MURA 56

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PARAXIAL LENS-ACTION IN ELECTRIC ION OPTICS OR GEOMETRICAL OPTICS

-- useful for extended (diffuse or thick) lenses --

1. Basic Equations

Least Action	Fermat's Principle
$\Delta \int p\, d\ell = 0$	$\Delta \int n\, d\ell = 0$

These equations are formally identical, with n proportional to p or, classically, proportional to the square root of the kinetic energy.

2. Paraxial Approximation

For rays close to parallel with the z-axis.

Choose an axis location and coördinate directions so that

$$
\begin{aligned}
p\, d\ell &= p(1+x'^2+y'^2)^{1/2}\, dz \\
&\cong p\left[1+\frac{x'^2+y'^2}{2}\right] dz \\
&= (E-V)^{1/2}\left[1+\frac{x'^2+y'^2}{2}\right] dz \\
&= (-q\phi)^{1/2}\left[1+\frac{x'^2+y'^2}{2}\right] dz \\
&= (-q\phi_o(z))^{1/2}\left[1+\frac{\phi_{xx}x^2+\phi_{yy}y^2}{4\phi_o}\right]\left[1+\frac{x'^2+y'^2}{2}\right] dz
\end{aligned}
$$

with the potential, ϕ, measured from a point where the kinetic energy is zero.

$$\frac{d}{dz}\left(\phi_o^{1/2} x'\right) = \frac{\phi_{xx}}{2\phi_o^{1/2}}\, x \, .$$

$$n\, d\ell \cong n_o(z)\left[1+\frac{n_{xx}x^2+n_{yy}y^2}{2n_o}\right]\cdot\left[1+\frac{x'^2+y'^2}{2}\right] dz$$

with $n_{xx} \equiv \left.\frac{\partial^2 n}{\partial x^2}\right|_o$, etc., and axies such that $n_x = n_y = n_{xy} = 0$.

Note: With axial symmetry $n_{xx} = n_{yy}$.

$$\frac{d}{dz}(n_o x') = n_{xx}\, x \, .$$

3. Character of General Solutions

Use, for brevity, the symbol n in place of n_o.

Given two solutions, $x_1(z)$, $x_2(z)$;

$$\frac{d}{dz}\begin{vmatrix} x_1 & x_2 \\ n x_1' & n x_2' \end{vmatrix} = \begin{vmatrix} x_1' & x_2' \\ n x_1' & n x_2' \end{vmatrix} + \begin{vmatrix} x_1 & x_2 \\ (n x_1')' & (n x_2')' \end{vmatrix}$$

$$= \quad 0 \quad + \begin{vmatrix} x_1 & x_2 \\ n_{xx}x_1 & n_{xx}x_2 \end{vmatrix},$$ by virtue of the fact that x_1 and x_2 satisfy the differential eqn.

$$= 0 \, ;$$

i.e., $\begin{vmatrix} x_1 & x_2 \\ n x_1' & n x_2' \end{vmatrix} \equiv n \cdot (\text{Wronskin})$ is constant.

(a) Corrollary:

With particular solutions $\begin{matrix} x_1(0) = 1 & x_2(0) = 0 \\ x_1'(0) = 0 & x_2'(0) = 1 \end{matrix}$,

the general solution is

$$x = x_o\, x_1(z) + x_o'\, x_2(z)$$
$$x' = x_o\, x_1'(z) + x_o'\, x_2'(z) \quad ;$$

i.e.,

$$\begin{pmatrix} x \\ x' \end{pmatrix} = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \begin{pmatrix} x_o \\ x'_o \end{pmatrix} \quad \text{with} \quad \begin{pmatrix} A & B \\ C & D \end{pmatrix} = \begin{pmatrix} x_1 & x_2 \\ x_1' & x_2' \end{pmatrix} \quad \text{and} \quad n \begin{vmatrix} A & B \\ C & D \end{vmatrix} = \begin{matrix} \text{const.} \\ = n_o \end{matrix}$$

or

$$\begin{pmatrix} x \\ nx' \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} x_o \\ (nx')_o \end{pmatrix} \quad \text{with} \quad \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} A & B/n_o \\ nC & nD/n_o \end{pmatrix} \quad \text{and} \quad \begin{vmatrix} a & b \\ c & d \end{vmatrix} = \frac{n}{n_o} \begin{vmatrix} A & B \\ C & D \end{vmatrix} = 1.$$

4. Relation between (transverse) Linear- and Angular-Magnification

For conjugate focii, $B = 0$ (to insure x independent of x_o').

