Coherent states in quantum physics: an overview

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Abstract. Various generalisations (group theoretical, nonlinear, discrete, ...) of Glauber-Sudarshan coherent states are presented in a unified way, with their statistical properties and their role in Quantum Mechanics, Quantum Optics, Signal Analysis, and quantization with or without Planck constant.

1. Introduction

Coherent states are emblematic objects of the quantum formalism. From the seminal papers of Schrödinger (1926) [1], Feynman (at a certain extent, see [2]), Klauder (1960) [3], Glauber (1963) [4], Sudarshan (1963) [5], one cannot count the world of articles, collection of works (e.g. [6]), books (e.g. [7, 8, 9], reviews (e.g. [10], since then devoted to this topic. Our purpose here is to present various families of coherent states in a concise mathematically unified way. In Sections 2, 3, 4 are presented the requisite mathematical material, functional analysis and group representation, which is initially common to signal analysis and quantum formalism and which become divided at a certain point to take into account the specificities of those two important branches of physics, mathematical physics, and applied mathematics. Then are presented three examples of implementation of this material:

(i) CS for Discrete cylinder (Fock-probabilistic approach), in Section 5

(ii) Affine CS for half-plane ("Wavelet" approach) in Section 6,

(iii) CS for Discrete cylinder (Group displacement approach *à la Perelomov*) in Section 7.

Some hints about future directions will be given in the conclusion (Section 8).

2. Preamble: Quantum formalism with or without $\hbar$

2.1. Resolution of the identity

The mathematics of signal analysis and quantum formalism share a common guideline, namely the resolution of the identity.

Given a measure space (X,μ) and a (separable) Hilbert space $\mathcal{H}$, one says that the bounded operator-valued function

$$X \ni x \mapsto \mathsf{M}(x) \text{ acting in } \mathcal{H}\,, \tag{1}$$

resolves the identity operator $\mathbb{1}_{\mathcal{H}}$ in $\mathcal{H}$ with respect to the measure μ if

$$\int_X \mathsf{M}(x)\,\mathrm{d}\mu(x) = \mathbb{1}_{\mathcal{H}} \tag{2}$$

holds in a weak sense.

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2.2. From Signal Analysis to Quantum Formalism

In signal analysis, *analysis* and *reconstruction* are grounded in the application of (2) on a signal, i.e., a vector in $\mathcal{H}$

$$\mathcal{H} \ni |s\rangle \overset{\text{reconstruction}}{=} \int_X \overbrace{\mathsf{M}(x)|s\rangle}^{\text{analysis}} \, \mathrm{d}\mu(x) \,, \tag{3}$$

In quantum formalism, *integral quantization* is grounded in the linear map of a function on the set X to an operator in $\mathcal{H}$

$$f(x) \mapsto \int_X f(x)\mathsf{M}(x) \, \mathrm{d}\mu(x) = A_f \,, \quad 1 \mapsto \mathbb{1}_{\mathcal{H}} \,. \tag{4}$$

2.3. Probabilistic content of integral quantization: semi-classical portraits

If the bounded operators $\mathsf{M}(x)$ in (2) are nonnegative, i.e., $\langle\phi|\mathsf{M}(x)|\phi\rangle \geq 0$ for almost all $x \in X$ and for all $\phi \in \mathcal{H}$, one says that they form a (normalised) positive operator-valued measure (POVM) on X. If they are further unit trace-class, i.e. $\mathrm{tr}(\mathsf{M}(x)) = 1$ for all $x \in X$, which means that the operators $\mathsf{M}(x)$ are density operators, then the map

$$f(x) \mapsto \check{f}(x) := \mathrm{tr}(\mathsf{M}(x)A_f) = \int_X f(x') \, \mathrm{tr}(\mathsf{M}(x)\mathsf{M}(x')) \, \mathrm{d}\mu(x') \tag{5}$$

is a local averaging of the original $f(x)$ (which can very singular, like a Dirac) with respect to the probability distribution on X,

$$x' \mapsto \mathrm{tr}(\mathsf{M}(x)\mathsf{M}(x')) \,. \tag{6}$$

This “semi-classical portrait” $\check{f}(x)$ is expected to be a regularisation of $f(x)$.

