Localizations with Noncompact Transverse Spaces and Covert Symmetry Breaking in Supergravity

Christopher W. Erickson

Imperial College London

Department of Physics

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Supervised by: Prof. Kellogg S. Stelle

Abstract

The focus of my doctoral work has been answering the question: Can lower-dimensional effective gravitational theories be found in a higher-dimensional theory with a non-compact transverse space? To answer this question this thesis is divided into two parts. First, I explore supergravity solutions on warped product manifolds and argue that they correspond to solutions of a modified Laplacian. I pay special attention to Type III † solutions, or solutions characterized by the presence of non-constant transverse zero modes, and emphasize that these are the only higher dimensional solutions corresponding to localized sources that yield effectively lower-dimensional physics when the transverse space has infinite volume. Second, I derive the lower dimensional effective field theory about such backgrounds. I discover that these effective field theories have covert symmetry breaking, spontaneous breaking of gauge symmetry which only appears at quartic order. I show this explicitly for $D = d + 1$ scalar electrodynamics with any boundary condition that corresponds to a non-constant transverse zero mode. The mathematical prerequisite for both of these conclusions is Sturm–Liouville theory with precise manipulations of Green's formula. To support this I derive the fundamental conclusions of Sturm–Liouville theory for a restricted class of operator, that is the Laplacian, which relaxes some requirements for permissible boundary conditions of Sturm–Liouville theory. The answer to my focal question is: Yes; however, the lower-dimensional theory has novel corrections which were previously unexplored, and further research is indicated.

Declaration of originality

I declare that the work contained in this thesis is my own, except where there are references to others works, or works done in collaboration with others. The work presented in this thesis was carried out between November 2016 and December 2021 at Imperial College London under the supervision of Professor K.S.Stelle. This thesis has not been submitted for a degree or diploma at any other university.

The work presented in sections 4, 5, and appendix E is based on [51]. This work was authored by me with Rahim Leung and Kellogg Stelle and published in the Journal of High Energy Physics. Specifically the calculation of the reduction of the Salam–Sezgin, Cvetic–Gibbon–Pope model to a five dimensional field theory, its equations of motion and their simplifications, and geodesic equation were performed by Rahim Leung.

The work presented in section 6 is based on [49] and [50]. These works were authored with Alexander Harrold, Rahim Leung, and Kellogg Stelle and with Rahim Leung and Kellogg Stelle, respectively. The calculation of the Hamiltonian of the free Covertly Symmetry Broken theory was performed by Kellogg Stelle.

The work presented in appendix D was performed in collaboration with Alexander Harrold and first appeared in his thesis [67].

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1 Introduction

1.1 Quantum Gravity

Perturbative gravity in $d = 1 + 3$ dimensions is a unitary, local, and renormalizable theory, pick two.

The study of the renormalization group flow of perturbatively nonrenormalizable theories is the purview of the asymptotic safety program. Theoretically the flow of operators at low energy need not dictate their flow at arbitrary energies. Thus theories with irrelevant operators may still flow from a well-defined ultraviolet fixed point [95]. This approach relies on the putative existence of non-Gaussian fixed points, which are manifestly strongly coupled and there is no guarantee that the corrections to this theory define a unitary theory. However, in principle the asymptotic safety program allows for a circumvention of the requirement that the renormalization group flow is determined only by the flow of relevant, marginally relevant, and exactly marginal operators, which might otherwise flummox the pursuit of quantum gravity since.

Contrastingly, gravity with curvature squared corrections is known to be local, perturbatively renormalizable, but not unitary [118]. The addition of quartic (in derivatives) covariant terms stabilizes the renormalization group flow. However, the cost of this is that the theory is equivalent to one with standard (i.e. two derivative) propagators with negative kinetic signs [119]. The background Minkowski space solution to these theories may be unstable[1], and could decay to some unknown lower-energy state.

Adding infinite derivative corrections to remove singularities from gravitational theories can define a perturbatively superrenormalizable and unitary theory [90]. However, these corrections break locality as the value of an analytic function anywhere is precisely known by its infinite derivative expansion at a point. Perhaps the most well studied nonlocal field theory that includes gravity is string theory.[2]

In this work we will focus on the low energy limit of string theory, supergravity, and discuss some of its features as a putative theory of our universe.

1.2 Dimensional Reduction

Famously, one challenge in applying string theory to make physical predictions is that criticality requires we consider string theory in ten (target-space) dimensions.

A standard method is to require that six of the dimensions of the target space are compact, and usually small. That is, we consider a solution on a higher-dimensional manifold $\mathcal{M}_H$ which has, as a sub-manifold,

[1] It is known that the Minkowski solution can be perturbatively stable in ghostly theories when the ghosts are scalar degrees of freedom [28]. However, this is not known for spin two degrees of freedom.

[2] More properly, when we discuss string theory as a nonlocal field theory it might be best to say we are studying string field theory.

a maximally-symmetric world-volume $\mathcal{M}_l$ with a compact transverse space $\mathcal{M}_t$ ($\mathcal{M}_H = \mathcal{M}_l \times_W \mathcal{M}_t$)[3]. On these space-times we write our higher-dimensional fields as sums of products of functions on the world-volume and the transverse space. Under a restricted ansatz at perturbative order, the higher-dimensional equations of motion of such a theory are identical to lower-dimensional equations of motion. Furthermore, when we require that our transverse space be a compact group and assume that our higher-dimensional fields respect not only the maximal symmetry of the world-volume, but also the group symmetry, we find that our perturbative finding extends to all orders in untruncated fields.

Fundamentally, this is a resurrection of the work of Kaluza and Klein, who first studied a reduction from $\mathbb{R}^{1,4}$ and $\mathbb{R}^{1,3} \times \mathcal{S}^1$, respectively. When the transverse space is a group manifold many essential results from Fourier analysis applied group theory may be used. We expand our higher-dimensional fields into an infinite tower of lower-dimensional fields beginning with massless fields which are constant in the transverse space. The extra degrees of freedom are suppressed by a large mass-squared and controlled by the length scale of the transverse space (the radius of $\mathcal{S}^1$). Mathematically, any combination of the massless (constant) degrees of freedom cannot interact with a single heavy field. Physically, processes involving only the lightest fields at energies lower than the mass of the next heaviest particles cannot excite the heavy particles. This is an example of a so-called consistent truncation.[4]

Consistent truncations give us an exact embedding of (some of) our lower-dimensional field theories into our higher-dimensional field theories. Mathematically this is very powerful. We can use the finite number of ten-dimensional field supergravity theories, each a consistent truncation of the unique eleven-dimensional supergravity theory, combined with our knowledge of six-dimensional group manifolds to focus our study of four-dimensional supergravity theories. Similarly we can use the space of theories to predict properties of possible transverse spaces.

1.3 Localization of Solutions

Consistent truncations, however, are in a sense sparse in the space of possible dimensional reductions, as group manifolds are in a sense sparse in the space of manifolds. In this work we shall study the effective dimensional reduction of solutions which correspond to localized[5] sources in the higher-dimensional theory. We will argue that there are two possible origins for such a consistent truncation.

One possibility is to disregard all dependence on the transverse dimension, regardless of whether it is compact or not. Lower-dimensional sources in such a setting are 'spokes' in the higher dimension. Such consistent truncations, when the transverse space is noncompact, can yield lower-dimensional effective field

[3] Throughout this text $\times_W$ indicates a warped product background.
[4] Which we shall discuss further in the introductory section on effective field theories.
[5] In the case of Type I reductions these are notably not localized. This is the essential contrasting point.

theories with noncompact gauge symmetries (such as those studied in [74]). These models frequently contain arbitrarily light 'heavy' fields, meaning that there is no energy scale at which the lower-dimensional theory can approximate the full higher-dimensional theory without large corrections.

Alternatively, for some compact spaces, we can instead consider the lower-dimensional effects of a genuinely localized higher-dimensional source. In this case the solution appears lower-dimensional (that is, approximating the truncated theory) at large separation from the source and higher-dimensional nearby to the source. These solutions still contain corrections to the consistently truncated theory; however, at some small (relative to the mass of the heavy particles) energy scale, or long length scale, these corrections are dominated by the solutions to the lower-dimensional theory. Therefore no matter how sensitive to these corrections we choose to be,[6] there exists a limit in which we cannot detect them. A notable restriction to these theories, however, is that they will only effectively reproduce massless lower-dimensional solutions when the transverse space is compact.

The alternative to a consistent truncation is an inconsistent truncation. In this work we shall refer to a would-be inconsistent truncation to contrast it mathematically with the requirements of a consistent truncation, but we will study the theory without actually truncating the heavy fields. That is, we study the setting of an inconsistent truncation, not the act. In such a setting the massless (or lightest) lower-dimensional fields interact with the heavier lower-dimensional fields so that the heavier fields cannot be set to zero for any but the background (vacuum) solution. Corrections to such a theory, however, can be quantitatively similar to the case discussed in the previous paragraph. That is, there can still be an energy scale at which the full theory is well approximated by a lower-dimensional theory.

A theory can prohibit a consistent truncation for (at least) two reasons. First, because some of the lightest transverse components of the fields in the higher dimension manifestly excite the heavy fields due to their tensorial structure. Second, because the transverse dependence of the world-volume light fields is non-constant in the transverse space. As such, understanding such inconsistent truncations is essential to understanding many physically interesting transverse spaces, such as reductions on Calabi–Yau manifolds, or reductions with noncompact transverse spaces.

Further, there are cosmological reasons to look at reductions outside the context of compact transverse spaces. Some cosmological models require a de Sitter vacuum. Sometimes these vacua can be approximate, sometimes they must be exact. Such exact vacua cannot be uplifted to ten-dimensional solutions with compact transverse spaces.

Some works relax the assumption of no dependence on the transverse space to attempt to provide an alternative to compactification. Most notably, Randall and Sundrum studied such an example [114]. In

[6] Perhaps "no matter how sensitive current experiments are".

their example the space is the Poincaré patch of AdS_5 with Neumann boundary conditions at some positive radius with a reflection, the transverse space is defined by the coordinate of the Poincaré radius. This space, however, has finite transverse volume. While they cut the transverse space so that the remaining radius had infinite extent as a variable, upon $\mathbb{Z}_2$ reflecting an infinite volume of AdS has been removed. This implies that their transverse space affords a normalizable constant zero mode, similar to a consistent truncation. However, their problem also affords transverse modes that give rise to massive lower-dimensional modes with arbitrarily small mass. Indeed considering only gravitons, their space allows a consistent truncation mathematically, but not physically.

In this work we will argue that both of these properties, the presence of a normalizable transverse zero mode and the presence of a mass gap between the said zero mode and the next massive mode, are required if corrections to the low-energy limit of the theory are to be exponentially and not just polynomially suppressed. That is, for one plus three dimensions where the 'native' solution to the Laplace equation is r^{-1} as a function of world-volume radius (r), corrections to the Randall–Sundrum model are of order r^{-7} while corrections to the Kaluza–Klein model are of order $r^{-1}\exp(-r)$ (C.F. [51].

1.4 Effective Field Theories

A correspondence between a subset of solutions of two different field theories can provide an incomplete comparison between those two theories. For instance, Einstein's equations in a space with a transverse circle share some, but not all, solutions to a space with a transverse interval and Neumann–Neumann boundary conditions. A more systematic method for contrasting theories can be accomplished by studying the full dynamics of the theory at nonlinear order or at perturbative order, when the higher-dimensional equations of motion are interpreted as lower-dimensional equations of motion. In the case of Einstein's equation mentioned above one would find that these theories are equivalent after considering a (physically) consistent truncation, but differ when one includes heavy fields.

The study of the dynamics of a theory as opposed to solutions to a theory is the second primary focus of this work. Our taxonomy can positively prove that a system with a given boundary condition does not permit any consistent truncation, but it only provides weak evidence that a system permits a consistent truncation. We further argue that some forms of boundary conditions generically will permit consistent truncations (Type I reductions). However, including additional degrees of freedom beyond the graviton, or the graviton's supermultiplet in the context of a supergravity theory, require additional care.

Effective field theory is a term with an impressive panoply of definitions which all center around the theme of extracting a theory that represents low-energy predictions out of high-energy dynamics. The example of renormalization group flow, mentioned in passing above, is a primary example in quantum field theory, however it also has a precise meaning when studying purely classical theories.

Effective field theories are, in this work, the lower-dimensional theory which is observed when studying physical processes below a certain energy. In the context of a consistent truncation the heavy fields simply cannot be excited by configurations of light fields in the classical theory. In the context of an inconsistent truncation the heavy fields are excited, but are integrated out by substituting their (approximately algebraic) on-shell value into the action or the light field's equations of motion. This procedure requires inverting the heavy field's effective propagator in the lower-dimensional space. As a result the lower-dimensional effective field theory has higher-derivative corrections, where the order of the derivatives is suppressed by the powers of the mass of the heavy particles.

In the context of consistent truncations of gauge theories gauge symmetry is maintained within the untruncated degrees of freedom. That is, since, or perhaps due to the same reason that, the heavy fields cannot be excited by light fields alone if they are vanishing and do not have any inhomogenous transformation under some symmetry transformation they cannot be transformed to nonzero values by these transformations. This is not true in the setting of an inconsistent truncation. Specifically, we will find that the heavy fields, even when they are unexcited, can receive a nonvanishing contribution at homogenous order from the transformation of the light fields.

However, one feature of dimensional reductions is that, as the higher-dimensional fields are expanded into lower-dimensional fields, so are the higher-dimensional gauge parameters expanded into lower-dimensional gauge parameters. We may therefore resolve this issue of 'turning on' the heavy fields by turning them back off again by using the transformation where they transform inhomogeneously. This has knock on effects at higher orders for the light fields. This is equivalent to a perturbative redefinition of the light fields and, when we develop a method to solve this generically, we find it also applies to consistent truncations and can be used to find standard dimensional reduction ansatz such as the Kaluza–Klein ansatz.

This does not change, however, that the type of effective field theories we study in this work have spontaneously broken gauge symmetries at higher order. This kind of surreptitious symmetry breaking, we describe as covert symmetry breaking was first noted in [49], and building towards understanding why this occurs at the level of effective field theories is the focus of the second half of this work.

1.5 Our Results

To construct solutions on the higher-dimensional theory we require an extension of Fourier analysis to Sturm–Liouville theory, which is the subject of section 2. Sturm–Liouville theory is an originally physically motivated study used for modal simplification of theories in many branches of physics. For example, the time-independent Schrödinger problem, the heat equation, and (as here) solutions to separable partial differential equations, are all Sturm–Liouville problems. The central result of Sturm–Liouville theory is in defining the spectrum of the Sturm–Liouville operator and showing that the eigenvalues of the Sturm–Liouville operator form a basis for L_2 functions on the relevant space. The standard proofs of Sturm–Liouville theory apply to finite spaces where the spectrum is discrete. We, however, require an understanding of the same results in the context of semi-infinite spaces; additionally the semi-infinite case can easily be extended to the infinite case.

Such extensions of Sturm–Liouville theory are well studied in the literature; however, here we summarize how they can all be studied in the same context. Furthermore, we are specifically interested in when the spectrum contains only positive eigenvalue modes, zero modes and positive eigenvalue modes, negative modes, etc. We present a summary of the central proofs of Sturm–Liouville theory with special attention on zero modes with a proof of under which conditions the transverse problem will have negative eigenvalue modes. Specifically we restrict the space of Sturm–Liouville problems to those defined by Laplace operators, and, in this context, find the generic solutions to the zero eigenvalue problem. Next we argue how choosing a specific function to lie within the basis defines boundary conditions that define a basis. Then we show how simply studying that mode will elucidate whether the transverse spectrum will contain negative eigenvalue modes.

Following this, in section 3, we study an immediate extension of Sturm–Liouville theory, Green functions. Manipulations of Green's formula allow us to find many useful tools which we will apply to both separable partial differential equations, in our search for solutions, and to effective field theories. We give a manipulation of an augmented Green function that allows us to calculate quantities defined by sums of nonzero eigenmodes which we will use in later sections to calculate corrections to effective field theories.

We then turn our focus to said field theories directly. In section 4 we give a recount of the argument due to Bachas and Estes that allows us to apply Sturm–Liouville theory, specifically its implications for separable partial differential equations, to supergravity. We then augment that argument with a study of Kerr–Schild perturbations and quantify how corrections to our linearized limit arise in gravitational theory, and argue how they might be controlled by considering extremal (i.e. supersymmetric) objects without affecting the leading order of our conclusion. We then present a taxonomy of the solutions to separable partial differential equations, and then review how this taxonomy applies to supergravity. A summary of these findings is presented in table 1, in the introduction to section 4.

After presenting our taxonomy we study specific species of such reductions. Specifically, in section 5, we focus on perturbations about the Salam-Sezgin lift, as three of the four types of solutions exist in this background. We first justify our use of the Bachas and Estes equation in the context of traceful operators, then find solutions of Type II and Type III†, then estimate corrections to Newton's constant in the context of Type III† solutions explicitly.

Next we study inconsistent truncations in the context of scalar field theories. In section 6 we introduce the 'dimensional reduction square' and introduce the necessary requirements to integrate over the transverse space in a higher-dimensional action and find an equivalent lower-dimensional action, the leading components of which can be seen as an effective field theory. Of essential importance is the nature of a boundary term at the boundaries of the transverse dimension and the use of a Sturm–Liouville basis which is complete with regards to the relevant transverse inner product. For the sake of completeness we show how (classically) different equivalent higher-dimensional boundary terms may be found. We then calculate the corrections to the lower-dimensional field theory originating from the non-constant zero mode and impossibility of a consistent truncation, focusing on the effects on the lower-dimensional interaction terms and of integrating out the massive fields in orders of both light fields and derivatives.

In section 7 we then apply a similar scrutiny to gravitational theories with consistent trunctations, specifically a similar setting to Kaluza–Klein theory, gravity on a transverse interval with Neumann-Neumann boundary conditions. We investigate the boundary term and discuss its relationship to the Gibbons–Hawking–York boundary term at both exact and perturbative order. We next diagonalize the (perturbative) free degrees of freedom of the lower-dimensional effective field theory, then diagonalize the lower-dimensional effective transformations of these degrees of freedom. Upon doing so we discover that this procedure derives the Kaluza–Klein ansatz ab initio. We finish the section by presenting the technique for diagonalizing the lower-dimensional effective transformations which we dub the recursion equation.

After studying effective field theories in the context of inconsistent truncations (specifically due to non-constant zero modes) and effective field theories in the context of higher-dimensional gauge transformations, we combine these two issues in section 8. We discover that the gauge symmetry of the lower-dimensional field theory is spontaneously broken, but only at quartic order (in the action) in interactions. This 'covert' symmetry breaking is a generic feature of non-constant zero modes in the dimensional reduction of gauge theories. However, we focus our study on the easiest case, scalar electrodynamics in one plus three dimensions cross a transverse interval. We calculate the lower-dimensional degrees of freedom at the free level, present three possible field redefinitions for the scalar sector, calculate the corrections for interacting out the heavy fields, and show how the field redefinitions and effects of integrating out the heavy fields conspire with Green function identities to preserve 'Stueckelberged' gauge invariance.

We include a brief discussion of covert symmetry breaking in the context of self-interacting field theories in section 9. We contrast our finding of covert symmetry breaking in scalar electrodynamics, specifically we note that the some of the possible cancellation of terms which include the lower-dimensional Stueckelbergs are not possible in the context of a massless gauge field. We conclude this section with a brief discussion of how these theories can be related to well-studied Kaluza–Klein by viewing them as different points in the space of possible boundary conditions.

Finally, we summarize what open questions remain in this work in section 10. Specifically, we describe how the recursion equation may be solved for Yang–Mills theory with a restricted set of possible gauge manifolds. We describe how corrections to non-extremal black hole solutions in gravity may be calculated using a generalization of our technique for integrating out higher-dimensional fields. Lastly we discuss the physical meanings of covert symmetry breaking in terms of possible corrections to Einstein's equations, what a novel seagull coefficient might predict, and how to falsify this possibility.

2 Sturm–Liouville Theory

This is the first of two sections discussing our exceptional form of Sturm–Liouville theory. We have included it first, to treat the exceptional limits of certain facts we require that fall outside the standard conditions, and therefore outside the standard proofs, and second to combine the study of Sturm–Liouville theory on finite, semi-infinite, and infinite domains, with regularized and non-regularized behavior at the boundaries of the domain, and with special and general boundary conditions.

In this section we will focus on the Sturm–Liouville theory of a one-dimensional Laplacian. We will focus on zero modes, complete bases, spectra, and resolutions of the identity. We will not cover Green functions, overlap integrals, or other applications of Sturm–Liouville theory to separable partial differential equations, which are the focus of the following section.

One advantage of focusing on Laplacians is that the zero-mode solutions to the theory are much simplified. We will solve the zero-mode eigenvalue equation for a Laplacian exactly for one solution, and define the other (non-constant) zero-mode solution by invoking a single anti-derivative, that is, by quadrature.[7] Laplace operators are a specification of the generic Sturm–Liouville problem (see further equation (2.1.12)), but general enough for this work.

The overall goal of this section is to demonstrate, for our purposes, the equivalence of selecting a zero mode (or other lightest mode) and selecting boundary conditions, show the generic argument behind Sturm–Liouville theories, and build up all spectra and bases we will use in later sections. The organization of this section is as follows.

1. We will introduce the Sturm–Liouville problem, manipulate it into a simplified Laplacian form, define our inner product space, and clarify the relationship between self-adjointness and boundary terms.

2. We will study the how the finitude of the domain affects the spectrum of the Sturm–Liouville problem.

3. We will discuss the interplay of zero mode solutions, boundary conditions, and negative eigenvalues.

4. We will introduce the main result of Sturm–Liouville theory we require, the resolution of the identity.

5. We will study the relationship between augmented inner products, and derivative bases.

6. Finally, we will discuss explicit bases for several problems.

    - We will derive all bases with zero modes for the flat interval.

[7]Since the Laplacian defines a second-order ordinary differential equation, it (locally) annihilates two linearly independent functions, or zero modes.

- We will derive the S-wave bases for $\mathbb{R}^n$.

- We will derive all bases for AdS$_5$'s transverse dimension which contain zero-modes.

- We will derive two bases for a special type of Pöschl–Teller potential.

2.1 The Sturm–Liouville Problem

2.1.1 The Laplace Eigenvalue Equation

In this work we will use Sturm–Liouville theory to solve separable partial differential equations and to simplify higher-dimensional equations of motion and actions into lower-dimensional effective equations of motion and actions. As such we will consider Sturm–Liouville theory where our differential operator is a Laplacian, that is [100]

$$\Delta(\cdot) = \nabla^2(\cdot) = \frac{1}{\sqrt{g}}\partial_a\left(\sqrt{g}g^{ab}\partial_b(\cdot)\right) \ . \tag{2.1.1}$$

Generically speaking, almost all quantities pertaining to differential operators (and matrix operators) can be calculated from algorithms beginning with solutions to the eigenvalue equation [69], that is

$$\Delta f_\omega = -\omega^2 f_\omega \ . \tag{2.1.2}$$

Furthermore, we will only require S-wave expansions of our fields in a single extra coordinate,[8] therefore if we consider equation (2.1.1) acting on a function of only one variable t we have:

$$\Delta f(t) = A(t)f''(t) + B(t)f'(t) \ . \tag{2.1.3}$$

We want to apply two simplifications to this operator. First, we want to prove this is equivalent to a generic Sturm–Liouville form [18] (without a homogeneous function) that is we want to define $p(t)$ and $w(t)$ so that

$$A(t){\partial_t}^2(\cdot) + B(t)\partial_t(\cdot) = \frac{p(t)}{w(t)}{\partial_t}^2(\cdot) + \frac{p'(t)}{w(t)}\partial_t(\cdot) \ . \tag{2.1.4}$$

[8]Formally we would say that an S-wave is defined at a point by a coordinate patch where one of the coordinates is proper distance from that point [52] (for example, consider radius for radial or spherical coordinates [102]). However this formality is mostly unnecessary and for the spacetimes we will discuss in detail the transverse coordinate our solutions depend upon will be given explicitly.

Using the equivalence of the coefficients of the relevant powers of ${\partial_t}^n$ to we define two equations.

$$A(t) = \frac{p(t)}{w(t)} \ , \tag{2.1.5}$$

$$B(t) = \frac{p'(t)}{w(t)} \ . \tag{2.1.6}$$

We solve the quadratic term (equation (2.1.5)) for $w(t)$ in terms of $p(t)$ and $A(t)$ and find

$$w(t) = \frac{p(t)}{A(t)} \ . \tag{2.1.7}$$

We solve the linear term for $p(t)$ in terms of $B(t)$ and $w(t)$, then substitute our solution for $w(t)$ into our solution for $p(t)$ to find

$$\frac{p'(t)}{p(t)} = \frac{B(t)}{A(t)} \ . \tag{2.1.8}$$

From this we solve $p(t)$ and $w(t)$ explicitly

$$p(t) = \exp\left(\int_0^t \frac{B(s)}{A(s)} ds\right) \ , \tag{2.1.9}$$

$$w(t) = \frac{1}{A(t)} \exp\left(\int_0^t \frac{B(s)}{A(s)} ds\right) \ . \tag{2.1.10}$$

Therefore our Laplacian can be stated as an ordinary differential operator of the form

$$\Delta(\cdot) = \frac{1}{w(t)} \left(\partial_t p(t) \partial_t(\cdot)\right) \ . \tag{2.1.11}$$

Here t is our putative coordinate.[9]

We find that this agrees with the definition of Sturm–Liouville problems

$$\partial_t \left(p(t)\partial_t g_\omega(t)\right) + q(t) g_\omega(t) = -\omega^2 w(t) g_\omega(t) \ , \tag{2.1.12}$$

given $q(t) = 0$. Here $p(t)$ and $w(t)$ are nowhere vanishing.[10] However, this is still an overgeneralized problem. To simplify further we define

$$t = s(z) \ , \tag{2.1.13}$$

then

$$dt = s'(z) dz \qquad \Rightarrow \qquad \frac{1}{s'(z)} = \frac{dz}{dt} \ . \tag{2.1.14}$$

[9] More properly, t is an inverse patch of $t : \mathbb{R} \to \mathcal{D}$. However, we will simply refer to t or z as the coordinates of our space [91].
[10] We will discuss domains shortly.

We may expand the generic Sturm–Liouville problem[11]

$$\frac{1}{s'(z)w(s(z))}\partial_z\left(\frac{p(s(z))}{s'(z)}\partial_z g_\omega(s(z))\right)+\frac{q(s(z))}{w(s(z))}g_\omega(s(z))=-\omega^2 g_\omega(s(z))\ . \tag{2.1.15}$$

Given this, we identify

$$\mu(z)=w(s(z))s'(z)=\frac{p(s(z))}{s'(z)}\ . \tag{2.1.16}$$

Therefore when $q(t)=0$ and if the first-order nonlinear ordinary differential equation

$$s'(z)=\sqrt{\frac{p(s(z))}{w(s(z))}}\ , \tag{2.1.17}$$

has a solution, then the Sturm–Liouville problem may be stated as

$$\frac{1}{\mu(z)}\partial_z\left(\mu(z)\partial_z f_\omega(z)\right)=-\omega^2 f_\omega(z)\ , \tag{2.1.18}$$

where $f_\omega(z)=g_\omega(s(z))$.

That is, we have proven that we may choose coordinates so that

$$\Delta(\cdot)=\frac{1}{\mu(z)}\partial_z\left(\mu(z)\partial_z(\cdot)\right)={\partial_z}^2(\cdot)+\left(\partial_z\log\left(\mu(z)\right)\right)\partial_z(\cdot)\ . \tag{2.1.19}$$

Here $\mu(z)$ is a real differentiable function with support on all interior points of some domain $z\in\mathcal{D}$, $\mu\in\mathcal{C}^1(\mathcal{D})$. $\mu(z)$ is sometimes called the weight function; we will call it the measure. It may diverge, or vanish at $\partial\mathcal{D}$ and later we will take two derivatives of its anti-derivative as well as derivatives of $1/\mu(z)$, so it must be differentiable and nonvanishing on the interior of $\mathcal{D}$.

By the intermediate value theorem, $\mu(z)$ is either a positive or negative function; we choose it to be a positive function. $\mathcal{D}$ is some one-dimensional connected manifold with boundary, that is it is either a finite, semi-infinite, or infinite interval or a circle ($\mathcal{S}^1$).[12]

To summarize, in this work, we will study this form of the Sturm–Liouville problem (with $q=0$ and $p=w$). Note that we may freely shift or rescale our coordinates as $z\to m\,z+c$. Given this, we may always choose z between -1 and 1 or 0 and 1, when our domain is finite, between 0 and ∞ or 1 and ∞, when our domain is semi-infinite, or between $-\infty$ and ∞, when our domain is infinite. Restating this, our domain is one of $z\in(-1,\,1)\,,\ (0,\,1)\,,\ (0,\,\infty)\,,\ (1,\,\infty)\,,$ or $(-\infty,\,\infty)$. We will treat all of these cases simultaneously

[11] We include $q(t)\neq 0$ for the sake of anyone who might require such a term.

[12] $\mathcal{S}^1$ affords only a single basis, which is usually described as 'the' Fourier basis [61]. We will exclude it until we come to describe bases for the interval.

where possible and describe the consequences of solving Sturm–Liouville problems on different domains in section 2.2.

2.1.2 The Inner Product Space and Boundary Conditions

Our domain will be equipped with an inner product

$$\langle f\ ,\ g\rangle = \int_{\mathcal{D}} \mu(z) f(z) g(z) dz\ . \tag{2.1.20}$$

Since $\mathcal{D}$ is a one-dimensional connected manifold it has an upper and lower boundary.[13] We say $z \to u^-$, l^+ for these boundaries, respectively.[14] Therefore first-order differential boundary conditions on some function f are defined by two operators[15]

$$a\, f'(z) + b\, f(z)\Big|_{z\to l^+} = x\ , c\, f'(z) + d\, f(z)\Big|_{z\to u^-} = y\ . \tag{2.1.21}$$

Focusing on the lower boundary, for nonvanishing x, our lower boundary condition is named general or inhomogeneous. For vanishing x our boundary condition is named special or homogeneous.[16] Independently, for nonvanishing a and b our boundary condition is named mixed or Robin, we will use Robin. When a vanishes we name our condition Dirichlet, and when b vanishes we name our condition Neumann [65]. The case of $a = 0$, $b = 1$, $x = 1$, for instance, is a general Dirichlet boundary condition.

Some functions require a more nuanced treatment. For instance, we may only require that our function be finite at the boundary.

$$a\, f'(z) + b\, f(z)\Big|_{z\to l^+} < \infty\ . \tag{2.1.22}$$

In such a case we will say we are requiring normalizability, which will be essential in the context of functions of semi-infinite or infinite domains. When we want to specify that our conditions require equality to some finite value, we will call our conditions exact. When we do not state otherwise the reader can assume that any boundary condition is exact.

Furthermore, if $l = 0$ and $f(z) = \log(z)$, there is no first-order differential operator defined in terms of finite, constant, not all vanishing, a, b, and x for which

$$\lim_{z\to 0^+} \left(a\ (\partial_z \log(z)) + b \log(z) - x\right) \to 0\ . \tag{2.1.23}$$

[13] In the case where $\mathcal{D} = \mathcal{S}^1$ this analysis is still valid, however, the boundary terms vanish for periodic functions per se.

[14] We do not restrict that $u < \infty$ or $l > -\infty$, which is why we use the arrow notation ($\to$) throughout.

[15] We use limits throughout, $z \to l^+$, for instance, because this is a strong enough requirement for our arguments, and because $f(z)$ or $f'(z)$ may both diverge, but we will require specific behavior from their asymptotes.

[16] The boundary conditions on perturbations of fields are generically special.

Nevertheless, we will need to define boundary conditions that allow for such singular functions. To handle such cases we first regularize l, considering the behavior of our function near the boundary $z \to (l+\epsilon)^+$ with $\epsilon \in \mathbb{R}^+$. Next, we choose constants so that our boundary condition is well defined

$$a\frac{1}{\epsilon} + b\log(\epsilon) - x = 0\ . \tag{2.1.24}$$

There are obviously infinitely many choices for a, b, and x. For this example, however, let us choose special boundary conditions, $x = 0$, and conditions so that the a and b are both finite for any selection of ϵ. One such choice is

$$a = \epsilon\ \log(\epsilon)\ , \qquad b = -1\ , \qquad x = 0\ . \tag{2.1.25}$$

Alternatively, we could have stated our initial boundary conditions as

$$z\ \log(z)\, f'(z) - f(z)\Big|_{z\to 0^+} = 0\ . \tag{2.1.26}$$

This is fundamentally a shorthand, indicating we study the behavior of the regularized Sturm–Liouville problem in the limit where the parameter of the regularization vanishes. We name such a requirement a regularized boundary condition. For equation (2.1.26) specifically we have a regularized special Robin boundary condition.

2.1.3 Self-Adjointness, Boundary Terms, and Orthogonality

A operator A on an inner product space, such as the space of functions on $\mathcal{D}$, is said to be adjoint to another operator $A^\dagger$ when [18]

$$\langle f\ ,\ A(g)\rangle = \left\langle g\ ,\ A^\dagger(f)\right\rangle\ . \tag{2.1.27}$$

Using integration by parts we find

$$\begin{aligned}\langle f\ ,\ \Delta(g)\rangle &= \int_{\mathcal{D}} \mu(z)f(z)\frac{1}{\mu(z)}\partial_z\left(\mu(z)\partial_z g(z)\right)dz \\ &= \int_{\mathcal{D}} \mu(z)g(z)\frac{1}{\mu(z)}\partial_z\left(\mu(z)\partial_z f(z)\right)dz + \mu(z)\left(f(z)\partial_z g(z) - g(z)\partial_z f(z)\right)\Big|_{\partial\mathcal{D}} \\ &= \langle g\ ,\ \Delta(f)\rangle + \mu(z)\left(f(z)\,g'(z) - f'(z)\,g(z)\right)\Big|_{\partial\mathcal{D}}\ .\end{aligned} \tag{2.1.28}$$

This teaches us two things. First, the adjoint of an operator is not always defined, since our boundary terms $\mu(z)\left(f(z)\partial_z g(z) - g(z)\partial_z f(z)\right)\Big|_{\partial\mathcal{D}}$ may not necessarily vanish. Second, when our Laplacian has an adjoint, its adjoint will be itself. In this case we say the Laplacian is self-adjoint.

