Covariant Constructive Gravity

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Upward, not Northward

— Edwin A. Abbott, *Flatland: A Romance of Many Dimensions*

Abstract

Matter theories are not predictive if they couple to geometry with unknown dynamics: it is not possible to anticipate how matter behaves in the future without knowing how the geometry evolves. This thesis studies the completion of such matter theories to predictive theories of matter *and* gravity. Einstein solved this problem for Maxwell's electrodynamics by providing the Einstein equations. Indeed, a recurring theme of the work presented here is that general relativity is recovered as far as metric theories are concerned.

We start with the definition of two axioms that guide the completion of matter theories to predictive theories: gravity must be generally covariant and causally compatibility with the matter theory. Both axioms are brought into precise mathematical form. The foundation for the mathematical formulation is Lagrangian field theory on the jet bundle, where the geometry may be given by fields of arbitrary tensorial nature or, by extension of the approach, even nontensorial fields. In particular, the geometry need not be metric.

From the mathematical definitions follow partial differential equations and algebraic equations whose solutions yield candidate gravitational Lagrangians. This finding reduces the task of completing a matter theory with a gravitational theory to a computational problem, which we state in the condensed form of an algorithm. Since the algorithm provides a construction procedure for gravitational theories on the basis of general covariance, it shall bear the name *covariant constructive gravity*.

Applying the construction algorithm to Maxwell's electrodynamics reproduces general relativity. Theories beyond Maxwell and Einstein turn out harder to construct, such that we need to reduce the complexity of the problem in order to arrive at *some* physical implications. One possibility is to make a perturbation ansatz, which transforms the problem into simple linear algebra. Using this ansatz, we derive the second-order gravitational field equations for a birefringent generalisation of Maxwell's electrodynamics and consider the binary star as a prototypical example. Interesting phenomenology is obtained as result: a modification of Kepler's third law, the emission of massive gravitational waves, and a modified inspiral curve. These predictions demonstrate the predictive power of covariant constructive gravity—given a generalisation of Maxwell's electrodynamics, it is possible to derive gravitational implications.

The second approach is symmetry reduction, which is shown to yield the Friedmann equations if applied to a metric theory with cosmological symmetry. We sketch the application to nonmetric theories, but leave the implementation open for future research.

Kurzzusammenfassung

Materietheorien, denen Geometrie mit unbekannter Dynamik zugrunde liegt, sind nicht prädiktiv. Sie können das Verhalten von Materie in der Zukunft nicht vorhersagen, denn die Entwicklung der Geometrie ist nicht bekannt. In dieser Dissertation soll die Vervollständigung von solchen Materietheorien zu prädiktiven Theorien von Materie *und* Gravitation untersucht werden. Einstein löste dieses Problem für die Maxwellsche Elektrodynamik, indem er die Einsteinschen Feldgleichungen postulierte. Auch im Folgenden wird die Allgemeine Relativitätstheorie erneut hergeleitet werden, wann immer metrische Theorien besprochen werden.

Zu Beginn werden die beiden Axiome präsentiert, welche die Vervollständigung von Materietheorien leiten: Die Gravitationstheorie muss allgemein kovariant sein und eine zur Materietheorie kompatible Kausalität aufweisen. Mittels Lagrange-Feldtheorie auf Jetbündeln gelingt eine präzise mathematische Definition beider Axiome, wobei die Geometrie durch beliebige tensorielle Felder gegeben sein kann – sogar eine Erweiterung zu nicht-tensoriellen Feldern ist möglich. Insbesondere muss die Geometrie nicht zwingend metrisch sein.

Die mathematische Formulierung der Axiome impliziert sowohl partielle Differentialgleichungen als auch algebraische Gleichungen, deren Lösungen potentielle Lagrange-Dichten der Gravitation sind. Damit reduziert sich das Problem der Vervollständigung von Materietheorien mittels einer geeigneten Gravitationstheorie auf ein reines Rechenproblem, welches in Form eines Algorithmus angegeben werden kann. Dieser konstruktive Zugang zu modifizierten Gravitationstheorien, der auf dem kovarianten Lagrange-Formalismus beruht, wird *Kovariante Konstruktive Gravitation* genannt.

Angewandt auf die Maxwellsche Elektrodynamik, reproduziert der Algorithmus die Allgemeine Relativitätstheorie. Theorien jenseits von Maxwell und Einstein sind weniger trivial zu konstruieren, weshalb Methoden zur Reduktion der Komplexität erforderlich sind, um überhaupt physikalische Schlüsse ziehen zu können. Ein Störungsansatz reduziert das Problem auf Lineare Algebra. Mittels dieses Ansatzes lässt sich die zweite Störungsordnung der gravitativen Feldgleichungen für eine doppelbrechende Erweiterung der Maxwellschen Elektrodynamik herleiten. In diesem Beispiel stellt ein Doppelsternsystem interessante Phänomenologie zur Schau: ein modifiziertes drittes Keplersches Gesetz, die Abstrahlung von massiven Gravitationswellen, sowie eine veränderte Dynamik der Orbitperiode aufgrund der Strahlungsverluste. Solche Ergebnisse verdeutlichen die Vorhersagekraft der Kovarianten Konstruktiven Gravitation – aus einer Verallgemeinerung der Elektrodynamik folgen gravitative Phänomene.

Des Weiteren ist es möglich, mittels Symmetriereduktion eine Gravitationstheorie mit begrenzter Gültigkeit herzuleiten. Am Beispiel einer metrischen Theorie mit kosmologischer Symmetrie werden die Friedmann-Gleichungen wiederentdeckt. Die Anwendung dieser Vorgehensweise auf nichtmetrische Theorien wird nur skizziert, ihre Implementierung bleibt ein offenes Forschungsgebiet.

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Thank you, Frédéric, for introducing me to your way of thinking about physics. Having a clear mathematical foundation is fundamental for asking the right questions later on—but the mathematics does never become an end in itself. Together with the whole Constructive Gravity Group, we embraced this principle and delivered interesting results using innovative methods, despite facing challenging conditions. It has been great working together with Maximilian, Florian, Hans-Martin and all the Bachelor and Master students. I especially enjoyed the deep dive with Tobias into the covariant approach and the computer-aided solution methods. What we two achieved in relatively short time makes me proud.

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I am very grateful to Prof. Gerd Leuchs from the Max Planck Institute for the Science of Light for supervising my PhD.

Finally, and above all, I am indebted to my friends and family for their unconditional support throughout this challenging time. Without my parents, nothing of this would have been possible. Without you, Jennifer, it could not have been this enjoyable.

Contributions

Where not stated otherwise, this thesis contains original research. Parts have been researched in close collaboration with Tobias Reinhart and published as Ref. [1], building up on previous results [2].

Some of the calculations involved in Chap. 6 have already been performed for Ref. [3]. The remainder of this chapter, excluding the results on radiation loss, is published as Ref. [4], albeit in a more compact form.

Many calculations rely on heavy use of computer algebra. Two Haskell packages have been developed specifically with this purpose in mind: `sparse-tensor` [5], which originates from joint development with Tobias Reinhart, and `safe-tensor` [6]. Haskell code for the example in Chap. 6 that makes use of these packages is available as Ref. [7].

Complete sections paraphrasing content of Refs. [1] and [4] are introduced as such and typeset in *italic style*. They may, however, contain additional, previously unpublished details.

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[1] Nils Alex and Tobias Reinhart. "Covariant constructive gravity: A step-by-step guide towards alternative theories of gravity". In: *Physical Review D* 101.8 (Apr. 2020), p. 084025. DOI: `10.1103/physrevd.101.084025`

[2] Tobias Reinhart and Nils Alex. "Covariant Constructive Gravity". In: *Proceedings of the Fifteenth Marcel Grossman Meeting on General Relativity (in press)*. arXiv: `arXiv:2009.07540 [gr-qc]`

[3] Nils Alex. "Solutions of gravitational field equations for weakly birefringent spacetimes". In: *Proceedings of the Fifteenth Marcel Grossman Meeting on General Relativity (in press)*. arXiv: `arXiv:2009.07540 [gr-qc]`

[4] Nils Alex. "Gravitational radiation from birefringent matter dynamics". In: *Physical Review D* 102.10 (Nov. 2020), p. 104017. DOI: `10.1103/physrevd.102.104017`

[5] Tobias Reinhart and Nils Alex. *sparse-tensor: typesafe tensor algebra library [Hackage]*. Aug. 2019. URL: `https://hackage.haskell.org/package/sparse-tensor`

[6] Nils Alex. *safe-tensor: dependently typed tensor algebra (version v0.2.1.0) [Zenodo]*. Version v0.2.1.0. Sept. 2020. DOI: `10.5281/zenodo.4030851`

[7] Nils Alex. *area-metric-gravity: 3+1 split of area metric gravity to second order [Zenodo]*. Sept. 2020. DOI: `10.5281/zenodo.4032251`

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1 Introduction to the constructive gravity programme

Space tells matter how to move
Matter tells space how to curve
— John Archibald Wheeler,
Gravitation (1973)

1.1 The rôle of gravity in physics

As the title suggests, this thesis is primarily concerned with *gravity*. In the ensemble of physical theories, gravity plays a special rôle. It serves a different purpose than the theories we will call *matter theories*. The latter are subject to direct observations: photons—quanta of the electromagnetic field—hit the observers retina, allowing her to make inferences about the source of the particles. Another example are charged fermions, again quanta of a corresponding matter field, which induce signals in a semiconductor detector. Specific signatures in the signals may be associated with certain events that contributed to the production of the incident fermions, such that the statistics of these observations is able to falsify hypotheses about the underlying mechanisms.

How does gravity fit into this picture? The revolution of a binary star about its centre of mass, commonly known to be caused by gravity, is not observed directly. Neither are its gravitational spin-up and eventual merger. Rather, the stars emit photons that are picked up by the astronomer, who concludes details about the trajectories. When the LIGO and Virgo Collaborations announced the first observation of gravitational waves [8], the ground-breaking detection was earth-bound, but in a certain sense not *direct*: it amounts to the analysis of interference patterns from photons that bounced off of mirrors at the end of the detector arms. General relativity predicts that these arms should expand and contract under the influence of incident gravitational waves. Eventually, the signature in the interference pattern was found to match the predictions for a binary black hole merger.

From this point of view, gravity merely sets the stage for the propagation of matter fields. This is witnessed by the matter dynamics, for example Maxwell's equations for

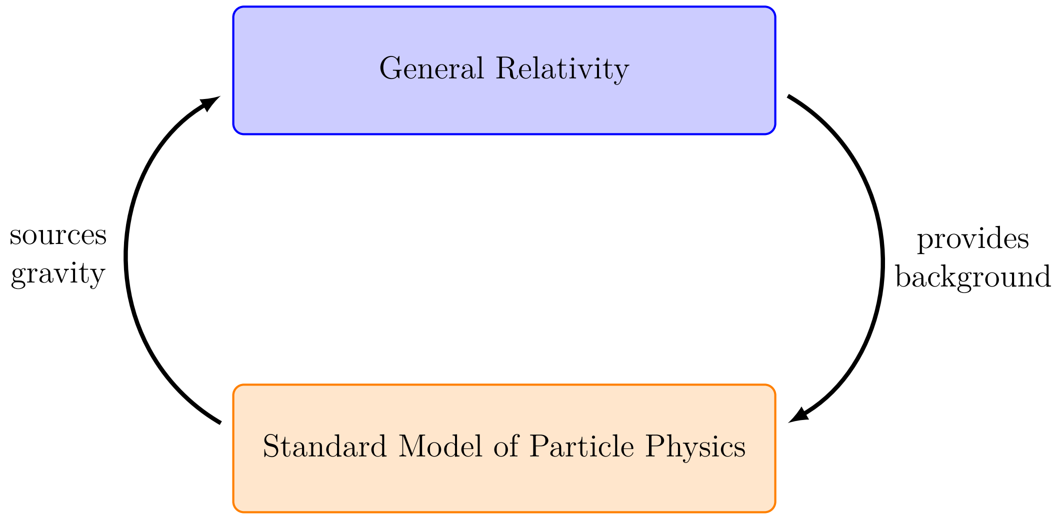

Figure 1.1: Interplay of the standard model theories and general relativity. Matter content sources the gravitational field equations. The gravitational field, in turn, provides the background on which matter fields propagate. Together, this yields a highly accurate fundamental description of the universe.

the electromagnetic potential. These are derived from the action functional

$$S_{\text{Maxwell}}[A] = \int d^4x \sqrt{-g} g^{ac} g^{bd} F_{ab} F_{cd},$$

which depends on the potential A via the field strength tensor $F = dA$. Maxwell's theory of the electromagnetic field has been a huge success as it is the foundation of many applications throughout science. The quantum field theories for the electromagnetic field, together with similar gauge theories and the fermonic sector, form the standard model of particle physics (SMPP), which is widely regarded as the most precisely tested physical theory[1]. Still, these matter theories presuppose knowledge of the spacetime metric g which enters the action for the electromagnetic field above and contributes to other theories of the SMPP in a similar way. Consequently, the SMPP alone lacks *predictivity*: collecting initial data of all physical fields is not enough for the physicist in order to determine the fields in the future, since the metric tensor has to be specified externally.

One of the many great contributions by Einstein was the prescription of field equations that govern the dynamics of the metric tensor. [11] This theory is called *general relativity*

[1] For example, the magnetic moment of the electron has been measured as $g/2 = 1.001\,159\,652\,180\,73(28)$. [9] Its value as proposed by quantum electrodynamics has been calculated as $g/2 = 1.001\,159\,652\,182\,03(73)$. [10] Both the experimentally measured value and the value calculated from quantum electrodynamics agree to more than 12 significant figures.

and may be derived from the Einstein-Hilbert action functional

$$S_{\text{Einstein-Hilbert}}[g] = \int d^4x \sqrt{-g} R.$$

Einstein's theory provides the missing link between matter and gravity, completing the SMPP to the joint model of SMPP and general relativity sketched in Fig. 1.1, which is now predictive. It has also been verified numerous times, both via astronomical observations and *in terra*[2] experiments, albeit to a lesser degree of certainty[3].

Of course, the division of physical theories into matter theories and gravity is only a *metaphysical* notion. Both make testable predictions about the outcome of experiments; both have been shown to accurately describe reality in a variety of circumstances. But exactly in this metaphysical idea lies the mindset of *constructive gravity*, which seeks to address the search for other, hopefully *more* complete pictures of matter and gravity.

1.2 Modified gravity from refined matter theories

Under certain assumptions, Einstein's general relativity is the unique theory that completes the SMPP to a predictive theory of matter and gravity. [14, 15, 16] The only two unknown parameters that need to be fixed by measurements are Newton's gravitational constant and the cosmological constant. This remarkable finding constrains the search for modified theories of gravity: if the mentioned assumptions are taken for granted, the standard model of particle physics can only be completed by general relativity. It is, however, well established that the joint theory of the SMPP and Einstein gravity cannot be universal, due to several inconsistencies.

One example are extreme circumstances, such as the beginning of the universe or the presence of black holes, where the whole formalism breaks down. [17] This is one of the justifications for the efforts of finding a quantum theory of gravity.

Even in more benign situations, the observations do not always coincide with the predictions from the SMPP and general relativity. The observed rotation curves of galaxies, for example, do not match the expectations calculated from the visible matter distributions. Starting from a certain minimum distance from the galaxy centre, stars rotate with higher velocities than expected. [18, 19, 20] This discrepancy generally increases with the radius. All proposed solutions[4] that may cure this inconsistency have

[2] earthbound

[3] Only four significant figures of the gravitational constant are known. [12] Measurements with higher precision yield conflicting results. [13]

[4] See e.g. [21, 22].

one thing in common—they modify or extend the currently accepted theories of matter and gravity.

There are more observations that demonstrate the need for modifications. [21] However such adjustments play out, they are constrained by the uniqueness of Einstein's general relativity in one of the following ways:

1. Additional or modified matter fields that make use of the same metric tensor g as the existing matter theories will still couple to general relativity.
2. Additional or modified matter fields that couple to a nonmetric geometry—e.g. two metric tensors or a tensor of higher rank—render general relativity as theory for a single metric tensor meaningless. A completely new description of gravity is needed, *which may be subject to similar uniqueness theorems.*
3. Modifications to general relativity itself are incompatible with the uniqueness theorem. This means that either the assumptions from which uniqueness follows must be dropped or that the matter theories have to be modified accordingly—if at all possible.

All three approaches are pursued, as they should be for a systematic search of modified theories. Constructive gravity, the subject of this thesis, is a framework for the structured treatment of approach number two. In most regards, its assumptions are very conservative, as it tries to deviate only ever so slightly from the established models. For example, where standard general relativity is restricted to field equations of second derivative order, constructive gravity keeps this restriction. This is not because other efforts are not deemed worthwhile—they certainly are, but different approaches towards modified gravity research should be explored *ceteris paribus*, only making one change at a time. The focus of constructive gravity lies on novel matter theories coupling to nonmetric geometries and the corresponding gravitational implications *within the existing meta-theory of classical physics.* Most importantly, because the framework is kept so close to the standard models, a similar uniqueness theorem can be derived for nonmetric geometries. It will not be as strong as for the SMPP and general relativity, but nevertheless provide a useful parameterisation of modified theories of gravity that fall into the second category.

As far as the framework is concerned, any matter theory that is formulated as classical field theory is fair game. The relevance of constructive gravity, however, crucially depends on the kind of matter theory that is proposed. A complete overhaul of physics is generally not desired—the existing models work very well in certain sectors. Any new theory must reproduce this phenomenology in order to be epistemically significant. For this reason, constructive gravity is considered a tool that guides the derivation of modified gravity from *refined* matter theories.

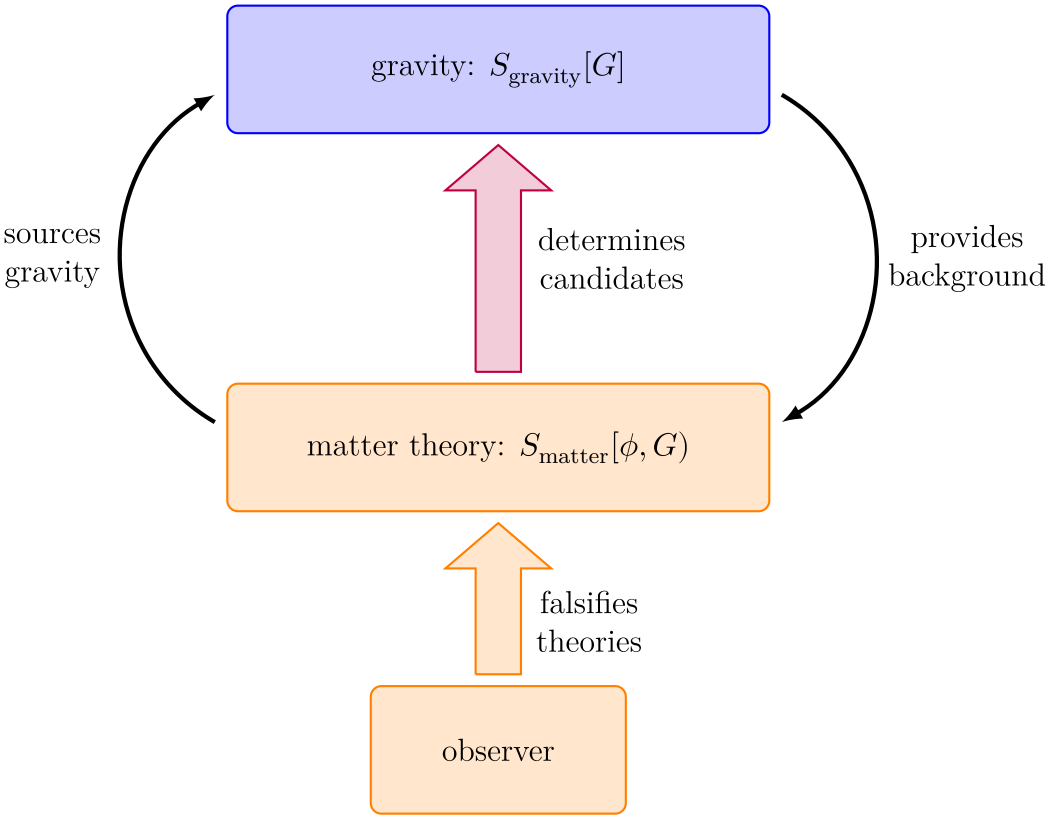

Figure 1.2: Rationale of constructive gravity. The matter theory S_{matter}, which couples the matter field ϕ to some geometry G, determines the structure of the gravitational theory S_{gravity}. In general, this theory will not be unique but parameterised by a set of constants or functions, which results in multiple candidates. Via the interdependence of matter and geometry, each candidate yields phenomenology that can be used by the observer for tests of the theory.

1.3 Canonical and covariant approaches to constructive gravity

The rationale of constructive gravity is pictured in Fig. 1.2. A matter theory, prescribed by the action $S_{\text{matter}}[\phi, G)$ serves as input. The round bracket next to the geometry G indicates that the action functional depends on G only locally, i.e. not via a derivative, while the matter fields ϕ enter with derivatives, typically of first order.

After successful application of constructive gravity, the corresponding gravity action $S_{\text{gravity}}[G]$ that closes the matter theory to a predictive theory of matter *and* gravity is obtained. Staying very close to the established formalism of the SMPP and general relativity, this action is assumed to be of second derivative order in the geometry G, with derived field equations also of second derivative order. In general, the gravity action is not unique, it depends on unknown parameters or functions.

This completed theory may now be used for the bread and butter business of theoretical physics: making predictions about the outcome of measurements. Comparisons with experimentally obtained data will restrict the parameter ranges. If the measurements turn out to be incompatible for all choices of parameters, the theory is falsified.

The essence of constructive gravity is the derivation of the gravity action from the matter action, i.e. the step

$$S_{\text{matter}}[\phi, G) \Rightarrow S_{\text{gravity}}[G].$$

Effectively, this amounts to a generalisation of the uniqueness theorems for general relativity, which can be interpreted as derivations of Einstein gravity from Maxwell electrodynamics (or, more generally, the SMPP).

Canonical constructive gravity (also called *canonical gravitational closure*) [23, 24, 25, 26] is the first approach that follows this pattern and is based on the work of Hojman, Kuchař, and Teitelboim (HKT) [15]. HKT showed that the ADM formulation[5] of general relativity is the unique representation of the so-called hypersurface deformation algebra[6]. The canonical approach to constructive gravity considers the hypersurface deformation algebra in a frame that corresponds to an observer subject to matter dynamics. Crucial for the definition of observer frames is the *principal polynomial* of the matter field equations, which captures the causality of the evolution of matter fields (see Sect. 2.4). It then imposes this algebra onto the constraint algebra of the canonical formulation of the unknown gravitational theory. This amounts to a system of functional differential equations, which is subsequently transformed into an infinite system of linear partial differential equations. These equations are called the *construction equations* or *closure equations*. Any solution to this system is a candidate Lagrangian for gravity.

In line with the already known results, general relativity has been shown to be the unique solution to the closure equations of canonical constructive gravity if the procedure is applied to Maxwell's electrodynamics as underlying matter theory. [25] However, based on previous work concerning matter theories that couple to arbitrary tensorial geometries [28, 29], the framework has been developed to be applicable to a wide range of matter theories. Two important examples of modified gravitational theories have been derived in linearised form. The first is *area metric gravity* [4], which completes

[5]The ADM formulation [27] is a canonical (i.e. Hamiltonian) formulation of general relativity. Due to the diffeomorphism invariance of general relativity, the canonical formulation is a constrained Hamiltonian system.

[6]A hypersurface is an embedding of a three-dimensional manifold in the four-dimensional spacetime manifold. Different embeddings are related via deformations. The actions of such deformations on hypersurface functionals form an algebra, the *hypersurface deformation algebra*. See also [15].

a birefringent generalisation of Maxwell's electrodynamics[7]. Although only valid for weak gravitational fields, this theory based on a physically well motivated refinement of Maxwell's electrodynamics offers interesting phenomenology when studying quantum electrodynamics [30], gravitational lensing [31], or gravitational waves [3]. *Bimetric gravity* [32, 33], the completion of a theory with two Klein-Gordon fields coupling to two different metric tensors[8], is a second example. Another interesting sector of solutions is the cosmological sector, which has also been studied in the past. [34]

Covariant constructive gravity[9] is a complementary approach. More in the tradition of Lovelock's proof [14, 35, 36] for the uniqueness of the Einstein-Hilbert Lagrangian, it derives the gravitational theory that completes a given matter theory by imposing two conditions directly on the spacetime formulation of gravity—hence the attribute "covariant". The two conditions will be called the *axioms* of covariant constructive gravity. Informally, they may be formulated as:

1. The dynamical laws that govern the gravitational field are generally covariant, i.e. are independent of the choice of a coordinate system.
2. The causality of the gravitational field equations is compatible with the causality of the matter field equations. In particular, a consistent co-evolution of all physical fields is guaranteed.

The motivation for these axioms is twofold. Firstly, they once again enforce the principle that any modified theory of gravity should be close, formally, to general relativity—which implements general covariance and has the same causality as the SMPP. This suggests that also modified theories of gravity should be independent of any coordinate choice and at least be *compatible* with the matter causality.

Secondly, the approach should complement the canonical framework. General covariance in the spacetime formulation is the equivalent for the conditions placed on the constraint algebra in the canonical formulation. As the second axiom, causal compatibility has been chosen because canonical constructive gravity claims to achieve something similar: the observer frame is constructed using the principal polynomial of the matter theory, which is why the hypersurface deformation algebra expressed in this frame contains terms related to matter causality. These terms carry over to the gravitational constraint algebra via the canonical construction procedure. This is often interpreted as the gravitational theory inheriting the causality of the matter theory. But there is little reason to believe so—the fact that the gravitational constraint algebra shares these terms with the hypersurface deformation algebra just means that both are expressed using the same frame. Whether this frame bears any significance for the gravitational theory is

[7] See Sect. 3.3.
[8] Or other theories that somehow make use of *two* metric tensors, see Sect. 3.4.
[9] First proposed in Ref. [1].

an unrelated question. In Ref. [37], it has been found[10] that the canonical constraint algebra of *any* diffeomorphism invariant theory implements the hypersurface deformation algebra, supporting the suspicion that causality may be unrelated. Even though matching causalities are not enforced by the canonical approach, it is still a sensible requirement for a theory of gravity that closes matter theories, so it is explicitly included as second axiom.

1.4 Outline

This thesis is dedicated to the development of the covariant approach. It aims to provide a complete picture of the current state of research, from the foundations to the construction procedure to testable predictions for an exemplary theory. At the end, every part of Fig. 1.2 will have been addressed.

After this chapter has introduced the rationale of covariant constructive gravity, Chap. 2 will be concerned with the mathematical foundations. We will walk through Lagrangian field theory in the jet bundle formulation, which allows a precise definition of the first axiom as equivariance condition for the Lagrangian with respect to spacetime diffeomorphisms. This condition is locally equivalent to a system of first-order, linear partial differential equations, the *equivariance equations*. For the derivation, we will make a detour to the global version of the Lagrangian variation problem, which comes with the definition of the so-called *Cartan form*. The Cartan form allows a quite elegant presentation of Noether's first and second theorem—especially the second theorem will prove to be useful later on. Afterwards, we introduce the notions necessary for a formulation of the second axiom. Causal compatibility will be phrased as conditions on certain geometric constructs that arise from the causality of the field equations.

In Chap. 3, we turn to the implementation of the axioms. While deriving the mathematical formalism, we already laid out most of the implementation details, so it suffices to give a short summary. This will be in the form of a *construction algorithm*. We then consider three examples of matter theories and discuss how the construction algorithm could be applied. The first example will be Maxwell's electrodynamics, for which we recover general relativity as the unique solution. The other two examples, a birefringent generalisation of Maxwell's electrodynamics and a bimetric Klein-Gordon theory, have no known generic solution, but we can nevertheless derive some interesting results.

Chap. 4 explores the *perturbative* application of the construction algorithm. The idea is that for weak gravitational fields it suffices to derive a truncated power series of the gravitational Lagrangian. We will see that with this approach the equivariance equations assume a particularly simple form, such that basic linear algebra suffices in order to derive

[10] As a preliminary result, for theories of first derivative order.

a perturbative equivalent of the “full” gravitational theory. This perturbative treatment of the equivariance equations is backed by strong results from the theory of partial differential equations. Also the second axiom is approachable by perturbation theory, such that we can close the chapter presenting a perturbative version of the construction algorithm.

While perturbative covariant constructive gravity amounts almost entirely to solving linear equations, its application in practice would still be quite laborious without help of the computer—the systems we have to deal with are simply too complex and large. In order to enable us to tackle the problem in a later chapter, we use Chap. 5 for the introduction of computational methods that have been developed with perturbative constructive gravity in mind. These are essentially two programmes written in the functional programming language Haskell: one for the generation of the perturbation ansatz and another one for the set-up and solution of the perturbative equivariance equations.

In Chap. 6, we will finally put covariant constructive gravity to the test. Coming back to the birefringent generalisation of Maxwell’s electrodynamics, we apply the perturbative construction procedure—using the computational methods introduced before—and obtain second-order gravitational field equations that close this novel matter theory for sufficiently weak fields. A thorough analysis of the linearised (*viz.* first-order) field equations already reveals interesting properties such as the emergence of *massive* gravitational waves. Afterwards, we will include the second-order terms and study the emission of gravitational waves from a binary star in this modified theory of gravity—focusing on the gravitational radiation emitted into the far zone, its effect on test matter, and the corresponding radiation loss of the binary system itself.

Another option to circumvent the complexity of the “full” construction algorithm, next to perturbation theory, is *symmetry reduction*. Chap. 7 shortly presents a way in which symmetry reduction may be performed within the framework of covariant constructive gravity. We will see that this procedure, when applied to metric theories with cosmological symmetry, indeed reproduces the Friedmann equations. How this may be extended to theories like birefringent electrodynamics is discussed, but the implementation is considered far beyond of our scope.

We finish the thesis with a discussion of the findings and proposals for future research.

2 The axioms of covariant constructive gravity

In this chapter, we will cast the two axioms of covariant constructive gravity in precise mathematical language. The basis for our theory is Lagrangian field theory defined in terms of jet bundles, whose basic notions we collect first. In particular, we will encounter the Cartan form, which is central for the global, coordinate-independent treatment of the Lagrangian variation problem. With this machinery at hand, we define the first axiom as equivariance condition for the Lagrangian with respect to spacetime diffeomorphisms. We then derive a few consequences that follow from this axiom: most importantly, a system of partial differential equations for a local representative of the Lagrangian, but also the existence of a stress-energy-momentum tensor. The former is fundamental for an algorithmic approach towards implementing the axiom, the latter is very useful for proving Noether's theorems in this setting. At the end of this chapter, we show how the relevant objects for a mathematical formulation of the second axiom are constructed. Finally, we present the second axiom using the terminology established so far.

The two axioms of covariant constructive gravity have been motivated in Sect. 1.3. Let us recall the informal definitions as phrased in Ref. [1].

Axiom I (diffeomorphism invariance)**.** *"The dynamical laws that govern gravity are invariant under spacetime diffeomorphisms." [1]*

Axiom II (causal compatibility)**.** *"Provided that spacetime is additionally inhabited by matter fields, their dynamics is causally compatible with the gravitational dynamics." [1]*

For a more precise formulation of the axioms, which will enable us to derive their consequences for gravitational theories coupled to novel matter, an introduction of the basic concepts of Lagrangian field theory is in order.

2.1 Lagrangian field theory

For the purpose of the present work, a *Lagrangian field theory* will be a geometric formulation of certain conditions on sections $\sigma \in \Gamma(\pi)$—called *fields*—of some bundle $E \xrightarrow{\pi} M$. These conditions select the physical realisations of fields admissible by the theory and constitute the *dynamical laws*. The bundle π shall be constructed from a tensor bundle, i.e. be a sub-bundle of some tensor bundle $T^r_s M$. It is possible to extend the framework to include other bundles, with the caveat that a lift of the action of the diffeomorphism group on M to E may have to be specified manually. Although not relevant for much of the development of the theory, the base manifolds to be considered later for concrete examples will be spacetime manifolds of dimension 4.

Example 2.1.1. *Two examples for Lagrangian field theories are*

- *Einstein gravity on the symmetric sub-bundle of $T^2_0 M$ of inverse metric tensors*[1] *with dynamical laws given by the Einstein equations, and*
- *Maxwell electrodynamics on the bundle T^*M of potential one-forms with dynamical laws given by the Maxwell equations.*

Both theories are Lagrangian because they derive their dynamical laws in a certain geometric manner. The mechanism will be explained in the following, but first, let us fix some of the notation involved.

A bundle is denoted as $E \xrightarrow{\pi} M$, where E is the total space, M is the base manifold, π is the submersion. As a shorthand, it is common to write just π—it is then understood that total space and base manifold are domain and co-domain of π, respectively. The dimension of M is written as n, the dimension of a typical fibre F of π as m. Coordinate functions on E are denoted by (x^i, u^A). Such coordinates extend to the kth jet bundle $J^k E \xrightarrow{\pi_k} M$ over π as $(x^i, u^A, u^A_{i_1}, u^A_{i_1 i_2}, \dots, u^A_{i_1 \dots i_k})$. The literature on jet bundles mostly employs multi-indices for higher-order jet bundles (see e.g. Ref. [38]), which is certainly the right approach for studying the properties of jet bundles, but for practical calculations on the second-order jet bundle performed below the intertwiner technique (see Def. 2.2.4) will be used. This technique is equally able to take care of ambiguities regarding symmetric indices. Prolongations of sections σ are denoted with $j^k(\sigma)$, projections between jet bundles of different orders with $\pi_{k,k'}(\sigma)$. The latter are submersions in their own right and thus also define bundle projections. Total derivatives are written as D_i. Throughout, the Einstein summation convention is used.

[1]The equivalent formulation as a theory for a metric tensor field takes place on the bundle $T^0_2 M$.

Definition 2.1.2 (Lagrangian [38]). *Let M be a smooth manifold of dimension n and $E \xrightarrow{\pi} M$ a smooth fibre bundle over M with typical fibre F. A* Lagrangian $\mathscr{L}$ *of order k is an element*

$$\mathscr{L} \in \textstyle\bigwedge_0^n \pi_k. \tag{2.1}$$

In other words, $\mathscr{L}$ is a horizontal n-form on the k-th-order jet bundle π_k of π.

Assuming M to be orientable with volume form $\Omega \in \Lambda^n M$, a Lagrangian $\mathscr{L}$ is equivalently characterised by its Lagrangian density $L \in C^\infty(J^k\pi)$,

$$\mathscr{L} = L\pi_k^*\Omega. \tag{2.2}$$

The claim of (2.2) becomes apparent in local coordinates, where a horizontal n-form on π_k appears as

$$\textstyle\bigwedge_0^n \pi_k \ni \mathscr{L} = \mathscr{L}(x^i, u^A, u_i^A, \dots)dx^1 \wedge \dots \wedge dx^n. \tag{2.3}$$

From (2.3) and (2.2), it is also clear how the notion of a Lagrangian as a horizontal n-form captures in a geometric way the notion used elsewhere as a bundle map $J^kE \to \Lambda^n M$ (see [38, 39, 1]).

From now on, we will consider smooth, orientable base manifolds M and smooth bundles over the base manifold. Depending on the context, the symbol Ω will denote either the form on M or the pullback to various bundles over M.

Definition 2.1.3 (local action functional [38]). *Given a Lagrangian $\mathscr{L} \in \bigwedge_0^n \pi_k$ and a compact n-dimensional submanifold $C \subset M$, the* local action functional *is defined as the map*

$$\sigma \mapsto S[\sigma] = \int_C (j^k\sigma)^*\mathscr{L} \tag{2.4}$$

for all local sections σ of π with support on C.

Lagrangian field theory now stipulates that sections $\sigma \in \Gamma(\pi)$ are physical if they are extremals of the action functional. The well-known *Euler-Lagrange equations* from the calculus of variations provide a necessary condition in local coordinates which such sections must satisfy.

Proposition 2.1.4 ([38]). *Let $L \in C^\infty(J^k\pi)$ be a Lagrangian density. Let C be a compact submanifold of M and σ be a local section of π such that the local action functional $S[\sigma]$ is defined. If σ is an extremal of S, it satisfies the* Euler-Lagrange equations

$$(j^{2k}\sigma)^* \left(\sum_{l=0}^{k} (-1)^l D_{i_1} \cdots D_{i_l} \frac{\partial L}{\partial u^A_{i_1 \cdots i_l}} \right) = 0. \tag{2.5}$$

Proof. See [38]. □

The intrinsic equivalent to the Euler-Lagrange equations in local coordinates introduces a new object, the *Cartan form*, which plays a central rôle in the geometrisation of Lagrangian field theory.

Proposition 2.1.5 ([38])**.** *Given a Lagrangian $\mathscr{L} = L\Omega \in \bigwedge_0^n \pi_k$, there exists an n-form $\Theta_L \in \bigwedge_0^n \pi_{2k-1,k-1} \cap \bigwedge_1^n \pi_{2k-1}$, such that, globally, the variation of the Lagrangian is given by*

$$\delta L = \pi^*_{2k,k}\left(dL \wedge \Omega\right) + d_h \Theta_L \tag{2.6}$$

and extremals of $\mathscr{L}$ are extremals of Θ_L in the sense that

$$\pi^*_{2k-1,k}(j^k\sigma)^*\mathscr{L} = (j^{2k-1}\sigma)^*\Theta_L. \tag{2.7}$$

Such a form Θ_L is called a Cartan form*.*

Proof. See [38]. □

This definition generalises the local derivation of (2.5): the variation δL is obtained by lifting $d\mathscr{L}$ to π_{2k} and cancelling nonhorizontal terms (over E) by adding a derivative, which corresponds to repeated integrations by parts.

A possible[2] coordinate expression for Θ_L is [38]

$$\Theta_L = L\Omega + \sum_{s=0}^{k-1} \sum_{l=0}^{k-s-1} (-1)^l D_{i_1} \cdots D_{i_l} \left(\frac{\partial L}{\partial u^A_{ji_1 \dots i_l p_1 \dots p_s}} \right) \psi^A_{p_1 \cdots p_s} \wedge \left(i_{\partial_j} \Omega \right), \tag{2.8}$$

where the forms $\psi^A_{p_1 \cdots p_s} = du^A_{p_1 \cdots p_s} - u^A_{p_1 \cdots p_s q} dx^q$ span the contact system of π_k (see [38]).

Straight-forward application of (2.6) to (2.8) yields the well-known coordinate expression

$$\delta L = \left(\sum_{l=0}^{k} (-1)^l D_{i_1} \cdots D_{i_l} \frac{\partial L}{\partial u^A_{i_1 \cdots i_l}} \right) u^A \wedge \Omega \tag{2.9}$$

for the Euler-Lagrange form, reconciling the intrinsic formulation using the Cartan form with the explicitly coordinate-dependent[3] formulation using the Euler-Lagrange equations.

In later sections, we will restrict our attention to Lagrangians of second derivative order. As it turns out, the Cartan form for such a theory is *unique*.

[2]Generally, Θ_L is not uniquely defined. [38]
[3]Which is not to say *ill-defined*.

Proposition 2.1.6 ([38])**.** *The Cartan form is unique for second-order Lagrangians.*

Proof. See [38]. □

2.2 Axiom I: diffeomorphism invariance

In the language of jet bundles, the first axiom can be formalised as equivariance condition under a certain group action on the Lagrangian. The group in question is the diffeomorphism group $\mathrm{Diff}(M)$, acting on M by function application. By virtue of the pushforward-pullback construction, sub-bundles of tensor bundles carry a canonical action of $\mathrm{Diff}(M)$ as bundle automorphisms, denoted as $\varphi_E \in \mathrm{Aut}(E)$ for every $\varphi \in \mathrm{Diff}(M)$. We call this the *lift* of the diffeomorphism φ. This action, in turn, lifts naturally to the jet bundles over E.

Definition 2.2.1 (prolongation of morphisms [38])**.** *Let $E \xrightarrow{\pi_E} M$ and $H \xrightarrow{\pi_H} N$ be two bundles. The kth-order jet bundle lift of a bundle morphism (F, f) from π_E to π_H is the unique bundle morphism $(j^k(F), f)$ from $J^k\pi_E$ to $J^k\pi_H$ such that for any section ϕ of π_E the identity $j^k(F) \circ j^k\phi \circ f^{-1} = j^k(F \circ \phi \circ f^{-1})$ holds.*

A proof for the existence and uniqueness of this construction can be found in Ref. [38]. With the notion of the *lift* of a bundle automorphism at hand, we now give the first axiom a precise meaning.

Definition 2.2.2 (diffeomorphism invariant theory)**.** *A Lagrangian field theory is called* diffeomorphism invariant *if its Lagrangian $\mathscr{L} \in \bigwedge_0^n \pi_k$ is invariant with respect to the lifted action of* $\mathrm{Diff}(M)$ *on J^kE, i.e. if for all $\varphi \in \mathrm{Diff}(M)$*

$$j^k(\varphi_E)^* \mathscr{L} = \mathscr{L}. \tag{2.10}$$

This definition applies not only to tensor bundles and the diffeomorphism group—all we need is a well-defined action as bundle automorphism. For tensor bundles, however, there is always the canonical action built from the pushforward action on tangent vectors

$$\varphi_* \colon T_pM \to T_{\varphi(p)}M \tag{2.11}$$

and the pullback action on cotangent vectors[4]

$$(\varphi^*)^{-1} \colon T_p^*M \to T_{\varphi(p)}^*M. \tag{2.12}$$

[4]The action is inverted in order to still define a covariant functor, in the sense that it maps from the cotangent space at p to the cotangent space at $\varphi(p)$.

Using a coordinate chart (U, x) containing p and (V, y) containing $\varphi(p)$, the bundle automorphisms act on coordinate-induced component functions of vector fields as

$$(\varphi_* X)^j_{(y)}(y(\varphi(p))) = \left.\frac{\partial(y^j \circ \varphi)}{\partial x^i}\right|_p \cdot X^i_{(x)}(x(p)) \tag{2.13}$$

and on component functions of covector fields as[5]

$$((\varphi^{-1})^* \omega)_{(y)j}(y(\varphi(p))) = \left.\frac{\partial(x^i \circ \varphi^{-1})}{\partial y^j}\right|_{\varphi(p)} \cdot \omega_{(x)i}(x(p)). \tag{2.14}$$

We wish to encode (2.10) as local conditions on the Lagrangian density. To this end, consider a coordinate representation $\mathscr{L} = L_{(x)} d^n x$.

Proposition 2.2.3. *Let $\mathscr{L} = L_{(x)} d^n x$ be a coordinate representation of a diffeomorphism invariant Lagrangian, induced by a coordinate chart (U, x) on M. It follows from the invariance condition* (2.10) *that $L_{(x)}$ is* diffeomorphism equivariant, *i.e. it holds for all $\varphi \in \mathrm{Diff}(M)$ that, over the intersection of U and $\varphi(U)$,*

$$L_{(x)} \circ j^k_{(x)}(\varphi_E) = |d\varphi_{(x)}|^{-1} L_{(x)}. \tag{2.15}$$

$|d\varphi_{(x)}|$ denotes the determinant of the Jacobian of φ in terms of the coordinate chart (U, x).

Proof. The result follows from the coordinate expression (2.14) for the pullback of one forms, which extends to horizontal forms on the jet bundle. □

Covariant constructive gravity derives its calculational power from the observation that the infinitesimal version of (2.15) is equivalent to a system of linear partial differential equations (PDEs) for the Lagrangian density L. For the derivation of this theorem and the remainder of the chapter, we will work on the second jet bundle, as we are ultimately interested in investigating field theories of second derivative order. We will also drop the chart label from coordinate-dependent quantities. In order to lighten the notation, partial derivatives of functions on J^2M are denoted $L_{,m}$ for derivatives with respect to coordinates on M and $L_{:A}$, ${L_{:A}}^p$, ${L_{:A}}^{pq}$ for derivatives with respect to fibre coordinates.

[5]As for any group homomorphism, we have $(\varphi^*)^{-1} = (\varphi^{-1})^*$.

Since the bundles in question are (sub-bundles of) tensor bundles $T^m_n M$, it is possible to restrict to coordinates which are linear on the fibres. The *intertwiner technique*[6] relates such coordinates to coordinates on $T^m_n M$ itself by virtue of two special bundle morphisms.

Definition 2.2.4 (intertwiners)**.** *Let* $E \xrightarrow{\pi} M$ *be a sub-bundle of* $T^m_n M$*. A pair of vector bundle morphisms* (I, J)*,*

$$\begin{aligned} I \colon E &\to T^m_n M, \\ J \colon T^m_n M &\to E, \end{aligned} \tag{2.16}$$

which cover id_M *and satisfy* $J \circ I = \mathrm{id}_E$ *is called a pair of* intertwiners *for* π*.*

It follows from the property $J \circ I = \mathrm{id}_E$ that J is a *surjection* and I is an *injection*. Expressed in adapted coordinates, it is clear how the coordinate representations of I and J relate fibre coordinates to each other,

$$\begin{aligned} u^{a_1 \dots a_m}_{b_1 \dots b_n} &= I^{a_1 \dots a_m}_{b_1 \dots b_n A} \cdot u^A, \\ u^A &= J^{A b_1 \dots b_n}_{a_1 \dots a_m} \cdot u^{a_1 \dots a_m}_{b_1 \dots b_n}, \\ \delta^B_A &= I^{a_1 \dots a_m}_{b_1 \dots b_n A} \cdot J^{B b_1 \dots b_n}_{a_1 \dots a_m}. \end{aligned} \tag{2.17}$$

Concrete implementations of I and J will be introduced in Chap. 3. Intertwiners for the symmetric sub-bundle of $T^0_2 M$ are used to deduplicate second-order derivative indices by defining

$$\begin{aligned} u^A_I &= J^{ij}_I u^A_{ij}, \\ u^A_{ij} &= I^I_{ij} u^A_I. \end{aligned} \tag{2.18}$$

Proceeding to derive the infinitesimal version of axiom I, we first need to specify what is meant by *infinitesimal*. As the symmetry group in question is the diffeomorphism group on the base manifold, the infinitesimal equivalent is the corresponding Lie algebra $\Gamma(TM)$ of sections of the tangent bundle over M. The Lie bracket is given by the Lie bracket of vector fields. In a given coordinate chart, an element $\xi \in \Gamma(TM)$ defines an infinitesimal diffeomorphism as

$$x^i \mapsto x^i + \xi^i. \tag{2.19}$$

From (2.13) and (2.14), we know how an infinitesimal diffeomorphism (2.19) acts on vectors and covectors. Dropping chart labels because everything takes place in the chart (U, x), the actions are given by

$$X^i \mapsto X^i + X^j \xi^i_{,j} \tag{2.20}$$

[6] First described for a similar setting in Ref. [40], later introduced in the context of covariant constructive gravity [1].

and

$$\omega_i \mapsto \omega_i - \omega_j \xi^j_{,i}. \tag{2.21}$$

On higher-ranked tensor bundles, the action generalises to

$$T^A \mapsto T^A + C^A{}_B{}^n{}_m T^B \xi^m_{,n}, \tag{2.22}$$

with $C^i{}_j{}^n{}_m = \delta^i_m \delta^n_j$ for the special case of vectors and $C_i{}^{jn}{}_m = -\delta^n_i \delta^j_m$ for covectors. From this, we can read off the Lie algebra morphism which maps vector fields on M to vector fields on E and is induced by the group homomorphism from $\mathrm{Diff}(M)$ to $\mathrm{Aut}(E)$:

$$\begin{aligned} \Gamma(TM) &\to \Gamma(TE) \\ \xi &\mapsto \xi_E := \xi^m \partial_m + C^A{}_B{}^n{}_m u^B \xi^m_{,n} \partial_A. \end{aligned} \tag{2.23}$$

The constant coefficients $C^A{}_B{}^n{}_m$ will be called *Gotay-Marsden coefficients* after the authors of Ref. [39], where this formalism is developed in a more general setting—but only for the first jet bundle. In the language of this reference, a tensor field theory is of *differential index* 1.

As it turns out, the map $\xi \mapsto \xi_E$ indeed defines a homomorphism between Lie algebras.

Proposition 2.2.5. *The map* (2.23) *is a Lie algebra homomorphism, i.e. it holds for all* $\xi, \psi \in \Gamma(TM)$ *that*

$$[\xi_E, \psi_E] = [\xi, \psi]_E. \tag{2.24}$$

Proof. The map $\xi \to \xi_E$ is the differential of the lift $\phi \to \phi_E$ of diffeomorphisms $\varphi \in \mathrm{Diff}(M)$ to vector bundle automorphisms $\varphi_E \in \mathrm{Aut}(E)$. This lift is a Lie group homomorphism, i.e. $(\phi \circ \psi)_E = \phi_E \circ \psi_E$. Eq. (2.24) follows from the functoriality of the Lie algebra. Note that $\mathrm{Diff}(M)$ is not a finite-dimensional Lie group, so the results from the theory of finite-dimensional Lie groups are not applicable as they are. See e.g. [41].

□

A useful corollary of the homomorphism property of the Lie algebra lift $\xi \mapsto \xi_E$, which will play a rôle in Sect. 4.2, is the following fact about Gotay-Marsden coefficients.

Corollary 2.2.6. *The Gotay-Marsden coefficients* $C^A{}_B{}^n{}_m$ *corresponding to a tensor field theory satisfy the relation*

$$C^A{}_B{}^n{}_m C^B{}_C{}^q{}_p - C^A{}_B{}^q{}_p C^B{}_C{}^n{}_m = C^A{}_C{}^q{}_m \delta^n_p - C^A{}_C{}^n{}_p \delta^q_m. \tag{2.25}$$

Proof. Expanding (2.24) and making use of the coordinate expression (2.23) yields the identity

$$\left[C^A{}_B{}^n{}_m C^B{}_C{}^q{}_p - C^A{}_B{}^q{}_p C^B{}_C{}^n{}_m - (C^A{}_C{}^q{}_m \delta^n_p - C^A{}_C{}^n{}_p \delta^q_m)\right] \xi^m_{,n} \psi^p_{,q} = 0, \tag{2.26}$$

from which the result follows, as ξ and ψ can be chosen arbitrarily. □

If required for calculations, Gotay-Marsden coefficients are easily expressed using intertwiners.

Proposition 2.2.7. *Let $E \xrightarrow{\pi} M$ be a sub-bundle of the tensor bundle $T^r_s M$ with a pair (I, J) of intertwiners. If the tensors in E are purely contravariant, i.e. $s = 0$, the Gotay-Marsden coefficients are*

$$C^A{}_B{}^n{}_m = r \cdot I_B^{a_1 \dots a_{r-1} n} J^A_{a_1 \dots a_{r-1} m}. \tag{2.27}$$

If the tensors in E are purely covariant, i.e. $r = 0$, the Gotay-Marsden coefficients are

$$C_A{}^{Bn}{}_m = -s \cdot I^B_{a_1 \dots a_{s-1} m} J_A^{a_1 \dots a_{s-1} n}. \tag{2.28}$$

Note that in the latter case of a purely covariant tensor bundle, a fibre coordinate function is denoted by u_A with a lower index.

The Gotay-Marsden coefficients are the defining objects for the PDE version of the invariance of L under infinitesimal diffeomorphisms. This constitutes the central result concerning the first axiom and shall be proved in the following.