2.4. Coherent states

When the operators $\mathsf{M}(x)$ are one-rank density operators, i.e., are orthogonal projectors

$$\mathsf{M}(x) = |x\rangle\langle x| \,, \quad |x\rangle \in \mathcal{H} \,, \quad \int_X |x\rangle\langle x| \, \mathrm{d}\mu(x) = \mathbb{1}_{\mathcal{H}} \,, \tag{7}$$

one says that the states form an overcomplete family in $\mathcal{H}$. They are naturally given the name of (generalized or in a wide sense) coherent states, since they share this fundamental overcompleteness property with the most emblematic coherent states in quantum mechanics, namely the Schrödinger ones for which

$$X = \mathbb{R}^2 = \{(q,p) \,, \, q,p \in \mathbb{R}\} \,, \quad \mathrm{d}\mu(x) = \frac{\mathrm{d}q \, \mathrm{d}p}{2\pi} \,, \quad \mathcal{H} = L^2(\mathbb{R}) \,, \tag{8}$$

or, equivalently in quantum optics, the Glauber-Sudarshan ones for which

$$X = \mathbb{C} = \{\alpha\} \,, \quad \mathrm{d}\mu(x) = \frac{\mathrm{d}^2\alpha}{\pi} \,, \quad \mathcal{H} = \text{ Fock space} \,. \tag{9}$$

3. Coherent states from a Fock-probabilistic approach

3.1. Construction

These CS are built as Fock-like Hilbertian expansions (e.g. Glauber-Sudarshan). Let us pick in the “large” Hilbert space $L^2(X, \mathrm{d}\mu(x))$ an orthonormal set $\mathcal{O} = \{\phi_i(x)\}_{i\in\mathcal{I}}$ satisfying

$$0 < \mathcal{N}(x) = \sum_{i\in\mathcal{I}} |\phi_i(x)|^2 < \infty \quad \text{(a.e.)} \,, \tag{10}$$

Precisely, let $\mathcal{H}$ be a ("companion") Hilbert space with orthonormal basis $\{|e_i\rangle\}$ in one-to-one correspondence $\{|e_i\rangle \leftrightarrow \phi_i\}_{i\in\mathcal{I}}$ with the elements of $\mathcal{O}$. There results a family $\mathcal{C}$ of unit CS vectors $|x\rangle$ in $\mathcal{H}$. These vectors are labelled by elements of X and resolve the identity operator in $\mathcal{H}$:

$$X \ni x \mapsto |x\rangle = \frac{1}{\sqrt{\mathcal{N}(x)}} \sum_{i\in\mathcal{I}} \overline{\phi_i(x)} |e_i\rangle \in \mathcal{H} \,. \tag{11}$$

$$\langle x|x\rangle = 1 \,, \quad \int_X |x\rangle\langle x| \, \mathrm{d}\nu(x) = \mathbb{1}_{\mathcal{H}} \,, \quad \mathrm{d}\nu(x) = \mathcal{N}(x) \, \mathrm{d}\mu(x) \,. \tag{12}$$

3.2. Probabilistic duality

There is, underlying the construction, a deep Bayesian content [11, 9], based or not on experimental evidences or on selective information choice (see [12] for examples and references in quantum optics and quantum measurement), namely, an interplay between

(i) the set of probability distributions on the classical measure space (X, μ),

$$x \mapsto |\phi_i(x)|^2 \quad \text{with} \quad \int_X |\phi_i(x)|^2 \, \mathrm{d}\mu(x) = 1 \,, \tag{13}$$

whose parameter is $i \in \mathcal{I}$,

(ii) and the discrete set of probability distributions on the labelling set $\mathcal{I}$

$$i \mapsto |\phi_i(x)|^2/\mathcal{N}(x) \quad \text{with} \quad \frac{1}{\mathcal{N}(x)} \sum_i |\phi_i(x)|^2 = 1 \,, \tag{14}$$

whose parameter is $x \in X$.

4. CS and beyond from UIR's of groups

4.1. Resolution of identity

This time, the resolution of the identity is provided by Schur's Lemma, an essential ingredient of irreducibility of group representations. Let G be a Lie group with left Haar measure $\mathrm{d}\mu(g)$, and let $g \mapsto U(g)$ be a unitary irreducible representation (UIR) of G in a Hilbert space $\mathcal{H}$. Let M be a bounded operator on $\mathcal{H}$. Suppose that the operator

$$R := \int_G \mathsf{M}(g) \, \mathrm{d}\mu(g) \,, \quad \mathsf{M}(g) := U(g) \, \mathsf{M} \, U^\dagger(g) \tag{15}$$

is defined in a weak sense. From the left invariance of $\mathrm{d}\mu(g)$ R commutes with all operators $U(g)$, $g \in G$, and so from Schur's Lemma, $R = c_{\mathsf{M}} \mathbb{1}_{\mathcal{H}}$ with

$$c_{\mathsf{M}} = \int_G \mathrm{tr} \left(\rho_0 \, \mathsf{M}(g) \right) \, \mathrm{d}\mu(g) \,, \tag{16}$$

where the unit trace positive operator ρ_0 is chosen in order to make the integral convergent. The resolution of the identity follows:

$$\int_G \mathsf{M}(g) \, \mathrm{d}\nu(g) = \mathbb{1}_{\mathcal{H}} \,, \quad \mathrm{d}\nu(g) := \mathrm{d}\mu(g)/c_{\mathsf{M}} \,. \tag{17}$$