To ensure that the adjoint of our Laplacian is well-defined we may restrict (the functions in) the inner product space. For instance, if we consider only functions that vanish on the boundaries, that is special Dirichlet functions, then our boundary terms vanish. However, there is a more direct method for understanding the restriction of our domain. Given any function $f(z)$, can simply define the (generically regularized) special (generically Robin)[17] boundary condition

$$\left(\mu(z)f(z)\partial_z - \mu(z)\left(\partial_z f(z)\right)\right) g(z)\Big|_{\partial\mathcal{D}} = 0\,. \tag{2.1.29}$$

Since our boundary term is antisymmetric in $f \leftrightarrow g$, this operator always annihilates f. The space of functions on $\mathcal{D}$ which obey this boundary condition compose the "self-adjoint domain of Δ which contains $f(z)$," or the space of real functions g on $\mathcal{D}$ which obey

$$\langle g\,\Delta\, f\rangle = \langle f\,\Delta\, g\rangle\ . \tag{2.1.30}$$

Consider any two eigenfunctions of Δ, f_α and f_β, which obey the same special boundary conditions ($\alpha^2 \neq \beta^2$). Given this we know that they lie within self-adjoint domain, since the boundary terms from integration by parts vanish. For instance, consider the boundary term at the lower boundary, it is[18]

$$\mu\left(f_\alpha\partial_z f_\beta - f_\beta\partial_z f_\alpha\right)\Big|_{z\to l^+} = \mu\left(f_\alpha\left(-\frac{b}{a}f_\beta\right) - \left(-\frac{b}{a}f_\alpha\right)f_\beta\right)\Big|_{z\to l^+} = 0\,. \tag{2.1.31}$$

We take their inner product[19]

$$\langle f_\alpha\,,\ f_\beta\rangle = -\frac{1}{\beta^2}\langle f_\alpha\,,\ \Delta\, f_\beta\rangle = -\frac{1}{\beta^2}\langle f_\beta\,,\ \Delta\, f_\alpha\rangle = \frac{\alpha^2}{\beta^2}\langle f_\alpha\, f_\beta\rangle\ . \tag{2.1.32}$$

Additionally, we have $\frac{\alpha^2}{\beta^2} \neq 1$.[20] Therefore the inner product of two distinct eigenvectors in the same basis is proportional to itself multiplied by non-unitary constant, ergo it must vanish.

In this work we will always consider Sturm–Liouville systems where our eigenvalues are distinct, however, in the case where the eigenvalues coincide we may freely define, via the Gram-Schmidt process, new eigenfunctions $\tilde{f}_\alpha$ and $\tilde{f}_\beta$ where their inner product vanishes, and the space of functions spanned by f_α and f_β versus $\tilde{f}_\alpha$ and $\tilde{f}_\beta$ is identical.[21]

[17] We specify the behavior for generic μ and f. However, many examples, including usually the first examples given when defining an explicit basis, such as $\mu = f = 1$, will define special Neumann conditions or some other simplified case. In this text, however, we require the fully general case.

[18] If $a = 0$, then we have a special Dirichlet condition and $f_\alpha\big|_{z\to l^+}$ vanishes.

[19] Note, since eigenvalues are unique, we may always choose $\alpha = 0$, if either eigenvalue vanishes.

[20] Since α and β are unique, without loss of generality we may allow only α to vanish.

[21] The case of degenerate eigenvalues is important for $\mathcal{D} = \mathcal{S}^1$, but does not matter in the rest of the cases we discuss in this section.

Our functions may now be normalized so that either

$$\int_{\mathcal{D}} \mu f_\alpha f_\beta dz = \delta_{\alpha\beta} \ , \tag{2.1.33}$$

where $\delta_{\alpha\beta}$ is the Kroneker delta or

$$\int_{\mathcal{D}} \mu f_\alpha f_\beta dz = \delta(\alpha^2 - \beta^2) \ , \tag{2.1.34}$$

where $\delta(\alpha - \beta)$ is a Dirac delta distribution over the spectrum of Δ. The first choice of normalization is guaranteed when f_α and f_β are in $L_2(\mu, \mathcal{D})$ and the second choice will be derived directly given some restrictions on μ in section 2.2.2. The essential point is that we also know that if the inner product of any two eigenfunctions with distinct eigenvalues vanish and they obey the same boundary conditions at one boundary then they must also obey the same boundary conditions to one another at the opposite boundary.

Furthermore in section 2.2.1 we prove that our self-adjoint domain of functions, which we define by imposing our boundary conditions, only permits functions with unique eigenvalues under the Laplacian. That is we must have a sequence such as πn for $n \in \mathbb{Z}^{\geq 0}$, as opposed to $2\pi n$ for $n \in \mathbb{Z}^{\geq 0}$ where there are two eigenfunctions where $n > 0$, which is the case for $\mathcal{S}^1$. The most important immediate consequence of this is we may have at most one eigenfunction that has support on all of $\mathcal{D}$. This is since any two functions which vanish nowhere (and are therefore each positive or negative definite) must have nonzero inner product, and therefore cannot be orthogonal to one another.

2.1.4 Discrete Zeros

An immediate extension of the fact that eigenfunctions with different eigenvalues must have a vanishing inner product is that we may have at most exactly one zero mode eigenfunction that has support on all of $\mathcal{D}$. Furthermore we can show that solutions with larger eigenvalue ${\omega_0}^2 < {\omega_1}^2$ 'oscillate faster.'

Stating that a solution oscillates more or less quickly has a colloquial meaning and a precise meaning. A precise definition of this requires changing variables to Poincaré phase space and applying the comparison theorem for ODEs and is too long and technical even for this section.[22] A more complete discussion is included in [18]. However, the precise meaning implies the colloquial meaning, which states that faster oscillating solutions have strictly more zeros within $\mathcal{D}$.[23]

This colloquial definition is enough to imply if we have a (normalizable) eigenfunction with zero eigenvalue that has a zero ($f_0(z_0) = 0$) then the self-adjoint domain of our Laplacian will have exactly one negative

[22] Additionally, this section is about showing each step in a novel and efficient way given a only a minor restriction of the Sturm–Liouville problem, and we have found no such novel or efficient way.

[23] The use of 'strictly' here requires that both eigenvalues belong to solutions that are within the same self-adjoint domain, the precise meaning makes this clearer.

eigenvalue mode (that is $-\omega^2 > 0$). We will later argue that any solution with zero eigenvalue can have at most one zero. Therefore if we have a zero-mode we may have at most one negative eigenvalue mode. Similarly negative eigenvalue modes may have at most one zero, therefore we may have at most two negative eigenvalue modes.

In this work we will state all boundary conditions explicitly and will therefore not require the generic mathematical argument that the eigenfunction corresponding to the n^{th} eigenvalue has exactly n zeroes. This exact proof is shown in [18].

2.2 Domains and Spectra

Our Laplacian's eigenvalue equation, (2.1.18), is a second-order linear differential equation. Therefore it has two linearly independent solutions

$$f_\omega(z) = A_\omega \zeta_\omega(z) + B_\omega \xi_\omega(z) \ . \tag{2.2.1}$$

Applying our (exact) boundary conditions (equation (2.1.21)) we will first restrict the space of nontrivial solutions from $\mathbb{R}^2 \setminus \{(0,0)\}$, to $\mathbb{R} \setminus \{0\}$, by setting either of our boundary conditions. The remaining freedom in our space of solutions will be set by normalization, which we discuss later. Setting our second (exact) boundary condition will then restrict the set of allowed eigenvalues.[24] How this restriction occurs cannot be discussed in a domain agnostic way, that is ignoring whether $\mathcal{D}$ is finite, semi-infinite, or infinite. We will divide and conquer each case now.

2.2.1 The Spectra of Finite $\mathcal{D}$

To set our boundary conditions on a finite domain we must calculate the value of the boundary operator on both ζ_ω and ξ_ω at both boundaries independently. That is, we denote

$$\begin{aligned} (a\,\partial_z + b)\,\zeta_\omega\Big|_{z\to l^+} &= Z^l_\omega \ , \qquad & (a\,\partial_z + b)\,\xi_\omega\Big|_{z\to l^+} &= X^l_\omega \ , \\ (c\,\partial_z + d)\,\zeta_\omega\Big|_{z\to u^-} &= Z^u_\omega \ , \qquad & (c\,\partial_z + d)\,\xi_\omega\Big|_{z\to u^-} &= X^u_\omega \ , \end{aligned} \tag{2.2.2}$$

when these limits exist. We want to solve

$$(a\,\partial_z + b)\,f_\omega\Big|_{z\to l^+} = x \ , \tag{2.2.3}$$

[24] Heuristically, this is why the weaker conditions we require of eigenfunctions on infinite and semi-infinte domains allow for continua of permissible eigenvalues.

we may simplify our right hand side as

$$(a\,\partial_z + b)\, f_\omega \Big|_{z\to l^+} = A_\omega\, Z^l_\omega + B_\omega\, X^l_\omega \ . \tag{2.2.4}$$

This is solved by[25]

$$A_\omega = -B_\omega \frac{X^l_\omega}{Z^l_\omega} + \frac{x}{Z^l_\omega} \ . \tag{2.2.5}$$

Note, this implies that, if an arbitrary eigenfunction with eigenvalue ω obeys the boundary condition, then it is, up to a scale, the unique eigenfunction which obeys that boundary condition. This because we have a linear relationship between A_ω and B_ω and therefore have a unique solution. Restated, we cannot have degenerate eigenvalues.[26] If we then substitute this solution into our upper boundary condition we have

$$A_\omega Z^u_\omega + B_\omega X^u_\omega = \frac{Z^u_\omega}{Z^l_\omega} x - B_\omega \frac{Z^u_\omega}{Z^l_\omega} X^l_\omega + B_\omega X^u_\omega = y \ . \tag{2.2.6}$$

Rearranging this we find

$$\left(X^u_\omega - \frac{Z^u_\omega}{Z^l_\omega} X^l_\omega \right) B_\omega = y - \frac{Z^u_\omega}{Z^l_\omega} x \ . \tag{2.2.7}$$

When we consider special boundary conditions ($x = y = 0$) the right hand side of equation (2.2.7) vanishes and it is only solved when B_ω, and by extension A_ω, vanishes, or when

$$\frac{Z^u_\omega}{Z^l_\omega} = \frac{X^u_\omega}{X^l_\omega} \ . \tag{2.2.8}$$

The above calculation predicated that neither mode (ζ_ω or ξ_ω) obeyed the boundary condition by itself. Again, we consider special boundary conditions. When one of these constants vanish, say $Z^l_\omega = 0$, our lower boundary condition becomes

$$B_\omega X^l_\omega = 0 \ . \tag{2.2.9}$$

Considering the case of nonvanishing X^l_ω, our lower boundary condition imposes that $B_\omega = 0$. Substituting this into our upper (again, special) boundary condition we learn

$$A_\omega Z^u_\omega = 0 \ , \tag{2.2.10}$$

[25] The reader may wonder why we are using this combination rather than the Wronskian [18,103]. It is because this is different from the case of a Wronskian both in simplicity, since we only have two conditions, and generality, since we allow x, $y \neq 0$.

[26] This logic does not imply that eigenvalues of Sturm–Liouville problems which are periodic cannot have degenerate eigenvalues (in fact, they generically do).

which only has a trivial solution for $Z^u_\omega = 0$.

One of our boundary conditions sets the ratio of our solutions. The other boundary condition imposes a constraint on the value of Z^u_ω and X^u_ω which is not generically solved, and is therefore a constraint our eigenvalue. We can show that this constraint cannot have two solutions that are perturbatively close together are disallowed. That is, suppose that we may Taylor expand our solutions in their eigenvalue[27]

$$f_{\omega+\epsilon} = f_\omega + \epsilon F_\omega + \mathcal{O}(\epsilon^2) \ . \tag{2.2.11}$$

Since f_ω and $f_{\omega+\epsilon}$ obey our boundary conditions F_ω must also obey our boundary conditions. Applying our eigenvalue equation we have

$$\Delta f_\omega + \epsilon \Delta F_\omega = -\left(\omega+\epsilon\right)^2 \left(f_\omega + \epsilon F_\omega\right) = -\omega^2 f_\omega - 2\omega\epsilon f_\omega - \omega^2 \epsilon F_\omega + \mathcal{O}(\epsilon^2) \ . \tag{2.2.12}$$

We may cancel the leading terms on the left and right hand side. Further, taking the inner product of each side with f_ω we have

$$\epsilon \left\langle f_\omega \ , \ \Delta(F_\omega)\right\rangle = -2\omega\epsilon \left\langle f_\omega \ , \ f_\omega\right\rangle - \omega^2 \epsilon \left\langle f_\omega \ , \ F_\omega\right\rangle + \mathcal{O}(\epsilon^2) \ . \tag{2.2.13}$$

Since both f_ω and F_ω obey the same exact boundary conditions they lie within the same self-adjoint domain of Δ,[28] therefore

$$\epsilon \left(\left\langle F_\omega \ , \ \Delta(f_\omega)\right\rangle + \omega^2 \left\langle f_\omega \ , \ F_\omega\right\rangle\right) = -2\omega\epsilon \left\langle f_\omega \ , \ f_\omega\right\rangle + \mathcal{O}(\epsilon^2) \ . \tag{2.2.14}$$

The left hand side of this expression vanishes, which makes the entire expression a contradiction except when $\omega = 0$.[29]

Ergo either our eigenfunctions which obey the boundary condition are generically not analytic in ω or every eigenvalue is separated from all other eigenvalues by some minimum value.[30] That is, we have shown that in the case of exact boundary conditions at both boundaries, the spectrum of the Laplacian must be discrete.

[27] It is shown that these functions generically are analytic in ω in the Poincaré phase space analysis in [18].

[28] This is false when we only require normalizability, which we shall leverage to define delta distribution orthonormality.

[29] Furthermore corrections to our eigenfunctions which do not have a Taylor expansion (e.g. of the form $\frac{1}{\epsilon}$) are disallowed except at $\omega = 0$ since, as is argued in [18], our solutions must also be differentiable with respect to ω for all ω.

[30] This is not a lower bound on the separation for arbitrary ω, therefore the spectrum may have a limit point, but may not contain that limit point (except $\omega = 0$).

2.2.2 The Spectrum of Semi-Infinite $\mathcal{D}$

In this work, we will only consider μ on semi-infinite domains which obey several simplifying conditions. First, we will consider only μ that are exponential or subexponential. That is, as $z \to \infty$, we have

$$\mu(z) \sim \exp(\lambda z) \;, \tag{2.2.15}$$

$$\mu(z) \prec \exp(z) \;. \tag{2.2.16}$$

Here $r \in \mathbb{R}^+$ and we use the equivalence symbol ($\sim$) and precedes symbol ($\prec$) to imply specific limits. In the limit $z \to u^+$ we say[31]

$$f(z) \sim g(z) \qquad \Leftrightarrow \qquad \lim_{z \to u^+} \frac{f(z)}{g(z)} \to c \;, \tag{2.2.17}$$

$$f(z) \prec g(z) \qquad \Leftrightarrow \qquad \lim_{z \to u^+} \frac{f(z)}{g(z)} \to 0 \;. \tag{2.2.18}$$

Additionally, we only consider μ whose derivative is proportional to μ in the exponential case, and μ whose derivative is subdominant to μ in the subexponential case

$$\mu(z) \sim \exp(\lambda z) \qquad \Rightarrow \qquad \partial_z \mu(z) \sim \mu(z) \;, \tag{2.2.19}$$

$$\mu(z) \prec \exp(z) \qquad \Rightarrow \qquad \partial_z \mu(z) \prec \mu(z) \;. \tag{2.2.20}$$

Given such a measure we may simplify our eigenvalue equation for $z \to \infty$; we define

$$f_\omega(z) = \frac{1}{\sqrt{\mu(z)}} l_\omega(z) \;. \tag{2.2.21}$$

Substituting this into equation (2.1.18) we have

$$\frac{1}{\sqrt{\mu(z)}} \left({\partial_z}^2 + \frac{\left(\partial_z \mu(z)\right)^2 - 2{\partial_z}^2 \mu(z)}{4\mu(z)} \right) l_\omega(z) = -\frac{1}{\sqrt{\mu(z)}} \omega^2 l_\omega(z) \;. \tag{2.2.22}$$

When $\mu(z) \sim \exp(\lambda z)$ we have

$$\left({\partial_z}^2 - \frac{\lambda^2}{4} \right) l_\omega \sim -\omega^2 l_\omega(z) \;. \tag{2.2.23}$$

Alternately, when $\mu(z) \prec \exp(z)$ we find

$${\partial_z}^2 l_\omega(z) \sim -\omega^2 l_\omega(z) \;. \tag{2.2.24}$$

[31] We return to the case of an arbitrary limit because these symbols are well-defined even in such domains.

These two cases may be considered simultaneously, simply allowing $\lambda = 0$ in the subexponential case.

Disregarding zero modes ($\omega^2 = 0$), as they will be discussed in section 2.3, in either case we have, when $\omega^2 - \frac{1}{4}\lambda^2 < 0$,

$$l_\omega(z) \sim A_\omega \exp\left(-\sqrt{\frac{1}{4}\lambda^2 - \omega^2}\, z\right) + B_\omega \exp\left(\sqrt{\frac{1}{4}\lambda^2 - \omega^2}\, z\right) \,, \tag{2.2.25}$$

when $\omega^2 - \frac{1}{4}\lambda^2 = 0$,

$$l_\omega(z) \sim A_\omega + B_\omega z \,, \tag{2.2.26}$$

and, when $\omega^2 - \frac{1}{4}\lambda^2 > 0$,

$$l_\omega(z) \sim A_\omega \cos\left(\sqrt{\omega^2 - \frac{1}{4}\lambda^2}\, z\right) + B_\omega \sin\left(\sqrt{\omega^2 - \frac{1}{4}\lambda^2}\, z\right) \,. \tag{2.2.27}$$

Therefore we have exponentially growing and falling modes with subcritical eigenvalues, that is eigenvalues smaller than our critical value $\omega^2 < \frac{1}{4}\lambda^2$. We have affine modes with critical eigenvalues, where $\omega^2 = \frac{1}{4}\lambda^2$. Also we have oscillating modes with supercritical eigenvalues, that is eigenvalues larger than our critical value $\omega^2 > \frac{1}{4}\lambda^2$.

Subcritical eigenvalues follow the same prescription as the discrete spectrum of a finite domain. In our section discussing zero modes we will note that for orthonormalizablity they will require exact boundary conditions. Furthermore, we note that for a mode to have a finite $L_2(\mathcal{D})$ norm, the mode must be exponentially suppressed as $z \to \infty$, since

$$\int_0^\infty \mu {f_\omega}^2 dz \sim \int_0^\Lambda {l_\omega}^2 dz + \int_\Lambda^\infty \left(A_\omega \exp\left(-\sqrt{\frac{1}{4}\lambda^2 - \omega^2}\, z\right) + B_\omega \exp\left(\sqrt{\frac{1}{4}\lambda^2 - \omega^2}\, z\right)\right)^2 dz \,, \tag{2.2.28}$$

up to corrections for regularizing our integral with a large spacial cutoff Λ. This diverges for any $B_\omega > 0$. Therefore these modes must obey special Dirichlet boundary conditions as $z \to \infty$. Similarly we learn that non-zero eigenvalue critical modes may never be normalizable in the semi-infinite case. Therefore non-zero critical modes cannot be in our spectrum.

For supercritical modes we must impose an exact lower boundary condition and normalizability at the upper boundary. The lower boundary condition sets the ratio of A_ω and B_ω. Since any combination of sines and cosines may be simplified into a single cosine with some shift in the argument [127] we have

$$l_\omega(z) \sim C_\omega \cos\left(\sqrt{\omega^2 - \frac{1}{4}\lambda^2}\, z + \delta_\omega\right) \,. \tag{2.2.29}$$

Here δ_ω is some z independent constant, the details of which are unimportant. We consider any two such solutions and take a regularized inner product

$$\int_l^R \mu(z) f_\omega(z) f_\sigma(z) dz \, . \tag{2.2.30}$$

Inserting Δ and integrating by parts we find

$$\int_l^R \mu(z) f_\omega(z) f_\sigma(z) dz = \mu(z) \frac{f_\omega(z)\partial_z f_\sigma(z) - f_\sigma(z)\partial_z f_\omega(z)}{\omega^2 - \sigma^2}\bigg|_{z\to R^+} \, . \tag{2.2.31}$$

Furthermore we may apply $f_\omega = \frac{1}{\sqrt{\mu}} l_\omega$ to simplify

$$\int_l^R \mu(z) f_\omega(z) f_\sigma(z) dz = \frac{l_\omega(z)\partial_z l_\sigma(z) - l_\sigma(z)\partial_z l_\omega(z)}{\omega^2 - \sigma^2}\bigg|_{z\to R^+} \, . \tag{2.2.32}$$

Then we may apply our approximate forms to find

$$\int_l^R \mu(z) f_\omega(z) f_\sigma(z) dz \sim C_\omega C_\sigma \frac{-\tilde{\sigma}\cos(\tilde{\omega} z + \delta_\omega)\sin(\tilde{\sigma} z + \delta_\sigma) + \tilde{\omega}\cos(\tilde{\sigma} z + \delta_\sigma)\sin(\tilde{\omega} z + \delta_\omega)}{\omega^2 - \sigma^2}\bigg|_{z\to R^+} \, . \tag{2.2.33}$$

Here we have used the effective frequency $\tilde{\omega} = \sqrt{\omega^2 - \frac{1}{4}\lambda^2}$. Applying trigonometric identities we learn

$$\int_l^R \mu(z) f_\omega(z) f_\sigma(z) dz \sim \frac{C_\omega C_\sigma}{2}\left(\frac{\sin\left(\left(\tilde{\omega} - \tilde{\sigma}\right) R + \delta_\omega - \delta_\sigma\right)}{\omega - \sigma} + \frac{\sin\left(\left(\tilde{\omega} + \sigma\right) R + \delta_\omega + \delta_\sigma\right)}{\omega + \sigma}\right) \, . \tag{2.2.34}$$

Therefore the integral of any two of these modes oscillates, and the integral of any individual mode will never converge. However, if we assume that δ_ω is a function with a Taylor expansion $\delta_{\omega+\epsilon} = \delta_0 + \delta_1 \omega + \ldots$, then we may expand $\sigma = \omega + \epsilon$ in the small ϵ limit as[32]

$$\frac{C_\omega C_\sigma}{2}\frac{\sin\left(\left(\tilde{\omega} - \tilde{\sigma}\right) R + \delta_\omega - \delta_\sigma\right)}{\omega - \sigma} = \frac{C_\omega C_\sigma}{2}\frac{\sin\left(\left(\frac{R}{\sqrt{\omega^2 - \frac{1}{4}\lambda^2}} + \delta_1\right)\epsilon + \mathcal{O}\left(\epsilon^2\right)\right)}{\epsilon} \, . \tag{2.2.35}$$

We note this is proportional to a sine representation of the Dirac delta distribution [97, 121]

$$\delta(x) = \lim_{R\to\infty} \frac{\sin(Rx)}{\pi x} \, . \tag{2.2.36}$$

Therefore the inner product of any two supercritical modes which obey the same boundary condition is proportional to a delta distribution. We will derive precisely how they can be normalized in our explicit

[32]This is required since our functions are also differentiable in ω.

cases.

In summary we have found that the spectrum of our Laplacian on a semi-infinite domain may have subcritical eigenvalue bound states, analogous to the spectrum of a finite domain, and will have supercritical eigenvalue delta function orthonormalizable bound states. We stress that this is shown generically, and furthermore, critical states can not be part of the spectrum, with the notable exception of zero modes, which we have not yet treated.

2.2.3 The Spectrum of Infinite $\mathcal{D}$

Fortunately, for the sake of simplicity, the case of infinite $\mathcal{D}$ can largely be treated as an extension of the case of semi-infinite $\mathcal{D}$. We will consider the same restrictions on μ as $z \to \pm\infty$ as for $z \to \infty$ for semi-infinite domain. That is μ may not grow faster than an exponential in either direction.

Next we separate our problem at $z = 0$ considering either functions that vanish there, or functions whose derivatives vanish there.[33] Consider some arbitrary eigenvalue ω; our functions are

$$\Delta\zeta_\omega(z) = -\omega^2\zeta_\omega(z) \ , \qquad \Delta\xi_\omega(z) = -\omega^2\xi_\omega(z) \ , \tag{2.2.37}$$

$$\partial_z\zeta_\omega(z)\bigg|_{z=0} = 0 \ , \qquad \xi^\omega_z(z)\bigg|_{z=0} = 0 \ . \tag{2.2.38}$$

A mode is now only part of our discrete spectrum when some linear combination of ζ_ω and ξ_ω is normalizable on the entire domain. Similarly, a mode is now part of our continuum only when either it has finite norm on one half of the domain and is oscillatory on the other half, or is oscillatory on both halves of the domain. Our asymptotic analysis of our modes for large $|z|$, equation (2.2.29), is still valid. However, the asymptotic frequencies for small $z \ll 0$ and large $z \gg 0$, may differ.

The overlap integral of two modes with arbitrary eigenvalue will therefore be made of a sum of two delta distributions, one from the $z > 0$ and $z < 0$ domain each. However, the sum of two delta distributions can be simplified to a single delta distribution. That is, suppose we have two functions $A_\sigma(\omega)$ and $B_\sigma(\omega)$ which have support everywhere except $\omega = \sigma$, then

$$\int_0^\infty \left(\delta\left(A_\sigma(\omega)\right) + \delta\left(B_\sigma(\omega)\right)\right) d\omega = \int_{\mathcal{I}_1} \frac{\delta(x)}{|A'_\sigma(\omega)|} dx + \int_{\mathcal{I}_2} \frac{\delta(y)}{|B'_\sigma(\omega)|} dy \ . \tag{2.2.39}$$

[33] In the case that μ is an even function therefore $f_\omega(-z)$ is also a solution, this is simply the statement that we may take the even and odd parts of any solution and that will define a solution. When μ is not even this is still possible.

Here we have substituted $x = A_\sigma(\omega)$ and $y = B_\sigma(\omega)$. The domains of our new integrals ($\mathcal{I}_1$ and $\mathcal{I}_2$) contain $x = 0$ and $y = 0$, respectively. Given this, we have

$$\int_0^\infty \left(\delta\left(A_\sigma(\omega)\right) + \delta\left(B_\sigma(\omega)\right)\right) d\omega = \frac{1}{|A'_\sigma(\sigma)|} + \frac{1}{|B'_\sigma(\sigma)|} \; . \tag{2.2.40}$$

Similarly, if we integrate over any domain that does not include $\omega = \sigma$, the integral of the sum of our distributions vanish. Therefore we may state

$$\delta\left(A_\sigma(\omega)\right) + \delta\left(B_\sigma(\omega)\right) = \left(\frac{1}{|A'_\sigma(\sigma)|} + \frac{1}{|B'_\sigma(\sigma)|}\right) \delta(\omega - \sigma) \; . \tag{2.2.41}$$

2.3 Zero-Modes and Negative Eigenvalue-Modes

Let us find the solutions to the zero eigenvalue equation. We have

$$\frac{1}{\mu}\left(\partial_z \mu\, \partial_z f_0(z)\right) = 0 \; . \tag{2.3.1}$$

We may factorize the Laplacian into 'inside' and 'outside' parts, that is $\frac{1}{\mu}\partial_z$ and $\mu\,\partial_z$. The 'outside' operator has a simple kernel

$$\frac{1}{\mu}\partial_z g(z) = 0 \qquad \Rightarrow \qquad g(z) = c \; . \tag{2.3.2}$$

Since Δ annihilates $f_0(z)$, we have that the 'inside' operator action on $f_0(z)$ must lie within the kernel of the 'outside' operator. That is

$$\mu\,\partial_z f_0(z) = c \; . \tag{2.3.3}$$

Here $c \in \mathbb{R}$ is undetermined. Since $\mu\,\partial_z$ is a linear operator, the scale of c is irrelevant. Therefore, there are only two cases we need to consider; $c = 0$, and $c = 1$. In the former case we have that f_0 lies within the kernel of the 'inside' operator; that f_0 is constant. In the later case we may use quadrature to find ξ_0. We call these solutions

$$\zeta_0(x) = 1 \; , \qquad \xi_0(z) = \partial_z{}^{-1}\frac{1}{\mu(z)} \; . \tag{2.3.4}$$

Generically inverse derivatives are only defined up to a constant. That is, we must be clearer about definition of ξ_0. First, consider the case where μ does not vanish at the lower boundary, that is $\lim_{z\to l^+}\mu(z) > 0$. In this case we may explicitly define our inverse derivative as

$$\xi_0(z) = \int_l^z \frac{1}{\mu(s)} ds \; . \tag{2.3.5}$$

This vanishes as $z \to l^+$. Additionally, since μ is a positive function, equation (2.3.5) defines a function which is positive and monotonically increasing everywhere within $\mathcal{D}$. Similarly, if μ defines a function that is nonvanishing as $z \to u^-$, then, we may define a negative but monotonically increasing ξ_0. In these two cases ξ_0 will vanish at the one of the boundaries by construction.

When μ vanishes on the boundary faster than $\mu \sim \rho$, ξ_0 will diverge at that boundary. Specifically ξ_0 will diverge to negative infinity when μ vanishes at the lower boundary and to positive infinity when μ diverges at the lower boundary. For example, consider $\mu(z) = z^2$ for $z \in (0, \infty)$, then, since z^2 vanishes at the lower boundary, but diverges at the upper boundary, we should choose

$$\xi_0(z) = -\int_z^\infty \frac{1}{s^2} ds = -\frac{1}{z} \ . \tag{2.3.6}$$

If μ vanishes at both boundaries, $\xi_0(z)$ may diverge to negative infinity at the lower boundary and positive infinity at the upper boundary, and therefore must vanish at some interior point. In this case we can choose any internal point, say, $s = 0$ in the case of a finite or infinite interval and $s = 1$ in the case of a semi-infinite interval and uniquely define ξ_0. Note, no possible measure can create a zero mode that vanishes on both boundaries, since f_0 is monotonically increasing.

With these solutions explicitly defined we write the most general f_0 as

$$f_0(z) = a + b\, \xi_0(z) \ . \tag{2.3.7}$$

With this we define the generically regularized special boundary condition

$$\mu(z) f_0(z) g'(z) - b g(z) \Big|_{z \to l^+,\, u^-} = 0 \ . \tag{2.3.8}$$

The eigenfunctions in the same basis as $\zeta_0(z)$ and $\xi_0(z)$ we call $\zeta_\omega(z)$ and $\xi_\omega(z)$, respectively. We note, since selecting one function in the self-adjoint domain of Δ is equivalent to selecting boundary conditions, we now have a putative one parameter family of boundary conditions that contain a zero mode, given by the ratio of $\frac{a}{b}$.

However, normalizability is not universally satisfied by these states. Essentially, the constant zero mode may only be normalizable when our space has finite volume. That is we may choose a so that

$$\int_{\mathcal{D}} \mu(z) \left(a \zeta_0(z)\right)^2 dz = 1 \ , \tag{2.3.9}$$

only when the integral of $\mu(z)$ over $\mathcal{D}$ is finite, or

$$\int_{\mathcal{D}} \mu(z) dz < \infty \ . \tag{2.3.10}$$

Therefore we must consider the cases of finite and infinite volume separately. Similarly, $\xi_0(z)$ may have finite or infinite norm. There are, therefore, four possibilities:

- ζ_0 and ξ_0 normalizable,
- ζ_0 normalizable and ξ_0 nonnormalizable,
- ζ_0 nonnormalizable and ξ_0 normalizable,
- ζ_0 and ξ_0 nonnormalizable.

All four cases can be seen with different selections of subdomains of a transverse AdS $_N$ (sections 2.6.3–2.6.6).

We pause momentarily to investigate the case of two normalizable zero modes. Even when both ζ_0 and ξ_0 are normalizable some selections of the ratio $\frac{a}{b}$ will define a zero-mode that vanishes somewhere on $\mathcal{D}$, therefore define spectra that contain negative eigenvalue modes, as explained in section 2.1.4. By construction ξ_0 vanishes on one boundary and is either positive or negative. For illustration, consider the case where it vanishes at the lower boundary and is positive. In this case, if $\inf(\xi_0) > -\frac{a}{b} > 0$, then we have z_0 where $f_0(z_0) = 0$, given by

$$\xi_0(z_0) = -\frac{b}{a} \ . \tag{2.3.11}$$

However, outside this interval, $\frac{a}{b} \in (inf(\xi_0) \, , \, 0)$, there are two interesting critical points, which are $\frac{a}{b} \to 0^+$ and $\frac{a}{b} \to \infty$. In these two limits we have a zero mode which is constant, and a zero mode which vanishes at the lower boundary, respectively. All choices of ratio except $\frac{a}{b} \to \infty$, correspond to non-constant zero modes. However, for all examples where we have calculated, the 'pure ξ_0' selection, $\frac{a}{b} \to 0^+$, has the maximum gap between the zero mode and the next eigenvalue.[34] We shall show this explicitly for the flat transverse interval in section 2.6.1.

2.4 Resolving the Identity and Orthonormalization

The primary reason we study Sturm–Liouville theory is the 'resolution of the identity.' That is, if we have a complete set of eigenfunctions $\{f_\omega\}$ obeying some boundary condition, then these modes form a basis for

[34]Given a spectrum that does not contain any negative eigenvalue modes.

$L_2(\mu,\,\mathcal{D})$ functions obeying that boundary condition. Or, for any f in $L_2(\mu,\,\mathcal{D})$ we have n_ω so that

$$f(z) = n_\omega f_\omega(z)\ . \tag{2.4.1}$$

Here we sum over the repeated 'index' ω. Since $f(z)$ has finite norm, and, as we have already argued, our f_ω are orthonormalizable, for a complete orthonormalized set $\{f_\omega\}$

$$\frac{f_\omega(z)f_\omega(s)}{\langle f_\omega\ ,\ f_\omega\rangle} = \frac{\delta(z-s)}{\mu(z)}\ . \tag{2.4.2}$$

Restated, the sum over our orthonormalized eigenmodes form the identity with respect to our inner product $\langle\cdot\,\cdot\rangle$. However, as stated before, normalization has different meanings in the context of discrete modes and continuum modes. Zero modes and discrete modes must have finite norm to contribute to this sum. Continuum modes must be normalized so that the delta distribution over their eigenvalues is orthonormalized with respect to ω, not the effective frequency $\tilde{\omega} = \sqrt{\omega^2 - \frac{1}{4}\lambda^2}$.

Showing that a complete set of eigenvalues forms a complete basis is a logical necessity for our work. However, again, we have found no more efficient method of showing this than an appeal to Poincaré coordinates, therefore we refer the reader to [18] for further inspection.