Theorem 2.2.8. *Let L be the local Lagrangian density of a second derivative order Lagrangian $\mathscr{L} = L d^4x \in \bigwedge^n_0 \pi_2$. If $\mathscr{L}$ is diffeomorphism invariant, i.e. satisfies the invariance condition* (2.10) *of the first axiom of covariant constructive gravity, its local representation L satisfies a system of first-order linear partial differential equations given by*

$$\begin{aligned}
0 &= L_{,m} && \text{(2.29a)}\\
0 &= L_{:A} C^A{}_B{}^n{}_m u^B + L_{:A}{}^p \left[C^A{}_B{}^n{}_m \delta^q_p - \delta^A_B \delta^q_m \delta^n_p\right] u^B{}_q \\
&\quad + L_{:A}{}^I \left[C^A{}_B{}^n{}_m \delta^J_I - 2\delta^A_B J^{pn}_I I^J_{pm}\right] u^B{}_J + L\delta^n_m && \text{(2.29b)}\\
0 &= L_{:A}{}^{(p|} C^A{}_B{}^{|n)}{}_m u^B + L_{:A}{}^I \left[C^A{}_B{}^{(n}{}_m 2 J^{p)q}_I - \delta^A_B J^{pn}_I \delta^q_m\right] u^B{}_q && \text{(2.29c)}\\
0 &= L_{:A}{}^I C^A{}_B{}^{(n}{}_m J^{pq)}_I u^B. && \text{(2.29d)}
\end{aligned}$$

Proof. Given a vector field X on E, the lift to the total space J^2E of the second jet bundle is uniquely defined. [38] Applying this lift to the vector field ξ_E corresponding to $\xi \in \Gamma(TM)$ yields the vector field

$$\begin{aligned}\xi_{J^2E} := \xi^m \partial_m & \\ & + C^A{}_B{}^n{}_m u^B \xi^m_{,n} \partial_A + C^A{}_B{}^n{}_m u^B_p \xi^m_{,n} \partial_A{}^p - u^A_m \xi^m_{,p} \partial_A{}^p \\ & + C^A{}_B{}^n{}_m u^B_I \xi^m_{,n} \partial_A{}^I - 2 J^{nr}_I I^J_{mr} u^A_J \xi^m_{,n} \partial_A{}^I \\ & + C^A{}_B{}^n{}_m u^B \xi^m_{,np} \partial_A{}^p + 2 C^A{}_B{}^n{}_m J^{pq}_I u^B_p \xi^m_{,nq} \partial_A{}^I - J^{pq}_I u^A_m \xi^m_{,pq} \partial_A{}^I \\ & + C^A{}_B{}^n{}_m J^{pq}_I u^B \xi^m_{,npq} \partial_A{}^I . \end{aligned} \tag{2.30}$$

Like before, the map $\xi \mapsto \xi_{J^2E}$ constitutes a Lie algebra morphism from $\Gamma(TM)$ to $\Gamma(TJ^2E)$.

Assuming that (2.15) holds, we obtain the infinitesimal version by acting on L with ξ_{J^2E} for the left-hand side and approximating $|d\varphi|^{-1}$ as $1 - \xi^m_{,m}$ for the right-hand side. Equating both sides yields

$$\begin{aligned} 0 = L_{,m}\xi^m & \\ & + \{ L_{:A} C^A{}_B{}^n{}_m u^B + L_{:A}{}^p \left[C^A{}_B{}^n{}_m \delta^q_p - \delta^A_B \delta^q_m \delta^n_p \right] u^B{}_q \\ & \quad + L_{:A}{}^I \left[C^A{}_B{}^n{}_m \delta^J_I - 2 \delta^A_B J^{pn}_I I^J_{pm} \right] u^B{}_J + L \delta^n_m \} \xi^m_{,n} \\ & + \{ L_{:A}{}^p C^A{}_B{}^n{}_m u^B + L_{:A}{}^I \left[C^A{}_B{}^n{}_m 2 J^{pq}_I - \delta^A_B J^{pn}_I \delta^q_m \right] u^B{}_q \} \xi^m_{,np} \\ & + L_{:A}{}^I C^A{}_B{}^n{}_m J^{pq}_I u^B \xi^m_{,npq} . \end{aligned} \tag{2.31}$$

Since (2.31) holds for any $\xi \in \Gamma(TM)$, the individual contributions for ξ, $\partial\xi$, $\partial\partial\xi$, $\partial\partial\partial\xi$ are satisfied separately. □

On a four-dimensional spacetime manifold, Thm. 2.2.8 yields a system of 140 linear PDEs of first order for the Lagrangian density L. Any diffeomorphism invariant tensor field theory of second derivative order must satisfy this system and, conversely, any solution to the system provides a candidate for a diffeomorphism invariant theory. Thus, the search for such theories has been reduced to the mathematical task of solving PDEs of a certain (simple!) form. The only ingredients which depend on the specific theory at hand are the Gotay-Marsden coefficients, such that it is possible to derive certain properties of the system without knowledge of the concrete tensor bundle.

The literature on this kind of PDEs is very extensive (see e.g. [42]) with many applications throughout science. There are strong results on the properties and solutions which provide a good basis for our work with Eqns. (2.29a)–(2.29d) in the following.

The PDE system will be referred to as *equivariance equations* from now on. In a similar form, these equations already appear in Ref. [39] during the derivation of conservation laws arising from diffeomorphism invariance. As they were not meant to be solved for the Lagrangian density, the presentation is not as explicit as here. Also note that Ref. [39] considers theories of arbitrary differential index but only the first jet bundle, whereas the present derivation takes place on the second jet bundle but is restricted to a differential index of 1, i.e. tensor field theories of second derivative order. The extension to theories with arbitrary differential index is possible—there will be a series of Gotay-Marsden coefficients

$$\xi_E = \xi^m \partial_m + C^A{}_{Bm}\xi^m + C^A{}_B{}^n{}_m \xi^m_{,n} + C^A{}_B{}^{np}{}_m \xi^m_{,np} + ... \tag{2.32}$$

which follow from the action of the diffeomorphism group on the bundle. [39]

2.3 Noether theorems

Diffeomorphism invariance of the Lagrangian as required by the first axiom results in a number of interesting properties of the theory. Among these are identities for the Euler-Lagrange equations and conservation laws for the dynamics given by the Euler-Lagrange equations, which are examples for the well-known Noether theorems. In analogy to the derivation for theories of first derivative order in Ref. [39], we shall now prove a version of the Noether theorems for the second-order formalism developed above.

The first step is to realise that the Cartan form for a diffeomorphism invariant Lagrangian is itself diffeomorphism invariant.

Proposition 2.3.1 ([43, 44])**.** *Let $\mathscr{L}$ be a diffeomorphism invariant Lagrangian, i.e. $j^k(\varphi_E)^*\mathscr{L} = \mathscr{L}$. Any corresponding Cartan form Θ_L satisfies the diffeomorphism invariance condition*

$$j^{2k-1}(\varphi_E)^*\Theta_L = \Theta_L. \tag{2.33}$$

Proof. See [43, 44]. □

Using the infinitesimal version $\mathcal{L}_{\xi_{J^{2k-1}E}}\Theta_L = 0$ of the diffeomorphism invariance of Θ_L, the first Noether theorem follows as a direct consequence.

Theorem 2.3.2 (first Noether theorem [39])**.** *Let $\mathscr{L}$ be a diffeomorphism invariant Lagrangian and Θ_L a corresponding Cartan form. For any lifted generator $\xi_{J^{2k-1}E}$ of the diffeomorphism action and any section $\sigma \in \Gamma(\pi)$ satisfying the Euler-Lagrange equations $j^{2k}(\sigma)^*(\delta L) = 0$, it follows that the current defined as*

$$j(\sigma) = (j^{2k-1}\sigma)^*\iota_{\xi_{J^{2k-1}E}}\Theta_L \tag{2.34}$$

is a closed differential form, i.e.

$$0 = dj(\sigma). \tag{2.35}$$

Proof. Applying the Cartan formula to the infinitesimal diffeomorphism invariance condition for Θ_L (which follows from Prop. 2.3.1) gives

$$0 = \mathcal{L}_{\xi_{J^{2k-1}E}}\Theta_L = d\,\iota_{\xi_{J^{2k-1}E}}\Theta_L + \iota_{\xi_{J^{2k-1}E}}d\Theta_L \tag{2.36}$$

such that

$$\begin{aligned} d\left((j^{2k-1}\sigma)^*\iota_{\xi_{J^{2k-1}E}}\Theta_L\right) &= (j^{2k-1}\sigma)^*(d\,\iota_{\xi_{J^{2k-1}E}}\Theta_L) \\ &= -(j^{2k-1}\sigma)^*(\iota_{\xi_{J^{2k-1}E}}d\Theta_L). \end{aligned} \tag{2.37}$$

One of the defining properties of the Cartan form Θ_L is that extremals of $\mathscr{L}$ are extremals of Θ_L. Because σ, satisfying the Euler-Lagrange equations, is an extremal of $\mathscr{L}$, it also satisfies the condition [45]

$$0 = (j^{2k-1}\sigma)^*(\iota_\Xi d\Theta_L) \tag{2.38}$$

for extremals of Θ_L. The condition holds for arbitrary vector fields Ξ on $J^{2k-1}E$, including the vector fields $\xi_{J^{2k-1}E}$. □

Eq. (2.35) defines a current which is conserved *on shell*, i.e. whenever the Euler-Lagrange equations hold. Thm. 3.1 of Ref. [39] already shows—for theories defined on the first jet bundle—how the current arises from the so-called *stress-energy-momentum* tensor. This result can now be generalised to tensor field theories of second derivative order. For the following calculations, we introduce the abbreviations $\omega = d^n x$, $\omega_i = \iota_{\partial_i}\omega$, $\omega_{ij} = \iota_{\partial_i}\iota_{\partial_j}\omega$, and so forth.

Theorem 2.3.3 (Gotay-Marsden stress-energy-momentum tensor)**.** *Let Θ_L be the diffeomorphism invariant Cartan form corresponding to a diffeomorphism invariant Lagrangian of second derivative order ($k = 2$). For any local section σ of the underlying bundle, there exists a unique $(1,1)$-tensor density $\mathcal{T}(\sigma)$ on the base manifold M such that for all vector fields ξ with compact support on M and embedded hypersurfaces $i_\Sigma \colon \Sigma \to M$*

$$\int_\Sigma i_\Sigma^* j(\sigma) = \int_\Sigma \mathcal{T}^{\ n}_{m}(\sigma)\xi^m\omega_n. \tag{2.39}$$

The tensor density $\mathcal{T}(\sigma)$ is called the Gotay-Marsden stress-energy-momentum (SEM) tensor density*.*

Proof. With the Cartan form for a second-derivative-order theory being uniquely defined (see Prop. 2.1.6), there is always the coordinate expression (2.8). Setting $k = 2$ and making use of intertwiners for second-derivative indices, this expression reads

$$\begin{aligned}\Theta_L &= L\Omega \\ &\quad + \frac{\partial L}{\partial u^A_j}(du^A - u^A_q dx^q)\wedge\omega_j - D_i\frac{\partial L}{\partial u^A_I}J^{ji}_I(du^A - u^A_q dx^q)\wedge\omega_j \\ &\quad + \frac{\partial L}{\partial u^A_I}J^{jp}_I(du^A_p - u^A_J I^J_{pq}dx^q)\wedge\omega_j.\end{aligned} \tag{2.40}$$

Note that, because the Cartan form is horizontal over the first jet bundle, there is no appearance of the forms du^A_{ij} and du^A_{ijk} in the coordinate expression for Θ_L. Thus, the pairing with ξ_{J^3E} for the calculation of $j(\sigma)$ makes use only of the coefficients ξ^m, ξ^A, and ξ^A_i. Performing the pairing and the subsequent pullback with respect to the prolongation of σ, the current is obtained as

$$\begin{aligned}j(\sigma) &= L\xi^j\omega_j \\ &\quad + L_{:A}{}^j(\xi^A - \sigma^A_{,q}\xi^q)\omega_j - D_i L_{:A}{}^I J^{ji}_I(\xi^A - \sigma^A_{,q}\xi^q)\omega_j \\ &\quad + L_{:A}{}^I J^{jp}_I(\xi^A_p - \sigma^A_{,J}I^J_{pq}\xi^q)\omega_j,\end{aligned} \tag{2.41}$$

where L and its derivatives are to be understood as being evaluated at prolongations of the section σ.

Using $\xi^A = C^A{}_B{}^n{}_m u^B\xi^m_{,n}$ and $\xi^A_p = D_p(C^A{}_B{}^n{}_m u^B\xi^m_{,n}) - u^A_m\xi^m_{,p}$ from Eq. (2.30) yields the current in its expanded form, which is

$$\begin{aligned}j(\sigma) &= [L\delta^n_m - L_{:A}{}^n\sigma^A_{,m} + D_i L_{:A}{}^I J^{in}_I\sigma^A_{,m} - L_{:A}{}^I J^{np}_I I^J_{mp}\sigma^A_{,J}]\xi^m\omega_n \\ &\quad + [C^A{}_B{}^n{}_m(L_{:A}{}^j\sigma^B - D_i L_{:A}{}^I J^{ij}_I\sigma^B + L_{:A}{}^I J^{jp}_I\sigma^B_{,p}) - L_{:A}{}^I J^{jn}_I\sigma^A_{,m}]\xi^m_{,n}\omega_j \\ &\quad + [L_{:A}{}^I J^{jp}_I C^A{}_B{}^n{}_m\sigma^B]\xi^m_{,np}\omega_j.\end{aligned} \tag{2.42}$$

The key to proving the identity (2.39) is to express the integrals with contributions from $\xi^m_{,n}$ and $\xi^m_{,np}$ as volume integrals using Gauss's theorem and to then repeatedly simplify the integrand by employing the diffeomorphism equivariance equations (2.29) and a variant of integration by parts, in analogy to the operations performed in Ref. [39] for first-order theories. This reference also explains how the region V has to be chosen in the following.

Applying this procedure to the terms containing two derivatives of ξ eliminates these terms, at the cost of new terms containing lower derivatives:

$$\begin{aligned}
&\int_\Sigma L_{:A}{}^I J_I^{jp} C^A{}_B{}^n{}_m \sigma^B \xi^m_{,np} \omega_j \\
&= \int_V D_j [L_{:A}{}^I J_I^{jp} C^A{}_B{}^n{}_m \sigma^B \xi^m_{,np}] \omega \\
&= \int_V \{ D_j [L_{:A}{}^I J_I^{jp} C^A{}_B{}^n{}_m \sigma^B] \xi^m_{,np} + \underbrace{L_{:A}{}^I J_I^{jp} C^A{}_B{}^n{}_m \sigma^B \xi^m_{,npj}}_{=0 \ (2.29\text{d})} \} \omega \\
&= \int_V \{ D_p (D_j [L_{:A}{}^I J_I^{jp} C^A{}_B{}^n{}_m \sigma^B] \xi^m_{,n}) - D_p D_j [L_{:A}{}^I J_I^{jp} C^A{}_B{}^n{}_m \sigma^B] \xi^m_{,n} \} \omega \\
&= \int_V \{ D_p (D_j [L_{:A}{}^I J_I^{jp} C^A{}_B{}^n{}_m \sigma^B] \xi^m_{,n}) - D_n (D_p D_j [L_{:A}{}^I J_I^{jp} C^A{}_B{}^n{}_m \sigma^B] \xi^m) \\
&\qquad + \underbrace{D_n D_p D_j [L_{:A}{}^I J_I^{jp} C^A{}_B{}^n{}_m \sigma^B]}_{=0 \ (2.29\text{d})} \xi^m \} \omega \\
&= \int_\Sigma \{ D_j [L_{:A}{}^I J_I^{jp} C^A{}_B{}^n{}_m \sigma^B] \xi^m_{,n} \omega_p - D_p D_j [L_{:A}{}^I J_I^{jp} C^A{}_B{}^n{}_m \sigma^B] \xi^m \omega_n \}
\end{aligned} \tag{2.43}$$

The original contributions from (2.42) together with the new contributions from (2.43) containing first derivatives of ξ combine to

$$\begin{aligned}
\int_\Sigma [\underbrace{L_{:A}{}^j C^A{}_B{}^n{}_m \sigma^B + 2 L_{:A}{}^I J_I^{jp} C^A{}_B{}^n{}_m \sigma^B_{,p} - L_{:A}{}^I J_I^{jn} \sigma^A_{,m}}_{:= S_m^{jn}}] \xi^m_{,n} \omega_j
&= \int_V D_j [S_m^{jn} \xi^m_{,n}] \omega \\
&= \int_V \{ D_j S_m^{jn} \xi^m_{,n} + \underbrace{S_m^{jn} \xi^m_{,jn}}_{=0 \ (2.29\text{c})} \} \omega \\
&= \int_V \{ D_n (D_j S_m^{jn} \xi^m) - \underbrace{D_n D_j S_m^{jn}}_{=0 \ (2.29\text{c})} \xi^m \} \omega \\
&= \int_\Sigma D_j S_m^{jn} \xi^m \omega_n .
\end{aligned} \tag{2.44}$$

Putting together (2.42)–(2.44) finally gives

$$\begin{aligned}\int_\Sigma i^*_\Sigma j(\sigma) = &\int_\Sigma \{L\delta^n_m - L_{:A}{}^n\sigma^A_{,m} - 2L_{:A}{}^I J^{np}_I I^J_{mp}\sigma^A_{,J} + (L_{:A}{}^I\sigma^B_{,I} \\ &+ L_{:A}{}^j\sigma^B_{,j} + D_j L_{:A}{}^j\sigma^B - D_j D_p L_{:A}{}^I J^{jp}_I \sigma^B) C^A{}_B{}^n{}_m\}\xi^m\omega_n \\ \overset{(2.29b)}{=}&\int_\Sigma \{-[L_{:A} - D_p L_{:A}{}^p + D_p D_q L_{:A}{}^I J^{pq}_I] C^A{}_B{}^n{}_m \sigma^B\}\xi^m\omega_n,\end{aligned} \tag{2.45}$$

from which the Gotay-Marsden stress-energy-momentum tensor density is easily read off as

$$\mathcal{T}^{\ n}_{m}(\sigma) = -\frac{\delta L}{\delta u^A} C^A{}_B{}^n{}_m \sigma^B, \tag{2.46}$$

where $\frac{\delta L}{\delta u^A} = L_{:A} - D_p L_{:A}{}^p + D_p D_q L_{:A}{}^I J^{pq}_I$ denotes the variational derivative. □

According to (2.46), the Gotay-Marsden stress-energy-momentum tensor density vanishes identically *on shell*, i.e. for sections solving the Euler-Lagrange equations—a recurring theme in the analysis of generally covariant theories. The first-order version of Thm. 2.3.3, proven in Ref. [39], settled a long-standing debate about SEM tensors densities by providing a definition which is based on Noether theory and naturally satisfies a *generalised Belinfante-Rosenfeld formula.*[7]

In addition, the Gotay-Marsden SEM tensor density lends itself for a concise formulation of Noether's second theorem, based on a previous result [39] for first-order theories.

Theorem 2.3.4 (second Noether theorem)**.** *Consider a second-derivative-order Lagrangian with local representative L on a tensor field bundle. If the Lagrangian is invariant with respect to diffeomorphisms, the corresponding Gotay-Marsden stress-energy-momentum tensor density $\mathcal{T}^n_m$ satisfies the differential relation*

$$D_n \mathcal{T}^n_m = \frac{\delta L}{\delta u^A} u^A{}_m. \tag{2.47}$$

Proof. Starting from the first expression for $\mathcal{T}^n_m$ obtained in Eq. (2.45), which is

$$\begin{aligned}\mathcal{T}^n_m = &L\delta^n_m - L_{:A}{}^n u^A{}_m - 2L_{:A}{}^I J^{np}_I I^J_{mp} u^A{}_J \\ &+ [L_{:A}{}^I u^B{}_I + L_{:A}{}^j u^B{}_j + D_j L_{:A}{}^j u^B - D_j D_p L_{:A}{}^I J^{jp}_I u^B] C^A{}_B{}^n{}_m,\end{aligned} \tag{2.48}$$

[7] For general relativity, the Belinfante-Rosenfeld formula [46, 47] relates the SEM tensor density obtained from Noether theory by considering translations to the Hilbert SEM tensor density, which is defined as the source density of the Einstein equations. [39] This comes with a seemingly *ad hoc* symmetrisation of the Noether SEM tensor density. The generalised Belinfante-Rosenfeld formula [39] relates the Gotay-Marsden SEM tensor density (2.46) to the Noether SEM tensor density without such choices, just by considering currents and spacetime diffeomorphisms.

the identity (2.47) follows via a direct computation of the divergence. Two terms in the intermediate result are reduced using the equivariance equations (2.29c) and (2.29d). □

With the Gotay-Marsden SEM tensor density replaced by its definition, Eq. (2.47) indeed reveals the differential relation

$$-D_n\left(\frac{\delta L}{\delta u^A}C^A{}_B{}^n{}_m u^B\right) = \frac{\delta L}{\delta u^A}u^A{}_m \tag{2.49}$$

for the Euler-Lagrange equations, which is exactly the statement of Noether's second theorem. The identity holds *off shell*, i.e. for any section of the tensor bundle regardless of whether it satisfies the Euler-Lagrange equations.

2.4 Axiom II: causal compatibility

This section follows very closely Sect. II.B of Ref. [1].

For the mathematical formulation of the second axiom, we utilise the close relation of the causal structure of field equations to the short-wavelength limit of the theory. [28, 26] First, we restrict to Lagrangians which are degenerate *in the sense that the Euler-Lagrange equations—although defined on J^4E—depend only on second derivatives and lower, i.e.*

$$\delta L = \pi^*_{2k,k}\delta\tilde{L} \tag{2.50}$$

for $k = 2$. This makes the theory immune from Ostrogradsky instabilities [48], which afflict theories of higher derivative orders. In addition, the formalism is being kept very close to Einstein gravity, whose Lagrangian is likewise degenerate—so we are still right on track in sticking closely to the established formalism and just inject different matter dynamics at the very beginning. Given the Euler-Lagrange equations $E_A = 0$ (henceforth called field equations*) of the degenerate second-order theory, we enter the limit of short wavelengths by considering the Wentzel-Kramers-Brillouin (WKB) ansatz for a local section σ of π*

$$\sigma^A(x^m) = \mathfrak{Re}\{\mathrm{e}^{\frac{\mathrm{i}S(x^m)}{\lambda}}[a^A(x^m) + \mathcal{O}(\lambda)]\}. \tag{2.51}$$

Evaluating the field equations at this ansatz and taking the limit $\lambda \to 0$ gives to leading order

$$\underbrace{\left(\frac{\partial E_A}{\partial u^B_I}\right) J^{ij}_I k_i k_j}_{T_{AB}(k)} a^B = 0, \tag{2.52}$$

which depends on the wavefront S only via the wave covector $k = -dS$. Eq. (2.52) is a linear equation for the amplitudes a^B with coefficients from the $r \times r$ matrix $T_{AB}(k)$, where r denotes the fibre dimension of the theory. This matrix, called the principal symbol *of the field equations, plays an important rôle in the short-wavelength limit: if the theory admits solutions with nontrivial amplitudes* a^B*, the principal symbol* $T_{AB}(k)$ *must necessarily be noninjective. By virtue of this condition, the principal symbol selects the physically admissible wave covectors in the WKB ansatz. As a square matrix is noninjective if and only if its determinant vanishes, admissible wave covectors can equivalently be characterised by a vanishing condition on the determinant of* $T_{AB}(k)$*.*

There is, however, a problem with this approach: in the presence of gauge symmetries, there are nontrivial solutions equivalent to the trivial solution $a^B = 0$*. These solutions will also be contained in the kernel of the principal symbol, rendering the naïve conditions on wave covectors formulated above meaningless. More specifically, assuming a gauge symmetry with* s*-dimensional gauge orbits, there are exactly* s *independent functions* $\chi^A_{(i)}(k)$ *which are equivalent to the trivial solution and span an* s*-dimensional subspace of the kernel of* $T_{AB}(k)$*. In order to allow for solutions which are* not *equivalent to the trivial solution, the kernel needs to be of dimension greater or equal than* $s+1$*.*

In the case of a diffeomorphism invariant theory, we have $s = 4$ *and it follows from the equivariance equation* (2.29d) *that*

$$0 = T_{AB}(k) C^B{}_C{}^n{}_i u^C k_n =: T_{AB}(k)\chi^B_{(i)}(k). \tag{2.53}$$

The condition that the kernel of the principal symbol be of dimension $s+1$ *or higher is equivalent to imposing that the order-*s *adjugate matrix*

$$Q^{A_1 \dots A_s B_1 \dots B_s}(k) := \frac{\partial^s \det(T_{AB}(k))}{\partial T_{A_1 B_1}(k) \dots T_{A_s B_s}(k)} \tag{2.54}$$

vanish.[8] *In this situation, where we have a square* $r \times r$ *matrix with* s *vectors* $(\chi_{(i)})_{i=1\dots s}$ *spanning a subspace of the kernel, we can use the general result [49, 26]*

$$Q^{A_1 \dots A_s B_1 \dots B_s}(k) = \epsilon^{\mu_1 \dots \mu_s} \epsilon^{\nu_1 \dots \nu_s} \left[\prod_{i=1}^{s} \chi^{A_i}_{(\mu_i)}\right] \left[\prod_{j=1}^{s} \chi^{B_j}_{(\nu_j)}\right] \mathcal{P}(k) \tag{2.55}$$

to arrive at the so-called principal polynomial $\mathcal{P}(k)$*.*

Definition 2.4.1 (principal polynomial [26])**.** *Consider a bundle* $E \xrightarrow{\pi} M$ *with fibre dimension* r *and a Lagrangian field theory on a jet bundle over* π *that results in Euler-Lagrange equations of second derivative order. Assume the* s *vectors* $(\chi^A_{(i)}(k))_{i=1\dots s}$ *to be*

[8] The vanishing of the order-s adjugate matrix is equivalent to the vanishing of all order-s subdeterminants, which are obtained by removing all possible combinations of s rows and and s columns from the matrix and calculating the determinant of each such reduced matrix. This is why the adjugate matrix is of dimension $\binom{r}{s} \times \binom{r}{s}$ for theories with fibre dimension r and s-dimensional gauge symmetries.

generators of the gauge transformations of the theory. In particular, the $\chi^A_{(i)}$ *span the left and right kernel of the principal symbol* $T_{AB}(k)$*. Choosing* s *rows and columns of* T *such that the order-*s *adjugate matrix entry* $Q^{A_1\dots A_s B_1\dots B_s}$ *does not vanish, we define the* principal polynomial *as the quotient*

$$\mathcal{P}(k) = \frac{Q^{A_1\dots A_s B_1\dots B_s}}{\epsilon^{\mu_1\dots\mu_s}\epsilon^{\nu_1\dots\nu_s}\left[\prod_{i=1}^{s}\chi^{A_i}_{(\mu_i)}\right]\left[\prod_{j=1}^{s}\chi^{B_j}_{(\nu_j)}\right]}. \tag{2.56}$$

The principal polynomial is a homogeneous polynomial of order $2r-4s$ *in the components* k_a *of the wave covector and has—as is clear from the derivation above—the important property that in order for an ansatz* (2.51) *to describe a nontrivial solution in the short-wavelength limit the wave covector* $k = -dS$ *must be a root of* $\mathcal{P}$*. Thus, the complete information about the propagation of waves in the infinite frequency limit is encoded in the principal symbol. This is an example for the more general result that the eligibility of a theory as a* physically relevant *theory hinges on properties of* $\mathcal{P}$*. More specifically, it has been shown that a theory can only be* predictive, interpretable, *and* quantizable *if the principal polynomial satisfies certain algebraic conditions, which further propagate to conditions on the underlying geometry. [28, 29]*

The principal polynomial is also closely related to the Cauchy problem of the field equations, as a Cauchy problem can only be well-posed within a region of M *if* $\mathcal{P}$ *restricts to a* hyperbolic[9] *polynomial in this region. Furthermore, given a theory with hyperbolic principal polynomial, admissible initial data hypersurfaces are characterised by the condition that the surface normal be hyperbolic with respect to* $\mathcal{P}$*. [50, 51] Predictivity is the* raison d'être *for physical theories, which is why we will restrict our attention to tensor field theories with hyperbolic principal polynomials.*

Two geometric objects are important for the formulation of the axiom of causal compatibility: the vanishing set $V_p \in T^*_pM$ *of* $\mathcal{P}$ *and the set* $C_p \in T^*_pM$ *of all hyperbolic covectors with respect to* $\mathcal{P}$*. Both sets are defined at each point and thus form distributions* V *and* C *on* M*. The vanishing set* V_p *consists of all admissible wave covectors in the infinite frequency limit, restricting the propagation directions of fields in spacetime. The set* $C_p \in T^*_pM$*, on the other hand, contains the information about possible choices of initial data hypersurfaces. It constitutes a convex cone [52] and is commonly called the* hyperbolicity cone *[28, 29].*

Let us now consider the situation where a theory for some matter field coupled to geometry has been prescribed, say on a bundle $E_{grav} \oplus_M J^1E_{mat}$*, and the principal polynomial* $\mathcal{P}_{mat}$ *is hyperbolic. Both distributions* V_{mat} *and* C_{mat} *exist and they contain*

[9] A homogeneous polynomial $\mathcal{P}$ of degree d is hyperbolic if there exists a covector h such that $\mathcal{P}(h) \neq 0$ and any shifted covector $h + \lambda w$ intersects the vanishing set of $\mathcal{P}$ exactly d times. Such a covector h is said to be hyperbolic with respect to $\mathcal{P}$.

all relevant information about the causality of the matter theory. The objective of covariant constructive gravity is to close the matter theory by providing a dynamical theory of the geometry, defined on the bundle J^2E_{grav}*. As a result, we obtain distributions* V_{grav} *and* C_{grav} *of vanishing sets and hyperbolicity cones for the gravitational theory. The principle of causal compatibility between matter theory and gravitational theory now mandates following relation between both pairs of distributions.*

Definition 2.4.2 (causally compatible gravitational closure)**.** *Consider two bundles* $E_{grav} \stackrel{\pi_{grav}}{\longrightarrow} M$ *and* $E_{mat} \stackrel{\pi_{mat}}{\longrightarrow} M$ *and a Lagrangian matter field theory on* $E_{grav} \oplus_M J^1E_{mat}$ *whose Euler-Lagrange equations are linear in the matter field. The corresponding principal polynomial* $\mathcal{P}_{mat}$ *shall be hyperbolic and thus defines the vanishing set distribution* V_{mat} *and the hyperbolicity cone distribution* C_{mat}*. We say that a gravitational Lagrangian field theory on* J^2E_{grav} *with Euler-Lagrange equations of second derivative order, a principal polynomial* $\mathcal{P}_{grav}$*, and distributions* V_{grav}, C_{grav} *is* causally compatible *with the matter field theory if*

$$C_{grav} = C_{mat} \quad \text{and} \quad V_{mat} \subseteq V_{grav}. \tag{2.57}$$

The first condition immediately implies that $\mathcal{P}_{grav}$ *is hyperbolic as well. Furthermore, it ensures that both theories share their initial value surfaces and allow for a unified observer definition [28, 29]. As recent measurements showed with a high degree of certainty that gravitational waves propagate at the speed of light [53], we include the second condition for the distribution of vanishing sets into the definition of causal compatibility. It requires that wave covectors of the matter theory are admissible wave covectors of the gravitational theory, but leaves open the possibility for different modes of propagation.*

Before closing this section about the second axiom, a remark about its practical implications is in order. As we will see during the perturbative implementation of the covariant constructive gravity programme, the requirement of diffeomorphism invariance alone already restricts the principal polynomial of the gravitational field equations quite a lot, such that up to the third iteration of the perturbative construction procedure we will not need to enforce Eq. (2.57) explicitly—at least for our chosen example. For the nonperturbative construction of general relativity, the condition will not be needed at all. This hints at the promising possibility that the second axiom may actually be weakened by the extent to which it may already be implied by the first axiom.

3 The construction algorithm

Having introduced the axioms of covariant constructive gravity and cast them in precise mathematical language, we consolidate the results and state the algorithm for the construction of modified gravity Lagrangians from novel matter theories. After a discussion about practical implications in general, we proceed with sketching the application to a few examples.

3.1 General formulation

The results obtained so far allow us to formulate a comprehensive algorithm for the construction of gravitational Lagrangians, which has already been presented in Ref. [1]. These Lagrangians are the most general conceivable Lagrangians within our formalism that satisfy both axioms of covariant constructive gravity. All that has to be provided is a matter theory that couples to geometry and the algorithm will yield all candidates for gravitational theories that determine the so far undetermined dynamics of the gravitational field, finally giving the theory predictive power. In this sense, the task of searching for modified gravitational theories boils down to the solution of PDE systems to ensure general covariance and of algebraic equations to match the causalities.

Algorithm 1: Gravitational closure using covariant constructive gravity [1]

Data: Geometry bundle $E_{\text{grav}} \xrightarrow{\pi_{\text{grav}}} M$, matter bundle $E_{\text{mat}} \xrightarrow{\pi_{\text{mat}}} M$, Lagrangian matter field theory on $E_{\text{grav}} \oplus_M J^1 E_{\text{mat}}$ with linear field equations

Result: Most general diffeomorphism invariant and causally compatible gravitational Lagrangian field theory on $J^2 E_{\text{grav}}$

compute the Gotay-Marsden coefficients (2.27) for E_{grav}
set up the equivariance equations (2.29a)–(2.29d)
solve the equivariance equations for the gravitational Lagrangian density L_{grav}
compute the Euler-Lagrange equations (2.5) corresponding to L_{grav}
restrict the gravitational theory to second-derivative-order field equations
calculate the principal polynomials (2.56) $\mathcal{P}_{\text{grav}}$ and $\mathcal{P}_{\text{mat}}$
solve the causal compatibility conditions $C_{\text{grav}} = C_{\text{mat}}$ and $V_{\text{grav}} \subseteq V_{\text{mat}}$

Let us comment on the algorithm step by step: the first step, calculating the Gotay-Marsden coefficients, is trivial. The coefficients follow from the prescribed or inherited action of diffeomorphisms on the geometry bundle. For purely covariant or contravariant tensor bundles, Prop. 2.2.7 already gives the final expression. These coefficients have to be inserted into Eqns. (2.29a)–(2.29d) in order to execute step 2. As a result, we obtain a system of linear, first-order partial differential equations for the Lagrangian density L with coefficients that are linear in the independent variables. More precisely, the PDEs are of the form

$$0 = A_i^j x^i u_{,j} + Bu, \tag{3.1}$$

u is the dependent variable, x^i are the independent variables and the coefficients A_i^j and B are constants. Conceptually, much about the solutions of such PDEs is known [42], although it is in most cases practically infeasible to solve the system, due to its sheer size and the number of independent variables. However, as we will see in Chap. 6, the system admits a property called *involutivity*, from which we can infer strong results about solutions and derive a perturbative solution strategy.

For a known solution, it is only a matter of applying Eq. (2.5) to the Lagrangian in order to compute the Euler-Lagrange equations for step 4. Restricting to second-derivative-order field equations, as required by step 5, could be done now by imposing that higher-derivative-order terms vanish. In practice, however, such restrictions will be placed at an earlier stage, in order to rule out higher orders from the beginning. A similar pattern emerges for steps 6 and 7: placing restrictions on the computed entities is possible, but may be hard to enforce after the fact. So it is worth keeping this requirement in mind early on.

3.2 Example: Einstein gravity

As already outlined in the introduction, Einstein gravity can be thought of as the gravitational closure of Maxwell electrodynamics in four dimensions. This theory provides dynamics for sections A in the bundle T^*M of one-forms, parameterised with sections g in the bundle $S(T_0^2 M)$ of contravariant[1] symmetric tensors of rank two. A is commonly known as the *electromagnetic potential*, g as the *metric tensor*. The dynamics of the electromagnetic field is given by the Lagrangian density

$$L_{\text{Maxwell}} = \sqrt{-\det g} g^{ac} g^{bd} F_{ab} F_{cd}, \tag{3.2}$$

[1]Many descriptions regard the covariant inverse metric as fundamental. In this case, the contravariant metric tensor appearing in the Lagrangian density (3.2) is the inverse of the metric field. Both descriptions will yield slightly differing intermediate results during the construction procedure, but are fundamentally equivalent.

where the electromagnetic potential enters via the field strength tensor $F = dA$ and we write "$\det g$" for the determinant of the *covariant* metric tensor, which is the inverse of the contravariant metric tensor chosen here as fundamental field.

We now collect the ingredients for the construction algorithm. The fibre dimension of the bundle $S(T^2_0M)$ is 10, such that a suitable pair of intertwiners (I, J) between this bundle and the unrestricted tensor bundle T^2M is given by

$$I_1^{ab} = \begin{pmatrix} 1&0&0&0\\0&0&0&0\\0&0&0&0\\0&0&0&0 \end{pmatrix}^{ab}, I_2^{ab} = \begin{pmatrix} 0&1&0&0\\1&0&0&0\\0&0&0&0\\0&0&0&0 \end{pmatrix}^{ab}, I_3^{ab} = \begin{pmatrix} 0&0&1&0\\0&0&0&0\\1&0&0&0\\0&0&0&0 \end{pmatrix}^{ab} I_4^{ab} = \begin{pmatrix} 0&0&0&1\\0&0&0&0\\0&0&0&0\\1&0&0&0 \end{pmatrix}^{ab},$$
$$I_5^{ab} = \begin{pmatrix} 0&0&0&0\\0&1&0&0\\0&0&0&0\\0&0&0&0 \end{pmatrix}^{ab}, I_6^{ab} = \begin{pmatrix} 0&0&0&0\\0&0&1&0\\0&1&0&0\\0&0&0&0 \end{pmatrix}^{ab}, I_7^{ab} = \begin{pmatrix} 0&0&0&0\\0&0&0&1\\0&0&0&0\\0&1&0&0 \end{pmatrix}^{ab}, I_8^{ab} = \begin{pmatrix} 0&0&0&0\\0&0&0&0\\0&0&1&0\\0&0&0&0 \end{pmatrix}^{ab},$$
$$I_9^{ab} = \begin{pmatrix} 0&0&0&0\\0&0&0&0\\0&0&0&1\\0&0&1&0 \end{pmatrix}^{ab}, I_{10}^{ab} = \begin{pmatrix} 0&0&0&0\\0&0&0&0\\0&0&0&0\\0&0&0&1 \end{pmatrix}^{ab}, \tag{3.3}$$

and

$$J^1_{ab} = \begin{pmatrix} 1&0&0&0\\0&0&0&0\\0&0&0&0\\0&0&0&0 \end{pmatrix}_{ab}, J^2_{ab} = \begin{pmatrix} 0&\frac{1}{2}&0&0\\\frac{1}{2}&0&0&0\\0&0&0&0\\0&0&0&0 \end{pmatrix}_{ab}, J^3_{ab} = \begin{pmatrix} 0&0&\frac{1}{2}&0\\0&0&0&0\\\frac{1}{2}&0&0&0\\0&0&0&0 \end{pmatrix}_{ab} J^4_{ab} = \begin{pmatrix} 0&0&0&\frac{1}{2}\\0&0&0&0\\0&0&0&0\\\frac{1}{2}&0&0&0 \end{pmatrix}_{ab},$$
$$J^5_{ab} = \begin{pmatrix} 0&0&0&0\\0&1&0&0\\0&0&0&0\\0&0&0&0 \end{pmatrix}_{ab}, J^6_{ab} = \begin{pmatrix} 0&0&0&0\\0&0&\frac{1}{2}&0\\0&\frac{1}{2}&0&0\\0&0&0&0 \end{pmatrix}_{ab}, J^7_{ab} = \begin{pmatrix} 0&0&0&0\\0&0&0&\frac{1}{2}\\0&0&0&0\\0&\frac{1}{2}&0&0 \end{pmatrix}_{ab}, J^8_{ab} = \begin{pmatrix} 0&0&0&0\\0&0&0&0\\0&0&1&0\\0&0&0&0 \end{pmatrix}_{ab},$$
$$J^9_{ab} = \begin{pmatrix} 0&0&0&0\\0&0&0&0\\0&0&0&\frac{1}{2}\\0&0&\frac{1}{2}&0 \end{pmatrix}_{ab}, J^{10}_{ab} = \begin{pmatrix} 0&0&0&0\\0&0&0&0\\0&0&0&0\\0&0&0&1 \end{pmatrix}_{ab}. \tag{3.4}$$

The intertwiner I distributes the ten degrees of freedom for a symmetric tensor of dimension 4 across the components of a generic rank-2 tensor

$$I^{ab}(c_A u^A) = \begin{pmatrix} c_1 & c_2 & c_3 & c_4 \\ c_2 & c_5 & c_6 & c_7 \\ c_3 & c_6 & c_8 & c_9 \\ c_4 & c_7 & c_9 & c_{10} \end{pmatrix}^{ab}, \tag{3.5}$$

while J projects such symmetrically distributed components back to the ten degrees of freedom, discarding possible antisymmetric contributions. Note that I and J could also be chosen such that the matrix representations coincide, using factors of $\frac{1}{\sqrt{2}}$ for off-diagonal entries in both intertwiners. This has the apparent advantage that I and J do not need to be distinguished from each other. However, a disadvantage of using them interchangeably is that this would obscure the different rôles that I and J play, especially if they are used not only in setting up the equivariance equations, but also for manipulating them. The irrational coefficients like $\frac{1}{\sqrt{2}}$ would also further complicate the computer-aided treatment introduced in Chap. 5, which for purely rational intertwiners yields purely rational results.

Prop. 2.2.7 yields the Gotay-Marsden coefficients from (I, J) as

$$C^A{}_B{}^n{}_m = 2 I^{pn}_B J^A_{pm}. \tag{3.6}$$

Contracting these coefficients with I and J leads to the spacetime expression

$$C^{ab}{}_{cd}{}^n{}_m = 2\delta^{(a}_m \delta^{b)}_{(c} \delta^n_{d)}, \tag{3.7}$$

which serves as a good sanity check: contracting again with a metric g and the derivatives of a vector field ξ results in the well-known transformation of g w.r.t. infinitesimal diffeomorphism generated by ξ,

$$C^{ab}{}_{cd}{}^n{}_m g^{cd} \xi^m_n = 2 g^{n(a} \xi^{b)}_{,n}. \tag{3.8}$$

The second ingredient is the principal polynomial of electrodynamics, which reduces to[2] the homogeneous quadratic polynomial [28]

$$\mathcal{P}_{\text{Maxwell}}(k) = g(k, k). \tag{3.9}$$

From this result follows the standard notion of causality in relativity: light rays with codirection k are constrained to the vanishing set V and, thus, satisfy $g(k, k) = 0$. The wave covectors related to massive observers lie within the hyperbolicity cone C, which

[2]Computing the principal polynomial may lead to a result of higher degree than (3.9). For the second axiom of covariant constructive gravity, however, only the reduced form without repeating factors is of relevance—because the causal structure is already determined by the reduced polynomial [28].

restricts them to $g(k,k) > 0$ (adopting the *mostly minus* convention $(+---)$ for the signature of the metric). For more details, we refer the reader to the theory developed in Refs. [28, 29, 23, 26] and the corresponding examples.

Before proceeding, let us emphasise that there are only two things needed from the matter theory, which is Maxwell electrodynamics in this case:

1. the Gotay-Marsden coefficients ${C^A}_B{}^n{}_m = 2I_B^{pn} J^A_{pm}$ and
2. the principal polynomial $\mathscr{P}_{\text{Maxwell}}(k) = g(k,k)$.

The equivariance equations (2.29a)–(2.29d) for the metric gravitational Lagrangian are a system of 140 PDEs for one variable dependent on 154 independent variables[3]. Because the system admits the aforementioned property called involutivity, which will play a major rôle in Chap. 4 and therefore will be considered in more detail there, we can make use of a very strong result about the solutions of this system [42]: there are $154-140 = 14$ functions ψ_α of the independent variables, such that any solution of the homogeneous system, denoted here as

$$0 = A^{Ij} u_{,j}, \tag{3.10}$$

is of the form $f(\psi_1, \dots, \psi_{14})$ for any suitably differentiable function f. Any particular solution ω of the inhomogeneous system

$$0 = A^{Ij} u_{,j} + B^I \tag{3.11}$$

yields, by virtue of the product rule, the general form of a solution,

$$u = \omega \cdot f(\psi_1, \dots, \psi_{14}). \tag{3.12}$$

Now, the dynamics of general relativity as derived by Einstein are given by the manifestly diffeomorphism equivariant Einstein-Hilbert Lagrangian density

$$L_{\text{Einstein-Hilbert}} = \frac{1}{2\kappa}\sqrt{-\det g}(R - 2\Lambda), \tag{3.13}$$

from which we readily recognise two solutions,

$$\omega = \sqrt{-\det g} \quad \text{and} \quad \psi_1 = R. \tag{3.14}$$

The constants κ and Λ are known as gravitational constant and cosmological constant, respectively, and R is the Ricci scalar curvature. Together with the homogeneous solution $\psi_1 = R$, the remaining 13 homogeneous solutions $\psi_2, \dots, \psi_{14}$ are known in the literature as the fourteen *curvature invariants* [54, 55].

[3] The dimension of the second jet bundle over the metric bundle is $4+10+4\times 10+\binom{4+2-1}{2}\times 10 = 154$.

While the system of equivariance equations alone admits a multitude of solutions (3.12), it has been shown by Lovelock [14, 35, 36] that only Einstein's general relativity (3.13) admits second-derivative-order field equations. Step 5 of the construction algorithm therefore restricts the gravitational theory closing Maxwell electrodynamics to general relativity with its two undetermined constants *exactly*. The causality conditions do not have to be implemented anymore, since they follow trivially—the causal structures of Maxwell electrodynamics and Einstein gravity coincide.

There is another interesting result that follows from the equivariance equations. Restricting to the zeroth jet bundle and switching from abstract indices to indices inherited from the tangent bundle, we retrieve the equivariance equations for a density $\omega(x^i, g^{ab})$ as

$$0 = \omega_{,m} \quad \text{and} \quad 0 = 2\frac{\partial \omega}{\partial g^{am}} g^{an} + \delta^n_m \omega. \tag{3.15}$$

If we solve the first equation by restricting further to $\omega = \omega(g^{ab})$ and manipulate the second equation by contraction with the covariant metric, we obtain

$$\frac{\partial \omega}{\partial g^{ab}} = -\frac{1}{2} g_{ab} \omega. \tag{3.16}$$

This equation is obviously symmetric in the indices and therefore boils down to a system of 10 PDEs for the function ω of the 10 independent variables g^{ab}. As the system is completely determined, the known solution $\omega = \sqrt{-\det g}$, which can be easily verified by straightforward differentiation, is the *unique* solution. Using our framework, we thus have provided a derivation of the well-known fact that the only scalar densities that can be constructed from the metric tensor are powers of the metric determinant.

The same result holds for the equivariance equations restricted to the first jet bundle, which are

$$0 = L_{,m}, \tag{3.17a}$$
$$0 = 2L_{:am} g^{an} + 2L_{:am}{}^{p} g^{an}_{,p} - L_{:ab}{}^{n} g^{ab}_{,m}, \tag{3.17b}$$
$$0 = L_{:am}{}^{(p} g^{n)a}. \tag{3.17c}$$

Eq. (3.17c) is a system of 40 individual equations for the 40 derivatives of L with respect to the first derivatives of the metric tensor. The rank of this subsystem is full[4], which completely eliminates any possible dependence of L on the first derivatives of g. The remaining system is equivalent to the zeroth-order system (3.15) with the unique solution $L = \sqrt{-\det g}$, demonstrating with a very quick derivation that there is no nontrivial

[4]If in doubt, such statements concerning our linear PDE systems can be verified without much computational effort by evaluation at randomly chosen points in the jet bundle. At worst, the rank at such points will be *less* than at a generic point.

diffeomorphism equivariant Lagrangian density of first derivative order for the metric tensor.

Before proceeding with the next example, it should be emphasised that the insights gained about metric gravitational theory compatible with Maxwell electrodynamics are not new as fare as the *results* are concerned. Rather, we have seen how the developed framework readily reproduces the known results without much effort and yet again confirms earlier derivations.

3.3 Example: area metric gravity

As a first example for a modified theory of gravity that follows from covariant constructive gravity, we consider *area metric gravity.* The starting point is a generalisation of Maxwell electrodynamics.

Definition 3.3.1 (generalized linear electrodynamics)**.** *Let M be a four-dimensional spacetime manifold. The bundle E_{area}, constructed as a subbundle of $T^4_0 M$ by imposing the linear conditions*

$$G^{abcd} = G^{cdab} = -G^{bacd} \tag{3.18}$$

on the tensor components, is called the area metric bundle. Given a scalar density ω of weight 1, sections G of this bundle serve as coefficients for the Lagrangian density of generalised linear electrodynamics (GLED),

$$L_{GLED} = \omega_G G^{abcd} F_{ab} F_{cd}. \tag{3.19}$$

It is easy to see that GLED is a generalisation of Maxwell electrodynamics by setting[5]

$$G^{abcd} = g^{ac}g^{bd} - g^{ad}g^{bc} + \frac{1}{\sqrt{-\det g}}\epsilon^{abcd} \quad \text{and} \quad \omega_G = \left(\frac{1}{24}\epsilon_{abcd}G^{abcd}\right)^{-1} \tag{3.20}$$

in the GLED Lagrangian density (3.19), which reproduces the Maxwell Lagrangian density (3.2). Not restricting the area metric field to the specific form (3.20) but leaving all 21 independent components unconstrained yields, of course, a more general theory.

GLED as generalisation of Maxwell electrodynamics is the result of an axiomatic approach to classical electrodynamics called *premetric electrodynamics* [56, 57]. This approach makes a few assumptions like conversation of charge and magnetic flux, the existence of

[5]Note that ω_G as defined in Eq. (3.20) is a valid scalar density of weight 1 not only for this special choice of G, but also for general area metric fields. Without loss of generality, we will keep making use of this density—any other density is obtained by multiplication of ω_G with a scalar.

a Lorentz force law, and a superposition principle. As a consequence, the indeterminates of such a theory are reduced to the so-called *constitutive tensor* χ, which is already known from electrodynamics in media, but now also determines the behaviour of the electromagnetic field in *in vacuo*. In our language, χ is the area metric tensor G.

The causality of GLED crucially depends on the area metric via the principle polynomial [58]

$$\mathcal{P}_{\mathrm{GLED}}(k) = -\frac{1}{24}\omega_G^2 \epsilon_{mnpq}\epsilon_{rstu} G^{mnra} G^{bpsc} G^{dqtu} k_a k_b k_c k_d, \tag{3.21}$$

which is generally irreducible and of rank 4. Consequently, the null surfaces are no longer metric light cones, but more complex quartic surfaces. For example, $\mathcal{P}_{\mathrm{GLED}}$ could factor into the product of two metrics, in which case the vanishing set at a point would be the union of two metric light cones with different opening angles. In this example, the phase velocity of a wave depends on the light cone in which the wave covector lies. The two options can be seen as new polarisation degree of freedom, such that the speed of light is determined by the polarisation—an effect commonly known as *birefringence*. While in classical electrodynamics this is only possible in nonlinear media, GLED allows for birefringence *in vacuo*.

Just like in Maxwell electrodynamics, where only metrics of Lorentzian signature meet the requirement of a hyperbolic principal polynomial, GLED only satisfies certain conditions regarding its causality—like hyperbolicity of the principal polynomial—if the area metric belongs to certain algebraic *subclasses*. [28, 29] The constructions that follow respect this requirement. In fact, we will work in a perturbative setting where the area metric to zeroth order belongs to an appropriate subclass. Perturbations must be such that the subclass does not change—akin to signature change in general relativity, which is also mostly excluded.

Much of the remainder of this thesis is dedicated to the application of the construction algorithm to GLED, which should yield the gravitational theory completing general linear electrodynamics to a predictive theory of matter *and* gravity. This new theory shall bear the name *area metric gravity*.

We again start with the definition of suitable intertwiners. It is often useful to interpret the components G^{abcd} of an area metric, which consist of two antisymmetric pairs and is symmetric in these pairs, as symmetric 6 by 6 matrix

$$G^{[ab][cd]} = \begin{pmatrix} G^{0101} & G^{0102} & G^{0103} & G^{0112} & G^{0113} & G^{0123} \\ \cdot & G^{0202} & G^{0203} & G^{0212} & G^{0213} & G^{0223} \\ \cdot & \cdot & G^{0303} & G^{0312} & G^{0313} & G^{0323} \\ \cdot & \cdot & \cdot & G^{1212} & G^{1213} & G^{1223} \\ \cdot & \cdot & \cdot & \cdot & G^{1313} & G^{1323} \\ \cdot & \cdot & \cdot & \cdot & \cdot & G^{2323} \end{pmatrix}^{[ab][cd]} . \tag{3.22}$$

Intertwiners can then be chosen such that I distributes abstract components $G^1, \dots, G^{21}$ over such a matrix, i.e.

$$I_1^{[ab][cd]} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}^{[ab][cd]}, \quad I_2^{[ab][cd]} = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}^{[ab][cd]}, \tag{3.23}$$

and so on. The surjections J project back to abstract indices, where the multiplicities are either 4 for components like G^{0123} or 8 for components like G^{0101}:

$$J^1_{[ab][cd]} = \begin{pmatrix} \frac{1}{4} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}_{[ab][cd]}, \quad J^2_{[ab][cd]} = \begin{pmatrix} 0 & \frac{1}{8} & 0 & 0 & 0 & 0 \\ \frac{1}{8} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}_{[ab][cd]}, \tag{3.24}$$

et cetera.