4.2. Covariant quantization from square integrable UIRs (e.g. affine group)

Let G be a group owing a square-integrable UIR U. Let ρ be an "admissible" density operator, that means that

$$c(\rho) := \int_G \mathrm{d}\mu(g)\, |\mathrm{tr}\,(\rho\, U(g))\,|^2 < \infty\,. \tag{18}$$

The resolution of the identity then is obeyed by the family $\rho(g) = U(g)\rho U^\dagger(g)$.

In particular, for one-rank $\rho = |\psi\rangle\langle\psi|$ (ψ is said *fiducial* vector) the set

$$\{|g\rangle_\psi = U(g)|\psi\rangle\,,\, g \in G\} \tag{19}$$

forms an overcomplete family of CS labelled by $g \in G$.

Condition (18) allows the *covariant* integral quantization of complex-valued functions on the group

$$f \mapsto A_f = \int_G \rho(g)\, f(g)\, \mathrm{d}\nu(g) \quad \text{(quantization)}\,, \tag{20}$$

$$U(g) A_f U^\dagger(g) = A_{\mathcal{U}(g)f} \quad \text{(covariance)}\,, \tag{21}$$

where $(\mathcal{U}(g)f)(g') := f(g^{-1}g')$. It results a generalization of the Berezin or heat kernel transform on G for defining the semi-classical portrait of A_f (or f):

$$\check{f}(g) := \int_G \mathrm{tr}(\rho(g)\, \rho(g'))\, f(g')\, \mathrm{d}\nu(g')\,. \tag{22}$$

4.3. Covariant quantization from square integrable UIR w.r.t. a subgroup (e.g. Weyl Heisenberg, Galileo, Poincaré groups)

In the absence of square-integrability of a UIR over G, it might exist a definition of square-integrability of this UIR with respect to a left coset manifold $X = G/H$, where H is a closed subgroup of G, equipped with a quasi-invariant measure ν (see [8] and references therein). For a global Borel section $\sigma : X \to G$ of the group, let ν_σ be the unique quasi-invariant measure defined by

$$\mathrm{d}\nu_\sigma(x) = \lambda(\sigma(x), x)\, \mathrm{d}\nu(x),\ , \tag{23}$$

where $\lambda(g, x)\mathrm{d}\nu(x) = \mathrm{d}\nu(g^{-1}x),\ \ \forall g \in G$.

For a so-called square integrable UIR $U \bmod(H)$ there exists admissible $\rho \bmod(H)$, in the sense that

$$c_\rho := \int_X \mathrm{tr}\,(\rho\, \rho_\sigma(x))\ \mathrm{d}\nu_\sigma(x) < \infty\,. \tag{24}$$

With $\rho_\sigma(x) = U(\sigma(x))\rho U(\sigma(x))^\dagger$ we have the resolution of the identity and the resulting quantization

$$f \mapsto A_f = \frac{1}{c_\rho}\int_X f(x)\, \rho_\sigma(x)\, \mathrm{d}\nu_\sigma(x) = \mathbb{1}_{\mathcal{H}}\,. \tag{25}$$

In order to establish the covariance of this quantization let us consider the sections $\sigma_g : X \to G,\ \ g \in G$, which are covariant translates of σ under g:

$$\sigma_g(x) = g\sigma(g^{-1}x) = \sigma(x)h(g, g^{-1}x)\,. \tag{26}$$

where h is the cocycle:

$$g\sigma(x) = \sigma(gx)h(g, x) \quad \text{with} \quad h(g, x) \in H\,. \tag{27}$$

With $\mathrm{d}\nu_{\sigma_g}(x) := \lambda(\sigma_g(x), x)\,\mathrm{d}\nu$ one defines

$$\rho_{\sigma_g}(x) = U(\sigma_g(x))\rho U(\sigma_g(x))^\dagger \,. \tag{28}$$

Hence, for a UIR U which square integrable $\mathrm{mod}(H,\sigma)$, a general covariance property of the integral quantization $f \mapsto A_f$ reads as

$$U(g)A_f U(g)^* = A^{\sigma_g}_{\mathcal{U}_l(g)f}\,, \quad A^{\sigma_g}_f := \frac{1}{c_\rho}\int_X \rho_{\sigma_g}(x) f(x)\mathrm{d}\nu_{\sigma_g}(x)\,, \tag{29}$$

where $(\mathcal{U}_l(g)f)(x) = f\left(g^{-1}x\right)$. Similar results hold true by replacing ρ by more general bounded operators M provided integrability and weak convergence hold in the above expressions.