2.5 Augmented Inner Products and Augmented Derivative Bases

We will also need to study a very closely related inner product space to $L_2(\mu,\,\mathcal{D})$. Specifically, if we consider any special boundary conditions defined as in equation (2.1.21), we have an augmented inner product

$$(f\ ,\ g) = \int_{\mathcal{D}} \mu(z)f(z)g(z)dz + \frac{c}{d}\mu(z)f(z)g(z)\Big|_{z\to u^+} - \frac{a}{b}\mu(z)f(z)g(z)\Big|_{z\to l^-}\ . \tag{2.5.1}$$

Given any two basis elements f_ω and f_σ which obey our boundary condition we have

$$\begin{aligned}(f'_\omega\ ,\ f'_\sigma) &= \int_{\mathcal{D}} \mu(z)f'_\omega(z)f'_\sigma(z)dz + \frac{c}{d}\mu(z)f'_\omega(z)f'_\sigma(z)\Big|_{z\to u^+} - \frac{a}{b}\mu(z)f'_\omega(z)f'_\sigma(z)\Big|_{z\to l^-} \\ &= -\int_{\mathcal{D}} \mu(z)f_\omega(z)\Delta f_\sigma(z)dz + \mu(z)\left(f_\omega(z) + \frac{c}{d}\partial_z f_\omega(z)\right)f'_\sigma(z)\Big|_{z\to u^+} - \mu(z)\left(f_\omega(z) + \frac{a}{b}\partial_z f_\omega(z)\right)f'_\sigma(z)\Big|_{z\to l^-} \\ &= -\langle f_\omega\ ,\ \Delta f_\sigma\rangle = \sigma^2\langle f_\omega\ ,\ f_\sigma\rangle\ .\end{aligned} \tag{2.5.2}$$

Therefore our derivative basis inherits orthogonality from the $L_2(\mu,\,\mathcal{D})$ inner product under our augmented inner product. This has a notable sub-case, when we have a zero mode.

When we have a non-constant zero mode in the basis defined by our boundary conditions the augmented inner product of f_0' with itself vanishes

$$(f_0'\,,\ f_0') = -\langle f_0\,,\ \Delta f_0\rangle = 0\ . \tag{2.5.3}$$

In this case our augmented inner product may be simplified to read

$$(f\,,\ g) = \int_{\mathcal{D}} \mu(z)f(z)g(z)dz + b\left(\mu(z)f_0(z)\right)f(z)g(z)\Big|_{z\to l^-}^{z\to u^+}\ , \tag{2.5.4}$$

where $f_0'(z) = \frac{b}{\mu(z)}$. We will also want to find an additional function, which we name ζ. (Note, it is without any subscripts or superscripts.) ζ obeys

$$\frac{1}{\mu(z)}\partial_z\left(\mu(z)\zeta(z)\right) = f_0(z)\ . \tag{2.5.5}$$

This mode is orthogonal under our augmented product to the derivative of any non zero mode, since

$$(f_{\varpi}'\,,\ \zeta) = -\left\langle f_{\varpi}\,,\ \frac{1}{\mu}\partial_z\left(\mu\zeta\right)\right\rangle = -\langle f_{\varpi}\,,\ f_0\rangle = 0\ . \tag{2.5.6}$$

Furthermore, since the solution to the homogeneous part of (2.5.5) is $\frac{1}{\mu}$, given an arbitrary solution $\overline{\zeta}$ we may define $\zeta = \overline{\zeta} + \frac{A}{\mu}$ which still solves (2.5.5). In the case where each of these quantities is finite we have

$$(\zeta\,,\ \zeta) = \int_{\mathcal{D}}\mu(z)\overline{\zeta}(z)^2dz + 2A\int_{\mathcal{D}}\overline{\zeta}(z) + A^2\int_{\mathcal{D}}\frac{1}{\mu(z)}dz + b\left(\mu(z)f_0(z)\overline{\zeta}(z)^2 + 2Af_0\overline{\zeta}(z) + A^2\frac{f_0(z)}{\mu(z)}\right)\Bigg|_{z\to l^+}^{z\to u^-} = 0\ . \tag{2.5.7}$$

Requiring this vanishes generically defines a quadratic equation for A. Therefore we may choose ζ so that its augmented inner product with itself vanishes. Given this we have

$$(f_0\,,\ f_0) = 0\ , \qquad (\zeta\,,\ \zeta) = 0 \qquad (\zeta\,,\ f_0) = 2Z\ . \tag{2.5.8}$$

Here our newly defined constant generically does not vanish; $2Z \neq 0$. Applying the Gram–Schmidt process to f_0 and ζ is tantamount to diagonalizing the matrix

$$M = \begin{pmatrix} 0 & 2Z \\ 2Z & 0 \end{pmatrix}\ . \tag{2.5.9}$$

This will always generate a negative norm state.[35]

The relationship between these two bases is best analogized with the relationship between lightcone coordinates and Minkowski coordinates in a Minkowski space time [133]. In this work we will always select f_0 and ζ coordinates when they are available to us.

We select our orthogonalized basis as

$$\alpha(z) = A_1\xi_0'(z) + A_2\zeta(z) \ , \qquad \beta(z) = B_1\xi_0'(z) + B_2\zeta(z) \ . \tag{2.5.10}$$

Here we would like

$$(\alpha,\ \alpha) = -1 \ , \qquad (\beta,\ \beta) = 1 \ , \qquad (\alpha,\ \beta) = 0 \ . \tag{2.5.11}$$

The equivalent of equation (2.4.2) is

$$-\alpha(z)\alpha(s) + \beta(z)\beta(s) - \xi_{\overline{\omega}}'(z)\xi_{\overline{\omega}}'(s) = \frac{\hat{\delta}(z,s)}{\mu(z)} \ . \tag{2.5.12}$$

Here $\overline{\omega} > 0$ is all positive eigenvalues and we have invoked a modified delta distribution which obeys

$$\left(\frac{\hat{\delta}}{\mu},\ f\right) = f \ , \tag{2.5.13}$$

as opposed to the standard Dirac delta distribution which obeys

$$\left\langle\frac{\delta}{\mu},\ f\right\rangle = f \ . \tag{2.5.14}$$

2.6 Explicit Bases for Common Laplacians

Let us apply these techniques to several examples of physical interest.

To reiterate how we accomplish this, recall that, in section 2.1.3, we argued that selecting a single mode required to be within a basis is equivalent to selecting boundary conditions. Similarly in section 2.3 we give a methodology for finding zero modes. Therefore if we find the space of zero modes and specify that we want to study the problem containing that zero mode, that is sufficient information to find the spectrum of the problem. Our algorithm for studying these spaces is to find the generic zero mode solutions and find their $L_2(\mu, \mathcal{D})$ norm (specifically when it is finite). The spaces at question may either have one parameter family of normalizable zero modes, a unique normalizable zero mode, or no zero modes.

[35]The sign of Z is irrelevant. What is actually important in this case is not that the right hand side of equation (2.5.5) be f_0, but that it is proportional to f_0. Therefore we rescale the right hand side of equation (2.5.5) so that $Z = 1$.

When we have a one parameter family of zero modes, some of those zero modes will have a zero on the interior of $\mathcal{D}$. In this case we will compare the set or parameters (boundary conditions) that allow for such a zero mode and compare these parameters with the eigenvalue equation found by applying equation (2.2.8). We will confirm that our space contains a single negative eigenvalue mode precisely when the zero mode has a zero on the interior of $\mathcal{D}$. We will then study the value of the first positive eigenvalue and argue that it is maximized for the boundary conditions that select a 'pure ξ_0' zero mode. On spaces that do not contain any zero mode we will simply give the spectrum for a Neumann basis, for illustration.

Specifically, in this section we will study:

1. The interval $\mathcal{D} = (-1, 1)$ with $\mu = 1$, which has a one-parameter family of zero modes.

2. S-waves on $\mathbb{R}^n$ which have no zero modes.

3. AdS$_N$ where the only dependence is on the Poincaré basis, which has the same Laplacian as[36] $\mathbb{R}^n$.

   - We will study the whole of AdS$_N$ which has no zero modes.

   - We will study an interval of AdS$_N$ which[37] has a one parameter family of zero modes.

   - We will study the 'upper half' of AdS$_N$ which has a constant zero mode.

   - We will study the 'lower half' of AdS$_N$ which has a non-constant zero mode.

4. We will study the Pöschl–Teller potential which has a non-constant zero mode.

2.6.1 All Bases with Zero Modes for the Flat Interval

Let's begin with the simplest case. Take the measure $\mu(z) = 1$, with Laplacian $\Delta = {\partial_z}^2$, and the domain $\mathcal{D} = (-1, 1)$. The generic zero mode is

$$f_0(z) = a + b\,(z+1)\ . \tag{2.6.1}$$

Here we have used the definition $\xi_0(z) = \int_{-1}^{z} 1\, ds$. We normalize this mode by requiring

$$2a^2 + 4ab + \frac{8}{3}b^2 = 1\ . \tag{2.6.2}$$

Our zero mode is zero at precisely the endpoint when

$$f_0(-1) = a = 0\ , \qquad f_0(1) = a + 2b = 0\ . \tag{2.6.3}$$

[36]with particular negative values of n

[37]All domains allow for a subdomain which have one parameter family of zero modes.

Taking our first condition we find $a = 0$ and $b = \pm\sqrt{3/8}$. Taking our second condition we find $a = \sqrt{3/2}$ and $b = -\sqrt{3/8}$ or $a = -\sqrt{3/2}$ and $b = \sqrt{3/8}$. If we imagine traveling along our configuration space from $a = 0$ and $b = \sqrt{3/8}$ in the direction of increasing a we have a nonvanishing zero mode, and therefore zero is the smallest eigenvalue in our spectrum. Our starting point, when $a = 0$ and $b = \sqrt{3/8}$ is 'pure ξ_0'. When $b = 0$ and $a = \sqrt{1/2}$ our solution is 'pure ζ_0', which is the symmetric solution (that is, symmetric under exchange of $z \to -z$). Beyond the 'pure ζ_0' point, all zero modes are just the reflection of a zero mode we have already seen during our travel from $a = 0$ to $b = 0$.

Similarly, if we travel from $a = 0$ to $b = \sqrt{3/8}$ in the direction of decreasing a, we find vanishing zero modes. When $a = -b = \sqrt{3/2}$ we find that $f_0(z)$ is antisymmetric, $f_0(z) \propto z$. Therefore to study the entire space we only need to consider possibilities between $a = -b$ and $b = 0$.

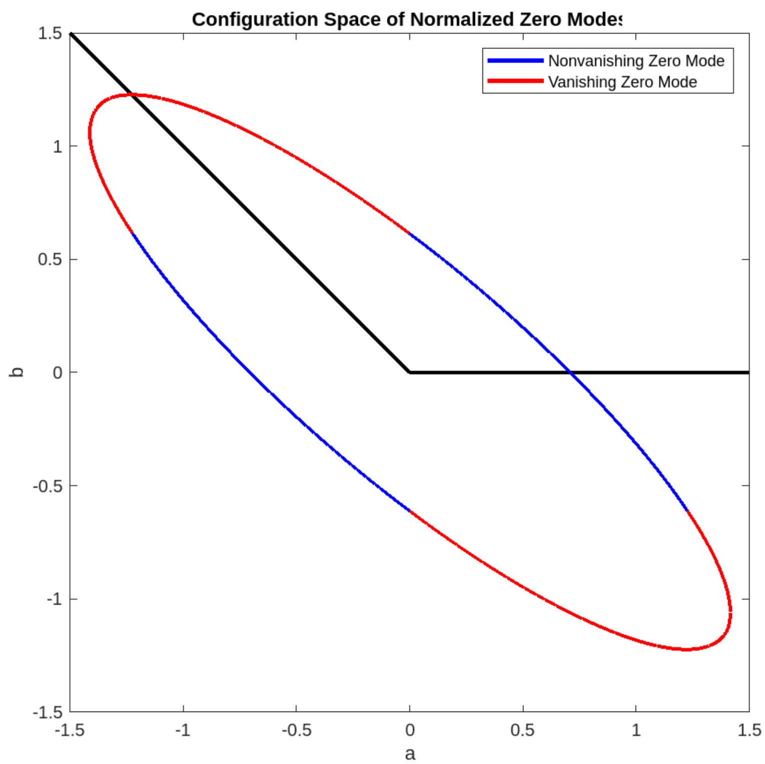

Figure 1: The configuration space of normalized zero modes on the interval.

In figure (1) we have a plotted on the x-axis and b on the y-axis and highlighted the values for which we have a nonvanishing zero mode (blue), where it is the lowest eigenvalue, and where we have a vanishing zero mode (red) where the spectrum contains a negative eigenvalue. The wedge of configuration space between $a = -b$ and $a = 0$ represents all states, up to exchange of $z \to -z$ and, independently $f_0(z) \to -f_0(z)$, which define problems with identical spectra.

The generic negative eigenvalue modes are

$$f_\omega(z) = A_\omega \cosh(\omega z) + B_\omega \sinh(\omega z) \ , \tag{2.6.4}$$

normalized by requiring

$$A_\omega{}^2 - B_\omega{}^2 + \frac{\cosh(\omega)\sinh(\omega)}{\omega}\left(A_\omega{}^2 + B_\omega{}^2\right) = 1 \ , \tag{2.6.5}$$

and the generic positive eigenvalue modes are

$$f_\omega(z) = A_\omega \cos(\omega z) + B_\omega \sin(\omega z) \ , \tag{2.6.6}$$

normalized by requiring

$$A_\omega{}^2 + B_\omega{}^2 + \frac{\cos(\omega)\sin(\omega)}{\omega}\left(A_\omega{}^2 - B_\omega{}^2\right) = 1 \ . \tag{2.6.7}$$

The boundary conditions that allow for a zero mode are

$$\left(\frac{f_0(-1)}{f_0'(-1)}\partial_z - 1\right) f(z)\bigg|_{z=-1} = 0 \ , \qquad \left(\frac{f_0(1)}{f_0'(1)}\partial_z - 1\right) f(z)\bigg|_{z=1} = 0 \ , \tag{2.6.8}$$

or, in terms of a and b

$$\left(\frac{a}{b}\partial_z - 1\right) f(z)\bigg|_{z=-1} = 0 \ , \qquad \left(\frac{a+2b}{b}\partial_z - 1\right) f(z)\bigg|_{z=1} = 0 \ . \tag{2.6.9}$$

This condition breaks down when $b = 0$, or for the symmetric zero mode. This is consistent with Neumann-Neumann boundary conditions and we may treat this case as a limit of the general case.

Applying our lower boundary condition to our negative and positive eigenvalue modes we learn

$$\frac{a}{b}\left(A_\omega \omega \sinh(-\omega) + B_\omega \omega \cosh(-\omega)\right) - \left(A_\omega \cosh(-\omega) + B_\omega \sinh(-\omega)\right) = 0 \ , \tag{2.6.10}$$

$$\frac{a}{b}\left(-A_\omega \omega \sin(-\omega) + B_\omega \omega \cos(-\omega)\right) - \left(A_\omega \cos(-\omega) + B_\omega \sin(-\omega)\right) = 0 \ , \tag{2.6.11}$$

respectively. We may simplify these conditions as

$$\frac{A_\omega}{B_\omega} = \frac{a\omega\cosh(\omega) + b\sinh(\omega)}{a\omega\sinh(\omega) + b\cosh(\omega)} \ , \tag{2.6.12}$$

$$\frac{A_\omega}{B_\omega} = -\frac{a\omega\cos(\omega) + b\sin(\omega)}{a\omega\sin(\omega) - b\cos(\omega)} \ . \tag{2.6.13}$$

The same manipulation at our upper boundary gives us

$$\frac{A_\omega}{B_\omega} = -\frac{(a+2b)\omega\cosh(\omega) - b\sinh(\omega)}{(a+2b)\omega\sinh(\omega) - b\cosh(\omega)} , \tag{2.6.14}$$

$$\frac{A_\omega}{B_\omega} = \frac{(a+2b)\omega\cos(\omega) - b\sin(\omega)}{(a+2b)\omega\sin(\omega) + b\cos(\omega)} . \tag{2.6.15}$$

Combining these we have the condition on our eigenvalues

$$\frac{a\omega\cosh(\omega) + b\sinh(\omega)}{a\omega\sinh(\omega) + b\cosh(\omega)} = -\frac{(a+2b)\omega\cosh(\omega) - b\sinh(\omega)}{(a+2b)\omega\sinh(\omega) - b\cosh(\omega)} , \tag{2.6.16}$$

$$\frac{a\omega\cos(\omega) + b\sin(\omega)}{b\cos(\omega) - a\omega\sin(\omega)} = \frac{(a+2b)\omega\cos(\omega) - b\sin(\omega)}{(a+2b)\omega\sin(\omega) + b\cos(\omega)} . \tag{2.6.17}$$

The condition on our negative eigenvalues is defined by a hyperbolic function and on our positive eigenvalues by an oscillating function. We find it particularly aesthetic when we set our zero mode normalization condition (equation (2.6.2)) and we study our positive eigenvalue equation as the condition $f_a(x) = 0$ (this is illustrated in figure (3))

$$f_a(x) = 6(-1 + a(-a + \sqrt{6-3a^2}))\omega\cos(2\omega) + (3 + a(-a + \sqrt{6-3a^2})(-3+4\omega^2))\sin(2\omega) . \tag{2.6.18}$$

Our negative eigenvalue equation may have one or no solutions, and our positive eigenvalue equation will always have infinitely many solutions [57]. Of particular note are the critical points where our negative eigenvalue condition becomes, for $a = -b$, $b = 0$, and $a = 0$ respectively

$$(1+\omega^2)\tanh(2\omega) = 2\omega . \tag{2.6.19}$$

$$\omega^2\sinh(2\omega) = 0 , \tag{2.6.20}$$

$$\tanh(2\omega) = 2\omega . \tag{2.6.21}$$

The former of these equations has solution $\omega \cong 1.9968$ and the latter two have no positive solutions. For $a = -b$, $b = 0$, and $a = 0$ our positive eigenvalue condition becomes

$$(1-\omega^2)\tan(2\omega) = 2\omega . \tag{2.6.22}$$

$$\omega^2\sin(2\omega) = 0 , \tag{2.6.23}$$

$$\tan(2\omega) = 2\omega \; . \tag{2.6.24}$$

In the case of pure ζ_0 our solutions are the roots of the sine function; $\omega = \pi n$. In the case of pure ξ_0 we cannot generically give a closed solution to this transcendental equation [23].

We can numerically solve equation (2.6.16) in the region of our configuration space where we have a vanishing zero mode (the red region in figure (1)).

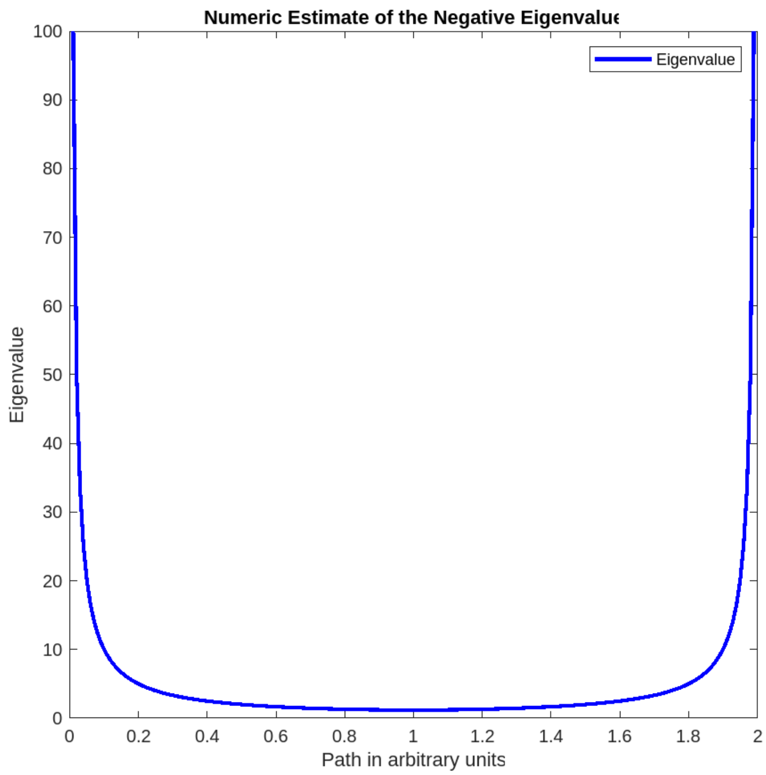

Figure 2: A numeric estimate of the negative eigenvalue.

In figure (2) we have a plotted our numeric estimate of the eigenvalue along a path from $a = 0$ and $b = 1$ to $a = -\sqrt{3/2}$ and $b = \sqrt{3/8}$. It achieves a minimum at $a = -b$ which is at 1 in the figure. We estimate this to have a value of $\omega \cong 1.9968$. An important note is that the negative eigenvalue becomes arbitrarily large as we approach a configuration with a nonvanishing zero mode.

Additionally, the maximum gap between the eigenvalue of our zero mode and the next positive mode is achieved when $a = -b$. The maximum gap for modes with no negative eigenvalue modes is achieved by 'pure ξ_0', when our zero mode vanishes on one boundary.

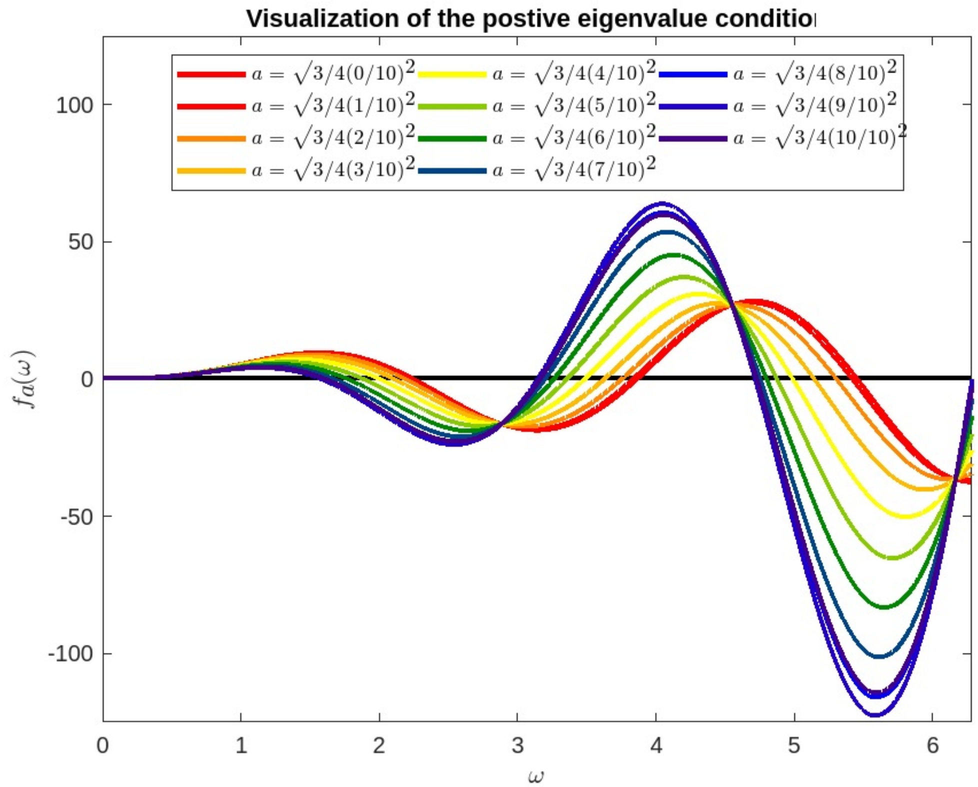

Figure 3: The spectrum of the Laplacian on a flat interval for different boundary conditions.

2.6.2 S-Wave Bases for $\mathbb{R}^n$

The purpose of this section is mainly to contrast the case of two normalizable zero modes against the case of no normalizable zero modes, to demonstrate precisely how these modes may be normalized, and to establish explicit bases of spaces with only positive eigenvalues so that we may exhibit how to calculate a Green function using integrals over special functions in the next section (Section 3). To wit, this section will be much briefer in analysis, and focus entirely on the normalization of these modes.

The radial metric for $\mathbb{R}^n$ is

$$ds_n{}^2 = dz^2 + z^2 d\,\Omega_{n-1}{}^2\,, \tag{2.6.25}$$

where $d\Omega_{n-1}{}^2$ is the metric for the $(n-1)$-dimensional round sphere $\mathcal{S}^{n-1}$. The associated measure (after integrating over the angular coordinates) is

$$V_{n-1}\mu(z) = \int_{\mathcal{S}^{n-1}} \sqrt{detg}\; d\Omega_{n-1} = V_{n-1}\, z^{n-1} = V_\alpha\, z^\alpha\,. \tag{2.6.26}$$

Here V_{n-1} is the surface area of the unit $\mathcal{S}^{n-2}$ sphere [20]. Our domain is semi-infinite, $\mathcal{D} = (0, \infty)$, and the associated (angle independent) Laplacian is

$$\Delta(\cdot) = {\partial_z}^2(\cdot) + \frac{\alpha}{z}\partial_z(\cdot) \; . \tag{2.6.27}$$

The generic zero mode solution is

$$f_0(z) = a - b\frac{z^{1-\alpha}}{\alpha - 1} \; . \tag{2.6.28}$$

There exist no α, a, and b, for which the integral $\int_0^\infty \mu {f_0}^2 dz$ converges [61], therefore the spectrum of $\mathbb{R}^n$ never contains a zero mode.[38] The spectrum does always contain modes with arbitrarily small eigenvalue, but not with zero eigenvalue. Our generic positive eigenvalue modes are

$$f_\omega(z) = A_\omega z^{\frac{1-\alpha}{2}} J_{\frac{\alpha-1}{2}}(\omega z) + B_\omega z^{\frac{1-\alpha}{2}} Y_{\frac{\alpha-1}{2}}(\omega z) \; . \tag{2.6.29}$$

Here J_σ and Y_σ are Bessel functions of the first and second type [18,61], respectively, of order σ. If we select that our solutions are regular as $z \to 0^+$ then our solutions become purely Bessel functions of the first type, or $B_\omega = 0$. Next, if we substitute the asymptotic value of these Bessel functions we find

$$f_\omega(z) = A_\omega z^{\frac{1-\alpha}{2}} \left(\sqrt{\frac{2}{\pi \omega z}} \cos\left(\omega z - \frac{\alpha \pi}{4}\right) + \mathcal{O}\left(\frac{1}{z^{-1}}\right) \right) \; . \tag{2.6.30}$$

This agrees with equation (2.2.29) with $C_\omega = \sqrt{\frac{2}{V_\alpha \pi \omega}} A_\omega$, $r = 0$, and $\delta_\omega = -\frac{\alpha\pi}{4}$. For normalization we require $C_\omega = 2$; therefore we select

$$A_\omega = \sqrt{2 V_\alpha \pi \omega} \; . \tag{2.6.31}$$

2.6.3 AdS $_N$

In the Poincaré coordinate patch the metric on AdS $_N$ is [12,107]

$$ds_{\text{ADS}_N}{}^2 = \frac{1}{z^2}\left(-dt^2 + {ds_{N-2}}^2 + dz^2\right) \; . \tag{2.6.32}$$

Here t is a timelike coordinate, ds_{N-2} is the $(N-2)$-dimensional Euclidean metric, and $z \in (0, \infty)$ is the Poincaré radius. If we ignore dependence on t and x momentarily we have $\mu = \frac{1}{z^N}$ and

$$\Delta(\cdot) = {\partial_z}^2(\cdot) - \frac{N}{z}\partial_z(\cdot) \; . \tag{2.6.33}$$

[38] We may always restrict to a subset of our space such as an annalus, a disc, or an annalus with infinite outer radius of $\mathbb{R}^n$, but the entire space allows for no normalizable zero mode.

We may analogize this to the radius dependent part of $\mathbb{R}^n$ with the dimension $N = -\alpha = -n+1$. Thus N-dimensional anti de Sitter space's Laplacian agrees with the $(-n+1)$-dimensional Euclidean Laplacian (compare equations (2.6.27) and (2.6.33). The 'negative' dimension does not affect the non-normalizability of zero modes, or the behaviour of the spectrum, therefore this problem maps exactly onto the problem described in 2.6.2.

2.6.4 An Interval of AdS $_N$

This is a generalization of the problem studied in [113]. We may force our spectrum to contain a zero mode if we consider only a subset of AdS $_N$ where $z \in (A, B)$ with $A, B \in \mathbb{R}^+$, $A < B$, then our generic zero mode becomes normalizable

$$\begin{aligned} \langle f_0\, f_0 \rangle &= \int_A^B \frac{1}{z^N} \left(a + b \frac{z^{N+1}}{N+1} \right)^2 dz \\ &= \frac{A^{1-N} - B^{1-N}}{N-1} a^2 + \frac{B^2 - A^2}{N+1} ab + \frac{B^{N+3} - A^{N+3}}{(N+1)^2 (N+3)} b^2 \,. \end{aligned} \tag{2.6.34}$$

This is analogous to the system analyzed in 2.6.1. We have a one-dimensional space of boundary conditions for which the spectrum of our problem contains a zero mode.

Since z^{N+1} is a positive increasing function, our zero mode will only have a zero ($f_0(z_0) = 0$) when

$$A < \left(\left| \frac{a}{b} \right| (N+1) \right)^{\frac{1}{N+1}} < B \,. \tag{2.6.35}$$

Outside of this range of values for our ratio $\frac{a}{b}$, the associated zero mode has support on $z \in (A, B)$, and therefore 0 is the minimum eigenvalue in our spectrum.

Following equation (2.3.8), the boundary conditions our zero mode obeys is

$$\left(\left(a z^N + \frac{b}{N+1} z^{2N+1} \right) \partial_z - b \right) g(z) \Big|_{z \to A^-,\, B^+} = 0 \,. \tag{2.6.36}$$

Applying this condition to an arbitrary positive eigenvalue solution, generically given in (2.6.29), we find that neither the Bessel function of the first or second kind satisfies these boundary conditions alone, and that the spectrum is given by a highly complex transcendental equation given by ratios of Bessel functions analogous to equation (2.6.11).

Of special interest are the cases when $B \to \infty$ and $A \to 0^+$. In the first case we have

$$\langle f_0,\, f_0 \rangle = \frac{B^{N+3}}{(N+1)^2 (N+3)} b^2 + \mathcal{O}\left(B^2\right) \,, \tag{2.6.37}$$

and in the the second we have

$$\langle f_0\, f_0 \rangle = -\frac{A^{1-N}}{1-N} a^2 + \mathcal{O}\left(A^2\right) \ . \tag{2.6.38}$$

In these two limits the Sturm–Liouville problem has a zero mode if we require $b = 0$ and $a = 0$, respectively.

2.6.5 The Upper Half of AdS $_N$

This is the general case of the problem studied in [114]. In the limit where we allow our subinterval of AdS $_N$ to extend to $r \to \infty$ our boundary conditions become normal Neumann conditions at both $z \to A^-$ and $z \to \infty$. Our zero mode is normalized given

$$b = 0 \ , \qquad a = \pm\sqrt{\frac{A^{1-N}}{N-1}} \ , \tag{2.6.39}$$

and we have normalizable positive eigenvalue modes for any positive real number. These are given by equation (2.6.29) with A_ω obeying (up to normalization)

$$A_\omega J_{\frac{1-N}{2}}(\omega A) + 1\ Y_{\frac{1-N}{2}}(\omega A) = 0 \ . \tag{2.6.40}$$

Interestingly, this case is one of a transverse space where our coordinate patch has infinite range, but the space itself has finite volume. That is

$$\int_A^\infty \frac{1}{Z^N} dz = \frac{A^{1-N}}{N-1} \ . \tag{2.6.41}$$

2.6.6 The Lower Half of AdS $_N$

Since AdS $_N$ has infinite volume and the upper half of AdS $_N$ has finite volume (see equation (2.6.41)), the remaining space, $z \in (0, B)$ has infinite volume, however it still has a normalizable zero mode. That is

$$f_0 = a + b\frac{z^{1-\alpha}}{\alpha - 1} \ , \qquad a = 0 \ , \qquad b = \pi\sqrt{\frac{B^{N+3}}{(N+1)^2(N+3)}} \ . \tag{2.6.42}$$

Here our boundary conditions on our massive modes selects $A_\omega = 0$ and we then require

$$\lim_{z\to 0^+} A_\omega J_{\frac{1-N}{2}}(z) + B_\omega Y_{\frac{1-N}{2}}(z) = 0 \qquad \Leftrightarrow \qquad A_\omega \sin\left(\frac{\pi}{2}N\right) - B_\omega \cos\left(\frac{\pi}{2}N\right) = 0 \ . \tag{2.6.43}$$

2.6.7 The Pöschl–Teller Potential

Of essential importance to our studies is the 'Crampton–Pope–Stelle' problem [32], which has the measure

$$\mu(\rho) = \sinh(2\rho) \; . \tag{2.6.44}$$

Note we have chosen $\rho \in \mathcal{D} = (0,\, \infty)$ as our coordinate in place of z for historical reasons. We have the associated eigenvalue problem

$$\left(\partial_\rho{}^2 + 2\coth(2\rho)\partial_\rho\right) f_\omega(\rho) = -\omega^2 f_\omega(\rho) \; . \tag{2.6.45}$$

We can Schrödingerize [32,51] this problem with the substitution $f_\omega(\rho) = \mu(\rho)^{-\frac{1}{2}} l_\omega(\rho)$, where $l_\omega(\rho)$ solves

$$\left(\partial_\rho{}^2 + \coth^2(2\rho) - 2\right) l_\omega(\rho) = -\omega^2 l_\omega(\rho) \; . \tag{2.6.46}$$

We can shift our eigenvalue and our coordinates ($y = 2\rho$) as

$$-\frac{1}{4}\left(\partial_\rho{}^2 + \coth^2(2\rho) - 1\right) l_\omega(\rho) = -\left(\partial_y{}^2 + \frac{1}{4}\mathrm{csch}^2(y)\right)\tilde{l}_\omega(y) = \tilde{\omega}^2 \tilde{l}_\omega(y) = \frac{\omega^2 - 1}{4}\tilde{l}_\omega(y) \; . \tag{2.6.47}$$

This is a special form of the more general Schrödinger problem with the Pöschl–Teller potential [11,75,110] ($\psi_E(y) = \tilde{l}_\omega(y)$, $\tilde{\omega}^2 = E$):

$$-\partial_y{}^2 \psi_E(y) + V(y)\psi_E(y) = E\psi_E(y) \; , \tag{2.6.48}$$

$$V(y) = -\lambda(\lambda+1)\mathrm{sech}^2(y) - \nu(\nu+1)\mathrm{csch}^2(y) \; . \tag{2.6.49}$$

This is an attractive potential when either $\nu(\nu+1) > 0$ or when $-\lambda(\lambda+1) > \nu(\nu+1)$.