As usual, the Gotay-Marsden coefficients follow from Prop. 2.2.7. Since the fibre dimension of $T^4_0 M$ is 4 and the tensors are purely contravariant, the coefficients are

$$C^A{}_B{}^n{}_m = 4 I_B^{pqrn} J^A_{pqrm}, \tag{3.25}$$

or, using the spacetime representation,

$$C^{abcd}{}_{efgh}{}^n{}_m = 4 \delta^{[a|}_m \delta^n_{[e} \delta^{|b]}_{f]} \delta^{[c}_{[g} \delta^{d]}_{h]} \Big|_{\substack{[ab]\leftrightarrow[cd] \\ [ef]\leftrightarrow[gh]}}, \tag{3.26}$$

where $e_{XY}\big|_{X\leftrightarrow Y} = \frac{1}{2} e_{XY} + \frac{1}{2} e_{YX}$ denotes idempotent symmetrisation of the expression e in X and Y.

Having computed the intertwiners and Gotay-Marsden coefficients, the equivariance equations (2.29a)–(2.29d) are ready to be set up. Since the second area metric jet bundle is of dimension

$$\dim(J^2 E_{\text{area}}) = 4 + 21 + 4 \times 21 + \binom{4+2-1}{2} \times 21 = 319, \tag{3.27}$$

the system of equivariance equations consists of 140 linear, first-order PDE for one function of 319 independent variables. The claim of covariant constructive gravity is that solutions to this system are candidates for gravitational Lagrangians. Unfortunately, it is computationally infeasible to present such a solution[6] and, unlike for metric gravity, there are no known curvature invariants for us to rely on.

Therefore, we will resort to perturbation theory in order to derive results for weak gravitational fields in Chap. 6 and also shortly explore the possibility of directly solving the cosmological sector of area metric gravity in Chap. 7.

3.4 Example: bimetric gravity

A lot of work has already been done in order to answer the question: *What would gravity look like if there were two metrics instead of one?* From the perspective of covariant constructive gravity (and gravitational closure in general), this question is meaningless without reference to a bimetric matter action. The question should rather be: *How can matter theories that couple to two different metrics be completed by a bimetric gravitational theory?*

Let us consider two examples for bimetric matter theories. The first theory prescribes the dynamics for two scalar fields, each field coupling to its own metric.

Definition 3.4.1 (bimetric Klein-Gordon theory)**.** *Let M be a four-dimensional spacetime manifold. The bundle $E_{bimetric} = S(T^2_0 M) \oplus S(T^2_0 M)$ constructed as the direct sum of two metric bundles is called the bimetric bundle. Sections (g, h) of this bundle serve as coefficients for the Lagrangian density of the* bimetric Klein-Gordon theory

$$L_{2KG} = \sqrt{-\det g}\, g^{ab}\phi_{,a}\phi_{,b} - m_\phi^2\phi^2 + \sqrt{-\det h}\, h^{ab}\psi_{,a}\psi_{,b} - m_\psi^2\psi^2, \tag{3.28}$$

where ϕ and ψ are smooth functions $\phi, \psi\colon M \to \mathbb{R}$ called scalar fields *with nonnegative* masses *m_ϕ and m_ψ.*

As second example, we use a generalisation of the Proca theory, a theory for a *massive* electromagnetic potential.

Definition 3.4.2 (bimetric Proca theory [32])**.** *Consider again the bundle $E_{bimetric}$. Sections (g, h) of this bundle together with a scalar density $\omega_{(g,h)}$ constructed from g and*

[6] Strictly speaking, two solutions are known: the scalar density ω_G defined in Eq. (3.20) and a different choice for ω_G which is computed from the determinant of the 6×6 matrix (3.22), which is also a valid scalar density. However, these solutions do not depend on derivatives of the area metric and as such would not yield *dynamic* field equations if used as Lagrangian density.

h serve as coefficients for the Lagrangian density of the bimetric Proca theory

$$L_{bi\text{-}Proca} = \omega_{(g,h)} \left(-g^{ac} g^{bd} F_{ab} F_{cd} + m^2 h^{ab} A_a A_b\right) \tag{3.29}$$

for an electromagnetic potential one-form A with field strength $F = dA$.

Both theories couple matter fields to geometry defined on E_{bimetric}. Consequently, the first steps in executing the construction algorithm are identical: define suitable intertwiners, calculate Gotay-Marsden coefficients, set up and solve the equivariance equations. A possible choice of intertwiners is to just reuse the intertwiners (3.3) and (3.4) defined for the metric bundle. Representing elements of the unrestricted bundle $T^2_0 M \oplus T^2_0 M$ as two matrices, $I^{ab}_1, \dots, I^{ab}_{10}$ distribute the 10 degrees of freedom for the first metric over the first matrix, while $I^{ab}_{11}, \dots, I^{ab}_{20}$ distribute the 10 degrees of freedom for the second metric over the second matrix. The intertwiner J is defined equivalently.

Considering how one metric transforms with respect to diffeomorphisms,

$$g^A \mapsto g^A + C^A{}_B{}^n{}_m \xi^m_{,n}, \tag{3.30}$$

we can reuse the Gotay-Marsden coefficients (3.6) for a single metric and obtain the transformation behaviour of two metrics as

$$\begin{pmatrix} g^A \\ h^B \end{pmatrix} \mapsto \begin{pmatrix} g^A \\ h^B \end{pmatrix} + \begin{pmatrix} C^A{}_C{}^n{}_m \xi^m_{,n} & 0 \\ 0 & C^B{}_D{}^n{}_m \xi^m_{,n} \end{pmatrix} \begin{pmatrix} g^C \\ h^D \end{pmatrix}. \tag{3.31}$$

The matrix introduced in this equation constitutes the Gotay-Marsden coefficients for the bimetric bundle. A lighter notation is to just write G^A for the bimetric field, where indices A range from 1 to 20 and split into two ranges, denoted by $\bar{A}$ (from 1 to 10) and $\bar{\bar{A}}$ (from 11 to 21), respectively. The original metrics g and h are included in G as $G^{\bar{A}}$ and $G^{\bar{\bar{A}}}$. Using this notation, the matrix in Eq. (3.31) is a block matrix representation of the bimetric Gotay-Marsden coefficients $C^A{}_B{}^n{}_m$: the coefficients are zero if A and B come from different ranges, while for the same ranges, they amount to the metric coefficients.

With the Gotay-Marsden coefficients at hand, the equivariance equations (2.29a)–(2.29d) follow as usual. This time, the bimetric bundle is of dimension

$$\dim(J^2 E_{\text{bimetric}}) = 4 + 20 + 4 \times 20 + \binom{4+2-1}{2} \times 20 = 304, \tag{3.32}$$

making the system a PDE system with 140 equations for one function dependent on 304 independent variables. The same remarks as for the construction of area metric gravity apply: it is notoriously hard to solve such a system *exactly*, but the method offers a lot of potential for perturbative or symmetry-reduced solutions. In the context of canonical gravitational closure, the former has already been pursued successfully, at least to second order in the perturbation expansion. [32, 33]

Some solutions are, of course, already known: the metric determinants are diffeomorphism invariant densities and the fourteen curvature invariants for each metric are diffeomorphism invariant scalars, i.e. solutions to the homogeneous system. This gives generic solutions of the form

$$\begin{aligned} &\sqrt{-\det g}\cdot f(\psi_1^{(g)},\dots,\psi_{14}^{(g)},\psi_1^{(h)},\dots,\psi_{14}^{(h)}) \\ \text{or}\quad &\sqrt{-\det h}\cdot f(\psi_1^{(g)},\dots,\psi_{14}^{(g)},\psi_1^{(h)},\dots,\psi_{14}^{(h)}). \end{aligned} \tag{3.33}$$

More scalars come easily to mind, like the contraction $g^{ab}h_{ab}$ (using the inverse h_{ab} of h^{ab}) and the ratio $\frac{\sqrt{-\det g}}{\sqrt{-\det h}}$. Adding these to the 28 curvature invariants, a more generic solution would be

$$\sqrt{-\det h}\cdot f(\psi_1^{(g)},\dots,\psi_{14}^{(g)},\psi_1^{(h)},\dots,\psi_{14}^{(h)},g^{ab}h_{ab},\frac{\sqrt{-\det g}}{\sqrt{-\det h}}). \tag{3.34}$$

From the strong results about such system, which will be proven in the next chapter, we know that this premature analysis is by no means exhaustive—the number of functionally independent scalars that can be constructed from a bimetric tensor and its derivative up to second order must be $304-140=164$.

By the second axiom of covariant constructive gravity, the space of admissible Lagrangians will be smaller than the solution space of the PDE system we just discussed. The input we need from the matter theory is the principal polynomial. Quite surprisingly, the polynomials of the bimetric Klein-Gordon theory and the bimetric Proca theory coincide, given by the expression

$$\mathcal{P}_{\text{bimetric}}(k)=g(k,k)h(k,k). \tag{3.35}$$

This is an intuitive result for the bimetric Klein-Gordon field, where the field equations for both scalar fields do not couple. For the bimetric Proca theory, however, a naïve inspection of the field equations seems to suggest that the principal polynomial is just given by the first metric which provides the coefficients for the kinetic term. Only after the field equations have been brought into involutive form, new equations emerge which ultimately yield the principal polynomial (3.35). [32]

As a consequence of this coincidence, the gravitational theories that are eligible as completions for both discussed bimetric matter theories are the same. This also restricts the causally relevant sectors of both theories to the sector where g and h are Lorentzian metrics with overlapping hyperbolicity cones—only then the product of both metrics is a hyperbolic polynomial.

4 Perturbative construction of gravitational theories

Following the presentation of the axioms of covariant constructive gravity and the construction algorithm, we now develop a perturbative approach for the implementation of both axioms. The equivariance equations turn out to lend themselves to an iterative solution strategy where the expansion coefficients of a power series ansatz are determined iteratively, power by power. A first approximation of the gravitational theory valid for weak fields is obtained already after the second iteration, which yields a quadratic Lagrangian with linear field equations. In a sense, this is the free theory without self-coupling. In order to investigate the lowest-order effects of gravitational self-coupling, which we will dare in the subsequent chapter, the next order is indispensable. Therefore, after establishing the general principle, we focus on the perturbation theory up to third order in the Lagrangian.

The development in this chapter follows closely the presentation in Ref. [1], but is at times more detailed.

4.1 Perturbative implementation of axiom I

Let us state again, for reference, the equivariance equations (2.29b)–(2.29d) *we are going to solve perturbatively:*

$$\begin{aligned}
0 &= L_{:A} C^{A}{}_{B}{}^{n}{}_{m} u^{B} + L_{:A}{}^{p} \left[C^{A}{}_{B}{}^{n}{}_{m} \delta^{q}_{p} - \delta^{A}_{B} \delta^{q}_{m} \delta^{n}_{p} \right] u^{B}{}_{q} \\
&\quad + L_{:A}{}^{I} \left[C^{A}{}_{B}{}^{n}{}_{m} \delta^{J}_{I} - 2 \delta^{A}_{B} J^{pn}_{I} I^{J}_{pm} \right] u^{B}{}_{J} + L \delta^{n}_{m}, \\
0 &= L_{:A}{}^{(p|} C^{A}{}_{B}{}^{|n)}{}_{m} u^{B} + L_{:A}{}^{I} \left[C^{A}{}_{B}{}^{(n}{}_{m} 2 J^{p)q}_{I} - \delta^{A}_{B} J^{pn}_{I} \delta^{q}_{m} \right] u^{B}{}_{q}, \\
0 &= L_{:A}{}^{I} C^{A}{}_{B}{}^{(n}{}_{m} J^{pq)}_{I} u^{B}.
\end{aligned}$$

The first equivariance equation $0 = L_{,m}$ *has been omitted, because from now on we will consider it solved by restricting the problem to Lagrangian densities L that depend only on the fibre coordinates.*

Perturbation theory starts with choosing an expansion point $p \in J^2E$. Let p have fibre coordinates $(N^A, N^A{}_p, N^A{}_I)$. The deviation of any point $q \in J^2E$ with fibre coordinates $(G^A, G^A{}_p, G^A{}_I)$ is then defined as the difference

$$(H^A, H^A{}_p, H^A{}_I) := (G^A - N^A, G^A{}_p - N^A{}_p, G^A{}_I - H^A{}_I). \tag{4.1}$$

Around p, this results in the formal power series ansatz

$$\begin{aligned} L = a &+ a_A H^A + a_A{}^p H^A{}_p + a_A{}^I H^A{}_I \\ &+ a_{AB} H^A H^B + a_{AB}{}^p H^A H^B{}_p + a_{AB}{}^I H^A H^B{}_I \\ &+ a_A{}^p{}_B{}^q H^A{}_p H^B{}_q + a_A{}^p{}_B{}^I H^A{}_p H^B{}_I + a_A{}^I{}_B{}^J H^A{}_I H^B{}_J \\ &+ a_{ABC} H^A H^B H^C + \dots, \end{aligned} \tag{4.2}$$

which is called formal *because at this point there is no assumption about the convergence of the power series. We do, however, make two assumptions about admissible expansion points, in order to justify the interpretation of perturbatively constructed theories as valid theories for* weak *gravitational fields.*

1. *The expansion point represents a* flat *instance of the gravitational field, i.e. both $N^A{}_p$ and $N^A{}_I$ vanish.*
2. *At the expansion point, the matter theory that is used to bootstrap the construction procedure reduces to a theory that is equivalent to a matter theory on Minkowski spacetime.*

Both restrictions for p ensure that the limit of weak gravitational fields can match our observations for situations with weak gravity: matter fields couple to flat geometry in the sense that there are coordinate charts where the geometric coefficients are constant and this geometry is determined by the Minkowski metric. Curvature effects and effects from non-Lorentzian geometry are expected to arise as deviations *from this ground state. After all, this is just another incarnation of the correspondence principle for modified gravity.*

The first assumption is easily implemented: in the chosen coordinate chart, p takes the form $(N^A, 0, 0)$. The best way to make sense of the second assumption is by considering a few examples:

Example 4.1.1 (flat Lorentzian expansion points). *Let us choose appropriate expansion points $(N^A, 0, 0)$ for GLED and bimetric theories introduced in Sections 3.3 and 3.4, respectively. In order to satisfy the second assumption made for expansion points, we construct N^A from the Minkowski metric $\eta^{ab} = \mathrm{diag}(1, -1, -1, -1)^{ab}$ in the following ways.*

1. *For bimetric theories, a suitable expansion point is given by* $N^{\bar{A}} = J^{\bar{A}}_{ab}\eta^{ab}$ *and* $N^{\bar{\bar{A}}} = J^{\bar{\bar{A}}}_{ab}\eta^{ab}$*, i.e. setting both metrics* g *and* h *equal to* η*. Where a scalar density is needed, we choose* $\omega = \sqrt{-\det\eta} = 1$*. This choice reduces the bimetric Klein-Gordon theory to the standard Klein-Gordon theory for two scalar fields on Minkowski spacetime*
$$L_{2KG}\big|_N = \eta^{ab}\phi_{,a}\phi_{,b} - m_\phi^2\phi^2 + \eta^{ab}\psi_{,a}\psi_{,b} - m_\psi^2\psi^2. \tag{4.3}$$
Similarly, the refined Proca theory reduces to the standard Proca theory
$$L_{bi\text{-}Proca}\big|_N = -\eta^{ac}\eta^{bd}F_{ab}F_{cd} + m^2\eta^{ab}A_aA_b. \tag{4.4}$$

2. *For GLED, we choose the expansion point* $N^A = J^A_{abcd}(\eta^{ac}\eta^{bd} - \eta^{ad}\eta^{bc} + \epsilon^{abcd})$*. Using the density* $\omega_G = (\frac{1}{24}\epsilon_{abcd}G^{abcd})^{-1}$*, which at* $G^A = N^A$ *results in* $\omega_N = 1$*, the Lagrangian density for GLED reduces to*
$$L_{GLED} = 2\eta^{ac}\eta^{bd}F_{ab}F_{cd}, \tag{4.5}$$
i.e. Maxwell electrodynamics on Minkowski spacetime.[1]

Both choices of expansion points ensure that the perturbatively constructed gravitational theories provide to zeroth order in the deviation a background on which known physics is reproduced. Novel physics—the coupling of matter fields to nonmetric geometries and the self-coupling of such geometries—should emerge as effect of first and higher orders in the deviation from the Minkowski background.

Having defined an expansion point, the equivariance equations can—in principle—be solved perturbatively by repeating the following process: all equations in the system contain derivatives of first order, so the expansion coefficient a *of zeroth order remain undetermined. In order to determine the expansion coefficients* a_A*,* ${a_A}^p$*, and* ${a_A}^I$*, substitute the formal power series ansatz (4.2) for the Lagrangian density* L *in the equivariance equations, evaluate the result at* N *(i.e. set the deviation* H *to zero) and solve the resulting linear equations for the first-order coefficients. Next, differentiate each PDE once with respect to every independent variable, substitute again the power series ansatz, evaluate at* N *and solve the linear system for the expansion coefficients of second order. Now repeat this process of differentiation, substitution, evaluation, and solving of linear equations* ad infinitum*—or up to the desired perturbation order.*

[1] The term $\epsilon^{abcd}F_{ab}F_{cd}$ is a surface term and thus does not contribute to the field equations. As such, it has been omitted in Eq. (4.5).

4.2 Involution analysis

For the previously described perturbative solution process to play out as desired, the PDE system must observe an important property: we need to be sure that each step really determines all expansion coefficients for the corresponding order to the extent that this is possible[2]*. This is not always guaranteed, as a simple example demonstrates.*

Example 4.2.1 (integrability conditions [59])**.** *Consider the linear, first-order PDE system*

$$\begin{aligned} u_{,z} + y u_{,x} &= 0, \\ u_{,y} &= 0 \end{aligned} \tag{4.6}$$

for one function u which depends on three independent variables x, y, z. Making a formal power series ansatz, inserting this ansatz into the system (4.6)*, and evaluating at the expansion point yields a linear system of rank two for the three expansion coefficients of first order.*

There are, however, hidden equations governing the first order, which emerge after differentiating the first equation with respect to y and the second equation with respect to z and x. This gives new PDEs

$$\begin{aligned} u_{,yz} + y u_{,xy} + u_{,x} &= 0, \\ u_{,xy} &= 0, \\ u_{,yz} &= 0. \end{aligned} \tag{4.7}$$

The second derivatives in the first equation can be cancelled using the second and third equation, yielding the first-order PDE $u_{,x} = 0$. Such a new equation that is algebraically independent of the original PDEs (4.6) *is called an* integrability condition*. Including it in the first-order system and simplifying a bit, we get*

$$\begin{aligned} u_{,x} &= 0, \\ u_{,y} &= 0, \\ u_{,z} &= 0. \end{aligned} \tag{4.8}$$

Only after performing this procedure of making explicit the hidden first-order PDEs, we know for sure that the expansion coefficients of first order are determined already after the first iteration. In this case, the resulting linear system has full rank, leaving no coefficient undetermined.

[2]Of course, the equivariance equations, in general, will not determine *all* expansion coefficients. Rather, the solutions will be parameterised exactly by the coefficients that cannot be determined.

Luckily, the system of equivariance equations suffers none of these maladies, as we will prove in the following. The mathematical framework that allows to make such a statement is involution theory *[42]. Within this framework, a PDE of order q is defined geometrically as fibred submanifold $R_q \subseteq J^q E$ of some jet bundle manifold $J^q E$.*[3] *A local coordinate representation of R_q yields a system of equations, more closely resembling what a PDE looks like in the nongeometric picture. Note that we deliberately call R_q a partial differential* equation*, rather than using the plural, as this approach makes no difference between systems or scalar equations. A solution to a PDE is just a local section σ of E such that the image of $j^q\sigma$ is contained within R_q.*

Two geometric operations will be performed repeatedly on PDEs for their involution analysis: prolongation *and* projection*. The former maps a PDE R_q to some PDE R_{q+r} by, in local coordinates, adding to R_q all possible derivatives of order r of the individual equations—the equivalent geometric construction is a bit more involved. On the contrary, it is simpler to define projections geometrically, which is as bundle projections $R^{(r)}_{q-r} = \pi_{q,q-r}(R_q)$. Using a local representation of R_q, the projection is performed by eliminating derivatives of higher orders using only algebraic manipulations such that equations of order $q-r$ remain. The maximal set of such equations is a representation of R_{q-r}. For linear systems, the task of projecting a PDE to lower order is solved by linear algebra with tools like Gaussian elimination and has already been demonstrated earlier in Example 4.2.1.*

With this, the main result can be established.

Theorem 4.2.2 (formal integrability of equivariance equations)**.** *The equivariance equations are a formally integrable partial differential equation R_q with $q = 1$, i.e. it holds for all $r > 0$ that $R^{(1)}_{q+r} = R_{q+r}$.*

Formal integrability as defined in Thm. 4.2.2 captures in geometric terms the requirement a PDE must satisfy in order for the iterative solution strategy to succeed. Otherwise, a truncated power series solution—which will later serve as approximate solution for weak gravitational fields—could never be trusted, as prolongations of the PDE to higher orders could always yield additional constraints on the coefficients of the truncated series.

The equivariance equations fall into such a simple category that their formal integrability can be proven in a very straightforward way. According to Example 2.3.12 of Ref. [42], the possible integrability conditions for a PDE of order $q = 1$ for a single dependent variable are given by a certain commutator of the local PDE representatives. See [42] for the details. Adapting this technique to the system of equivariance equations, we can prove that the integrability conditions are already contained in the system to begin with.

[3]Note that this jet bundle is *not* the jet bundle on which the Lagrangian density is defined. For the equivariance equations, the order q is 1 and the underlying bundle E is $J^2 E_{\text{geometry}}$.

Proof of Thm. 4.2.2. The system (2.29a)–(2.29d) *of equivariance equations is equivalent (by its derivation) to the equation*

$$0 = \xi_{J^2E} L + L \cdot \xi^m_{,m} \tag{4.9}$$

for all vector fields ξ over M. With ξ_{J^2E} we denoted the lift to the second jet bundle over the field bundle. Applying the same technique as in Example 2.3.12 of Ref. [42], we generate all possible integrability conditions by acting with a second vector field ψ_{J^2E} on Eq. (4.9) *and subtracting the same equation with the rôles of ξ and ψ interchanged. These conditions turn out to be*

$$\begin{aligned} 0 &= [\xi_{J^2E}, \psi_{J^2E}] L + L \cdot [\xi, \psi]^m_{,m} \\ &\quad - \left(\psi_{J^2E} L + L \cdot \psi^m_{,m}\right) \xi^n_{,n} \\ &\quad + \left(\xi_{J^2E} L + L \cdot \xi^m_{,m}\right) \psi^n_{,n} \\ &= [\xi, \psi]_{J^2E} L + L \cdot [\xi, \psi]^m_{,m} \\ &\quad - \left(\psi_{J^2E} L + L \cdot \psi^m_{,m}\right) \xi^n_{,n} \\ &\quad + \left(\xi_{J^2E} L + L \cdot \xi^m_{,m}\right) \psi^n_{,n}, \end{aligned} \tag{4.10}$$

which is a linear combination of equations that are already contained in the system. Note how the Lie algebra homomorphism property of the vector field lift is crucial for this result.

Since it is impossible to generate integrability conditions that are not already present in the system, the equivariance equations are formally integrable. □

For more involved PDEs, formal integrability is in practice proven by showing that the system is involutive, *from which formal integrability follows. This comes with an algebraic condition on the PDE, which boils down to calculating a matrix rank for our particular PDE. However, it has still to be checked that a single prolongation does not generate new integrability condition—which amounts to the calculation above. So in this case, nothing would be gained by pursuing this approach. See [42, 37] for a proof of involutivity.*

4.3 Lorentz invariant ansätze

In order to implement the second assumption for suitable expansion points in the power series ansatz, we choose coordinate representations where the flat geometry is Lorentz invariant, i.e. satisfies the Lorentz invariance conditions

$$0 = N^A C^B{}_A{}^n{}_m (K_{(i)})^m_n, \tag{4.11}$$

where $K_{(i)}$ *are the 6 generators* $\{\eta^{m[r}\delta^{s]}_n \mid r < s\}$ *of the Lorentz group. This special symmetry of the expansion point carries over to the equivariance equations and causes rank defects, for example in the second equivariance equation* (2.29b). *At a generic (non-Lorentz invariant) point* $p = (x^i, M^A, 0, 0)$*, it reads*

$$0 = L_{:A}|_p C^A{}_B{}^n{}_m M^B + L|_p \delta^n_m \tag{4.12}$$

and is, in general, of rank 16. Evaluating the same equation at $p_0 = (x^i, N^A, 0, 0)$ *and contracting with Lorentz generators* $K_{(i)}$*, we obtain the 6 vanishing linear combinations*

$$\begin{aligned} 0 &= L_{:A}|_{p_0} C^A{}_B{}^n{}_m N^B (K_{(i)})^m_n + L|_{p_0} \delta^n_m (K_{(i)})^m_n \\ &= 0. \end{aligned} \tag{4.13}$$

In the end, only 10 equations remain linearly independent. While at the first glance this seems to reduce the number of determinable expansion coefficients, the converse is actually true: consider again the second equivariance equation and calculate the prolongation with respect to the variables u^A*. Evaluated at* p_0*, this gives*

$$0 = L_{:A:B}|_{p_0} C^B{}_C{}^n{}_m N^C + L_{:B}|_{p_0} C^B{}_A{}^n{}_m + L_{:A}|_{p_0} \delta^n_m, \tag{4.14}$$

which contracted with the Lorentz generators reduces again to first-order equations

$$0 = L_{:B}|_{p_0} C^B{}_A{}^n{}_m (K_{(i)})^m_n. \tag{4.15}$$

Comparing this equation with Eq. (4.11) *emphasises its significance: it mandates Lorentz invariance of the expansion coefficients* $L_{:B}|_{p_0} = a_B$.

Similar results hold for all other expansion coefficients and are obtained exactly the same way: calculate prolongations of the second equivariance equation, evaluate at the Lorentz invariant expansion point, and contract with the Lorentz generators. While this yields new independent equations of order q *by prolongation to order* $q + 1$ *and subsequent projection, it is important not to conflate the Lorentz invariance conditions on expansion coefficients with integrability conditions from involution theory. The former are an artefact of the expansion point with additional symmetries and are as such only valid exactly there, while an integrability condition would not be restricted to singular points.*

When solving the equivariance equations iteratively, we could just include the Lorentz invariance conditions and solve them together with the original equations. A better way is to exploit the nature of the additional conditions and implement Lorentz invariance of the expansion coefficients before *substituting the ansatz in the equivariance equations. For example, working on the metric bundle, rather than including the 60 equations*

$$a_B C^B{}_A{}^n{}_m (K_{(i)})^m_n \tag{4.16}$$

for the 10-dimensional ansatz a_B, we implement Lorentz invariance by reducing a_B to the ansatz

$$a_B = c \cdot J_B^{ab} \eta_{ab} \tag{4.17}$$

with just one undetermined coefficient. Not only did we get by without adding equations to the system, but we reduced the number of unknowns significantly.

A particular reduction we can perform right now is to set expansion coefficients with an odd *number of indices to zero.*[4] *Assuming that the number of indices on the geometry is even*[5]*, this removes all coefficients with odd total number of derivatives from the ansatz, e.g.* ${a_A}^p{}_B{}^I = 0$.

4.4 Perturbative implementation of axiom II

Before deriving consequences from the second axiom for the perturbatively constructed solutions, we can already infer restrictions on the perturbation ansatz. As the matter Lagrangians considered here depend on the geometry only locally, and so do the corresponding principal polynomials, a matching gravitational polynomial must also depend on the geometry locally, i.e. not via derivatives. In order to enforce this, we remove ansätze with a total number of derivatives greater than two and obtain the general ansatz

$$\begin{aligned} L = a + a_A H^A + {a_A}^I {H^A}_I + a_{AB} H^A H^B + {a_{AB}}^I H^A {H^B}_I + {a_A}^p{}_B{}^q {H^A}_p {H^B}_q \\ + a_{ABC} H^A H^B H^C + {a_{ABC}}^I H^A H^B {H^C}_I + {a_{AB}}^p{}_C{}^q H^A {H^B}_p {H^C}_q + \dots \,. \end{aligned} \tag{4.18}$$

As discussed before, all expansion coefficients are Lorentz invariant.

Now, consider a solution of the equivariance equations for the ansatz (4.18)*, truncated at order q. The corresponding field equations will be of order $q-1$ and the principal symbol, consequently, of order $q-2$. The second axiom of covariant constructive gravity is implemented perturbatively by matching the expansion*

$$\mathcal{P}_{mat} = (P^{(0)}_{mat}) + (P^{(1)}_{mat})_A H^A + \dots + (P^{(q-2)}_{mat})_{A_1 \dots A_{q-2}} H^{A_1} \dots H^{A_{q-2}} + \mathcal{O}(q-1) \tag{4.19}$$

of the matter polynomial with the expansion

$$\mathcal{P}_{grav} = (P^{(0)}_{grav}) + (P^{(1)}_{grav})_A H^A + \dots + (P^{(q-2)}_{grav})_{A_1 \dots A_{q-2}} H^{A_1} \dots H^{A_{q-2}} + \mathcal{O}(q-1) \tag{4.20}$$

[4]For field bundles that are defined as proper subbundle of some “unrestricted” tensor bundle, the number of indices refers to the rank of the latter.

[5]Otherwise, we would not be able to define a Lorentz invariant expansion point to begin with.

of the gravitational polynomial. Note that a "match" does not necessarily mean that both polynomials coincide, but rather that the causalities are compatible in the sense of Def. 2.4.2.

While the expansion (4.19) *generally follows from a closed form for the matter principal polynomial, we only have the currently constructed orders of the gravitational Lagrangian at our disposal when calculating terms from Eq.* (4.20)*. The process to arrive at the gravitational polynomial from there by expanding the definition* (2.56)*, however, is straightforward. We restrict our attention to the order $q-1=2$ in the field equations, which is the maximum order we will consider for a concrete example later, but the calculations can be generalised to higher orders if necessary.*

The principal polynomial was defined in Def. 2.4.1 as the quotient of a nonvanishing entry from the order-s adjugate $Q(k)$ corresponding to the symbol $T(k)$ and an expression built from the generators $\chi_{(i)}(k)$ of the gauge symmetry,

$$\mathcal{P}(k) = \frac{Q^{A_1\cdots A_s B_1 \cdots B_s}}{\epsilon^{\mu_1\cdots\mu_s}\epsilon^{\nu_1\cdots\nu_s}\left[\prod_{i=1}^{s}\chi^{A_i}_{(\mu_i)}\right]\left[\prod_{j=1}^{s}\chi^{B_j}_{(\nu_j)}\right]}. \tag{4.21}$$

We start the expansion of Eq. (4.21) *with separating the perturbation orders in the vectors $\chi_{(n)}(k)$ as*

$$\chi^A_{(n)}(k) = C^A{}_B{}^m{}_n N^B k_m + C^A{}_B{}^m{}_n H^B k_m =: (\chi^{(0)})^A_n(k) + (\chi^{(1)})^A_{Bn}(k) H^B. \tag{4.22}$$

From there, the denominator in Eq. (4.21)*, which will be abbreviated as $f^{A_1\cdots A_s B_1\cdots B_s}(k)$ in the following, can be expanded into*

$$f^{A_1\cdots A_s B_1\cdots B_s}(k) = (f_{(0)})^{A_1\cdots A_s B_1\cdots B_s}(k) + (f_{(1)})_C^{A_1\cdots A_s B_1\cdots B_s}(k) H^C + \mathcal{O}(2). \tag{4.23}$$

For the numerator, we choose a submatrix $T^{A_1\cdots A_s B_1\cdots B_s}(k)$ of the principal symbol $T(k)$ which is of full rank, i.e. has a nonvanishing determinant.[6] *The determinant of this submatrix will be (up to, possibly, an irrelevant sign) the entry of the adjugate matrix entering the principal polynomial definition* (4.21)*. Recalling the expansion of the matrix*

[6] This sounds like a hard problem in practice, but turns out to be quite feasible. While the matrices contain symbolic entries given by undetermined expansion coefficients of the Lagrangian density and covector components k_a, ranks can actually be calculated using randomly drawn numeric values for the symbolic entries. In the worst case, we introduce additional linear dependencies and obtain a lower rank. If the rank obtained by such a calculation is maximal, on the other hand, we have nothing to worry about and can trust the result. For the examples encountered later on, it is possible to perform all calculations with arbitrary precision arithmetic on integers and use fraction-free Gaussian elimination, yielding results without any numerical instabilities.

determinant

$$\det(A+\epsilon B) = \det(A)\det(I+\epsilon A^{-1}B) = \det(A)(1+\epsilon\,\mathrm{Tr}(A^{-1}B)) + \mathcal{O}(\epsilon^2), \tag{4.24}$$

and expanding the submatrix of the principal symbol as

$$T^{A_1\ldots A_s B_1\ldots B_s}(k) = (T_{(0)})^{A_1\ldots A_s B_1\ldots B_s}(k) + (T_{(1)})_C^{A_1\ldots A_s B_1\ldots B_s}(k)H^C + \mathcal{O}(2), \tag{4.25}$$

we arrive at the expansion of $Q^{A_1\ldots A_s B_1\ldots B_s}(k)$,

$$\begin{aligned}\pm Q^{A_1\ldots A_s B_1\ldots B_s}(k) =& \det(T^{A_1\ldots A_s B_1\ldots B_s}(k))\\ =& \det((T_{(0)})^{A_1\ldots A_s B_1\ldots B_s}(k))\\ &\times[1+(T_{(0)})^{A_1\ldots A_s B_1\ldots B_s}(k)^{-1}(T_{(1)})_C^{A_1\ldots A_s B_1\ldots B_s}(k)H^C]\\ &+\mathcal{O}(2)\\ =&(D_{(0)})^{A_1\ldots A_s B_1\ldots B_s}(k) + (D_{(1)})_C^{A_1\ldots A_s B_1\ldots B_s}(k)H^C.\end{aligned} \tag{4.26}$$

The last equality introduces abbreviations $D_{(0)}$ *and* $D_{(1)}$ *for the expansion coefficients of* Q*. In order to take the quotient of* Q *and* f*, it remains to calculate the multiplicative inverse*

$$\begin{aligned}f^{A_1\ldots A_s B_1\ldots B_s}(k)^{-1} =&(f_{(0)})^{A_1\ldots A_s B_1\ldots B_s}(k)^{-1}\\ &\times[1-(f_{(0)})^{A_1\ldots A_s B_1\ldots B_s}(k)^{-1}(f_{(1)})_C^{A_1\ldots A_s B_1\ldots B_s}(k)H^C]\\ &+\mathcal{O}(2).\end{aligned} \tag{4.27}$$

Finally, the product of Q *and* f^{-1} *yields the expansion*

$$\pm\mathcal{P}(k) = P^{(0)}(k) + (P^{(1)})_C(k)H^C + \mathcal{O}(2), \tag{4.28}$$

of the principal polynomial with coefficients

$$\begin{aligned}P^{(0)}(k) &= \frac{(D_{(0)})^{A_1\ldots A_s B_1\ldots B_s}(k)}{(f_{(0)})^{A_1\ldots A_s B_1\ldots B_s}(k)},\\ P_C^{(1)}(k) &= \frac{(D_{(1)})_C^{A_1\ldots A_s B_1\ldots B_s}(k) - (f_{(1)})_C^{A_1\ldots A_s B_1\ldots B_s}(k)\cdot P^{(0)}(k)}{(f_{(0)})^{A_1\ldots A_s B_1\ldots B_s}(k)}.\end{aligned} \tag{4.29}$$

The thus obtained relevant order of the gravitational principal polynomial may be compared with an expansion of the principal polynomial originating from the matter theory. Focusing not on an exact correspondence, but rather on the perturbative version of axiom II to second order,

$$C_{mat} = C_{grav} + \mathcal{O}(2) \quad \textit{and} \quad V_{mat} \subseteq V_{grav} + \mathcal{O}(2), \tag{4.30}$$

restricts the perturbative solution of the equivariance equations to the causally compatible sector.

4.5 The perturbative construction algorithm

Having elaborated in detail all the steps necessary in order to construct approximate solutions to the equivariance equations and causal compatibility conditions, it is worthwhile to take a step back and collect the results in the form of a concise algorithm.

Algorithm 2: Perturbative gravitational closure using covariant constructive gravity

Data: *Geometry bundle* $E_{grav} \xrightarrow{\pi_{grav}} M$, *matter bundle* $E_{mat} \xrightarrow{\pi_{mat}} M$, *Lagrangian matter field theory on* $E_{grav} \oplus_M J^1 E_{mat}$ *with linear field equations, expansion order* $q \geq 2$, *Lorentz invariant expansion point* $(N^A, 0, 0)$

Result: *Truncated power series of the most general diffeomorphism invariant and causally compatible gravitational Lagrangian field theory on* $J^2 E_{grav}$

compute the Gotay-Marsden coefficients (2.27) *for* E_{grav}
construct a basis for the Lorentz invariant expansion coefficients in the ansatz (4.18)
calculate prolongations up to order q *of the equivariance equations* (2.29b)–(2.29d)
evaluate the prolongations at the expansion point N
solve the resulting linear system for the expansion coefficients
compute the expansion of the gravitational principal symbol $T_{grav}(k)$
choose a submatrix $T^{A_1 \ldots A_4 B_1 \ldots B_4}(k)$ *of* $T_{grav}(k)$ *which is of full rank*
compute the expansion (4.26) *of the submatrix determinant (the* numerator*)*
compute the expansion (4.23) *of* $f^{A_1 \ldots A_4 B_1 \ldots B_4}(k)$ *(the* denominator*)*
from the numerator and denominator, compute the expansion (4.29) *of* $\mathcal{P}_{grav}(k)$
expand $\mathcal{P}_{mat}$ *up to order* $q-2$
impose $C_{mat} = C_{grav}$ *and* $V_{mat} \subseteq V_{grav}$ *up to order* $q-2$

The perturbative approach has reduced most of the task of closing a matter field theory with a diffeomorphism invariant and causally compatible gravitational theory to linear algebra—at the cost, of course, that the resulting theory is only an approximation for weak gravitational fields. This approximation, however, is final in the following sense: because the equivariance equations have been proven to be formally integrable, we can be sure that the truncated power series obtained from the algorithm is as definite as it gets. [42] No prolongation of the equivariance equations to orders higher than q will yield new restrictions on the expansion coefficients up to order q. It is still not possible to make a statement about the convergence of the formal power series, so it remains unclear whether this procedure would yield an exact solution if—somehow—performed up to $q = \infty$.

Also note that the expansions of the objects relevant for the calculation of $\mathcal{P}_{grav}(k)$ *in steps 6–10 have only been stated explicitly for the case* $q = 3$*. This does not take away from the generality of the algorithm for higher orders, as the necessary expansions follow the same pattern: essentially, one has to consider an expression of the form*

$$\frac{\det(A + \epsilon B)}{a + \epsilon b} \tag{4.31}$$

and expand to whichever order in ϵ *is desired.*

Let us close with a list of the equivariance equations and their first prolongations evaluated at an expansion point $N = (N^A, 0, 0)$*. We will perform the construction algorithm for the order* $q = 3$ *in Chap. 6, so it shall suffice to limit ourselves to this order here as well. We use the reduced power series ansatz Eq.* (4.18)*. The unprolonged equivariance equations evaluated at* N *are*

$$\begin{aligned} 0 &= a_A {C^A}_B{}^n{}_m N^B + a\delta^n_m, \\ 0 &= a_A{}^I {C^A}_B{}^{(n}{}_m J_I^{pq)} N^B. \end{aligned} \tag{4.32}$$

The first prolongations evaluate to

$$\begin{aligned} 0 &= a_A {C^A}_B{}^n{}_m + 2a_{AB} {C^A}_C{}^n{}_m N^C + a_B \delta^n_m, \\ 0 &= a_A{}^I [{C^A}_B{}^n{}_m \delta^J_I - 2\delta^A_B J_I^{pn} I^J_{pm}] + a_{AB}{}^J {C^A}_C{}^n{}_m N^C + a_B{}^J \delta^n_m, \\ 0 &= 2a_A{}^{(p}{}_B{}^{q|} {C^A}_C{}^{|n)}{}_m N^C + a_A{}^I [{C^A}_B{}^{(n}{}_m 2J_I^{p)q} - \delta^A_B J_I^{pn} \delta^q_m], \\ 0 &= a_{BA}{}^I {C^A}_C{}^{(n}{}_m J_I^{pq)} N^C + a_A{}^I {C^A}_B{}^{(n}{}_m J_I^{pq)}, \end{aligned} \tag{4.33}$$

and the second prolongations finally yield

$$\begin{aligned} 0 &= 2a_{AC} {C^A}_B{}^n{}_m + 2a_{AB} {C^A}_C{}^n{}_m + 6a_{ABC} {C^A}_D{}^n{}_m N^D + 2a_{BC} \delta^n_m, \\ 0 &= 2a_A{}^p{}_C{}^r [{C^A}_B{}^n{}_m \delta^q_p - \delta^A_B \delta^n_p \delta^q_m] + 2a_{AB}{}^q{}_C{}^r {C^A}_D{}^n{}_m N^D + 2a_B{}^q{}_C{}^r \delta^n_m, \\ 0 &= a_{CA}{}^I [{C^A}_B{}^n{}_m \delta^J_I - 2\delta^A_B J_I^{pn} I^J_{pm}] + 2a_{ACB}{}^J {C^A}_D{}^n{}_m N^D + a_{CB}{}^J \delta^n_m, \\ 0 &= 2a_{CA}{}^{(p}{}_B{}^{q|} {C^A}_D{}^{|n)}{}_m N^D + a_{CA}{}^I [{C^A}_B{}^{(n}{}_m 2J_I^{p)q} - \delta^A_B J_I^{pn} \delta^q_m], \\ 0 &= 2a_{BCA}{}^I {C^A}_D{}^{(n}{}_m J_I^{pq)} N^D + a_{CA}{}^I {C^A}_B{}^{(n}{}_m J_I^{pq)}. \end{aligned} \tag{4.34}$$

5 Computational methods for perturbative constructive gravity

The results from the previous chapter provide us with a comprehensive algorithm for the perturbative construction of gravitational theories. While consisting almost entirely of linear algebra, the execution of the algorithm is not feasible without the help of the computer. Therefore, we dedicate this section to the presentation of two Haskell libraries: the first one, `sparse-tensor`, implements the generation of Lorentz invariant perturbation ansätze. The second library, `safe-tensor`, is designed for safe and efficient evaluation and solution of the equivariance equations.

5.1 Ansatz generation

A central finding of Chap. 4 is that the perturbation ansätze inherit the Lorentz invariance of the expansion point. This has important practical ramifications: for example, instead of the 10 coefficients a_A in the expansion of a metric Lagrangian, we can just work with the one-dimensional Lorentz invariant coefficient $c \cdot J_A^{ab}\eta_{ab}$. That means, before even considering the equivariance equations, the dimensionality of the ansatz can already be reduced a lot.

It can be shown that a constant Lorentz invariant tensor, say T^{abcd}, is comprised of the Minkowski metric η and the totally antisymmetric symbol ϵ[1], such that for this example

$$T^{abcd} = A \cdot \epsilon^{abcd} + B \cdot \eta^{ab}\eta^{cd} + C \cdot \eta^{ac}\eta^{bd} + D \cdot \eta^{ad}\eta^{bc}. \tag{5.1}$$

The coefficients A, B, C, D can be chosen freely, leaving us with 4 degrees of freedom instead of 64. If the tensor shall have certain symmetries, e.g. the symmetries of an area metric tensor, we find an ansatz by applying the symmetry projections to the generic rank-4 ansatz (5.1), which yields in this case

$$S^{abcd} = A \cdot \epsilon^{abcd} + \frac{C - D}{2}\left(\eta^{ac}\eta^{bd} - \eta^{ad}\eta^{bc}\right). \tag{5.2}$$

[1] See e.g. [60, 61].

Two coefficients A and $\frac{C-D}{2}$ would parameterise such an ansatz.

In order to execute the perturbative construction algorithm, we need to find a basis for the ansätze (4.18) up to the desired perturbation order. This is, in principle, achieved by listing all possible products of ϵ and η and assigning to each term a unique coefficient. Each product will contain at most one ϵ, because the product of two ϵ symbols amounts to a linear combination of products of Minkowski metrics.

The ansätze we want to construct exhibit certain symmetries. Some stem from the field bundle itself (e.g. the symmetry of a metric or the symmetries of an area metric), but there are also symmetries inherited from second derivatives or products of perturbations. Consider, for example, the area metric ansatz

$$a_{ABC}{}^{I} H^A H^B H^C{}_I. \tag{5.3}$$

Expressed using spacetime indices, this ansatz reads

$$a_{abcd\ efgh\ pqrs}{}^{ij} H^{abcd} H^{efgh} H^{pqrs}{}_{ij}. \tag{5.4}$$

Of course, the individual index sets $abcd$, $efgh$, and $pqrs$ inherit the area metric symmetries from the perturbation H. The indices i, j are symmetric due to the commutativity of partial derivatives. The product of H^{abcd} and H^{efgh} enforces a block symmetry of the ansatz under the exchange of the index sets $abcd$ and $efgh$. We construct such an ansatz like before, by applying the respective projections to the ansatz, which collapses many individual terms with different coefficients to symmetric terms sharing a common prefactor. Note that we deal with the mixed index positions by constructing a purely covariant ansatz and raising the derivative indices using an η afterwards, e.g.

$$a_{ab}{}^{ij} H^{ab}{}_{ij} = \eta^{ii'} \eta^{jj'} \tilde{a}_{abi'j'} H^{ab}{}_{ij}. \tag{5.5}$$

One thing has not been considered so far: it is not clear, *a priori*, whether the constructed ansätze really form a *basis*. We need to be sure that a representation like Eq. (5.1) uniquely determines the ansatz. In general, this will *not* be the case, as the ansatz

$$\begin{aligned} T^{abcdef} = A_1 \cdot \epsilon^{abcd}\eta^{ef} + A_2 \cdot \epsilon^{abce}\eta^{df} + A_3 \cdot \epsilon^{abcf}\eta^{de} + A_4 \cdot \epsilon^{abde}\eta^{cf} + \dots \\ \dots + A_{16} \cdot \eta^{ab}\eta^{cd}\eta^{ef} + A_{17} \cdot \eta^{ab}\eta^{ce}\eta^{df} + \dots \end{aligned} \tag{5.6}$$

for a rank-6 tensor demonstrates. The 15 terms of the type $\epsilon^{abcd}\eta^{ef}$ are linearly dependent via the identity

$$0 = 5\epsilon^{[abcd}\eta^{e]f} = \epsilon^{abcd}\eta^{ef} - \epsilon^{abce}\eta^{df} - \epsilon^{abed}\eta^{cf} - \epsilon^{aecd}\eta^{bf} - \epsilon^{ebcd}\eta^{af}. \tag{5.7}$$

Because of this circumstance, we cannot consider two ansatz terms distinct just because their representations as linear combinations of ϵ and η products differ. Rather, we need to inspect the actual *components* of the tensors in order to make a decision. For the ansatz in Eq. (5.6), this would mean that we evaluate the 4^6 components T^{abcdef}, which gives 4096 linear combinations of the 30 coefficients $A_1 \dots A_{30}$. An ansatz without linearly dependent terms would exhibit 30 linearly independent combinations, which could be checked by calculating the rank of the 4096×30 matrix representing the linear combinations—it should be equal to 30. In this case, it will be less than 30 because we already know of at least one linear dependence. Gaussian elimination of the matrix tells us which coefficients can be used as basis: exactly those whose corresponding column contains, for some row, the first nonzero entry in this row. The other coefficients are linearly dependent on the basis coefficients and can thus safely be set to zero.

Let us demonstrate this reduction of linearly dependent ansatz coefficients with the help of an example. Pretend that, after evaluation of a tensor with four indices, the matrix

$$\begin{array}{c} \\ 0000 \\ 0101 \\ 0123 \end{array} \begin{array}{c} \begin{array}{cccc} A & B & C & D \end{array} \\ \begin{pmatrix} 1 & 1 & -2 & 0 \\ 0 & 2 & -6 & -4 \\ 3 & 0 & 3 & 1 \end{pmatrix} \end{array} \tag{5.8}$$

is obtained. In practice, matrices will often reduce to such simple forms, because they contain many zero or duplicate rows that can be removed. Gaussian elimination may yield (depending on the pivoting)

$$\begin{array}{c} \begin{array}{cccc} A & B & C & D \end{array} \\ \begin{pmatrix} 3 & 0 & 3 & 1 \\ 0 & 2 & -6 & -4 \\ 0 & 0 & 0 & \frac{5}{3} \end{pmatrix} \end{array}, \tag{5.9}$$

from which we read off the linearly independent columns A, B, and D. The superfluous ansatz coefficient C can be set to zero.

The Haskell package `sparse-tensor`[2] exports the module `Math.Tensor.LorentzGenerator`, which implements the procedure outlined above. Haskell is a purely functional language with lazy semantics by default. [62] In practice, this means that the programmer does not modify state but composes expressions, which are evaluated only when asked for. Consider, for example, a routine that sums up the elements of an array. First, let us look at an implementation in C.

[2] See [5]. The source code is publicly available at `https://github.com/TobiReinhart/sparse-tensor`.

```c
int sum(int array[], int length) {
  int result = 0;

  for (int i = 0; i < length; ++i) {
    result += array[i];
    // perform_side_effect();      <-- possible side effect!
  }

  return result;
}
```

Listing 5.1: C implementation of the `sum` function.

Note how state—in the form of the `result` variable—is created, modified, and eventually returned. At any point of the programme, it is possible to perform arbitrary side effects, which could modify the input data, alter the local state (consisting of counter variable `i` and result variable `result`), print something to the user's screen, and so on.

In Haskell, on the other hand, a naïve[3] implementation of the `sum` function reads quite differently.

```haskell
1 sum :: [Int] -> Int
2 sum xs = go 0 xs
3   where
4     go acc [] = acc
5     go acc (y:ys) = go (acc+y) ys
```

Listing 5.2: Haskell implementation of the `sum` function.

The `sum` function in Listing 5.2 demonstrates how functional programming approaches certain tasks. The input is a `List` of integers, a functional data structure that matches either the empty list `[]` or an integer appended to some list, e.g. `5 : xs`. Data is *consumed* by matching on patterns and results are *produced* by building up expressions, in this case repeated applications of the `(+)` function in line 5. It is, by design, impossible to slip in side effects, which is why functions in Haskell are *pure*. This leads to the important property called *referential transparency*, meaning that expressions can be replaced by their values without changing the behaviour of the programme.

Because of its purity and, importantly, the powerful type system based on System F [63], Haskell allows to write programmes that are both efficient and safe. As we will

[3] Performance considerations put aside.

see, the objects with which we are concerned have natural representations as functional data types and the manipulations that need to be performed translate into efficient, pure functions operating on these types.

We will only sketch the implementation of the ansatz generation procedure outlined above. Performance optimisations like strictness annotations and unpacking are not given explicitly. For all details, see Ref. [37] and the documentation [5] of the package. There are more differences between the presentation here and the production code, which have been introduced deliberately for lighter reading.

As already mentioned, an ansatz has a representation as a functional data structure. Let us begin with the individual η and ϵ tensors in Listing 5.3.

```
-- data type representing an \eta^{a b} tensor
data Eta     = Eta     Char Char           deriving (Eq, Ord)
-- data type representing an \epsilon^{a b c d} tensor
data Epsilon = Epsilon Char Char Char Char deriving (Eq, Ord)
```

Listing 5.3: Haskell representation of η and ϵ tensors.

These types are, essentially, named wrappers for the index labels. We also need a type that represents a coefficient. For our purposes, an integer prefactor (because we will never perform division) and a variable label, also an integer, will suffice. See Listing 5.4.

```
-- data type representing a coefficient c * A_i
data Coeff = Coeff Int Int
```

Listing 5.4: Haskell representation of a scaled ansatz coefficient.

Now, the central type for the generation of ansätze is a list of trees, called *forest*. From now on, we leave ϵ tensors out of the picture. As they appear at most once, we will always sort the trees such that an ϵ—if present—is the root. In everything that follows, a distinction has to be made when operating on the roots of ansatz trees, but everything else concerns only trees of η tensors. With this caveat, the data type is as shown in Listing 5.5.

```
data Forest a b = Forest [(a, Forest a b)] | Leaf b
type Ansatz = Forest Eta Coeff
```

Listing 5.5: Haskell representation of an ansatz consisting only of η tensors.