5. CS for discrete cylinder through the Fock-probabilistic approach

5.1. Coherent states

Let us illustrate the construction of CS along the approach described in Section 3 with the "semi-discrete" cylinder as the measure space X. Details are found in the recent [13]. Precisely the measure space (X,μ) is defined as

$$X_{\mathrm{cyl}} = \mathbb{S}^1 \times \mathbb{Z} = \{x = (\phi, \ell)\,,\, \phi \in [0, 2\pi)\,,\, \ell \in \mathbb{Z}\}\,, \quad \int_X \mathrm{d}\mu(x) := \frac{1}{2\pi}\sum_{\ell\in\mathbb{Z}}\int_0^{2\pi}\mathrm{d}\phi\,. \tag{30}$$

The orthonormal set $\mathcal{O}$ in $L^2(X_{\mathrm{cyl}}, \mathrm{d}\mu)$ is chosen to be

$$\mathcal{O} = \{\Upsilon_n\,,\, n \in Z\}\,, \quad \Upsilon_n(\phi,\ell) = \frac{e^{-\ell^2/2}}{\sqrt{\varpi}}\, e^{-\frac{1}{2}n^2} e^{n(\ell + \mathrm{i}\phi)}\,, \tag{31}$$

where $\varpi = \sum_{n\in\mathbb{Z}} e^{-n^2} \approx 1.776372$. With this choice the coherent states read:

$$|\phi,\ell\rangle = \frac{e^{-\ell^2/2}}{\sqrt{\varpi}}\sum_{n\in\mathbb{Z}} e^{-n^2/2} e^{n(\ell - \mathrm{i}\phi)}|e_n\rangle\,. \tag{32}$$

The states $\{|e_n\rangle\,,\, n \in \mathbb{Z}\}$ can be any orthonormal basis in the companion Hilbert space $\mathcal{H}$, for instance, the eigenstates $|n\rangle$ of the angular momentum operator $\hat{L} = -\mathrm{i}\partial/\partial\phi$, which read as $e^{in\phi} \equiv \langle\phi|n\rangle$ in the (Fourier) Hilbert space $L^2\left(\mathbb{S}^1, \frac{\mathrm{d}\phi}{2\pi}\right)$.

As expected from their construction the states (32) resolve the identity in $\mathcal{H}$

$$\frac{1}{2\pi}\sum_{\ell\in\mathbb{Z}}\int_0^{2\pi}\mathrm{d}\phi\,|\phi,\ell\rangle\langle\phi,\ell| = \mathbb{1}_{\mathcal{H}}\,. \tag{33}$$

5.2. Stellar representation of quantum states from semi-discrete cylinder CS

To every normalized state $|\psi\rangle = \sum_n \psi_n|n\rangle \in \mathcal{H}$ corresponds its *stellar representation*, i.e., defined as its phase space representation through its projections on the CSs (32):

$$|\psi\rangle = \frac{1}{2\pi}\sum_{\ell\in\mathbb{Z}}\int_0^{2\pi}\mathrm{d}\phi\,\Psi(\phi,\ell)\,|\phi,\ell\rangle\,, \tag{34}$$

$$\Psi(\phi,\ell) = \langle\phi,\ell|\psi\rangle = \frac{e^{-\ell^2/2}}{\sqrt{\varpi}}\sum_{n\in\mathbb{Z}} e^{-\frac{1}{2}n^2}\psi_n\, e^{nz} \equiv e^{-\ell^2/2}\widetilde{\psi}(z)\,. \tag{35}$$

The interest of such a decomposition is that the components ψ_n are reasonably expected to be experimentally accessible/maniplable. Note that the associated probability distribution on $X_{\rm cyl}$ is the so-called Husimi Q representation of ψ: $Q_\psi(\phi,\ell) = |\Psi(\phi,\ell)|^2$.

The zeros of the holomorphic part $\widetilde{\psi}(z)$ of $\psi(\phi,\ell)$ constitute the *Majorana constellation on the cylinder*. According to Cauchy's argument principle, the number r_ψ of zeros of $\psi(z)$ ("stellar rank") inside a simple closed contour C, counted with multiplicity, is equal to

$$r_\psi = \frac{1}{2i\pi}\oint_C \frac{\partial_z\widetilde{\psi}(z)}{\widetilde{\psi}(z)}dz\,. \tag{36}$$

and the Hadamard factorization theorem ensures that $\widetilde{\psi}(z) = \prod_{m=0}^{r_\psi}(z-z_m)$ where z_m are the zeros of $\widetilde{\psi}(z)$.