After a further substitution of either $u = \tanh(y)$,

$$\partial_u\left(\left(1-u^2\right)\partial_u \psi_E(u)\right) + \left(\lambda(\lambda+1) + \frac{\nu(\nu+1)}{u^2} + \frac{E}{1-u^2}\right)\tilde{\psi}_E(u) = 0 \; , \tag{2.6.50}$$

or a substitution of $v = \coth(y)$

$$\partial_v\left(\left(1-v^2\right)\partial_v \psi_E(v)\right) - \left(\nu(\nu+1) + \frac{\lambda(\lambda+1)}{v^2} - \frac{E}{1-v^2}\right)\tilde{\tilde{\psi}}_E(v) = 0 \; , \tag{2.6.51}$$

we find we may solve this problem exactly when either $\lambda = 0,\, -1$ or $\nu = 0,\, -1$. In these cases we have

$$\tilde{\psi}_E(u) = A_E P_\lambda^{\sqrt{-E}}(u) + B_E Q_\lambda^{\sqrt{-E}}(u) \ , \tag{2.6.52}$$

$$\tilde{\tilde{\psi}}_E(v) = A_E \mathcal{P}^{\sqrt{-E}}_{-\frac{1}{2}\left(1-\sqrt{1-4\nu(\nu+1)}\right)}(v) + B_E \mathcal{Q}^{\sqrt{-E}}_{-\frac{1}{2}\left(1-\sqrt{1-4\nu(\nu+1)}\right)}(v) \ . \tag{2.6.53}$$

Here $P_\mu^\nu(u)$ and $Q_\mu^\nu(u)$ and $\mathcal{P}_\mu^\nu(v)$ and $\mathcal{Q}_\mu^\nu(v)$ are associated Legendre functions of the first and second kind with degree μ and order ν with principle domain from $u \in (0,\, 1)$ and $v \in (1,\, \infty)$, respectively.

For this work we are focusing on the zero mode sector. When we set $\omega = 0$ in the Crampton–Pope–Stelle problem we find $E = -\frac{1}{4}$, $\lambda(\lambda+1) = 0$, and $\nu(\nu+1) = \frac{1}{4}$. This gives us

$$l_0(\rho) = A_{\frac{1}{4}} \mathcal{P}_0^{\frac{1}{2}}(\coth(2\rho)) + B_{\frac{1}{4}} \mathcal{Q}_0^{\frac{1}{2}}(\coth(2\rho)) \ . \tag{2.6.54}$$

Fortunately, in this special case, we do not need these associated Legendre functions, as we can appeal to Whipple formulae [20] to simplify

$$\begin{aligned} f_\omega(\rho) = \frac{1}{\sqrt{\sinh(2\rho)}} l_\omega(\rho) = \frac{1}{\sqrt{\sinh(2\rho)}} \Big(& \tilde{A}_{\tilde{\omega}} \sqrt{\sinh(2\rho)} \mathcal{P}_{-\frac{1}{2}\left(1-\sqrt{1-\omega^2}\right)}(\cosh(2\rho) \\ & + \tilde{B}_{\tilde{\omega}} \sqrt{\sinh(2\rho)} \mathcal{Q}_{-\frac{1}{2}\left(1-\sqrt{1-\omega^2}\right)}(\cosh(2\rho)) \Big) \ . \end{aligned} \tag{2.6.55}$$

This nicely agrees with our standard method for finding the zero mode solutions[39]

$$f_0(\rho) = a - b\frac{1}{2}\log\left(\tanh(\rho)\right) \ . \tag{2.6.56}$$

Here $a = \tilde{A}_{\frac{1}{2}}$ and $b = \frac{1}{2} B_{\frac{1}{4}}$.

If we take any finite section of $\rho \in (0,\, \infty)$ we will find the situation as for the finite interval, where there are two normalizable zero modes. Of special physical interest is when the we consider the full transverse domain. We note that the inner product of the constant zero mode and itself, and the inner product of the constant zero mode and the nonconstant zero mode both diverge,

$$\int_0^\infty \mu(\rho) d\rho \to \infty \ , \tag{2.6.57}$$

$$\int_0^\infty \mu(\rho)\xi_0(\rho) d\rho \to \infty \ . \tag{2.6.58}$$

[39]Here $\xi_0(\rho) = \int_\rho^\infty \frac{1}{\mu(\eta)} d\eta = \frac{1}{2}\log \circ \tanh(\rho)$, following section 2.3.

However the inner product of the nonconstant zero mode with itself converges,

$$\int_0^\infty \mu(\rho)\xi_0(\rho)^2 d\rho = \int_1^\infty \frac{1}{8}\mathcal{Q}_0(z)^2 dz = \frac{\pi^2}{48} \, . \tag{2.6.59}$$

That is, when we consider the full domain the constant zero mode becomes nonnormalizable, however the nonconstant zero mode remains normalizable. The exponential growth of the measure exponentially suppresses the nonconstant zero mode so that the integral of the square of the nonconstant zero mode converges.

Therefore if we choose $a = 0$ and $b = 4\frac{\sqrt{3}}{\pi}$ we have a normalizable zero mode on the entire infinite volume domain. No other set of boundary conditions will yield a normalizable zero mode state. Furthermore, since $\sinh(2\rho) \sim \frac{1}{2}\exp(2\rho)$ we only have scattering states when $\omega > 1$. This is a gapped system with a normalizable zero mode and an infinite volume space.

One wrinkle that must be addressed however, is the diversity of possible lowest eigenvalue states. We note that we have when $\omega^2 < 1$ we have

$$\int_0^\infty \mu(\rho)\xi_\omega(\rho)^2 d\rho \cong \int_1^\infty \frac{1}{8}\mathcal{Q}_{-\frac{1}{2}\left(1-\sqrt{1-\omega^2}\right)}(z)^2 dz = \frac{\psi'\left(\frac{1}{2}\left(\sqrt{1-\omega^2}+1\right)\right)}{8\sqrt{1-\omega^2}} < \infty \, . \tag{2.6.60}$$

Each of these $\xi_\omega(\rho)$ with $\omega^2 < 1$ are positive definite, therefore we may have only one such mode in our spectrum. Therefore any boundary condition that selects any $\xi_\omega(\rho)$ will have qualitatively similar scattering states, however will define a unique bound state.

Similarly, we may have seen this behavior by appealing to the asymptotic form of the Pöschl–Teller problem, since

$$V(y) = \frac{1}{4}\frac{1}{y^2} + \mathcal{O}\left(\frac{1}{y}\right) \, . \tag{2.6.61}$$

The $a\frac{1}{y^2}$ potential has a variable number of bound states depending on what the coefficient is [14,27,29,35,85]. The case where $a = \frac{1}{4}$ is a boundary case, where the system has only a single bound state. Therefore we need not calculate the remainder of the spectrum explicitly. This agrees with our argument from the perspective of Sturm–Liouville theory.

What are the boundary conditions that give us these bound states? Following equation (2.3.8) the boundary condition that selects $\xi_\omega(\rho)$ as its lowest lying mode is

$$\left(\mu(\rho)f_\omega(\rho)\partial_\rho - b\right)g(\rho)\Big|_{\rho\to 0^+,\,\infty} = 0 \, . \tag{2.6.62}$$

Inserting the asymptotic forms of each of these, as well as factorizing an overall b we have

$$\left(\left(H_{\frac{1}{2}\left(\sqrt{1-\omega^2}-1\right)} + \log(\rho)\right) \rho\, \partial_\rho - 1\right) g(\rho)\Big|_{\rho\to 0^+} = 0 \;, \tag{2.6.63}$$

at the lower boundary and either special Dirichlet or Neumann conditions at the upper boundary (both conditions are true for all normalizable states). Here H_α is the harmonic number [99] which agrees with the partial sums of the harmonic series when α is a positive integer. Notably, we cannot impose special Neumann conditions at the lower boundary and keep a normalizable bound state. This is obvious from the perspective of Sturm–Liouville theory, since the constant zero mode obeys special Neumann at the lower boundary, but is not normalizable. This is an interesting case for the supercritical eigenmodes ($\omega^2 > 1$), however, as this is the case where $f_\omega(\rho)$ becomes 'pure $\mathcal{P}$'. Finally, we note, that no basis can be made to obey special Dirichlet conditions at the lower boundary. This does not mean that sourced equations such as

$$\frac{1}{\mu(\rho)}\partial_\rho\left(\mu(\rho)\partial_\rho g(\rho)\right) = 1 \;, \tag{2.6.64}$$

do not have solutions that obey special Dirichlet conditions. In fact we know

$$g(\rho) = \frac{1}{2}\log\left(\cosh(\rho)\right) \;, \tag{2.6.65}$$

is such a solution. However, it does mean that finding such solutions requires considerations other than Sturm–Liouville theory.

To summarize, the Crampton–Pope–Stelle problem maps neatly onto the Pöschl–Teller potential, which is, for 'nice' values of its configuration space a solved problem. The particular limit of the Pöschl–Teller problem we are studying has exactly one, known, bound state, and a mass gap with qualitatively similar scattering states for all boundary conditions that give bound states.

3 Green Functions

After our interlude at the end of the last section describing several explicit Sturm–Liouville bases, we return to a generic discussion of the applications of Sturm–Liouville theory. In the last section we derived why there is a resolution of the identity and what properties the transverse basis obeys. In this section we will derive what is possible given a resolution of the identity. The central understanding of Sturm–Liouville theory is that the action of a linear operator on its eigenvectors uniquely defines the operator [47, 92], and that understanding any higher behavior of the operator may be done by further manipulation of said eigenvectors, especially the ones in its kernel [69].

This section is organized as follows

1. We begin by finding Green functions for arbitrary Laplacians and their relation to fundamental solutions for those Laplacians.
   - We find the Green function using the kernel of the Laplacian and the Dirac delta distribution.
   - We argue the fundamental solution for spaces is given by the Green function in radial coordinates.
   - We give the Green function by appealing to the resolution of the identity (Green's formula).
2. We next find several useful formulae for sums of overlap integrals (products of more than two modes).
   - We find for spaces with a constant zero mode almost all relevant overlap integrals vanish.
   - We find for other spaces sums of overlap integrals are given by integrals of Green functions.
   - We calculate some of these sums explicitly for a few cases relevant to later sections.
3. We use Green's formula to give solutions to partial differential equations on product spaces.
   - We describe the structure of this process for arbitrary separable operators.
   - We describe the results of this study when one of our component spaces is odd-dimensional $\mathbb{R}^n$.
   - We discuss how to relate Green functions with differing boundary conditions.
   - We apply our recently described technique to different solutions of $\mathbb{R}^n \times (-l,\, l)$.
   - We apply the technique again to a space with a transverse Pöshl–Teller problem.
   - We discuss the implications of this technique for different spaces with or without mass gaps.

3.1 Finding Green Functions

For any linear second-order ordinary differential operator, Δ, which is self-adjoint with respect to some measure $L_2(\mu, \mathcal{D})$, we have an associated Green function $G(z\,,\ s)$, which obeys [66]

$$\Delta G(z\,,\ s) = \frac{\delta(z-s)}{\mu(z)}\ . \tag{3.1.1}$$

Here, $\delta(z-s)$ is the Dirac delta distribution.

Throughout this section all functions of a pair of coordinates on the same space, generically r and t or z and s are symmetric under exchange of functions on that space unless otherwise stated ($r \leftrightarrow t$ and/or $z \leftrightarrow s$).

The existence of said Green function is equivalent to the proof that the differential equation we solve using the Green function exists. In this view we may construct our Green function by knowing the kernel of Δ, which is the focus of the next section 3.1.1. Alternatively we can appeal to the fact that Δ is a linear operator between two Hilbert spaces [112] and therefore has an inverse image, which is tantamount to a Green function. For this inverse image to define an inverse function (which is still not the Green function) we must have some method of selecting a unique member of the inverse image. Here we can again use that linear ordinary differential equations have unique solutions. This is a consequence of the Picard–Lindelöf theorem [30]. This argument misses that the Green function is necessarily a function, for that to be true the inverse image must be differentiable with respect to changes in the source function.

$G(z,\ s)$ can be defined for any set of boundary conditions that, as we shall argue, does not permit a zero mode. One method for finding $G(z,\ s)$ is appealing to the kernel of our differential operator.

3.1.1 From the Kernel

By the definition of a delta distribution, we have, for any $\epsilon \in \mathbb{R}^+$ [97]

$$\begin{aligned}
\int_{s-\epsilon}^{s+\epsilon} \mu(z)\Delta G(z\,,\ s)dz &= 1\ ,\\
\int_{l}^{s-\epsilon} \mu(z)\Delta G(z\,,\ s)dz &= 0\ ,\\
\int_{s+\epsilon}^{u} \mu(z)\Delta G(z\,,\ s)dz &= 0\ .
\end{aligned} \tag{3.1.2}$$

We know the zero modes associated with Δ (given in equation (2.3.7)). Given boundary conditions for our Green function to obey we may find G explicitly. Begin by noting

$$\Delta G(z\,,\ s)\big|_{z\neq s} = 0 \qquad \Rightarrow \qquad G(z\,,\ s) = \begin{cases} a + b\,\xi_0(z) & z < s \\ c + d\,\xi_0(z) & z > s \end{cases} . \tag{3.1.3}$$

Applying our boundary conditions sets the ratio between a and b (for our lower boundary condition), then c and d (for our upper boundary condition). We may set an additional condition between our constants by requiring symmetry under exchange of $z \leftrightarrow s$. We obtain a final condition on our constants via integration. That is

$$\int_{s-\epsilon}^{s+\epsilon} \mu(z)\Delta G(z\,,\ s)dz = 1\ . \tag{3.1.4}$$

Here $\epsilon \in \mathbb{R}^+$ is some infinitesimal. Integrating by parts once we find

$$\int_{s-\epsilon}^{s+\epsilon} \mu(z)\Delta G(z\,,\ s)dz = \mu(z)\partial_z G(z\,,\ s)\Big|_{z=s-\epsilon}^{z=s+\epsilon} - \int_{s-\epsilon}^{s+\epsilon} \mu(z)\left(\partial_z 1\right)\left(\partial_z G(z\,,\ s)\right)dz\ . \tag{3.1.5}$$

This integrand vanishes, and applying our definitions of ζ_0, ξ_0, and G; we have

$$\int_{s-\epsilon}^{s+\epsilon} \mu(z)\Delta G(z\,,\ s)dz = d - b = 1\ . \tag{3.1.6}$$

These four constraints on four conditions uniquely define G, and generically have a solution unless the boundary conditions permit a zero mode.

Given these definitions we may write the most generic Green function as

$$G(z\,,\ s) = A + B\,\xi_0(z) + \begin{cases} \xi_0(s) & z < s \\ \xi_0(z) & z > s \end{cases} . \tag{3.1.7}$$

3.1.2 Regular Versus Fundamental Solutions and Radial Problems

Solutions are stated to be regular at any given point if they are C^1, that is if $\lim_{z\to p} f_\omega(z) = f_\omega(p)$, and $\lim_{z\to p} \partial_z f_\omega(z) = \partial_z f_\omega(z)\big|_{z=p}$. On the boundary, a solution is regular if both these limits exist and are finite when approached from the interior of the domain [98].

Fundamental solutions are solutions defined by a point source at the origin.[40] In our context they are

[40] The origin can be defined arbitrarily for some spaces, such as $\mathbb{R}^N$. However for the example of CPS, $\rho = 0$ indicates a physically significant location of 'waist' and therefore the fundamental solution is specifically defined by a boundary condition at that point.

most easily understood by comparison to the Green function. For instance, there exists a fundamental zero-mode solution to the Laplacian $\Delta = \partial_z{}^2$ on the interval $\mathcal{D} = (-1, 1)$, which obey Dirichlet-Dirichlet conditions:

$$f_{\text{fundamental}}(z) = |z| - 1 \ . \tag{3.1.8}$$

A solution is said to be fundamental at the boundary when the boundary coincides with the origin and the solution is singular [105]. That is, either the limit for the value of the function or the limit for the derivative of the function diverges.

In this work fundamental solutions will primarily be relevant in the case of a 'radial' Laplacian, for instance, when our measure $\mu(z)$ vanishes at the lower boundary and diverges at the upper boundary. The introductory examples might be when $\mu(z) = z^2$ or $\mu(z) = z^3$, as is the case for the S-wave modes of the three- and four-dimensional radial Laplacians, respectively (equation (2.6.27)). In these cases for G to define a function continuous at all points in the full space-time its derivative must vanish at $z = 0$. That is, G must obey Neumann conditions at $z = 0$, as illustrated in figure (4).

Furthermore, in these cases, we may restrict G to vanish at the boundary $z \to \infty$. This requires $b = 0$

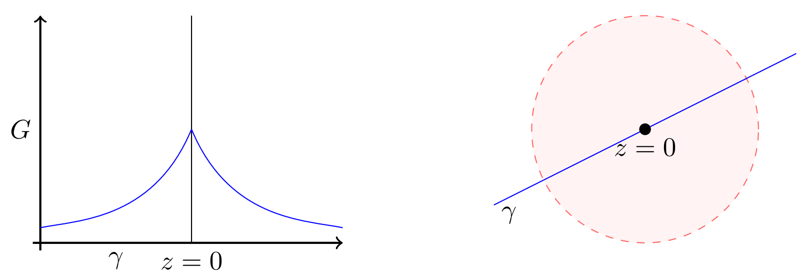

Figure 4: The value of a scalar function along a path through the origin

and $c = 0$. The normalization condition, equation (3.1.6), requires then $d = 1$. Together we have

$$G(z\,,\ s) = \begin{cases} \xi_0(s) & z < s \\ \xi_0(z) & z > s \ . \end{cases} \tag{3.1.9}$$

In such a case, when $s \to 0$, $\xi_0(z)$ is proportional to the Green function, and ξ_0 diverges at the lower boundary. In such a case we say that this mode constitutes a fundamental S-wave solution.

In the context of a field theory, this simply implies that the higher-dimensional action requires a boundary term to support the equations of motion associated with the zero mode (section 6.3). This is likenable to the electric field in the presence of an additional point source, the electron, at the level of the equations of motion.

3.2 Green's Formula

If we consider a Laplacian and boundary conditions that do not permit a zero mode, then we have an associated spectrum and orthonormalized eigenfunctions $f_\omega(z)$ ($\omega > 0$). These eigenfunctions form a resolution of the identity (equation (2.4.2)). From this we may define an additional sum, and, through the action of the Laplacian on said sum find

$$\Delta\left(-\frac{f_\omega(z)f_\omega(s)}{\omega^2}\right) = f_\omega(z)f_\omega(s) = \frac{\delta(z-s)}{\mu(z)} \; . \tag{3.2.1}$$

Here we sum or integrate over the repeated Sturm–Liouville eigenvalue, as necessary. Note that Δ only acts on z so $\Delta f_\omega(s) = 0$. We notice that this sum solves the same distributional requirement that defines the Green function. Therefore the Green function, given this boundary condition, is

$$G(z\,,\ s) = -\frac{f_\omega(z)f_\omega(s)}{\omega^2} \; . \tag{3.2.2}$$

We may immediately extend this identity to any case that includes a zero mode in two independent manners. First, we note that in the presence of a zero mode, we may still define our sum, but only over nonzero eigenvalues $\overline{\omega} > 0$. That is, we have

$$\overline{G}(z\,,\ s) = -\frac{f_{\overline{\omega}}(z)f_{\overline{\omega}}(s)}{\overline{\omega}^2} \; . \tag{3.2.3}$$

Acting on this with the Laplacian (and adding $0 = f_0(z)f_0(s) - f_0(z)f_0(s)$) we note

$$\Delta\overline{G}(z\,,\ s) = f_{\overline{\omega}}(z)f_{\overline{\omega}}(s) + f_0(z)f_0(s) - f_0(z)f_0(s) = \frac{\delta(z-s)}{\mu(z)} - f_0(z)f_0(s) \; . \tag{3.2.4}$$

Therefore, if we can solve

$$\Delta Z(z\,,\ s) = f_0(z)f_0(s) \; , \tag{3.2.5}$$

given symmetry in the exchange $z \leftrightarrow s$, then we have

$$\overline{G}(z\,,\ s) + Z(z\,,\ s) = G(z\,,\ s) \; , \tag{3.2.6}$$

where $\overline{G}(z\,,\ s)$ obeys our boundary conditions even though they permit a zero mode.

This is perhaps best understood with an example. Consider the case of the flat interval $\mathcal{D} = (-1, 1)$.

The most general associated Green function is

$$G(z\,,\ s) = a + bz + \frac{1}{2}|z - s|\ . \tag{3.2.7}$$

No choice of a and b, even as generic functions of s will make this G obey Neumann-Neumann conditions. However, if we consider the associated $Z(z\,,\ s)$

$$\Delta Z(z\,,\ s) = \frac{1}{2} \qquad \Rightarrow \qquad Z(z\,,\ s) = -\frac{z^2 + s^2}{4}\ , \tag{3.2.8}$$

then the associated $\overline{G}(z\,,\ s)$, which obeys Neumann-Neumann conditions is (for some $a \in \mathbb{R}$)

$$\overline{G}(z\,,\ s) = a + \frac{1}{2}|z - s| - \frac{z^2 + s^2}{4}\ . \tag{3.2.9}$$

We may find a by calculating the explicit value of the sum over our nonzero spectrum. In this case[41]

$$\frac{f_{\overline{\omega}}(0) f_{\overline{\omega}}(0)}{\overline{\omega}^2} = -\sum_{n=1}^{\infty} \frac{1}{4\pi^2 n^2} = -\frac{1}{6}\ . \tag{3.2.10}$$

Explicitly, we have the sum

$$-\sum_{n=1}^{\infty} \left(\frac{\cos(\pi n s)\cos(\pi n z)}{\pi^2 n^2} + \frac{\sin\left(\pi\left(n - \frac{1}{2}\right)s\right)\sin\left(\pi\left(n - \frac{1}{2}\right)z\right)}{\pi^2\left(n - \frac{1}{2}\right)^2} \right) = -\frac{1}{6} + \frac{1}{2}|z - s| - \frac{z^2 + s^2}{2}\ . \tag{3.2.11}$$

Note, while we introduce this modified Green function in this context for the purpose of defining handling zero modes, this technique may be expanded to excluding any specific mode from the sum. Restated, we may always define a modified Green function $\overline{G}$ by removing a single mode, such as the lightest nonzero mode. This technique is especially powerful for finding sums of overlap integrals.

The second method of including a zero mode into our Green function is to shift our Laplacian by a constant, and define

$$\left(\Delta - \sigma^2\right) G^{\sigma}(z\,,\ s) = \frac{\delta(z - s)}{\mu(z)}\ . \tag{3.2.12}$$

Then we have, even in the case where our spectrum contains a zero mode (so long as $\sigma^2 \neq -\omega^2$ for any ω^2 in our spectrum)

$$G^{\sigma}(z\,,\ s) = -\frac{f_\omega(z) f_\omega(s)}{\sigma^2 + \omega^2}\ . \tag{3.2.13}$$

[41] All such sums over Fourier modes on the interval I have come across have been related to integer values of the Riemann zeta function [7]. This one is an example of the Basel problem [6]. For more generic values of the Riemann function, see [78].

This is especially useful in the context of using Sturm–Liouville theory to solve partial differential equations.

Finally, we may combine these methods simultaneously, both shifting the Laplacian and omitting a light mode. In which case we have the sum

$$\overline{G}^{\sigma}(z\,,\ s) = -\frac{f_{\overline{\omega}}(z)f_{\overline{\omega}}(s)}{\sigma^2+\overline{\omega}^2}\ . \tag{3.2.14}$$

If we act on this sum with the shifted Laplacian we see

$$\left(\Delta-\sigma^2\right)\overline{G}^{\sigma}(z\,,\ s) = f_{\overline{\omega}}(z)f_{\overline{\omega}}(s) = \frac{\delta(z\,,\ s)}{\mu(z)} - f_0(z)f_0(s)\ . \tag{3.2.15}$$

Therefore, if we can solve

$$\left(\Delta-\sigma^2\right)Z^{\sigma}(z\,,\ s) = -f_0(z)f_0(s) \qquad \Leftrightarrow \qquad \Delta\left(Z^{\sigma}(z\,,\ s)-Z(z\,,\ s)\right) = \sigma^2 Z^{\sigma}(z\,,\ s)\ , \tag{3.2.16}$$

given symmetry in the exchange $z \leftrightarrow s$, then we have

$$\overline{G}^{\sigma}(z\,,\ s) + Z^{\sigma}(z\,,\ s) = G^{\sigma}(z\,,\ s)\ . \tag{3.2.17}$$

3.3 Sums of Overlap Integrals

In the context of field theory we will define a lightest mode, frequently a zero mode, and speak of corrections to its behavior which arise for 'integrating out' couplings to the heavy modes. This may happen in two ways.

3.3.1 Consistent Truncations

In the first case many of our possible couplings, given by overlap integrals, vanish. We consider some lightest mode, with eigenvalue l, and heavy modes, with eigenvalues $\overline{\omega}$.

For a consistent truncation to be possible we require the triple overlap integrals vanish when there is exactly one heavy mode,

$$\int_{\mathcal{D}} \mu(z)f_l(z)f_l(z)f_{\overline{\omega}}(z)dz = I_{ll\overline{\omega}} = 0\ . \tag{3.3.1}$$

However, the triple overlap integrals with two heavy modes (with eigenvalues $\overline{\omega}$ and $\overline{\sigma}$) do not vanish,[42]

$$\int_{\mathcal{D}} \mu(z)f_l(z)f_{\overline{\omega}}(z)f_{\overline{\sigma}}(z)dz = I_{l\overline{\omega}\overline{\sigma}} \neq 0\ . \tag{3.3.2}$$

[42] Our analysis does not rely on $I_{l\overline{\omega}\overline{\sigma}}$ being nonzero for every pair of $\overline{\omega}$ and $\overline{\sigma}$, simply that it have nonzero support, or be nonzero for some combination thereof.

We will use $I_{l\overline{\omega}\dots}$ interchangeably with the explicit integrals.

In these cases we will justify, at the level of the field theory, setting the degrees of freedom associated with the heavy modes to zero, or 'truncating' the theory. Mathematically we may do this for some set of modes $\overline{\omega}$ so that all overlaps of l and any modes we keep $\overline{\sigma}$ vanish with single modes we have truncated $\overline{\omega}$. For example we may truncate odd modes but keep even modes. Physically, this is only justified when we truncate all but the lightest mode.

These are called consistent truncations, and usually occur when our lightest mode is a constant. In this Neumann-Neumann case the I_{lXY} is simply proportional to I_{XY}, or just the standard inner product. Since our modes are orthornormalized, the inner product of any heavy mode with the lightest zero mode vanishes, therefore $I_{ll\overline{\omega}} = 0$.

We may actually prove that the Neumann-Neumann condition is the only case where you can have a consistent truncation. We begin by noting we may always expand $f_0(z)^2$ in terms of our basis

$$f_0(z)^2 = f_0(z) \int_{\mathcal{D}} \mu(s) f_0(s)^3 ds + f_{\overline{\omega}}(z) \int_{\mathcal{D}} \mu(s) f_0(s)^2 f_{\overline{\omega}}(s) ds \; . \tag{3.3.3}$$

If $I_{ll\overline{\omega}} = 0$ for all $\overline{\omega} > l$ then we have

$$f_0(z)^2 = a f_0(z) \; , \tag{3.3.4}$$

for some $a \in \mathbb{R}$. Therefore

$$\partial_z f_0(z)^2 = \partial_z a f_0(z) \qquad \Rightarrow (2 f_0(z) - a) \left(\partial_z f_0(z) \right) = 0 \; . \tag{3.3.5}$$

Therefore either $f_0(z) = \frac{a}{2}$ or $\partial_z f_0(z) = 0$ for all z. Both of these conditions require $f_0(z)$ to be constant.[43]

3.3.2 Inconsistent Truncations

In the case where, due to field theoretic reasons, we cannot truncate our heavy modes $\overline{\omega} \neq l$ we may still integrate them out. To integrate these fields out we will require squares of overlap integrals, or these squares divided by the relevant eigenvalues. That is,[44]

$$X = {I_{ll\overline{\omega}}}^2 \; , \tag{3.3.6}$$

$$Y = -\frac{{I_{ll\overline{\omega}}}^2}{\overline{\omega}^2} \; . \tag{3.3.7}$$

[43]Hypothetically, in a situation like this, we simply need that the support of our two conditions is disjoint. However, since the implications of these two conditions agree, that implies that both conditions are true for all $z \in \mathcal{D}$.

[44]We do not sum over explict indices (l or 0). So ${I_{ll\overline{\omega}}}^2 = \sum_{\overline{\omega}>l} I_{ll\overline{\omega}} I_{ll\overline{\omega}}$.

We recognize the modified Green function (equation (3.2.9)) through the following manipulation[45]

$$
\begin{aligned}
X &= \int_{\mathcal{D}} \mu(z) f_l(z) f_l(z) f_{\overline{\omega}}(z) dz \int_{\mathcal{D}} \mu(s) f_l(s) f_l(s) f_{\overline{\omega}}(s) ds \\
&= \int_{\mathcal{D}} \int_{\mathcal{D}} \mu(z)\mu(s) f_l(z)^2 f_l(s)^2 \left(f_{\overline{\omega}}(z) f_{\overline{\omega}}(s)\right) dz ds \\
&= \int_{\mathcal{D}} \int_{\mathcal{D}} \mu(z)\mu(s) f_l(z)^2 f_l(s)^2 \left(\frac{\delta(z-s)}{\mu(z)} - f_l(z) f_l(s)\right) dz ds \\
&= \int_{\mathcal{D}} \mu(z) f_l(z)^4 dz - \left(\int_{\mathcal{D}} \mu(z) f_l(s)^3 dz\right)^2 .
\end{aligned}
\tag{3.3.8}
$$

$$
\begin{aligned}
Y &= -\frac{\int_{\mathcal{D}} \mu(z) f_l(z) f_l(z) f_{\overline{\omega}}(z) dz \int_{\mathcal{D}} \mu(s) f_l(s) f_l(s) f_{\overline{\omega}}(s) ds}{\overline{\omega}^2} \\
&= \int_{\mathcal{D}} \int_{\mathcal{D}} \mu(z)\mu(s) f_l(z)^2 f_l(s)^2 \left(-\frac{f_{\overline{\omega}}(z) f_{\overline{\omega}}(s)}{\overline{\omega}^2}\right) dz ds \\
&= \int_{\mathcal{D}} \int_{\mathcal{D}} \mu(z)\mu(s) f_l(z)^2 f_l(s)^2 \overline{G}(z\,,\ s) dz ds .
\end{aligned}
\tag{3.3.9}
$$

Similarly, this can be extended to the case with an additional shift in the denominator

$$
Y_M = -\frac{{I_{ll\overline{\omega}}}^2}{M^2 + \overline{\omega}^2} = \int_{\mathcal{D}} \int_{\mathcal{D}} \mu(z)\mu(s) f_l(z)^2 f_l(s)^2 \overline{G}^M(z\,,\ s) dz ds . \tag{3.3.10}
$$

This constant will be useful when calculating corrections in our field theory in sections 6 and 8. However, we need not know the value of any overlap integral or set of overlap integrals so long as we may explicitly calculate our modified Green function.

3.3.3 Explicit Corrections from Inconsistent Truncations

Since the value of these overlap integrals and sums of overlap integrals is relevant in the context of field theory we will give their values for each of the spaces on which we have described Sturm–Liouville bases in section 2.6.

The Flat Interval

The augmented Green function for the most general zero mode ($f_0(z) = az + b(z+1)$, equation (2.6.1) and $2a^2 + 4ab + \frac{8}{3}b^2 = 1$, equation (2.6.2)) for the flat interval is[46]

$$
\overline{G}(z\,,\ s) = K(z\,,\ s) + Z(z\,,\ s) + \frac{1}{2}|z-s| , \tag{3.3.11}
$$

[45] Assuming uniform convergence of the sums and integrals.

[46] To find the explicit value of $K(z\,,\ s)$ we must apply that a and b are related by normalization, thus the reason only b appears in $K(z\,,\ s)$'s definition.

$$Z(z\,,\ s) = \frac{1}{6}\left(3(a+b)^2\left(s^2+z^2\right)+b(a+b)(s+z)^3+b^2sz\left(s^2+z^2\right)\right)\ , \tag{3.3.12}$$

$$K(z\,,\ s) = \frac{\sqrt{3-2\,b^2}\left(b^2-d\right)(z-s)}{\sqrt{6}\,b}+\frac{1}{6}\left(2-\frac{3}{b^2}\right)d+d\,s\,z\ . \tag{3.3.13}$$

Note, the additional constant d is fictitious. However, to calculate $\overline{G}(z\,,\ s)$ explicitly here we applied our boundary conditions, and d is not constrained by the boundary conditions alone. Two limits in which we have calculated d are $b \to 0$ and $a \to 0$. When $b \to 0$, to make our generic form of $\overline{G}(z\,,\ s)$ limit to equation (3.2.11), we have $d \to \frac{1}{3}b^2$. When $a \to 0$, $d = \frac{7}{16}$.

The arbitrary overlap integral of only the zero mode is

$$I_{(n)} = I_{00\ldots0} = \int_{-1}^{1} f_0(z)^n dz = \frac{(a+2b)^{n+1}-a^{n+1}}{(n+1)b}\ . \tag{3.3.14}$$

Of special note are $I_{(3)} = \frac{(a+2b)^4-a^4}{4b}$ and $I_{(4)} = \frac{(a+2b)^5-a^5}{5b}$. We note $I_{(3)} = 0$ when $a = -b$. Therefore, from equation (3.3.8), we have

$$X = {I_{00\overline{n}}}^2 = I_{(4)} - I_{(3)}^2 = \frac{(a+2b)^5-a^5}{5b} - \left(\frac{(a+2b)^4-a^4}{4b}\right)^2\ . \tag{3.3.15}$$

We further note the three interesting limits of this expression: when $b \to 0$

$$X = \frac{1}{3}b^2 + \mathcal{O}(b^4)\ , \tag{3.3.16}$$

when $a \to 0$ we have

$$X = \frac{9}{160} + \frac{3}{80}\sqrt{\frac{3}{2}}a - \frac{3a^2}{160} + O\left(a^3\right)\ , \tag{3.3.17}$$

and when $a \to -b+\epsilon$ we have

$$X = \frac{9}{10} - 3\epsilon^2 - \mathcal{O}(\epsilon^4)\ . \tag{3.3.18}$$

That is, we note that the size of this overlap is at minimum[47] in the Neumann-Neumann case, has a linear term when we have 'pure ξ_0' boundary condition, and is at maximum in the antisymmetric case when our zero mode solution is an odd function. Similarly we compute

[47]That is, we can prove that it is a local minimum, and assuming it is monotonically increasing everywhere else, it is also a global minimum.

$$
\begin{aligned}
Y = -\frac{2}{189b^2}\Big(&63a^6b^2 + 378a^5b^3 + 1071a^4b^4 - 126a^4b^2d - 126a^4b^2 + 189a^4d + 1764a^3b^5 \\
&- 504a^3b^3d - 504a^3b^3 + 756a^3bd + 1764a^2b^6 - 1008a^2b^4d - 756a^2b^4 + 1260a^2b^2d \\
&+ 1008ab^7 - 1008ab^5d - 504ab^5 + 1008ab^3d + 252b^8 - 392b^6d - 144b^6 + 336b^4d\Big) \ .
\end{aligned} \tag{3.3.19}
$$

This vanishes ($Y = 0$) in the Neumann-Neumann case (when $b \to 0$) which is what we expect for a consistent truncation. When $a = 0$ we have $Y = -\frac{57}{244}$.