We will always keep the forests sorted in two ways: the list of trees `[(Eta, Forest Eta Coeff)]` is sorted, meaning that e.g. η^{ab} comes before η^{cd}, but also all η tensors appearing in the inner forest must come after the outer η tensor—so it is forbidden to insert an η^{ab} below a node η^{cd}.

```
Eta 'a' 'b'
 |
 +---- Eta 'c' 'd'
 |      |
 |      +---- Eta 'e' 'f' - Coeff 1 1
 |
 +---- Eta 'c' 'e'
 |      |
 |      +---- Eta 'd' 'f' - Coeff 1 2
 |
 +---- Eta 'c' 'f'
        |
        +---- Eta 'd' 'e' - Coeff 1 3

Eta 'a' 'c'
 |
 +---- Eta 'b' 'd'
 |      |
 |      +---- Eta 'e' 'f' - Coeff 1 4
 |
 +---- Eta 'b' 'e'
 |      |
 |      +---- Eta 'd' 'f' - Coeff 1 5
 |
 +---- Eta 'b' 'f'
        |
        +---- Eta 'd' 'e' - Coeff 1 6
...
```

Listing 5.6: First 6 η-only terms of an ansatz tensor with 6 indices.

An example representation of the first 6 η-only terms for the ansatz

$$A_1 \cdot \eta^{ab}\eta^{cd}\eta^{ef} + \cdots + A_6 \cdot \eta^{ac}\eta^{bf}\eta^{de} \tag{5.10}$$

is given in Listing 5.6. Such a sorted tree is easily traversed for updates, insertions, deletions, *et cetera*. Also, the evaluation of specific components is greatly simplified: `Eta 'a' 'b'` only has to be evaluated once, and very importantly, for components where $\eta^{ab} = 0$, the whole tree can be discarded. Let us give one example for an operation on ansatz forests, namely the sum of two ansätze in Listing 5.7.

```haskell
addAnsatz :: Ansatz -> Ansatz -> Ansatz
addAnsatz (Leaf coeff1) (Leaf coeff2) = Leaf (addCoeffs coeff1 coeff2)
  where
    addCoeffs :: Coeff -> Coeff -> Coeff
    addCoeffs (Coeff c1 var1) (Coeff c2 var2)
      | var1 == var2 = Coeff (c1+c2) var1
      | otherwise    = error "adding distinct variables"
addAnsatz (Forest fs1) (Forest fs2) = Forest (addForests fs1 fs2)
  where
    addForests :: [(Eta, Forest Eta Coeff)]
               -> [(Eta, Forest Eta Coeff)]
               -> [(Eta, Forest Eta Coeff)]
    addForests [] ys = ys
    addForests xs [] = xs
    addForests (x:xs) (y:ys) =
        case fst x `compare` fst y of
          LT -> x : addForests xs (y:ys)
          EQ -> let innerAnsatz = addAnsatz (snd x) (snd y)
                in  (fst x, innerAnsatz) : addForests xs ys
          GT -> y : addForests (x:xs) ys
addAnsatz _ _ = error "cannot add incompatible ansätze"
```

Listing 5.7: Sum of two ansatz forests.

The occurrence of `error` functions means that the function `addAnsatz` is *partial*, i.e. does not compute an output for every input. This could be cured by refining the return type of the function, but doing so is not really necessary for this use case, as the input is under our control: we will neither add incompatible ansätze, nor will two leaves with distinct variables be added. The curious reader may be referred to the next section, where we actually introduce methods for catching such runtime errors already at the type level.

Ansatz generation proceeds as follows: starting with the empty forest, we consider each possible ansatz term separately, one at a time, for example $\eta^{ab}\eta^{cd}\eta^{ef}$. For each such term, it is first checked whether the ansatz already contains the term, utilising fast lookup in the sorted forest. If it is contained, we can discard the term and proceed with the next one. If, on the other hand, the term is *new*, it is assigned a new variable, symmetrised, and added to the ansatz.

When all possible ansatz terms have been added (or discarded, for that matter), the linear dependencies are identified and removed, like explained before. For the linear algebra part, the package `hmatrix` [64] is used together with a custom implementation of Gaussian elimination that is tested for sufficient stability.

For the handling of larger ansätze—up to 18 indices at the time of writing, which is enough for fourth-order area metric Lagrangians—a second mode has been implemented. In order for the matrix not to become too large, its is checked *before insertion* whether a given symmetrised term would be linearly dependent on the already existing terms. This entails keeping track of the evaluation matrix, as re-evaluating the ansatz tensors for each term that is added would be too expensive. However, we do not have to perform Gaussian elimination, because it is not necessary to identify which column would introduce a rank defect—it is always the new one, because we ensure that the matrix rank is maximal with our construction. So, fast and numerically stable singular value decomposition can be used for computing ranks.

Overall, the second method is a bit slower for ansätze with 14 indices (needed for third-order area metric Lagrangians) than the first method, taking a couple of seconds to compute on modern workstation hardware. For the fourth-order ansätze, however, it is the only option. With all the optimisation work that has been done, like exploiting the symmetries in order to reduce the number of terms that are even considered for insertion or, likewise, reducing the number of index combinations to be probed (see [37] or the source code and documentation [5]), the computation times have been reduced drastically. The largest fourth-order ansätze with 18 indices are computed within about three hours, using three gigabytes of memory. To the knowledge of the author, the methods developed for the canonical approach [65] (to which the method presented here is applicable as well) do not achieve this efficiency.

We will encounter the generated ansätze in Chap. 6 when constructing perturbative area metric gravity. But first, let us walk through the second Haskell package developed in the course of this thesis.

5.2 Equivariance equations

In principle, `sparse-tensor` provides the machinery for setting up and solving the equivariance equations. It even contains some safeguards against composing tensors of incompatible ranks, but not nearly enough in order to safely mirror Eqns. (4.32)–(4.34) in a Haskell programme. For this purpose, the package `safe-tensor`[4] has been developed, which implements index-based tensor calculus as known from mathematical physics. `safe-tensor` makes it comparably easy and, above all, *safe* to perform all kinds of operations on tensors, including transpositions of indices, contractions, symmetrisations, tensor products, and tensor sums.

[4]See [6]. The source code is publicly available at `https://github.com/nilsalex/safe-tensor`, the package is also available via hackage at `https://hackage.haskell.org/package/safe-tensor`.

Central for the design of the tensor type provided by the package is the *generalised rank* of a tensor. The tensors we deal with, a good example being the Gotay-Marsden coefficients ${C^A}_B{}^n{}_m$, can be considered as multilinear maps over *different* vector spaces. Applying this interpretation to C, we get a map

$$C\colon V^*_{\text{area}} \times V_{\text{area}} \times V^* \times V \to \mathbb{R} \tag{5.11}$$

with V_{area} being a fibre of the area metric bundle and V a tangent space to the base manifold.

Concrete calculations employ a basis $(e_i)_{i=1\dots n}$ of V and a corresponding dual basis $(\epsilon^i)_{i=1\dots n}$ of V^*, where n denotes the dimension of the base manifold. Such bases carry over to fibres like V_{area} or V_{metric}. Representations such as ${C^A}_B{}^n{}_m$ for C are understood in terms of these bases. For the definition of the generic rank of these representations, we assign each type of vector space a label, e.g. ST[5] for V and STArea for V_{area}. The indices corresponding to each space and the dual complete the list of labels to the generic rank. For the example of the Gotay-Marsden coefficients, we have

$$\text{rank}({C^A}_B{}^n{}_m) = \{(\underbrace{\text{ST}}_{\text{label}}, \underbrace{4}_{\text{dimension}}, \underbrace{\{n\}}_{\text{contravariant}}, \underbrace{\{m\}}_{\text{covariant}}), (\text{STArea}, 21, \{A\}, \{B\})\}. \tag{5.12}$$

Note that the contravariant and covariant indices are each provided as *set*, i.e. they cannot contain duplicates and have no specific order.[6] It is permitted, however, for the set of covariant indices and the set of contravariant indices to have a nonempty intersection—these are candidates for contractions.

Let us consider more examples:

$$\text{rank}(\eta^{ab}) = \{(\text{ST}, 4, \{a, b\}, \{\})\} \tag{5.13}$$
$$\text{rank}(\eta^{ba}) = \{(\text{ST}, 4, \{a, b\}, \{\})\} \tag{5.14}$$
$$\text{rank}(\eta_{\heartsuit}) = \{(\text{STSym2}, 10, \{\}, \{\heartsuit\})\}^7 \tag{5.15}$$
$$\text{rank}({C^B}_A{}^p{}_p N^A) = \{(\text{ST}, 4, \{p\}, \{p\}), (\text{STArea}, 21, \{A, B\}, \{A\})\} \tag{5.16}$$

The contraction of a rank is obtained by removing duplicate indices. If as a result there are no indices associated to a vector space, it is also removed from the generalised rank. Revisiting the previous example (5.16), application of the contraction yields

$$\text{contract}(\text{rank}({C^B}_A{}^p{}_p N^A)) = \{(\text{STArea}, 21, \{B\}, \{\})\}. \tag{5.17}$$

[5] Meaning: tangent space to spacetime.
[6] For this reason, we are allowed to sort the index lists in our implementation, which results in more efficient operations.
[7] Index labels can be arbitrary!

The rules of the tensor calculus we are going to implement can now be stated using the generalised rank of a tensor. Note that some rules already follow from the definition of a generalised rank, but are stated again for completeness.

1. Each tensor carries a generalised rank as defined above.
2. Transpositions of indices corresponding to the *same* set do not change the generalised rank, i.e. $\mathrm{rank}(T^{Aab}) = \mathrm{rank}(T^{Aba})$. Transpositions of indices corresponding to different sets (that is, transpositions across different vector spaces or covariant *and* contravariant indices) are not defined.
3. Contractions are always allowed. If for some vector space the intersection of covariant and contravariant indices is nonempty, the rank is reduced as described. Otherwise, a contraction has no effect.
4. Taking the product of two tensors merges both ranks and is thus only allowed if the two tensors do not share indices corresponding to the same vector space in the same (upper or lower) position. For example
$$\mathrm{rank}(T^{Aab}S_a^{Bp}) = \{(\mathrm{ST}, 4, \{abp\}, \{a\}), (\mathrm{STArea}, 21, \{A, B\}, \{\})\}. \tag{5.18}$$
5. Adding and subtracting two tensors is only allowed if both ranks coincide. The result has, of course, the same rank.

Listing 5.8 contains the definition of the `Rank` type as used by the `safe-tensor` package. The function `sane` can be specialised to the type `sane :: Rank -> Bool`. It provides a check for whether a given generalised rank satisfies all constraints and can be used to decide whether a certain tensor can be defined or, more importantly, whether a certain operation would yield a tensor of invalid rank and is thus forbidden.

Listing 5.8: Generalised rank type implementation in Haskell and the corresponding validity check. The type `Rank` and the function `sane` are the foundation for the tensor type to be defined later. `Rank` contains all the information of a generalised rank: vector space labels, dimensions, and index lists. `sane` ensures that the constraints are satisfied: the list of sub-ranks for the individual vector spaces must be strictly ascending, but also the index lists itself. The listing is printed on the next page. Note that the `safe-tensor` package [6] does not export a module called `Rank`, but the definitions are part of a larger module.

```haskell
module Rank where

-- type for nonempty lists: data NonEmpty a = a :| [a]
import Data.List.NonEmpty (NonEmpty ((:|)))
-- type-level natural numbers and symbols
import Data.Singletons.TypeLits (Nat, Symbol)

-- vector space, contains a label vId and the dimension vDim
data VSpace a b = VSpace {vId :: a, vDim :: b} deriving (Ord, Eq)

-- index list, is either of mixed type or purely co/contravariant
data IList a
  = ConCov (NonEmpty a) (NonEmpty a)
  | Cov (NonEmpty a)
  | Con (NonEmpty a)
  deriving (Ord, Eq)

-- generalised rank, a list of vector spaces with assoc. index lists
type GRank s n = [(VSpace s n, IList s)]

-- generalised rank used for type-level computations
type Rank = GRank Symbol Nat

-- check whether a generalised rank is valid
sane :: (Ord a, Ord b) => [(VSpace a b, IList a)] -> Bool
sane [] = True
sane [(_, is)] = isAscendingIList is
sane ((v, is) : (v', is') : xs) =
  v < v' && isAscendingIList is && sane ((v', is') : xs)

-- index lists are strictly ascending if the nonempty lists are
isAscendingIList :: Ord a => IList a -> Bool
isAscendingIList (ConCov x y) =
  isAscending x && isAscending y
isAscendingIList (Con x) = isAscending x
isAscendingIList (Cov x) = isAscending x

isAscending :: Ord a => NonEmpty a -> Bool
isAscending (x :| []) = True
isAscending (x :| (y : ys)) =
  x < y && isAscending (y :| ys)
```

In order to make such decisions at the *type* level, we leverage the machinery provided by the `singletons` package [66]. With the help of the metaprogramming technique Template Haskell [67], `singletons` serves us twofold: first it lifts the function `sane` to `type Sane :: forall a b. [(VSpace a b, IList a)] -> Bool`, which may be interpreted as a function at the type level.

We use the type `Sane` together with a second type `TailR` derived from the function `tailR`[8] for the definition of the `Tensor` type in Listing 5.9.

```
-- (...) skipping some language extensions
module Math.Tensor.Safe where
-- (...) skipping some imports

data Tensor :: Rank -> Type -> Type where
  ZeroTensor :: forall (r :: Rank) v.  Sane r ~ 'True =>
                Tensor r v
  Scalar     :: forall v.
                !v -> Tensor '[] v
  Tensor     :: forall (r :: Rank) (r' :: Rank) v.
                (Sane r ~ 'True, TailR r ~ r') =>
                [(Int, Tensor r' v)] -> Tensor r v

deriving instance Eq v   => Eq (Tensor r v)
deriving instance Show v => Show (Tensor r v)

instance NFData v => NFData (Tensor r v) where
    rnf ZeroTensor   = ()
    rnf (Scalar v)   = rnf v
    rnf (Tensor ts)  = rnf ts

instance Functor (Tensor r) where
    fmap _ ZeroTensor   = ZeroTensor
    fmap f (Scalar s)   = Scalar (f s)
    fmap f (Tensor ts)  = Tensor (fmap (fmap (fmap f)) ms)
```

Listing 5.9: The `Tensor` GADT and its instances.

The `Tensor` type is a so-called *generalised algebraic datatype*, short GADT, which means that the type of each constructor can be specified explicitly. There are three of such constructors:

[8] This function yields, for a nonempty rank, the “tail” of the rank after removing the first index.

- `ZeroTensor` yields the zero tensor for any valid `Rank` type. It is useful to introduce this special value, because it allows to short-circuit a lot of calculations: for example, when this constructor is encountered while calculating a sum, it can be ignored. For a product, on the other hand, we can automatically return the result `ZeroTensor` and need not inspect the second factor.
- `Scalar` as the base case for the recursive definition of a tensor wraps for the *empty* `Rank` type `'[]` a value of type `v`. It should be interpreted as the result of a "fully applied" tensor.
- `Tensor` is the recursive case. It is constrained to valid `Rank` types and is existentially quantified by the existence of a second `Rank` type, which is constrained to be the "tail" of the first rank. For this constructor, the value amounts to a list `[(Int, Tensor r' v)]` of index values and associated tensors with lesser rank `r'`. It should be understood as *partial* application of a tensor, by inserting all possible basis vectors/covectors in the "first available slot" of the tensor (which is, after all, a multilinear map) and collecting the resulting subtensors.

A few instances have been defined for the `Tensor` type. `Eq` and `Show` have generic implementations, while the `NFData` implementation just amounts to recursive evaluation. The `Functor` instance is the first manipulation we define for the `Tensor` type: it allows to apply functions directly to the values of the tensor, for example scalar multiplication as defined in Listing 5.10.

```
scalarMult :: forall r v. Num v => v -> Tensor r v -> Tensor r v
scalarMult s = fmap (s*)
```

Listing 5.10: Scalar multiplication leveraging the `Functor` instance.

Tensors of the same rank are added by merging the tensors, performing a recursive addition whenever an index is present in both summands. The requirement that both ranks coincide is encoded as constraint on the types. See Listing 5.11 for the implementation. In Listing 5.12, the addition of two tensors is demonstrated using the interactive repl[9] *ghci*. The tensor `delta_ab` represents the Kronecker delta δ^a_b, while `delta_ac` represents δ^a_c. Consequently, the expression `delta_ab &+ delta_ac` is ill-typed. On the other hand, `delta_ab &+ delta_ab` is well-typed and yields the expected result $2 \cdot \delta^a_b$.

[9]read-eval-print loop

```haskell
(&+) :: forall (r :: Rank) (r' :: Rank) v.
        ((r ~ r'), Num v) =>
        Tensor r v -> Tensor r' v -> Tensor r v
(&+) ZeroTensor t = t
(&+) t ZeroTensor = t
(&+) (Scalar s) (Scalar s')   = Scalar (s + s')
(&+) (Tensor xs) (Tensor xs') = Tensor xs''
  where
    xs'' = unionWith (&+) xs xs'

    unionWith :: (a -> a -> a) -> [(Int, a)] -> [(Int, a)] ->
                 [(Int, a)]
    unionWith f [] ys = ys
    unionWith f ys [] = ys
    unionWith f ys@((iy,vy):ys') zs@((iz,vz):zs') =
      case iy `compare` iz of
        LT -> (iy,vy) : unionWith f ys' zs
        EQ -> (iy,f vy vz) : unionWith f ys' zs'
        GT -> (iz,vz) : unionWith f ys zs'
```

Listing 5.11: Recursive addition of tensors.

```
> let delta_ab = delta :: Tensor '[ '( 'VSpace "ST" 4, 'ConCov ("a"
↪  ':| '[]) ("b" ':| '[]))] Int
> let delta_ac = delta :: Tensor '[ '( 'VSpace "ST" 4, 'ConCov ("a"
↪  ':| '[]) ("c" ':| '[]))] Int
> delta_ab + delta_ab == fmap (2*) delta_ab
True
> delta_ab + delta_ac
<interactive>:6:1: error:
    • Couldn't match type ‘"b"’ with ‘"c"’ arising from a use of ‘&+’
    • In the expression: delta_ab &+ delta_ac
      In an equation for ‘it’: it = delta_ab &+ delta_ac
```

Listing 5.12: Addition of tensors in the interactive repl *ghci*. The addition of two tensors with the same rank produces a result, while addition of tensors with different ranks yields a type error.

More intricate operations make use of the second feature from the `singletons` package, which are the singleton types that are generated for our lifted rank types. Singleton types are inhabited by only one value. As such, they are able to bridge the gap between compile time and run time, which are usually separate phases in Haskell. See [66] for more details. With singletons, we can implement e.g. typesafe tensor multiplication as sketched in Listing 5.13.

```
(&*) :: forall (r :: Rank) (r' :: Rank) (r'' :: Rank) v.
        (Num v, 'Just r'' ~ MergeR r r', SingI r, SingI r') =>
        Tensor r v -> Tensor r' v -> Tensor r'' v
(&*) = mult (sing :: Sing r) (sing :: Sing r')

mult :: forall (r :: Rank) (r' :: Rank) (r'' :: Rank) v.
        (Num v, 'Just r'' ~ MergeR r r') =>
        Sing r -> Sing r' -> Tensor r v -> Tensor r' v -> Tensor r'' v
mult _ _ (Scalar s) (Scalar t)   = Scalar (s*t)
mult _ _ (Scalar s) t@(Tensor _) = fmap (s*) t
mult _ _ t@(Tensor _) (Scalar s) = fmap (*s) t
mult sr sr' (Tensor ms) (Tensor ms') = _ -- omitted
mult sr sr' ZeroTensor ZeroTensor =
  case saneMergeRProof sr sr' of
    Sub Dict -> ZeroTensor
-- more ZeroTensor cases omitted
```

Listing 5.13: Typesafe tensor multiplication implemented using `singletons`. The ranks must satisfy a `SingI` constraint. With this constraint, the singleton values can be retrieved and passed to the implementation of the multiplication function.

The tensor multiplication makes us of a new function, `mergeR`, which takes two ranks and returns the rank of the tensor product—if the ranks allow to take this product. Lifted to the type level, this encodes the requirement that the ranks be compatible in the constraint `'Just r'' ~ MergeR r r'`. The `SingI` instances are used in order to retrieve the singleton values corresponding to the rank type. These are passed to the implementation function `mult`.

The simplest cases of the `mult` function are the cases matching on `ZeroTensor`—they just yield a `ZeroTensor`. However, it first has to be proven that the rank `r''` satisfies the constraint `Sane r'' ~ 'True`. This is the job of the pattern match on the result of `saneMergeRProof sr sr'`. For the time being, this proof (and all other proofs) are implemented by coercion, trusting in this case the function `mergeR`. In principle, it is possible to have Haskell check such proofs, although its capabilities in this regard are limited, as Haskell is not *total*.

The product of two `Scalar` values is equally as simple, as it yields the product of the wrapped numerical values. Multiplication of a `Scalar` with a `Tensor` and *vice versa* amounts to scalar multiplication introduced above. For the remaining product of two `Tensor` values, which is omitted above because it is quite lengthy[10], we inspect the foremost indices of both tensors. If the *left* tensor has a "lesser" index, we descend into the subtensors of the left tensor and multiply each subtensor with the right tensor. If the foremost index of the *right* tensor is "lesser", we proceed the other way around. Eventually, one of the base cases matching on a `Scalar` is reached.

Transpositions of indices and contractions are implemented similarly by descending into the relevant subtensors and manipulating the functional data structure appropriately. With the module `Math.Tensor.Basic`, the `safe-tensor` package exports all necessary basic tensors for setting up the perturbative equivariance equations 4.32–4.34 for metric and area metric theories—including Kronecker deltas, bundle intertwiners, Gotay-Marsden coefficients, Minkowski metrics, and Levi-Civita symbols.

Because handling the refined tensor type defined in the `Math.Tensor.Safe` module is at times quite unwieldy, the package also provides an *opaque* variant. Values of this opaque type are constructed from a tensor that has a generalised rank, but the rank cannot be extracted—it is *hidden*. The opaque type is exported by the `Math.Tensor` module, see Listing 5.14 for the definition.

```haskell
-- (...) skipping some language extensions
module Math.Tensor where
-- (...) skipping some imports

data T :: Type -> Type where
  T :: forall (r :: Rank) v. SingI r => Tensor r v -> Tensor r v
```

Listing 5.14: Opaque tensor type with existentially quantified rank.

With the opaque type `T`, tensor operations are always well-typed, but may not always yield a result because of rank mismatches. This is implemented utilising the `MonadError` type class, for example in the definition of tensor addition presented in Listing 5.15.

[10] See [6] for the complete implementation.

```haskell
(.+) :: (Eq v, Num v, MonadError String m) => T v -> T v -> m (T v)
(.+) o1 o2 =
  case o1 of
    T (t1 :: Tensor r1 v) ->
      case o2 of
        T (t2 :: Tensor r2 v) ->
          let sr1 = sing :: Sing r1
              sr2 = sing :: Sing r2
          in case sr1 %~ sr2 of
               Proved Refl ->
                 case sSane sr1 %~ STrue of
                   Proved Refl ->
                     return $ T (t1 &+ t2)
                   Disproved _ ->
                     throwError "Rank of summands is not sane."
               Disproved _ ->
                 throwError "Generalised tensor ranks do not match."
```

Listing 5.15: Addition of opaque tensors.

Finally, let us discuss how equivariance equations can be set up and solved with this package. There is a compatibility layer `safe-tensor-sparse-tensor-compat` [6], which uses the ansatz generation capabilities from `sparse-tensor` to provide ansätze for the construction of area metric gravity Lagrangians. The scalar type of these ansätze is not a plain numeric type but amounts to linear combinations of the ansatz coefficients. Data types and functions dealing with such linear combinations are provided by the module `Math.Tensor.LinearAlgebra`. Using the ansätze, the predefined basic tensors (such as Kronecker deltas, intertwiners, etc.), and the various tensor operations, all equivariance equations can be composed as given by Eqns. (4.32)–(4.34).

Having composed the equations, it is just a matter of evaluating all components in order to retrieve the linear system that determines the ansatz coefficients. With all basic tensors, intertwiners, and Gotay-Marsden coefficients being purely rational, the linear system itself contains only rational numbers. `safe-tensor` can also perform the last step, which is *solving* the linear system. This is done using fraction-free Gaussian elimination using 64-bit integers. Each solution is verified afterwards using rank computations by numerically stable singular value decomposition—eliminating worries that integer overflows may have invalidated the result.

6 Application: gravitational radiation from birefringent matter dynamics

So far, we have developed the general framework of covariant constructive gravity and derived a perturbative equivalent. A few examples illustrated the constructions, but the presentation focused on broad applicability to various geometries, without any specific bundle or matter theory in mind. In this chapter, we shift our focus and consider in depth the application of the framework to generalised linear electrodynamics, a birefringent generalisation of Maxwell electrodynamics introduced in Chap. 3. Applying the perturbative construction procedure to third order yields gravitational field equations to second order. We will carefully analyse a $3+1$ split for the linear part of this theory and restrict to a certain sector with, in a very specific sense, physically sane phenomenology. Afterwards, we solve the two-body problem to first order and obtain the orbits of a binary system in area metric gravity. Building up on this solution, the second order of the field equations is used to derive the emission of gravitational radiation from the binary system and the radiative loss, which causes spin-up of the system. The binary star subject to area metric gravity turns out to exhibit qualitatively new behaviour as compared to Einstein gravity, e.g. additional massive modes of gravitational radiation and a modification of Kepler's third law.

To a large extent, the work presented in this chapter has been published as Ref. [4]. The results on radiation loss are not part of this publication.

6.1 Construction of third-order area metric Lagrangians

The matter theory in question is generalised linear electrodynamics (GLED) as defined in Def. 3.3.1 with the Lagrangian density

$$L_{GLED} = \omega_G G^{abcd} F_{ab} F_{cd},$$

where we choose without loss of generality the scalar density

$$\omega_G = \left(\frac{1}{24} \epsilon_{abcd} G^{abcd} \right)^{-1}. \tag{6.1}$$

The principal polynomial of GLED is quartic and takes the form

$$\mathcal{P}_{GLED}(k) = -\frac{1}{24}\omega_G^2 \epsilon_{mnpq}\epsilon_{rstu} G^{mnra} G^{bpsc} G^{dqtu} k_a k_b k_c k_d.$$

As appropriate Lorentz invariant expansion point constructed from the Minkowski metric η*, we already determined in Example 4.1.1*

$$N^A = J^A_{abcd}(\eta^{ac}\eta^{bd} - \eta^{ad}\eta^{bc} + \epsilon^{abcd}). \tag{6.2}$$

Before solving the system of equivariance equations perturbatively around N*, let us reconsider the reduced power series ansatz* (4.18)*. In addition to dropping terms with a total number of derivatives that is odd or greater than 2, and dropping non-Lorentz invariant expansion coefficients, we can also discard the linear term* $a_A H^A$*. This term would yield a constant in the Euler-Lagrange equations, causing the flat expansion point* N *to no longer constitute a solution to the vacuum field equations. However, the perturbation ansatz stipulates that we perturb around a solution of the field equations. Since it is obvious that Eq.* (4.32) *implies from vanishing coefficients* a_A *that also the coefficient* a *vanishes, we readily drop both and make the further reduced ansatz*

$$\begin{aligned} L = {} & a_A{}^I H^A{}_I \\ & + a_{AB} H^A H^B + a_A{}^p{}_B{}^q H^A{}_p H^B{}_q + a_{AB}{}^I H^A H^B{}_I \\ & + a_{ABC} H^A H^B H^C + a_{AB}{}^p{}_C{}^q H^A H^B{}_p H^C{}_q + a_{ABC}{}^I H^A H^B H^C{}_I \\ & + \mathcal{O}(H^4). \end{aligned} \tag{6.3}$$

6.1.1 Solving axiom I

Step one of the perturbative construction algorithm consists in computing the Gotay-Marsden coefficients for the gravitational bundle. For area metric gravity, we found in Sect. 3.3

$$C^A{}_B{}^n{}_m = 4 I_B^{pqrn} J^A_{pqrm},$$

which followed from the general result (2.27) *for purely contravariant tensor bundles.*

Proceeding with step two, we need to construct a basis for the Lorentz invariant expansion coefficients

$$(a_A{}^I, a_{AB}, a_A{}^p{}_B{}^q, a_{AB}{}^I, a_{ABC}, a_{AB}{}^p{}_C{}^q, a_{ABC}{}^I) \tag{6.4}$$

in the ansatz (6.3)*. This task is solved using the Haskell library* `sparse-tensor` *[5] discussed in Chap. 5. The result is a basis of dimension 237, enumerated in full in Appendix A and summarised in Table 6.1. It should be emphasised that the requirement of Lorentz invariance, which is not a direct stipulation but follows via the equivariance*

coefficient	dimension	gravitational constants
$a_A{}^I$	3	$(e_{38}, \dots, e_{40})$
a_{AB}	6	$(e_1, \dots, e_6)$
$a_A{}^p{}_B{}^q$	15	$(e_7, \dots, e_{21})$
$a_{AB}{}^I$	16	$(e_{22}, \dots, e_{37})$
a_{ABC}	15	$(e_{41}, \dots, e_{55})$
$a_{AB}{}^p{}_C{}^q$	110	$(e_{56}, \dots, e_{165})$
$a_{ABC}{}^I$	72	$(e_{166}, \dots, e_{237})$

Table 6.1: Summary of the Lorentz invariant expansion coefficients for the area metric gravity ansatz (6.3) obtained from the Haskell library `sparse-tensor` [5]. The dimension is the number of linearly independent basis tensors returned from the computer program. Assigning labels from 1 to 237 to all basis tensors, an ansatz is represented by real numbers $e_1 \dots e_{237}$ using its unique basis decomposition. These numbers parameterise the gravitational theory and are thus referred to as *gravitational constants*. For a complete picture of the decomposition of ansätze using basis tensors, refer to Appendix A or the computer code in Ref. [7].

equations from a physically motivated assumption about the expansion point, drastically reduces the dimensionality of the ansatz from

$$210 + \frac{21 \cdot 22}{2} + 21 \cdot 210 + \frac{84 \cdot 85}{2} + \frac{21 \cdot 22 \cdot 23}{6} + \frac{21 \cdot 22}{2} \cdot 210 + 21 \cdot \frac{84 \cdot 85}{2} = 133672$$

to only 237. In principle, the correctness of the ansatz can be verified by showing that it is the most generic solution to the ansatz equations.[1] *All we have to show is that the dimensionality of the ansatz equals the corank of the linear system of ansatz equations. For the ansatz including third-order coefficients, the system is quite large—considering that the coefficient space is already of dimension 133672—such that, on standard hardware, the rank cannot be computed naïvely by storing the matrix in memory and using methods like singular value decomposition or fraction-free Gaussian elimination. It is rather easy, however, to use the aforementioned methods and work out the corank of the linear system determining the Lorentz invariant ansatz coefficients to* second *order, as the dimension of this ansatz space is only* $210 + \frac{21 \cdot 22}{2} + 21 \cdot 210 + \frac{84 \cdot 85}{2} = 8421$. *Confirming the number of obtained basis ansätze up to second order, the corank of the corresponding system is indeed 40.*

With the 237 ansatz coefficients at hand, solving the equivariance equations as required

[1] Eq. (4.15) and similar.

for step five is only a matter of inserting the ansatz in the system and its first two prolongations as displayed in Eqns. (4.32)–(4.34)*, extracting a system of linear equations for the gravitational constants, and solving this system. This task is again performed using efficient computer algebra, implemented in the Haskell library* `safe-tensor`*, which is introduced in Chap. 5. The procedure is roughly as follows: a compatibility layer with* `sparse-tensor` *is used in order to construct the ansatz tensors and make them available as* `Tensor` *types with generalised rank (see Sect. 5.2). Together with predefined tensors like Kronecker deltas, intertwiners, Gotay-Marsden coefficients, or the Minkowski metric, the ansatz tensors are used in order to construct the (prolonged) equivariance equations evaluated at N. Each tensorial equation is a value of type* `Tensor` *and, as such, can be evaluated into a list of its components. Every component is a linear equation for the 237 gravitational constants. Collecting all components for all tensorial equations, we obtain a matrix representing the linear system for the constants* $e_1 \ldots e_{237}$*. The system is small enough to be brought into reduced row echelon form applying fraction-free Gaussian elimination and backward substitution using 64-bit integers*[2]*, which yields a solution that parameterises the constants with a few remaining indeterminate gravitational constants. As an example for the process, let us walk through the solution for the linear expansion coefficient* ${a_A}^I$*.*

Example 6.1.1 (solution of the equivariance equations to first order)**.** *Having set* $a_A = 0$*, the remaining expansion coefficient for the linear order is* ${a_A}^I$*, which is determined in part by the second unprolonged equation* (4.32)*. A suitable basis for this coefficient is*

$$ {a_A}^I = J_A^{abcd} J^I_{pq} [e_1 \cdot \eta_{ac}\eta_{bd}\eta^{pq} + e_2 \cdot \eta_{ac}\delta^p_b\delta^q_d + e_3 \cdot \epsilon_{abcd}\eta^{pq}] \tag{6.5} $$

with three gravitational constants e_1, e_2, e_3*. Inserting this ansatz into the unprolonged equation*

$$ 0 = {a_A}^I {C^A}_B{}^{(n}{}_m J^{pq)}_I N^B =: T^{npq}_m \tag{6.6} $$

yields a tensorial equation $0 = T^{npq}_m$ *with 256 components. Each component is of the form*

$$ 0 = c_1 \cdot e_1 + c_2 \cdot e_2 + c_3 \cdot e_3. \tag{6.7} $$

The collection of all components is a system of 256 linear equations for three variables. A lot of these equations are redundant, because they are trivial or linearly dependent. A naïve reduction by eliminating trivial equations and choosing only one representative for equations that are multiples of each other already reduces the system to the single equation

$$ 0 = 2e_1 + e_2 + 4e_3. \tag{6.8} $$

Setting e.g. $e_2 = -2e_1 - 4e_3$ *solves the equivariance equation for the coefficient* ${a_A}^I$*,*

[2]Exploiting the observation we made earlier that, using intertwiners with purely rational components, all coefficients in the system remain rational.

leaving it parameterised by two gravitational constants e_1 *and* e_3*.*

Applied to the whole system of equivariance equations, we obtain a parameterisation of the solution (displayed for the first two orders in Appendix B) by 50 independent gravitational constants. A subset of 16 constants governs linearised *area metric gravity via the quadratic Lagrangian density, from which—as we will encounter later—only 10 independent linear combinations play a rôle for the Euler-Lagrange equations. The procedure outlined here is implemented in Haskell using the aforementioned libraries. Source code and results are published as Ref. [7].*

6.1.2 Solving axiom II

The pedestrian approach towards implementing causal compatibility of the just constructed gravitational theory with GLED is to carefully execute steps 6–12 of the perturbative construction algorithm. This way, we obtain an approximation of the area metric gravity principal polynomial and have to match the causal structure with a first-order expansion of the GLED principal polynomial. While entirely feasible, this approach is less illustrative than the constructive *approach we employ instead. The underlying realisation behind this technique is that the diffeomorphism invariance of the gravitational theory dramatically restricts the possible principal polynomials. In fact, we will see that for third-order area metric Lagrangians, the admissible principal polynomials are already causally compatible with the corresponding expansion of the GLED polynomial.* There is no causality mismatch left to be fixed.

To this end, recall the GLED polynomial (3.21)*, which using the scalar density* (6.1) *assumes the form*

$$\mathcal{P}_{GLED}(k) = -\frac{1}{\frac{1}{24}(\epsilon_{abcd}G^{abcd})^2}\epsilon_{mnpq}\epsilon_{rstu}G^{mnra}G^{bpsc}G^{dqtu}k_a k_b k_c k_d. \tag{6.9}$$

Expanding this expression to linear order in the perturbation yields

$$\begin{aligned}\mathcal{P}_{GLED}(k) &= \left\{\left[1 - \frac{1}{24}\epsilon(H)\right]\eta(k,k) + \frac{1}{2}H(k,k)\right\}^2 + \mathcal{O}(H^2)\\ &= [P^{(\leq 1)}_{GLED}]^2 + \mathcal{O}(H^2),\end{aligned} \tag{6.10}$$

where the abbreviations

$$\epsilon(H) = \epsilon_{abcd}H^{abcd} \quad \textit{and} \quad H(k,k) = \eta_{ac}H^{abcd}k_b k_d \tag{6.11}$$

have been introduced. In the following, we will also make use of the contraction

$$\eta(H) = \eta_{ac}\eta_{bd}H^{abcd}. \tag{6.12}$$

Up to first order, we find that the GLED polynomial factors into the square of a metric polynomial $P^{(\leq 1)}_{GLED}$*. This has a remarkable consequence: for weak gravitational fields, where the approximation to first order is sufficiently good, the physics of point particles adhering to GLED dynamics is indistinguishable from the Maxwellian setting with a metric perturbation* h *by virtue of the identification*

$$h^{ab} = \left[1 - \frac{1}{24}\epsilon(H)\right]\eta^{ab} + \frac{1}{2}\eta_{cd}H^{acbd} = (P^{(\leq 1)}_{GLED})^{ab}. \tag{6.13}$$

This effect only holds in the limit of geometric optics—the GLED field equations do not *reduce to Maxwell equations with a metric perturbation. Consequently, even to first order in the area metric perturbation, nonmetric effects can be observed. An in-depth study of classical and quantum electrodynamics on weakly birefringent backgrounds based on exactly this realisation has been conducted in Ref. [30].*

We will now proceed to show that the possible principal polynomials arising from third-order area metric gravity Lagrangians as constructed in the previous section are only mildly more general than the effectively quadratic first-order GLED polynomial (6.10)*. This issue is approached by first considering the corresponding Euler-Lagrange equations.*

Proposition 6.1.2. *Let* $E \xrightarrow{\pi} M$ *be a sub-bundle of some tensor bundle over* M*. Consider a Lagrangian field theory on* $J^2\pi$ *that is degenerate in the sense that the Euler-Lagrange equations are of second derivative order, i.e. are also defined on* $J^2\pi$*. If the Lagrangian field theory is diffeomorphism invariant with respect to the diffeomorphism action on the second jet bundle, it follows that the Euler-Lagrange equations are diffeomorphism equivariant. In particular, a local representation of the Euler-Lagrange equations*

$$E_A = L_{:A} - D_p L_{:A}{}^p + I^{pq}_I D_p D_q L_{:A}{}^I \tag{6.14}$$

exhibits the transformation behaviour

$$\delta_\xi E_A = -E_A \xi^m_{,m} - E_B C^B{}_A{}^n{}_m \xi^m_{,n}, \tag{6.15}$$

where $C^B{}_A{}^n{}_m$ *are the Gotay-Marsden coefficients corresponding to the field bundle. In other words, the Euler-Lagrange equations transform as tensor density of weight 1.*

Proof. *The claim follows from expanding the left-hand side of Eq.* (6.15) *as*

$$\delta_\xi E_A = E_{A:B}\delta_\xi u^B + E_{A:B}{}^p \delta_\xi u^B{}_p + E_{A:B}{}^I \delta_\xi u^B{}_I, \tag{6.16}$$

then replacing E_A *with its definition* (6.14) *and simplifying the result using the equivariance of the Lagrangian density* L*. Rather than performing this tedious calculation, we can alternatively consider the geometric definition* (2.6) *of the Euler-Lagrange form and deduce that it must transform covariantly (for a contravariant tensor bundle) with density weight of one, i.e. according to the local expression* (6.15)*.* □

This transformation behaviour carries over to the principal symbol of the Euler-Lagrange equations, which is also a tensor density of weight 1.

Proposition 6.1.3. *Consider the same Lagrangian field theory as in Prop. 6.1.2. The principal symbol*

$$T_{AB}(k) = E_{A:B}{}^{I} J_I^{pq} k_p k_q \tag{6.17}$$

of the corresponding Euler-Lagrange equations E_A*, where* $k \in T^*M$ *denotes a covector, transforms as a tensor density of weight one, i.e. an infinitesimal diffeomorphism acts as*

$$\delta_\xi T_{AB}(k) = -T_{AB}(k)\xi^m_{,m} - T_{CB}(k)C^C{}_A{}^n{}_m \xi^m_{,n} - T_{AC}(k)C^C{}_B{}^n{}_m \xi^m_{,n}. \tag{6.18}$$

Proof. The idea of the proof is as before: we insert the just proven transformation behaviour of the Euler-Lagrange equations E_A *and of covectors* k*, which is*

$$\delta_\xi k_a = -k_m \xi^m_{,a}, \tag{6.19}$$

into the transformation

$$\begin{aligned}\delta_\xi T_{AB}(k) = {} & (T_{AB}(k))_{:C}\delta_\xi u^C + (T_{AB}(k))_{:C}{}^{p}\delta_\xi u^C{}_p + (T_{AB}(k))_{:C}{}^{I}\delta_\xi u^C{}_I \\ & + \frac{\partial T_{AB}}{\partial k_a}(k)\delta_\xi k_a.\end{aligned} \tag{6.20}$$

This time, the calculation is rather trivial and the claim (6.17) *follows almost immediately.* □

We are now in a position to prove the first part of the central result, which is that the principal polynomial of area metric gravity is a scalar density. Note that we restrict our considerations to the case of a principal symbol that is independent of the derivatives of the derivatives of the gravitational field, as otherwise the causality could not be matched anyway (see Sect. 4.4).

Theorem 6.1.4. *Let* π *be the area metric bundle. Consider a degenerate Lagrangian field theory with a principal symbol that is independent of the derivatives of the area metric field. The principal polynomial* $\mathcal{P}(k)$ *corresponding to the symbol, as defined in*

Def. 2.4.1 is a scalar density of weight 57, i.e. transforms locally under infinitesimal spacetime diffeomorphisms as

$$\delta_\xi \mathcal{P}(k) = -57 \cdot \mathcal{P}(k)\xi^m_{,m}. \tag{6.21}$$

Proof. *From the transformation behaviour of area metric tensors and covectors, it follows that an infinitesimal diffeomorphism acts on generators $\chi^A_{(i)}(k) = C^A{}_B{}^n{}_i u^B k_n$ of gauge transforms as*

$$\delta_\xi \chi^A_{(i)}(k) = C^A{}_B{}^n{}_m \chi^B_{(i)}(k)\xi^m_{,n} - \chi^A_{(m)}(k)\xi^m_{,i}. \tag{6.22}$$

Now calculating the transformation behaviour of the principal polynomial numerator $Q^{A_1\dots A_4 B_1\dots B_4}$ (dropping the covector k from the notation) we obtain

$$\begin{aligned}
\delta_\xi Q^{A_1\dots A_4 B_1\dots B_4} &= \delta_\xi \frac{\partial^4 \det T}{\partial T_{A_1B_1}\dots \partial T_{A_4B_4}} \\
&= \delta_\xi \left[\frac{4}{21!}\epsilon^{A_1\dots A_{21}}\epsilon^{B_1\dots B_{21}} T_{A_5B_5}\dots T_{A_{21}B_{21}}\right] \\
&= \frac{1}{17!}\epsilon^{A_1\dots A_{21}}\epsilon^{B_1\dots B_{21}}[\delta_\xi T_{A_5B_5}]T_{A_6B_6}\dots T_{A_{21}B_{21}} \\
&= -17\cdot \delta_\xi Q^{A_1\dots A_4B_1\dots B_4}\xi^m_{,m} \\
&\quad - \frac{17}{17!}\epsilon^{A_1\dots A_{21}} C^A{}_{A_5}{}^n{}_m \epsilon^{B_1\dots B_{21}} T_{AB_5}\dots T_{A_{21}B_{21}}\xi^m_{,n} \\
&\quad - \frac{17}{17!}\epsilon^{A_1\dots A_{21}}\epsilon^{B_1\dots B_{21}} C^B{}_{B_5}{}^n{}_m T_{A_5B}\dots T_{A_{21}B_{21}}\xi^m_{,n}.
\end{aligned} \tag{6.23}$$

This is further simplified using the identity $0 = \epsilon^{[A_1\dots A_{21}}X^{A]\dots}$, from which we derive after a few index relabellings

$$\begin{aligned}
0 &= 22\cdot \epsilon^{[A_1\dots A_{21}} C^{A]}{}_{A_5}{}^n{}_m \epsilon^{B_1\dots B_21} T_{AB_5}T_{A_6B_6}\dots T_{A_{21}B_{21}}\xi^m_{,n} \\
&= 17\cdot \epsilon^{A_1\dots A_{21}} C^A{}_{A_5}{}^n{}_m \epsilon^{B_1\dots B_{21}} T_{AB_5}T_{A_6B_6}\dots T_{A_{21}B_{21}}\xi^m_{,n} \\
&\quad - C^A{}_A{}^n{}_m \epsilon^{A_1\dots A_{21}}\epsilon^{B_1\dots B_{21}} T_{A_5B_5}\dots T_{A_{21}B_{21}}\xi^m_{,n} \\
&\quad + \epsilon^{AA_2A_3A_4\dots A_{21}} C^{A_1}{}_A{}^n{}_m \epsilon^{B_1\dots B_{21}} T_{A_5B_5}T_{A_6B_6}\dots T_{A_{21}B_{21}}\xi^m_{,n} \\
&\quad + \epsilon^{A_1AA_3A_4\dots A_{21}} C^{A_2}{}_A{}^n{}_m \epsilon^{B_1\dots B_{21}} T_{A_5B_5}T_{A_6B_6}\dots T_{A_{21}B_{21}}\xi^m_{,n} \\
&\quad + \epsilon^{A_1A_2AA_4\dots A_{21}} C^{A_3}{}_A{}^n{}_m \epsilon^{B_1\dots B_{21}} T_{A_5B_5}T_{A_6B_6}\dots T_{A_{21}B_{21}}\xi^m_{,n} \\
&\quad + \epsilon^{A_1A_2A_3A\dots A_{21}} C^{A_4}{}_A{}^n{}_m \epsilon^{B_1\dots B_{21}} T_{A_5B_5}T_{A_6B_6}\dots T_{A_{21}B_{21}}\xi^m_{,n}.
\end{aligned} \tag{6.24}$$

Applying the same technique to the index set $[B_1 \dots B_{21}B]$ and carrying out the contraction $C^A{}_A{}^n{}_m = 21\cdot\delta^n_m$, the identity can be applied to the second and third terms in Eq. (6.23)*,* *such that we finally obtain*

$$
\begin{aligned}
\delta_\xi Q^{A_1\dots A_4 B_1\dots B_4} = &-59\cdot Q^{A_1\dots A_4 B_1\dots B_4}\xi^m_{,m}\\
&+C^{A_1}{}_{A}{}^{n}{}_{m}Q^{AA_2A_3A_4B_1\dots B_4}\xi^m_{,n}+C^{A_2}{}_{A}{}^{n}{}_{m}Q^{A_1AA_3A_4B_1\dots B_4}\xi^m_{,n}\\
&+C^{A_3}{}_{A}{}^{n}{}_{m}Q^{A_1A_2AA_4B_1\dots B_4}\xi^m_{,n}+C^{A_4}{}_{A}{}^{n}{}_{m}Q^{A_1A_2A_3AB_1\dots B_4}\xi^m_{,n}\\
&+C^{B_1}{}_{B}{}^{n}{}_{m}Q^{A_1\dots A_4BB_2B_3B_4}\xi^m_{,n}+C^{B_2}{}_{B}{}^{n}{}_{m}Q^{A_1\dots A_4B_1BB_3B_4}\xi^m_{,n}\\
&+C^{B_3}{}_{B}{}^{n}{}_{m}Q^{A_1\dots A_4B_1B_2BB_4}\xi^m_{,n}+C^{B_4}{}_{B}{}^{n}{}_{m}Q^{A_1\dots A_4B_1B_2B_3B}\xi^m_{,n}.
\end{aligned}
\tag{6.25}
$$

A similar calculation, this time using the identity $0=\epsilon^{[a_1a_2a_3a_4}X^{a]\dots}$, *yields the transformation of the denominator* $f^{A_1\dots A_4B_1\dots B_4}$,

$$
\begin{aligned}
\delta_\xi f^{A_1\dots A_4 B_1\dots B_4} &= \delta_\xi\left[\epsilon^{a_1\dots a_4}\epsilon^{b_1\dots b_4}\prod_{i=1}^{4}\chi^{A_i}_{(a_i)}\chi^{B_i}_{(b_i)}\right]\\
&=-2\cdot f^{A_1\dots A_4 B_1\dots B_4}\xi^m_{,m}\\
&\quad+C^{A_1}{}_{A}{}^{n}{}_{m}f^{AA_2A_3A_4B_1\dots B_4}\xi^m_{,n}+C^{A_2}{}_{A}{}^{n}{}_{m}f^{A_1AA_3A_4B_1\dots B_4}\xi^m_{,n}\\
&\quad+C^{A_3}{}_{A}{}^{n}{}_{m}f^{A_1A_2AA_4B_1\dots B_4}\xi^m_{,n}+C^{A_4}{}_{A}{}^{n}{}_{m}f^{A_1A_2A_3AB_1\dots B_4}\xi^m_{,n}\\
&\quad+C^{B_1}{}_{B}{}^{n}{}_{m}f^{A_1\dots A_4BB_2B_3B_4}\xi^m_{,n}+C^{B_2}{}_{B}{}^{n}{}_{m}f^{A_1\dots A_4B_1BB_3B_4}\xi^m_{,n}\\
&\quad+C^{B_3}{}_{B}{}^{n}{}_{m}f^{A_1\dots A_4B_1B_2BB_4}\xi^m_{,n}+C^{B_4}{}_{B}{}^{n}{}_{m}f^{A_1\dots A_4B_1B_2B_3B}\xi^m_{,n}.
\end{aligned}
\tag{6.26}
$$

Putting both numerator and denominator together proves the claim

$$\delta_\xi\mathcal{P}(k) = -57\cdot\mathcal{P}(k)\xi^m_{,m}. \tag{6.27}$$

□

An equivalent formulation of the fact that $\mathcal{P}(k)$ *is a density of weight 57 is that the symmetric coefficients*[3] $P^{a_1\dots a_{26}}$ *constitute a tensor density of the same weight, i.e. live on the bundle of symmetric tensor densities of contravariant rank 26 with weight 57. For this geometry, the equivariance equations on the "zeroth jet bundle" (since the polynomial must not depend on derivatives of the geometry) are*

$$
\begin{aligned}
P^{a_1\dots a_{26}}{}_{,m} &= 0,\\
P^{a_1\dots a_{26}}{}_{:A}C^{A}{}_{B}{}^{n}{}_{m}u^B &= -57\cdot P^{a_1\dots a_{26}}\delta^n_m + 26\cdot P^{n(a_1\dots a_{25}}\delta^{a_{26})}_m.
\end{aligned}
\tag{6.28}
$$

The second part of the central result follows from these equations. All we have to do is construct the perturbative solution to first order and see that it is impossible not *to have*

[3] Recall that the principal polynomial for area metric gravity is homogeneous and of degree 26.

the causality match GLED causality to the same order.

Theorem 6.1.5. *Let $\mathcal{P}_{area}$ be the principal polynomial of area metric gravity as considered in Thm. 6.1.4. To first order in the expansion $G = N + H$ of the area metric field, where N is the Lorentz invariant expansion point* (6.2)*, $\mathcal{P}_{area}$ is equivalent to the GLED principal polynomial $\mathcal{P}_{GLED}$ in the sense that*

$$\mathcal{P}_{area} = [\omega P^{(\leq 1)}_{GLED}]^{13} + \mathcal{O}(H^2), \tag{6.29}$$

where ω denotes a density of weight $\frac{57}{13}$ on the area metric bundle and $P^{(\leq 1)}_{GLED}$ is the expansion of the GLED polynomial to first order. In particular, to first order in the perturbation, both principal polynomials describe the same null surfaces and hyperbolicity cones.