Examples of stellar representations of simple states are shown in Figure 1 (extracted from [13]).

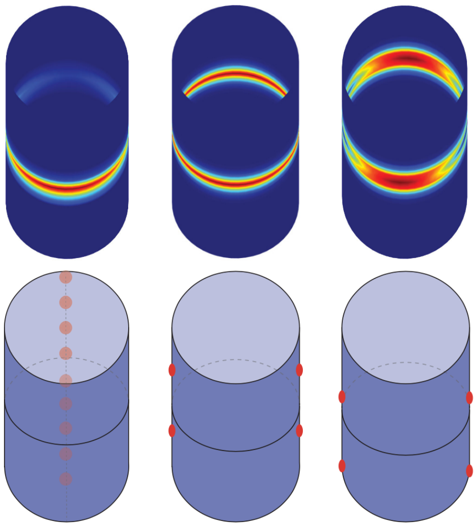

Figure 1. Extracted from [13]. Density plots of the Husimi distribution (upper panel) and associated constellations on the cylinder (lower panel) for (from left to right) a coherent state $|\phi_0,\ell_0\rangle$ (with $\ell_0 = 1$ and $\phi_0 = 0$), an odd cat, and an even cat (both with $\ell = 1$and $\phi_0 = 0$). See details in [13].

6. Affine CS for the half-plane:"wavelet" approach

6.1. Coherent states

This second example deals with coherent states built from one square-integrable representation of the affine group $G = \mathrm{Aff}_+(\mathbb{R})$ of the line:

$$\mathrm{Aff}_+(\mathbb{R}) = \{(a,b)\,,\, a>0\,,\, b\in\mathbb{R}\}\,,\quad \mathbb{R}\ni x\mapsto (a,b)\cdot x = ax+b\,. \tag{37}$$

Instead of the above (a,b) coordinates we here adopt the phase-space-like ones defined by $q = 1/a$, $p = b$. In (quantum) cosmology q can describe a scaling factor (or a volume) while p is its conjugate expansion $\propto \dot{q}$ (e.g. see the recent [14] and references therein).

Hence the open right half-plane (Poincaré) $\mathsf{H} := \{(q,p)\,,\, q>0\,,\, p\in\mathbb{R}\} = \mathbb{R}_+\times\mathbb{R}$ has the affine group structure, exemplified by its left action on itself,

$$(q,p)\,(q_0,p_0) = \left(qq_0, \frac{p_0}{q}+p\right)\,,\quad \mathrm{I_d} = (1,0)\,,\quad (q,p)^{-1} = \left(\frac{1}{q}, -qp\right)\,, \tag{38}$$

and the corresponding left invariant measure $\mathrm{d}\mu(q,p) = \mathrm{d}q\,\mathrm{d}p$. The group $\mathrm{Aff}_+(\mathbb{R})$ owns two square-integrable UIRs. One chooses the following one.

$$L^2(\mathbb{R}_+, \mathrm{d}x) \ni \psi(x) \mapsto (U(q,p)\psi)(x) = \frac{e^{\mathrm{i}px}}{\sqrt{q}}\,\psi\left(\frac{x}{q}\right) . \tag{39}$$

Picking a normalised $\psi \in L^2(\mathbb{R}_+, \mathrm{d}x) \cap L^2(\mathbb{R}_+, \mathrm{d}x/x)$ (the "mother wavelet" in continuous wavelet analysis [8]), the construction (19) yields the corresponding Affine CS (ACS):

$$\psi \mapsto |q,p\rangle_\psi = U(q,p)|\psi\rangle . \tag{40}$$

With the shortened notation $|q,p\rangle_\psi = |q,p\rangle$ the ACS-integral quantization of functions (or distributions) f on H is given by:

$$f(q,p) \mapsto A_f = \frac{1}{2\pi c_{-1}} \int_{\mathbb{R}_+\times\mathbb{R}} \mathrm{d}q\,\mathrm{d}p\, f(q,p)\,|q,p\rangle\langle q,p| \,, \quad c_\gamma(\psi) := \int_0^\infty |\psi(x)|^2\,\frac{\mathrm{d}x}{x^{2+\gamma}} \,, \tag{41}$$

with $A_1 = \mathbb{1}_{\mathcal{H}}$. This map is covariant with respect to U:

$$U(q_0,p_0)A_f U^\dagger(q_0,p_0) = A_{\mathfrak{U}(q_0,p_0)f} \,, \tag{42}$$

with

$$\left(\mathfrak{U}(q_0,p_0)f\right)(q,p) = f\left((q_0,p_0)^{-1}(q,p)\right) = f\left(\frac{q}{q_0}, q_0(p-p_0)\right) . \tag{43}$$

The symmetry property (42) means that no point in the phase space Π_+ is privileged. Precisely, the choice of the origin $(1,0) \in \Pi_+$ for the affine geometry of Π_+ is totally arbitrary, and this is reflected in (42).

