \rangle_\psi$ with typical characteristics of the 2nd order phase transition, let's turn to the conductivity issue. Optical conductivity can be computed by adding small perturbations to the transverse components of Maxwell field. The simplest choice is [11]

$$\delta A = A_x(r) e^{-i\omega t} dx \quad \rightsquigarrow \quad \delta A_m = (0, A_x(r) e^{-i\omega t}, 0, 0) \,, \tag{24}$$

that leads to the linearized Maxwell equation

$$A_x''(r) + \frac{f'}{f} A_x'(r) + \left(\frac{\omega^2}{f^2} - \frac{2\psi^2}{f} \right) A_x(r) = 0 \,. \tag{25}$$

Specifying the near the horizon BC (no outgoing radiation at the horizon)

$$A_x(r \to r_H) \sim f(r) e^{-i\omega / 4 r_H} \,, \tag{26}$$

and the near the boundary expansion

$$A_x(r \to \infty) = A_x^{(0)}(r) + \frac{A_x^{(1)}(r)}{r} + \ldots \,, \tag{27}$$

which, in accordance to general expansion (2), sets $A_x = A_x^{(0)}$ and $\langle J_x \rangle = A_x^{(1)}$, one may solve eq. (25) numerically (see footnote 9 for the code source). Then, the conductivity value comes from the Ohm's law in the Kirchhoff reformulation[11]

$$\sigma = \frac{\langle J_x \rangle}{E_x} = -\frac{\langle J_x \rangle}{\partial_0 A_x} = \frac{\langle J_x \rangle}{i\omega A_x} = -i \frac{A_x^{(1)}}{\omega A_x^{(0)}} \,. \tag{28}$$

Results of numerical simulations are presented on two plots, Fig. 3-4.

According to the results in Fig. 4, the lowest value of the imaginary part of σ can be found near $\omega / T_c \approx 8$. It turns out that a robust feature, that holds in AdS$_4$ s-wave superconductors for all scalar condensate scalings $\Delta \geq \Delta_{BF}$, is that [24]

$$\frac{\omega}{T_c} \approx 8 \pm 8\% \,. \tag{29}$$

Measurements of this ratio in the real High-T superconductors give roughly this value [25].

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[10] One can freely get the "C.P. Herzog's Mathematica notebook for ArXiv:0803.3295" with code and results of numerical simulations by means of any search engine - Google, Yahoo, etc.

[11] The same result follows from the Kubo's formula of the linear response theory

$$\sigma = - \lim_{\omega \to 0} \frac{\Im G^R(\omega)}{\omega}$$

with the current-current retarded Green function $G^R(\omega)$ [23].

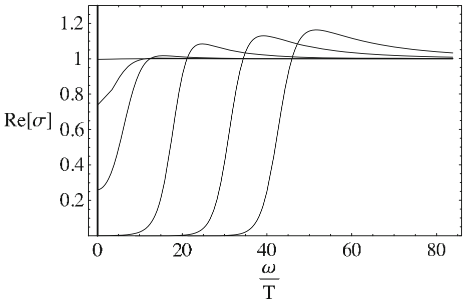

Figure 3. A gap formation in the real part of the conductivity under lowering temperature below the T_c. Taken from [11].

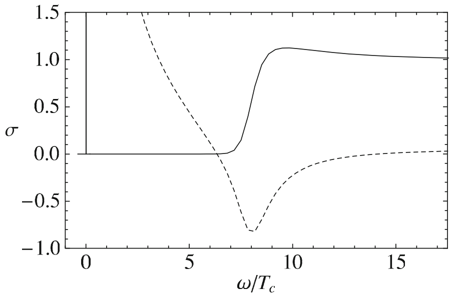

Figure 4. The low temperature limit of the optical conductivity. The solid line is the real part, the dashed line is the imaginary part. Taken from [24].

3 Summary

To summarize, the simple AdS/CFT setup of [11] leads to the second order phase transition from normal to superconducting state, with forming the energy gap. Critical exponent near the phase transition point coincides with that of the BCS theory, the gap is formed in the similar way. These results initiate the CMP community people to ask:"The HSC approach reproduced the known results of the BCS, so what? What the difference between the HSC approach and the standard BCS mechanism then?" At first site, qualitatively, the HSC picture looks very close to the results of the BCS theory. It is very expected, since the simplest model of the HSC [11] deals with s-wave superconductivity; the BCS theory deals with this case too. But quantitatively, there are principle differences between two approaches. The universal relation of the BCS theory, corresponding to (29), is $\omega/T_c \sim 3.5$. This value is twice less than the true experimental value, obtained, say, for cuprates. Mechanisms of forming the bosonic-type condensate, triggering the phase transition, are also different for both approaches. In the BCS theory it comes from the fermions pairing due to the electron-phonon interaction; on the HSC side the condensate forms by means of bosonic fields. Inclusion of fermions into the AdS/CMT setup leads to new objects - "strange metals" [26], [27], whose properties are defined by a holographic non-Fermi liquid. The strange metal state shows a "local quantum criticality", the property which has been experimentally discovered in the late of 80th (see, e.g., [28], [29] for review).

Could we believe in HSC results, having just one coincidence with experimental data as (29)? Could it be an accident? Critics from authors of [30] is appreciated in this respect, and initiates searching for new, may be more powerful arguments in favor of the "strange, but magic" relation between String theory and Condensed Matter Physics.

4 Epilogue

Since the discovery in 2008, the HSC approach has been extended to studies of Holography in disordered systems (like glasses, insulators, semiconductors) (see, e.g., [31]), to non-static and non-equilibrium dynamics within the HSC [32], [33], to the HSC models in periodic potentials, modeling the lattice [34], and to dissipative non-linear dynamics in Holography, which may be considered as first steps to chaos [35]. The list of achievements in the AdS/CMT during the last five years is more comprehensive of course, so there is the strong believe that one day the famous phase diagrams of cuprates, Fig. 5, becomes tractable and clear on the theory side.

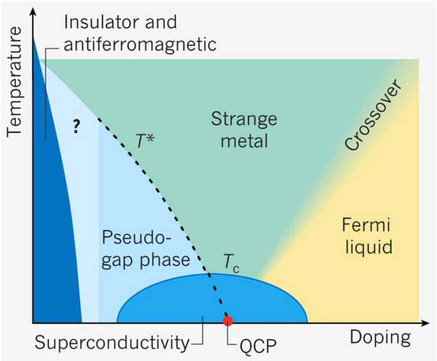

Figure 5. Phase diagram of cuprates.

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