Proof. *Knowing that the principal polynomial of area metric gravity transforms as a density of weight 57, we can construct possible candidates by solving the equivariance equations* (6.28)*. To this end, we make the ansatz*

$$\begin{aligned}\mathcal{P}_{area}(k) = \eta(k,k)^{13} &\\ + A \cdot \epsilon(H)\eta(k,k)^{13} + B \cdot \eta(H)\eta(k,k)^{13} + C \cdot H(k,k)\eta(k,k)^{12} &\\ + \mathcal{O}(H^2). &\end{aligned} \tag{6.30}$$

An overall factor would be irrelevant, so it has already been dropped when setting the coefficient of the constant term to 1. The generality of the ansatz can, as always, be verified by calculating the corank of the ansatz equations, which will yield 4—the number of ansatz tensors in Eq. (6.30)*. Evaluating the equivariance equation at the ansatz and contracting the 26 symmetric indices with covector components, for the sake of a cleaner presentation, yields an equation where we can cancel a common factor of $\eta(k,k)^{12}$. The remaining equation has a covariant and a contravariant spacetime index, such that a decomposition into the trace*

$$0 = [24A + 12B + 3C + 57 - \frac{13}{2}]\delta^n_m \tag{6.31}$$

and the tracefree part

$$0 = [4C - 26][\eta^{na}\delta^b_m k_a k_b - \frac{1}{4}\delta^n_m \eta(k,k)] \tag{6.32}$$

lends itself for a first attempt to retrieve scalar equations from the system. As it turns *out, these two equations are already maximal. Parameterising the solution with B yields*

$$\begin{aligned}\mathcal{P}_{area}(k) &= \eta(k,k)^{13} \\ &\quad - \frac{35}{12}\epsilon(H)\eta(k,k)^{13} + B(\eta(H) - \frac{1}{2}\epsilon(H))\eta(k)^{13} + \frac{13}{2}H(k,k)\eta(k,k)^{12} \\ &\quad + \mathcal{O}(H^2) \\ &= \left\{\left[1 - \frac{35}{12\cdot 13}\epsilon(H) + \frac{B}{13}\left(\eta(H) - \frac{1}{2}\epsilon(H)\right)\right]\eta(k,k) + \frac{1}{2}H(k,k)\right\}^{13} \\ &\quad + \mathcal{O}(H^2),\end{aligned} \tag{6.33}$$

where for the last equality we completed the thirteenth power as

$$1 + \epsilon = \left(1 + \frac{1}{13}\epsilon\right)^{13} + \mathcal{O}(\epsilon^2). \tag{6.34}$$

In order to relate the quadratic polynomial that determines the first order of $\mathcal{P}_{area}(k)$ to $\mathcal{P}^{(\leq 1)}_{GLED}$ via a scalar density, as claimed in Eq. (6.29)*, we consider the equivariance equations*

$$\begin{aligned}\omega_{,m} &= 0, \\ \omega_{:A}C^{A}{}_{B}{}^{n}{}_{m}u^{B} &= -\frac{57}{13}\,\omega\delta^{n}_{m}\end{aligned} \tag{6.35}$$

for such a density ω of weight $\frac{57}{13}$. This time, the Lorentz invariant ansatz is just

$$\omega = 1 + A\cdot\epsilon(H) + B\cdot\eta(H) + \mathcal{O}(H^2) \tag{6.36}$$

and reduces the equivariance equations to the single condition

$$24A + 12B = -\frac{57}{13}, \tag{6.37}$$

such that the most general scalar density of weight $\frac{57}{13}$ is to first order given by

$$\omega = 1 - \frac{57}{13\cdot 24}\epsilon(H) + B[\eta(H) - \frac{1}{2}\epsilon(H)] + \mathcal{O}(H^2). \tag{6.38}$$

The result now follows from multiplication of $P^{(\leq 1)}_{area}$ with ω, which yields exactly the area metric gravity polynomial (6.33)*. To first order, the principal polynomial of area metric gravity is determined by a quadratic polynomial which reduces to the quadratic first-order GLED polynomial* up to a factor. *Because such an overall factor is irrelevant for vanishing sets and hyperbolicity cones, the polynomials must be considered identical for the purpose of comparing their causal structure.* □

Having fixed the causality of third-order perturbative area metric gravity—by proof, rather than by explicit calculation—the construction procedure up to this order is completed. Third-order area metric gravity[4] *is determined by the ansatz* (6.3) *which is constructed from the Lorentz-invariant basis tensors* (A.1)–(A.7). *From the 237 gravitational constants—the coefficients in the basis expansion—50 constants turn out to be independent, 10 of which govern the linearised field equations. The relations between gravitational constants are collected in Appendix B. In the following, we will examine the linear theory, which forms the basis for predicting first-order and, later on, second-order effects of area metric gravity.*

6.1.3 3+1 decomposition

As remarked in Sect. 3.3, the expansion point should be an area metric of a certain subclass in order to guarantee hyperbolicity of the GLED principal polynomial—which encompasses, by the previously proven result, hyperbolicity of the second-order area metric gravity field equations. Indeed, N is of subclass I according to the classification in Ref. [68]. Thus, we can turn to a $3+1$ formulation, starting with the definition of a slicing.

Definition 6.1.6 (slicing)**.** *Consider a spacetime manifold M of dimension four. Any diffeomorphism*

$$\phi\colon \Sigma \times \mathbb{R} \to M \tag{6.39}$$

from a three-dimensional spatial manifold Σ and the reals to M is called a slicing of M.

Such a slicing always exists, as we only consider matter theories that have a well-defined initial value problem. It is, however, not unique: any diffeomorphism $\psi\colon M \to M$ yields a new slicing $\tilde{\phi} = \psi \circ \phi$. Since the spatial manifold is of dimension three and not four, working with slicings comes with new indices running from one to three. These will be denoted with lowercase Greek letters, while lowercase Latin letters represent spacetime indices running from zero to three.

Every tangent space $T_{\phi(s,\lambda)}M$ has a holonomic basis

$$\frac{\partial}{\partial x^a} = \left(\frac{\partial}{\partial t}, \frac{\partial}{\partial x^\alpha}\right), \tag{6.40}$$

where the vectors on the right are understood as pushforwards of holonomic basis vectors on $T_s\Sigma$ and $T_\lambda\mathbb{R}$. The same construction yields a holonomic basis

$$dx^a = (dt, dx^\alpha) \tag{6.41}$$

[4]With second-order field equations and, therefore, a principal polynomial of first order.

for the cotangent spaces $T^*_{\phi(s,\lambda)}M$. The bundle π_{area}, constructed as subbundle of T^4_0M, inherits a $3+1$ split from the decomposition of tangent and cotangent spaces, and so does the second jet bundle of π_{area}.

Based on a slicing, we now introduce an observer definition[5] for arbitrary tensor theories. Only the principal polynomial is needed for this notion.

Definition 6.1.7 (observer frame, lapse and shift)**.** *Let P be the principal polynomial of a field theory on a tensor bundle. An observer frame consists of a nonholonomic frame*

$$(T, e_\alpha = \frac{\partial}{\partial x^\alpha}) \tag{6.42}$$

and a dual coframe

$$(n = \lambda \cdot dt, \epsilon^\alpha), \tag{6.43}$$

where the temporal direction and codirection must satisfy[6]

$$P(n) = 1 \quad \textit{and} \quad T = \frac{1}{\deg P}\frac{DP(n)}{P(n)}. \tag{6.44}$$

In the following, we assume $P(n) = 1$ to be solved by choosing an appropriate basis on $T\mathbb{R}$ and setting $\lambda = 1$.

The holonomic time direction $\frac{\partial}{\partial t}$ decomposes in the observer frame as

$$\frac{\partial}{\partial t} = NT + N^\alpha \frac{\partial}{\partial x^\alpha} \tag{6.45}$$

with the lapse N and shift N^α.

Essential for the $3+1$ split is the parameterisation of the geometry with quantities an observer can measure in her frame, as well as lapse and shift. For example, using the completeness relation

$$\mathrm{id} = T \otimes n + e_\alpha \otimes \epsilon^\alpha = \frac{1}{N}\frac{\partial}{\partial t} \otimes n - \frac{1}{N} N^\alpha e_\alpha \otimes n + e_\alpha \otimes \epsilon^\alpha, \tag{6.46}$$

a vector field v decomposes as

$$v = v \circ \mathrm{id} = v(n)\, T + v(\epsilon^\alpha)\, e_\alpha. \tag{6.47}$$

[5] see also Ref. [23]

[6] DP denotes the formal derivative of P as a polynomial.

The holonomic components are thus determined by lapse N, shift N^α, and the observer quantities $v(n)$ and $v(\epsilon^\alpha)$ as

$$v(dt) = v(n) \quad \textit{and} \quad v(dx^\alpha) = -\frac{1}{N} N^\alpha v(n) + v(\epsilon^\alpha). \tag{6.48}$$

Obviously, the information contained in N, N^α, $v(n)$, and $v(\epsilon^\alpha)$ is redundant—four holonomic components are represented using 8 observer quantities. This is where the frame conditions (6.44) *come into play: consider the decomposition of the area metric field into [23]*

$$G(dt, dx^\alpha, dt, dx^\beta) = \frac{1}{N^2} G(n, \epsilon^\alpha, n, \epsilon^\beta), \tag{6.49}$$

$$G(dt, dx^\alpha, dx^\beta, dx^\gamma) = -\frac{2}{N^2} G(n, \epsilon^\alpha, n, \epsilon^{[\gamma}) N^{\beta]} + \frac{1}{N} G(n, \epsilon^\alpha, \epsilon^\beta, \epsilon^\gamma), \tag{6.50}$$

$$\begin{aligned} G(dx^\alpha, dx^\beta, dx^\gamma, dx^\delta) = {} & \frac{4}{N^2} N^{[\alpha} G(n, \epsilon^{\beta]}, n, \epsilon^{[\delta}) N^{\gamma]} + \frac{2}{N} N^{[\alpha} G(n, \epsilon^{\beta]}, \epsilon^\gamma, \epsilon^\delta) \\ & + \frac{2}{N} N^{[\gamma} G(n, \epsilon^{\delta]}, \epsilon^\alpha, \epsilon^\beta) + G(\epsilon^\alpha, \epsilon^\beta, \epsilon^\gamma, \epsilon^\delta). \end{aligned} \tag{6.51}$$

So far, the situation seems to be similar—21 area metric components are determined by 21 observer quantities plus lapse and shift. The difference to the decomposition of a vector is that the frame conditions (6.44) *depend—via the principal polynomial—on the area metric, which introduces dependencies among area metric, lapse, and shift. To formulate these conditions, it is more convenient to redefine the observer quantities as [23]*

$$\begin{aligned} \hat{G}^{\alpha\beta} &= -G(n, \epsilon^\alpha, n, \epsilon^\beta), \\ \hat{G}^\alpha{}_\beta &= \frac{1}{2} (\omega_{\hat{G}})^{-1} \epsilon_{\beta\mu\nu} G(n, \epsilon^\alpha, \epsilon^\mu, \epsilon^\nu) - \delta^\alpha{}_\beta, \\ \hat{G}_{\alpha\beta} &= \frac{1}{4} (\omega_{\hat{G}})^{-2} \epsilon_{\alpha\mu\nu} \epsilon_{\beta\rho\sigma} G(\epsilon^\mu, \epsilon^\nu, \epsilon^\rho, \epsilon^\sigma), \end{aligned} \tag{6.52}$$

with the spatial density

$$\omega_{\hat{G}} = \sqrt{\det \hat{G}^{\cdot\cdot}}. \tag{6.53}$$

By definition, $\hat{G}^{\alpha\beta}$ and $\hat{G}_{\alpha\beta}$ are symmetric. The frame conditions (6.44) *translate into the two additional properties [23]*

$$0 = \hat{G}^\alpha{}_\alpha \quad \textit{and} \quad 0 = \hat{G}^{\mu[\alpha} \hat{G}^{\beta]}{}_\mu, \tag{6.54}$$

i.e. $\hat{G}^\alpha{}_\beta$ is tracefree and symmetric with respect to $\hat{G}^{\alpha\beta}$. In total, lapse and shift and the observer quantities $\hat{G}^{\alpha\beta}$, $\hat{G}^\alpha{}_\beta$, $\hat{G}_{\alpha\beta}$ have $1 + 3 + 6 + 5 + 6 = 21$ degrees of freedom, such that they are in one-to-one correspondence with the area metric field G^{abcd}. Note the similarity to the $3 + 1$ decomposition of the metric tensor g^{ab} into shift N^α, lapse N, *and spatial metric* $\hat{g}^{\alpha\beta}$*—the purely temporal and spatiotemporal components of the metric are parameterised only by shift and lapse, due to the frame conditions* (6.44)*.*

Around the perturbation point N*, the area metric observer quantities expand as*

$$\begin{aligned} N &= 1 + A, \\ N^\alpha &= b^\alpha, \\ \hat{G}^{\alpha\beta} &= \gamma^{\alpha\beta} + h^{\alpha\beta}, \\ \hat{G}^\alpha{}_\beta &= k^\alpha{}_\beta, \\ \hat{G}_{\alpha\beta} &= \gamma_{\alpha\beta} + l_{\alpha\beta}. \end{aligned} \tag{6.55}$$

With γ *we denote the positive-definite spatial part of the Minkowski metric, i.e.* $\eta^{\alpha\beta} = -\gamma^{\alpha\beta}$*. From now on, spatial indices are raised and lowered at will using* γ *and its inverse. The perturbations* A*,* b*,* h*,* k*, and* l *are again in one-to-one correspondence with the 21 perturbations* H*, by virtue of*

$$\begin{aligned} H^{0\alpha0\beta} &= 2A\gamma^{\alpha\beta} - h^{\alpha\beta}, \\ H^{0\alpha\beta\gamma} &= -A\epsilon^{\alpha\beta\gamma} + 2b^{[\beta}\gamma^{\gamma]\alpha} + \frac{1}{2}\epsilon^{\alpha\beta\gamma}\gamma_{\mu\nu}h^{\mu\nu} + \epsilon^{\mu\beta\gamma}k^\alpha{}_\mu, \\ H^{\alpha\beta\gamma\delta} &= 2\gamma^{\alpha[\gamma}\gamma^{\delta]\beta}\gamma_{\mu\nu}h^{\mu\nu} + \epsilon^{\mu\alpha\beta}\epsilon^{\nu\gamma\delta}l_{\mu\nu}. \end{aligned} \tag{6.56}$$

A set of perturbations that is more convenient to work with is given by the linear combinations

$$u^{\alpha\beta} = h^{\alpha\beta} - l^{\alpha\beta}, \quad v^{\alpha\beta} = h^{\alpha\beta} + l^{\alpha\beta}, \quad w^{\alpha\beta} = 2k^{\alpha\beta}. \tag{6.57}$$

Using these fields rather than the original ones, the field equations assume a particularly simple form. In fact, we find in Sect. 6.1.4 that this choice yields decoupled *equations for the individual fields.*

Area metric gravity as constructed in the framework of covariant constructive gravity is—by the first axiom—diffeomorphism invariant. For the linear theory, this invariance manifests itself in the presence of a gauge symmetry

$$H'^A = H^A + C^A{}_B{}^n{}_m N^B \xi^m_{,n} \tag{6.58}$$

generated by vector fields $\xi \in \Gamma(TM)$*. As a result, the Euler-Lagrange equations are underdetermined, as solutions can only be obtained up to a gauge transform.*

In order to have a determined system for our following analysis, we fix the gauge by reducing the number of perturbation fields in a way that can always be reproduced using appropriate gauge transforms. The tool that makes the gauge fixing quite straightforward

perturbation kind	dof per field	fields	total dof
scalar	1	$A, \tilde{U}, \tilde{V}, V, W$	5
transverse vector	2	$B^\alpha, U^\alpha, W^\alpha$	6
transverse traceless tensor	2	$U^{\alpha\beta}, V^{\alpha\beta}, W^{\alpha\beta}$	6

Table 6.2: The 17 gauge-fixed degrees of freedom (dof) in linearised area metric gravity. Transverse vectors are divergence free, i.e. satisfy $0 = \partial_\alpha U^\alpha$. Transverse traceless vectors are symmetric, tracefree, and divergence free, i.e. $0 = U^{[\alpha\beta]}$, $0 = \gamma_{\alpha\beta} U^{\alpha\beta}$, and $0 = \partial_\alpha U^{\alpha\beta}$. Together with the four gauge-fixed fields $B = 0$, $V^\alpha = U^\alpha$, and $U = -V$, the area metric perturbation in this particular gauge is reproduced using Eq. (6.56).

is Helmholtz' theorem[7]*, which allows us to decompose the spatial vector field b into a so-called longitudinal scalar B and a divergence-free transverse vector B^α satisfying $\partial_\alpha B^\alpha = 0$ as*

$$b^\alpha = \partial^\alpha B + B^\alpha. \tag{6.59}$$

Applied to a tensor of rank 2, the Helmholtz theorem yields a decomposition

$$u^{\alpha\beta} = U^{\alpha\beta} + 2\partial^{(\alpha} U^{\beta)} + \gamma^{\alpha\beta}\tilde{U} + \Delta^{\alpha\beta} U. \tag{6.60}$$

In this decomposition, $U^{\alpha\beta}$ is the transverse traceless (TT) tensor satisfying $\partial_\alpha U^{\alpha\beta} = 0$ and $\gamma_{\alpha\beta} U^{\alpha\beta} = 0$. The vector U^α is again a transverse vector, U and $\tilde{U}$ are scalars, and $\Delta_{\alpha\beta} = \partial_\alpha \partial_\beta - \frac{1}{3}\gamma_{\alpha\beta}\Delta$, with the Laplacian Δ, denotes the traceless Hessian. The same decomposition

$$v^{\alpha\beta} = V^{\alpha\beta} + 2\partial^{(\alpha} V^{\beta)} + \gamma^{\alpha\beta}\tilde{V} + \Delta^{\alpha\beta} V \tag{6.61}$$

applies to $v^{\alpha\beta}$. Being traceless, the field $w^{\alpha\beta}$ is missing the trace scalar $\tilde{W}$, but otherwise admits a similar deconstruction into transverse traceless tensor $W^{\alpha\beta}$, transverse vector W^α, and longitudinal scalar W. At last, we have the lapse perturbation A, which is already a scalar.

Explicitly carrying out the gauge transform (6.58) *and carefully inspecting the components of H'^A, we find that the vector field ξ can always be chosen such that the four gauge conditions*

$$0 = B, \quad 0 = U^\alpha - V^\alpha, \quad 0 = U + V \tag{6.62}$$

are satisfied [65]. This choice reduces the degrees of freedom to 17, which are summarised in Table 6.2.

Let us briefly collect the results of a similar decomposition and gauge fixing for metric

[7] The Helmholtz theorem is only valid for certain classes of functions. Applicability to linearised area metric gravity, i.e. sufficiently well-behaved perturbations, is assumed.

gravity perturbed around the Minkowski metric. This will be of use later when we compare area metric gravity with metric gravity and highlight the differences. The metric tensor has 10 degrees of freedom and, as already remarked, decomposes into shift N^α, lapse N, and spatial metric $\hat{g}^{\alpha\beta}$ by virtue of the relations

$$\begin{aligned} g(dt, dt) &= \frac{1}{N^2}, \\ g(dt, dx^\alpha) &= -\frac{N^\alpha}{N^2}, \\ g(dx^\alpha, dx^\beta) &= \frac{N^\alpha N^\beta}{N^2} - \hat{g}^{\alpha\beta}. \end{aligned} \tag{6.63}$$

Around η, the observer quantities expand as

$$\begin{aligned} N &= 1 + A, \\ N^\alpha &= b^\alpha, \\ \hat{g}^{\alpha\beta} &= \gamma^{\alpha\beta} + \varphi^{\alpha\beta}. \end{aligned} \tag{6.64}$$

Like before, we use the Helmholtz theorem to write

$$b^\alpha = \partial^\alpha B + B^\alpha \tag{6.65}$$

and

$$\varphi^{\alpha\beta} = E^{\alpha\beta} + 2\partial^{(\alpha} V^{\beta)} + C\gamma^{\alpha\beta} + \Delta^{\alpha\beta} D. \tag{6.66}$$

A possible choice of gauge conditions is to set B, D, and V^α to zero, leaving us with 6 degrees of freedom in the fields A, B^α, C, and $E^{\alpha\beta}$.

6.1.4 Linearised field equations

Applying the $3+1$ decomposition of the area metric field to the Lagrangian density constructed in Sect. 6.1.1 yields an expression that is determined only by lapse, shift, and observer quantities. The corresponding field equations are obtained by the variations

$$\frac{\delta L}{\delta N}, \; \frac{\delta L}{\delta N^\alpha}, \; \frac{\delta L}{\delta G^{\alpha\beta}}, \; \frac{\delta L}{\delta G^\alpha{}_\beta}, \; \frac{\delta L}{\delta G_{\alpha\beta}} \tag{6.67}$$

with respect to all of these fields—as opposed to the "single" variation

$$\frac{\delta L}{\delta G^{abcd}} \tag{6.68}$$

with respect to the area metric in the spacetime picture.

For the linearised field equations, we automatically obtain a Helmholtz decomposition of the Euler-Lagrange equations: the variation with respect to the lapse N is a scalar and contains only contributions from scalars, the variation with respect to the shift N^α is a vector and contains only contributions from vectors. The same holds for the scalar and vector constituents of the observer quantities $\widehat{G}$. Also the variations with respect to transverse traceless tensors are again tensors and only comprised of tensors. As a result, the field equations already decouple to a large extent. For this reason, the terminology of the individual scalar, vector, and tensor fields as modes *of the gravitational field is justified.*

Performing the $3+1$ *split is a computationally heavy task. Essentially, the perturbation* (6.56) *has to be inserted into the ansätze* (A.1)–(A.7)*, the resulting expression must be simplified, then varied with respect to the different modes, and simplified again. In order to gain confidence in the result, speed up the computation, and—very importantly—have a calculation that can be reproduced and amended, the task has been offloaded to the computer algebra system* `cadabra` *[69, 70]. The code is available at Ref. [7].*

The result of this computation finally yields the field equations of perturbative area metric gravity in a gauge-fixed $3+1$ *setting. Of the 16 undetermined gravitational constants k_i that determine the expansion coefficients e_i (see Appendix B), ten independent linear combinations s_i (listed in Appendix C) make up the linearised field equations* (C.2)–(C.4)*.*

An important sanity check is provided by the second Noether theorem (2.47)

$$0 = D_n \mathcal{T}^n_m - \frac{\delta L}{\delta u^A} u^A{}_m = -D_n\Big[\frac{\delta L}{\delta u^A} C^A{}_B{}^n{}_m u^B\Big] - \frac{\delta L}{\delta u^A} u^A{}_m, \tag{6.69}$$

whose expansion around N amounts to

$$0 = -\Big[D_n \frac{\delta L}{\delta u^A}\Big]_{N+H} C^A{}_B{}^n{}_m N^B + \mathcal{O}(H^2). \tag{6.70}$$

Inverting the relation (6.56) *between spacetime area metric and observer fields, we can make use of the chain rule in order to express the variations with respect to the area metric in terms of variations with respect to the observer quantities. This renders the perturbative expansion of the Noether theorem in the particularly simple form*

$$0 = \partial_t \frac{\delta L}{\delta A} - \partial_\alpha \frac{\delta L}{\delta b_\alpha} \quad \textit{and} \quad 0 = \partial_t \frac{\delta L}{\delta b^\alpha} - 4\partial_\beta \frac{\delta L}{\delta u_{\alpha\beta}}, \tag{6.71}$$

which is indeed satisfied by the system (C.2)–(C.4)*. As a consequence of the diffeomorphism invariance of the theory, the field equations have four dependencies among themselves. This is, of course, expected—not only from the Noether theorem, but also from the fact that gauge-fixing the observer quantities by constraining four fields reduces the 21 unknowns* *by four. In order for the system of 21 field equations not to be overdetermined, it must express additional dependencies. These considerations are reminiscent of the rich field of constraint analysis*[8]*, which is predominantly studied in the Hamiltonian picture and also plays a rôle in canonical constructive gravity. For some results in the context of covariant constructive gravity, limited to first-derivative-order theories, see Ref. [37].*

While the Noether identities are expected and, in fact, indispensable, a thorough analysis of the linearised field equations reveals further properties that are impossible to reconcile with our premises. After all, the axioms of covariant constructive gravity are only necessary *conditions for a theory to be viable. Any such constructed theory needs to be further specified by finding appropriate values for the gravitational constants. This also applies to Einstein gravity—the Newtonian and cosmological constants only match observations for specific ranges, where some possibilities like a negative Newtonian constant can be dismissed outright.*

The first restriction of the area metric gravity parameter range we will make is to match the weak gravitational field sourced by a point mass with a modest generalisation of the Einstein equivalent. More specifically, we consider the gravitational field sourced by a point mass M which is at rest at the coordinate origin and thus describes the worldline

$$\gamma^a(\lambda) = \lambda\delta^a_0. \tag{6.72}$$

If the point particle M is an idealisation of a matter field that obeys GLED dynamics, its action is given by [28, 29]

$$S_{matter}[\gamma] = -M \int d\lambda \mathcal{P}_{GLED}(\mathcal{L}^{-1}(\dot{\gamma}(\lambda)))^{-\frac{1}{4}}, \tag{6.73}$$

where $\mathcal{L}^{-1}$ is the inverse of the Legendre map associated with the principal polynomial. In the Einstein equivalent, this action coincides with the common notion of the length of the particle worldline as measured using the covariant metric tensor. The full expansion for arbitrary curves γ is employed in the following section, it suffices here to consider the special case (6.72) *and find the only nonvanishing contribution*

$$\frac{\delta S_{matter}}{\delta A(x)} = -M\delta^{(3)}(x). \tag{6.74}$$

With the matter distribution being stationary, we consider a stationary ansatz for the solution to the field equations by assuming that the time derivatives of the gravitational field vanish. Using the source (6.74) *as the left-hand side of the linearised field equations* (C.2)–(C.4) *yields vector and tensor equations that are trivially sourced by zero and as*

[8] See e.g. [71, 72].

such only admit the trivial solution $B^\alpha = U^\alpha = W^\alpha = 0$ *and* $U^{\alpha\beta} = V^{\alpha\beta} = W^{\alpha\beta} = 0$.[9] *The scalar equations take the form*

$$E_i^{(scalar)} = M\delta^{(3)}(x)\delta_i^0 + \sum_j [a_{ij}S_j + b_{ij}\Delta S_j + c_{ij}\Delta\Delta S_j] \tag{6.75}$$

for constant coefficients a_{ij}, b_{ij}, c_{ij} *and scalar fields* $S_{(i)}$.

As solution to the scalar equations we obtain[10] *certain combinations of long-ranging Coulomb solutions* $\propto \frac{1}{r}$ *and short-ranging Yukawa solutions* $\propto \frac{1}{r}\mathrm{e}^{-\mu r}$. *While the coefficients of these combinations depend in an intricate way on the gravitational constants and are impossible to present in general, it* is *feasible to make a generic argument concerning the phenomenology of the linearised result:* the solution to the scalar field equations corresponds to the linearised Schwarzschild solution of general relativity for a central mass M corrected by short-ranging Yukawa potentials if and only if two linear conditions on the gravitational constants s_i hold.

This statement concerns the metric limit of area metric gravity, which is reached using the metrically induced area metric (3.20). *Inserting the metric* $3+1$ *decomposition* (6.63) *and its perturbative expansion* (6.64) *in the expression for the induced area metric yields*

$$\begin{aligned} \hat{G}^{\alpha\beta} &= \hat{g}^{\alpha\beta} = \gamma^{\alpha\beta} + \varphi^{\alpha\beta}, \\ \hat{G}^{\alpha}{}_{\beta} &= 0, \\ \hat{G}_{\alpha\beta} &= (\hat{g}^{-1})_{\alpha\beta} \approx \gamma_{\alpha\beta} - \varphi_{\alpha\beta}, \end{aligned} \tag{6.76}$$

from which we read off the induced perturbations

$$u^{\alpha\beta} = 2\varphi^{\alpha\beta}, \quad v^{\alpha\beta} = 0, \quad w^{\alpha\beta} = 0. \tag{6.77}$$

If the metric perturbation is now given by the expansion of the Schwarzschild solution [73] to first order,

$$A \propto \frac{1}{r} \quad \text{and} \quad \varphi^{\alpha\beta} = 2A\gamma^{\alpha\beta}, \tag{6.78}$$

the metrically induced area metric scalar fields amount to first order to

$$\begin{aligned} V = W = \tilde{V} &= 0, \\ \tilde{U} &= 4A, \\ A &\propto \frac{1}{r}. \end{aligned} \tag{6.79}$$

[9] See [3], where it is shown how the Fourier transform yields a linear system of full rank. Maple code for this calculation is available at Ref. [7].

[10] See [3] and the Maple code at Ref. [7].

The condition stated above requires that the area metric deviations from these fields amount to short-ranging Yukawa corrections, i.e. informally

$$\begin{aligned} 4A - \tilde{U} &= (\textit{Yukawa corrections}), \\ V &= (\textit{Yukawa corrections}), \\ W &= (\textit{Yukawa corrections}), \\ \tilde{V} &= (\textit{Yukawa corrections}). \end{aligned} \tag{6.80}$$

These conditions are equivalent to the vanishing of the linear combinations

$$s_1 + 4s_4 = 0 \quad \textit{and} \quad s_6 = 0, \tag{6.81}$$

which we from now on implement, reducing the number of first-order gravitational constants by two to eight. Thus, we have ruled out the possibility of deviating too much[11] *from Einstein gravity already in the regime of weak birefringence and restricted perturbative area metric gravity to a phenomenologically plausible sector. In this subtheory, the scalar fields around a point mass reduce to*

$$\begin{aligned} V(x) &= 0, \\ W(x) &= 0, \\ \tilde{U}(x) &= \frac{M}{4\pi r}[\alpha - (\beta + \frac{3}{4}\gamma)\mathrm{e}^{-\mu r}], \\ \tilde{V}(x) &= \frac{M}{4\pi r}[\frac{1}{4}\gamma\mathrm{e}^{-\mu r}], \\ A(x) &= \frac{M}{4\pi r}[\frac{1}{4}\alpha + \frac{1}{4}\beta\mathrm{e}^{-\mu r}], \end{aligned} \tag{6.82}$$

where we redefined the relevant gravitational constants using the more convenient set

$$\begin{aligned} \mu^2 &= \frac{8s_1 s_{39}}{9s_1^2 - 24s_1 s_3 + 8s_1 s_{37} + 16s_3^2}, \\ \alpha &= \frac{1}{2s_1}, \\ \beta &= \frac{(3s_1 + 4s_3)^2}{6s_1(9s_1^2 - 24s_1 s_3 + 8s_1 s_{37} + 16s_3^2)}, \\ \gamma &= \frac{-8(3s_1 + 4s_3)}{6(9s_1^2 - 24s_1 s_3 + 8s_1 s_{37} + 16s_3^2)}. \end{aligned} \tag{6.83}$$

With the reduction from ten to eight gravitational constants, the linearised field equations

[11] In the specific sense explained above.

assume a simpler form. There are reduced scalar field equations

$$
\begin{aligned}
\left[\frac{\delta L}{\delta u^{\alpha\beta}}\right]^{S\text{-}TF} &= \Delta_{\alpha\beta}\Big[s_1 A - \frac{s_1}{4}\tilde{U} + s_3\tilde{V} - \frac{s_1}{4}\ddot{V} + \frac{s_1}{12}\Delta V\Big],\\
\left[\frac{\delta L}{\delta v^{\alpha\beta}}\right]^{S\text{-}TF} &= \Delta_{\alpha\beta}\Big[s_{11}\Box V + s_{13}V + s_{14}\Box W + s_{16}W\Big],\\
\left[\frac{\delta L}{\delta w^{\alpha\beta}}\right]^{S\text{-}TF} &= \Delta_{\alpha\beta}\Big[s_{14}\Box V + s_{16}V - s_{11}\Box W - s_{13}W\Big],\\
\left[\frac{\delta L}{\delta u^{\alpha\beta}}\right]^{S\text{-}TR} &= \gamma_{\alpha\beta}\Big[-\frac{2s_1}{3}\Delta A - \frac{s_1}{2}\ddot{\tilde{U}} + \frac{s_1}{6}\Delta\tilde{U} + (-\frac{3s_1}{4} + s_3)\ddot{\tilde{V}} - \frac{2s_3}{3}\Delta\tilde{V}\\
&\quad + \frac{s_1}{3}\Delta\ddot{V} - \frac{s_1}{18}\Delta\Delta V\Big],\\
\left[\frac{\delta L}{\delta v^{\alpha\beta}}\right]^{S\text{-}TR} &= \gamma_{\alpha\beta}\Big[(-s_1 + \frac{4s_3}{3})\Delta A + (-\frac{3s_1}{4} + s_3)\ddot{\tilde{U}} - \frac{2s_3}{3}\Delta\tilde{U}\\
&\quad + s_{37}\ddot{\tilde{V}} - (\frac{3s_1}{2} - 2s_3 + s_{37})\Delta\tilde{V} + s_{39}\tilde{V}\\
&\quad + (\frac{s_1}{2} - \frac{2s_3}{3})\Delta\ddot{V} + \frac{2s_3}{9}\Delta\Delta V\Big],\\
\left[\frac{\delta L}{\delta b^{\alpha}}\right]^{S} &= \partial_\alpha\partial_t\Big[-2s_1\tilde{U} + (-3s_1 + 4s_3)\tilde{V} + \frac{2s_1}{3}\Delta V\Big],\\
\frac{\delta L}{\delta A} &= -2s_1\Delta\tilde{U} + (-3s_1 + 4s_3)\Delta\tilde{V} + \frac{2s_1}{3}\Delta\Delta V,
\end{aligned}
\tag{6.84}
$$

vector field equations

$$
\begin{aligned}
\left[\frac{\delta L}{\delta u^{\alpha\beta}}\right]^{V} &= \frac{s_1}{2}\partial_t\partial_{(\alpha}\Big[2B_{\beta)} + \dot{U}_{\beta)}\Big],\\
\left[\frac{\delta L}{\delta v^{\alpha\beta}}\right]^{V} &= 2\partial_{(\alpha}\Big[s_{11}\Box U_{\beta)} + s_{13}U_{\beta)} + s_{14}\Box W_{\beta)} + s_{16}W_{\beta)}\Big],\\
\left[\frac{\delta L}{\delta w^{\alpha\beta}}\right]^{V} &= 2\partial_{(\alpha}\Big[s_{14}\Box U_{\beta)} + s_{16}U_{\beta)} - s_{11}\Box W_{\beta)} - s_{13}W_{\beta)}\Big],\\
\left[\frac{\delta L}{\delta b^{\alpha}}\right]^{V} &= s_1\Delta\Big[2B_\alpha + \dot{U}_\alpha\Big],
\end{aligned}
\tag{6.85}
$$

and traceless tensor field equations

$$
\begin{aligned}
\left[\frac{\delta L}{\delta u^{\alpha\beta}}\right]^{TT} &= \frac{s_1}{4}\Box U_{\alpha\beta}, \\
\left[\frac{\delta L}{\delta v^{\alpha\beta}}\right]^{TT} &= s_{11}\Box V_{\alpha\beta} + s_{13}V_{\alpha\beta} + s_{14}\Box W_{\alpha\beta} + s_{16}W_{\alpha\beta}, \\
\left[\frac{\delta L}{\delta w^{\alpha\beta}}\right]^{TT} &= s_{14}\Box V_{\alpha\beta} + s_{16}V_{\alpha\beta} - s_{11}\Box W_{\alpha\beta} - s_{13}W_{\alpha\beta}.
\end{aligned}
\tag{6.86}
$$

The second observation we want to make concerns the subset

$$
\begin{aligned}
\left[\frac{\delta L}{\delta v^{\alpha\beta}}\right]^{S\text{-}TF} &= \Delta_{\alpha\beta}\Big[s_{11}\Box V + s_{13}V + s_{14}\Box W + s_{16}W\Big], \\
\left[\frac{\delta L}{\delta w^{\alpha\beta}}\right]^{S\text{-}TF} &= \Delta_{\alpha\beta}\Big[s_{14}\Box V + s_{16}V - s_{11}\Box W - s_{13}W\Big]
\end{aligned}
\tag{6.87}
$$

of the reduced scalar equations (6.84), *whose pattern is repeated in the vector equations* (6.85) *for the modes* U^α *and* W^α *as well as in the tensor equations* (6.86) *for the modes* $V^{\alpha\beta}$ *and* $W^{\alpha\beta}$. *Linear combinations of these equations* in vacuo *yield the equivalent system*

$$
\begin{aligned}
0 &= \Box V + \nu^2 V + \sigma W, \\
0 &= \Box W + \nu^2 W - \sigma V,
\end{aligned}
\tag{6.88}
$$

with constants

$$
\nu^2 = \frac{s_{11}s_{13} + s_{14}s_{16}}{s_{11}^2 + s_{14}^2} \quad \textit{and} \quad \sigma = \frac{s_{11}s_{16} - s_{13}s_{14}}{s_{11}^2 + s_{14}^2}. \tag{6.89}
$$

Performing a spatial Fourier transform of the vacuum scalar equations (6.88), *we can translate them into a system of linear, first-order ordinary differential equations for the modes* $\tilde{v}(t,k)$ *and* $\tilde{w}(t,k)$

$$
\frac{d}{dt}\begin{pmatrix}\tilde{v}\\ \tilde{w}\\ \dot{\tilde{v}}\\ \dot{\tilde{w}}\end{pmatrix} = \begin{pmatrix} 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 1\\ -(k^2+\nu^2) & -\sigma & 0 & 0\\ \sigma & -(k^2+\nu^2) & 0 & 0\end{pmatrix}\begin{pmatrix}\tilde{v}\\ \tilde{w}\\ \dot{\tilde{v}}\\ \dot{\tilde{w}}\end{pmatrix}. \tag{6.90}
$$

What is now interesting about this system are the eigenvalues of the time evolution, which are the four complex roots

$$
\lambda_k = \pm\mathrm{i}\sqrt{(k^2+\nu^2) \pm \mathrm{i}\sigma}. \tag{6.91}
$$

Most importantly, there are always λ_k *such that* $\mathrm{Re}(\lambda_k) > 0$ unless σ *vanishes. As a consequence, there will always be diverging modes under time evolution if* σ *is not zero.*

This finding is not restricted to the scalar modes we analysed, but also holds for the vector and transverse traceless tensor modes that are coupled in the same way. Such a theory would not only be physically implausible, *it would be fundamentally broken. We set σ to zero by imposing the additional condition*

$$s_{11}s_{16} - s_{13}s_{14} = 0 \tag{6.92}$$

and have thus reduced linearised area metric gravity to a theory parameterised by seven remaining gravitational constants, of which there are five combinations that determine the results obtained above: the two constants μ and ν appear as masses *in wave equations and screened Poisson equations, respectively, and three constants α, β, and γ further parameterise the linearised Schwarzschild solution.*

Note that with $\sigma = 0$ the wave equations for W, V, U^α, V^α, $U^{\alpha\beta}$, $V^{\alpha\beta}$, and $W^{\alpha\beta}$ decouple, *e.g. the system of transverse traceless tensor equations can be transformed by taking linear combinations into*

$$\begin{aligned} \left[\frac{\delta L}{\delta u^{\alpha\beta}}\right]^{TT} &= \frac{s_1}{4}\Box U_{\alpha\beta}, \\ \frac{s_{11}}{s_{11}^2+s_{14}^2}\left[\frac{\delta L}{\delta v^{\alpha\beta}}\right]^{TT} + \frac{s_{14}}{s_{11}^2+s_{14}^2}\left[\frac{\delta L}{\delta w^{\alpha\beta}}\right]^{TT} &= \Box V_{\alpha\beta} + \nu^2 V_{\alpha\beta}, \\ \frac{s_{14}}{s_{11}^2+s_{14}^2}\left[\frac{\delta L}{\delta v^{\alpha\beta}}\right]^{TT} - \frac{s_{11}}{s_{11}^2+s_{14}^2}\left[\frac{\delta L}{\delta w^{\alpha\beta}}\right]^{TT} &= \Box W_{\alpha\beta} + \nu^2 W_{\alpha\beta}. \end{aligned} \tag{6.93}$$

Similar decoupled wave equations are obtained for the mentioned vector and scalar modes. It is also possible to find a linear combination of scalar field equations (6.84) *such that the mode $\tilde{V}$ obeys a massive wave equation*[12]

$$\textit{(source terms)} = \Box\tilde{V} + \mu^2\tilde{V}. \tag{6.94}$$

Counting the wave equations we already found, there are at least 13 propagating degrees of freedom. This is already the maximum number, because our system for 17 degrees of freedom must exhibit four constraint equations arising from the gauge symmetry. In fact, the four remaining degrees B^α, $\tilde{U}$, and A are determined by field equations with less than two time derivatives, as can be read off from Eqns. (6.84)–(6.86)*. Such equations as part of an initial value problem are usually associated with constraints, as they are not capable to* evolve *initial data, but only to* constrain *it.*

Summing up, the phenomenologically relevant subsector of linearised area metric gravity admits two massless propagating degrees of freedom in the form of the tensor mode

[12]Not denoting linear combinations of Lagrangian variations explicitly but just referring to them as *source terms.*

$U^{\alpha\beta}$. Furthermore, there are 11 massive propagating degrees of freedom with mass μ, represented by the fields W, V, $\tilde{V}$, U^{α}, W^{α}, $V^{\alpha\beta}$, and $W^{\alpha\beta}$. The remaining four degrees of freedom A, $\tilde{U}$, and B^{α} do not propagate but follow from constraints.

This again constitutes an important sanity check: the count of propagating degrees of freedom is as expected and yields $21 - 2 \times 4 = 13$, just like in general relativity where we have $10 - 2 \times 4 = 2$ degrees of freedom. In the latter theory, only the transverse traceless part of the spatial metric tensor propagates and does so according to a massless wave equation. For area metric gravity, the only massless propagating modes turn out to be the transverse traceless tensor $U^{\alpha\beta}$, which is exactly the perturbation induced by the propagating metric modes *(see* (6.77)*).*

All other modes, which are not inducible by the propagating metric modes, follow massive wave equations with mass μ. In the next section, it will become clear that the generation of such modes from matter distributions is suppressed, e.g. a binary star only radiates on nonmetric tensor modes or on vector or scalar modes when its angular frequency exceeds a certain threshold. This is another realisation of the correspondence principle, which demands that Einstein gravity approximate area metric gravity in certain limits.

6.2 The binary star

As example for a matter distribution that gravitates according to area metric gravity, we consider a binary star. The system shall be approachable without too much computational effort, while at the same time exhibiting exciting new physics beyond Einstein gravity—a configuration of two point masses that circle each other turns out to meet both requirements. First, let us introduce a method to construct a solution up to the second perturbation order.

6.2.1 Iterative solution strategy for gravitational field equations

Covariant constructive gravity closes matter theories by providing previously unknown dynamics for geometry to which the matter field couples. Let ϕ be the matter field in question, coupling locally to a geometric field G. Starting from the matter action[13] *$S_{matter}[\phi, G)$, the closure procedure yields the joint action*

$$S[G,\phi] = S_{gravity}[G] + \kappa S_{matter}[\phi, G), \tag{6.95}$$

[13] Round parentheses indicate local dependencies.

where $S_{gravity}$ is the action of the constructed theory compatible with the matter theory. The constant κ controls the scale of coupling between both fields. Abbreviated as

$$e[G] = \frac{\delta S_{grav}}{\delta G}, \quad T[\phi, G) = \frac{\delta S_{mat}}{\delta G}, \quad f[\phi, G) = \frac{\delta S_{mat}}{\delta \phi}, \tag{6.96}$$

the variations with respect to the matter field and the gravitational field yield the Euler-Lagrange equations

$$e[G] = -\kappa T[\phi, G) \quad \textit{and} \quad f[\phi, G) = 0. \tag{6.97}$$

Such a tightly coupled system is hard to solve in general. Fortunately, it is not our objective to obtain exact solutions—we have expanded the field equations up to second order and only seek to derive effects up to this finite order. Proceeding similarly as in Ref. [74], a solution is constructed iteratively by expanding the geometry formally as

$$G = N + \sum_{k=1}^{\infty} \kappa^k H_{(k)}. \tag{6.98}$$

Truncations of Eq. (6.98) *at order k yield approximations $G_{(k)}$ of the geometry. The constituents e and T of the Euler-Lagrange equations expand as*

$$\begin{aligned} e[N+H] &= e_{(0)} + e_{(1)}[H] + e_{(2)}[H] + \mathcal{O}(H^3), \\ T[\phi, N+H) &= T_{(0)}[\phi] + T_{(1)}[\phi, H) + \mathcal{O}(H^2), \end{aligned} \tag{6.99}$$

where H contributes linearly to the first-order terms and quadratically to the second-order terms. We now solve the equations for the gravitational field up to second order by considering the orders zero to two in κ.

For the zeroth *iteration, the Euler-Lagrange equations* (6.97) *are evaluated at $G_{(0)} = N$, resulting in the equation*

$$e[N] = e_{(0)} = 0. \tag{6.100}$$

This just enforces that the expansion point N must solve the gravitational field equations in vacuo. *Since we explicitly consider this condition when perturbatively constructing theories, Eq.* (6.100) *is solved trivially.*

Proceeding with the first *iteration, we evaluate at $G_{(1)} = N + \kappa H_{(1)}$. Since $e_{(0)} = 0$ already holds from the previous iteration, the first of the two equations simplifies to*

$$e_{(1)}[H_{(1)}] = -T_{(0)}[\phi]. \tag{6.101}$$

Figuratively speaking, the first correction of the gravitational field is sourced by the matter content on a flat background. Having solved this equation for $H_{(1)}$, the perturbation may *be used in order to solve the second equation*

$$f[\phi, G_{(1)}) = 0 + \mathcal{O}(\kappa^2). \tag{6.102}$$

The interpretation is similar: a deviation from the flat gravitational field, caused by the presence of matter, makes also the matter field deviate from its unperturbed configuration.

The second *iteration yields an equation for the second-order perturbation $H_{(2)}$ by inserting $G_{(2)} = N + \kappa H_{(1)} + \kappa^2 H_{(2)}$ in the first field equation and simplifying using the lower-order equations. We obtain the result*

$$e_{(1)}[H_{(2)}] = -\kappa^{-1} T_{(0)}[\phi] - T_{(1)}[\phi, H_{(1)}) - e_{(2)}[H_{(1)}] + \mathcal{O}(k), \tag{6.103}$$

where it has to be noted that ϕ, having been fixed in Eq. (6.102)*, has a dependence on $\kappa H_{(1)}$. Therefore, contributions from $T_{(0)}[\phi]$ must only be considered up to order κ^1 and contributions from $T_{(1)}[\phi, H_{(1)})$ only up to order κ^0.*

The second-order perturbation $H_{(2)}$ is thus sourced by both the first-order deviations of the gravitational field and the induced motion of the matter field, as will become clear when explicitly solving the binary star in the following section. Aborting the iterative solution procedure at this point, we have found the approximation

$$G_{(2)} = N + \kappa H_{(1)} + \kappa^2 H_{(2)} \tag{6.104}$$

of the geometry G coupled to ϕ and, as a bonus, the trajectory of the matter field ϕ on the linearised background $G_{(1)}$.

6.2.2 Solution in Einstein gravity

Before proceeding to make use of the iterative solution strategy and solving the binary star in area metric gravity, let us consider the same problem in Einstein gravity. We will, of course, only reproduce well-established results, but also gain confidence in the approach and become acquainted with the calculations. It is also advantageous to have the metric theory at hand in order to distinguish the uniquely area metric features later on. State-of-the-art methods derived from Einstein gravity (see e.g. Ref. [74]) extend to higher perturbation orders and much more complex matter configurations than the relatively simple case considered here, but they are not applicable to area metric gravity. Rather, we make use of our hand-crafted approach that accommodates nonmetric geometries just as well.

A binary star consists of two slowly moving point masses m_i describing two worldlines $\gamma_i \colon \mathbb{R} \to M$. The metric field is a section g of the metric bundle and defines the matter *action* S_{matter} *via the length functional*[14]

$$S_{matter}[\gamma_{(1)},\gamma_{(2)},g] = \sum_{i=1,2} m_i c \int d\lambda \sqrt{g^{-1}(\dot\gamma_{(i)}(\lambda),\dot\gamma_{(i)}(\lambda))}. \tag{6.105}$$

Einstein gravity completes Eq. (6.105) *to a predictive theory by providing dynamics for the metric* g *in terms of the Einstein-Hilbert action*

$$S_{gravity}[g] = \frac{c^3}{16\pi G}\int d^4x \sqrt{-\det g} R. \tag{6.106}$$

We use the parameterisation $\gamma^0_{(i)}(\lambda) = c\lambda$ *and obtain by variation the Euler-Lagrange equations*

$$\sqrt{-\det g}\left[R^{ab} - \frac{1}{2}g^{ab}R\right] = \frac{8\pi G}{c^3}\sum_{i=1,2} m_i \delta^{(3)}(\vec x - \vec\gamma_{(i)}(t))\frac{\dot\gamma^a_{(i)}\dot\gamma^b_{(i)}}{\sqrt{g^{-1}(\dot\gamma_{(i)},\dot\gamma_{(i)})}} \tag{6.107}$$

and

$$0 = \ddot\gamma^a_{(i)} + \Gamma^a{}_{bc}\dot\gamma^b_{(i)}\dot\gamma^c_{(i)}. \tag{6.108}$$

The first equation (6.107) *consists of the densitised Einstein tensor on the left-hand side and the stress-energy-momentum tensor of the point particle on the right-hand side. Eq.* (6.108) *is the geodesic equation on a pseudo-Riemannian manifold with the Christoffel symbols* $\Gamma^a{}_{bc}$*. Using the slow-motion condition*

$$\frac{1}{c}\dot\gamma^\alpha_{(i)} \ll 1, \tag{6.109}$$

the geodesic equation simplifies to

$$\dot\gamma^0_{(i)} = c \quad \textit{and} \quad \frac{1}{c^2}\ddot\gamma^\alpha_{(i)} = -\Gamma^\alpha{}_{00}. \tag{6.110}$$

In order to construct the second-order solution for the metric tensor in this setting, we expand g *as*

$$g^{ab} = \eta^{ab} + h^{ab} = \eta^{ab} + Gh^{ab}_{(1)} + G^2h^{ab}_{(2)} + \mathcal{O}(G^3), \tag{6.111}$$

using the Newtonian constant G *as coupling constant. Adopting the* $3+1$ *decomposition* (6.63)–(6.66) *for the metric tensor as well as the gauge* $B = D = 0$ *and* $V^\alpha = 0$*, the perturbation is given by*

$$h^{00} = -2A, \quad h^{0\alpha} = B^\alpha, \quad h^{\alpha\beta} = -E^{\alpha\beta} - \gamma^{\alpha\beta}C, \tag{6.112}$$

[14] From now on, we do not use geometrised units but state every occurrence of the speed of light c and Newton's constant G explicitly.

with scalar modes A and C, vector modes B^α, and transverse traceless tensor modes $E^{\alpha\beta}$.

Since the variation $e[g]$ of the Einstein-Hilbert Lagrangian with respect to the metric tensor is given by the Einstein tensor, which is derived from the Riemann curvature tensor, the zeroth-order equation $e[\eta] = 0$ is already solved—the flat Minkowski metric η has zero curvature.

The first-order equation

$$e_{(1)}[h_{(1)}] = -T_{(0)}[\gamma_{(1)}, \gamma_{(2)}] \tag{6.113}$$

is obtained from the full Euler-Lagrange equations (6.107) *using the well-known expansion of the Einstein tensor to linear order*[15] *for the left-hand side and—since the right-hand side already contains a factor G—the zeroth order of the matter distribution. Split into spatial and temporal components, we get*

$$\begin{aligned} e^{00}_{(1)}[h] &= \Delta C, \\ e^{0\alpha}_{(1)}[h] &= -\frac{1}{2}\Delta B^\alpha - \partial^\alpha \dot{C}, \\ e^{\alpha\beta}_{(1)}[h] &= -\frac{1}{2}\Box E^{\alpha\beta} + \partial^{(\alpha}\dot{B}^{\beta)} + \gamma^{\alpha\beta}\left[\ddot{C} - \frac{2}{3}\Delta(-A + \frac{1}{2}C)\right] + \Delta^{\alpha\beta}\left[-A + \frac{1}{2}C\right], \end{aligned} \tag{6.114}$$

and the only nonzero contribution[16]

$$\begin{aligned} T^{00}[\gamma_{(1)}, \gamma_{(2)}] = &\ -\frac{8\pi}{c^2}\sum_{i=1,2} m_i \delta^{(3)}(\vec{x} - \vec{\gamma}_{(i)}(t)) \\ =: &\ -\frac{8\pi}{c^2}\rho(\vec{x}, t), \end{aligned} \tag{6.115}$$

such that the first iteration boils down to the Poisson equation

$$\Delta C_{(1)} = \frac{8\pi}{c^2}\sum_{i=1,2} m_i \delta^{(3)}(\vec{x} - \vec{\gamma}_{(i)}(t)). \tag{6.116}$$

The remaining equations are not sourced by matter and, thus, yield the trivial results $A_{(1)} = \frac{1}{2}C_{(1)}$ and $B^\alpha_{(1)} = 0$. For $E^{\alpha\beta}_{(1)}$, we obtain the massless wave equation in vacuo*,*

$$0 = \Box E^{\alpha\beta}_{(1)}, \tag{6.117}$$

which we solve by setting $E^{\alpha\beta}_{(1)}$ to zero.[17]

[15] The prefactor given by the metric determinant is irrelevant: it expands as $1 + \frac{1}{2}\eta_{\alpha\beta}h^{\alpha\beta}$ and contributes only to zeroth order, because the expansion of the Einstein tensor has *no* zeroth order.