Magnification $= \frac{\Delta x}{\Delta x_o} = A$

Angular Mag. $= \frac{\Delta x'}{\Delta x'_o} = D$

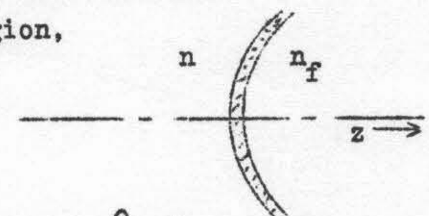
Product, $\frac{\Delta x}{\Delta x_o} \frac{\Delta x'}{\Delta x'_o} = AD = \begin{vmatrix} A & B \\ C & D \end{vmatrix}$ (since $B = 0$)

$= n_o/n$;

i.e., $n\, \Delta x\, \Delta x' = n_o\, \Delta x_o\, \Delta x'_o$.

5. Solution of Differential Equation by Impulse Approximation

If $n_{xx} = 0$, save at a narrowly-localized region,



$\Delta x \doteq 0$ and

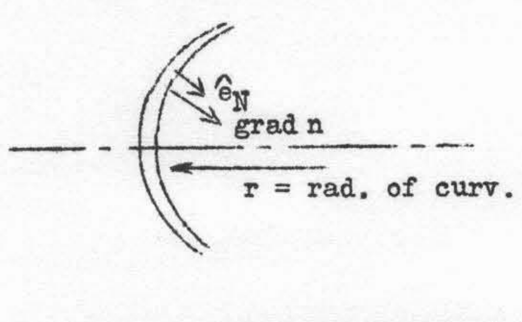
$\Delta(nx') = x \cdot \int n_{xx}\, dz$;

i.e., $\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ \int n_{xx} dz & 1 \end{pmatrix}$, $\begin{pmatrix} A & B \\ C & D \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ (1/n_f)\int n_{xx} dz & n_o/n_f \end{pmatrix}$

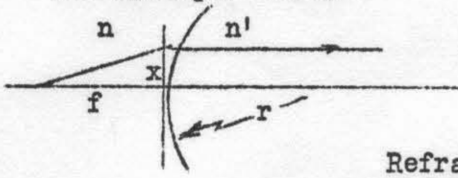
$$= \begin{pmatrix} 1 & 0 \\ -P/n_f & n_o/n_f \end{pmatrix} \quad \text{with} \quad P \equiv -\int n_{xx}\, dz\,.$$

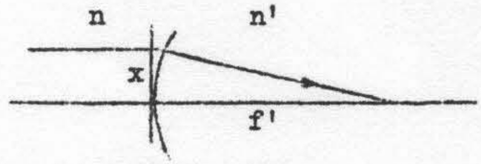
6. Evaluation, and Interpretation, of P for Transmission through one Surface

$$P = -\int n_{xx}\, dz \doteq -\int \frac{1}{x}\, n_x\, dz$$
$$= -\int \frac{1}{x}\, |\text{grad}\, n| \cos(x, N)\, dz$$
$$\doteq \int \frac{|\text{grad}\, n|}{r}\, dz = \frac{n' - n}{r}\,.$$



(a) Interpretation





Refracting surface characterized by

$$\begin{pmatrix} 1 & 0 \\ -P/n' & n/n' \end{pmatrix}$$

For situation depicted in drawing at left above

$$-\frac{P}{n'}x + \frac{n}{n'}x'_{\text{init}} = 0$$

$$f = \frac{x}{x'_{\text{init}}} = \frac{n}{P}$$

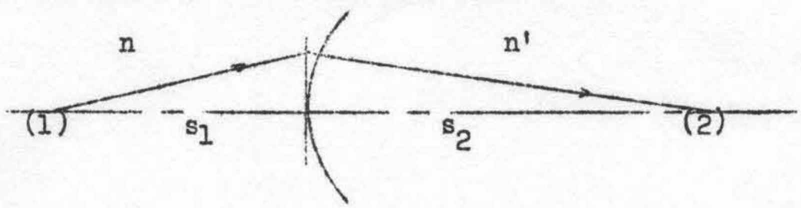
For situation depicted in drawing at right above