3.4 Green Functions and Solutions to Partial Differential Equations

A second use of Green functions is finding explicit solutions to our field theory, specifically in finding solutions to the sourced Laplace equation on product spaces.

Consider a higher-dimensional Laplacian[48] which can be expressed as a separable sum

$$
\Delta = \Delta_r + \Delta_z \ . \tag{3.4.1}
$$

For S-wave expansions in both spaces, we can choose coordinates (as in section 2.1.1) so that

$$
\Delta_r = \frac{1}{m(r)}\partial_r m(r) \partial_r \ , \qquad \Delta_z = \frac{1}{\mu(z)}\partial_z \mu(z) \partial_z \ . \tag{3.4.2}
$$

Our goal is to compute a higher-dimensional Green function

$$
(\Delta_r + \Delta_Z)\, G(r\,,\ t\,,\ z\,,\ s) = \frac{\delta(r-t)\delta(z-s)}{m(r)\mu(z)} \ . \tag{3.4.3}
$$

3.4.1 Green's Formula for Separable Operators

To accomplish this we may find two independent bases and modify Green's formula for a single basis. Alternatively we can find one basis and one augmented Green functions and give a related modification. In the end the formulae are related.

For the sake of illustration our bases are

$$
\Delta_r g_\sigma(r) = -\sigma^2 g_\sigma(r) \ , \qquad \Delta_z f_\omega(z) = -\omega^2 f_\omega(z) \ , \tag{3.4.4}
$$

[48] For the remainder of the text we call the total space the higher-dimensional space the space with coordinate r the lower-dimensional space and the space with coordinate z the transverse space.

and our augmented Green functions are

$$\left(\Delta_r - \omega^2\right) H^\omega(r-t) = \frac{\delta(r-t)}{m(r)} \ , \qquad \left(\Delta_z - \sigma\right)^2 K^\sigma(z\,,\ s) = \frac{\delta(z-s)}{\mu(z)} \ . \tag{3.4.5}$$

These are, of course, related through the formula for our augmented Green function (equation (3.2.13)). All of these relate to the total Green function via the diagram:

$$\begin{tikzcd}[column sep=tiny]
 & -\dfrac{g_\sigma(r)g_\sigma(t)f_\omega(z)f_\omega(s)}{\sigma^2+\omega^2} \arrow[dl] \arrow[dr] & \\
= H^\omega(r-t)f_\omega(z)f_\omega(s) \arrow[dr] & & = g_\sigma(r)g_\sigma(t)K^\sigma(z\,,\ s) \arrow[dl] \\
 & = G(r\,,\ t\,,\ z\,,\ s)\ . &
\end{tikzcd}$$

Figure 5: The relationship between Sturm–Liouville bases and higher-dimensional Green functions on a product space.

3.4.2 Laplace Transformations and Odd-Dimensional Real Space

In section 2.6.2 we found the general solution for the S-wave Laplace equation in $\mathbb{R}^n$. Recall that Bessel functions of the first type are the regular solutions, and Bessel functions of the second type are the fundamental solutions. In odd dimensions these simplify significantly. That is, in odd dimensions our regular solutions (Bessel functions of the first type) are polynomial functions of world-volume radius multiplied by an exponential decay.[49]

We consider a product space such as $\mathcal{M} = \mathbb{R}^{2n+1} \times_W \mathcal{M}_z$ then apply the equation in figure (5). If we integrate over the spectrum of real space first, setting $t = 0$, we find our Green functions become

$$G(r\,,\ z\,,\ s) = \frac{1}{V_{2n} r^{2n-2}} \int \exp(-\omega r)(a + b\,\omega\, r + c\,\omega^2 r^2 + \ldots) f_\omega(z) f_\omega(s) d\omega \ . \tag{3.4.6}$$

Here a, b, c, etc. are the relevant triangle coefficients of the Bessel polynomials [61].

Therefore we understand that our Green function is given by a Laplace transformation of the Sturm–Liouville basis of our transverse space [61]. Furthermore, since

$$\int_0^\infty \exp(-\omega r)\omega^{\alpha-1} d\omega = \Gamma(\alpha) r^{-\alpha} \ , \tag{3.4.7}$$

[49]There is a similar generalization for even dimension as well. However, we are principally interested in the case of $\mathbb{R}^3$ for physical reasons, therefore we only present this simple generalization for the sake of brevity.

we expect each of these terms beyond the zeroth-order (a in equation (3.4.6)) to be subdominant to the fundamental solution of the Laplace equation in $\mathbb{R}^{2n+1}$. For instance, in three real-space dimensions the fundamental solution is $\frac{1}{r}$ and these Laplace transforms generically begin at $\mathcal{O}\left(\frac{1}{r^2}\right)$.

3.4.3 Long Distance Mirrors

Understanding how Green functions, with different boundary conditions but on the same product space, are related provides an interesting window into the relationship between special and general boundary conditions.

Suppose we know one Green function on a product space exactly,[50] $G^K(r\,,\ t\,,\ z\,,\ s)$, and we want to know a separate Green function on the same space which obeys different boundary conditions, $G^T(r\,,\ t\,,\ z\,,\ s)$. Further, for illustration, we will consider that our Green functions obey the same upper boundary condition, but different boundary condition at the lower boundary of our space, $z \to l^+$ and all r. That is we will have some operator O^l which annihilates G^T at the lower boundary

$$O^l G^T(r\,,\ t\,,\ z\,,\ s)\Big|_{z\to l^+} = 0\ . \tag{3.4.8}$$

We may always define an interpolating function, $F^T(r\,,\ t\,,\ z\,,\ s)$, such that

$$G^T(r\,,\ t\,,\ z\,,\ s) = G^K(r\,,\ t\,,\ z\,,\ s) + F^T(r\,,\ t\,,\ z\,,\ s)\ . \tag{3.4.9}$$

Since both G^T and G^K solve the sourced Laplace equation, that is

$$\Delta G^T(r\,,\ t\,,\ z\,,\ s) = \Delta G^K(r\,,\ t\,,\ z\,,\ s) = \frac{\delta(r-t)\delta(z-s)}{m(r)\mu(z)}\ , \tag{3.4.10}$$

F^T will solve the unsourced Laplace equation, that is

$$\Delta F^T = \Delta\left(G^T - G^K\right) = 0\ . \tag{3.4.11}$$

Furthermore since G^T obeys special boundary conditions and G^K obeys different boundary conditions F^T will obey generic boundary conditions

$$O^l F^T\Big|_{z\to l^+} = O^l\left(G^T - G^K\right)\Big|_{z\to l^+} = -O^l G^K\Big|_{z\to l^+} \neq 0\ . \tag{3.4.12}$$

[50]As we will argue in section 3.4.5, we actually only need known this Green function approximately. However, for illustration we say this is the 'known Green function.'

Similarly, we may define $f^{\omega}(r-t)$ as

$$f^{\omega}(r\,,\;t\,,\;s)=\int_{\mathcal{D}}\mu(z)f_{\omega}(z)F^{T}(r\,,\;t\,,\;z\,,\;s)dz\;. \tag{3.4.13}$$

Here f_{ω} are the eigenfunctions of Δ_z which obey our boundary conditions defined by O^l, that is $O^l f_{\omega}(z)\big|_{z\to l^+}=0$. Given this definition we apply our full Laplacian and simplify as

$$\begin{aligned}
&\int_{\mathcal{D}}\mu(z)f_{\omega}(z)\Delta F^{T}(r-t,\,z\,,\;s)dz\\
&=\Delta_r\int_{\mathcal{D}}\mu(z)f_{\omega}(z)F^{T}(r-t,\,z\,,\;s)dz+\int_{\mathcal{D}}\mu(z)f_{\omega}(z)\Delta_z F^{T}(r-t,\,z\,,\;s)dz\\
&=\Delta_r f^{\omega}(r-t,\,s)+\int_{\mathcal{D}}\mu(z)F^{T}(r-t,\,z\,,\;s)\Delta_z f_{\omega}(z)dz\\
&\qquad+\mu(z)\left(f_{\omega}(z)\partial_z F^{T}(r-t,\,z\,,\;s)-F^{T}(r-t,\,z\,,\;s)\partial_z f_{\omega}(z)\right)\Big|_{z\to l^+}\;.
\end{aligned} \tag{3.4.14}$$

Furthermore if we suppose $O^l=\partial_z+b$, then we may simplify our boundary term as

$$\begin{aligned}
&\mu(z)\left(f_{\omega}(z)\partial_z F^{T}(r-t,\,z\,,\;s)-F^{T}(r-t,\,z\,,\;s)\partial_z f_{\omega}(z)\right)\Big|_{z\to l^+}\\
&=\mu(z)\big(f_{\omega}(z)\partial_z F^{T}(r-t,\,z\,,\;s)+bf_{\omega}(z)F^{T}(r-t,\,z\,,\;s)\\
&\qquad-F^{T}(r-t,\,z\,,\;s)\partial_z f_{\omega}(z)-bF^{T}(r-t,\,z\,,\;s)f_{\omega}(z)\big)\Big|_{z\to l^+}\\
&=\mu(z)\left(f_{\omega}(z)O^l F^{T}(r-t,\,z\,,\;s)-F^{T}(r-t,\,z\,,\;s)O^l f_{\omega}(z)\right)\Big|_{z\to l^+}\\
&=-\mu(z)f_{\omega}(z)O^l G^{K}(r-t,\,z\,,\;s)\Big|_{z\to l^+}\;.
\end{aligned} \tag{3.4.15}$$

Therefore our Laplace equation on F^T implies the following equation for f^{ω}

$$\Delta_r f^{\omega}(r-t,\,s)-\omega^2 f^{\omega}(r-t,\,s)=\mu(z)f_{\omega}(z)O^l G^{K}(r-t,\,z\,,\;s)\Big|_{z\to l^+}\;. \tag{3.4.16}$$

Therefore, given any sourced product Laplace equation, and any known Green function G^K we may find the Green function which obeys different boundary conditions G^T by solving a series of sourced Laplace equations in one of the product spaces.

3.4.4 Long Distance Mirrors and a Transverse Interval

Consider the Green function on $\mathbb{R}^3 \times (-l, l)$, which solves[51]

$$\left(\partial_r^2 + \frac{2}{r}\partial_r + \partial_z^2\right) G^T(r,z) = \frac{\delta(r)\delta(z)}{4\pi r^2} \; . \tag{3.4.17}$$

If the transverse interval is infinitely large ($l \to \infty$), then the exact solution, which is Dirichlet in all directions at infinity, is given by [18]

$$G^K(r,z) = -\frac{1}{4\pi^2\left(r^2+z^2\right)} \; . \tag{3.4.18}$$

On a transverse interval, however, we require different boundary conditions. One possible set of boundary conditions which trivializes this problem is

$$G^K(r,z)\Big|_{z\to\pm l} = -\frac{1}{4\pi^2\left(r^2+l^2\right)} \; . \tag{3.4.19}$$

We are, however, more interested in special Neumann conditions. That is,

$$\partial_z G^T(r,z)\Big|_{z=\pm l} = 0 \; . \tag{3.4.20}$$

The zero modes of relevant bases are given in section 2.6.1. Defining $F^T = G^T - G^K$ and expanding in the Neumann-Neumann basis we have

$$F^t(r,z) = \frac{1}{\sqrt{2l}} f^0(r) + \frac{1}{\sqrt{l}} \sum_{n\neq 0} \sin\left(\frac{\pi n}{2l} z + \frac{\pi}{4}\left(1+(-1)^n\right)\right) f^n(r) \, . \tag{3.4.21}$$

Note, we have chosen to label our functions f^n by the number of their eigenvalue, as opposed to the actual value of the eigenvalue.

Following the analysis of the previous section (equation (3.4.16)) we have

$$\left(\partial_r^2 + \frac{2}{r}\partial_r\right) f^0(r) = \sqrt{\frac{l}{2}} \frac{1}{\pi^2\left(r^2+l^2\right)^2} \; , \tag{3.4.22}$$

$$\left(\partial_r^2 + \frac{2}{r}\partial_r - \frac{\pi^2 n^2}{4l^2}\right) f^n(r) = 0 \; , \quad n \text{ odd} \, , \tag{3.4.23}$$

$$\left(\partial_r^2 + \frac{2}{r}\partial_r - \frac{\pi^2 n^2}{4l^2}\right) f^n(r) = -\frac{1}{\sqrt{l}} \frac{4l}{4\pi^2\left(r^2+l^2\right)^2} \; , \quad n \text{ positive, even} \, , \tag{3.4.24}$$

[51] For the sake of brevity we take $t = 0$, $s = 0$.

for our zero mode, odd modes, and even modes ($n > 0$), respectively. For regularity at $r = 0$ and for vanishing of F^T at infinity, we impose Neumann-Dirichlet boundary conditions on the modes,

$$\partial_r f^n(r)\Big|_{r=0} = 0 \,, \qquad f^n(r)\Big|_{r\to\infty} = 0 \,. \tag{3.4.25}$$

Of greatest interest is the zero mode. The solution to (3.4.22) obeying the above boundary conditions is

$$f^0(r) = -\sqrt{\frac{2}{l}}\frac{\tan^{-1}(\frac{r}{l})}{4\pi^2 r} \,. \tag{3.4.26}$$

From this, we observe that f^0 encodes a lower-dimensional behavior. When $r \to \infty$, one has asymptotically

$$f^0(r) = -\sqrt{\frac{2}{l}}\frac{1}{4\pi r} + \mathcal{O}\left(\frac{1}{r^2}\right) \,. \tag{3.4.27}$$

As $r \to \infty$ the modes f^n with $n \neq 0$ are exponentially suppressed. Therefore, in the large r regime, the zero mode, f^0, encodes the leading behaviour of the full solution $F^T(r,z)$ [43]. Recalling the relation (3.4.9) between the homogeneous solution $F^T(r,z)$ and the full Green function $G^T(r,z)$, we find that for large r,

$$G^T(r,z) = \frac{1}{\sqrt{2l}}f^0(r) + \mathcal{O}\left(\frac{1}{r^2}\right) = -\frac{1}{4\pi l}\frac{1}{r} + \mathcal{O}\left(\frac{1}{r^2}\right) \,. \tag{3.4.28}$$

3.4.5 Long Distance Mirrors, Real Space, and Mass Gaps

With a few general arguments based on the long distance mirror technique we can may actually find one additional condition that the transverse Laplacian for the higher-dimensional Green function to agree with the lower-dimensional Green function in r space. Not only must we have a transverse zero mode but the sourcing for that zero mode from any other 'mirror source,' (that is boundary term generated by integration by parts in terms of G^K) must converge to zero 'quickly enough' that the lower-dimensional fundamental solution remains dominant at world-volume infinity. To properly quantify the meaning of 'quickly enough,' we may scan the space of sourced $\mathbb{R}^{\alpha+1}$. However, first let us understand how, given a zero-mode, this 'quickly enough' condition arises.

First, we note that, the only context in the presence of a non-compact (i.e. infinite volume) transverse space which permits a zero mode is when that zero mode obeys special Robin conditions. Therefore only the non-derivative 'mirror terms' matter. That is, in the sourcing for the zero-mode's equation, following equation (3.4.16), we have

$$\Delta_r f^0(r) \propto G^K(r,\, Z) + \text{subleading} \,. \tag{3.4.29}$$

To expound further, the mirror terms on the right hand side will comprise of G^K near its singularity multiplied by some function of Z and derivatives of G^K with respect to z multiplied by some function of z and G^K. Since derivatives generically decrease the order of a function (except for superexponential functions which we disregard), and the only r dependence is through G^K we have

$$G^K(r,\, z\,,\ s) \gg \partial_z\, G^K(r,\, z\,,\ s)\ . \tag{3.4.30}$$

Restated, the non-derivative mirror terms dominate.

Next, we note that any positive-definite sourcing on the right hand side of f^0's equation will excite effectively lower-dimensional behavior. Consider the example of a step function source in three world-volume spacial dimensions

$$\left(\partial_r{}^2 + \frac{2}{r^2}\right) f^0(r) = \theta\left(\frac{1}{2} - r\right) \qquad \Rightarrow \qquad f^0(r) = \left(\frac{r^2}{6} - \frac{1}{8}\right)\theta\left(\frac{1}{2} - r\right) - \frac{1}{24r}\theta\left(r - \frac{1}{2}\right)\ . \tag{3.4.31}$$

Regardless of the form of a source, unless it is oscillatory, we will always have either effectively lower-dimensional behavior or behavior dominant to that away from the source.

Therefore, when $r \gg 1$ we require only

$$\Delta_r f^{\text{out}}(r) = \frac{k}{r^{\alpha-1}} + \text{subleading}\ . \tag{3.4.32}$$

If we suppose that $G^K \sim \frac{1}{r^\beta}$, then we have

$$\left(\partial_r{}^2 + \frac{\alpha}{r}\partial_r\right) f^{\text{out}}(r) = \frac{1}{r^\beta} \qquad \Rightarrow \qquad f^{\text{out}}(r) = \frac{k}{r^{\alpha-1}} + \frac{1}{(2-\beta)(\alpha-\beta+1)r^{\beta-2}}\ . \tag{3.4.33}$$

Therefore we require $\beta > \alpha + 1$. Or that

$$G^K(r,\, z\,,\ s) \prec \frac{1}{r^{\alpha+1}}\ . \tag{3.4.34}$$

That is, when this bound is reached G^T will have additional effects at leading order.

4 Taxonomy of Brane Gravity Localizations

In this section we will develop a taxonomy of different solutions to product space Green functions based on the properties of the transverse space [51]. The overall motivation for this taxonomy is to catalogue what effective lower-dimensional gravitational physics might arise in spacetimes with differing transverse spaces. There are three methodologies for finding the lower-dimensional effective physics from a higher-dimensional system.

For the first type of solution in our taxonomy, we may, by fiat, ignore almost all dependence on the transverse dimensions. Specifically we state that the transverse dependence of the perturbation of our background fields will be a warped constant in the transverse direction and any tensors (such as the metric or form fields) with components in the transverse directions are unperturbed.

For our second type of solution, we we may insist on observing only perturbations that both couple to a truly isolated (or nearly isolated) source(s) in the higher dimension and are regular at all other points in the full spacetime. When the transverse space has infinite volume all such perturbations are dominated by the 'native' solution in the higher dimension. This is similar to the relation of G^K and G^T in section 3.4.4, whic are asymptotically $\frac{1}{r}$ and $\frac{1}{r^2}$, respectively

For our third type of solution, we may either insist that our transverse space is finite due to periodicity, or otherwise impose boundary conditions. As we shall argue in our section on Kerr–Schild perturbations [79], these are in some respect similar requirements from the perspective of graviton self-interaction [41,42]. If we consider solutions on the universal cover of the transverse space, periodic solutions are simply the interaction of infinitely many 'colinear' point charges, or 'sandwiching the charge between two mirrors', while other boundary conditions are the presence of similar boundary objects.

For the sake of completeness for our taxonomy we will define a fourth type of solution. In these spaces the properties of the transverse Laplacian require we consider a finite transverse domain, but otherwise they are comparable to our third type of solution. We do not have any examples that correspond to this type, so we will only mention the possibility briefly.

However, we will find that, in the case of an infinite volume transverse space, only when our boundary conditions allow for non-constant zero mode may we have lower-dimensional effective physics which agrees with the 'native' lower-dimensional effective physics. Furthermore we will find that, for the transverse measures with a supergravity origin considered in this work, the universal cover of the transverse space always has infinite volume.

To make our argument we will first analyze solutions with a maximally symmetric world-volume on a warped product background, then specifize this case to a four-dimensional Minkowski world-volume. However, as we argued in section 3.4.2, these arguments extend to arbitrary even real dimensions, and with a careful study of integrals of Bessel functions, one may extend our arguments here to odd dimensions with $d > 3$. We expect the generic structure of our taxonomy to hold for non-Minkowski world volumes.

To give the summary at the top, see table 1, where we summarize the types of solutions, their boundary conditions, possible volumes for the transverse space, the presence of zero modes, the presence of a mass gap, an overview of the lower-dimensional EFT (specifically if it is massless/massive/stable and whether there is any symmetry breaking), and finally give several known examples.

Table 1: Taxonomy of Brane Gravity Localizations

Type	BC	Volume	Zero Modes	Mass Gap	EFT	Example
Type I	N/A	Infinite	Constant	N/A[52]	Uncoupled[53]	SS-CGP[54]
Type II	N-N[55]	Infinite	None	N/A	Massive	$\mathbb{R}^{1,4} \to \mathbb{R}^{1,3}$
Type II*	N-N	Finite	Constant	Yes	Massless	Unknown
Type III	R-R[56]	Either	None	N/A	Massive	SS-CGP[57]
Type III*	N-N	Finite	Constant	No	Massless	RS II
Type III†	R-R[58]	Either	Nonconstant	Yes	Massless CSB	CPS
Type III$^\flat$	R-R	Either	Nonconstant	Unknown	Unstable	Unknown
Type IV(All)	As III	As III	As III	As III	As III	Unknown

4.1 World Volume Gravitons on Product Spaces

The basis of our taxonomy is a study of the higher-dimensional scalar wave equation. We start by noticing, according to Bachas and Estes, the higher-dimensional Einstein equations allow for a lower-dimensional graviton when the transverse space contains a zero mode [10]. The assumptions necessary to isolate this graviton's wave equation precluded sourced gravitational solutions [67]. However, in all cases where we have relaxed the assumptions the leading component of the full solution is still determined by the graviton's wave equation [51]. This is consistent with other studies of perturbations of all gravitation degrees of freedom in warped product backgrounds [46].

Let us consider the physics of gravitational perturbations (in fact arbitrary perturbations) around a warped maximally symmetric product space $\mathcal{M}_H = \mathcal{M}_l \times_W \mathcal{M}_t$. Here $\mathcal{M}_l$ is the maximally symmetric

[52] Higher modes are explicitly excluded.
[53] Zero mode cannot couple to bulk matter.
[54] To 6-dimensional Salam–Sezgin Supergravity
[55] Neumann-Neumann
[56] Robin-Robin
[57] General
[58] Generally, sometimes Robin-Dirichlet

product space and our metric is [40]

$$ds^2 = \exp\left(2A(Z)\right) g_{\mu\nu}(x) dx^\mu dx^\nu + g_{ab}(Z) dZ^a dZ^b \ , \tag{4.1.1}$$

where X^M are the coordinates and indices on our total space, x^μ are the coordinates and indices on a maximally symmetric space such as $\mathbb{R}^{1,3}$ or AdS, and Z^a are the coordinates and indices on the tranvserse space which need not be compact. When we have precisely one non-compact coordinate we will call it z.

We of course may also consider the product metric [72] related to our metric via Weyl transformation [128]

$$g_{MN} = \exp\left(2A(Z)\right) \overline{g}_{MN} \ , \tag{4.1.2}$$

$$\overline{g}_{MN} dX^M dX^N = \overline{g}_{\mu\nu}(x) dx^\mu dx^\nu + \exp\left(-2A(Z)\right) g_{ab}(Z) dZ^a dZ^b = \overline{g}_{\mu\nu}(x) dx^\mu dx^\nu + \overline{g}_{ab}(Z) dZ^a dZ^b \ . \tag{4.1.3}$$

Here we have the Riemann tensor associated with the Levi–Civita connection of $\overline{g}_{\mu\nu}$ which obeys [82]

$$\overline{R}_{\mu\nu\rho\sigma} = k\left(\overline{g}_{\mu\rho}\overline{g}_{\nu\sigma} - \overline{g}_{\mu\sigma}\overline{g}_{\nu\rho}\right) \ . \tag{4.1.4}$$

Here $k = -1,\, 0,\, 1$ for anti-de Sitter, Minkowski, or de Sitter space, respectively [39]. We can relate the Einstein, Riemann, and Ricci tensors [126], and Ricci scalar associated with the Levi–Civita connections of g_{MN} (∇_M) and $\overline{g}_{MN}$ ($\overline{\nabla}_M$) via the relation [111]

$$\begin{aligned} G_{MN} =& \overline{R}_{MN} - \left(\overline{\nabla}^2 A\right)\overline{g}_{MN} - (D-2)\left(\overline{\nabla}_M\overline{\nabla}_N A - \left(\overline{\nabla}_M A\right)\left(\overline{\nabla}_N A\right) + \left(\overline{\nabla} A\right)^2 \overline{g}_{MN}\right) \\ &- \frac{1}{2}\overline{R}\,\overline{g}_{MN} + (D-1)\left(\overline{\nabla}^2 A\right)\overline{g}_{MN} + \frac{1}{2}(D-2)(D-1)\left(\overline{\nabla} A\right)^2 \overline{g}_{MN} \ . \end{aligned} \tag{4.1.5}$$

Here D is the dimension of the total spacetime.

If we consider the $\mu\nu$ components of the Einstein equations we may apply the independence of $A(Z)$ from our world-volume coordinates to simplify the left-hand side.

$$G_{\mu\nu} + \Lambda g_{\mu\nu} = \overline{R}_{\mu\nu} - \frac{1}{2}\overline{R}\,\overline{g}_{\mu\nu} + (D-2)\left(\left(\overline{\nabla}^2 A\right) + \frac{1}{2}(D-3)\left(\overline{\nabla} A\right)^2\right)\overline{g}_{\mu\nu} + \Lambda \exp(2A)\,\overline{g}_{\mu\nu} = \frac{\kappa^2}{2} T_{\mu\nu} \ . \tag{4.1.6}$$

Here Λ is our cosmological constant and T_{MN}, our energy-momentum tensor, is given by the variation of our matter Lagrangian density [15]

$$T_{MN} = \frac{2}{\sqrt{-g}} \frac{\partial}{\partial g^{MN}} \left(\sqrt{-g}\mathcal{L}_{\text{Matter}}\left(\sqrt{-\det(\overline{g}_{\mu\nu})},\ \overline{g}_{mn},\ A,\ \Phi\right)\right) \ . \tag{4.1.7}$$

Here we have given the most general form of $\mathcal{L}_{\text{Matter}}$, given that it is invariant under transformation which respect the maximal symmetry of our world-volume [10], which implies that it can only be dependent on $\overline{g}_{\mu\nu}$ through its determinant. Here all other dependence is on the warp factor (A), the transverse metric ($\overline{g}_{mn}$), and other higher-dimensional fields Φ,[59] which themselves must respect the world-volume's maximal symmetry.

If we calculate $T_{\mu\nu}$ explicitly we have

$$T_{\mu\nu} = \exp(2A) \left(\mathcal{L}_{\text{Matter}} + 2\sqrt{-\det(\overline{g}_{\mu\nu})}\mathcal{L}^{(1,\,0,\,0)}_{\text{matter}}\right)\overline{g}_{\mu\nu} = \mathcal{T}\left(\sqrt{-\det(\overline{g}_{\mu\nu})},\, A,\, \overline{g}_{mn}\right)\overline{g}_{\mu\nu}\ . \qquad (4.1.8)$$

Here the essential detail is that $\mathcal{T}$, which is given by tracing over the world-volume components of $T_{\mu\nu}$, captures all behavior of the world-volume components of $T_{\mu\nu}$, and that it only depends on the world-volume components of $\overline{g}_{MN}$ through the trace.

Tracing the world-volume components of our Einstein equations we find[60]

$$\left(1-\frac{d}{2}\right)\overline{R}_{\mu}{}^{\mu} - \frac{d}{2}\overline{R}_{m}{}^{m} + d(D-2)\left(\left(\overline{\nabla}^2 A\right) + \frac{1}{2}(D-3)\left(\overline{\nabla} A\right)^2\right) + d\Lambda\exp(2A) = d\frac{\kappa^2}{2}\mathcal{T}\ . \qquad (4.1.9)$$

Here d is the dimension of our world-volume, and $\overline{R}_{\mu}{}^{\mu}$ is the Ricci scalar associated with $\overline{g}_{\mu\nu}$, which is the same as $\overline{R}_{\mu\nu}\overline{g}^{\mu\nu}$ because of $\overline{g}_{MN}$'s block diagonal form. Similarly $\overline{R}_{m}{}^{m}$ is the Ricci scalar associated with $\overline{g}_{mn}$. By maximal symmetry we have $\overline{R}_{\mu}{}^{\mu} = k(d-1)d$. We further simplify our Einstein equations as

$$-\frac{1}{2}(d-2)(d-1)\,k - \frac{1}{2}\overline{R}_{m}{}^{m} + (D-2)\left(\left(\overline{\nabla}^2 A\right) + \frac{1}{2}(D-3)\left(\overline{\nabla} A\right)^2\right) + \Lambda\exp(2A) = \frac{\kappa^2}{2}\mathcal{T}\ . \qquad (4.1.10)$$

We now consider the following perturbation of our metric

$$\hat{g}_{\mu\nu}(x,\, z)dX^{\mu}dX^{\nu} = \left(\overline{g}_{\mu\nu}(x) + \mathcal{H}_{\mu\nu}(x,\, Z)\right)dX^{\mu}dX^{\nu}\ , \qquad (4.1.11)$$

$$g_{MN}dX^{M}dX^{N} = \exp(2A(Z))\hat{g}_{\mu\nu}(x,\, Z)dx^{\mu}dx^{\nu} + g_{ab}(x,\, Z)dZ^{a}dZ^{b}\ , \qquad (4.1.12)$$

where all other components of the metric and all other fields are left unperturbed. If we insist that $\mathcal{H}_{\mu\nu}$ is transverse with respect to the lower-dimensional Levi–Civita connection and traceless, that is

$$\overline{g}^{\rho\mu}\overline{\nabla}_{\rho}\mathcal{H}_{\mu\nu} = 0\ , \qquad \overline{g}^{\mu\nu}\mathcal{H}_{\mu\nu} = 0\ , \qquad (4.1.13)$$

[59] This may include the embedding coordinates of a brane X^M, etc. This calculation does not require any specific physical meaning for Φ, simply that Φ not explictly depend on $\overline{g}_{\mu\nu}$.

[60] Unless explicitly stated we choose $\overline{\nabla}^2 = \overline{\nabla}_M\overline{\nabla}^M$ and $(\overline{\nabla}A)\cdot\overline{\nabla} = (\overline{\nabla}_M A)\,\overline{\nabla}^M$.

then the world-volume (that is $M = \mu$, $N = \nu$) first-order perturbation (eliminating all terms proportional to a transverse derivative of the trace of our metric perturbation) of our Einstein equation becomes

$$\begin{aligned}\delta\overline{G}_{\mu\nu} = &-\frac{1}{2}\left(\overline{\nabla}^2 + (d-2)(\overline{\nabla}A)\cdot\overline{\nabla}\right)\mathcal{H}_{\mu\nu} - \overline{R}_{\mu}{}^{\rho}{}_{\nu}{}^{\sigma}\mathcal{H}_{\rho\sigma} + \overline{R}_{(\mu}{}^{\sigma}\mathcal{H}_{\nu)\sigma} \\ &-\frac{1}{2}\overline{R}_m{}^m\mathcal{H}_{\mu\nu} + (D-2)\left(\left(\overline{\nabla}^2 A\right) + \frac{1}{2}(D-3)\left(\overline{\nabla}A\right)^2\right)\mathcal{H}_{\mu\nu} + \Lambda\exp(2A)\mathcal{H}_{\mu\nu} = \frac{\kappa^2}{2}\mathcal{T}\mathcal{H}_{\mu\nu}\ .\end{aligned} \tag{4.1.14}$$

If we now apply our background equations of motion we find[61]

$$-\frac{1}{2}\left(\overline{\nabla}^2 + (d-2)(\overline{\nabla}A)\cdot\overline{\nabla}\right)\mathcal{H}_{\mu\nu} = 0\ . \tag{4.1.15}$$

We will call this the Bachas and Estes equation, which is the conformal translation of their equation (2.23) in [10]. Furthermore, all other equations of motion are trivially solved with no additional conditions on $\mathcal{H}_{\mu\nu}$.

If we now constrain $\overline{g}_{\mu\nu} = \eta_{\mu\nu}$ and $\mathcal{H}_{\mu\nu}$ to have dependence only on z, our noncompact transverse coordinate, instead of Z^a, all of our transverse coordinates, we have

$$(\Box + \Delta_z)\,\mathcal{H}_{\mu\nu}(x,\, z) = 0\ . \tag{4.1.16}$$

Therefore our metric perturbation solves a scalar wave equation.

Bachas and Estes extend this argument further by considering the kinetic term for our metric perturbation at the level of the action. Our above equation of motion is generated in the bulk by the action[62]

$$\mathcal{S}_{\text{Perturbation, Free}} = \int_{\mathcal{M}_l\times_W\mathcal{D}}\sqrt{-\det\overline{g}_{\mu\nu}}\,\mu(z)\left(-\frac{1}{2}\partial_\sigma\mathcal{H}_{\mu\nu}\partial^\sigma\mathcal{H}^{\mu\nu} - \frac{1}{2}\partial_z\mathcal{H}_{\mu\nu}\partial^z\mathcal{H}^{\mu\nu}\right)d^dx dz\ . \tag{4.1.17}$$

Here we have assumed all contributions from the angular coordinates of Z^a are finite and normalized by the choice of $\mu(z) = \int_{\mathcal{M}_{\text{angular}}}\sqrt{d}^{D-d-1}Z_{\text{angular}}$. Given this, if we assume a separable solution

$$\mathcal{H}_{\mu\nu}(x,\, z) = h_{\mu\nu}(x)\,\psi(z)\ , \tag{4.1.18}$$

then we can integrate over the transverse space to find a lower-dimensional effective field theory ($\mathcal{S}_{\text{Perturbation, Free}} =$)

$$\int_{\mathcal{M}_l}\sqrt{-\det\overline{g}_{\mu\nu}}\left(-\frac{1}{2}\partial_\sigma h_{\mu\nu}\partial^\sigma h^{\mu\nu}\left(\int_{\mathcal{M}_t}\mu\psi^2 d^{D-d}Z\right) - \frac{1}{2}h_{\mu\nu}h^{\mu\nu}\left(\int_{\mathcal{M}_t}\mu\left(\partial_z\psi\right)^2 d^{D-d}Z\right)\right)d^dx\ . \tag{4.1.19}$$

[61]Our warp factor, A, is generically a non-constant function in the transverse direction, but is not perturbed.
[62]This is, of course, only true up to corrections from the boundary.