[16] Implementing the slow-motion condition.

[17] Allowing for nonvanishing solutions would place the binary star not on a flat background but on a

Solving Eq. (6.116) *yields the linearised solution*

$$E_{(1)}^{\alpha\beta} = 0, \quad B_{(1)}^{\alpha} = 0, \quad A_{(1)} = \frac{1}{c^2}\phi, \quad C_{(1)} = \frac{2}{c^2}\phi, \tag{6.118}$$

effectively composed of one scalar field, the Newtonian potential

$$\begin{aligned} \phi(\vec{x},t) =& -\int d^3\vec{y}\frac{\rho(\vec{y},t)}{|\vec{x}-\vec{y}|} \\ =& -\frac{m_1}{|\vec{x}-\vec{\gamma}_{(1)}(t)|} - \frac{m_2}{|\vec{x}-\vec{\gamma}_{(2)}(t)|}. \end{aligned} \tag{6.119}$$

According to the iterative solution procedure, the worldlines $\gamma_{(i)}$ *can now be fixed by solving their equations of motion* (6.108) *on the linearised background* (6.118)*. These equations are governed by the Christoffel symbols, which expand as*

$$\Gamma^{\alpha}{}_{00} = -\frac{1}{2}\partial^\alpha h^{00} - \frac{1}{c}\dot{h}^{\alpha 0} + \mathcal{O}(h^2), \tag{6.120}$$

such that on the linearised background provided by the first iteration

$$\begin{aligned} \ddot{\gamma}_{(i)}^{\alpha} =& -c^2\Gamma^{\alpha}{}_{00} \\ =& -G\partial^\alpha\phi + \mathcal{O}(G^2). \end{aligned} \tag{6.121}$$

After all, slowly moving matter obeys—to first order—the Newtonian laws of gravity!

The equations of motion come with the same inconsistencies that plague Newtonian gravity: as is obvious from the formula (6.119)*, the potential sourced by a point mass diverges at the very location of the mass itself. Consequently, whenever a particle "feels" its own field, which is certainly the case in Eq.* (6.121)*, infinities are involved. The culprit is the idealisation of the matter distribution as point masses. One of the remedies pointed out in Ref. [74] is to forgo this idealisation and model the stars as extended fluids—taking the limit of negligible extension where necessary. Alternatively, the diverging integrals may be regularised, which has the same impact on the results. Effectively, both approaches are implemented the same way: we keep the point mass idealisation but discard diverging integrals, i.e. when solving for the trajectory of the first particle, the diverging term*

$$m_1\int d^3\vec{y}\frac{\delta^{(3)}(\vec{y}-\vec{\gamma}_{(1)}(t))}{|\vec{y}-\vec{\gamma}_{(1)}(t)|} \tag{6.122}$$

background filled with gravitational radiation. As long as this radiation is weak enough in order not to interfere with the second-order field equations, it can be included without affecting the phenomenology. For simplicity, however, it is customary to choose the zero solution.

does not contribute. This also holds, mutatis mutandis, *for the second particle.*

With this regularisation, the stars are subject to the equations of motion

$$\ddot{\gamma}^\alpha_{(i)} = -G \sum_{j \neq i} m_j \frac{\gamma^\alpha_{(i)} - \gamma^\alpha_{(j)}}{|\vec{\gamma}_{(i)} - \vec{\gamma}_{(j)}|^3}, \tag{6.123}$$

which is the centuries-old Kepler problem. The solutions are given by the various conic sections, depending on the initial conditions. We will consider the bound states and within this sector the configurations with exactly circular orbits. As it will turn out, the additional complexity introduced by eccentricities is immaterial for at least some of the new effects that come with the area metric generalisation. In this configuration, the bodies have constant separation r and move on trajectories

$$\vec{\gamma}_{(1)}(t) = \frac{m_2}{m} r\vec{n}, \quad \vec{\gamma}_{(2)}(t) = -\frac{m_1}{m} r\vec{n}, \tag{6.124}$$

where $m = m_1 + m_2$ denotes the total mass. The vector $\vec{n}$ is one of the three basis vectors

$$\vec{n} = \begin{pmatrix} \cos\omega t \\ \sin\omega t \\ 0 \end{pmatrix}, \quad \vec{\lambda} = \begin{pmatrix} -\sin\omega t \\ \cos\omega t \\ 0 \end{pmatrix}, \quad \vec{e}_z = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \tag{6.125}$$

that span the orbit-adapted *frame [74]. Both masses reside in the orbital plane spanned by $\vec{n}$ and $\vec{\lambda}$, to which $\vec{e}_z$ is perpendicular. The frame rotates around the axis $\vec{e}_z$ with angular frequency ω according to Kepler's third law*

$$\omega^2 = \frac{Gm}{r^3}. \tag{6.126}$$

Based on this configuration of matter content and gravitational field, the second iteration yields the corrections sourced by both the first-order gravitational field itself and by the influence of the gravitational field on the masses. We are only concerned with the propagating degrees of freedom, as our interest lies in radiation emitted into the far zone, so it suffices to consider the purely spatial part from Eq. (6.103)[18]

$$e^{\alpha\beta}_{(1)}[h_{(2)}] = -G^{-1} T^{\alpha\beta}_{(0)}[\gamma_{(1)}, \gamma_{(2)}] - T^{\alpha\beta}_{(1)}[\gamma_{(1)}, \gamma_{(2)}, h_{(1)}) - e^{\alpha\beta}_{(2)}[h_{(1)}] + \mathcal{O}(G). \tag{6.127}$$

Again the functionals $e^{\alpha\beta}_{(.)}$ can be read off from the left-hand side of the full Euler-Lagrange equations (6.107) *and the functionals $T^{\alpha\beta}_{(.)}$ follow from the right-hand side.*

The contribution from $e^{\alpha\beta}_{(1)}[h_{(2)}]$ is already known from Eq. (6.114). *Its projection to the*

[18] Note that the labels on the worldlines $\gamma_{(i)}$ do not denote perturbation orders but the individual stars.

transverse traceless tensor part is given by

$$e_{(1)}^{\alpha\beta}[h_{(2)}]^{TT} = -\frac{1}{2}\Box E_{(2)}^{\alpha\beta}. \tag{6.128}$$

We find that the first-order matter functional $T_{(1)}^{\alpha\beta}[\gamma_{(1)}, \gamma_{(2)}, h_{(1)})$ does not contribute, because each derivative of a spatial trajectory comes with a factor ω, such that the whole functional is proportional to $\omega^2 \propto G$. This is already of higher order than considered in the second iteration equation (6.127)*. Reading off the term $T_{(0)}^{\alpha\beta}[\gamma_{(1)}, \gamma_{(2)}]$ and projecting onto the transverse traceless tensor mode, we arrive at the intermediate expression*

$$\Box E^{\alpha\beta} = -\frac{16\pi}{Gc^4}\left[\sum_{i=1,2} m_i \delta^{(3)}(\vec{x} - \vec{\gamma}_{(i)}(t))\dot{\gamma}_{(i)}^{\alpha}\dot{\gamma}_{(i)}^{\beta}\right]^{TT} + 2e_{(2)}^{\alpha\beta}[h_{(1)}]^{TT}. \tag{6.129}$$

It remains to derive the contribution from $e_{(2)}^{\alpha\beta}[h_{(1)}]$. This is the first and only time where the second order of the Einstein field equations is needed. Thankfully, the field equations have to be evaluated at the result of the first iteration, $h_{(1)}$, which assumes a particularly simple form where all fields are derived from only the Newtonian potential ϕ. Evaluation of the Einstein field equations at this solution yields the transverse traceless tensor part

$$e_{(2)}^{\alpha\beta}[h_{(1)}]^{TT} = -\frac{2}{c^4}\left[\partial^\alpha\phi\partial^\beta\phi - 2\partial^\alpha(\phi\partial^\beta\phi)\right]^{TT}. \tag{6.130}$$

We are thus left with the wave equation (6.129)*, which is of the kind*

$$\Box\psi(\vec{x}, t) = 4\pi\varphi(\vec{x}, t). \tag{6.131}$$

Such an equation is solved by convolution of the source with the retarded Green's function

$$\psi(\vec{x}, t) = \int d^3\vec{y}\frac{\varphi(\tau, \vec{y})}{|\vec{x} - \vec{y}|}, \tag{6.132}$$

where the source is evaluated at the retarded time

$$\tau = t - \frac{1}{c}|\vec{x} - \vec{y}|. \tag{6.133}$$

For radiation into the far zone, we are only interested in the result at points in spacetime with $R = |\vec{x}| \gg r$. A first approximation in this regime is given by the zeroth order $\vec{x} - \vec{y} \approx R$, which yields the simplified integral

$$\psi(\vec{x}, t) = \frac{1}{R}\int d^3\vec{y}\varphi(\tau, \vec{y}), \tag{6.134}$$

where from now on $\tau = t - R/c$. This approximation is valid to lowest order, because for the first part of the source (the first summand in Eq. (6.129)*), the integration variable $\vec{y}$ is confined to the matter distribution, a region of radius r, such that*

$$|\vec{x} - \vec{y}| \leq |\vec{x}| + |\vec{y}| \leq |\vec{x}| + r = R(1 + \frac{r}{R}) \xrightarrow{\frac{r}{R} \to 0} R. \tag{6.135}$$

For the second part, the source occupies an unbounded region but decreases in magnitude with $|\vec{y}|^{-4}$, allowing for a similar argument.

The integrals that remain to be evaluated are

$$K^{\alpha\beta} = \int d^3\vec{y} \sum_{i=1,2} m_i \delta^{(3)}(\vec{y} - \vec{\gamma}_{(i)}(\tau)) \dot{\gamma}^{\alpha}_{(i)} \dot{\gamma}^{\beta}_{(i)} \tag{6.136}$$

and, after dropping a boundary term,

$$U^{\alpha\beta} = \int d^3\vec{y} \partial^\alpha \phi \partial^\beta \phi. \tag{6.137}$$

Evaluating $K^{\alpha\beta}$, whose integrand is a simple delta distribution, gives

$$K^{\alpha\beta} = \frac{G\eta m^2}{r} \lambda^\alpha \lambda^\beta. \tag{6.138}$$

Here, the reduced mass

$$\eta = \frac{m_1 m_2}{m^2} \tag{6.139}$$

makes its first appearance. In order to evaluate the second integral, we first substitute the Newtonian potential with the unevaluated integral expression (6.119)*, such that*

$$U^{\alpha\beta} = \int d^3\vec{y} \int d^3\vec{y'} \int d^3\vec{y''} \frac{\rho(\vec{y'})\rho(\vec{y''})}{|\vec{y} - \vec{y'}|^3 |\vec{y} - \vec{y''}|^3} (y^\alpha - y'^\alpha)(y^\beta - y''^\beta). \tag{6.140}$$

The integration over $\vec{y}$ now yields

$$U^{\alpha\beta} = 2\pi \int d^3\vec{y'} \rho(\vec{y'}) \int d^3\vec{y''} \frac{\rho(\vec{y''})}{|\vec{y'} - \vec{y''}|} \left[\gamma^{\alpha\beta} - \frac{(y'^\alpha - y''^\alpha)(y'^\beta - y''^\beta)}{|\vec{y'} - \vec{y''}|^2} \right], \tag{6.141}$$

which corresponds to the repeated application of delta distributions. Making sure not to include diverging terms, as explained earlier, the evaluation yields

$$U^{\alpha\beta} = \frac{4\pi\eta m^2}{r} [\gamma^{\alpha\beta} - n^\alpha n^\beta]. \tag{6.142}$$

We finally put together both parts with the proper constants and the prefactor of $1/R$. *The result is the lowest nontrivial order of the gravitational field that is radiated away into the far zone from a binary star in circular motion,*

$$G^2 h_{(2)}^{\alpha\beta} = \frac{4\eta}{c^4 R} \frac{(Gm)^2}{r} [\lambda^\alpha \lambda^\beta - n^\alpha n^\beta]^{TT}, \tag{6.143}$$

parameterised by properties of the matter distribution (total mass m*, reduced mass* η*, separation* r*), the speed of light* c*, and Newton's gravitational constant* G*. This is in exact accordance with the literature [74] and, of course, no surprise. Contemporary methods employ what is called post-Minkowskian and post-Newtonian theory [74], which provide a framework for more complex calculations. However, the pedestrian approach presented here is derived from the same full theory and is thus equally valid.*

The strength of this solution procedure is that it does not presuppose knowledge of the exact (i.e. unperturbed) dynamics and is not restricted to metric theories. Both properties are important for the analysis of the binary star in area metric gravity, a nonmetric theory of gravity for which there are no known exact dynamics. Even though we followed a top-down approach and derived the perturbative expansion of the Einstein field equations from its full form, it would have been possible to construct this expansion from the bottom up, order by order. In the following section, this will be the only *option.*

Finally, it should be noted (see also the discussion in Ref. [74]) that the radiation emitted by the binary star is indeed an effect of second order. The presence of the masses alone sources a gravitational field, which to first order is given by the Newtonian potential. Under the influence of this field, the masses are confined to circular orbits—a first-order effect. This *refined motion, together with the first order of the gravitational field*[19]*, is the source of the gravitational radiation produced in the second iteration. Knowledge of the first-order field equations is not sufficient in order to derive the result* (6.143)*, contrary to the impression that the reading of derivations in the older literature might leave [75]. Whoever arrives at the conclusion* (6.143)*, or its generalisation for more general matter configurations called* quadrupole formula*, using only the linearised gravitational field equations either did so out of pure luck, by silently slipping in knowledge about the second order, or by having inadvertently constructed this order during the process. If, for example, the derivation involves some basic assumptions about the theory, such as restrictions concerning derivative orders, and also diffeomorphism invariance, it is no surprise that a correct formula may be obtained—after all, as discussed in Chap. 3, Einstein gravity is* unique *if certain assumptions are met.*

[19] Via the contribution $e_{(2)}[h_{(1)}]$, where the second order of the field equations enters.

6.2.3 Solution in area metric gravity

In the area metric gravity scenario, the point masses are subject to the action

$$S_{matter}[\gamma_{(1)}, \gamma_{(2)}, G) = \sum_{i=1,2} m_i c \int d\lambda \mathcal{P}_{GLED}(L^{-1}(\dot{\gamma}_{(i)}(\lambda)))^{-\frac{1}{4}}, \tag{6.144}$$

which we already encountered when discussing the linearised Schwarzschild solution. This time, the masses are not at rest, such that the "full" linearised expression for generic worldlines is needed. It comes in very handy that the GLED polynomial is to first order equivalent to the quadratic polynomial (see Eq. (6.10)*)*

$$P^{(\leq 1)}_{GLED}(k) = [1 - \frac{1}{24}\epsilon(H)]\eta(k,k) + \frac{1}{2}H(k,k), \tag{6.145}$$

which using the $3+1$ *split introduced in Sect. 6.1.3 decomposes into*

$$P^{(\leq 1)}_{GLED}(k) = \eta(k,k) + [-2A](k_0)^2 + [-2b^\alpha]k_0 k_\alpha + [-\frac{1}{2}u^{\alpha\beta} - \frac{1}{2}\gamma_{\mu\nu}v^{\mu\nu}\gamma^{\alpha\beta}]k_\alpha k_\beta. \tag{6.146}$$

Since the causality is effectively metric, the integrand in the point particle action (6.144) *is given by the inverse of this metric. [28, 29] To linear order, the inverse is calculated as*

$$[\eta + h]^{-1}{}_{ab} = \eta_{ab} - \eta_{ap}\eta_{pq}h^{pq} + \mathcal{O}(h^2), \tag{6.147}$$

such that we obtain the linearised action

$$S_{matter}[\gamma_{(1)}, \gamma_{(2)}, N+H) = \sum_{i=1,2} m_i c \int d\lambda \Big\{ \eta_{ab}\dot{\gamma}^a_{(i)}\dot{\gamma}^b_{(i)} + 2A\dot{\gamma}^0_{(i)}\dot{\gamma}^0_{(i)} - 2b_\alpha \dot{\gamma}^0_{(i)}\dot{\gamma}^\alpha_{(i)} + \left[\frac{1}{2}u_{\alpha\beta} + \frac{1}{2}\gamma^{\mu\nu}v_{\mu\nu}\gamma^{\alpha\beta}\right]\dot{\gamma}^\alpha_{(i)}\dot{\gamma}^\beta_{(i)} \Big\} + \mathcal{O}(H^2). \tag{6.148}$$

In addition to the linearised matter action (6.148)*, we also need the gravitational action expanded to third order in the area metric perturbation. Sect. 6.1 was dedicated to the construction of third-order area metric Lagrangian densities* $\mathcal{L} = Ld^4x$*. The result of this construction procedure will be used here in the action*

$$S_{gravity}[N+H] = \frac{c^3}{16\pi G}\int d^4x \mathcal{L} + \mathcal{O}(H^4). \tag{6.149}$$

Like before, the zeroth iteration is already solved by construction—the flat instance N *of the area metric field solves the vacuum field equations.*

Due to the slow-motion condition, the first-order field equations

$$e_{(1)}[H_{(1)}] = -T_{(0)}[\gamma_{(1)}, \gamma_{(2)}], \tag{6.150}$$

which were derived in Sect. 6.2.1, are only sourced by the variation of the matter action (6.148) *with respect to the lapse, via the equation*

$$\left(\frac{\delta S_{matter}}{\delta A}\right)_{(1)} [H_{(1)}] = c\rho(\vec{x}, t). \tag{6.151}$$

This is similar to the stationary case considered in Sect. 6.1.4, with the difference that the delta distribution is not centred at the origin but given as

$$\rho(\vec{x}, t) = \sum_{i=1,2} m_i \delta^{(3)}(\vec{x} - \vec{\gamma}_{(i)}(t)). \tag{6.152}$$

Since the vector and transverse traceless tensor equations are not sourced at all, we solve these by setting the respective modes to zero. Again, it is possible to add background radiation to the solution, as long as it remains negligible. The scalar modes are solved by superposition of the linearised Schwarzschild solutions (6.82)*, in the integral representation as*

$$\begin{aligned} A_{(1)} &= -\frac{1}{c^2} \int d^3\vec{y} \rho(\vec{y}) \left[\frac{\alpha}{|\vec{x} - \vec{y}|} + \frac{\beta \mathrm{e}^{-\mu|\vec{x}-\vec{y}|}}{|\vec{x} - \vec{y}|}\right], \\ \tilde{V}_{(1)} &= -\frac{1}{c^2} \int d^3\vec{y} \rho(\vec{y}) \left[\frac{\gamma \mathrm{e}^{-\mu|\vec{x}-\vec{y}|}}{|\vec{x} - \vec{y}|}\right], \\ \tilde{U}_{(1)} &= 4A_{(1)} - (3 + 8\frac{\beta}{\gamma})\tilde{V}_{(1)}, \end{aligned} \tag{6.153}$$

or in the evaluated form

$$\begin{aligned} A_{(1)} &= -\frac{1}{c^2} \sum_{i=1,2} m_i \left[\frac{\alpha}{|\vec{x} - \vec{\gamma}_{(i)}(t)|} + \frac{\beta \mathrm{e}^{-\mu|\vec{x}-\vec{\gamma}_{(i)}(t)|}}{|\vec{x} - \vec{\gamma}_{(i)}(t)|}\right], \\ \tilde{V}_{(1)} &= -\frac{1}{c^2} \sum_{i=1,2} m_i \left[\frac{\gamma \mathrm{e}^{-\mu|\vec{x}-\vec{\gamma}_{(i)}(t)|}}{|\vec{x} - \vec{\gamma}_{(i)}(t)|}\right], \\ \tilde{U}_{(1)} &= 4A_{(1)} - (3 + 8\frac{\beta}{\gamma})\tilde{V}_{(1)}. \end{aligned} \tag{6.154}$$

The constants α, β, γ, and μ are the four relevant first-order gravitational constants (6.83) *for stationary or slowly moving matter configurations.*

For the second part of the first iteration, the matter trajectories have to be fixed. We again exploit the fact that the matter action is effectively metric, because as a consequence of this circumstance, the stars are to this order subject to the same geodesic equation

(6.121) *as in Einstein gravity. The Christoffel symbols are derived from the effective metric* (6.147) *with* $h^{00} = -2A$*, such that*

$$\ddot{\gamma}^{\alpha}_{(i)} = -c^2 G \partial^{\alpha} A_{(1)}, \tag{6.155}$$

where the integrals (6.153) *have to be regularised by, effectively, dropping the divergent terms (see Sect. 6.2.2).*

The equations of motion (6.155) *constitute a refined Kepler problem. Instead of the Newtonian potential* $\propto \frac{1}{r}$*, the stars move in modified potentials with additional Yukawa terms* $\propto \frac{1}{r}\mathrm{e}^{-\mu r}$*. Still, this potential has a spherical symmetry and circular orbits remain solutions to the geodesic equations. This is seen by making the ansatz* (6.124) *and solving for the angular frequency* ω*, which yields the refined relation*

$$\omega^2 = \frac{(G\alpha)m}{r^3}\left[1 + \frac{\beta}{\alpha}\mathrm{e}^{-\mu r}(1+\mu r)\right], \tag{6.156}$$

i.e. a modification of Kepler's third law.

Let us start solving the second iteration by considering the massless transverse traceless tensor mode $U^{\alpha\beta}_{(2)}$*. The contribution* $\mathrm{e}_{(1)}[H_{(2)}]$ *follows from the first of the linearised transverse traceless tensor field equations* (6.86)*, which reads*

$$\left[\frac{\delta L}{\delta u^{\alpha\beta}}\right]^{TT} = \frac{1}{8\alpha}\Box U_{\alpha\beta}. \tag{6.157}$$

As before, when solving the metric problem, the vectors tangent to the worldlines contribute a factor of $\sqrt{G}$ *each, such that there is no contribution from* $T_{(1)}[\gamma_{(1)}, \gamma_{(2)}, H_{(1)})$*, but only from the lower order* $T_{(0)}[\gamma_{(1)}, \gamma_{(2)}]$*. Evaluating this term by variation of the matter action* (6.148) *with respect to the field* $u^{\alpha\beta}$*, we obtain the equation*

$$-\frac{c^3 G}{16\pi}\frac{1}{8\alpha}\Box U^{\alpha\beta}_{(2)} = \left\{\frac{1}{4c}\sum_{i=1,2} m_i \delta^{(3)}(\vec{x} - \vec{\gamma}_{(i)}(t))\dot{\gamma}^{\alpha}_{(i)}\dot{\gamma}^{\beta}_{(i)} + \left(\frac{\delta S_{gravity}}{\delta u_{\alpha\beta}}\right)_{(2)}[GH_{(1)}]\right\}^{TT}. \tag{6.158}$$

The contribution from the second-order field equations is again calculated using `cadabra` *[69, 70]. The process is roughly as follows: first, the the third-order ansätze* (A.5)–(A.7) *and the corresponding coefficient relations are loaded into the programme. Then, the Lagrangian is decomposed into observer quantities, shift, and lapse, according to the* $3+1$ *decomposition introduced in Sect. 6.1.3. All fields, except for the field* $u^{\alpha\beta}$ *which will be* *varied, are replaced with the solution from the first iteration, using abbreviations*

$$X = \int d^3\vec{y}\rho(\vec{y})\frac{1}{|\vec{x}-\vec{y}|},$$
$$Y = \int d^3\vec{y}\rho(\vec{y})\frac{\mathrm{e}^{-\mu|\vec{x}-\vec{y}|}}{|\vec{x}-\vec{y}|}. \tag{6.159}$$

This simplifies the Lagrangian significantly, because it only depends on the scalar fields X and Y, as well as the field $u^{\alpha\beta}$. Performing the variation with respect to the remaining tensorial field, projecting the result onto the transverse traceless tensor mode, and further simplifying finally yields

$$\left(\frac{\delta S_{gravity}}{\delta u_{\alpha\beta}}\right)^{TT}_{(2)}[GH_{(1)}] = \frac{G}{16\pi c}[\alpha\partial^\alpha X\partial^\beta X + \beta\partial^\alpha Y\partial^\beta Y]^{TT}. \tag{6.160}$$

The `cadabra` *code can be found in Ref. [7].*

Being also a massless wave equation, the differential equation (6.158) *is solved like before in Sect. 6.2.2, by convolution of the right-hand side with the retarded Green's function of the d'Alembert operator. Taking the same limit for the far zone, the solution is*

$$U^{\alpha\beta}_{(2)} = -\frac{\alpha}{c^4R}\left[\frac{8}{G}K^{\alpha\beta} + \frac{2\alpha}{\pi}\Phi^{\alpha\beta}_{(0)} + \frac{2\beta}{\pi}\Phi^{\alpha\beta}_{(\mu)}\right]^{TT}, \tag{6.161}$$

with a kinetic term

$$K^{\alpha\beta} = \int d^3\vec{y}\sum_{i=1,2} m_i\delta^{(3)}(\vec{y}-\vec{\gamma}_{(i)}(\tau))\dot{\gamma}^\alpha_{(i)}\dot{\gamma}^\beta_{(i)} \tag{6.162}$$

and the potential terms

$$\Phi_{(\mu)\alpha\beta} = \int d^3\vec{y}\int d^3\vec{y'}\int d^3\vec{y'}'\rho(\vec{y'})\rho(\vec{y'}')\left(\partial_\alpha\frac{\mathrm{e}^{-\mu|\vec{z}|}}{|\vec{z}|}\right)_{\vec{z}=\vec{y}-\vec{y'}}\left(\partial_\beta\frac{\mathrm{e}^{-\mu|\vec{z}|}}{|\vec{z}|}\right)_{\vec{z}=\vec{y}-\vec{y'}'}. \tag{6.163}$$

Working out the integrals results in a first prediction for the gravitational radiation produced by a binary star subject to area metric gravity. On the massless transverse traceless tensor mode, radiation is emitted into the far zone $R \gg r$ according to the formula

$$G^2U^{\alpha\beta}_{(2)} = -\frac{8\eta}{c^4R}\frac{(G\alpha m)^2}{r}[1+f(r)][\lambda^\alpha\lambda^\beta - n^\alpha n^\beta]^{TT}, \tag{6.164}$$

where the correction term $f(r)$ is given by

$$f(r) = \frac{\beta}{\alpha}(1 + \mu r)\mathrm{e}^{-\mu r}. \tag{6.165}$$

In order to point out the significance of Eq. (6.164)*, let us come back to the metric radiation formula* (6.143) *for the modes $E^{\alpha\beta}$. The area metric result amounts to the metric result up to a correction proportional to $f(r)$, which—being of Yukawa type—falls off exponentially with the separation r. In formulæ,*

$$G^2 U^{\alpha\beta}_{(2)} = 2(\alpha G)^2 E^{\alpha\beta}_{(2)}[1 + f(r)]. \tag{6.166}$$

Considering that the area metric perturbation induced by the metric perturbation (6.143) *has the only nonvanishing modes $U^{\alpha\beta}_{(2)} = 2E^{\alpha\beta}_{(2)}$, we arrive at the remarkable conclusion that—on the metrically inducible modes and in the far zone—the radiation emitted by a binary star in circular motion is qualitatively the same, but quantitatively* refined. *Both Kepler's third law and the amplitude of the emitted waves pick up Yukawa corrections, which originate from the presence of mass terms in the scalar field equations for stationary and slowly moving sources. These corrections can become arbitrarily small—by restricting the parameter range or considering large enough radii r. In this sense, gravitational radiation as predicted in Einstein gravity is contained within the area metric result.*

The remaining propagating modes of the area metric perturbation have all shown to be governed by massive wave equations (see Sect. 6.1.4). Let us first consider the traceless modes, i.e. the massive transverse traceless tensors, the vectors, and the traceless scalars. Since the coefficients in the relevant wave equations (6.84)*–*(6.86) *are the same for all of these modes, regardless of whether the equations are of scalar, vector, or tensor type, all propagating traceless modes can be considered on the same footing.*

We define the tracefree auxiliary field $\tilde{v}^{\alpha\beta} = v^{\alpha\beta} - \frac{1}{3}\gamma^{\alpha\beta}\gamma_{\mu\nu}v^{\mu\nu}$. Taking appropriate linear combinations of the linearised field equations (see e.g. (6.93)*), the modes $\tilde{v}^{\alpha\beta}$ decouple from the modes $w^{\alpha\beta}$, such that the left-hand side $\mathrm{e}_{(1)}[H_{(2)}]$ of the second iteration equations is given by $\Box\tilde{v}^{\alpha\beta}_{(2)} + \nu^2\tilde{v}^{\alpha\beta}_{(2)}$ and $\Box w^{\alpha\beta}_{(2)} + \nu^2 w^{\alpha\beta}_{(2)}$, respectively. Because the linearised matter action* (6.148) *does only depend on the trace of $v^{\alpha\beta}$ and is entirely independent of $w^{\alpha\beta}$, there is no matter contribution to the second iteration. A calculation of the contribution $\mathrm{e}_{(2)}[H_{(1)}]$, employing the previously outlined* `cadabra`*-based technique, yields the wave equations*

$$\begin{aligned} \Box\tilde{v}^{\alpha\beta}_{(2)} + \nu^2\tilde{v}^{\alpha\beta}_{(2)} &= \delta[\partial^\alpha X \partial^\beta Y]^{TF}, \\ \Box w^{\alpha\beta}_{(2)} + \nu^2 w^{\alpha\beta}_{(2)} &= \epsilon[\partial^\alpha X \partial^\beta Y]^{TF}. \end{aligned} \tag{6.167}$$

The label $[\cdot]^{TF}$ *denotes the idempotent projection*

$$[t^{\alpha\beta}]^{TF} = t^{(\alpha\beta)} - \frac{1}{3}\gamma_{\mu\nu}t^{\mu\nu}\gamma^{\alpha\beta} \tag{6.168}$$

onto the tracefree symmetric part. Both δ *and* ϵ *are combinations of gravitational constants that include genuine third-order constants, i.e. coefficients for the third order in the area metric Lagrangian expansion that are not solely determined by second-order coefficients.*[20]

Solving the wave equations (6.167) *is again a matter of convoluting the source terms with a retarded Green's function. This time, the differential equation is of the kind*

$$(\Box + m^2)\psi(x) = 4\pi\varphi(x) \tag{6.169}$$

and thus solved by the massive propagator

$$G_{ret}(x, y) = \theta(x^0 - y^0)\int \frac{d^3\vec{k}}{(2\pi)^3}\frac{\sin[\omega_k(x^0 - y^0)]}{\omega_k}\mathrm{e}^{\mathrm{i}\vec{k}\cdot(\vec{x}-\vec{y})}, \tag{6.170}$$

where θ *is the Heaviside step function and the massive dispersion relation*

$$\omega_k = \sqrt{|\vec{k}|^2 + m^2} \tag{6.171}$$

holds. The convolution integrals work out differently this time, depending on the value of $\omega_0 := 2\omega$.

1. ***nonradiating solution**:* $\omega_0 < c\nu$

   The gravitational fields decay exponentially with distance R *from the binary system, e.g. in the orbit-adapted frame they are given as*

$$G^2\tilde{v}^{\alpha\beta}_{(2)} = \frac{\delta\eta}{c^4 R}\frac{(Gm)^2}{r}g(r)\left[3\mathrm{e}^{-\tilde{\nu}R}\begin{pmatrix}\cos\omega_0 t & \sin\omega_0 t & 0\\ \sin\omega_0 t & -\cos\omega_0 t & 0\\ 0 & 0 & 0\end{pmatrix} + \mathrm{e}^{-\nu R}\begin{pmatrix}\frac{1}{2} & & \\ & \frac{1}{2} & \\ & & -1\end{pmatrix}\right]^{\alpha\beta} \tag{6.172}$$

[20] Earlier, the contrary was true: for the massless mode, the second iteration equation—although derived from the third-order expansion—was completely determined by second-order coefficients of the Lagrangian expansion. The perturbative equivariance equations can, and in general will, in each iteration determine some higher-order coefficients by lower-order coefficients.

with the abbreviations

$$\begin{aligned} g(r) &= \frac{1-[1+\mu r+\frac{1}{3}(\mu r)^2]\mathrm{e}^{-\mu r}}{(\mu r)^2}, \\ \tilde{\nu} &= \sqrt{\nu^2-\left(\frac{\omega_0}{c}\right)^2}. \end{aligned} \tag{6.173}$$

For the traceless modes $w^{\alpha\beta}$*, we obtain the same solution, but with prefactor* ϵ *instead of* δ*. Note that the oscillating part has a "direct" dependence on the coordinate time* t*. Characteristic behaviour of a radiating solution would be a dependence through the retarded time* τ*.*

2. ***radiating solution****:* $\omega_0 > c\nu$

   The nonoscillating part of the previous solution (6.172) *is not affected. Since it decreases exponentially with* R *and we are interested in the far zone* $R \gg r$*, it will be dropped from now on—being shadowed by another contribution that is proportional to* $\frac{1}{R}$*. This term* radiates *according to*

$$G^2\tilde{v}^{\alpha\beta}_{(2)} = \frac{3\delta\eta}{c^4 R}\frac{(Gm)^2}{r}g(r)[n^\alpha n^\beta - \lambda^\alpha\lambda^\beta], \tag{6.174}$$

   where the phase of the orbit-adapted frame is now $\frac{\varphi}{2}$ *with*

$$\varphi = \omega_0 t - \sqrt{\omega_0^2-(c\nu)^2}\frac{R}{c} =: \omega_0 t - \tilde{\omega}\frac{R}{c}. \tag{6.175}$$

   Earlier, for the massless modes, we had $\varphi = \omega_0\tau$ *or, equivalently,* $\frac{\varphi}{2} = \omega\tau$*. Again, the solution for* $w^{\alpha\beta}$ *is the same up to the prefactor of* δ*, which has to be replaced with* ϵ*. The dependence on coordinate time is only via a retardation term* (6.175)*.*

The fact that radiation is "switched on" only above a certain angular frequency threshold is an expected and welcome property. It is expected *because of the mass* ν *in the wave equations. An analogy would be a massive particle in relativistic quantum field theory, which requires a minimum energy—its mass—in order to be created. Earlier results [76] in area metric gravity discovered a similar behaviour for electromagnetically bound binaries, which has now been shown to extend to gravitationally bound systems, where radiation is an effect of gravitational self-coupling. The result is certainly encouraging for the viability of area metric gravity, as it once again keeps the theory very close to Einstein gravity and introduces only modest modifications, assuming that parameters are chosen appropriately. Without this property, it would be impossible to reconcile area metric gravity with the metric theory, ruling it out as a candidate for a modified theory of gravity. On the other hand, we observe a new* quality*: propagation of massive gravitational waves* *on nonmetric*[21] *modes.*

One propagating degree of freedom has not been considered so far. The scalar degree of freedom $\tilde{V}$ *obeys a massive wave equation* (6.94)*, which is of mass* μ*, but comes with an additional complexity: it is a linear combination of scalar field equations, such that the second iteration equation takes the form*

$$\Box\tilde{V}_{(2)} + \mu^2\tilde{V}_{(2)} = -\gamma\left[\frac{1}{4}\rho_A - \left(1 + \frac{3}{4}\frac{\gamma}{\beta}\right)\rho_u + \frac{\gamma}{4\beta}\rho_v\right]. \tag{6.176}$$

The three source terms ρ_u*,* ρ_v*, and* ρ_A *denote the right-hand sides of the second iteration equations* (6.103) *that originate from variations of the actions with respect to* $u^{\alpha\beta}$*,* $v^{\alpha\beta}$*, and* A*, respectively. Except for the* A *variation, the trace has to be taken afterwards. For our previous results, we only picked up zeroth-order contributions from the variations of the matter action, such as*

$$\left(\frac{\delta S_{matter}}{\delta u^{\alpha\beta}}\right), \tag{6.177}$$

because the fields would couple to the spatial components of the particle worldline tangents. These come with factors of $\sqrt{G}$*, rendering the first-order contributions a higher order than considered for the second iteration. For the variation with respect to the lapse, this is* not *the case, as an expansion of the GLED polynomial to second order shows. The lapse perturbation comes with terms proportional to* $[\dot{\gamma}^0_{(i)}]^4 = c^4 = \mathcal{O}(G^0)$*, which illustrates that we have to expect contributions* $T_{(1)}[\gamma_{(1)}, \gamma_{(2)}, H_{(1)})$ *in the second iteration equation* (6.176)*.*

Unfortunately, the second-order expansion of the GLED principal polynomial is not effectively metric anymore. The straightforward way of applying the Legendre transform—lowering indices with the help of the covariant metric—is not available in this case. In fact, there is no closed expression for the Legendre transform corresponding to a quartic polynomial. [29] While it is certainly possible to treat the problem perturbatively, it is considered out of scope for the purpose of this thesis. After all, we do not seek to derive a comprehensive solution, but rather wish to demonstrate the ramifications of novel matter dynamics on the gravitational phenomenology. The results for the massless transverse traceless tensor modes, as well as the massive tracefree modes, already allow to make nontrivial predictions concerning the binary star, such that more extensive knowledge of the remaining scalar modes is deemed dispensable.

[21] In the sense of not inducible by a metric tensor.

6.3 Phenomenology of area metric gravitational radiation

The quintessence of Sect. 6.2 is the prediction of gravitational waves emitted by a binary star subject to area metric gravity. While compatible with the behaviour of Einstein gravity in certain limits, the area metric result offers new features such as a modification of Kepler's third law, which determines the angular frequency, or radiation on massive modes. All of these effects, however, concern the gravitational field *and are thus inaccessible to direct observations. This is because the geometric fields only play an auxiliary rôle in the ensemble of physical fields. Observable effects of gravity involve the* matter fields *whose dynamics are governed by the geometry in question.*

In order to derive observable predictions from the previous results, we will first consider a distribution of test matter and study the effect of a passing gravitational wave. This yields the usual deformations known as geodesic deviation, possibly amended by novel deformation patterns. A more direct effect of the radiation that is emitted by a binary star is its energy loss, which makes the binaries reduce their distance and spin faster as the system loses energy through radiation. This will serve as second prediction.

6.3.1 Effect on test matter

Let us probe the gravitational field using an arrangement of matter called a geodesic sphere*. It is composed of freely falling point masses that are, at least initially, distributed spherically on a spatial hypersurface. The masses are* test masses, *which is to say that their gravitational field is negligible compared to the field we like to probe, the incident gravitational wave. To first order, the dynamics of point masses in area metric gravity are effectively metric (see Eq.* (6.10)*). The standard procedures from metric gravity for studying the motion of point masses are thus applicable, including the geodesic deviation equation [75]*

$$\frac{1}{c^2}\ddot{X}^\alpha = -R^\alpha{}_{0\beta 0}X^\beta \tag{6.178}$$

for the spatial deviation vector $\vec{X}$*. Applying the* $3+1$ *split of a metric tensor (see Eqns.* (6.63) *and* (6.64)*) to the effective metric, the Riemann tensor* $R^\alpha{}_{0\beta 0}$ *can be expanded to linear order, such that the deviation equation assumes the form*

$$\ddot{X}^\alpha = -\frac{1}{2}[\ddot{\varphi}^\alpha{}_\beta + c(\partial_\beta \dot{b}^\alpha + \partial^\alpha \dot{b}_\beta) + 2c^2\partial^\alpha\partial_\beta A]X^\beta. \tag{6.179}$$

For small perturbations, the deviation due to purely spatial fields $\varphi^{\alpha\beta}$ *is integrated as*

$$X^\alpha(t) = X^\alpha(0) - \frac{1}{2}\varphi^\alpha{}_\beta(t)X^\beta(0). \tag{6.180}$$

This constitutes the starting point for the following predictions.

In Sect. 6.2, all modes of gravitational radiation—except the trace modes—have been found to be proportional to the various projections of

$$M^{\alpha\beta} = \begin{pmatrix} \cos(\varphi) & \sin(\varphi) & 0 \\ \sin(\varphi) & -\cos(\varphi) & 0 \\ 0 & 0 & 1 \end{pmatrix}^{\alpha\beta} \tag{6.181}$$

onto the transverse traceless tensor, vector, and scalar traceless subspaces. The phase φ *is either given by* $\varphi = 2\omega\tau$ *for massless modes or Eq.* (6.175) *for massive modes.*

The tensor $M^{\alpha\beta}$ *is still expressed in the* orbit-adapted *frame* $(\vec{e}_x, \vec{e}_y, \vec{e}_z)$*, whose orientation is determined by the orbital plane. When considering the effect on test matter, it is more instructive to switch to the* detector-adapted *frame* $(\vec{e}_X, \vec{e}_Y, \vec{e}_Z)$ *[74], which is oriented such that the Z-axis points from the barycentre of the binary star to the test matter distribution. A simple rotation around the x-axis transforms between both frames[22], such that the detector-adapted frame is expressed in terms of the orbit-adapted frame as*

$$\vec{e}_X = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \vec{e}_Y = \begin{pmatrix} 0 \\ \cos\iota \\ -\sin\iota \end{pmatrix}, \vec{e}_Z = \begin{pmatrix} 0 \\ \sin\iota \\ \cos\iota \end{pmatrix}. \tag{6.182}$$

The angle ι *measures the inclination of the orbital plane as seen from the XY-plane.[23]*

Transforming $M^{\alpha\beta}$ *accordingly and projecting onto the several modes, we obtain the transverse traceless tensor mode*

$$M^{\mathrm{TT}} = \begin{pmatrix} \frac{1}{2}(1+\cos^2\iota)\cos\varphi & \cos\iota\sin\varphi & 0 \\ \cos\iota\sin\varphi & -\frac{1}{2}(1+\cos^2\iota)\cos\varphi & 0 \\ 0 & 0 & 0 \end{pmatrix}, \tag{6.183}$$

the vector mode

$$M^{\mathrm{V}} = \begin{pmatrix} 0 & 0 & \sin\iota\sin\varphi \\ 0 & 0 & -\cos\iota\sin\iota\cos\varphi \\ \sin\iota\sin\varphi & -\cos\iota\sin\iota\cos\varphi & 0 \end{pmatrix}, \tag{6.184}$$

[22] As the circular orbit is completely isotropic with respect to rotations around the z-axis, we can always make this choice. Otherwise, we would need to parameterise the frame using *two* angles, i.e. first perform a rotation around the z-axis and only afterwards around the x-axis. [74]

[23] Put differently, ι is the angle between the Z-axis (pointing towards the test matter distribution) and the z-axis (the rotation axis of the binary star).

and the traceless scalar mode

$$M^{\mathrm{S-TF}} = \sin^2 \iota \cos\varphi \begin{pmatrix} \frac{1}{2} & 0 & 0 \\ 0 & \frac{1}{2} & 0 \\ 0 & 0 & -1 \end{pmatrix}. \tag{6.185}$$

Out of these, the TT part (6.183) *is already known from general relativity, where we found as only radiating mode into the far zone the transverse traceless tensor perturbation*

$$\varphi^{\alpha\beta} = \underbrace{\frac{4\eta}{c^4 R}\frac{(Gm)^2}{r}}_{:=2d}(M^{\mathrm{TT}})^{\alpha\beta}.$$

Consequently, the right-hand side of the geodesic deviation equation (6.179) is *purely spatial and we obtain the deviation*

$$X^\alpha(t) - X^\alpha(0) = -d \times (M^{\mathrm{TT}})^\alpha{}_\beta(t) X^\beta(0), \tag{6.186}$$

oscillating with the phase $\varphi(t) = 2\omega\tau = 2\omega(t - \frac{R}{c})$*. Geodesic spheres are thus deformed in both lateral directions, but not in the direction of the incident wave. The deformation is volume-preserving, because there is a phase shift of* $\frac{\pi}{2}$ *between the lateral axes—elongation into one direction comes with contraction into the other direction. As a result, the sphere assumes the form of an* ellipsoid *with rotating axes, while its dimension in the Z-direction stays constant.*

Predictions of this effect exist since the inception of general relativity. [77, 78] The confirmation of metric geodesic deviation due to incident gravitational radiation in 2015 by the LIGO and Virgo collaborations [8] marks the culmination of 100 years of general relativity research.

As far as massless modes of radiation are concerned, geodesic deviation in area metric *gravity looks roughly the same: from Eq.* (6.146)*, we know that the spatial part* $\varphi^{\alpha\beta}$ *of the effective metric that constitutes the linearised principal polynomial and enters the deviation equation* (6.179) *amounts to*

$$\varphi^{\alpha\beta} = \frac{1}{2}u^{\alpha\beta} + \frac{1}{2}\gamma_{\mu\nu}v^{\mu\nu}\gamma^{\alpha\beta}. \tag{6.187}$$

If only the massless TT mode is "switched on", i.e. the angular frequency of the binary star is below the threshold determined by the masses of the nonmetric modes, Eq. (6.164) *gives*

$$\varphi^{\alpha\beta} = 2d[1 + f(r)](M^{\mathrm{TT}})^{\alpha\beta}. \tag{6.188}$$

In such a case, the area metric deviation coincides with the metric deviation up to a correction factor $1 + f(r)$*. For large radii, but also for appropriate choices of gravitational* *constants, the factor becomes arbitrarily close to* 1*. In this limit, the metric result is recovered. Otherwise, area metric gravity introduces a correction factor into the otherwise unchanged effect—this could be called a* quantitative *refinement.*

More intrusive modifications would come from vector or scalar contributions to Eq. (6.179)*. Taking a closer look at the corresponding deformation matrices* (6.184)*–*(6.185)*, it is apparent what these could entail: more interesting patterns of deformations that include contractions and expansions in the direction of the incident wave—an effect that is unknown from Einstein gravity. However, if nonmetric deformations can be observed at all, they are well-hidden in the scalar modes, as a substitution of the area metric perturbations in Eq.* (6.179) *shows. The vector contribution from $\varphi^{\alpha\beta}$ is cancelled by the shift perturbation b^α using the vector field equations* (6.85)*. What remains are the massless TT perturbations $U^{\alpha\beta}$, whose effect has already been discussed, and a scalar contribution. Without more knowledge of the radiation on* all *scalar modes, it is not possible to give a definite prediction about the effect of scalar waves on test matter.*

6.3.2 Binary star spin-up

While the gravitational radiation that causes geodesic deviation is produced as second-order effect, the deviation *per se* has only been studied to first order in the previous section. Similar results would hold for incident waves that have their origin in linearised gravity, like the radiation that is emitted from nongravitationally bound systems [76].

An effect that cannot be observed in a solely linearised setting is *radiation reaction*: during the iterative solution procedure, the matter trajectories have been fixed to first order, which provides the background for the second-order gravitational field we were interested in. This is by no means necessary—it is possible to solve for higher orders n before fixing the matter fields, which then enables the prediction of the gravitational field to order $n+1$. The modern treatment of post-Newtonian and post-Minkowskian general relativity proceeds in exactly this way. [74] Doing so, the matter trajectories accumulate corrections, which are *backreactions* from the gravitational field sourced by the matter content itself. In the context of a gravitationally bound matter distribution which emits gravitational radiation to second order, these backreactions are often referred to as radiation reaction.

In general relativity, the aforementioned modern perturbative treatment yields detailed predictions for the deviation of a binary system from Kepler orbits. These calculations are quite intricate, taking into account not only higher perturbation orders but also the internal structure of the binaries. For a tentative qualitative result, however, we do not need to go there. Noether's second theorem (see Thm. 2.3.4) provides us with a tool that resembles energy conservation equations.[24] Loosely speaking, gravitational waves radiate away energy from the system, decimating the radius of the Kepler orbits, which results—via Kepler's third law—in the binary star *spinning up*. The qualitative analysis from "energy conservation" yields a rate of change for the angular frequency but cannot predict how exactly the trajectories are affected, e.g. how the phase shifts. Still, the prediction[25] for the orbital period decrease dP/dt of the Hulse-Taylor pulsar PSR B1913+6 is in very strong agreement with the measurement, as the ratio amounts to [79]

$$\frac{(dP/dt)_{\text{observed}}}{(dP/dt)_{\text{predicted}}} = 0.9983 \pm 0.0016. \tag{6.189}$$

Again, let us first illustrate the calculation for metric gravity before diving into area metric gravity. The total Lagrangian density of Maxwell electrodynamics and Einstein gravity is

$$L = L_{\text{matter}} + L_{\text{gravity}}. \tag{6.190}$$

Following a normal coordinate argument[26] from Ref. [39], the second Noether identity (2.47) reduces to the vanishing of the divergence of the Gotay-Marsden stress-energy-momentum tensor density, i.e.

$$0 = \partial_n \left[\mathcal{T}^n_m(g)\right]. \tag{6.191}$$

If the section g of the metric bundle satisfies the Euler-Lagrange equations, the integral equation

$$\begin{aligned} 0 &= \int_\Sigma \partial_0 \left[C^A{}_B{}^0{}_0 g^B \frac{\delta L}{\delta g^A} \right] d^3x \\ &= \int_\Sigma \partial_0 \left[C^A{}_B{}^0{}_0 g^B \frac{\delta L_{\text{matter}}}{\delta g^A} \right] d^3x + \int_\Sigma \partial_0 \left[C^A{}_B{}^0{}_0 g^B \frac{\delta L_{\text{gravity}}}{\delta g^A} \right] d^3x \end{aligned} \tag{6.192}$$

over a spatial slice Σ holds. Renaming the first term and making use of the Noether

[24] The notion of *energy* in general relativity and, more specifically, *energy conservation* is subject to many debates. General relativity does not exhibit the kind of time-translation symmetry that is usually the justification for the definition of energy. In the setting considered here, where the matter content is localised to a specific region of spacetime and the geometry is asymptotically flat, such a notion can be recovered from symmetries that hold asymptotically. [27] For our purposes, we do not rely on the interpretation of certain quantities as energies or momenta. They are just derived quantities from the fundamental fields. Changes in these quantities are interesting insofar as they pertain to changes in the underlying fields.

[25] Considering, of course, the eccentric case with the appropriate parameters.

[26] For the metric tensor bundle, there are always local coordinates such that $g^{ab}{}_{,p} = 0$. In this coordinate chart, the divergence of the SEM tensor density vanishes. Being a tensor density, it follows that the components of the divergence vanish in *any* coordinate chart.

identity (6.191) for the second term, this yields

$$\begin{aligned} 0 &= \partial_0 \underbrace{\int_\Sigma C^A{}_B{}^0{}_0 g^B \frac{\delta L_{\text{matter}}}{\delta g^A} d^3x}_{=:\mathcal{H}_{\text{matter}}} + \int_\Sigma \partial_0 \left[C^A{}_B{}^0{}_0 g^B \frac{\delta L_{\text{gravity}}}{\delta g^A} \right] d^3x \\ &= \frac{1}{c}\dot{\mathcal{H}}_{\text{matter}} - \int_\Sigma \partial_\alpha \left[C^A{}_B{}^\alpha{}_0 g^B \frac{\delta L_{\text{gravity}}}{\delta g^A} \right] d^3x. \end{aligned} \tag{6.193}$$

Finally, we apply the Gauß theorem to the second integral, such that

$$\dot{\mathcal{H}}_{\text{matter}} = c \int_{S_\infty} C^A{}_B{}^\alpha{}_0 g^B \frac{\delta L_{\text{gravity}}}{\delta g^A} dS_\alpha, \tag{6.194}$$

which should be understood as the limit of surface integrals over a family of appropriate closed surfaces that approach infinity.