6.2. Semi-classical phase space portraits of quantum dynamics

The quantum states and their dynamics have phase space representations. Analogously to (34) the phase space portrait of a state $|\phi\rangle \in L^2(\mathbb{R}_+, \mathrm{d}x)$ is given by

$$\Phi(q,p) = \langle q,p|\phi\rangle . \tag{44}$$

The associated probability distribution on phase space (or Husimi Q representation) is given by

$$\rho_\phi(q,p) = \frac{1}{2\pi c_{-1}}|\Phi(q,p)|^2 . \tag{45}$$

Having at hand the (energy) eigenstates of some quantum Hamiltonian H, e.g. the ACS quantized A_{h} of a classical $\mathsf{h}(q,p)$, at our disposal, we can compute the time evolution

$$\rho_\phi(q,p,t) := \frac{1}{2\pi c_{-1}}|\langle q,p|e^{-\mathrm{i}\mathsf{H}t}|\phi\rangle|^2 \tag{46}$$

for any state $\phi \in L^2(\mathbb{R}_+, \mathrm{d}x)$.

6.3. A toy example: classical and quantum models for dust ball

Let us present one example excerpted from [15]. It is a simplified model for the dynamics of dust in Newtonian cosmology [16], viewed as a sphere of radius $q(t)$, mass M, in an infinite, expanding, homogeneous and isotropic universe filled with dust, i.e. a matter with negligible pressure compared with its energy density. Newtonian gravity is applicable in the case of weak gravity and not too large radius. Hence, the Hamiltonian for a probe unit mass at the surface of the sphere is Kepler-like:

$$\mathsf{H} = \frac{p^2}{2} - \frac{k}{q}\,, \quad k = GM\,. \tag{47}$$

Its ACS quantum version reads

$$A_{\mathsf{H}} = \frac{P^2}{2} + \frac{\hbar^2}{2}\frac{K_\psi}{Q^2} - \frac{1}{c_{-1}}\frac{k}{Q}\,, \quad K_\psi := \int_0^\infty (\psi'(x))^2\, x\frac{\mathrm{d}x}{c_{-1}} > 0\,. \tag{48}$$

Its phase space portrait at constant energy $A_{\mathsf{H}} = E$ is then given by:

$$E = \frac{p^2}{2} + \frac{\hbar^2}{2}\frac{K_\psi^{\mathrm{scl}}}{q^2} - \frac{k}{q}\,, \quad K_\psi^{\mathrm{scl}} = \int_0^\infty \left(\psi'(x)\right)^2 \left(1 + \frac{c_0}{c_{-1}}x\right)\mathrm{d}x\,. \tag{49}$$

In Figure 2 (extracted from [15]) is shown the phase space representation of the quantum dynamical behavior through the evolution of the density $\rho_\phi(q,p,t) = \rho_{q_0,p_0}(q,p,t) = \frac{1}{2\pi}\left|\langle q,p|e^{-\mathrm{i}A_{\mathsf{H}}t}|q_0,p_0\rangle\right|^2$ with initial ACS $|q_0 = 4, p_0 = 0\rangle$.

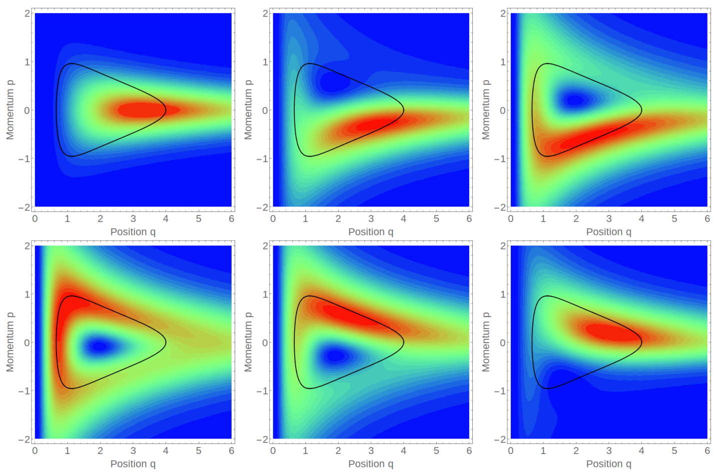

Figure 2. Excerpted from [15]. Thick curves stand for quantum phase-space trajectories with $G = \hbar = 1$ and $M = 2$. Increasing values of the density $\rho_\phi(q,p,t) = \rho_{q_0,p_0}(q,p,t) = \frac{1}{2\pi}\left|\langle q,p|e^{-\mathrm{i}A_{\mathsf{H}}t}|q_0,p_0\rangle\right|^2$ go from blue to red. Time is increasing from the top left to the bottom right.