Since both of the integrals over the transverse space are of a square, they both must have non-negative values. Furthermore we have

$$\int_{\mathcal{M}_t} \mu\,(\partial_z\psi)^2\, d^{D-d}Z = 0 \qquad \Rightarrow \qquad \partial_z\psi = 0\,. \tag{4.1.20}$$

Therefore we may only have solutions to $\mathcal{H}_{\mu\nu}$'s equation (4.1.16) that correspond to lower-dimensional massless gravitons when $\mathcal{H}$ is constant in the transverse directions, or when the total perturbation is conformally constant in the transverse directions. This implies that no space with a infinite volume transverse space may have a zero mode associated with the lower-dimensional effective field theory described by the action of equation (4.1.17).

Furthermore, when the metric perturbation solves equation (4.1.16) no other fields need be excited. This is since ${\mathcal{H}_\mu}^\mu = \overline{\nabla}^\mu\overline{\nabla}^\nu\mathcal{H}_{\mu\nu} = 0$, therefore there are no scalar or vector quantities sourced by $\mathcal{H}_{\mu\nu}$ that can excite any other field's equation of motion at this order (except perhaps a second spin-2 field in a bimetric theory). Lastly, this argument makes no assumptions about what the remaining field content of the theory. It therefore broadly applies to gravity and supergravity theories with or without branes spanning the transverse dimensions.

4.2 Tracing Out Corrections to World Volume Gravitons

While we will rely on the intuition of the previous section for describing our putative solutions, we will escape some of its more restrictive conclusions by relaxing its more restrictive assumptions.

First, we note that Bachas' and Estes' conclusion that the EFT is effectively described by only the gravitational perturbations because all other fields may be consistently truncated is violated at higher orders. For instance we could have an arbitrary scalar quantity sourced by $\mathcal{H}_{\mu\nu}\mathcal{H}^{\mu\nu}$ at quadratic order.

Second, their conclusion that their higher-dimensional bulk action associated with their higher-dimensional equations of motion determines the lower-dimensional EFT is valid, given a higher-dimensional bulk action with no boundary terms. This is a workable initial assumption, corrections to which compose of much the remaining text of this work. To summarize the main conclusion, however, we may always rely on a Sturm–Liouville decomposition at the level of the equations of motion, but must verify the correspondence of the action and equations of motion in the higher dimension to rely on the technique of integrating over the transverse dimension.

Third, the assumption that $\mathcal{H}_{\mu\nu}$ is transverse and traceless in the lower-dimensional sense has no basis in the higher-dimensional theory. It is valid to point out that such perturbations solve the higher-dimensional theory, but that does not imply that they correspond to valid lower-dimensional effective degrees of freedom.

Understanding corrections which arise from nonlinear interactions in the effective field theory is covered in Sections 6 and 8. Understanding corrections from boundary terms in the higher-dimensional action is covered in Section 6 and specifically for gravity in 7. Understanding corrections from relaxing transverseality or tracelessness is the focus of this section 4.2, the following section4.3 as well a primary focus in Section 5.

4.3 Kerr–Schild Perturbations and Self-Interaction

Do unsourced solutions of the linear order of a gravitational theory represent the leading order of solutions to the entire theory? We will present arguments that the answer to both parts of this question is affirmative.

The first part of this question, the relationship between the sourced and unsourced solutions, we will discuss at linear order in our section on specific solutions (Section (5). However, the question of sourcing gravitational theories at nonlinear order is an open question we will not discuss in higher precision.

The second part of this question, whether the corrections to the theory from interactions qualitatively change the solution is a similarly difficult question. We shall discuss this question in this section.

Let us begin by presenting the Kerr–Schild ansatz, then discuss our intuitive understanding. Here we will use only the total coordinates on our space X^M (we choose the same background as equation (4.1.2)). Consider a perturbation to the total metric [37, 124]

$$\hat{g}_{MN} = g_{MN} - 2\phi k_M k_N \ . \tag{4.3.1}$$

Here $\hat{g}_{MN}$ is the background metric, k_M is a null vector field

$$g_{MN} k^M k^N = 0 \ , \tag{4.3.2}$$

and ϕ is a scalar function. Given this perturbation we have the exact inverse metric[63]

$$\hat{g}^{MN} = g^{MN} + 2\phi k^M k^N \ , \tag{4.3.3}$$

and the perturbed Ricci tensor,

$$\hat{R}^M{}_N = \left(g^{MP} + 2\phi k^M k^P\right) R_{PN} - \nabla^P \nabla^M \left(\phi k_N k_P\right) - \nabla_P \nabla_N \left(\phi k^M k^P\right) + \nabla_P \nabla^P \left(\phi k^M k_N\right) \ . \tag{4.3.4}$$

[63] In principle, all indices are raised and lowered using the background metric g_{MN}, however, in this special case the quantities are identical regardless of which metric we use, since $k^N k_N = 0$.

If we further suppose that k^N is an autoparallelly transported vector field (i.e. a geodesic vector field) with respect to the background connection,

$$k^N \nabla_N k^M = 0 \ , \tag{4.3.5}$$

then equation (4.3.4) is exact and k is autoparallelly transported with respect to the perturbed connection, since

$$k^N \hat{\nabla}_N k^M = k^N \nabla_N k^M - 2k^M k^P k_N \phi \nabla_P k^N = 0 \ . \tag{4.3.6}$$

We shall call all such null vector fields Kerr–Schild vector fields, or just Kerr–Schild vectors, and call them perturbative or exact based on whether they satisfy the autoparallelly transported condition. We call ϕ a Kerr–Schild scalar.

We interpret equation (4.3.4) as an equation on only ϕ, since we began by determining k^N. This k^N defines the 'force lines' of some 'source' which is defined by the singularities of k^N. Once we have defined what these force lines are our choice of ϕ becomes very restricted (as we shall see). That is, once we have found how gravitons propagate in our space we can define exact solutions with either isolated or highly symmetric singularities.

4.3.1 Kerr-Schild Solutions in a Five-Dimensional Minkowski Space

Let us illustrate how solutions with isolated singularities can be made exact and how their leading order of behavior persists with interactions. If we consider $\mathcal{M} = \mathbb{R}^{1,4}$ and

$$ds^2 = -dt^2 + dR^2 + R^2 d\Omega_3{}^2 \ , \tag{4.3.7}$$

one choice of exact Kerr–Schild vector would be [37]

$$k = \partial_t + \partial_R \ . \tag{4.3.8}$$

Given this choice we further restrict our attention to our Kerr–Schild scalar having only radial dependence. We then find our Kerr–Schild scalar obeys two, in principle, independent equations when we solve all Einstein equations (given by the tt and one angular equation as representatives).

$$\left(\partial_R{}^2 + \frac{3}{R}\partial_R\right)\phi = 0 \ , \tag{4.3.9}$$

$$-\frac{2}{R}\left(\partial_R \phi + \frac{2}{R}\phi\right) = 0 \ . \tag{4.3.10}$$

These equations are actually degenerate (a solution to the first equation is manifestly a solution to the second), and select a unique solution (up to scale)

$$\phi \propto \frac{1}{R^2} \ . \tag{4.3.11}$$

Which agrees with one specific solution to the Bachas and Estes equation (4.1.16). To track what occurs at nonlinear order we first state the total line element

$$ds^2 = -\left(1 - \frac{2M}{R^2}\right) dt^2 - \frac{4M}{R^2} dtdR + \left(1 + \frac{2M}{R^2}\right) dR^2 + R^2 d{\Omega_3}^2 \ . \tag{4.3.12}$$

This line element, given the coordinate transformation

$$t = -\sqrt{2M} \ \mathrm{arctanh}\left(\frac{R}{\sqrt{2M}}\right) + \tau \ , \tag{4.3.13}$$

becomes the five-dimensional Scharzschild solution [115]. At long distance we have the same gravitational potential $\phi \sim \frac{1}{R^2}$ as we did for the Bachas and Estes equation (4.1.16).

We note two details. First, the exact solution must obey a condition beyond only solving the radial transverse Laplace equation. Therefore not all solutions to the Bachas and Estes equation are realized with the same Kerr–Schild vector k^M. In the next section we will investigate how this can be generalized. Second, finding the effective gravitational potential ($\frac{2M}{R^2}$) assumes a diagonal metric, which our ansatz violates. However, since the coordinate transformation to remove the off-diagonal ($dtdR$) is linear in t it leaves the potential unchanged. That is, to read off the potential from the Kerr–Schild ansatz we need only confirm that the off-diagonal terms of the metric are time independent.

4.3.2 Kerr–Schild Perturbations on Four-Dimensional Minkowski Cross a Circle

We now change coordinates from the unperturbed problem and write the background line element as

$$ds^2 = -dt^2 + dr^2 + r^2 d{\Omega_2}^2 + dz^2 \ . \tag{4.3.14}$$

Note, our original R component is now given as $R^2 = r^2 + z^2$. We leave the conditions of periodicity in z unstated momentarily. There are two notable Kerr–Schild vectors

$$k_1 = \partial_t + \partial_r \ , \tag{4.3.15}$$

$$k_2 = \partial_t + \frac{r}{\sqrt{r^2+z^2}}\partial_r + \frac{z}{\sqrt{r^2+z^2}}\partial_z \ . \tag{4.3.16}$$

Similar to the above, (assuming ϕ_1 is z independent) we must conclude

$$\phi_1 \propto \frac{1}{r} \ , \tag{4.3.17}$$

$$\phi_2 \propto \frac{1}{r^2+z^2} \ . \tag{4.3.18}$$

We recognize ϕ_1 as corresponding to a four-dimensional Schwarzschild solution, interpreted in five dimensions as a black string, and ϕ_2 is just a restatement of equation (4.3.11). Notably there is no way of enforcing periodicity on ϕ_2 since it obeys (4.3.10). However, we may superpose displaced copies of the singularity of ϕ_2 in the z direction. We consider a series of such solutions given by

$$k^n = \partial_t + \frac{r}{\sqrt{r^2+(z-2n)^2}}\partial_r + \frac{z-2n}{\sqrt{r^2+(z-2n)^2}}\partial_z \ , \tag{4.3.19}$$

$$\phi^n \propto \frac{1}{r^2+(z-2n)^2} \ , \tag{4.3.20}$$

$$\hat{g}_{\mu\nu} = g_{\mu\nu} + \sum_{n=-\infty}^{\infty} \phi^n {k^n}_\mu {k^n}_\nu \ . \tag{4.3.21}$$

Superposing these solutions gives only an approximate solution since $k^n \cdot k^m \neq 0$. However, to perturbative order, we may now construct the solution

$$\begin{aligned} ds^2 = &-\left(1 - 2M\sum_{n=-\infty}^{\infty}\frac{1}{r^2+(z-2n)^2}\right)dt^2 \\ &- 4M\sum_{n=-\infty}^{\infty}\frac{r}{(r^2+(z-2n)^2)^{\frac{3}{2}}}dtdr + \left(1+2M\sum_{n=-\infty}^{\infty}\frac{r^2}{(r^2+(z-2n)^2)^2}\right)dr^2 \\ &- 4M\sum_{n=\infty}^{\infty}\frac{z-2n}{(r^2+(z-2n)^2)^{\frac{3}{2}}}dtdz + 4M\sum_{n=\infty}^{\infty}\frac{r(z-2n)}{(r^2+(z-2n)^2)^2}drdz \\ &\left(1+2M\sum_{n=\infty}^{\infty}\frac{(z-2n)^2}{(r^2+(z-2n)^2)^2}\right)dz^2 + r^2 d{\Omega_2}^2 \ . \end{aligned} \tag{4.3.22}$$

Again, we may read off the potential as explained at the end of the previous section. We may compute the sum analytically to find

$$V = -2M \frac{\pi \coth\left(\frac{1}{2}(\pi r - i\pi z)\right) + \pi \coth\left(\frac{1}{2}(\pi r + i\pi z)\right)}{4r} \ . \tag{4.3.23}$$

Note that this function is manifestly periodic in $z \to z+2$. When $r \gg z$ we have

$$V = -\frac{\pi M}{r} \ , \tag{4.3.24}$$

which agrees with the perturbative finding.

To conclude, we find that we can find force lines corresponding to our solutions to the Bachas and Estes equation given we superpose solutions and sacrifice giving an exact answer. However, we can now estimate the size of the corrections to our solution as being of order $\sim k^n \cdot k^m \phi^n \phi^m$. That is, we know our solution is valid up to self-interaction of the source (or superposed sources) with itself (themselves), which should generically produce sub-leading effects. Furthermore this methodology may be used to find exact solutions when describing extremal objects (i.e. five-dimensional Reissner–Nordström black holes), where the large-r asymptotic behavior of the potential is the same [76].

4.4 The Taxonomy (of Scalar Solutions)

We will now elucidate our taxonomy, which can be understood on two separate levels, the level of what solutions exist to separable partial differential equations, and level of the time-independent solutions to the sourced perturbation of the Einstein equations. We have already argued that this should represent the leading behavior of our higher-dimensional theory (section 4.3).

Since we do not have examples of all possible μ and $\mathcal{D}$ in gravitational theories we will state the taxonomy at the level of solutions to equation (3.4.3) begun in (3.4.6) and define our taxonomy on these Green functions before extending our analysis to gravitational perturbations.

When our world-volume is $\mathbb{R}^{1,3}$ our world-volume Beltrami–Laplace operator is [89]

$$\Box = -{\partial_t}^2 + {\partial_r}^2 + \frac{2}{r}\partial_r + \frac{1}{r^2}\left({\partial_\theta}^2 + \cot(\theta)\partial_\theta + \frac{1}{\sin^2(\theta)}{\partial_\phi}^2\right) = -{\partial_t}^2 + \Delta_r + \frac{1}{r^2}\Delta_{\text{ang}} \ . \tag{4.4.1}$$

Here t is time, r is world-volume radius, θ azimuthal angle, and ϕ is polar angle. We are only interested in r dependence, and we denote $\Delta_r = {\partial_r}^2 + \frac{2}{r}\partial_r$. As derived in the equation in figure (5) we know that time-independent S-waves, solving

$$(\Delta_r + \Delta_z)\, G(r,\, z-s) = \frac{\delta(r)\delta(z-s)}{4\pi r^2 \mu(z)} \ , \tag{4.4.2}$$

are given by

$$G(r,\, z-s) = -\int_0^\infty \int \frac{1}{\omega^2+\sigma^2} \frac{\omega \sin(\omega r)}{r} f_\sigma(z) f_\sigma(s) \ . \tag{4.4.3}$$

However we still have to select boundary conditions on this $G(r, z-s)$ at the boundaries of our transverse space. We will argue that there are several natural choices for these boundary conditions that cause different leading behavior for our functions at large world-volume separation from our source. Cataloguing which of these boundary conditions (which we use to define different types) yield effectively lower-dimensional behavior and what modifications they require for equation (4.4.2) led us to recognize a generic pattern when handling non-compact transverse spaces, which we summarize here.

4.4.1 Type I Solutions

We begin with the exception to our analysis. Type I solutions are not solutions to equation (4.4.2) exactly. However, they arise as the limit of certain Type II-IV solutions and exact solutions to supergravity theories in the context of consistent truncations. In this section we present two interesting relations to the full space's Green function (equation (4.4.2)).

We note that a s, z independent version of our total Green function

$$\left(\left({\partial_r}^2 + \frac{2}{r}\partial_r\right) + \Delta_z\right) g(r) = -\frac{\delta(r)}{4\pi r^2} \ , \tag{4.4.4}$$

is a simple algebraic function[64]

$$g(r) = -\frac{1}{4\pi r} \ . \tag{4.4.5}$$

This $g(r)$ we dub a Type I solution. Generically, the Type I solution is any solution of either of the separated terms (Δ_r or Δ_z) in a separable Laplacian ($\Delta_r + \Delta_z$).

This is unenlightening; however, to relate this to the total Green function we note

$$\int_{\mathcal{D}} \mu(z)\, (\Delta_r + \Delta_z)\, G(r,\, z-s) dz = \frac{\delta(r)}{4\pi r^2} \ . \tag{4.4.6}$$

[64] Here $\mathbb{R}^{1,3}$ is simply used because it allows for an easy example where higher- and lower-dimensional behavior are clearly separated.

Simplifying the left hand side of this expression and equating to equation (4.4.4) we have

$$\Delta_r g(r) = \Delta_r \int_{\mathcal{D}} \mu(z) G(r, z-s) dz + \mu(z) \left(\partial_z G(r,\, z-s)\right)\Big|_{z\to l^+}^{z\to u^-} . \tag{4.4.7}$$

Dependent on our boundary conditions at $z \to l^+,\, u^-$ the boundary contribution will vanish and we will understand that $g(r)$ is given by a smeared source in the transverse space. We call such smeared sources 'radial black spokes.'

Alternatively we can consider the case when our transverse space has a finite volume, or

$$\int_{\mathcal{D}} \mu(z) dz = V < \infty. \tag{4.4.8}$$

Therefore the leading component of $G(r,\, z-s)$ with Neumann conditions at $z \to l^+,\, u^-$ is

$$G(r,\, z-s) = \frac{1}{V} g(r) + \mathcal{O}\left(V\right) \ . \tag{4.4.9}$$

Here the increase in order of the corrections (V) comes from the normalization of the heavy modes (V) and the increased eigenvalue $\left(\omega \sim \frac{1}{V}\right)$.

This allows for a clear explanation of how we might understand Type I solutions in the context of consistent truncations of the transverse modes. Ignoring any tensorial structure, we can understand Type I solutions as the dimensional reduction limit when the volume of the transverse space vanishes, or when the higher Fourier modes become suppressed by a factor of V. Alternatively, if we consider an arbitrary theory and regularize it so that the transverse volume is finite, truncate all transverse dependence, then remove the regulation, we understand that these transverse space independent, Type I, solutions are what remain.

4.4.2 Type II Solutions

Since we may always restrict our domain to some subdomain with finite volume by imposing boundary conditions at some point in the interior of our domain, to properly classify solutions outside of such ansätze we must study what happens when we consider replacing our domain with a larger domain.

That is, consider the measure

$$\mu(z) = (z-1)(z+1) \ , \tag{4.4.10}$$

with the domain $z \in \left(-\frac{1}{2},\, \frac{1}{2}\right)$. If we study the solution at $z \in \left(-1, -\frac{1}{2}\right) \cup \left(\frac{1}{2}, 1\right)$ we encounter singular points[65] above and below our domain. That is our differential equation becomes singular at $z = \pm 1$. As

[65] We consider both $\mu \to \infty$ and $\mu \to 0^+$ as singular.

we have shown, the spectrum of operators is qualitatively different on finite measures and semi-infinite and infinite measures. Therefore to study the more general case we separate problems that cannot be extended to a semi-infinite or infinite domain into Type IV in section 4.4.10.

By contrast to a problem where we encounter singularities, we consider the measure

$$\mu(z) = \frac{1}{\sqrt{z}}\ , \tag{4.4.11}$$

where $z \in (0,\, 1)$. In such a case if we impose that our solutions are Neumann at $z \to 0^+,\, 1^-$, then our eigenvalue equation has a normalizable zero mode. That is

$$\zeta_0(z) = \frac{1}{\sqrt{2}}. \tag{4.4.12}$$

However, the boundary at $z = 1$ can be replaced by a boundary at $z = u \in \mathbb{R}^+$ for any u and the differential equation will not become singular. Furthermore the Green function, or the solution to equation (4.4.2), that obeys Neumann boundary conditions at $z \to 1^-$ is actually: either sourced by a delta function at the image point:

$$(\Delta_r + \Delta_z)\, G(r,\, z-s) = \frac{\delta(r)}{4\pi r^2}\left(\frac{\delta(z-s)}{\mu(z)} + \frac{\delta(z-2+s)}{\mu(2-z)}\right)\ , \tag{4.4.13}$$

or the underlying theory requires a boundary term so that its extrema correspond to such solutions (cf. section 7.2). Solutions where we have imposed boundary conditions other than Neumann conditions at any interior point and other than Dirichlet as $z \to \pm\infty$ we dub Type III and will discuss them momentarily (section 4.4.5).

Given these restrictions we have only one final bifurcation. Either the measure has finite volume or not. In the case where the measure has finite volume

$$\int_{\mathcal{D}} \mu(z) dz < \infty\ , \tag{4.4.14}$$

and we consider a semi-infinite domain we have a constant normalizeable zero mode. We dub these solutions Type II*, and discuss them shortly in section 4.4.4.

4.4.3 Type II, General

All remaining problems we dub generic Type II. That is we have an infinite domain, or a semi-infinite domain and infinite transverse volume. In these cases our transverse Sturm–Liouville problem does not afford a constant transverse zero mode. We may have discrete normalizable transverse modes and we will have a spectrum of delta-distribution normalizeable transverse modes. Therefore our total Green function can be generically written as

$$G(r,\ z-s) = -\sum_{i=0}^{N} \frac{\exp(-\omega_i r)}{2\pi r} f_{\omega_i}(z) f_{\omega_i}(s) - \int_{\frac{\lambda}{2}}^{\infty} \frac{\exp(-\omega r)}{2\pi r} f_{\omega}(z) f_{\omega}(s) d\omega\ . \tag{4.4.15}$$

Here $\lambda \geq 0$ is as in equation (2.2.23). That is, $G(r,\ z-s)$ is composed of discrete asymptotic Yukawa modes [132] and a Laplace transform, as noted in equation (3.4.6). Further, as noted in section 3.4.2, the Laplace transform must always be subdominant to the Green function on $\mathbb{R}^3$. Ergo these solutions must be fundamentally higher-dimensional in their long range behavior.

4.4.4 Type II* Solutions

When the transverse space has finite volume then the transverse basis has a normalizable zero mode, and therefore the long range behavior of such solutions is effectively lower-dimensional. One possible example is when the transverse measure is given as

$$\mu(z) = \frac{\exp(-z)}{\sqrt{\pi z}}\ . \tag{4.4.16}$$

Given this the higher-dimensional solution is, when $r \gg 1$,

$$G(r,\ z-s) = -\frac{1}{4\pi r} - \frac{f_2(z)}{r^2} + \mathcal{O}\left(\frac{1}{r^3}\right)\ . \tag{4.4.17}$$

In these spaces both boundaries have finite proper distance from the source, so our reflecting boundary conditions concentrate the force back into the bulk of the spacetime. This is analogous to a compact extra dimension.

4.4.5 Type III Solutions

All solutions that are effectively lower-dimensional at long range that we have seen are caused by the boundary condition reflecting the effect of the source back into the space. Is this property universal? Is there an alternative?

The answer to both of these questions is yes, and understanding these solutions is inexorably linked to the boundary conditions we impose in the transverse space and where we impose those boundary conditions.

If we take a arbitrary extended space from any Type II example and impose Neumann conditions at points away from any singularities of the transverse measure and at finite values we will always generate a constant normalizable zero mode. We call such solutions Type III*. We note that we need not always impose that the boundaries be at finite values of z, if we chose a space that has finite volume over some semi-infinite domain, but infinite volume over the whole domain, such as for AdS $_5$.

Another alternative is considering a non-constant transverse zero mode. We are especially interested in this case, since it has the potential to produce an effectively lower-dimensional solution even when the transverse space has infinite volume. We call all systems where the transverse space has a non-constant transverse zero modes Type III † solutions.

One final possibility worth mentioning, mostly for the purpose of contrast to the general case, is the case where our transverse spectrum contains negative eigenvalue modes. We will call such cases Type III $^\flat$ solutions, and briefly discuss how such solutions appear from the perspective of the lower dimension.

4.4.6 Type III, General

Disregarding all systems that have transverse zero modes and all systems with negative eigenvalue modes, the general case for Type III solutions is the same as the general case for Type II solutions, that is effectively higher-dimensional.

These spaces may have Dirichlet boundary conditions at finite boundaries, such as the Dirichlet-Dirichlet solution with a transverse interval

$$G(r,\, z) = -\frac{\pi(\operatorname{csch}(\pi(r - iz)) + \operatorname{csch}(\pi(r + iz)))}{8r}\ , \tag{4.4.18}$$

which exponentially decays for large $r \gg 1$.

Alternatively these solutions may simply have an infinite volume transverse space despite the reflection of half of the space back on itself, such as $\mathcal{D} = (0,\, \infty)$, $\mu = 1$ with Neumann-Dirichlet conditions,

$$G(r,\, z - s) = -\frac{1}{2\pi^2}\left(\frac{1}{r^2 + (z - s)^2} + \frac{1}{r^2 + (z + s)^2}\right)\ . \tag{4.4.19}$$

This decays like the standard $(1 + 3)$-dimensional solution (that is, like $\frac{1}{r^2}$) for $r \gg 1$.

4.4.7 Type III * Solutions

All spaces allow for restricting the transverse domain so that both zero modes become normalizable. When, due to an internal boundary condition, we have a normalizable constant zero mode we call the Neumann-Neumann solution a Type III * solution. When we relax our internal boundary condition to a boundary condition at a singularity or $z \to \pm\infty$ we call these Type II * solutions.

The simplest example of at Type III * solution is $(-1,\, 1)$ with $\mu = 1$, where the Neumann-Neumann

Green function is

$$G(r,\, z) = -\frac{\pi \coth(\pi r - i\pi z) + \pi \coth(\pi r + i\pi z)}{8r} \ . \tag{4.4.20}$$

Which asymptotes to an effectively lower-dimensional solution as $r \gg 1$.

4.4.8 Type III† Solutions

If we pick boundary conditions so that our transverse space has a normalizable non-constant zero mode, we call our solution a Type III† solution. These solutions are of special interest, because they are the only type of solution where the transverse space can have infinite volume and we can still see effectively lower-dimensional physics.

These solutions are generically the hardest to give explicitly, since usually the solutions to the eigenvalue equation associated with the boundary conditions which allow for a non-constant transverse zero mode require solving some transcendental equation, such as that of equation (2.6.21). However, we can easily read off the leading component of the solution in the case of the interval with Dirichlet-Robin boundary conditions, which correspond to the solutions of equation (2.6.21) as

$$G(r,\, z - s) = -\frac{1}{4\pi}\frac{1}{r}(z+1)(s+1) + \mathcal{O}\left(\frac{1}{r^2}\right) \ . \tag{4.4.21}$$

We also immediately see a potential difficulty in stating the physical constants in the lower-dimensional effective field theory. Since the solutions all vary over the transverse space the effect on a higher-dimensional particle will change, from a lower-dimensional perspective, when the transverse position of that particle changes. We see several strategies for handling this issue in section 5.5.

4.4.9 Type III$^{\flat}$ Solutions

Many possible selections of boundary conditions allow for transverse zero modes. In these cases the leading component of the higher-dimensional solution will oscillate in the world-volume. This will cause the lower-dimensional effective field theory to become nonunitary, that is, to develop tachyonic degrees of freedom [21].

However, to continue the analysis of the leading component of these solutions we have the example of the transverse interval $\mathcal{D} = (-1,\, 1)$ with an antisymmetric zero mode $f_{(1)}(z) = \sqrt{\frac{3}{2}}z$. In this case we also have a negative eigenvalue mode $f_{(0)}(z) = k\cosh(\omega_0 z)$. Here

$$k = \left(\frac{2\omega_0 + \sinh(2\omega_0)}{2\omega_0}\right)^{-\frac{1}{2}} \ , \tag{4.4.22}$$

and ω_0 is the solution to the transcendental equation

$$\coth(\omega_0) = \omega_0 \ . \tag{4.4.23}$$

That is, $\omega_0 \cong 1.199678640257734\ldots$.

The leading component of the transverse Green function is dominated by the augmented world-volume Green function,

$$\left(\frac{1}{r^2}\partial_r r^2 \partial_r + \omega_0\right) G^{i\omega_0}(r) = \frac{\delta(r)}{4\pi r^2} \ , \tag{4.4.24}$$

multiplied by this transverse function. That is

$$G(r,\, z-s) = -\frac{k^2}{4\pi}\frac{\cos(\omega_0 r)}{r}\cosh(\omega_0 z)\cosh(\omega_0 s) - \frac{3}{8\pi}\frac{1}{r}zs + \mathcal{O}\left(\frac{1}{r^2}\right) \ . \tag{4.4.25}$$

While these create perfectly viable solutions to the Bachas and Estes equation, they will define theories with manifestly unstable vacua.

4.4.10 Type IV Solutions

As previously stated, Type IV solutions are those where the measure either vanishes or diverges at two finite values.[66] Like Type III solutions, Type IV solutions can be divided into cases:

1. General, where there are no normalizable zero modes (and all eigenvalues are positive),
2. *, where the constant zero mode is normalizable,
3. †, where the non-constant transverse zero mode is normalizable,
4. $^\flat$, where there are negative eigenvalues.

We emphasize the difference between our coordinates extending to infinite values and our measure being finite. Perhaps the most interesting example in this category is $\mathcal{D} = \left(0,\, \frac{\pi}{2}\right)$ and $\mu(z) = \tan(z)$. In this case the volume of the transverse space diverges, but the non-constant transverse wave function,

$$\xi_0(z) = \frac{2}{\sqrt{\zeta(3)}}\log\left(\sin(z)\right) \ , \tag{4.4.26}$$

is normalizeable.

The Green function on this space with the boundary conditions that select ξ_0 is an example of a Type

[66]We will give an example of this in principle, but we do not know of any gravitational solutions where the background has this property.

IV † solution, whose leading component

$$G(r,\, z-s) = -\frac{1}{4\pi}\frac{4}{\zeta(3)}\frac{1}{r}\log(\sin(z))\log(\sin(s)) + \frac{1}{r^2}\ , \tag{4.4.27}$$

diverges at $z,\, s \to 0^-$, but vanishes when $z,\, s \to \frac{\pi}{2}^-$.

We include a brief mention of this possibility for completeness; again, we do not know of any such gravitational backgrounds. However, from a mathematical standpoint they emphasize the difference between defining the coordinates on an infinite space, and the space having infinite volume. The volume of the transverse space is independent of the coordinates we choose, and even when we have a preferred set of coordinates, such as those that eliminate the constant term of the Laplacian, they do not define when the transverse space is 'non-compact'.

4.5 Type I: Ricci-Flat Branes and Radial Black Spokes

From our section on the generalizations of the Bachas and Estes equation (4.1.16) we understand that we cannot give full nonlinear solutions to Type III solutions with our present techniques. However, we understand from our section on Kerr-Schild that we can present exact Type I solutions, and even present an understanding of them at the level of the full field theory.

For supersymmetric brane solutions, whether the brane is resolved by transgression or not, it is possible to replace the Ricci-flat worldvolume (or effective worldvolume in the resolved case) by an arbitrary Ricci-flat manifold. This is also true for its transverse space. The metrics of these doubly-Ricci-flat branes are given by equation (4.1.1) where $\exp\left(\frac{2}{a}A\right)$ is a harmonic function on the transverse space. For a flat transverse space this is an example of a brane with Ricci-flat worldvolume as first explored in [24], which is a special case of a branes on branes construction, where one considers a consistent truncation to a supergravity theory on the lower-dimensional worldvolume [87].

It is not difficult to show that the solution (4.1.1), along with its appropriate scalars and fluxes, is supersymmetric provided that $g_{\mu\nu}$ and g_{ab} admit covariantly constant spinors with an appropriate projection condition. This is explored in Appendix A of [51]. From [55], the unique static, Ricci-flat, Lorentzian manifolds that admit covariantly constant spinors is Minkowski. For the transverse space, on the other hand, there are many options other than Euclidean space. In particular, depending on dimension, one can select Calabi–Yau, hyper-Kähler, $G_{(2)}$, or $Spin(7)$ manifolds [64, 70, 109].

We can infer many details of the effective field theory, including the number of preserved supercharges, which is determined by the number of singlets in the decomposition of the representation of the $Spin(D-d)$ spinor with respect to the holonomy group of the transverse space, where $D-d$ is the dimension of the transverse space. A more detailed acount of special holonomy manifolds and their relation to supersymmetry may be found in [58].

A compact Riemannian manifold without boundary only admits a constant solution to the Laplace equation. Therefore we may only study non-trivial A in the context of a non-compact transverse space. Due to the non-compactness of the transverse space, A will generically have a singularity. For example, consider when the transverse space is a conical Calabi–Yau space with metric

$$ds^2(CY) = dR^2 + R^2 ds^2(SE_{D-d-1})\,. \tag{4.5.1}$$

Here SE_{D-d-1} is a $(D-d-1)$-dimensional Sasaki-Einstein manifold with $D-d$ even.[67] If we consider A with only R-dependence, then

$$\exp\left(\frac{2}{a}A\right) = 1 + \frac{k}{R^{D-d-2}}\,. \tag{4.5.2}$$

This is singular at $R = 0$. For generic spaces this implies a curvature singularity of the solution. However, for the M2 and M5-branes, the $R = 0$ singularity is a horizon [3, 44, 63].

We are generically interested in doubly-Ricci-flat branes because they allow for effective gravitational physics on the brane worldvolume. As an example, one possible Ricci-flat worldvolume metric is the Schwarzschild black hole. In isotropic coordinates it is

$$ds^2 = -\left(\frac{1-\frac{M}{r^{d-3}}}{1+\frac{M}{r^{d-3}}}\right)^2 dt^2 + \left(1+\frac{M}{r^{d-3}}\right)^{\frac{4}{d-3}} \left(dr^2 + r^2 ds^2(\mathcal{S}^{d-2})\right)\,, \tag{4.5.3}$$

where d is the worldvolume dimension. This solution, is singular at $r = 0$, ignoring potential singularities due to A. However, this $r = 0$ singularity is 'smeared' everywhere in the transverse space. In the perturbative picture, a doubly-Ricci-flat brane with worldvolume given by equation (4.5.3) can be written as a perturbation of a doubly-Ricci-flat brane with a $\mathbb{R}^{1,d-1}$ worldvolume,

$$ds^2 = \exp(2A)\left(\eta_{\mu\nu}dx^\mu dx^\nu + M\mathcal{H}_{\mu\nu}dx^\mu dx^\nu\right) + g_{ab}(Z)dZ^a dZ^b + \mathcal{O}(M^2)\,, \tag{4.5.4}$$

with

$$\mathcal{H}_{00} = \frac{4}{r^{d-3}}\,, \quad \mathcal{H}_{ij} = \frac{4}{(d-3)r^{d-3}}\delta_{ij}\,, \tag{4.5.5}$$

[67]Taking $SE_{D-d-1} = \mathcal{S}^{D-d-1}$ gives the flat metric on $\mathbb{R}^{D-d}$.

where δ_{ij} is the flat metric on the $\mathbb{R}^{d-1}$ slice of $\mathbb{R}^{1,d-1}$. This perturbation is not traceless; however, it does obey the de Donder gauge

$$\partial^\mu \mathcal{H}_{\mu\nu} - \frac{1}{2}\partial_\nu \mathcal{H}^\rho{}_\rho = 0\,. \tag{4.5.6}$$

The stress tensor that sources this perturbed solution (4.5.4) has the form

$$T_{MN} = M\delta_{M0}\delta_{N0}f(Z)\frac{\delta(r)}{r^{n-2}}\,, \tag{4.5.7}$$

where $f(Z)$ is a smooth function on the transverse space. From the delta function structure of the stress tensor, we observe that the source of this solution is not localized in the higher dimension, but is spread out radially like a spoke.