The variations of both Lagrangians have already been worked out for Eq. (6.107). For the matter part, we obtain to lowest nontrivial order

$$\mathcal{H}_{\text{matter}} = 2 \int_\Sigma g^{0a} \frac{\delta L_{\text{matter}}}{\delta g^{0a}} d^3x \approx -\frac{1}{c}[mc^2 + \frac{1}{2} m\eta r^2 \omega^2] = \frac{1}{c}[E_0 + E] \tag{6.195}$$

with the constant energy $E_0 = -mc^2$ for the system at rest and the kinetic energy

$$E = -\frac{1}{2} m\eta r^2 \omega^2 = -\frac{1}{2}\eta \frac{Gm^2}{r}. \tag{6.196}$$

Thus, the left-hand side of the balance equation (6.194) is given as

$$\dot{\mathcal{H}}_{\text{matter}} = \frac{1}{c}\dot{E} = \frac{1}{2c}\eta G m^2 \frac{\dot{r}}{r^2}. \tag{6.197}$$

For the right-hand side, we have the full densitised Einstein tensor $\mathcal{G}^{ab} = \frac{16\pi G}{c^3} \frac{\delta L_{\text{gravity}}}{\delta g_{ab}}$ at our disposal. To lowest order, the integral amounts to

$$\begin{aligned} c \int_{S_\infty} C^A{}_B{}^\alpha{}_0 \frac{\delta L_{\text{gravity}}}{\delta g^A} g^B dS_\alpha &= \frac{c^4}{8\pi G} \int_{S_\infty} g_{0a} \mathcal{G}^{a\alpha} dS_\alpha \\ &\approx \frac{c^4}{8\pi G} \int_{S_\infty} \mathcal{G}^{0\alpha} dS_\alpha. \end{aligned} \tag{6.198}$$

Since the integral is evaluated at infinity, only the radiation part is relevant. But this part is given by transverse traceless tensor perturbations, such that divergences and traces of the spatial perturbation $h^{\mu\nu}$ can readily be dropped when extracting the lowest nonvanishing order of the integral (6.198). Doing so, we arrive at

$$\begin{aligned}\int_{S_\infty} \mathcal{G}^{0\alpha} dS_\alpha &= -\frac{1}{4}\int_{S_\infty} [\partial^\alpha h_{\mu\nu}\partial_0 h^{\mu\nu} + 2\partial_0\partial^\alpha h_{\mu\nu}h^{\mu\nu}]dS_\alpha \\ &= \frac{1}{4c^2}\int_{S_\infty} [\dot{h}_{\mu\nu}\dot{h}^{\mu\nu} + 2\ddot{h}_{\mu\nu}h^{\mu\nu}]dS \\ &= -\frac{1}{4c^2}\int_{S_\infty} \dot{h}_{\mu\nu}\dot{h}^{\mu\nu}dS + \frac{1}{4c^2}\int_{S_\infty} [h_{\mu\nu}h^{\mu\nu}]^{\cdot\cdot}dS,\end{aligned} \tag{6.199}$$

which we further simplified using the identity (letting $N^\alpha = x^\alpha/R$)

$$\partial_\alpha h^{\mu\nu} = -\frac{1}{c}N^\alpha \dot{h}^{\mu\nu} + \mathcal{O}\left(\frac{1}{R^2}\right) \tag{6.200}$$

for radiation terms $\propto \frac{1}{R}f(\omega(ct-R))$.

It is now time to take the concrete form of the metric perturbation into account. Earlier, we arrived at the result (6.143)

$$h^{\mu\nu} = \frac{4\eta}{c^4 R}\frac{(Gm)^2}{r}[n^\alpha n^\beta - \lambda^\alpha\lambda^\beta]^{\cdot}(\mathrm{TT})^{\mu\nu}{}_{\alpha\beta}, \tag{6.201}$$

where this time the projection onto the transverse traceless tensor mode is made explicit using the projector [74]

$$\begin{aligned}(\mathrm{TT})^{\mu\nu}{}_{\alpha\beta} &= P^{(\mu}{}_\alpha P^{\nu)}{}_\beta - \frac{1}{2}P^{\mu\nu}P_{\alpha\beta}, \\ P^\alpha{}_\beta &= \delta^\alpha_\beta - N^\alpha N_\beta.\end{aligned} \tag{6.202}$$

Under these circumstances, the contraction $h_{\mu\nu}h^{\mu\nu}$ is constant with respect to coordinate time and thus does not contribute to the integral. The vectors n and λ only depend on the radius R and coordinate time t, such that the angular dependence is completely contained within the TT projector. This reduces the integral to

$$\int_{S_\infty} \mathcal{G}^{0\alpha} dS_\alpha = -\frac{256\pi\eta^2(Gm)^5}{c^{10}r^5} n^\alpha n_\mu \lambda^\beta \lambda_\mu \langle (\mathrm{TT})^{\mu\nu}{}_{\alpha\beta}\rangle, \tag{6.203}$$

where only the angular average

$$\langle X\rangle := \frac{1}{4\pi}\int_S X d\Omega \tag{6.204}$$

of the projector remains to be calculated. Referring to Ref. [74] for the details, we just make use of the result

$$\langle (\mathrm{TT})^{\mu\nu}{}_{\alpha\beta} \rangle = \frac{2}{5}\delta_\alpha^{(\mu}\delta_\beta^{\nu)} \tag{6.205}$$

and arrive at

$$\int_{S_\infty} \mathcal{G}^{0\alpha} dS_\alpha = -\frac{256\pi}{5}\eta^2 \left(\frac{Gm}{c^2 r}\right)^5 . \tag{6.206}$$

Together with the left-hand side (6.197) of the balance equation (6.194), this yields a first approximation for the spin-up of a binary star due to radiation loss. The separation r of the stars decreases with the rate

$$\dot{r} = -\frac{64}{5}\eta c \left(\frac{Gm}{c^2 r}\right)^3 , \tag{6.207}$$

which translates into an increase of the angular frequency ω, according to Kepler's third law.

This result for the lowest-order radiation loss approximation is in agreement with the literature [74]. While we had prior knowledge of the full Lagrangian and the corresponding Einstein equations, the approach is not restricted to such theories. A perturbatively constructed third-order Lagrangian can be used just as well and will yield a comparable prediction of binary star spin-up in area metric gravity.

The area metric calculation starts out similarly. Even though the right-hand side of the Noether identity (2.47) does not vanish this time, it can be neglected because it is always of one order higher than the lowest order of the left-hand side, due to the appearance of $G^A_{,m} = H^A_{,m}$. Let us also consider only radiation on the massless TT mode $U^{\alpha\beta}$. This is sufficient in order to derive a nontrivial effect and it can be interpreted as the phase of binary spin-up during which the angular frequency is not yet high enough in order for the system to produce massive waves on the nonmetric modes.

As we did before when deriving Eq. (6.71), we start by inverting the relation (6.56) between the spacetime area metric perturbation and the perturbed observer quantities. For the matter contribution on the left-hand side of the balance equation, this allows us to calculate the variation by only varying with respect to A, as

$$C^A{}_B{}^0{}_0 G^B \frac{\delta L_{\text{matter}}}{\delta G^A} = -4\frac{\delta L_{\text{matter}}}{\delta A} + \mathcal{O}(H^2). \tag{6.208}$$

The second variation in question behaves similarly, reducing the relevant variation of the gravity Lagrangian to

$$C^A{}_B{}^\alpha{}_0 G^B \frac{\delta L_{\text{gravity}}}{\delta G^A} = 4\frac{\delta L_{\text{gravity}}}{\delta b_\alpha} + \mathcal{O}(H^2). \tag{6.209}$$

Roughly the same result as for metric gravity is obtained for the matter part

$$\mathcal{H}_{\text{matter}} = \frac{4}{c}[E_0 + E], \tag{6.210}$$

where $E_0 = -mc^2$ and

$$E = -\frac{1}{2}m\eta r^2\omega^2 = -\frac{1}{2}m\eta\frac{\alpha G m}{r}[1 + f(r)]. \tag{6.211}$$

For the gravity part, we again use `cadabra` [69, 70] in order to derive the contributions from the transverse traceless $U^{\alpha\beta}$ modes (6.164) to $\frac{\delta L_{\text{gravity}}}{\delta b^{\alpha}}$. This yields via a similar calculation as before the right-hand side

$$\begin{aligned} 4c\int_{S_\infty}\frac{\delta L_{\text{gravity}}}{\delta b_\alpha}dS_\alpha &= \frac{c^2}{4\pi G}\int_{S_\infty}\left[(-\frac{1}{8\alpha})\dot{U}_{\alpha\beta}\dot{U}^{\alpha\beta} + (\frac{1}{4\alpha} - 4k_{12})\left(U_{\alpha\beta}\dot{U}^{\alpha\beta}\right)^{\cdot}\right]dS \\ &= -\frac{c^2}{32\pi\alpha G}\int_{S_\infty}\dot{U}_{\alpha\beta}\dot{U}^{\alpha\beta}dS \\ &= -\frac{128}{5}m\eta^2\frac{(1+f(r))^3}{r^5}\frac{(\alpha G m)^4}{c^6} \end{aligned} \tag{6.212}$$

of the balance equations. Putting together both sides results in the rate of change

$$\left(\frac{1+f(r)}{r}\right)^{\cdot} = -\frac{\dot{r}}{r^2}[1 + f(r) - rf'(r)] = \frac{64}{5}\eta c\frac{1}{r^2}\left(\frac{\alpha G m}{c^2 r}[1 + f(r)]\right)^3. \tag{6.213}$$

As always, it is first very instructive to consider the limit $f(r) \to 0$ of Eq. (6.213), which reproduces the metric result

$$\dot{r} = -\frac{64}{5}\eta c\left(\frac{\alpha G m}{c^2 r}\right)^3. \tag{6.214}$$

This again shows the correspondence of both theories for a suitable parameter range. However, when the correction $f(r)$ is not negligible, area metric gravity introduces an interesting deviation from the binary star spin-up behaviour in metric gravity.

Fig. 6.1 shows the evolution of the orbital period, obtained by integrating Eq. (6.213) and applying the refined Kepler law (6.156). For our convenience, we set all constants and parameters of the system that are common to both area metric and metric gravity to 1, which makes the time scale somewhat arbitrary, but that should not concern us—most interesting is the deviation of the area metric prediction from the metric result. This deviation is controlled by the gravitational constants β and μ. With a coarse tuning of the constants, it is possible to achieve two things: bringing the prediction arbitrarily close

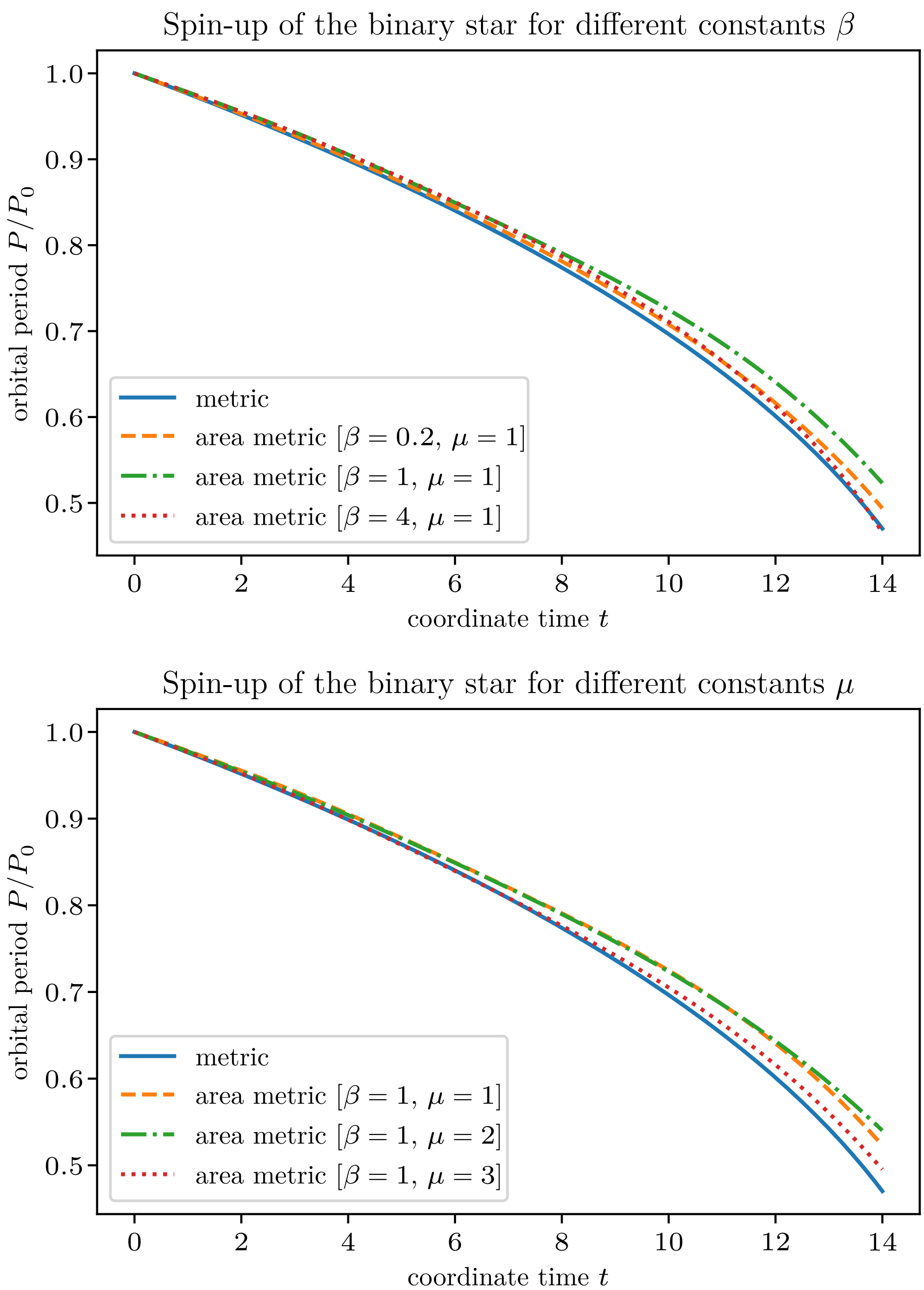

Figure 6.1: Binary star spin-up due to radiation loss. The orbital period decreases according to Eqns. (6.156) and (6.213), which have been numerically integrated using the `scipy` python package [80]. For reference, the metric result following (6.207) is shown as well. Note that in general a binary star will *not* spin up exactly like this, as the calculation neglects a lot of complications like eccentricities and is based only on the first approximation provided by the Noether theorem. Massive radiation modes are not considered as well. However, this analysis highlights the potential deviations of area metric gravity from Einstein gravity. For simplicity, $\alpha = G = \eta = m = c = 1$.

to the metric prediction—again an incarnation of the correspondence principle—but also generating hypotheticals that disagree with all observations. The latter are, of course, easily falsified.

It is very tempting to perform a finer tuning, i.e. try and fit the predictions to observational data and infer viable parameter ranges for β and μ. This is certainly an interesting approach, albeit with quite limited power at the current state: the analysis is very crude, with the ambition to derive *first* qualitative and quantitative implications of area metric gravity. Only binary stars without eccentricity have been considered. Also, the result (6.213) does not include massive modes of radiation, which further contribute to radiation loss once their generation threshold is reached.

Still, with our approach that tried to limit the computational complexity, we eventually derived novel, nontrivial behaviour of matter subject to area metric gravity.

7 Outlook: symmetry-reduced constructive gravity

In the previous chapters, we reduced the complexity of the covariant construction procedure by considering a perturbative equivalent. Consequently, the result was an approximation of the exact gravitational theory, valid for sufficiently weak gravitational fields. A second approach towards reducing the complexity of the equivariance equations is symmetry reduction, which assumes that the gravitational field exhibits certain symmetries. Ideally, these symmetries bring the construction equations into a much simpler form. The solutions of the reduced equations should be theories of gravity valid in this symmetry-reduced sector, comparable to the Friedmann equations for spatially homogeneous and isotropic Einstein gravity. In this chapter, we explore a possible approach towards symmetry-reduced covariant constructive gravity by reducing the bundle on which the procedure operates. Our main result will be that the Friedmann-Lemaître-Robertson-Walker (FLRW) model can be recovered without the need to know the full Einstein-Hilbert Lagrangian beforehand. The area metric equivalent will not be solved, only the construction equations are derived. For a more in-depth study of symmetry reduction in the context of canonical constructive gravity, see Ref. [34].

7.1 The cosmological bundle

The introduction of the metric cosmological bundle follows the presentation in Ref. [1].

For the purpose of developing a symmetry reduction strategy, let us consider the cosmological symmetry, which assumes spacetime to be spatially homogeneous and isotropic (see e.g. [81, 82]). It is well known what this entails for the metric field: implicitly, this symmetry comes with the assumption of a sliced spacetime, i.e. $M \cong \mathbb{R} \times \Sigma$*. In appropriate coordinates on the product manifold, the covariant metric tensor then reads [81, 83]*

$$g = dt \otimes dt - a(t)^2 \gamma, \tag{7.1}$$

where the spatial part is given by the positive scale factor $a(t) > 0$ *and a constant curvature metric* γ *on* Σ*.*[1]

[1]Restricting the topology of the spatial manifold Σ to either $\mathbb{R}^3$ or S^3.

More formally, we have a slicing $\phi\colon \mathbb{R}\times\Sigma\to M$, which induces embeddings

$$\begin{aligned}\phi_\lambda\colon \Sigma &\to M\\ p &\mapsto \phi_\lambda(p) := \phi(\lambda, p)\end{aligned} \tag{7.2}$$

of the spatial hypersurface Σ into the spacetime manifold M. Each slicing introduces a time coordinate $t := \pi_{\mathbb{R}}\circ\phi^{-1}$. The corresponding vector field ∂_t defines the spatial and spatiotemporal components of the metric tensor by virtue of the conditions

$$g(\partial_t, \partial_t) = 1 \quad \text{and} \quad dt(X) = 0 \Rightarrow g(\partial_t, X) = 0. \tag{7.3}$$

For the spatial components, we consider the pullback of the metric tensor onto the spatial slice Σ. This yields Riemannian 3-manifolds

$$(\Sigma, \gamma_\lambda := -\phi_\lambda^* g) \tag{7.4}$$

that are of constant curvature. For simplicity, let us restrict to zero *curvature manifolds, such that the spatial volume is determined only by the scale factor*

$$a(\lambda) := \sqrt{\det\gamma_\lambda}^{\frac{1}{3}}. \tag{7.5}$$

From these insights, we define the cosmological bundle *for metric gravity.*

Definition 7.1.1 (metric cosmological bundle)**.** *The cosmological bundle over a manifold M which captures the information of a spatially homogeneous and isotropic metric spacetime (M, g) is defined as*

$$E_{metric}^{(cosmological)} = TM \oplus_M \mathrm{Vol}^{\frac{1}{3}}(M), \tag{7.6}$$

i.e. the sum of the tangent bundle and the bundle of densities with weight $\frac{1}{3}$.

A similar definition can be given for the area metric bundle. It has been shown [34] that a spatially homogeneous and isotropic area metric manifold is determined by *two* spatial degrees of freedom, which are a density-valued scale factor and a second scalar-valued factor.[2] Consequently, the area metric cosmological bundle can be defined as follows.

[2]This already follows quite intuitively from the $3+1$ decomposition in Sect. 6.1.3. As opposed to metric gravity, where a single three-metric determines all spatial components, we now have *two* spatial metrics and one tracefree endomorphism which parameterise the 17 spatial degrees of freedom. If the fields are to be isotropic, they must be given by two scale factors for the metrics; the tracefree endomorphism must be zero. Instead of working with two density-valued functions, it is more convenient to use a redefinition where one of the functions becomes scalar-valued.

Definition 7.1.2 (area metric cosmological bundle)**.** *The cosmological bundle over a manifold M which captures the information of a spatially homogeneous and isotropic area metric spacetime* (M, G) *is defined as*

$$E_{area}^{(cosmological)} = TM \oplus_M \mathrm{Vol}^{\frac{1}{3}}(M) \oplus_M \mathrm{Scalar}(M), \tag{7.7}$$

i.e. the sum of the tangent bundle, the bundle of $\frac{1}{3}$-densities, and the line bundle.

7.2 Recovering the FLRW model

The recovering of the FLRW model follows the presentation in Ref. [1], but provides additional detail concerning the ansätze and the evaluation of the equivariance equations for the ansätze.

Having defined the metric cosmological bundle $TM \oplus_M \mathrm{Vol}^{\frac{1}{3}}(M)$, setting up the equivariance equations (2.29a)–(2.29d) *is just a matter of deriving the Gotay-Marsden coefficients. For vector fields, Prop. 2.2.7 yields*

$$C^{a}{}_{b}{}^{n}{}_{m} = \delta^a_m \delta^n_b, \tag{7.8}$$

while $\frac{1}{3}$-densities transform according to the Gotay-Marsden coefficients

$$C^{n}{}_{m} = -\frac{1}{3}\delta^n_m. \tag{7.9}$$

In order for the field equations to be of second derivative order with a principal polynomial that does not depend on derivatives of the geometry, we make the ansatz

$$\begin{aligned} L(a, \partial a, \partial\partial a, U, \partial U, \partial\partial U) = \quad & f_1(a) U^m U^n a_{,mn} && + f_2(a) U^m U^n_{,mn} \\ & + f_3(a) U^m U^n a_{,m} a_{,n} && + f_4(a) U^m U^n_{,n} a_{,m} \\ & + f_5(a) U^m U^n_{,m} a_{,n} && + f_6(a) U^m_{,m} U^n_{,n} \\ & + f_7(a) U^m_{,n} U^n_{,m} && + f_8(a) U^m a_{,m} \\ & + f_9(a) U^m_{,m} && + f_{10}(a). \end{aligned} \tag{7.10}$$

The functions $f_1, \dots, f_{10}$ are arbitrary functions of the scale factor. Any occurrence of of a vector field U would have to be contracted with a derivative of either a or U, such that with our causality restrictions it is only appropriate to include linear and quadratic terms in the ansatz.

Taking the trace of the equivariance equation (2.29b) restricts the functions $f_1, \dots, f_{10}$ to polynomials, as we obtain simple ordinary differential equations:

$$\begin{aligned}
0 &= 2f_1 - f_1' a & &\Rightarrow & f_1(a) &= \kappa_1 a^2 \\
0 &= 3f_2 - f_2' a & &\Rightarrow & f_2(a) &= \kappa_2 a^3 \\
0 &= f_3 - f_3' a & &\Rightarrow & f_3(a) &= \kappa_3 a \\
0 &= 2f_4 - f_4' a & &\Rightarrow & f_4(a) &= \kappa_4 a^2 \\
0 &= 2f_5 - f_5' a & &\Rightarrow & f_5(a) &= \kappa_5 a^2 \\
0 &= 3f_6 - f_6' a & &\Rightarrow & f_6(a) &= \kappa_6 a^3 \\
0 &= 3f_7 - f_7' a & &\Rightarrow & f_7(a) &= \kappa_7 a^3 \\
0 &= 2f_8 - f_8' a & &\Rightarrow & f_8(a) &= \kappa_8 a^2 \\
0 &= 3f_9 - f_9' a & &\Rightarrow & f_9(a) &= \kappa_9 a^3 \\
0 &= 3f_{10} - f_{10}' a & &\Rightarrow & f_{10}(a) &= \kappa_{10} a^3
\end{aligned} \tag{7.11}$$

Evaluation of the remaining equivariance equations (2.29b)–(2.29d) further narrows down the gravitational constants $\kappa_1, \dots, \kappa_{10}$, leaving us with four independent constants in the Lagrangian density

$$\begin{aligned}
L = \quad & \kappa_1 \times \Big[a^2 U^m U^n a_{,mn} + \frac{1}{3} a^3 U^m U^n_{,mn} \\
& \qquad + \frac{2}{3} a^2 U^m U^n_{,n} a_{,m} + a^2 U^m U^n_{,m} a_{,n} + \frac{1}{9} a^3 U^m_{,m} U^n_{,n} \Big] \\
& + \kappa_3 \times \Big[a U^m U^n a_{,m} a_{,n} + \frac{2}{3} a^2 U^m U^n_{,n} a_{,m} + \frac{1}{9} a^3 U^m_{,m} U^n_{,n} \Big] \\
& + \kappa_8 \times \underbrace{\Big[a^2 U^m a_{,m} + \frac{1}{3} a^3 U^m_{,m} \Big]}_{\text{boundary term}} \\
& + \kappa_{10} \times \Big[a^3 \Big],
\end{aligned} \tag{7.12}$$

where one constant, κ_8, only contributes to a boundary term, which will be dropped from now on.

Let us couple the metric to a matter field by adding to the Lagrangian density (7.12) *a matter Lagrangian*[3] L_{matter}. *Expressed in coordinates where $U^a = const$, variations with*

[3]Not a *density* for the purposes of this section. We will always make the "densitisation" using $\sqrt{-g} = a^3$ explicit.

respect to the fields a and U reproduce the well-known Friedmann equations *[84]*

$$\left(\frac{\dot{a}}{a}\right)^2 - \frac{\Lambda}{3} = \frac{\kappa}{3}\rho, \tag{7.13}$$
$$\frac{\ddot{a}}{a} - \frac{\Lambda}{3} = -\frac{\kappa}{6}(\rho + 3p), \tag{7.14}$$

with combinations κ and Λ of the gravitational constants $\kappa_1, \kappa_3, \kappa_{10}$ and the derivative $\dot{a} := U(a)$.

For the field equations (7.13) *and* (7.14) *we introduced the* energy density

$$\rho = \frac{1}{a^3}\left[-\frac{a}{3}\frac{\delta(a^3 L_{matter})}{\delta a} + U^p \frac{\delta(a^3 L_{matter})}{\delta U^p}\right] \tag{7.15}$$

and the pressure

$$p = \frac{1}{a^3}\left[\frac{a}{3}\frac{\delta(a^3 L_{matter})}{\delta a}\right]. \tag{7.16}$$

An example for a matter field that couples to the FLRW metric is a spatially homogeneous and isotropic scalar field ϕ in a potential V, with dynamics according to the action

$$S_{matter}[\phi] = \int \sqrt{-g}[g(d\phi, d\phi) - V(\phi)]d^4x = \int a^3[(U(\phi))^2 - V(\phi)]d^4x. \tag{7.17}$$

The corresponding energy density and pressure as defined in Eqns. (7.15) *and* (7.16) *are given by*

$$\rho = (\dot{\phi})^2 + V(\phi) \tag{7.18}$$
$$p = (\dot{\phi})^2 - V(\phi). \tag{7.19}$$

Together, energy density and pressure constitute the metric stress-energy tensor

$$T^{ab} = \frac{2}{\sqrt{-g}}\frac{\delta(\sqrt{-g}L_{matter})}{\delta g_{ab}} = (\rho + p)U^a U^b + p g^{ab}, \tag{7.20}$$

as can be verified by inserting the expressions (7.18) *and* (7.19) *for ρ and p in above equation.*

Summing up, we have found the Friedmann equations (7.13) *and* (7.14) *as the dynamical equations for the remaining degrees of freedom in spatially isotropic and homogeneous metric cosmology—without recurrence to the Einstein equation, just by performing the symmetry reduction* beforehand *and applying the covariant construction procedure to the reduced problem. The equations are parameterised by the gravitational constant κ and the cosmological constant Λ. All inferences that can be drawn from the Friedmann* *equations already follow from this simplified approach—demonstrating the potential of symmetry-reduced covariant constructive gravity for investigations into the cosmological sector of modified theories of gravity.*

7.3 Towards area metric cosmology

In principle, the same procedure can be applied to the cosmological bundle of area metric gravity, resulting in a parameterisation of all possible symmetry-reduced gravitational theories for the area metric tensor. The only new ingredient as compared to metric cosmology are the Gotay-Marsden coefficients for scalar fields, which are

$$C^{n}{}_{m} = 0. \tag{7.21}$$

This is not surprising at all—the Gotay-Marsden coefficients define the transformation behaviour with respect to spacetime diffeomorphisms. A scalar is, by definition, diffeomorphism *invariant* and the corresponding coefficients are thus zero. As a consequence, the functional form of the dependence on the scalar is much less restricted, i.e. a result equivalent to Eq. (7.11) cannot be derived. Any solution will contain undetermined *functions*, not only constants.

An in-depth study of the equivalent problem in *canonical* constructive gravity has been performed in Ref. [34].

8 Conclusions

In this thesis, the concept of covariant constructive gravity has been put on a solid mathematical footing. Lagrangian field theory on jet bundles turned out to be ideally suited for the definition of the general covariance axiom. The equivariance equations that follow from this axiom transform the implementation into a computational task, opening up the *constructive* pathway towards modified theories of gravity. Using the Cartan form that corresponds to a diffeomorphism invariant Lagrangian density, we have seen how general covariance implies a version of the first and second Noether theorem—important results that have proven very useful further down the line. Within this framework, the axiom of causal compatibility has been formulated in terms of additional algebraic conditions on the gravitational field equations.

From the equivariance equations and causality conditions, we derived a concise algorithm which guides the construction of novel gravitational theories that implement general covariance and are causally compatible to a given matter theory. Because it can be seen as generalisation of Lovelock's uniqueness theorem for Einstein gravity [14, 35, 36], we could show that this construction procedure applied to Maxwell's electrodynamics indeed reproduces metric gravity as derived by Einstein.

Of course, covariant constructive gravity would not be that interesting if it were just another tool that reproduces Einstein gravity. Its *raison d'être* is the derivation of *modified* theories of gravity that complete novel matter theories to predictive models of the universe. The remainder of the thesis was dedicated towards achieving this goal. First, we have discussed three examples of novel matter theories: birefringent electrodynamics and two bimetric theories. While it is straightforward to set up the construction procedure and derive general results concerning the solution space, *finding* these solutions in practice is notoriously hard and turned out not to be feasible for the examples in question.

Therefore, we investigated possibilities to arrive at results that are valid in certain specific settings, without the need to know the "full" solutions. Our main strategy was perturbation theory, which seeks to find solutions that are valid for small deviations of the gravitational field from a Lorentz invariant background. With a corresponding perturbation ansatz, the equivariance equations transform into a system of linear equations for the expansion coefficients. As a consequence of the Lorentz invariance of the background geometry, the expansion coefficients themselves are Lorentz invariant, which reduces their dimensionality a lot—before solving any equivariance equation.

Many of the computations that are necessary in order to construct Lorentz invariant ansätze and solve the perturbative equivariance equations have been delegated to the computer. For this purpose, two Haskell packages have been developed and presented in this thesis—with a focus on the package `sparse-tensor` which composes and solves the equivariance equations. Methods from functional programming lend themselves for an efficient and safe implementation of tensor algebra, enabling us to repeat and modify calculations whenever required, without having to redo them by hand.

Chap. 6 was the culmination of this thesis, where we put all pieces together and derived perturbative area metric gravity up to third perturbation order in the Lagrangian density. From this Lagrangian, the linearised gravitational field equations and their second perturbation order follows. Quite remarkably, the linearised field equations coincide with the equations derived in the canonical framework [65, 3]—with an important caveat: the field equations in the canonical picture as obtained by solving the canonical closure equations [26, 65] are *not* causally compatible with the matter theory, i.e. their principal polynomial does not reduce to the Minkowski metric.

In order to cure the causality, one of the *eleven* gravitational constants had to be fixed [3], reducing their number to *ten*, which then equals the number of constants obtained in the covariant framework. The reason for this mismatch is believed to lie in the so-called ansatz equations, which enforce Lorentz invariance of the perturbation ansatz. In perturbative covariant constructive gravity, these are already solved by considering only Lorentz invariant ansätze to begin with. Canonical constructive gravity, on the other hand, makes ansätze *after* performing the 3+1 split—effectively implementing a spatial SO(3) symmetry. But this is a *weaker* requirement than the spatiotemporal $\mathrm{SO}(1,3)$ symmetry that follows from the equivariance equations. In the case of the linearised field equations with causality mismatch, not all of the ansatz equations seem to be solved—one condition on the gravitational constants is not yet implemented. To remedy this, one has to find the equivalent of the ansatz equations in the canonical picture by prolonging and projecting the PDE or otherwise ensure Lorentz invariance of the ansätze.

The comparison demonstrates that the presence of matter causality in the canonical constraint algebra is not responsible for the causality of the gravitational theory. For the linearised field equations[1], diffeomorphism invariance actually constrains the gravity causality. Whether diffeomorphism invariance is enforced by imposing it directly on the Lagrangian density or by requiring the canonical constraint algebra to implement the hypersurface deformation algebra—in whichever frame—is secondary.

The ability to reason about the canonical closure programme using insights from covariant constructive gravity shows how both approaches complement each other. Comparing

[1] But also for the second-order equations, as shown in Sect. 6.1.

the results of covariant and canonical constructive gravity on the one hand increases the confidence, as they are so similar, but also provides impulses for improvements: the canonical approach should embrace Lorentz invariance and also reconsider its claims concerning causality, while the covariant approach could benefit from a canonical formulation. The multisymplectic framework based on the Cartan form [45] seems ideal for this task.

Building up on the third-order area metric Lagrangian, we inspected the binary star with circular orbits, one of the simplest conceivable systems. Thanks to this simplicity, however, it was possible to derive second-order effects that proved to be quite rich. A binary star in area metric gravity emits massive gravitational waves—in addition to the radiation already known from Einstein gravity. These new modes of radiation have the potential to induce novel deformation patterns in test matter distributions and to alter the spin-up behaviour of the binary star. For the massless modes of radiation that are also observed in metric gravity, we made use of the second Noether theorem and obtained a quantitative description of how a binary star is expected to decrease its orbital period as it emits gravitational waves. Fig. 6.1 shows a few exemplary cases, which demonstrate the deviations from Einstein gravity that are expected in area metric gravity.

These results should be understood as *conceptual*, because much more work would be needed for the prediction of the outcome of high-precision experiments. The ambition of this thesis was to demonstrate *in principle* the predictive power of covariant constructive gravity. Starting from a modification of Maxwell's electrodynamics—by allowing for birefringence *in vacuo*—it is possible to derive a compatible theory of gravity that prescribes the dynamics of the new geometry used by the such refined matter theory. The resulting gravitational theory has a limit where it corresponds to Einstein gravity, but it also allows for interesting deviations: massive gravitational waves that are emitted from a binary star which exceeds a certain angular velocity threshold, a modification of Kepler's third law, or a refined inspiral curve.

We finally explored the possibility of making similar predictions for symmetry-reduced theories—proposing an approach that meets the minimal requirement of reproducing metric cosmology. It will be exciting to see the application to novel matter theories.

The famous words by John Archibald Wheeler quoted at the beginning of Chap. 1 seem to apply not only at the level of field equations—where matter fields source gravitational fields, while gravitational fields determine the motion of matter fields—but also at the level of *theories*. The gravitational field equations themselves are, to a certain degree, determined by the dynamics of matter fields. Considering novel matter theories that couple to nonmetric geometries has gravitational implications, which covariant constructive gravity is able to quantify. Improving the predictions in order to make the constructed theories testable in practice should be at the centre of upcoming research. The standard model of particle physics and general relativity are not the definite models of the universe—covariant constructive gravity can further the search for alternatives.

A Ansätze for third-order area metric gravity Lagrangians

The following ansätze have been computed using the Haskell package `sparse-tensor` (see [5] and Chap. 5). Haskell code for generation and pretty printing as well as the ansätze in machine-readable form are available at Ref. [7].

- first order (constants e_{38}, e_{39}, e_{40}):

$$a_A{}^I H^A{}_I = \Big[e_{38} \cdot \eta_{ac}\eta_{bd}\eta_{pq} + e_{39} \cdot \eta_{ac}\eta_{bp}\eta_{dq} + e_{40} \cdot \epsilon_{abcd}\eta_{pq}\Big] \times \eta^{pr}\eta^{qs} H^{abcd}{}_{,rs} \quad \text{(A.1)}$$

- second order (constants $e_1, \dots, e_{37}$):

$$\begin{aligned} a_{AB} H^A H^B = \Big[& \\ & e_1 \cdot \eta_{ac}\eta_{bd}\eta_{eg}\eta_{fh} + e_2 \cdot \eta_{ac}\eta_{be}\eta_{dg}\eta_{fh} + e_3 \cdot \eta_{ae}\eta_{bf}\eta_{cg}\eta_{dh} \\ & + e_4 \cdot \eta_{ae}\eta_{bg}\eta_{cf}\eta_{dh} + e_5 \cdot \epsilon_{abcd}\eta_{eg}\eta_{fh} + e_6 \cdot \epsilon_{abef}\eta_{cg}\eta_{dh} \\ \Big] & \times H^{abcd} H^{efgh} \end{aligned} \quad \text{(A.2)}$$

$$\begin{aligned} a_A{}^p{}_B{}^q H^A{}_p H^B{}_q = \Big[& \\ & e_7 \cdot \eta_{ac}\eta_{bd}\eta_{pe}\eta_{fg}\eta_{hq} + e_8 \cdot \eta_{ac}\eta_{bd}\eta_{pq}\eta_{eg}\eta_{fh} + e_9 \cdot \eta_{ac}\eta_{bp}\eta_{de}\eta_{fg}\eta_{hq} \\ & + e_{10} \cdot \eta_{ac}\eta_{be}\eta_{dg}\eta_{pf}\eta_{hq} + e_{11} \cdot \eta_{ac}\eta_{be}\eta_{dg}\eta_{pq}\eta_{fh} + e_{12} \cdot \eta_{ac}\eta_{be}\eta_{dq}\eta_{pg}\eta_{fh} \\ & + e_{13} \cdot \eta_{ap}\eta_{be}\eta_{cf}\eta_{dg}\eta_{hq} + e_{14} \cdot \eta_{ap}\eta_{be}\eta_{cg}\eta_{dh}\eta_{fq} + e_{15} \cdot \eta_{ae}\eta_{bf}\eta_{cg}\eta_{dh}\eta_{pq} \\ & + e_{16} \cdot \epsilon_{abcd}\eta_{pe}\eta_{fg}\eta_{hq} + e_{17} \cdot \epsilon_{abcd}\eta_{pq}\eta_{eg}\eta_{fh} + e_{18} \cdot \epsilon_{abpe}\eta_{cf}\eta_{dg}\eta_{hq} \\ & + e_{19} \cdot \epsilon_{abpe}\eta_{cg}\eta_{dq}\eta_{fh} + e_{20} \cdot \epsilon_{abef}\eta_{cp}\eta_{dg}\eta_{hq} + e_{21} \cdot \epsilon_{abef}\eta_{cg}\eta_{dh}\eta_{pq} \\ \Big] & \times \eta^{pr}\eta^{qs} H^{abcd}{}_{,r} H^{efgh}{}_{,s} \end{aligned} \quad \text{(A.3)}$$

$$\begin{aligned} a_{AB}{}^I H^A H^B{}_I = \Big[& \\ & e_{22} \cdot \eta_{ac}\eta_{bd}\eta_{eg}\eta_{fh}\eta_{pq} + e_{23} \cdot \eta_{ac}\eta_{bd}\eta_{eg}\eta_{fp}\eta_{hq} + e_{24} \cdot \eta_{ac}\eta_{be}\eta_{dg}\eta_{fh}\eta_{pq} \\ & + e_{25} \cdot \eta_{ac}\eta_{be}\eta_{dg}\eta_{fp}\eta_{hq} + e_{26} \cdot \eta_{ac}\eta_{be}\eta_{dp}\eta_{fg}\eta_{hq} + e_{27} \cdot \eta_{ac}\eta_{bp}\eta_{dq}\eta_{eg}\eta_{fh} \\ & + e_{28} \cdot \eta_{ae}\eta_{bf}\eta_{cg}\eta_{dh}\eta_{pq} + e_{29} \cdot \eta_{ae}\eta_{bf}\eta_{cg}\eta_{dp}\eta_{hq} + e_{30} \cdot \eta_{ae}\eta_{bg}\eta_{cf}\eta_{dh}\eta_{pq} \\ & + e_{31} \cdot \epsilon_{abcd}\eta_{eg}\eta_{fh}\eta_{pq} + e_{32} \cdot \epsilon_{abcd}\eta_{eg}\eta_{fp}\eta_{hq} + e_{33} \cdot \epsilon_{abef}\eta_{cg}\eta_{dh}\eta_{pq} \\ & + e_{34} \cdot \epsilon_{abef}\eta_{cg}\eta_{dp}\eta_{hq} + e_{35} \cdot \epsilon_{abep}\eta_{cf}\eta_{dg}\eta_{hq} + e_{36} \cdot \epsilon_{abep}\eta_{cg}\eta_{dh}\eta_{fq} \\ & + e_{37} \cdot \epsilon_{efgh}\eta_{ac}\eta_{bd}\eta_{pq}\Big] \times \eta^{pr}\eta^{qs} H^{abcd} H^{efgh}{}_{,rs} \end{aligned} \quad \text{(A.4)}$$

- third order (constants $e_{41}, \dots, e_{237}$):

$$
\begin{aligned}
a_{ABC}H^A H^B H^C = \Big[\quad & e_{41} \cdot \eta_{ac}\eta_{bd}\eta_{eg}\eta_{fh}\eta_{ik}\eta_{jl} + e_{42} \cdot \eta_{ac}\eta_{bd}\eta_{eg}\eta_{fi}\eta_{hk}\eta_{jl} \\
& + e_{43} \cdot \eta_{ac}\eta_{bd}\eta_{ei}\eta_{fj}\eta_{gk}\eta_{hl} + e_{44} \cdot \eta_{ac}\eta_{bd}\eta_{ei}\eta_{fk}\eta_{gj}\eta_{hl} \\
& + e_{45} \cdot \eta_{ac}\eta_{be}\eta_{dg}\eta_{fi}\eta_{hk}\eta_{jl} + e_{46} \cdot \eta_{ac}\eta_{be}\eta_{di}\eta_{fg}\eta_{hk}\eta_{jl} \\
& + e_{47} \cdot \eta_{ae}\eta_{bf}\eta_{ci}\eta_{dj}\eta_{gk}\eta_{hl} + e_{48} \cdot \eta_{ae}\eta_{bf}\eta_{ci}\eta_{dk}\eta_{gj}\eta_{hl} \\
& + e_{49} \cdot \epsilon_{abcd}\eta_{eg}\eta_{fh}\eta_{ik}\eta_{jl} + e_{50} \cdot \epsilon_{abcd}\eta_{eg}\eta_{fi}\eta_{hk}\eta_{jl} \\
& + e_{51} \cdot \epsilon_{abcd}\eta_{ei}\eta_{fj}\eta_{gk}\eta_{hl} + e_{52} \cdot \epsilon_{abcd}\eta_{ei}\eta_{fk}\eta_{gj}\eta_{hl} \\
& + e_{53} \cdot \epsilon_{abef}\eta_{cg}\eta_{dh}\eta_{ik}\eta_{jl} + e_{54} \cdot \epsilon_{abef}\eta_{cg}\eta_{di}\eta_{hk}\eta_{jl} \\
& + e_{55} \cdot \epsilon_{abef}\eta_{ci}\eta_{dj}\eta_{gk}\eta_{hl}\Big] \times H^{abcd}H^{efgh}H^{ijkl}
\end{aligned}
\tag{A.5}
$$

$$
\begin{aligned}
& a_{AB}{}^{p}{}_{C}{}^{q} H^A H^B{}_p H^C{}_q = \Big[\\
& \quad e_{56} \cdot \eta_{ac}\eta_{bd}\eta_{eg}\eta_{fh}\eta_{pi}\eta_{jk}\eta_{lq} + e_{57} \cdot \eta_{ac}\eta_{bd}\eta_{eg}\eta_{fh}\eta_{pq}\eta_{ik}\eta_{jl} + e_{58} \cdot \eta_{ac}\eta_{bd}\eta_{eg}\eta_{fp}\eta_{hi}\eta_{jk}\eta_{lq} \\
& + e_{59} \cdot \eta_{ac}\eta_{bd}\eta_{eg}\eta_{fi}\eta_{hk}\eta_{pj}\eta_{lq} + e_{60} \cdot \eta_{ac}\eta_{bd}\eta_{eg}\eta_{fi}\eta_{hk}\eta_{pq}\eta_{jl} + e_{61} \cdot \eta_{ac}\eta_{bd}\eta_{eg}\eta_{fi}\eta_{hq}\eta_{pk}\eta_{jl} \\
& + e_{62} \cdot \eta_{ac}\eta_{bd}\eta_{ep}\eta_{fi}\eta_{gj}\eta_{hk}\eta_{lq} + e_{63} \cdot \eta_{ac}\eta_{bd}\eta_{ep}\eta_{fi}\eta_{gk}\eta_{hl}\eta_{jq} + e_{64} \cdot \eta_{ac}\eta_{bd}\eta_{ei}\eta_{fj}\eta_{gk}\eta_{hl}\eta_{pq} \\
& + e_{65} \cdot \eta_{ac}\eta_{be}\eta_{dg}\eta_{fh}\eta_{pi}\eta_{jk}\eta_{lq} + e_{66} \cdot \eta_{ac}\eta_{be}\eta_{dg}\eta_{fh}\eta_{pq}\eta_{ik}\eta_{jl} + e_{67} \cdot \eta_{ac}\eta_{be}\eta_{dg}\eta_{fp}\eta_{hi}\eta_{jk}\eta_{lq} \\
& + e_{68} \cdot \eta_{ac}\eta_{be}\eta_{dg}\eta_{fp}\eta_{hq}\eta_{ik}\eta_{jl} + e_{69} \cdot \eta_{ac}\eta_{be}\eta_{dg}\eta_{fi}\eta_{hk}\eta_{pj}\eta_{lq} + e_{70} \cdot \eta_{ac}\eta_{be}\eta_{dg}\eta_{fi}\eta_{hk}\eta_{pq}\eta_{jl} \\
& + e_{71} \cdot \eta_{ac}\eta_{be}\eta_{dg}\eta_{fi}\eta_{hq}\eta_{pk}\eta_{jl} + e_{72} \cdot \eta_{ac}\eta_{be}\eta_{dp}\eta_{fg}\eta_{hi}\eta_{jk}\eta_{lq} + e_{73} \cdot \eta_{ac}\eta_{be}\eta_{dp}\eta_{fg}\eta_{hq}\eta_{ik}\eta_{jl} \\
& + e_{74} \cdot \eta_{ac}\eta_{be}\eta_{dp}\eta_{fi}\eta_{gj}\eta_{hk}\eta_{lq} + e_{75} \cdot \eta_{ac}\eta_{be}\eta_{dp}\eta_{fi}\eta_{gk}\eta_{hl}\eta_{jq} + e_{76} \cdot \eta_{ac}\eta_{be}\eta_{dp}\eta_{fi}\eta_{gk}\eta_{hq}\eta_{jl} \\
& + e_{77} \cdot \eta_{ac}\eta_{be}\eta_{di}\eta_{fg}\eta_{hp}\eta_{jk}\eta_{lq} + e_{78} \cdot \eta_{ac}\eta_{be}\eta_{di}\eta_{fg}\eta_{hj}\eta_{pk}\eta_{lq} + e_{79} \cdot \eta_{ac}\eta_{be}\eta_{di}\eta_{fg}\eta_{hk}\eta_{pj}\eta_{lq} \\
& + e_{80} \cdot \eta_{ac}\eta_{be}\eta_{di}\eta_{fg}\eta_{hk}\eta_{pl}\eta_{jq} + e_{81} \cdot \eta_{ac}\eta_{be}\eta_{di}\eta_{fg}\eta_{hk}\eta_{pq}\eta_{jl} + e_{82} \cdot \eta_{ac}\eta_{be}\eta_{di}\eta_{fg}\eta_{hq}\eta_{pk}\eta_{jl} \\
& + e_{83} \cdot \eta_{ac}\eta_{be}\eta_{di}\eta_{fp}\eta_{gj}\eta_{hk}\eta_{lq} + e_{84} \cdot \eta_{ac}\eta_{be}\eta_{di}\eta_{fp}\eta_{gk}\eta_{hl}\eta_{jq} + e_{85} \cdot \eta_{ac}\eta_{be}\eta_{di}\eta_{fj}\eta_{gp}\eta_{hk}\eta_{lq} \\
& + e_{86} \cdot \eta_{ac}\eta_{be}\eta_{di}\eta_{fk}\eta_{gj}\eta_{hq}\eta_{pl} + e_{87} \cdot \eta_{ac}\eta_{be}\eta_{dq}\eta_{fg}\eta_{hp}\eta_{ik}\eta_{jl} + e_{88} \cdot \eta_{ac}\eta_{be}\eta_{dq}\eta_{fg}\eta_{hi}\eta_{pk}\eta_{jl} \\
& + e_{89} \cdot \eta_{ac}\eta_{bp}\eta_{dq}\eta_{eg}\eta_{fh}\eta_{ik}\eta_{jl} + e_{90} \cdot \eta_{ae}\eta_{bf}\eta_{cg}\eta_{dh}\eta_{pi}\eta_{jk}\eta_{lq} + e_{91} \cdot \eta_{ae}\eta_{bf}\eta_{cg}\eta_{dh}\eta_{pq}\eta_{ik}\eta_{jl} \\
& + e_{92} \cdot \eta_{ae}\eta_{bf}\eta_{cg}\eta_{dp}\eta_{hi}\eta_{jk}\eta_{lq} + e_{93} \cdot \eta_{ae}\eta_{bf}\eta_{cg}\eta_{dp}\eta_{hq}\eta_{ik}\eta_{jl} + e_{94} \cdot \eta_{ae}\eta_{bf}\eta_{cg}\eta_{di}\eta_{hp}\eta_{jk}\eta_{lq} \\
& + e_{95} \cdot \eta_{ae}\eta_{bf}\eta_{cg}\eta_{di}\eta_{hj}\eta_{pk}\eta_{lq} + e_{96} \cdot \eta_{ae}\eta_{bf}\eta_{cg}\eta_{di}\eta_{hk}\eta_{pj}\eta_{lq} + e_{97} \cdot \eta_{ae}\eta_{bf}\eta_{cg}\eta_{di}\eta_{hk}\eta_{pl}\eta_{jq} \\
& + e_{98} \cdot \eta_{ae}\eta_{bf}\eta_{cg}\eta_{di}\eta_{hk}\eta_{pq}\eta_{jl} + e_{99} \cdot \eta_{ae}\eta_{bf}\eta_{cg}\eta_{di}\eta_{hq}\eta_{pk}\eta_{jl} + e_{100} \cdot \eta_{ae}\eta_{bf}\eta_{cg}\eta_{dq}\eta_{hp}\eta_{ik}\eta_{jl} \\
& + e_{101} \cdot \eta_{ae}\eta_{bf}\eta_{cg}\eta_{dq}\eta_{hi}\eta_{pk}\eta_{jl} + e_{102} \cdot \eta_{ae}\eta_{bf}\eta_{cp}\eta_{di}\eta_{gj}\eta_{hk}\eta_{lq} + e_{103} \cdot \eta_{ae}\eta_{bf}\eta_{cp}\eta_{di}\eta_{gk}\eta_{hl}\eta_{jq} \\
& + e_{104} \cdot \eta_{ae}\eta_{bf}\eta_{cp}\eta_{di}\eta_{gk}\eta_{hq}\eta_{jl} + e_{105} \cdot \eta_{ae}\eta_{bf}\eta_{ci}\eta_{dj}\eta_{gp}\eta_{hk}\eta_{lq} + e_{106} \cdot \eta_{ae}\eta_{bf}\eta_{ci}\eta_{dj}\eta_{gk}\eta_{hl}\eta_{pq} \\
& + e_{107} \cdot \eta_{ae}\eta_{bf}\eta_{ci}\eta_{dk}\eta_{gp}\eta_{hj}\eta_{lq} + e_{108} \cdot \eta_{ae}\eta_{bg}\eta_{cf}\eta_{dh}\eta_{pi}\eta_{jk}\eta_{lq} + e_{109} \cdot \eta_{ae}\eta_{bg}\eta_{cf}\eta_{dh}\eta_{pq}\eta_{ik}\eta_{jl} \\
& + e_{110} \cdot \eta_{ae}\eta_{bp}\eta_{cf}\eta_{di}\eta_{gj}\eta_{hk}\eta_{lq} + e_{111} \cdot \eta_{ae}\eta_{bp}\eta_{cf}\eta_{di}\eta_{gk}\eta_{hq}\eta_{jl} + e_{112} \cdot \eta_{ae}\eta_{bi}\eta_{cf}\eta_{dj}\eta_{gp}\eta_{hk}\eta_{lq} \\
& + e_{113} \cdot \eta_{ae}\eta_{bi}\eta_{cf}\eta_{dj}\eta_{gk}\eta_{hl}\eta_{pq} + e_{114} \cdot \epsilon_{abcd}\eta_{eg}\eta_{fh}\eta_{pi}\eta_{jk}\eta_{lq} + e_{115} \cdot \epsilon_{abcd}\eta_{eg}\eta_{fh}\eta_{pq}\eta_{ik}\eta_{jl} \\
& + e_{116} \cdot \epsilon_{abcd}\eta_{eg}\eta_{fp}\eta_{hi}\eta_{jk}\eta_{lq} + e_{117} \cdot \epsilon_{abcd}\eta_{eg}\eta_{fi}\eta_{hk}\eta_{pj}\eta_{lq} + e_{118} \cdot \epsilon_{abcd}\eta_{eg}\eta_{fi}\eta_{hk}\eta_{pq}\eta_{jl} \\
& + e_{119} \cdot \epsilon_{abcd}\eta_{eg}\eta_{fi}\eta_{hq}\eta_{pk}\eta_{jl} + e_{120} \cdot \epsilon_{abcd}\eta_{ep}\eta_{fi}\eta_{gj}\eta_{hk}\eta_{lq} + e_{121} \cdot \epsilon_{abcd}\eta_{ep}\eta_{fi}\eta_{gk}\eta_{hl}\eta_{jq} \\
& + e_{122} \cdot \epsilon_{abcd}\eta_{ei}\eta_{fj}\eta_{gk}\eta_{hl}\eta_{pq} + e_{123} \cdot \epsilon_{abef}\eta_{cg}\eta_{dh}\eta_{pi}\eta_{jk}\eta_{lq} + e_{124} \cdot \epsilon_{abef}\eta_{cg}\eta_{dh}\eta_{pq}\eta_{ik}\eta_{jl} \\
& + e_{125} \cdot \epsilon_{abef}\eta_{cg}\eta_{dp}\eta_{hi}\eta_{jk}\eta_{lq} + e_{126} \cdot \epsilon_{abef}\eta_{cg}\eta_{dp}\eta_{hq}\eta_{ik}\eta_{jl} + e_{127} \cdot \epsilon_{abef}\eta_{cg}\eta_{di}\eta_{hp}\eta_{jk}\eta_{lq} \\
& + e_{128} \cdot \epsilon_{abef}\eta_{cg}\eta_{di}\eta_{hj}\eta_{pk}\eta_{lq} + e_{129} \cdot \epsilon_{abef}\eta_{cg}\eta_{di}\eta_{hk}\eta_{pj}\eta_{lq} + e_{130} \cdot \epsilon_{abef}\eta_{cg}\eta_{di}\eta_{hk}\eta_{pl}\eta_{jq} \\
& + e_{131} \cdot \epsilon_{abef}\eta_{cg}\eta_{di}\eta_{hk}\eta_{pq}\eta_{jl} + e_{132} \cdot \epsilon_{abef}\eta_{cg}\eta_{di}\eta_{hq}\eta_{pk}\eta_{jl} + e_{133} \cdot \epsilon_{abef}\eta_{cg}\eta_{dq}\eta_{hp}\eta_{ik}\eta_{jl} \\
& + e_{134} \cdot \epsilon_{abef}\eta_{cg}\eta_{dq}\eta_{hi}\eta_{pk}\eta_{jl} + e_{135} \cdot \epsilon_{abef}\eta_{cp}\eta_{di}\eta_{gj}\eta_{hk}\eta_{lq} + e_{136} \cdot \epsilon_{abef}\eta_{cp}\eta_{di}\eta_{gk}\eta_{hl}\eta_{jq} \\
& + e_{137} \cdot \epsilon_{abef}\eta_{cp}\eta_{di}\eta_{gk}\eta_{hq}\eta_{jl} + e_{138} \cdot \epsilon_{abef}\eta_{ci}\eta_{dj}\eta_{gp}\eta_{hk}\eta_{lq} + e_{139} \cdot \epsilon_{abef}\eta_{ci}\eta_{dj}\eta_{gk}\eta_{hl}\eta_{pq} \\
& + e_{140} \cdot \epsilon_{abef}\eta_{ci}\eta_{dk}\eta_{gp}\eta_{hj}\eta_{lq} + e_{141} \cdot \epsilon_{abep}\eta_{cf}\eta_{dg}\eta_{hi}\eta_{jk}\eta_{lq} + e_{142} \cdot \epsilon_{abep}\eta_{cf}\eta_{dg}\eta_{hq}\eta_{ik}\eta_{jl} \\
& + e_{143} \cdot \epsilon_{abep}\eta_{cf}\eta_{di}\eta_{gj}\eta_{hk}\eta_{lq} + e_{144} \cdot \epsilon_{abep}\eta_{cf}\eta_{di}\eta_{gk}\eta_{hl}\eta_{jq} + e_{145} \cdot \epsilon_{abep}\eta_{cf}\eta_{di}\eta_{gk}\eta_{hq}\eta_{jl} \\
& + e_{146} \cdot \epsilon_{abep}\eta_{cg}\eta_{dh}\eta_{fi}\eta_{jk}\eta_{lq} + e_{147} \cdot \epsilon_{abep}\eta_{cg}\eta_{dh}\eta_{fq}\eta_{ik}\eta_{jl} + e_{148} \cdot \epsilon_{abep}\eta_{cg}\eta_{di}\eta_{fj}\eta_{hk}\eta_{lq} \\
& + e_{149} \cdot \epsilon_{abep}\eta_{cg}\eta_{di}\eta_{fk}\eta_{hq}\eta_{jl} + e_{150} \cdot \epsilon_{abei}\eta_{cf}\eta_{dg}\eta_{hp}\eta_{jk}\eta_{lq} + e_{151} \cdot \epsilon_{abei}\eta_{cf}\eta_{dg}\eta_{hj}\eta_{pk}\eta_{lq} \\
& + e_{152} \cdot \epsilon_{abei}\eta_{cf}\eta_{dg}\eta_{hk}\eta_{pj}\eta_{lq} + e_{153} \cdot \epsilon_{abei}\eta_{cf}\eta_{dg}\eta_{hk}\eta_{pl}\eta_{jq} + e_{154} \cdot \epsilon_{abei}\eta_{cf}\eta_{dg}\eta_{hk}\eta_{pq}\eta_{jl} \\
& + e_{155} \cdot \epsilon_{abei}\eta_{cf}\eta_{dg}\eta_{hq}\eta_{pk}\eta_{jl} + e_{156} \cdot \epsilon_{abei}\eta_{cf}\eta_{dp}\eta_{gj}\eta_{hk}\eta_{lq} + e_{157} \cdot \epsilon_{abei}\eta_{cf}\eta_{dp}\eta_{gk}\eta_{hl}\eta_{jq} \\
& + e_{158} \cdot \epsilon_{abei}\eta_{cf}\eta_{dp}\eta_{gk}\eta_{hq}\eta_{jl} + e_{159} \cdot \epsilon_{abei}\eta_{cf}\eta_{dj}\eta_{gp}\eta_{hk}\eta_{lq} + e_{160} \cdot \epsilon_{abei}\eta_{cf}\eta_{dj}\eta_{gk}\eta_{hl}\eta_{pq} \\
& + e_{161} \cdot \epsilon_{abei}\eta_{cf}\eta_{dk}\eta_{gp}\eta_{hj}\eta_{lq} + e_{162} \cdot \epsilon_{abei}\eta_{cf}\eta_{dk}\eta_{gp}\eta_{hl}\eta_{jq} + e_{163} \cdot \epsilon_{abei}\eta_{cf}\eta_{dk}\eta_{gj}\eta_{hq}\eta_{pl} \\
& + e_{164} \cdot \epsilon_{abeq}\eta_{cf}\eta_{dg}\eta_{hp}\eta_{ik}\eta_{jl} + e_{165} \cdot \epsilon_{efgh}\eta_{ac}\eta_{bd}\eta_{pi}\eta_{jk}\eta_{lq} \\
& \Big] \times \eta^{pr}\eta^{qs} H^{abcd} H^{efgh}{}_{,r} H^{ijkl}{}_{,s} \qquad \text{(A.6)}
\end{aligned}
$$