$$x'_{\text{out}} = -\frac{P}{n'}x$$

$$f' = -\frac{x}{x'_{\text{out}}} = \frac{n'}{P}$$

$$\boxed{\frac{n}{f} = \frac{n'}{f'} = P = \frac{n'-n}{r}\ .}$$

7. <u>Thin-lens Action of such a Refracting Surface</u>



$\begin{pmatrix} x_2 \\ x'_2 \end{pmatrix}$ is obtained from $\begin{pmatrix} x_1 \\ x'_1 \end{pmatrix}$ by the matrix

$$\begin{pmatrix} 1 & s_2 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ -P/n' & n/n' \end{pmatrix}\begin{pmatrix} 1 & s_1 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 - s_2P/n' & s_1 + s_2n/n' - s_1s_2P/n' \\ -P/n' & n/n' - s_1P/n' \end{pmatrix};$$

for conjugate focii,

$$s_1 + s_2\frac{n}{n'} - s_1s_2\frac{P}{n'} = 0$$

$$\boxed{\frac{n}{s_1} + \frac{n'}{s_2} = P\ .}$$

[Since (Sect. 6) $P = n/f = n'/f'$, this is consistent in including the special cases $s_2 = \infty$, $s_1 = f$ and $s_1 = \infty$, $s_2 = f'$.]

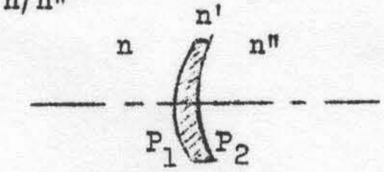
<u>Note</u>: Writing our result as $\frac{n}{s_1P} + \frac{n'}{s_2P} = 1$, we have $\frac{f}{s_1} + \frac{f'}{s_2} = 1$,

$s_1f' + s_2f - s_1s_2 = 0$, or $(s_1 - f)(s_2 - f') = ff'$, which is the NEWTONIAN form.

(a) Corrollary:

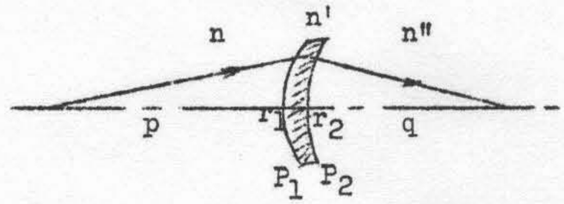
Two such refracting surfaces, separated by a negligible amount, are characterized by the matrix

$$\begin{pmatrix} 1 & 0 \\ -P_2/n'' & n'/n'' \end{pmatrix}\begin{pmatrix} 1 & 0 \\ -P_1/n' & n/n' \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ -(P_1+P_2)/n'' & n/n'' \end{pmatrix},$$



showing an over-all $\underline{P = P_1 + P_2}$;

i.e.,
$$\frac{n}{p} + \frac{n''}{q} = P = P_1 + P_2 .$$



If $n = n'' \equiv n_o$,

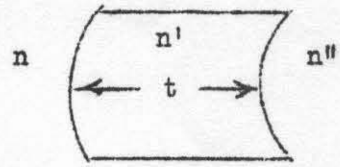
$$\frac{1}{p} + \frac{1}{q} = \frac{P}{n_o} = \frac{1}{f_1} + \frac{1}{f'_2} \equiv \frac{1}{F},$$

which is the usual thin-lens formula, where

$$\frac{1}{f_1} = \frac{P_1}{n_o} = \frac{n' - n_o}{n_o r_1}, \quad \frac{1}{f'_2} = \frac{P_2}{n''} = \frac{P_2}{n_o} = \frac{n_o - n'}{n_o r_2},$$

and $\frac{1}{F} = [(n'/n_o) - 1][\frac{1}{r_1} - \frac{1}{r_2}]$.

8. Thick-Lens -- two such surfaces separated by t



The lens is characterized by the matrix

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ -P_2/n'' & n'/n'' \end{pmatrix}\begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ -P_1/n' & n/n' \end{pmatrix}$$