4.6 Type II: Linearized Isolated Sources

If we source our theory at an isolated point then finding the exact solution becomes much more difficult. This can be explained in two related ways. The independent components (the brane and the black spoke) of the source of Type I solutions are generically invariant under the diffeomorphisms of the world-volume and the transverse space, respectively. A source point source breaks both symmetries simultaneously. Alternatively we may compare our theory to Kerr–Schild perturbations. While either source in Type I solutions can be described by a Kerr–Schild perturbation where the Kerr–Schild vector is dependent only on one of our constituent manifolds[68] the source to a Type II solution must vary in both spaces.

Significant quantities of ink have been spilled over considering the question of when the solutions we would identify as Type II solutions are stable at higher order, when they meaningfully approximate different physical phenomena, and how they may be approximated. Our goal in identifying them here is simply to restrict their possible asymptotic behavior purely from the study of the relevant Green function. However, the comparison to Kerr–Schild perturbations allows us to grasp some level of control of these solutions. First, we can understand the near field behavior of our solution. Second, we can intuitively see why such solutions will generically not give lower-dimensional behavior.

4.6.1 Near Field Behavior

To this first point, if we choose any arbitrary point in our spacetime we can freely construct a coordinate system given by the exponential map at that point. More formally we can construct Fermi normal coordinates at that point. In appendix C we demonstrate the generic algorithm for giving Fermi normal coordinates

[68] Therefore so is the Kerr–Schild scalar, but this is non-obvious and perhaps only generically true.

and we apply this algorithm for the spacetime of the dimensionally reduced Salam–Sezgin-Cvetic–Gibbons–Pope background. The essential argument is that at any (nonsingular) point our spacetime we can give coordinates in terms of the geodesic distance from that point. These coordinates are manifestly Kerr–Schild vectors. Moreover if parallel transport along the time-like coordinate of our Fermi normal is a geodesic curve at any point, our metric approximates the Minkowski metric at that point, therefore our Laplacian at that point approximates the flat Laplacian. From this we may approximate that the near field limit of a higher-dimensional point source.

This is perhaps no great innovation in physics, but it does allow us to see another route for corrections to our solutions. Since our time coordinate must be a geodesic, it will generically oscillate, and in a background with a brane source, it will generically be attracted to the brane and cause the brane itself to oscillate [56]. Note this is a generic property and we do not state that one can never have a stationary black-hole solution in the presence of a brane.

4.6.2 Far Field Behavior

To our second point, we know that the force lines of our solution will generically be geodesics away from that point. In these solutions all of these force lines must end on our source. If we consider a world-volume slice of our spacetime containing the source there are three possibilities. The forcelines can diverge away from our world-volume slice faster than linearly, in which case we expect exponential supression of gravity in the lower dimension (associated with a mass gap and no zero mode). The forcelines can diverge away from our world-volume slice linearly, in which case we expect an approximately massless higher-dimensional behavior ($\frac{1}{r^2}$ compared to $\frac{1}{r}$). The forcelines can concentrate towards the world-volume slice, which might lead to lower-dimensional behavior.

We note that, excluding Type II* solutions, we know that the effective behavior must be higher-dimensional. Therefore we can associate our last case with a finite volume transverse space.

The limitation of this speculation is that we cannot, at the level of the Kerr–Schild ansatz, rule out that the potential along our force-lines does not have some scalar dependence that invalidates our estimation of the far field potential. However, the comparison we want to draw, is between our understanding of world-volume gravitons, as per the Bachas and Estes equation, and the leading order of the full solution.

4.7 Type III: Linearized Source-Brane Interactions

By comparison, our boundary conditions in Type III solutions allow us to concentrate the effect of the source back into the spacetime. If we consider the example of AdS$_5$, we can calculate the leading order of the Type

II solution. The Green function associated with the Bachas and Estes operator is [51]

$$\left(\left({\partial_r}^2+\frac{2}{r}\partial_r\right)+\left({\partial_z}^2-\frac{3}{z}\partial_z\right)\right)G(r,z-s)=\frac{\Lambda^3 z^3\delta(r)\delta(z-s)}{4\pi s^3 r^2}\ . \tag{4.7.1}$$

We can verify that this has no associated zero modes for $z\in\mathcal{D}=(0,\infty)$. We can verify our assertion that this should approximate near-field limit of five-dimensional flat-space solution. That is, nearby to the source we have

$$G^K(r,z-s)=-\frac{\Lambda^3}{2\pi^2}\frac{1}{r^2+(z-s)^2}+\mathcal{O}\left(\frac{1}{\sqrt{r^2+(z-s)^2}}\right)\ . \tag{4.7.2}$$

Far from our source ($r>>1$ and $z=s$) we have

$$G^K(r,\,0)=-\frac{15\Lambda^3 s^7}{4\pi}\frac{1}{r^7}+\mathcal{O}\left(\frac{1}{r^8}\right)\ . \tag{4.7.3}$$

However, if we demand that out solution obeys Neumann conditions at $z=\frac{1}{k}$ then we find there is a transverse zero mode. Using our long-distance mirror technique we find that our related Green function G^T has the same short-range behavior and, when $r>>0$ we have

$$G^T(r,0)=-\frac{\Lambda^3 k}{2\pi r}+\mathcal{O}\left(\frac{1}{r^2}\right)\ . \tag{4.7.4}$$

The dynamics of this solution has been discussed at length [54, 129], and there are many problems to be solved and details to be studied about whether these solutions are stable, whether they are stationary, etc. However from our next perspective (in section 8) all such Type III * are consistent truncations, at least until you include additional degrees of freedom beyond the metric. Whereas when we consider Type III † solutions at the level of the field theory these are not inconsistent truncations.

However, these solutions require a more subtle treatment of what precisely the higher-dimensional theory is. In the case of the Randall–Sundrum II model, or the standard Kaluza–Klein reduction, we can view our theory as a perturbation about a background with features which compactify the higher-dimensional space. In the case of Randall–Sundrum II it is the brane by which they stitch the spacetime together, in Kaluza–Klein reductions it is the fact that one dimension is circular. The perturbations around these spaces are regular everywhere once this additional requirement is created.

Type III † reductions are, by necessity, not regular at some point, either at the boundary of the higher-dimensional space or at the interior or the higher-dimensional space. As we will explore in sections 6, 8, and 9, they require some novel boundary term, or additional source. Continuing the anology with AdS $_5$, when Randall and Sundrum reflected the spacetime at their fold they created a cusp in the warp factor of their background solution. Type III * solutions just change the world-volume metric underneath such a cusp. Type III † solutions create a cusp, and therefore either require an additional source or boundary term. For the sake of making the most favorable comparison to existing work, we will always consider this additional cusp to be caused by a boundary term which vanishes when the perturbation vanishes.

5 Species of Brane Gravity Localizations

5.1 The Geometry of the Salam–Sezgin Lift

The uplift of the Salam–Sezgin $\mathbb{R}^{1,3}\times\mathcal{S}^2$ solution into Type IIA supergravity was obtained in Reference [34]. In Einstein frame, this uplifted solution, the SS–CGP solution, is

$$\begin{aligned} ds^2_{10} &= \exp(2A)\left(\eta_{\mu\nu}dx^\mu dx^\nu + dy^2 + \frac{1}{4g^2}\big(d\psi + \mathrm{sech}2\rho\,(d\chi+\cos\theta\,d\varphi)\big)^2\right) + \frac{1}{g^2}\exp(6A)ds^2_{EH}\,, \\ B_{(2)} &= \frac{1}{4g^2}\big(d\chi + \mathrm{sech}2\rho\,d\psi\big)\wedge\big(d\chi+\cos\theta\,d\varphi\big)\,,\quad e^{2\phi}=\exp(-8A)\,,\quad A=\log\circ\cosh^{-4}(2\rho)\,, \end{aligned} \tag{5.1.1}$$

where g is a constant, and ds^2_{EH} is the metric on the four-dimensional Eguchi–Hanson space [22],

$$ds^2_{EH} = \cosh 2\rho\Big(d\rho^2 + \frac{1}{4}(\tanh 2\rho)^2(d\chi+\cos\theta\,d\varphi)^2 + \frac{1}{4}(d\theta^2+\sin^2\theta\,d\varphi^2)\Big)\,. \tag{5.1.2}$$

The coordinates take values in ranges $x^\mu\in\mathbb{R}^{1,3}$, $y\in[0,l_y)$, $\chi,\varphi\in[0,2\pi)$, $\psi\in[0,4\pi)$, $\theta\in[0,\pi]$ and $\rho\in[0,\infty)$. Since χ has a period of 2π (instead of a period of 4π) the boundary of the Eguchi–Hanson space at infinity is $\mathbb{RP}^3\cong\mathcal{S}^3/\mathbb{Z}_2$. Here $\mathcal{S}^3$ is realised as a Hopf fibration [73] over $\mathcal{S}^2$, with χ as the fibre coordinate. As $\rho=0$ the geometry of the Eguchi–Hanson space is approximately $\mathbb{R}^2\times\mathcal{S}^2$ for constant (θ,φ), with (ρ,χ) acting as plane polar coordinates on $\mathbb{R}^2$.

As explained in [32], the SS–CGP solution preserves eight supercharges, and has the form of an NS5-brane wrapped on $(y,\psi)\in T^2$ with an effective worldvolume $\mathbb{R}^{1,3}$ that has a singularity which is resolved by transgression. The function A, which is usually a harmonic function on the transverse space (Eguchi–Hanson in our case), is now a particular solution to the sourced Laplacian

$$\Delta_{EH}\exp(-8A) = -\frac{g^2}{2}(\mathcal{F}_{(2)})^2\,, \tag{5.1.3}$$

where Δ_{EH} is the Laplacian on Eguchi–Hanson space, and $\mathcal{F}_{(2)}$ is the field strength of the 1-form

$$\mathcal{A}_{(1)} = \mathrm{sech}2\rho\,(d\chi+\cos\theta\,d\varphi)\,, \tag{5.1.4}$$

and is the unique, anti-self-dual 2-form on Eguchi–Hanson space. Geometrically, this transgression is realised as a $U(1)$ fibration of the worldvolume over Eguchi–Hanson space with fibre coordinate ψ and connection $\mathcal{A}_{(1)}$. We will call this $U(1)$ bundle the transgression bundle. For generic values of ρ, this bundle is non-trivial with second Chern character

$$\int \mathrm{ch}_2(\mathcal{F}_{(2)}) = \frac{1}{2(2\pi)^2} \int \mathcal{F}_{(2)} \wedge \mathcal{F}_{(2)} = 1\,, \tag{5.1.5}$$

where the integral is over the Eguchi–Hanson space. There is a special limit of the SS–CGP solution which makes the connection to NS5-branes even more manifest. As $\rho \to \infty$, the field strength $\mathcal{F}_{(2)}$ vanishes, so the transgression bundle trivialises. In this limit, the solution asymptotes to the linear dilaton solution, which is the near-horizon limit of the NS5-brane. Consequently, there is also an enhancement of supersymmetry to sixteen supercharges in this limit.

It is worth mentioning that it is possible [32] to include an additional NS5-brane into the SS–CGP solution without breaking any more supersymmetry by adding to $\exp(-8A)$ a homogeneous solution to (5.1.3). Explicitly, one has

$$\exp(-8A) = \mathrm{sech}2\rho - k \log\tanh\rho\,, \tag{5.1.6}$$

where k is a positive constant that is proportional to the tension of the NS5-brane. The logarithmic behaviour of $\mathcal{H}$ for small ρ is indicative of the fact that the topology of the Eguchi–Hanson space is $\mathbb{R}^2 \times S^2$ in this neighbourhood. In order for this to remain a solution, the NSNS 2-form [93] is also modified to be [32]

$$B_{(2)} = \frac{1}{4g^2}\big((1+k)d\chi + \mathrm{sech}2\rho\, d\psi\big) \wedge \big(d\chi + \cos\theta\, d\varphi\big)\,. \tag{5.1.7}$$

We will not be studying this solution further in the present paper, but more information about it can be found in Reference [32].

5.2 Perturbations from the Salam–Sezgin Cvetic–Gibbons–Pope Background

One method of finding solutions to the perturbation problem about the Salam–Sezgin and Cvetic–Gibbons–Pope background is to find solutions to the perturbation problem of the five-dimensional theory

$$\mathcal{L}_5 = R - \frac{1}{2}\left(\nabla\Phi_i\right)^2 - \frac{1}{2}e^{\sqrt{2}\Phi_1}\left(\nabla\sigma\right)^2 - V\ , \tag{5.2.1}$$

obtained from Type I supergravity reduced on $T^3 \times S^2$, the details of which are presented in Appendix E.[69] Here, the scalar potential V is

$$V = 2g^2 e^{\sqrt{\frac{2}{5}}\Phi_2 - \frac{8}{\sqrt{15}}\Phi_3}\Big(e^{-\sqrt{2}\Phi_1} + \sigma^2 + \frac{1}{4}e^{\sqrt{2}\Phi_1}(\sigma^2-2)^2 - 4e^{-\sqrt{\frac{2}{5}}\Phi_2 + \sqrt{\frac{3}{5}}\Phi_3}\Big)\,. \tag{5.2.2}$$

[69]The two form in this solution does not originate from a matter field in Type I supergravity, but from the metric in the reduction, please refer to Appendix E for more detail.

The (background) Salam–Sezgin solution of Type I supergravity in five dimensions is

$$
\begin{aligned}
ds_5^2 = (\sinh 2\rho)^{\frac{2}{3}} \Big(\eta_{\mu\nu}dx^\mu dx^\nu + \frac{1}{g^2}d\rho^2\Big)\,, \qquad e^{-\sqrt{2}\Phi_1} = (\tanh 2\rho)^2\,,\\
e^{\sqrt{10}\Phi_2} = e^{\sqrt{15}\Phi_3} = (\sinh 2\rho)^2\,, \qquad \sigma = \sqrt{2}\mathrm{sech} 2\rho\,.
\end{aligned}
\tag{5.2.3}
$$

We note g is the (length)$^{-1}$ dimensional parameter characterising the scale of the $\mathcal{H}^{(2,2)}$ hyperbolic geometry [32]. We perturb about this background (5.2.3),

$$
g_{MN} = (\sinh 2\rho)^{\frac{2}{3}} \left(\overline{g}_{MN} + H_{MN}\right)\,, \quad \Phi_i = \overline{\Phi}_i + \phi_i\,, \quad \sigma = \overline{\sigma} + \Sigma\,,
\tag{5.2.4}
$$

where $\overline{g}$ is our conformally related metric described in (4.1.2). Here $\overline{g}$ is explicitly

$$
d\overline{s}_5^2 = \overline{g}_{MN}dX^M dX^N = \eta_{\mu\nu}dx^\mu dx^\nu + \frac{1}{g^2}d\rho^2\,.
\tag{5.2.5}
$$

$\overline{\Phi}_i$ and $\overline{\sigma}$ are the background values of the scalars. Here, we have used $X^M = (x^\mu, \rho)$. For notational convenience, we will define a function $A(\rho)$ by

$$
e^{2A(\rho)} = (\sinh 2\rho)^{\frac{2}{3}}\,,
\tag{5.2.6}
$$

and also make the coordinate rescaling $\rho \to z(\rho) = \rho/g$. In the $X^M = (x^\mu, z)$ coordinate system, $\overline{g}_{MN} = \eta_{MN}$, and the linearized Ricci tensor of (5.2.4) is given by

$$
\begin{aligned}
R_{MN} =& \frac{1}{2}\big(\partial^P\partial_M\mathcal{H}_{NP} + \partial^P\partial_N\mathcal{H}_{MP} - \Box_5\mathcal{H}_{MN} - \partial_M\partial_N\mathcal{H}\big) + \frac{3}{2}A'\big(\partial_M\mathcal{H}_{Nz} + \partial_N\mathcal{H}_{Mz} - \partial_z\mathcal{H}_{MN}\big)\\
&+ \Big(A'\Big(\partial^M\mathcal{H}_{Mz} - \frac{1}{2}\partial_z\mathcal{H}\Big) + \big(A'' + 3(A')^2\big)\mathcal{H}_{zz}\Big)\eta_{MN} - \big(A'' + 3(A')^2\big)\mathcal{H}_{MN} + \mathcal{O}(\mathcal{H}^2)\,,
\end{aligned}
\tag{5.2.7}
$$

where $\Box_5 = \eta^{MN}\partial_M\partial_N$, and $\mathcal{H} = \eta^{MN}\mathcal{H}_{MN}$. Using the definition of $A(z)$ given in (5.2.6), we find

$$
\begin{aligned}
R_{MN} =& \frac{1}{2}\big(\partial^P\partial_M\mathcal{H}_{NP} + \partial^P\partial_N\mathcal{H}_{MP} - \Delta_5\mathcal{H}_{MN} - \partial_M\partial_N\mathcal{H}\big) + g\coth(2gz)\big(\partial_M\mathcal{H}_{Nz} + \partial_N\mathcal{H}_{Mz}\big)\\
&+ \Big(\frac{2}{3}g\coth(2gz)\Big(\partial^M\mathcal{H}_{Mz} - \frac{1}{2}\partial_z\mathcal{H}\Big) + \frac{4g^2}{3}\mathcal{H}_{zz}\Big)\eta_{MN} - \frac{4g^2}{3}\mathcal{H}_{MN} + \mathcal{O}(\mathcal{H}^2)\,.
\end{aligned}
\tag{5.2.8}
$$

The operator Δ_5 in (5.2.8) is defined as

$$
\Delta_5 = \Box_5 + 2g\coth(2gz)\partial_z = \eta^{\mu\nu}\partial_\mu\partial_\nu + g^2\left(\partial_\rho^2 + 2\coth(2\rho)\partial_\rho\right)\,.
\tag{5.2.9}
$$

5.2.1 Scalar Equations

Before looking at $\mathcal{H}_{MN}$'s field equations, we will consider the equations for ϕ_i and Σ. At general order these are

$$\Box_5\Phi_{2,3} = \frac{\partial V}{\partial\Phi_{2,3}}\,, \quad \Box_5\Phi_1 = \frac{1}{\sqrt{2}}e^{\sqrt{2}\Phi_1}\big(\partial\sigma\big)^2 + \frac{\partial V}{\partial\Phi_1}\,, \quad \nabla_M\big(e^{\sqrt{2}\Phi_1}\partial^M\sigma\big) = \frac{\partial V}{\partial\sigma}\,. \tag{5.2.10}$$

We use that

$$\sqrt{-g} = e^{5A}\Big(1 + \frac{1}{2}\mathcal{H} + \mathcal{O}(\mathcal{H}^2)\Big)\,, \tag{5.2.11}$$

to expand our scalar equations at first order to find

$$\Box_5\Phi_i = e^{-2A}\Delta_5\overline{\Phi}_i + e^{-2A}\Big(\Delta_5\phi_i - \Big(\partial^M\mathcal{H}_{Mz} - \frac{1}{2}\partial_z\mathcal{H}\Big)\overline{\Phi}_i' - \mathcal{H}_{zz}\Delta_5\overline{\Phi}_i\Big)\,, \tag{5.2.12}$$

$$\begin{aligned}\nabla_M\big(e^{\sqrt{2}\Phi_1}\partial^M\sigma\big) = {}& e^{\sqrt{2}\,\overline{\Phi}_1-2A}\widetilde{\Delta}_5\overline{\sigma} + e^{\sqrt{2}\,\overline{\Phi}_1-2A}\Big(\widetilde{\Delta}_5\Sigma + \big(\sqrt{2}\phi_1 - \mathcal{H}_{zz}\big)\widetilde{\Delta}_5\overline{\sigma} \\ &- \Big(\partial^M\mathcal{H}_{Mz} - \frac{1}{2}\partial_z\mathcal{H}\Big)\overline{\sigma}' + \sqrt{2}\overline{\sigma}'\phi_1'\Big)\,,\end{aligned} \tag{5.2.13}$$

where the operator $\widetilde{\Delta}_5$ is defined as

$$\widetilde{\Delta}_5 = \Delta_5 + \sqrt{2}\,\overline{\Phi}_1'\partial_z = \Delta_5 - 8g\,\mathrm{csch}(4gz)\partial_z\,. \tag{5.2.14}$$

For the right hand side of the scalar equations, we have, to first order in perturbations,

$$\frac{\partial V}{\partial S_\alpha} = \frac{\partial V}{\partial S_\alpha}\bigg|_{\overline{S}} + \frac{\partial^2 V}{\partial S_\beta\partial S_\alpha}\bigg|_{\overline{S}}\delta S_\beta\,, \tag{5.2.15}$$

where $S_\alpha = \{\Phi_i, \sigma\}$, $\delta S_\alpha = \{\phi_i, \Sigma\}$, and $\overline{S}$ denote the scalars, scalar perturbations, and background scalars respectively. Note that there is no $\frac{1}{2}$ prefactor on the second derivative of the potential.

We also have

$$e^{\sqrt{2}\Phi_1}\big(\partial\sigma\big)^2 = e^{\sqrt{2}\,\overline{\Phi}_1-2A}\big((\overline{\sigma}')^2 + (\sqrt{2}\phi_1 - \mathcal{H}_{zz})(\overline{\sigma}')^2 + 2\overline{\sigma}'\Sigma'\big)\,. \tag{5.2.16}$$

Calculation shows that

$$\frac{\partial^2 V}{\partial\Phi_{2,3}\partial\Phi_1}\bigg|_{\overline{S}} = \frac{\partial^2 V}{\partial\Phi_{2,3}\partial\sigma}\bigg|_{\overline{S}} = 0\,. \tag{5.2.17}$$

Consequently, $\{\phi_1, \Sigma\}$ and $\{\phi_2, \phi_3\}$ are decoupled from each other at this order in perturbations. Explicitly, the scalar equations are

$$\phi_1: \quad \big(\Delta_5 - 8g^2\big)\phi_1 = -4g\mathrm{csch}(4gz)\Big(\sqrt{2}\mathcal{G}_z + 2\cosh(2gz)\big(\partial_z\Sigma + 2g\tanh(2gz)\Sigma\big)\Big)\,, \tag{5.2.18}$$

$$\Sigma: \quad \big(\widetilde{\Delta}_5 - 8g^2(\mathrm{sech}(2gz))^2\big)\Sigma = -2g\mathrm{sech}(2gz)\tanh(2gz)\big(\sqrt{2}\mathcal{G}_z - 2\big(\partial_z\phi_1 - 2g\tanh(2gz)\phi_1\big)\big)\,, \tag{5.2.19}$$

$$\phi_2: \quad \Delta_5\phi_2 - \frac{8g^2}{5}\phi_2 + \frac{32}{5}\sqrt{\frac{2}{3}}g^2\phi_3 = 2\sqrt{\frac{2}{5}}g\big(\coth(2gz)\mathcal{G}_z + 2g\mathcal{H}_{zz}\big)\,, \tag{5.2.20}$$

$$\phi_3: \quad \Delta_5\phi_3 - \frac{56g^2}{15}\phi_3 + \frac{32}{5}\sqrt{\frac{2}{3}}g^2\phi_2 = \frac{4g}{\sqrt{15}}\big(\coth(2gz)\mathcal{G}_z + 2g\mathcal{H}_{zz}\big)\,, \tag{5.2.21}$$

where for brevity, we define

$$\mathcal{G}_z = \partial^M\mathcal{H}_{Mz} - \frac{1}{2}\partial_z\mathcal{H}\,. \tag{5.2.22}$$

We can solve one of the $\{\phi_1, \Sigma\}$ equations and one of the $\{\phi_2, \phi_3\}$ equations by requiring

$$\Sigma = \sinh(2gz)\tanh(2gz)\phi_1\,, \quad \phi_3 = \sqrt{\frac{2}{3}}\phi_2\,. \tag{5.2.23}$$

The resulting equations are

$$\phi_1: \quad \big(\Box_5 + 2g\mathrm{csch}(4gz)(3\cosh(4gz) - 1)\partial_z + 8g^2\big)\phi_1 = -4\sqrt{2}g\mathrm{csch}(4gz)\mathcal{G}_z\,, \tag{5.2.24}$$

$$\phi_2: \quad \Delta_5\phi_2 + \frac{8g^2}{3}\phi_2 - 4\sqrt{\frac{2}{5}}g^2\mathcal{H}_{zz} = 2\sqrt{\frac{2}{5}}g\coth(2gz)\mathcal{G}_z\,. \tag{5.2.25}$$

The right hand side of the remaining scalar equations are proportional to $\mathcal{G}_z$. We recognize this to be the z-component of the de Donder combination [41, 116]. Since the supergravity equations are invariant under linearized diffeomorphisms [9]

$$\mathcal{H}_{MN} \mapsto \mathcal{H}_{MN} + \partial_{(M}\xi_{N)} + 2A'\xi_z\eta_{MN}\,, \quad \xi_M \coloneqq \eta_{MN}\xi^N\,, \tag{5.2.26}$$

with similar expressions for the transformations of ϕ_i and Σ, we can set $\mathcal{G}_z = 0$ as a gauge condition.

In this gauge, ϕ_1 decouples from the gravity sector. So, for simplicity, we will set $\phi_1 = 0$. The same is not true for ϕ_2, as it couples to $\mathcal{H}_{zz}$. For completeness, the equation for ϕ_2 in this gauge is

$$\Delta_5\phi_2 + \frac{8g^2}{3}\phi_2 - 4\sqrt{\frac{2}{5}}g^2\mathcal{H}_{zz} = 0\,. \tag{5.2.27}$$

5.2.2 Einstein Equations

Now, let us analyze the equations of motion for $\mathcal{H}_{MN}$. The linearized (trace-reversed) stress tensor $\theta^{(1)}_{MN}$, where $R^{(1)}_{MN} = \theta^{(1)}_{MN}$, is given by

$$\begin{aligned}\theta^{(1)}_{MN} &= \partial_{(M}\overline{\Phi}_2\partial_{N)}\delta\Phi_2 + \partial_{(M}\overline{\Phi}_3\partial_{N)}\delta\Phi_3 + \frac{e^{2A}}{3}\Big(\frac{\partial V}{\partial\Phi_2}\Big|_{\overline{S}}\delta\Phi_2 + \frac{\partial V}{\partial\Phi_3}\Big|_{\overline{S}}\delta\Phi_3\Big)\eta_{MN} + \frac{e^{2A}}{3}V\Big|_{\overline{S}}\mathcal{H}_{MN} \\ &= \frac{\sqrt{10}}{3}g\coth(2gz)\big(\delta_{Mz}\partial_N\phi_2 + \delta_{Nz}\partial_M\phi_2\big) + \frac{4\sqrt{10}}{9}g^2\phi_2\,\eta_{MN} - \frac{4g^2}{3}\mathcal{H}_{MN}\,, \end{aligned} \tag{5.2.28}$$

where we have used $\phi_1 = \Sigma = 0$ and the $\mathcal{G}_z = 0$ gauge.

For simplicity, we now use our remaining diffeomorphism invariance to set the full de Donder gauge

$$\partial^M\mathcal{H}_{MN} - \frac{1}{2}\partial_N\mathcal{H} = 0\,. \tag{5.2.29}$$

In this gauge, the linearized Ricci tensor given in (5.2.8) becomes

$$R^{(1)}_{MN} = -\frac{1}{2}\Delta_5\mathcal{H}_{MN} + g\coth(2gz)\big(\partial_M\mathcal{H}_{Nz} + \partial_N\mathcal{H}_{Mz}\big) + \frac{4g^2}{3}\mathcal{H}_{zz}\eta_{MN} - \frac{4g^2}{3}\mathcal{H}_{MN}\,, \tag{5.2.30}$$

and the Einstein equations now simplify to

$$\Delta_5\mathcal{H}_{MN} - 4g\coth(2gz)\partial_{(M}\mathcal{H}_{N)z} - \frac{8g^2}{3}\mathcal{H}_{zz}\eta_{MN} = -\frac{4\sqrt{10}}{3}g\coth(2gz)\delta_{z(M}\partial_{N)}\phi_2 - \frac{8\sqrt{10}}{9}g^2\phi_2\,\eta_{MN}\,. \tag{5.2.31}$$

Firstly, we examine the zz component of (5.2.31). It reads

$$\Delta_5\mathcal{H}_{zz} - \frac{8g^2}{3}\mathcal{H}_{zz} + \frac{8\sqrt{10}}{9}g^2\phi_2 = 4g\coth(2gz)\Big(\partial_z\mathcal{H}_{zz} - \frac{\sqrt{10}}{3}\partial_z\phi_2\Big)\,. \tag{5.2.32}$$

Recall that ϕ_2 obeys (5.2.27). Performing the field redefinitions

$$\mathcal{H}_{zz} = \frac{1}{\sqrt{2}}\phi + \varphi\,, \quad \phi_2 = \frac{3}{2\sqrt{5}}\phi\,, \tag{5.2.33}$$

we find that (5.2.27) and (5.2.32) become

$$\phi_2: \quad \Delta_5\phi = \frac{8\sqrt{2}}{3}g^2\varphi\,, \tag{5.2.34}$$

$$\varphi: \quad \overline{\Delta}_5\varphi = 0\,, \tag{5.2.35}$$

where

$$\overline{\Delta}_5 = \Box_5 - 2g\coth(2gz)\partial_z\,. \tag{5.2.36}$$

Next, we have the μz components of (5.2.31), which read

$$\Box_5 \mathcal{H}_{\mu z} = 2g\coth(2gz)\Big(\partial_\mu\varphi - \frac{1}{\sqrt{2}}\partial_\mu\phi\Big)\,. \tag{5.2.37}$$

Since ϕ and φ are fixed, if the operator $\Box_5$ is invertible (which it is in the case of time-independent solutions), the solution to $\mathcal{H}_{\mu z}$ is symbolically,

$$\mathcal{H}_{\mu z} = 2g\frac{1}{\Box_5}\coth(2gz)\Big(\partial_\mu\varphi - \frac{1}{\sqrt{2}}\partial_\mu\phi\Big)\,. \tag{5.2.38}$$

Finally, the $\mu\nu$ components of (5.2.31) are

$$\Delta_5\mathcal{H}_{\mu\nu} = 4g\coth(2gz)\partial_{(\mu}\mathcal{H}_{\nu)z} + \frac{8g^2}{3}\varphi\eta_{\mu\nu}\,. \tag{5.2.39}$$

Since Δ_5 is a linear operator, we can split $\mathcal{H}_{\mu\nu}$ into three parts

$$\mathcal{H}_{\mu\nu} = H_{\mu\nu} + K_{\mu\nu} + J\eta_{\mu\nu}\,, \tag{5.2.40}$$

where

$$\Delta_5 H_{\mu\nu} = 0\,, \tag{5.2.41}$$

$$\Delta_5 K_{\mu\nu} = 4g\coth(2gz)\partial_{(\mu}\mathcal{H}_{\nu)z}\,, \tag{5.2.42}$$

$$\Delta_5 J = \frac{8g^2}{3}\varphi\,. \tag{5.2.43}$$

As with the μz equation, all of the quantities on the right hand sides are known. In fact, (5.2.43) is equivalent to (5.2.34) with the choice $J = \phi/\sqrt{2}$. Thus, provided that appropriate boundary conditions are imposed, Δ_5 can be inverted to solve (5.2.41)-(5.2.43).

5.2.3 Time Independent $\mathcal{H}_{00}$

For time-independent solutions, it is clear from (5.2.38) that $\mathcal{H}_{0z} = 0$ is a solution, and consequently, we have $K_{00} = 0$ as a solution. Then, $\mathcal{H}_{00} = H_{00} - J = H_{00} - \phi/\sqrt{2}$, where, for completeness, H_{00} and ϕ satisfy

$$\Delta_5 H_{00} = 0\,, \quad \Delta_5 \phi = \frac{8\sqrt{2}}{3} g^2 \varphi\,, \quad \overline{\Delta}_5 \varphi = 0\,. \tag{5.2.44}$$

with operators Δ_5 and $\overline{\Delta}_5$ as defined in (5.2.9) and (5.2.36) respectively. For solutions that are also radially symmetric in $\mathbb{R}^{1,3}$, we have, recalling that $z = \rho/g$,

$$\Delta_5 = \partial_r{}^2 + \frac{2}{r}\partial_r + g^2\left(\partial_\rho{}^2 + 2\coth 2\rho\, \partial_\rho\right)\,, \quad \overline{\Delta}_5 = \partial_r{}^2 + \frac{2}{r}\partial_r + g^2\left(\partial_\rho{}^2 - 2\coth 2\rho\, \partial_\rho\right)\,, \tag{5.2.45}$$

with r the isotropic, spatial radius in $\mathbb{R}^{1,3}$. For simplicity, we will consider the case $\varphi = 0$.

In summary, we find that, as for Minkowski spacetime, the leading component of any perturbative solution for $\mathcal{H}_{00}$ is given by a Green function associated with Δ_5, the Crampton–Pope–Stelle operator [32].

5.3 Type II and Type III†: Green Functions for the CPS Operator

In section 5.4 we will argue the key to understanding the effective Newton potential is understanding the behavior of $\mathcal{H}_{00}$, which is given by a Green function of the CPS operator Δ_5. Since we are interested in computing Newton's constant, which arises from the interaction of a small test particle orbiting a massive source, we consider the sourced equation,

$$\Delta_5 G(r,\rho) = \frac{g\hat{\kappa}^2 M \delta(r)\delta(\rho)}{4\pi r^2 \mu(\rho)} = \frac{g\hat{\kappa}^2 M \delta(r)\delta(\rho)}{4\pi r^2 \sinh 2\rho}\,, \tag{5.3.1}$$

where $\hat{\kappa}^2$ is the five-dimensional Newton constant, M is the mass of the source, $\mu(\rho) = \sinh 2\rho$ is the appropriate measure for integrating over ρ, as seen from consideration of the $\mathcal{H}^2_{\mu\nu}$ terms in the perturbative action, and $V_N(r,\rho) = -2m_{\text{particle}} G(r,\rho)$ is the Newtonian potential. Eigenfunctions of this operator have previously been studied in [32]. There, time dependent solutions were found that localize gravity to four dimensions via a non-constant, normalizable zero mode ξ_0 of the ρ-dependent part of Δ_5,

$$\left(\partial_\rho^2 + 2\coth 2\rho\, \partial_\rho\right)\xi_0 = 0\,. \tag{5.3.2}$$

For convenience, we will call this ρ-dependent part the transverse operator Δ. The solution to (5.3.2) that is normalizable (and normalized) with respect to the measure $\mu(\rho)$ is

$$\xi_0 = \pm\frac{2\sqrt{3}}{\pi}\log\tanh\rho\,. \tag{5.3.3}$$

The existence of this normalizable zero mode is special. In many examples of non-compact, transverse geometries realized in supergravity, such as BPS branes [120], the zero modes of the associated transverse operator are non-normalizable, and the coupling of the lower-dimensional massless gravitational sector to all other modes in the effective field theory consequently vanishes. This is a consequence of the extended nature of the source in the higher dimension, as is the case with black spokes as discussed in section 4.5.