$$
\begin{aligned}
&a_{ABC}{}^{I} H^A H^B H^C{}_I = \Big[\\
&\quad e_{166} \cdot \eta_{ac}\eta_{bd}\eta_{eg}\eta_{fh}\eta_{ik}\eta_{jl}\eta_{pq} + e_{167} \cdot \eta_{ac}\eta_{bd}\eta_{eg}\eta_{fh}\eta_{ik}\eta_{jp}\eta_{lq} + e_{168} \cdot \eta_{ac}\eta_{bd}\eta_{eg}\eta_{fi}\eta_{hk}\eta_{jl}\eta_{pq} \\
&+ e_{169} \cdot \eta_{ac}\eta_{bd}\eta_{eg}\eta_{fi}\eta_{hk}\eta_{jp}\eta_{lq} + e_{170} \cdot \eta_{ac}\eta_{bd}\eta_{eg}\eta_{fi}\eta_{hp}\eta_{jk}\eta_{lq} + e_{171} \cdot \eta_{ac}\eta_{bd}\eta_{eg}\eta_{fp}\eta_{hq}\eta_{ik}\eta_{jl} \\
&+ e_{172} \cdot \eta_{ac}\eta_{bd}\eta_{ei}\eta_{fj}\eta_{gk}\eta_{hl}\eta_{pq} + e_{173} \cdot \eta_{ac}\eta_{bd}\eta_{ei}\eta_{fj}\eta_{gk}\eta_{hp}\eta_{lq} + e_{174} \cdot \eta_{ac}\eta_{bd}\eta_{ei}\eta_{fk}\eta_{gj}\eta_{hl}\eta_{pq} \\
&+ e_{175} \cdot \eta_{ac}\eta_{be}\eta_{dg}\eta_{fh}\eta_{ik}\eta_{jl}\eta_{pq} + e_{176} \cdot \eta_{ac}\eta_{be}\eta_{dg}\eta_{fh}\eta_{ik}\eta_{jp}\eta_{lq} + e_{177} \cdot \eta_{ac}\eta_{be}\eta_{dg}\eta_{fi}\eta_{hk}\eta_{jl}\eta_{pq} \\
&+ e_{178} \cdot \eta_{ac}\eta_{be}\eta_{dg}\eta_{fi}\eta_{hk}\eta_{jp}\eta_{lq} + e_{179} \cdot \eta_{ac}\eta_{be}\eta_{dg}\eta_{fi}\eta_{hp}\eta_{jk}\eta_{lq} + e_{180} \cdot \eta_{ac}\eta_{be}\eta_{dg}\eta_{fp}\eta_{hq}\eta_{ik}\eta_{jl} \\
&+ e_{181} \cdot \eta_{ac}\eta_{be}\eta_{di}\eta_{fg}\eta_{hk}\eta_{jl}\eta_{pq} + e_{182} \cdot \eta_{ac}\eta_{be}\eta_{di}\eta_{fg}\eta_{hk}\eta_{jp}\eta_{lq} + e_{183} \cdot \eta_{ac}\eta_{be}\eta_{di}\eta_{fg}\eta_{hp}\eta_{jk}\eta_{lq} \\
&+ e_{184} \cdot \eta_{ac}\eta_{be}\eta_{di}\eta_{fj}\eta_{gk}\eta_{hl}\eta_{pq} + e_{185} \cdot \eta_{ac}\eta_{be}\eta_{di}\eta_{fj}\eta_{gk}\eta_{hp}\eta_{lq} + e_{186} \cdot \eta_{ac}\eta_{be}\eta_{di}\eta_{fk}\eta_{gj}\eta_{hp}\eta_{lq} \\
&+ e_{187} \cdot \eta_{ac}\eta_{be}\eta_{di}\eta_{fk}\eta_{gl}\eta_{hp}\eta_{jq} + e_{188} \cdot \eta_{ac}\eta_{be}\eta_{di}\eta_{fp}\eta_{gj}\eta_{hk}\eta_{lq} + e_{189} \cdot \eta_{ac}\eta_{be}\eta_{di}\eta_{fp}\eta_{gk}\eta_{hl}\eta_{jq} \\
&+ e_{190} \cdot \eta_{ac}\eta_{be}\eta_{di}\eta_{fp}\eta_{gk}\eta_{hq}\eta_{jl} + e_{191} \cdot \eta_{ac}\eta_{be}\eta_{dp}\eta_{fg}\eta_{hq}\eta_{ik}\eta_{jl} + e_{192} \cdot \eta_{ac}\eta_{be}\eta_{dp}\eta_{fi}\eta_{gj}\eta_{hk}\eta_{lq} \\
&+ e_{193} \cdot \eta_{ac}\eta_{bi}\eta_{dk}\eta_{eg}\eta_{fp}\eta_{hq}\eta_{jl} + e_{194} \cdot \eta_{ac}\eta_{bi}\eta_{dk}\eta_{ej}\eta_{fp}\eta_{gl}\eta_{hq} + e_{195} \cdot \eta_{ae}\eta_{bf}\eta_{cg}\eta_{dh}\eta_{ik}\eta_{jl}\eta_{pq} \\
&+ e_{196} \cdot \eta_{ae}\eta_{bf}\eta_{cg}\eta_{dh}\eta_{ik}\eta_{jp}\eta_{lq} + e_{197} \cdot \eta_{ae}\eta_{bf}\eta_{ci}\eta_{dj}\eta_{gk}\eta_{hl}\eta_{pq} + e_{198} \cdot \eta_{ae}\eta_{bf}\eta_{ci}\eta_{dj}\eta_{gk}\eta_{hp}\eta_{lq} \\
&+ e_{199} \cdot \eta_{ae}\eta_{bf}\eta_{ci}\eta_{dk}\eta_{gj}\eta_{hl}\eta_{pq} + e_{200} \cdot \eta_{ae}\eta_{bg}\eta_{cf}\eta_{dh}\eta_{ik}\eta_{jl}\eta_{pq} + e_{201} \cdot \eta_{ae}\eta_{bg}\eta_{cf}\eta_{dh}\eta_{ik}\eta_{jp}\eta_{lq} \\
&+ e_{202} \cdot \eta_{ae}\eta_{bg}\eta_{ci}\eta_{dj}\eta_{fk}\eta_{hl}\eta_{pq} + e_{203} \cdot \eta_{ae}\eta_{bg}\eta_{ci}\eta_{dj}\eta_{fk}\eta_{hp}\eta_{lq} + e_{204} \cdot \eta_{ae}\eta_{bi}\eta_{cg}\eta_{dk}\eta_{fp}\eta_{hq}\eta_{jl} \\
&+ e_{205} \cdot \epsilon_{abcd}\eta_{eg}\eta_{fh}\eta_{ik}\eta_{jl}\eta_{pq} + e_{206} \cdot \epsilon_{abcd}\eta_{eg}\eta_{fh}\eta_{ik}\eta_{jp}\eta_{lq} + e_{207} \cdot \epsilon_{abcd}\eta_{eg}\eta_{fi}\eta_{hk}\eta_{jl}\eta_{pq} \\
&+ e_{208} \cdot \epsilon_{abcd}\eta_{eg}\eta_{fi}\eta_{hk}\eta_{jp}\eta_{lq} + e_{209} \cdot \epsilon_{abcd}\eta_{eg}\eta_{fi}\eta_{hp}\eta_{jk}\eta_{lq} + e_{210} \cdot \epsilon_{abcd}\eta_{eg}\eta_{fp}\eta_{hq}\eta_{ik}\eta_{jl} \\
&+ e_{211} \cdot \epsilon_{abcd}\eta_{ei}\eta_{fj}\eta_{gk}\eta_{hl}\eta_{pq} + e_{212} \cdot \epsilon_{abcd}\eta_{ei}\eta_{fj}\eta_{gk}\eta_{hp}\eta_{lq} + e_{213} \cdot \epsilon_{abcd}\eta_{ei}\eta_{fk}\eta_{gj}\eta_{hl}\eta_{pq} \\
&+ e_{214} \cdot \epsilon_{abef}\eta_{cg}\eta_{dh}\eta_{ik}\eta_{jl}\eta_{pq} + e_{215} \cdot \epsilon_{abef}\eta_{cg}\eta_{dh}\eta_{ik}\eta_{jp}\eta_{lq} + e_{216} \cdot \epsilon_{abef}\eta_{cg}\eta_{di}\eta_{hk}\eta_{jl}\eta_{pq} \\
&+ e_{217} \cdot \epsilon_{abef}\eta_{cg}\eta_{di}\eta_{hk}\eta_{jp}\eta_{lq} + e_{218} \cdot \epsilon_{abef}\eta_{cg}\eta_{di}\eta_{hp}\eta_{jk}\eta_{lq} + e_{219} \cdot \epsilon_{abef}\eta_{cg}\eta_{dp}\eta_{hq}\eta_{ik}\eta_{jl} \\
&+ e_{220} \cdot \epsilon_{abef}\eta_{ci}\eta_{dj}\eta_{gk}\eta_{hl}\eta_{pq} + e_{221} \cdot \epsilon_{abef}\eta_{ci}\eta_{dj}\eta_{gk}\eta_{hp}\eta_{lq} + e_{222} \cdot \epsilon_{abef}\eta_{ci}\eta_{dk}\eta_{gj}\eta_{hl}\eta_{pq} \\
&+ e_{223} \cdot \epsilon_{abei}\eta_{cf}\eta_{dj}\eta_{gk}\eta_{hl}\eta_{pq} + e_{224} \cdot \epsilon_{abei}\eta_{cf}\eta_{dj}\eta_{gk}\eta_{hp}\eta_{lq} + e_{225} \cdot \epsilon_{abei}\eta_{cf}\eta_{dk}\eta_{gj}\eta_{hl}\eta_{pq} \\
&+ e_{226} \cdot \epsilon_{abei}\eta_{cf}\eta_{dk}\eta_{gj}\eta_{hp}\eta_{lq} + e_{227} \cdot \epsilon_{abei}\eta_{cf}\eta_{dk}\eta_{gl}\eta_{hp}\eta_{jq} + e_{228} \cdot \epsilon_{abei}\eta_{cf}\eta_{dp}\eta_{gj}\eta_{hk}\eta_{lq} \\
&+ e_{229} \cdot \epsilon_{abei}\eta_{cf}\eta_{dp}\eta_{gk}\eta_{hl}\eta_{jq} + e_{230} \cdot \epsilon_{abei}\eta_{cf}\eta_{dp}\eta_{gk}\eta_{hq}\eta_{jl} + e_{231} \cdot \epsilon_{abep}\eta_{cf}\eta_{di}\eta_{gj}\eta_{hk}\eta_{lq} \\
&+ e_{232} \cdot \epsilon_{abij}\eta_{ce}\eta_{df}\eta_{gk}\eta_{hl}\eta_{pq} + e_{233} \cdot \epsilon_{abij}\eta_{ce}\eta_{df}\eta_{gk}\eta_{hp}\eta_{lq} + e_{234} \cdot \epsilon_{abij}\eta_{ce}\eta_{dk}\eta_{fp}\eta_{gl}\eta_{hq} \\
&+ e_{235} \cdot \epsilon_{abip}\eta_{ce}\eta_{df}\eta_{gj}\eta_{hk}\eta_{lq} + e_{236} \cdot \epsilon_{abip}\eta_{ce}\eta_{df}\eta_{gk}\eta_{hl}\eta_{jq} + e_{237} \cdot \epsilon_{ijkl}\eta_{ac}\eta_{bd}\eta_{eg}\eta_{fh}\eta_{pq} \\
&\Big] \times \eta^{pr}\eta^{qs} H^{abcd} H^{efgh} H^{ijkl}{}_{,rs}
\end{aligned}
\tag{A.7}
$$

B Solution of the equivariance equations

The following relations for the ansatz coefficients $e_1, \dots, e_{237}$ solve the perturbative equivariance equations (4.32)–(4.34) in terms of 50 indeterminate constants $k_1, \dots, k_{50}$. See [7] for Haskell code that yields this result.

- first and second order (constants $e_1, \dots, e_{40}$):

$$\begin{aligned}
e_1 &= k_1 \\
e_2 &= k_2 \\
e_3 &= -2k_1 - \frac{2}{3}k_2 \\
e_4 &= 4k_1 + \frac{1}{3}k_2 \\
e_5 &= k_3 \\
e_6 &= -3k_1 - \frac{1}{2}k_2 - 3k_3 \\
e_7 &= k_4 \\
e_8 &= k_5 \\
e_9 &= k_6 \\
e_{10} &= k_7 \\
e_{11} &= k_8 \\
e_{12} &= \frac{1}{2}k_6 + \frac{5}{8}k_7 \\
e_{13} &= -\frac{16}{3}k_4 + 16k_5 - \frac{7}{3}k_6 - \frac{5}{12}k_7 + \frac{4}{3}k_8 \\
e_{14} &= -\frac{8}{3}k_4 + 8k_5 - \frac{13}{6}k_6 - \frac{11}{24}k_7 + \frac{2}{3}k_8 \\
e_{15} &= k_4 - \frac{1}{8}k_6 - \frac{23}{32}k_7 - \frac{1}{2}k_8 \\
e_{16} &= k_9 \\
e_{17} &= k_{10}
\end{aligned}$$

$$
\begin{aligned}
e_{18} &= \frac{3}{2}k_4 + \frac{3}{4}k_6 - \frac{3}{16}k_7 + 3k_9 \\
e_{19} &= \frac{1}{2}k_4 + \frac{1}{4}k_6 - \frac{1}{16}k_7 + k_9 \\
e_{20} &= -\frac{1}{4}k_4 - \frac{1}{8}k_6 + \frac{1}{32}k_7 - \frac{1}{2}k_9 \\
e_{21} &= k_4 - 3k_5 + \frac{1}{4}k_6 - \frac{3}{16}k_7 - \frac{1}{2}k_8 + k_9 - 3k_{10} \\
e_{22} &= k_{11} \\
e_{23} &= k_{12} \\
e_{24} &= k_{13} \\
e_{25} &= k_{14} \\
e_{26} &= k_6 + \frac{3}{4}k_7 - k_{14} \\
e_{27} &= -k_4 + \frac{1}{2}k_7 \\
e_{28} &= \frac{5}{3}k_4 + \frac{5}{12}k_6 - \frac{25}{48}k_7 - 2k_{11} - k_{12} - \frac{2}{3}k_{13} - \frac{1}{4}k_{14} \\
e_{29} &= k_6 + \frac{3}{4}k_7 - k_{14} \\
e_{30} &= -\frac{4}{3}k_4 - \frac{5}{6}k_6 + \frac{1}{24}k_7 + 4k_{11} + 2k_{12} + \frac{1}{3}k_{13} + \frac{1}{2}k_{14} \\
e_{31} &= k_{15} \\
e_{32} &= k_{16} \\
e_{33} &= k_4 - \frac{1}{2}k_7 - 3k_{11} - \frac{1}{2}k_{13} - 6k_{15} \\
e_{34} &= \frac{1}{2}k_6 + \frac{3}{8}k_7 - \frac{3}{2}k_{12} - \frac{1}{2}k_{14} - 3k_{16} \\
e_{35} &= -2k_4 - k_6 + \frac{1}{4}k_7 \\
e_{36} &= -k_4 + \frac{1}{2}k_7 - \frac{3}{2}k_{12} - \frac{1}{2}k_{14} - 3k_{16} \\
e_{37} &= \frac{1}{12}k_4 + \frac{1}{12}k_6 + \frac{1}{48}k_7 - \frac{1}{8}k_{12} - \frac{1}{24}k_{14} + k_{15} + \frac{1}{4}k_{16} \\
e_{38} &= -2k_4 + k_7 \\
e_{39} &= -2k_6 - \frac{3}{2}k_7 \\
e_{40} &= k_4 + \frac{1}{2}k_6 - \frac{1}{8}k_7
\end{aligned}
\tag{B.1}
$$

- third order (constants $e_{41}, \dots, e_{237}$): see [7].

C Linearised field equations

From the 16 constants $k_1, \ldots, k_{16}$ that govern the second-order expansion of the area metric Lagrangian, only 10 linearly independent combinations contribute to the linearised Euler-Lagrange equations. A possible basis is given by the 10 gravitational constants s_i below.[1] These are obtained from a 3+1 and subsequent scalar-vector-tensor decomposition of the linearised field equations (see [7]).

$$
\begin{aligned}
s_1 &= 2k_6 + \frac{3}{2}k_7 \\
s_3 &= \frac{3}{2}k_6 + \frac{9}{8}k_7 - 6k_{12} - 2k_{14} \\
s_4 &= -\frac{1}{2}k_6 - \frac{3}{8}k_7 - \frac{1}{2}k_{14} \\
s_6 &= k_6 + \frac{3}{4}k_7 - 3k_{12} - k_{14} - 6k_{16} \\
s_{11} &= \frac{1}{2}k_6 + \frac{11}{8}k_7 + 2k_8 - 2k_{13} - \frac{1}{2}k_{14} \\
s_{13} &= -2k_2 \\
s_{14} &= -2k_4 + 24k_5 - k_6 - \frac{3}{4}k_7 + 4k_8 - 12k_9 + 24k_{10} - 24k_{11} - 6k_{12} - 4k_{13} \\
&\quad - 2k_{14} - 48k_{15} - 12k_{16} \\
s_{16} &= -24k_1 - 4k_2 - 24k_3 \\
s_{37} &= -24k_5 + 2k_6 + \frac{5}{2}k_7 - 4k_8 + 24k_{11} - 12k_{12} + 4k_{13} - 4k_{14} \\
s_{39} &= 24k_1 + 4k_2
\end{aligned}
\tag{C.1}
$$

[1] The constants are not labelled with consecutive numbers, because the labels reflect how they have been calculated: each constant from s_1 to s_{46} is the prefactor of a certain term in the scalar field equations, but also a linear combination of the 16 constants k_i. The subset (C.1) is a basis; every coefficient of the linearised field equations is a linear combination of the s_i.

With the linearly independent subset (C.1) of gravitational constants, the scalar field equations read

$$
\begin{aligned}
\left[\frac{\delta L}{\delta u^{\alpha\beta}}\right]^{\text{S-TF}} &= \Delta_{\alpha\beta}\left[s_1 A - \frac{s_1}{4}\tilde{U} + s_3\tilde{V} + s_4\ddot{V} - \frac{s_4}{3}\Delta V + s_6\ddot{W} - \frac{s_6}{3}\Delta W\right],\\
\left[\frac{\delta L}{\delta v^{\alpha\beta}}\right]^{\text{S-TF}} &= \Delta_{\alpha\beta}\Big[(s_1+4s_4)A + (\frac{s_1}{4}+s_4)\tilde{U} + (\frac{3s_1}{4}+3s_4)\tilde{V}\\
&\quad + s_{11}\ddot{V} - (\frac{s_1}{3}+\frac{4s_4}{3}+s_{11})\Delta V + s_{13}V + s_{14}\Box W + s_{16}W\Big],\\
\left[\frac{\delta L}{\delta w^{\alpha\beta}}\right]^{\text{S-TF}} &= \Delta_{\alpha\beta}\Big[4s_6 A + s_6\tilde{U} + 3s_6\tilde{V}\\
&\quad + (-s_6+s_{14})\ddot{V} - (\frac{s_6}{3}+s_{14})\Delta V + s_{16}V - (\frac{s_1}{4}+s_4+s_{11})\Box W - s_{13}W\Big],\\
\left[\frac{\delta L}{\delta u^{\alpha\beta}}\right]^{\text{S-TR}} &= \gamma_{\alpha\beta}\Big[-\frac{2s_1}{3}\Delta A - \frac{s_1}{2}\ddot{\tilde{U}} + \frac{s_1}{6}\Delta\tilde{U} + (-\frac{3s_1}{4}+s_3)\ddot{\tilde{V}} - \frac{2s_3}{3}\Delta\tilde{V}\\
&\quad + \frac{s_1}{3}\Delta\ddot{V} + \frac{2s_4}{9}\Delta\Delta V + \frac{2s_6}{9}\Delta\Delta W\Big],\\
\left[\frac{\delta L}{\delta v^{\alpha\beta}}\right]^{\text{S-TR}} &= \gamma_{\alpha\beta}\Big[(-s_1+\frac{4s_3}{3})\Delta A + (-\frac{3s_1}{4}+s_3)\ddot{\tilde{U}} - \frac{2s_3}{3}\Delta\tilde{U}\\
&\quad + s_{37}\ddot{\tilde{V}} - (\frac{3s_1}{2} - 2s_3 + s_{37})\Delta\tilde{V} + s_{39}\tilde{V}\\
&\quad + (\frac{s_1}{2}-\frac{2s_3}{3})\Delta\ddot{V} + (\frac{s_1}{6}+\frac{2s_3}{9}+\frac{2s_4}{3})\Delta\Delta V + \frac{2s_6}{3}\Delta\Delta W\Big],\\
\left[\frac{\delta L}{\delta b^{\alpha}}\right]^{\text{S}} &= \partial_\alpha\partial_t\left[-2s_1\tilde{U} + (-3s_1+4s_3)\tilde{V} + (\frac{4s_1}{3}+\frac{8s_4}{3})\Delta V + \frac{8s_6}{3}\Delta W\right],\\
\frac{\delta L}{\delta A} &= -2s_1\Delta\tilde{U} + (-3s_1+4s_3)\Delta\tilde{V} + (\frac{4s_1}{3}+\frac{8s_4}{3})\Delta\Delta V + \frac{8s_6}{3}\Delta\Delta W,
\end{aligned}
\tag{C.2}
$$

where the label (S-TF) denotes the projection of a tensor onto the tracefree scalar and (S-TR) means the projection onto the trace.

A subset of seven constants out of the ten constants s_i parameterizes the vector field equations

$$
\begin{aligned}
\left[\frac{\delta L}{\delta u^{\alpha\beta}}\right]^{\mathrm{V}} &= \partial_t \partial_{(\alpha}\left[s_1 B_{\beta)} - 2s_4 \dot{U}_{\beta)} - 2s_6 \epsilon_{\beta)}{}^{\mu\nu} U_{\mu,\nu} + 2s_6 \dot{W}_{\beta)} + (-\frac{s_1}{2} - 2s_4)\epsilon_{\beta)}{}^{\mu\nu} W_{\mu,\nu}\right], \\
\left[\frac{\delta L}{\delta v^{\alpha\beta}}\right]^{\mathrm{V}} &= \partial_{(\alpha}\Big[(-s_1 - 4s_4)\dot{B}_{\beta)} + 4s_6 \epsilon_{\beta)}{}^{\mu\nu} B_{\mu,\nu} \\
&\quad + (s_1 + 4s_4 + 2s_{11})\ddot{U}_{\beta)} + (-\frac{3s_1}{2} - 6s_4 - 2s_{11})\Delta U_{\beta)} + 2s_6 \epsilon_{\beta)}{}^{\mu\nu} \dot{U}_{\mu,\nu} + 2s_{13} U_{\beta)} \\
&\quad + 2s_{14} \Box W_{\beta)} + 2s_{16} W_{\beta)}\Big], \\
\left[\frac{\delta L}{\delta w^{\alpha\beta}}\right]^{\mathrm{V}} &= \partial_{(\alpha}\Big[4s_6 \dot{B}_{\beta)} + (s_1 + 4s_4)\epsilon_{\beta)}{}^{\mu\nu} B_{\mu,\nu} \\
&\quad + (2s_6 + 2s_{14})\ddot{U}_{\beta)} - 2s_{14}\Delta U_{\beta)} + (\frac{s_1}{2} + 2s_4)\epsilon_{\beta)}{}^{\mu\nu} \dot{U}_{\mu,\nu} + 2s_{16} U_{\beta)} \\
&\quad + (-\frac{3s_1}{2} - 6s_4 - 2s_{11})\Box W_{\beta)} - 2s_{13} W_{\beta)}\Big], \\
\left[\frac{\delta L}{\delta b^{\alpha}}\right]^{\mathrm{V}} &= \Delta\left[2s_1 B_\alpha - 4s_4 \dot{U}_\alpha - 4s_6 \epsilon_\alpha{}^{\mu\nu} U_{\mu,\nu} + 4s_6 \dot{W}_\alpha + (-s_1 - 4s_4)\epsilon_\alpha{}^{\mu\nu} W_{\mu,\nu}\right],
\end{aligned}
\tag{C.3}
$$

as well as the transverse traceless tensor field equations

$$
\begin{aligned}
\left[\frac{\delta L}{\delta u^{\alpha\beta}}\right]^{\mathrm{TT}} &= \frac{s_1}{4}\Box U_{\alpha\beta} \\
&\quad + (\frac{s_1}{4} + s_4)\ddot{V}_{\alpha\beta} + (\frac{s_1}{4} + s_4)\Delta V_{\alpha\beta} - 2s_6 \epsilon_{(\alpha}{}^{\mu\nu} \dot{V}_{\beta)\mu,\nu} \\
&\quad + s_6 \ddot{W}_{\alpha\beta} + s_6 \Delta W_{\alpha\beta} + (\frac{s_1}{2} + 2s_4)\epsilon_{(\alpha}{}^{\mu\nu} \dot{W}_{\beta)\mu,\nu}, \\
\left[\frac{\delta L}{\delta v^{\alpha\beta}}\right]^{\mathrm{TT}} &= (\frac{s_1}{4} + s_4)\ddot{U}_{\alpha\beta} + (\frac{s_1}{4} + s_4)\Delta U_{\alpha\beta} + 2s_6 \epsilon_{(\alpha}{}^{\mu\nu} \dot{U}_{\beta)\mu,\nu} \\
&\quad + (\frac{s_1}{4} + s_4 + s_{11})\Box V_{\alpha\beta} + s_{13} V_{\alpha\beta} + s_{14}\Box W_{\alpha\beta} + s_{16} W_{\alpha\beta}, \\
\left[\frac{\delta L}{\delta w^{\alpha\beta}}\right]^{\mathrm{TT}} &= s_6 \ddot{U}_{\alpha\beta} + s_6 \Delta U_{\alpha\beta} - (\frac{s_1}{2} + 2s_4)\epsilon_{(\alpha}{}^{\mu\nu} \dot{U}_{\beta)\mu,\nu} \\
&\quad + s_{14}\Box V_{\alpha\beta} + s_{16} V_{\alpha\beta} - (\frac{s_1}{4} + s_4 + s_{11})\Box W_{\alpha\beta} - s_{13} W_{\alpha\beta}.
\end{aligned}
\tag{C.4}
$$

References

[1] Nils Alex and Tobias Reinhart. "Covariant constructive gravity: A step-by-step guide towards alternative theories of gravity". In: *Physical Review D* 101.8 (Apr. 2020), p. 084025. DOI: 10.1103/physrevd.101.084025.

[2] Tobias Reinhart and Nils Alex. "Covariant Constructive Gravity". In: *Proceedings of the Fifteenth Marcel Grossman Meeting on General Relativity (in press)*. arXiv: arXiv:2009.07540 [gr-qc].

[3] Nils Alex. "Solutions of gravitational field equations for weakly birefringent spacetimes". In: *Proceedings of the Fifteenth Marcel Grossman Meeting on General Relativity (in press)*. arXiv: arXiv:2009.07540 [gr-qc].

[4] Nils Alex. "Gravitational radiation from birefringent matter dynamics". In: *Physical Review D* 102.10 (Nov. 2020), p. 104017. DOI: 10.1103/physrevd.102.104017.

[5] Tobias Reinhart and Nils Alex. *sparse-tensor: typesafe tensor algebra library [Hackage]*. Aug. 2019. URL: https://hackage.haskell.org/package/sparse-tensor.

[6] Nils Alex. *safe-tensor: dependently typed tensor algebra (version v0.2.1.0) [Zenodo]*. Version v0.2.1.0. Sept. 2020. DOI: 10.5281/zenodo.4030851.

[7] Nils Alex. *area-metric-gravity: 3+1 split of area metric gravity to second order [Zenodo]*. Sept. 2020. DOI: 10.5281/zenodo.4032251.

[8] B. P. Abbott *et al.* (LIGO Scientific Collaboration and Virgo Collaboration). "Observation of Gravitational Waves from a Binary Black Hole Merger". In: *Physical Review Letters* 116.6 (Feb. 2016), p. 061102. DOI: 10.1103/physrevlett.116.061102.

[9] D. Hanneke, S. Fogwell Hoogerheide and G. Gabrielse. "Cavity control of a single-electron quantum cyclotron: Measuring the electron magnetic moment". In: *Physical Review A* 83.5 (May 2011), p. 052122. DOI: 10.1103/physreva.83.052122.

[10] Tatsumi Aoyama et al. "Erratum: Tenth-order electron anomalous magnetic moment: Contribution of diagrams without closed lepton loops". In: *Physical Review D* 96.1 (July 2017), p. 019901. DOI: 10.1103/physrevd.96.019901.

[11] Albert Einstein. "Die Grundlage der allgemeinen Relativitätstheorie". In: *Annalen der Physik* 354.7 (1916), pp. 769–822. DOI: 10.1002/andp.19163540702.

[12] Peter J. Mohr, David B. Newell and Barry N. Taylor. "CODATA recommended values of the fundamental physical constants: 2014". In: *Reviews of Modern Physics* 88.3 (Sept. 2016), p. 035009. DOI: 10.1103/revmodphys.88.035009.

[13] Peter J. Mohr, Barry N. Taylor and David B. Newell. "CODATA recommended values of the fundamental physical constants: 2010". In: *Reviews of Modern Physics* 84.4 (Nov. 2012), pp. 1527–1605. DOI: 10.1103/revmodphys.84.1527.

[14] David Lovelock. "The uniqueness of the Einstein field equations in a four-dimensional space". In: *Archive for Rational Mechanics and Analysis* 33.1 (Jan. 1969), pp. 54–70. DOI: 10.1007/bf00248156.

[15] Sergio A. Hojman, Karel Kuchař and Claudio Teitelboim. "Geometrodynamics regained". In: *Annals of Physics* 96.1 (Jan. 1976), pp. 88–135. DOI: 10.1016/0003-4916(76)90112-3.

[16] S. Deser. "Self-interaction and gauge invariance". In: *General Relativity and Gravitation* 1.1 (1970), pp. 9–18. DOI: 10.1007/bf00759198.

[17] S. W. Hawking and G. F. R. Ellis. *The Large Scale Structure of Space-Time*. Cambridge University Press, May 1973. DOI: 10.1017/cbo9780511524646.

[18] K. C. Freeman. "On the Disks of Spiral and S0 Galaxies". In: *The Astrophysical Journal* 160 (June 1970), p. 811. DOI: 10.1086/150474.

[19] Vera C. Rubin and W. Kent Ford, Jr. "Rotation of the Andromeda Nebula from a Spectroscopic Survey of Emission Regions". In: *The Astrophysical Journal* 159 (Feb. 1970), p. 379. DOI: 10.1086/150317.

[20] Vera C. Rubin, Norbert Thonnard and W. Kent Ford, Jr. "Rotational properties of 21 SC galaxies with a large range of luminosities and radii, from NGC 4605 $R = 4\,\text{kpc}$ to UGC 2885 $R = 122\,\text{kpc}$". In: *The Astrophysical Journal* 238 (June 1980), p. 471. DOI: 10.1086/158003.

[21] Gianfranco Bertone, Dan Hooper and Joseph Silk. "Particle dark matter: evidence, candidates and constraints". In: *Physics Reports* 405.5-6 (Jan. 2005), pp. 279–390. DOI: 10.1016/j.physrep.2004.08.031.

[22] Kenath Arun, S. B. Gudennavar and C. Sivaram. "Dark matter, dark energy, and alternate models: A review". In: *Advances in Space Research* 60.1 (July 2017), pp. 166–186. DOI: 10.1016/j.asr.2017.03.043.

[23] Kristina Giesel et al. "Gravitational dynamics for all tensorial spacetimes carrying predictive, interpretable, and quantizable matter". In: *Physical Review D* 85.10 (May 2012), p. 104042. DOI: 10.1103/physrevd.85.104042.

[24] Frederic P. Schuller and Christof Witte. "How quantizable matter gravitates: A practitioner's guide". In: *Physical Review D* 89.10 (May 2014), p. 104061. DOI: 10.1103/physrevd.89.104061.

[25] Christof Witte. "Gravity actions from matter actions". PhD thesis. Mathematisch-Naturwissenschaftliche Fakultät I der Humboldt-Universität zu Berlin, 2014. eprint: https://edoc.hu-berlin.de/bitstream/handle/18452/17640/witte.pdf.

[26] Maximilian Düll et al. "Gravitational closure of matter field equations". In: *Physical Review D* 97.8 (Apr. 2018), p. 084036. DOI: 10.1103/physrevd.97.084036.

[27] R. Arnowitt, S. Deser and C. W. Misner. "Canonical Variables for General Relativity". In: *Physical Review* 117.6 (Mar. 1960), pp. 1595–1602. DOI: 10.1103/physrev.117.1595.

[28] Dennis Rätzel, Sergio Rivera and Frederic P. Schuller. "Geometry of physical dispersion relations". In: *Physical Review D* 83.4 (Feb. 2011), p. 044047. DOI: 10.1103/physrevd.83.044047.

[29] Sergio Rivera Hernández. "Tensorial spacetime geometries carrying predictive, interpretable and quantizablematter dynamics". PhD thesis. Mathematisch-Naturwissenschaftliche Fakultät der Universität Potsdam, 2012. eprint: https://d-nb.info/1026808375/34.

[30] Simon Grosse-Holz, Frederic P. Schuller and Roberto Tanzi. *Gravitational effects in birefringent quantum electrodynamics*. 21st Mar. 2017. arXiv: arXiv:1703.07183 [hep-ph].

[31] Frederic P. Schuller and Marcus C. Werner. "Etherington's Distance Duality with Birefringence". In: *Universe* 3.3 (July 2017), p. 52. DOI: 10.3390/universe3030052.

[32] Alexander Alfred Wierzba. "Parametrodynamics". MA thesis. Friedrich-Alexander-Universität Erlangen-Nürnberg, 2018.

[33] Udo Beier. "Perturbative gravitational closure of a bi-metric matter theory". MA thesis. Friedrich-Alexander-Universität Erlangen-Nürnberg, 2018.

[34] Maximilian Düll. "Gravitational closure of matter field equations General theory & symmetrization". PhD thesis. Combined Faculty of Natural Sciences and Mathematics of Heidelberg University, 2020. DOI: 10.11588/HEIDOK.00028630.

[35] David Lovelock. "The Einstein Tensor and Its Generalizations". In: *Journal of Mathematical Physics* 12.3 (Mar. 1971), pp. 498–501. DOI: 10.1063/1.1665613.

[36] David Lovelock. "The Four-Dimensionality of Space and the Einstein Tensor". In: *Journal of Mathematical Physics* 13.6 (June 1972), pp. 874–876. DOI: 10.1063/1.1666069.

[37] Tobias Reinhart. "Covariant Constructive Gravity". MA thesis. Friedrich-Alexander-Universität Erlangen-Nürnberg, 2019.

[38] D. J. Saunders. *The Geometry of Jet Bundles*. Cambridge University Press, Mar. 1989. DOI: 10.1017/cbo9780511526411.

[39] Mark J. Gotay and Jerrold E. Marsden. “Stress-Energy-Momentum Tensors and the Belinfante-Rosenfeld Formula”. In: *Mathematical Aspects of Classical Field Theory*. American Mathematical Society, Providence, RI, 1992, pp. 367–392. URL: https://www.cds.caltech.edu/~marsden/bib/1992/05-GoMa1992.

[40] David Anselm Reiß. *Covariance equations for tensors with symmetries:a general and practical approach via intertwiners*. BA Thesis. Friedrich-Alexander-Universität Erlangen-Nürnberg. 2014.

[41] Helge Glöckner and Karl-Hermann Neeb. *Infinite-Dimensional Lie Groups*. Springer-Verlag New York Inc., Jan. 2015. 400 pp. ISBN: 038709444X. eprint: http://www.les-mathematiques.net/phorum/file.php/2/2980/ch1vorl.pdf.

[42] Werner M. Seiler. *Involution*. Springer Berlin Heidelberg, 2010. DOI: 10.1007/978-3-642-01287-7.

[43] Ralph Abraham and Jerrold E. Marsden. *Foundations of Mechanics*. 364. Addison-Wesley Publishing Company, lnc., 1987. URL: https://authors.library.caltech.edu/25029/.

[44] Mark J. Gotay et al. “Momentum Maps and Classical Relativistic Fields. Part I: Covariant Field Theory”. In: (Jan. 1998). arXiv: physics/9801019 [math-ph].

[45] Mark J. Gotay. “A Multisymplectic Framework for Classical Field Theory and the Calculus of Variations”. In: *Mechanics, Analysis and Geometry: 200 Years After Lagrange*. Elsevier, 1991, pp. 203–235. DOI: 10.1016/b978-0-444-88958-4.50012-4.

[46] F. J. Belinfante. “On the current and the density of the electric charge, the energy, the linear momentum and the angular momentum of arbitrary fields”. In: *Physica* 7.5 (May 1940), pp. 449–474. DOI: 10.1016/s0031-8914(40)90091-x.

[47] Léon Jacques Henri Constant Rosenfeld. “Sur le tenseur d’impulsion-énergie”. In: *Mém. Acad. Roy. Belg.* 18 (1940), pp. 1–30.

[48] M. Ostrogradsky. “Mémoires sur les équations différentielles, relatives au problème des isopérimètres”. In: *Mem. Acad. St. Petersbourg* 6.4 (1850), pp. 385–517.

[49] Yakov Itin. “On light propagation in premetric electrodynamics: the covariant dispersion relation”. In: *Journal of Physics A: Mathematical and Theoretical* 42.47 (Oct. 2009), p. 475402. DOI: 10.1088/1751-8113/42/47/475402.

[50] Lars Hörmander. “The Cauchy problem for differential equations with double characteristics”. In: *Journal d’Analyse Mathématique* 32.1 (Dec. 1977), pp. 118–196. DOI: 10.1007/bf02803578.

[51] V. Ya. Ivrii and V. M. Petkov. “Necessary conditions for the Cauchy problem for non-strictly hyperbolic equations to be well-posed”. In: *Russian Mathematical Surveys* 29.5 (Oct. 1974), pp. 1–70. DOI: 10.1070/rm1974v029n05abeh001295.

[52] Lars Gårding. "An inequality for hyperbolic polynomials". In: *Journal of Mathematics and Mechanics* (1959), pp. 957–965. URL: https://www.jstor.org/stable/24900665.

[53] Xiaoshu Liu et al. "Measuring the speed of gravitational waves from the first and second observing run of Advanced LIGO and Advanced Virgo". In: *Physical Review D* 102.2 (July 2020), p. 024028. DOI: 10.1103/physrevd.102.024028.

[54] V. V. Narlikar and K. R. Karmarkar. "The scalar invariants of a general gravitational metric". In: *Proceedings of the Indian Academy of Sciences - Section A* 29.2 (Feb. 1949), p. 92. DOI: 10.1007/bf03171357.

[55] E. Zakhary and C. B. G. Mcintosh. "A Complete Set of Riemann Invariants". In: *General Relativity and Gravitation* 29.5 (May 1997), pp. 539–581. DOI: 10.1023/a:1018851201784.

[56] Yuri N. Obukhov and Friedrich W. Hehl. "Spacetime metric from linear electrodynamics". In: *Physics Letters B* 458.4 (July 1999), pp. 466–470. DOI: 10.1016/s0370-2693(99)00643-7.

[57] Friedrich W. Hehl and Yuri N. Obukhov. *Foundations of Classical Electrodynamics*. Progress in Mathematical Physics. Birkhäuser Boston, 2003. ISBN: 9780817642228. DOI: 10.1007/978-1-4612-0051-2.

[58] Yuri N. Obukhov, Tetsuo Fukui and Guillermo F. Rubilar. "Wave propagation in linear electrodynamics". In: *Physical Review D* 62.4 (July 2000), p. 044050. DOI: 10.1103/physrevd.62.044050.

[59] Werner M. Seiler. *Involution—the formal theory of differential equations and its applications in computer algebra and numerical analysis*. Habilitation Thesis. Universität Mannheim. 2002. URL: https://www.mathematik.uni-kassel.de/~seiler/Papers/habil.html.

[60] Helmer Aslaksen, Eng-Chye Tan and Chen-Bo Zhu. "Invariant theory of special orthogonal groups". In: *Pacific Journal of Mathematics* 168.2 (Apr. 1995), pp. 207–215. DOI: 10.2140/pjm.1995.168.207.

[61] C. Procesi. "The invariant theory of n × n matrices". In: *Advances in Mathematics* 19.3 (Mar. 1976), pp. 306–381. DOI: 10.1016/0001-8708(76)90027-x.

[62] Simon Marlow. *Haskell 2010 Language Report*. Tech. rep. 2010. URL: https://www.haskell.org/onlinereport/haskell2010/.

[63] Jean-Yves Girard. "Interprétation fonctionnelle et élimination des coupures de l'arithmétique d'ordre supérieur". PhD thesis. Université Paris 7, 1972. eprint: https://www.cs.cmu.edu/~kw/scans/girard72thesis.pdf.

[64] Alberto Ruiz. *hmatrix: Numerical Linear Algebra [Hackage]*. 2006. URL: https://hackage.haskell.org/package/hmatrix.

[65] Jonas Schneider et al. *Gravitational closure of weakly birefringent electrodynamics.* 13th Aug. 2017. arXiv: arXiv:1708.03870 [hep-th].

[66] Richard A. Eisenberg and Stephanie Weirich. “Dependently typed programming with singletons”. In: *ACM SIGPLAN Notices* 47.12 (Dec. 2012), pp. 117–130. DOI: 10.1145/2430532.2364522.

[67] Tim Sheard and Simon Peyton Jones. “Template meta-programming for Haskell”. In: *Proceedings of the 2002 Haskell Workshop, Pittsburgh.* Oct. 2002, pp. 1–16. URL: https://www.microsoft.com/en-us/research/publication/template-meta-programming-for-haskell/.

[68] Frederic P. Schuller, Christof Witte and Mattias N. R. Wohlfarth. “Causal structure and algebraic classification of non-dissipative linear optical media”. In: *Annals of Physics* 325.9 (Sept. 2010), pp. 1853–1883. DOI: 10.1016/j.aop.2010.04.008.

[69] Kasper Peeters. *Introducing Cadabra: a symbolic computer algebra system for field theory problems.* 25th Jan. 2007. arXiv: arXiv:hep-th/0701238 [hep-th].

[70] Kasper Peeters. “Cadabra2: computer algebra for field theory revisited”. In: *Journal of Open Source Software* 3.32 (Dec. 2018), p. 1118. DOI: 10.21105/joss.01118.

[71] Paul A. M. Dirac. “Generalized Hamiltonian Dynamics”. In: *Canadian Journal of Mathematics* 2 (1950), pp. 129–148. DOI: 10.4153/cjm-1950-012-1.

[72] James L. Anderson and Peter G. Bergmann. “Constraints in Covariant Field Theories”. In: *Physical Review* 83.5 (Sept. 1951), pp. 1018–1025. DOI: 10.1103/physrev.83.1018.

[73] Karl Schwarzschild. “Über das Gravitationsfeld eines Massenpunktes nach der Einsteinschen Theorie”. In: *Preussische Akademie der Wissenschaften, Sitzungsberichte* 1916 (part 1) (1916), pp. 189–196. URL: https://archive.org/details/sitzungsberichte1916deutsch/page/188/mode/2up.

[74] Eric Poisson and Clifford M. Will. *Gravity: Newtonian, Post-Newtonian, Relativistic.* Cambridge University Press, 2014. ISBN: 9781107032866. URL: https://www.cambridge.org/de/academic/subjects/physics/astrophysics/gravity-newtonian-post-newtonian-relativistic?format=HB&isbn=9781107032866.

[75] Charles W. Misner, K. S. Thorne and J. A. Wheeler. *Gravitation.* San Francisco: W. H. Freeman, 1973. ISBN: 978-0-691-17779-3.

[76] Moritz Möller. “The generation of gravitational waves by orbiting charges”. MA thesis. Friedrich-Alexander-Universität Erlangen-Nürnberg, 2018.

[77] Albert Einstein. “Näherungsweise Integration der Feldgleichungen der Gravitation”. In: *Preussische Akademie der Wissenschaften, Sitzungsberichte* 1916 (part 1) (1916), pp. 688–696. URL: https://archive.org/details/sitzungsberichte1916deutsch/page/688/mode/2up.

[78] Albert Einstein. “Über Gravitationswellen”. In: *Preussische Akademie der Wissenschaften, Sitzungsberichte* 1918 (part 1) (1918), pp. 154–167. URL: https://archive.org/details/mobot31753002089727/page/154/mode/2up.

[79] J. M. Weisberg and Y. Huang. “Relativistic measurements from timing the binary pulsar PSR B1913+16”. In: *The Astrophysical Journal* 829.1 (Sept. 2016), p. 55. DOI: 10.3847/0004-637x/829/1/55.

[80] Pauli Virtanen et al. “SciPy 1.0: fundamental algorithms for scientific computing in Python”. In: *Nature Methods* 17.3 (Feb. 2020), pp. 261–272. DOI: 10.1038/s41592-019-0686-2.

[81] Steven Weinberg. *Gravitation and Cosmology*. Wiley, New York, 1972.

[82] Robert M. Wald. *General Relativity*. University of Chicago Press, 1984. DOI: 10.7208/chicago/9780226870373.001.0001.

[83] M. O. Katanaev. “Killing vector fields and a homogeneous isotropic universe”. In: *Phys.Usp.59:689-700,2016; Usp.Fiz.Nauk 186:763-775,2016* 59.7 (Oct. 2016), pp. 689–700. DOI: 10.3367/UFNe.2016.05.037808.

[84] A. Friedman. “Über die Krümmung des Raumes”. In: *Zeitschrift für Physik* 10.1 (Dec. 1922), pp. 377–386. DOI: 10.1007/bf01332580.