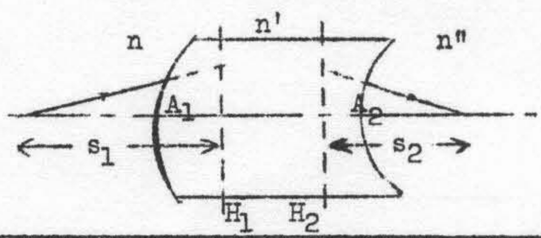
$$= \begin{pmatrix} 1 & (A-1)/C \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ C & n/n'' \end{pmatrix}\begin{pmatrix} 1 & (1/C)(D - n/n'') \\ 0 & 1 \end{pmatrix}$$ since $AD - BC = n/n''$ (Sect. 3),

where

$$A = 1 - \frac{t}{n'}P_1 \qquad B = n\frac{t}{n'}$$

$$C = -\frac{P_1 + P_2 - (t/n')P_1P_2}{n''} \equiv -\frac{P}{n''} \qquad D = \frac{n}{n''}[1 - \frac{t}{n'}P_2].$$

Thus, measuring distances to the principal planes H_1, H_2, the lens may be replaced in effect by a thin lens characterized by
$$\begin{pmatrix} 1 & 0 \\ C & n/n'' \end{pmatrix}.$$



In particular

$n/s_1 + n''/s_2 = P = P_1 + P_2 - (t/n')P_1P_2$, so that, for $s_2 = \infty$, $s_1 \equiv f = \frac{n}{P}$ and, for $s_1 = \infty$, $s_2 \equiv f' = n''/P$;

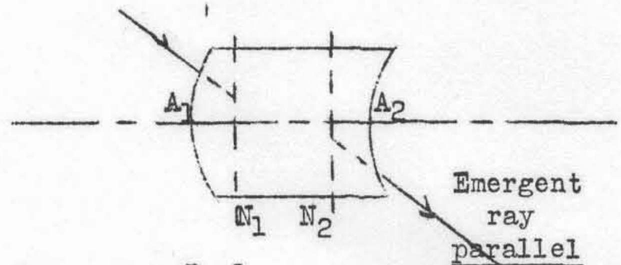
$H_2A_2 = \frac{A-1}{C} = n''\frac{t}{n'}\frac{P_1}{P} = t\frac{f'}{f'_1}$, since $f' = n''/P$ (see immediately above) and $f'_1 = n'/P_1$ (Sect. 6a) ;

$A_1H_1 = \frac{D - (n/n'')}{C} = n\frac{t}{n'}\frac{P_2}{P} = t\frac{f}{f_2}$, since, similarly, $f = n/P$ and $f_2 = n'/P_2$.

Note: Again, in a manner similar to that noted in Sect. 7, we may write our result in the NEWTONIAN form by measuring distances from the principal focii --

$$\frac{f}{s_1} + \frac{f'}{s_2} = 1, \quad \text{or} \quad (s_1 - f)(s_2 - f') = ff' .$$

9. Nodal Planes



For the rays illustrated,

$$x = Ax_o + Bx'_o$$

$$x' = x'_o = Cx_o + Dx'_o \quad ;$$

$$A_1N_1 = -\frac{x_o}{x'_o} = \frac{D-1}{C}, \quad N_2A_2 = \frac{x}{x'} = \frac{x}{x'_o} = B - A\frac{D-1}{C}$$

$$= \frac{A - (AD - BC)}{C} = \frac{A - n/n''}{C} .$$

Thus the nodal planes coincide with the principal planes when $n'' = n$ and otherwise are given in terms of the matrix elements by the foregoing relations to yield the formulas:

$$A_1N_1 = \frac{(n/n')tP_2 + n'' - n}{P} = \frac{(n/n')tP_2 + r_2P_2 + r_1P_1}{P}$$

$$N_2A_2 = \frac{(n''/n')tP_1 + n - n''}{P} = \frac{(n''/n')tP_1 - r_1P_1 - r_2P_2}{P} .$$

Note: $H_1N_1 = H_2N_2 = \frac{n''-n}{P} = f\,\frac{n''-n}{n} = f'\,\frac{n''-n}{n''}$.

10. Magnification

(a) Referring again to the thick lens of Sect. 8, when distances are measured to the principal planes the lens is characterized by

$$\begin{pmatrix} 1 & 0 \\ -P/n'' & n/n'' \end{pmatrix}$$

and the matrix connecting points in the image space with points in the object space is

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} = \begin{pmatrix} 1 & s_2 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ -P/n'' & n/n'' \end{pmatrix}\begin{pmatrix} 1 & s_1 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 - s_2P/n'' & s_1 + s_2n/n'' - s_1s_2P/n'' \\ -P/n'' & n/n'' - s_1P/n'' \end{pmatrix}.$$

(b) As indicated in Sect. 4, conjugate focal points are those for which $B = 0$ and the linear transverse magnification then is

$$\text{Magnification} = A = 1 - s_2\frac{P}{n''}$$

$$= \frac{s_1 - s_1s_2P/n''}{s_1} = -\frac{n}{n''}\frac{s_2}{s_1} .$$

[By a little algebra, this magnification may also be shown to be equal to $-\frac{\text{dist. from } N_2}{\text{dist. to } N_1} = -\frac{s_2 + N_2A_2 - H_2A_2}{s_1 + A_1N_1 - A_1H_1}$, as of course it must.]