7. CS for discrete cylinder: Perelomov group theoretical approach

7.1. Coherent states

This third example is extracted from the recent [17] (see for instance [18] for early works with similar formalism). Here the measure space (X,μ) is the same (up to the factor $1/2\pi$) as for Section 5:

$$X = \mathbb{S}^1 \times \mathbb{Z} = \{x = (\theta, m)\,,\ \theta \in [0, 2\pi)\,,\ m \in \mathbb{Z}\}\,, \quad \int_{\mathbb{S}^1\times\mathbb{Z}} \mathrm{d}\mu(\theta, m) = \sum_{m\in\mathbb{Z}} \int_0^{2\pi} \mathrm{d}\theta\,. \tag{50}$$

The semi-discrete Weyl-Heisenberg group $G = \mathrm{H}_1^{\mathrm{dc}}$ is the set of triplets $(s, \theta, m) \in \mathbb{R} \times \mathbb{S}^1 \times \mathbb{Z}\times$ equipped with the multiplication:

$$(s,\theta,m,)\,(s',\theta',m') = \left(s+s'+\frac{m\theta'-m'\theta}{2}, \theta+\theta' \bmod 2\pi, m+m'\right) . \tag{51}$$

The neutral element is $\mathrm{I_d} = (0,0,0)$ and the inverse is $(s,\theta,m)^{-1} = (-s, -\theta \bmod 2\pi, -m)$. The unitary irreducible representation of $\mathrm{H}_1^{\mathrm{dc}}$ acts on $\mathcal{H} = L^2(\mathbb{S}^1, \mathrm{d}\gamma)$ as:

$$(U(s,\theta,m)\psi)(\gamma) = e^{\mathrm{i}\,s} e^{-\mathrm{i}\,\frac{m\theta}{2}} e^{\mathrm{i}\,m\gamma}\psi(\gamma-\theta)\,. \tag{52}$$

It is square integrable on the right coset ($\sim$ phase space) $X = \mathbb{R}\backslash\mathrm{H}_1^{\mathrm{dc}} = \mathbb{S}^1 \times \mathbb{Z}$ of $\mathrm{H}_1^{\mathrm{dc}}$ with its center.

Let σ be the section $X \ni (\theta, m) \mapsto \sigma(\theta,m) = (0,\theta,m) \in \mathrm{H}_1^{\mathrm{dc}}$. The corresponding coherent states $|\theta,m\rangle_\phi$ are defined through the unitary action of a Weyl-like displacement operator $U(\theta,m) := V(0,\theta,m)$ on a normalised fiducial vector $\phi \in \mathcal{H} = L^2(\mathbb{S}^1, \mathrm{d}\gamma)$:

$$|\theta,m\rangle_\phi = U(\theta,m)|\phi\rangle\,, \quad (U(\theta,m)\phi)(\gamma) = e^{-\mathrm{i}\,\frac{m\theta}{2}}\, e^{\mathrm{i}\,m\gamma}\phi(\gamma-\theta)\,. \tag{53}$$

From Schur's Lemma the states $|\theta,m\rangle_\phi$ resolve the identity in $\mathcal{H}$:

$$\sum_{m\in\mathbb{Z}} \int_{\mathbb{S}^1} \mathrm{d}\theta\, |\theta,m\rangle_{\phi\phi}\langle m,\theta| = \mathbb{1}_{\mathcal{H}}\,. \tag{54}$$

Hence the states $|\theta,m\rangle_\phi$ is a coherent state (CS) for the group $\mathrm{H}_1^{\mathrm{dc}}$ in the sense given by Perelomov.

7.2. Semi-classical portrait of a state

The projection of $\psi \in \mathcal{H}$ on $|\theta,m\rangle_\phi$, namely

$${}_\phi\langle\theta,m|\psi\rangle \equiv \Psi_\phi(\theta,m) = \int_{\mathbb{S}^1} \mathrm{d}\,\gamma\, e^{\mathrm{i}\,\frac{m\theta}{2}}\, e^{-\mathrm{i}\,m\gamma}\overline{\phi(\gamma-\theta)}\psi(\gamma)\,, \tag{55}$$

is the phase space or momentum-angular position representation of ψ with respect to this family of coherent states.

For any pair of states $\phi, \psi \in L^2(\mathbb{S}^1, \mathrm{d}\gamma)$, with $||\phi|| = 1$, the map

$$L^2(\mathbb{S}^1,\mathrm{d}\gamma) \ni \psi \to {}_\phi\langle\theta,m|\psi\rangle = \Psi_\phi(\theta,m) \in L^2(\mathbb{S}^1\times\mathbb{Z}, \mathrm{d}\mu(\theta,m)) \tag{56}$$

satisfies the following properties:

(i) it is an isometry:

$$\|\psi\|^2_{L^2(\mathbb{S}^1)} = \int_{\mathbb{S}^1} \mathrm{d}\gamma\, |\psi(\gamma)|^2 = \sum_{m\in\mathbb{Z}} \int_{\mathbb{S}^1} \mathrm{d}\theta\, |\Psi_\phi(\theta,m))|^2 = \|\Psi_\phi\|^2_{L^2(\mathbb{S}^1\times\mathbb{Z})}\,, \tag{57}$$

(ii) it can be inverted on its range:

$$\psi(\gamma) = \sum_{m\in\mathbb{Z}} \int_{\mathbb{S}^1} \mathrm{d}\theta\, \Psi_\phi(\theta,m)\, |\theta,m\rangle_\phi(\gamma)\,, \tag{58}$$

(iii) its range is a reproducing kernel space:

$$\Psi_\phi(\theta,m) = \sum_{m'\in\mathbb{Z}} \int_{\mathbb{S}^1} \mathrm{d}\theta'\, K_\phi((\theta,m),(\theta',m'))\, \psi_\phi(\theta',m',)\,, \tag{59}$$

where the reproducing kernel K_ϕ is the two-point function (or overlap of two CSs) on the phase space $\mathbb{S}^1 \times \mathbb{Z}$:

$$K_\phi((\theta,m),(\theta',m')) = {}_\phi\langle\theta,m|\theta',m'\rangle_\phi\,. \tag{60}$$

7.3. CS quantization with a few examples

The CSs (53) together with the resolution of the unity (54) allow the integral quantization of functions (or distributions) f on $X = \mathbb{S}^1 \times \mathbb{Z}$:

$$f(\theta, m) \mapsto A_f = \sum_{m\in\mathbb{Z}} \int_{\mathbb{S}^1} \mathrm{d}\theta\, f(\theta, m)\, |\theta, m\rangle_{\phi\phi}\langle m, \theta|\,, \tag{61}$$

This map is covariant with respect to the action of the representation V, that is,

$$V\, A_f V^\dagger = A_{\mathcal{V}f} \tag{62}$$

where $(\mathcal{V}(s, m, \theta)f)(n, \varphi) = f(n - m, \varphi - \theta)$ is the induced action on the phase space. Again, this means that no point in the semi-discrete cylinder X is privileged.

We then derive from (61) the following quantizations of the angular momentum, of its square, and a function of the angle only.

- To $g(m) = m$ corresponds

$$A_m = -\mathrm{i}\frac{\partial}{\partial\gamma} + \langle m\rangle_{|\widehat{\psi}|^2} \equiv \hat{L} + \langle m\rangle_{|\widehat{\psi}|^2}\,, \tag{63}$$

  where $\hat{\psi}(m) = \langle e_m|\psi\rangle = \int_{\mathbb{S}^1} \mathrm{d}\gamma\, \frac{e^{-\mathrm{i}m\gamma}}{\sqrt{2\pi}}\, \psi(\gamma)$ and $\langle g(m)\rangle_{|\widehat{\psi}|^2} = \sum_{g(m)\in\mathbb{Z}} m\, |\hat{\psi}(m)|^2$. Note that if the fiducial ψ is chosen such that $|\widehat{\psi}(m)| = |\widehat{\psi}(-m)|$ then $\langle m\rangle_{|\widehat{\psi}|^2} = 0$.

- To $g(m) = m^2$ corresponds

$$A_{m^2} = \hat{L}^2 + 2\langle m\rangle_{|\widehat{\psi}|^2}\hat{L} + \langle m^2\rangle_{|\widehat{\psi}|^2}\,. \tag{64}$$

- To $h(\theta)$ corresponds the multiplication operator

$$\left(A_h\phi\right)(\gamma) = \left[\sum_{m\in\mathbb{Z}} \hat{h}(m)\, \langle e^{-\mathrm{i}m\gamma}\rangle_{|\widehat{\psi}|^2}\, e^{\mathrm{i}\,m\gamma}\right] \phi(\gamma)\,. \tag{65}$$

8. Conclusion

The world of coherent states in its various generalisations and uses is extremely populated. In this contribution, we have just given a flavour of this variety through the works of the author and collaborators, specifically three of them which have recently been published. For other recent ones, with applications in different domains (quantum optics, cosmology, curved manifolds, signal analysis...) see [19, 20, 21, 22, 23].

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