Likewise, Angular Magnification $= D = \frac{n}{n''} - s_1\frac{P}{n''} = -\frac{s_1}{s_2}$.

The product is seen to be n/n'', as indicated generally in Sect. 4.

(c) The longitudinal magnification, ds_2/ds_1, is obtained by differentiation of the identity $B \equiv s_1 + s_2\frac{n}{n''} - s_1 s_2 \frac{P}{n''} = 0$.
One thus obtains

$$\frac{ds_2}{ds_1} = -\frac{1 - s_2 P/n''}{\frac{n}{n''} - s_1 \frac{P}{n''}} \qquad \left(= -\frac{A}{D} \right)$$

$$= -\frac{s_2}{s_1}\,\frac{s_1 - s_1 s_2 P/n''}{\frac{n}{n''}s_2 - s_1 s_2 P/n''} = -\frac{n}{n''}\left(\frac{s_2}{s_1}\right)^2 .$$

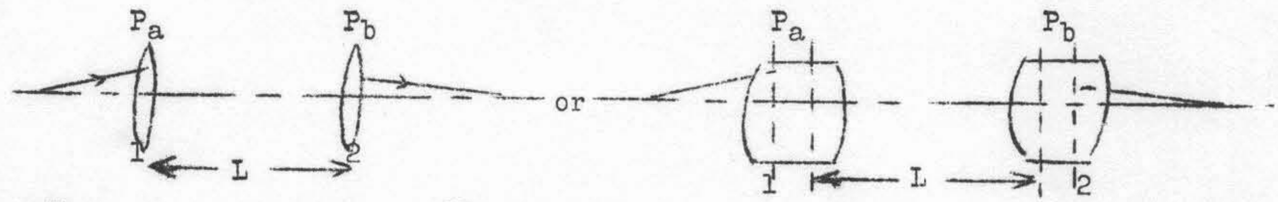
Note: A simple equivalent expression is obtained by differentiating the Newtonian expression noted in Sect. 8:

With $x_1 \equiv s_1 - f, \;\; x_2 \equiv s_2 - f', \;\; x_1 x_2 = ff'$,

whence $$ds_2/ds_1 \equiv dx_2/dx_1 = -x_2/x_1 .$$

11. Separated Pair of Lenses

For a pair of thin lenses, separated by a distance L, or for the equivalent situation involving thick lenses with L measured between the appropriate principal planes, all situated in a medium of index n_o,



$\binom{x_2}{x'_2}$ is obtained from $\binom{x_1}{x'_1}$ by the matrix

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ -P_b/n_o & 1 \end{pmatrix}\begin{pmatrix} 1 & L \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ -P_a/n_o & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ -1/F_b & 1 \end{pmatrix}\begin{pmatrix} 1 & L \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ -1/F_a & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 1 - L/F_a & L \\ -(1/F_a + 1/F_b - L/F_aF_b) & 1 - L/F_b \end{pmatrix} .$$

Just as with the case of a single thick lens (in medium n_o) -- cf. Sect. 8 -- the lens system is characterized by the focal length F given by

$$\frac{1}{F} = -C = \frac{1}{F_a} + \frac{1}{F_b} - \frac{L}{F_a F_b} ,$$

with principal (and nodal) planes situated to the right of 1 and to the left of 2 by the respective distances

Lens 1 to $H_1 = \frac{D-1}{C} = \frac{LF}{F_b}$, $\qquad H_2$ to Lens 2 $= \frac{A-1}{C} = \frac{LF}{F_a}$.

12. References

Schliermacher Equations -- M. Herzberger, Strahlenoptik (Springer, Berlin, 1931), pp. 62 ff.

Helmholtz-Lagrange Relation -- Herzberger (op. cit., p. 31); M. Merté, Hdbk. d. Physik XVIII (Springer, Berlin, 1927), Sect. 32, p. 48-49.