In this section, we will first inspect asymptotic solutions to (5.3.1) in order to understand the general behavior of the Green functions. Then, following [32], we will solve for the Green functions by expanding in a basis of eigenfunctions of the transverse operator Δ. There are two bases of eigenfunctions of interest which are distinguished by their boundary conditions in ρ. We will start with a mode decomposition where the Green function $G(r,\rho)$ vanishes at infinity and is continuous everywhere away from the source at the $(r,\rho) = (0,0)$ origin. We will find that this solution does not become effectively lower-dimensional (i.e. 4D) for a massless field, but instead becomes exponentially suppressed in the worldvolume radius r. We will secondly consider a mode decomposition that includes the zero mode (5.3.3) as found in [32], and will find that the corresponding solution then does effectively become lower-dimensional, but that it also has logarithmic structure as $\rho \to 0$. The relationship between these two cases is explained in more detail in Section 3.4.3.

5.3.1 Asymptotic Solutions

There are two main regimes where the Δ_5 operator simplifies greatly. The first is when $\rho \ll 1$, and the second is when $\rho \gg 1$. The relevant asymptotic expansions of the operator are

$$\Delta_5 = \partial_r^2 + \frac{2}{r}\partial_r + g^2\left(\partial_\rho^2 + \frac{1}{\rho}\partial_\rho + \frac{4}{3}\rho\,\partial_\rho + \mathcal{O}(\rho^3)\right)\,, \tag{5.3.4}$$

when ρ is small, and

$$\Delta_5 = \partial_r^2 + \frac{2}{r}\partial_r + g^2\left(\partial_\rho^2 + 2\,\partial_\rho + 4\exp(-2\rho)\partial_\rho + \mathcal{O}(\exp(-4\rho))\right)\,, \tag{5.3.5}$$

when ρ is large. Since we are interested in sources at $r = 0$ near the $\rho = 0$ submanifold, we should inspect the Green function in that limit. Specifically, by substituting the coordinate redefinition

$$R^2 = g^2 r^2 + \rho^2 \ , \qquad \theta = \arctan\left(\frac{\rho}{gr}\right) \ , \tag{5.3.6}$$

(5.3.4) becomes

$$\Delta_5 = g^2 \left(\partial_R^2 + \frac{4}{R}\partial_R + \frac{1}{R^2}\left(\partial_\theta^2 + (\cot(\theta) - 2\tan(\theta))\,\partial_\theta\right)\right) + \mathcal{O}(\rho) \ . \tag{5.3.7}$$

For θ-independent functions, this is just the Laplacian on $\mathbb{R}^5$. The precise normalization of the radially symmetric Green function on $\mathbb{R}^5$ is given by

$$\left(\partial_P^2 + \frac{4}{P}\partial_P\right)\frac{1}{2\pi^2 P^3} = -\delta^5\left(X^M\right) \ , \tag{5.3.8}$$

where $X \in \mathbb{R}^5$ and $P^2 = X \cdot X$. Defining $r^2 = (X^1)^2 + (X^2)^2 + (X^3)^2$ and $\rho^2 = (X^4)^2 + (X^5)^2$, we may integrate over the angular dimensions in (5.3.8) to find

$$\left(\partial_P^2 + \frac{4}{P}\partial_P\right)\frac{1}{2\pi^2 P^3} = -\frac{\delta(r)\delta(\rho)}{8\pi^2 r^2 \rho} \ . \tag{5.3.9}$$

Now, the right hand side of (5.3.1) in the $\rho \to 0$ limit reads

$$\frac{g\hat{\kappa}^2 M \delta(r)\delta(\rho)}{4\pi r^2 \sinh 2\rho} \sim \frac{g\hat{\kappa}^2 M \delta(r)\delta(\rho)}{8\pi r^2 \rho} + \mathcal{O}(\rho^2) \ . \tag{5.3.10}$$

Consequently, we expect the leading component of the Green function in the $R \to 0$ limit to be

$$G(r,\rho) = -\frac{g^4\hat{\kappa}^2 M}{2\pi\left(g^2r^2 + \rho^2\right)^{\frac{3}{2}}} + \mathcal{O}\left(\frac{1}{R^2}\right) \ . \tag{5.3.11}$$

There are two more regimes of interest. The first is when $r \gg 1$ and $\rho \ll 1$. For $r \gg 1$, the differential operator takes the same form as in (5.3.4). However, we are interested in solutions expanded as a Laurent series about $r = \infty$ [4]. As such, we may use separability to find the leading term, which can be expanded in inverse integer powers of r. We have

$$\Delta_5 f(r,\rho) = 0 \qquad \Rightarrow \qquad f(r,\rho) = \frac{A}{r} + \frac{B\log(\rho)}{r} + \mathcal{O}\left(\frac{1}{r^2}\right) \ . \tag{5.3.12}$$

The second regime is when $\rho \gg 1$. In this regime, the transverse operator Δ can be manipulated into the form of the Helmholtz operator [112] in leading-order by writing

$$\Delta_5 \left(\frac{\exp(-\rho)}{r} f(gr, \rho) \right) = 0 \qquad \Rightarrow \qquad \left(\partial_x^2 + \partial_\rho^2 - 1 \right) f(x, \rho) = 0 \ , \tag{5.3.13}$$

for $x = gr$. In this regime, we will be interested in the $f = \exp(-\rho)$ solution to (5.3.13), since this is the leading component of the $\frac{\xi_0(\rho)}{r}$ solution found in [32]:

$$\frac{\xi_0(\rho)}{r} \propto \frac{\log \tanh \rho}{r} = -\frac{2}{r} \exp(-2\rho) + \mathcal{O}\left(\exp(-4\rho)\right) \ . \tag{5.3.14}$$

Knowing the leading components of a Green function in asymptotic regimes, however, does not tell us how the solution for a given source near $(r, \rho) = (0, 0)$ evolves as it approaches infinity in various directions. We are left with the question: does the solution with leading behavior (5.3.11) asymptote to the $\frac{\xi_0(\rho)}{r}$ solution at large r? And if so, what is the coefficient of this term?

5.3.2 Type II Solutions from Green's Formula

To find the corresponding Type II solution (at least asymptotically) it is sufficient to find G^K which obeys special Neumann conditions at $\rho = 0$. That is

$$\begin{gathered} \partial_\rho G^K(r, \rho - \eta)\Big|_{\rho=0} \ , \qquad \partial_r G^K(r, \rho - \eta)\Big|_{r=0 \text{ and } \rho \neq \eta} = 0 \ , \\ G^K(r, \rho - \eta)\Big|_{r \to \infty} = 0 \ , \qquad G^K(r, \rho - \eta)\Big|_{\rho \to \infty} = 0 \ . \end{gathered} \tag{5.3.15}$$

The transverse basis we must choose for this is the 'pure ζ_0' basis, and is calculated explicitly in appendix B. For the sake of the reader we will restate equation (B.1.11). The elements of our basis ζ_ω, with $\omega > 1$, are Legendre functions of the second type [61] with an detailed normalization coefficient and complex order. It is

$$\zeta_\omega(\rho) = \sqrt{\frac{\pi\omega^2}{\sqrt{\omega^2 - 1}} \tanh\left(\frac{\pi}{2}\sqrt{\omega^2 - 1}\right)} \mathcal{P}_{-\frac{1}{2} + \frac{\sqrt{1-\omega^2}}{2}} \left(\cosh(2\rho)\right) \ . \tag{5.3.16}$$

Similarly the world-volume basis we must choose also obeys special Neumann as $r \to 0$, and is given in section 2.6.2. For $n = 3$ we have

$$f^\sigma(r) = \frac{\sin(\sigma r)}{\sqrt{2}\pi r} \ . \tag{5.3.17}$$

Applying Green's equation for product spaces, as given in figure (5), we have

$$G^K(r, \rho - \eta) = \int_0^\infty \int_1^\infty \frac{1}{\omega^2 + \sigma^2} \frac{\sigma \sin(\sigma r)}{2\pi^2 r} \zeta_\omega(\rho) \zeta_\omega(\eta) d\omega d\sigma \ . \tag{5.3.18}$$

As we established in section 3.4.2 we know that integrating over the world-volume eigenvalue (σ) simplifies this equation to

$$G^K(r,\rho-\eta) = -\int_1^{\omega} \frac{\exp(-\omega r)}{4\pi r}\zeta_\omega(\rho)\zeta_\omega(\eta)\, d\omega\ . \tag{5.3.19}$$

From this we know that the leading component of $G^K(r,\rho-\eta)$ is exponentially suppressed at large $r \gg 1$. In fact, since we know for no combination of ρ and η does the product of our transverse basis functions approximate a delta function in ω, we know

$$G^K(r,\rho-\eta) \prec \frac{\exp(-r)}{r}\ . \tag{5.3.20}$$

5.3.3 Type III† Solutions from Green's Formula

A similarly low precision estimate of the solutions with the boundary conditions[70] which permit our normalizable zero mode $\xi_0(\rho)$ (which we show to be normalizable in equation (2.6.59)) is sufficient to find the leading behavior of this solution as $r \gg 1$ as well. Furthermore, it is enough to demonstrate that this is a Type III† solution.

By definition the transverse basis in which we should expand our target Green function $G^T(r,\rho-\eta)$ contains ξ_0

$$\xi_0(\rho) = \frac{4\sqrt{3}}{\pi}\mathcal{Q}_0(\cosh(2\rho)) = \frac{\sqrt{12}}{\pi}\log\circ\tanh(\rho)\ . \tag{5.3.21}$$

As well as oscillating modes with $\omega > 1$ with

$$\xi_\omega(\rho) = b_\omega \mathcal{Q}_{-\frac{1}{2}+\frac{\sqrt{1-\omega^2}}{2}}(\cosh(2\rho))\ . \tag{5.3.22}$$

Repeating Green's equation for product spaces (figure (5)) we have

$$G^T(r,\rho-\eta) = \int_0^\infty \frac{1}{\sigma^2}\frac{\sigma\sin(\sigma r)}{2\pi^2 r}\xi_0(\rho)\xi_0(\eta) + \int_0^\infty\int_1^\infty \frac{1}{\omega^2+\sigma^2}\frac{\sigma\sin(\sigma r)}{2\pi^2 r}\xi_\omega(\rho)\xi_\omega(\eta)d\omega d\sigma\ . \tag{5.3.23}$$

For the same reasons as for equation (5.3.20) the double integral must be dominated by $\frac{\exp(-r)}{r}$. However, since our spectrum has a discrete zero mode we know

$$G^T(r,\rho-\eta) = -\frac{1}{4\pi r}\frac{12}{\pi}\log\circ\tanh(\rho)\log\circ\tanh(\eta) + \mathcal{O}\left(\frac{\exp(-r)}{r}\right)\ . \tag{5.3.24}$$

Notably, this function diverges logarithmically as $\rho,\eta \to 0$. What this means physically and possible motivations for this will be discussed in the context of Newton's constant in section 5.5. However, we could

[70] The details of the boundary conditions fit more neatly into section 5.3.4 and are given in equation (5.3.27).

choose to regularize our source around $\rho \sim 0$ with respect to some transverse function, $f(\rho)$. In this case the long-distance potential, $V^T(r,\eta)$, is

$$V^T(r,\eta) = \int_0^\infty \mu(\rho) f(\rho) G(r, \rho - \eta) d\rho \,. \tag{5.3.25}$$

If we choose to regularize our source with $f(\rho) = \xi_0(\rho)$, we have

$$V^T(r,\eta) = \int_0^\infty -\frac{1}{4\pi r}\sqrt{\frac{12}{\pi}} \log \circ \tanh(\eta) \,. \tag{5.3.26}$$

Similarly if we regularize the potential of our probe we have that the long-distance effect of a source on a probe (at leading order) is exactly given as $-\frac{1}{4\pi r}$ as a function of r (with some mass factor). Therefore we exactly reproduce the Newtonian limit at linear order.

5.3.4 Long Distance Mirrors and Transverse Pöschl–Teller Potential

We will now illustrate the relationship between these two solutions via the methodology of 3.4.3. We consider the Green function G^K on $\mathbb{R}^3 \times_W \mathcal{M}$ which obeys special Neumann conditions at $\rho = 0$. That is, we are interested in comparison between the regularity and boundary conditions which permit a zero mode, that is

$$\begin{gathered}
\left(\sinh(2\rho) \log \circ \tanh \rho \, \partial_\rho - 2\right) G^T(r, \rho - \eta)\Big|_{\rho=0} = 0 \,, \qquad \partial_r G^T(r, \rho - \eta)\Big|_{r=0 \text{ and } \rho \neq \eta} = 0 \,, \\
G^T(r, \rho - \eta)\Big|_{r \to \infty} = 0 \,, \qquad G^T(r, \rho - \eta)\Big|_{\rho \to \infty} = 0 \,.
\end{gathered} \tag{5.3.27}$$

In this case our interpolating function F^T solves

$$\left(\partial_r^2 + \frac{2}{r}\partial_r + \partial_\rho^2 + 2\coth(2\rho)\partial_\rho\right) F^T(r, \rho - \eta) = 0 \,. \tag{5.3.28}$$

F obeys the following general boundary conditions at $\rho = 0$,

$$\left(\sinh(2\rho) \log \circ \tanh \rho \, \partial_\rho - 2\right) F(r,\rho)\Big|_{\rho=0} = -\left(\sinh(2\rho) \log \circ \tanh \rho \, \partial_\rho - 2\right) G^K(r,\rho)\Big|_{\rho=0} \,. \tag{5.3.29}$$

From asymptotic analysis of our differential equation when $r \ll 1$ or $r \gg 1$, we know G^K approximately. We have, in the small r regime,

$$
\begin{aligned}
&-\left(\sinh(2\rho)\log\circ\tanh\rho\,\partial_\rho - 2\right) G^K(r,\rho)\Big|_{\rho=0} \\
&\quad = \frac{2}{\left(r^2+\eta^2\right)^{3/2}} - \left(\frac{6(\eta\log(\rho)+\eta)}{\left(r^2+\eta^2\right)^{5/2}}\rho + \mathcal{O}\left(\rho^2\right) + \mathcal{O}\left(\frac{1}{R}\right)\right)\Big|_{\rho=0} \\
&\quad = \frac{2}{\left(r^2+\eta^2\right)^{3/2}} + \mathcal{O}\left(\frac{1}{R}\right) .
\end{aligned}
\tag{5.3.30}
$$

The interpolating function F can be expanded in either of the $\{\zeta_\omega\}$ or $\{\xi_\omega\}$ bases from section 2.6.7. Choosing the $\{\xi_\omega\}$ basis for convenience, the Laplace equation (3.4.16) gives

$$
\left(\partial_r^2 + \frac{2}{r}\partial_r - \omega^2\right) f^\omega(r) = -\mu(\rho)\left(\xi_\omega(\rho)\partial_\rho F^T(r,\rho-\eta) - \left(\partial_\rho\xi_\omega\right) F^T(r,\rho-\eta)\right)\Big|_{\rho=0}^{\rho\to\infty} . \tag{5.3.31}
$$

Here, we have projected (5.3.28) into the $\{\xi_\omega\}$ basis and have integrated by parts as in the previous section. For the $\omega=0$ zero mode, this simplifies, in the $r\to 0$ limit, to

$$
\left(\partial_r^2 + \frac{2}{r}\partial_r\right) f^0(r) = \pm\frac{2\sqrt{3}}{\pi}\frac{2}{\left(\eta^2+r^2\right)^{\frac{3}{2}}} + \mathcal{O}\left(\frac{1}{R}\right) . \tag{5.3.32}
$$

We shall momentarily disregard the subleading $\mathcal{O}\left(\frac{1}{R}\right)$ corrections. The solution to 5.3.32 is then

$$
f^0(r) = \pm\frac{2\sqrt{3}}{\pi}\left(\frac{1}{\eta} - \frac{\sinh^{-1}\left(\frac{r}{\eta}\right)}{r}\right) + \frac{c_1}{r} + k_1 , \tag{5.3.33}
$$

where c_1 and k_1 are integration constants. Since f^0 must be regular at $r=0$, we must have $c_1 = 0$. Now we consider the $\mathcal{O}\left(\frac{1}{R}\right)$ corrections. Since the explicit form of G^N is not known, these cannot be written in closed form.

However, we do know that G^K must vanish as $r\to\infty$, at least exponentially fast. So, as $r\to\infty$, the zero mode of F must solve

$$
\left(\partial_r^2 + \frac{2}{r}\partial_r\right) f^0(r) = \pm\frac{A\exp(-r)}{r^2} , \tag{5.3.34}
$$

where A is some unspecified constant given by the asymptotic form of G^K. Assuming that f^0 vanishes when $r\to\infty$, the solution to this is

$$
f^0(r) = \pm A\left(\frac{\exp(-r)}{r} + \mathrm{Ei}(-r)\right) + \frac{c_2}{r} , \tag{5.3.35}
$$

where $\mathrm{Ei}(-r)$ is the exponential integral function, and c_2 is a constant. We now define

$$f^{\mathrm{in}} = \pm\frac{2\sqrt{3}}{\pi}\left(\frac{1}{\eta} - \frac{\sinh^{-1}\left(\frac{r}{\eta}\right)}{r}\right) + k_1\,, \quad f^{\mathrm{out}} = \pm A\left(\frac{\exp(-r)}{r} + \mathrm{Ei}(-r)\right) + \frac{c_2}{r}\,, \tag{5.3.36}$$

for inside and outside solutions, respectively. We assume some crossover point $r = l$ where $\frac{\exp(-r)}{r^2}$ becomes a better estimate of G^K than $\frac{1}{R^3}$ and require continuity of the functions,

$$f^{\mathrm{in}}(r)\Big|_{r=l} = f^{\mathrm{out}}(r)\Big|_{r=l}\,, \quad \partial_r f^{\mathrm{in}}(r)\Big|_{r=l} = \partial_r f^{\mathrm{out}}(r)\Big|_{r=l}\,. \tag{5.3.37}$$

Solving these junction conditions fixes the remaining constants k_1 and c_2:

$$\begin{aligned} k_1 &= \pm\left(\frac{2\sqrt{3}}{\pi\sqrt{l^2+\eta^2}} - \frac{2\sqrt{3}}{\pi\eta} + A\,\mathrm{Ei}(-l)\right)\,, \\ c_2 &= \pm\left(\frac{2\sqrt{3}l}{\pi\sqrt{l^2+\eta^2}} - \frac{2\sqrt{3}\sinh^{-1}\left(\frac{l}{\eta}\right)}{\pi} - Ae^{-l}\right)\,. \end{aligned} \tag{5.3.38}$$

The constant k_1 is irrelevant to the large r behavior, but c_2 gives the lower-dimensional behavior at large r. Specifically, when $\eta \ll 1$, one has

$$c_2 = \pm\frac{2\sqrt{3}}{\pi}\log(\eta) + h(l,\eta)\,, \tag{5.3.39}$$

where $h(l,\eta) = \mathcal{O}(\eta^0)$. The independence of l in the first term of c_2 shows that it is valid to estimate f^0 by matching f^{in} and f^{out}. Ignoring $h(l,\eta)$ since it is finite as $\eta \to 0^+$, we reconstruct the leading order of $F^T(r,\rho-\eta)$ when $r \gg 1$ by multiplying our solution for f^0 by the zero mode $\xi_0(\rho)$ to find

$$F^T(r,\rho-\eta) = \frac{12}{\pi^2}\log\tanh\rho\log(\eta)\frac{1}{r} + \mathcal{O}(\eta^0) + \mathcal{O}\left(\frac{1}{r^2}\right)\,, \tag{5.3.40}$$

which agrees with our preliminary analysis in the $\eta \to 0^+$ limit.

5.4 Inferring Newton's Constant for SS–CGP

Owing to the non-trivial nature of the SS–CGP background, it is difficult to solve for all components of a gravitational perturbation that is sourced at the $(r,\rho) = (0,0)$ origin. However, our goal is to understand whether brane-gravity localization is possible, and, to this end, we only need to compute the effective gravitational potential associated with the perturbation and from that infer the lower-dimensional Newton constant. The problem of defining a lower-dimensional Newton constant can be interpreted at the level of the field theory action by reading off the coupling of matter fields to the metric. However, one alternate method by which one might determine the Newton constant, and also determine the dimension to which the effective gravity corresponds, would be to measure the response of a test particle to a known source mass and so to infer the corresponding Newton constant by the way geodesics in spacetime are distorted by the gravitational perturbation.

If one considers a weak-field limit in the neighbourhood of a source mass for a perturbation caused by the source in a Minkowski spacetime, only the details of the time-time component[71] of the perturbation need be known [26]. We will show that this is also true in the SS–CGP background.

5.4.1 The SS–CGP Background

In this section, we will consider timelike geodesics on the SS–CGP background (5.1.1). The affinely parametrized geodesic equation for a path γ is given by [126]

$$\frac{d^2 Z^M}{d\tau^2} + \Gamma^M{}_{KL}(Z)\frac{dZ^K}{d\tau}\frac{dZ^L}{d\tau} = 0\,, \tag{5.4.1}$$

where $Z^M = (X^\mu, Y, P, \Theta, \Phi, \Sigma, \Psi)$ are the coordinates of the path γ, and τ is the proper time. We use capital letters here in order to avoid confusion with the global coordinates in (5.1.1).[72] This equation of motion gives extrema for the Lagrangian[73]

$$L = g_{MN}(Z)\frac{dZ^M}{d\tau}\frac{dZ^N}{d\tau}\,. \tag{5.4.2}$$

The isometry group of (5.1.1) is given by

$$\mathrm{Isom}_{10} = ISO(1,3) \times U(1)^3 \times SO(3)^2\,, \tag{5.4.3}$$

where the $U(1)^3$ corresponds to the 3-torus parametrized by (y, ψ, χ), and the $SO(3)^2$ is the isometry of the S^2 parametrized by (θ, φ). Using these isometries, we find that a partial solution to the geodesic equation is

$$Y = 0\,, \quad \Theta = \pi\,, \quad \Phi = 0\,, \quad \Sigma = 0\,, \quad \Psi = 0\,. \tag{5.4.4}$$

[71] This statement makes use of an implicit gauge. We give full details of our gauge below.
[72] $(X^\mu, Y, P, \Theta, \Phi, \Sigma, \Psi)$ correspond to $(x^\mu, y, \rho, \theta, \varphi, \chi, \psi)$.
[73] We use the form (5.4.2) of the particle Lagrangian in this discussion for simplicity, instead of the worldline reparametrization invariant proper-time action $\int \frac{d\tau(p)}{dp}\,dp$ involving a square root. The Lagrangian (5.4.2) can of course be obtained from the einbein form [25] $L_{\mathrm{BdVH}} = \frac{1}{2}(e^{-1} g_{MN}(Z)\frac{dZ^M}{d\tau}\frac{dZ^N}{d\tau} + m^2_{\mathrm{particle}} e)$ by choosing the reparametrization gauge $e = \frac{1}{2}$.

Simplifying the Lagrangian (5.4.2) for a solution that obeys conditions (5.4.4), we find

$$L = (\cosh 2P)^{1/4}\Big[\eta_{\mu\nu}\frac{dX^\mu}{d\tau}\frac{dX^\nu}{d\tau} + \frac{1}{g^2}\Big(\frac{dP}{d\tau}\Big)^2\Big] . \tag{5.4.5}$$

The equation of motion for $P(\tau)$ is then

$$\frac{d^2P}{d\tau^2} + \frac{1}{4}(\tanh 2P)\Big(\frac{dP}{d\tau}\Big)^2 - \frac{g^2}{4}(\tanh 2P)\eta_{\mu\nu}\frac{dX^\mu}{d\tau}\frac{dX^\nu}{d\tau} = 0 . \tag{5.4.6}$$

This admits the solution

$$P(\tau) = 0 . \tag{5.4.7}$$

The remaining equations for X^μ are the usual geodesic equations on $\mathbb{R}^{1,3}$. Remembering that we are looking for a timelike geodesic, the appropriate solution is

$$X^0 = \tau , \quad X^i = 0 . \tag{5.4.8}$$

To summarize, we find that the SS–CGP geometry admits the stable timelike geodesic

$$X^0 = \tau , \quad X^i = 0 , \quad Y = 0 , \quad P = 0 , \quad \Theta = \pi , \quad \Phi = 0 , \quad \Sigma = 0 , \quad \Psi = 0 . \tag{5.4.9}$$

This will be the starting point for the next section, where we look for a timelike geodesic on a perturbed SS–CGP geometry.

5.4.2 Perturbed SS–CGP Geodesics

We consider the perturbed geometry described by a metric $\hat{g}$, with

$$\hat{g}_{MN} = (\cosh 2\rho)^{1/4}(\bar{g}_{MN} + \mathcal{H}_{MN}) , \tag{5.4.10}$$

where $\bar{g}$ is the string-frame metric [32] on the SS–CGP background,[74] and $\mathcal{H}_{MN}$ is a perturbation. The perturbation that we are interested in is independent of the time coordinate x^0, and has components only along the x^μ and ρ directions, with $\mathcal{H}_{0i} = 0$. As such, the $U(1)^3 \times SO(3)^2$ isometry of the SS–CGP background is unbroken, and we have that

$$Y = 0 , \quad \Theta = \pi , \quad \Phi = 0 , \quad \Sigma = 0 , \quad \Psi = 0 \tag{5.4.11}$$

[74] The string-frame metric is related to the Einstein-frame metric by $ds^2_{\text{str}} = e^{\phi/2}ds^2_{\text{Ein}}$.

solves the perturbed geodesic equations. With this choice, the perturbed particle Lagrangian is

$$\begin{aligned} L &= \hat{g}_{MN}(Z)\frac{dZ^M}{d\tau}\frac{dZ^N}{d\tau} \\ &= (\cosh 2P)^{1/4}\Big[(\eta_{\mu\nu}+\mathcal{H}_{\mu\nu})\frac{dX^\mu}{d\tau}\frac{dX^\nu}{d\tau} + 2A_\mu\frac{dX^\mu}{d\tau}\frac{dP}{d\tau} + \frac{1}{g^2}(1+B)\Big(\frac{dP}{d\tau}\Big)^2\Big]\,, \end{aligned} \tag{5.4.12}$$

where we have defined

$$\mathcal{H}_{\mu\rho}(X^i,P) \equiv A_\mu(X^i,P)\,, \quad \mathcal{H}_{\rho\rho}(X^i,P) = \frac{1}{g^2}B(X^i,P)\,. \tag{5.4.13}$$

It is important to note that $\mathcal{H}_{\mu\nu}$, A_μ and B are functions of X^i and P only.

The resulting equations for X^μ and P are given by

$$\begin{aligned} &(\delta^\mu_\nu + \mathcal{H}^\mu{}_\nu)\frac{d^2X^\nu}{d\tau^2} + \frac{1}{2}\eta^{\mu\nu}\big(\partial_\sigma\mathcal{H}_{\lambda\nu} + \partial_\lambda\mathcal{H}_{\sigma\nu} - \partial_\nu\mathcal{H}_{\sigma\lambda}\big)\frac{dX^\sigma}{d\tau}\frac{dX^\lambda}{d\tau} \\ &+ A^\mu\Big(\frac{d^2P}{d\tau^2} + \frac{1}{2}(\tanh 2P)\Big(\frac{dP}{d\tau}\Big)^2\Big) + \Big(\partial_P A^\mu - \frac{1}{2g^2}\partial^\mu B\Big)\Big(\frac{dP}{d\tau}\Big)^2 \\ &+ \Big(F_\nu{}^\mu + \partial_P\mathcal{H}^\mu{}_\nu + \frac{1}{2}(\tanh 2P)(\delta^\mu_\nu + \mathcal{H}^\mu{}_\nu)\Big)\frac{dX^\nu}{d\tau}\frac{dP}{d\tau} = 0\,, \end{aligned} \tag{5.4.14}$$

and

$$\begin{aligned} &\frac{1}{g^2}(1+B)\Big(\frac{d^2P}{d\tau^2} + \frac{1}{4}(\tanh 2P)\Big(\frac{dP}{d\tau}\Big)^2\Big) + \frac{1}{2g^2}\partial_P B\Big(\frac{dP}{d\tau}\Big)^2 + A_\mu\frac{d^2X^\mu}{d\tau^2} \\ &+ \frac{1}{g^2}\partial_\mu B\frac{dX^\mu}{d\tau}\frac{dP}{d\tau} + \Big(\partial_{(\mu}A_{\nu)} - \frac{1}{2}\partial_P\mathcal{H}_{\mu\nu} - \frac{1}{4}\tanh 2P\,(\eta_{\mu\nu}+\mathcal{H}_{\mu\nu})\Big)\frac{dX^\mu}{d\tau}\frac{dX^\nu}{d\tau} = 0\,, \end{aligned} \tag{5.4.15}$$

where

$$\partial_\mu \equiv \frac{\partial}{\partial X^\mu}\,, \quad \partial_P \equiv \frac{\partial}{\partial P}\,, \quad F_{\mu\nu} \equiv 2\partial_{[\mu}A_{\nu]}\,, \tag{5.4.16}$$

and the μ,ν indices are raised by $\eta^{\mu\nu}$.

We now consider a deviation of the original timelike geodesic on the SS–CGP background as given in (5.4.9). We write

$$X^0 = \tau + \delta X^0\,, \quad X^i = \delta X^i\,, \quad P = \delta P\,. \tag{5.4.17}$$

We will treat δP and the τ-derivatives of these deviations as small (the Newtonian limit), and will only consider terms of order 1 in perturbations. Here, the considered perturbations include $\mathcal{H}_{\mu\nu}$, A_μ, and B, the τ-derivatives of δX^μ and δP, as well as δP itself; so we will neglect, for example, terms of the form

$$B\frac{d^2\delta P}{d\tau^2} = \mathcal{O}(\mathrm{pert}^2)\,. \tag{5.4.18}$$

The resulting linearized equations for X^μ are then

$$\frac{d^2\delta X^0}{d\tau^2} = 0\,, \quad \frac{d^2\delta X^i}{d\tau^2} = \frac{1}{2}\frac{\partial}{\partial\delta X^i}\mathcal{H}_{00}\,, \tag{5.4.19}$$

where we have used that $\mathcal{H}_{\mu\nu}$ is independent of X^0. The first equation allows us to set $\delta X^0 = 0$, and so $X^0 = \tau$ at least in this linearized regime. Thus, we are allowed to interpret τ as the underlying manifold's time, which we will write as $x^0 = t$. The δX^i equation is then Newton's equation, with a gravitational potential

$$V_N(\delta X^i, \delta P) = -2m_{\text{particle}}\mathcal{H}_{00}(\delta X^i, \delta P)\,, \tag{5.4.20}$$

where m_{particle} is the small mass of the test particle following the geodesic.

Finally, we also have the δP equation, which in our approximation reads

$$\frac{d^2\delta P}{dt^2} + \frac{g^2}{2}\delta P - \frac{g^2}{2}\frac{\partial}{\partial\delta P}\mathcal{H}_{00} = 0\,, \tag{5.4.21}$$

where we have used that A_μ is independent of X^0. Using the Newtonian potential defined in (5.4.20), and removing the δ's from δX^i and δP for convenience, we can rewrite these equations as

$$m_{\text{particle}}\frac{d^2X^i}{dt^2} = -\frac{\partial}{\partial X^i}V_N\,, \tag{5.4.22}$$

$$m_{\text{particle}}\Big(\frac{d^2P}{dt^2} + \frac{g^2}{2}P\Big) = -g^2\frac{\partial}{\partial P}V_N\,. \tag{5.4.23}$$

In conclusion, the leading effect of perturbations about the SS–CGP background is through $V_N \propto \mathcal{H}_{00}$.

5.5 The Worldvolume Newton Constant

Now that we have the effective Newton potentials for the Type I to Type III cases, we want to understand their physics. In particular, we are interested in studying whether these potentials have a lower-dimensional (four-dimensional) behaviour, and if they do, what is the effective, four-dimensional Newton constant. Let's begin by analyzing the Type I and II cases. Type I solutions (black spokes), as we recall, correspond to worldvolume Ricci-flat solutions. Although these solutions are clearly four-dimensional in nature and actually solve a full nonlinear self-interacting equation, they do not correspond to a specific four-dimensional Newton constant. This is because of the worldvolume 'trombone' symmetry that is inherent in the Ricci-flat family of solutions - any rescaling of the worldvolume metric by any positive constant remains a solution [33]. Due to the existence of this symmetry, there is no well-defined four-dimensional Newton's constant, as its value can always be scaled to a different value by a trombone transformation. The trombone symmetry, however, can be broken when we couple to external sources. In such cases, since our transverse space is non-compact, the usual argument of Reference [74] states that the four-dimensional Newton's constant vanishes. What this really means is that, in contrast to the Ricci-flat self interactions, coupling of the black spokes to external sources would be inherently higher-dimensional instead of four-dimensional.

As for Type II solutions, they clearly do not exhibit four-dimensional behaviour. From (B.4.5), we found that the large-distance behaviour of the Type II potential is of a form corresponding to massive gravity in 5 dimensions. We will not be examining this further, but will move on to the Type III solutions, which as we can see from (5.3.24), do indeed have four-dimensional behaviour at large r worldvolume distance. However, it is not immediately obvious what the four-dimensional Newton constant should be, as it appears to depend on the non-compact transverse coordinate ρ. In the following, we offer three different approaches to identifying an appropriate four-dimensional Newton constant.

5.5.1 Newton's Constant or the Gravitational Coupling κ in Type III

All three interpretive approaches centre on the geodesic equations derived in Section 5.4, which we will reproduce here in radial coordinates on the worldvolume,

$$R''(t) - \frac{{l_W}^2}{R(t)^3} = -\frac{6g\hat{\kappa}^2 M}{\pi^3 R(t)^2} \log\tanh\left(P(t)\right) \log\tanh(\eta) + \mathcal{O}\left(\frac{1}{R(t)^3}\right) \,, \tag{5.5.1}$$

$$P''(t) + \frac{g^2}{2}P(t) = \frac{12 g^3 \hat{\kappa}^2 M}{\pi^3 R(t)} \frac{\log\tanh(\eta)}{\sinh(2P(t))} + \mathcal{O}\left(\frac{1}{R(t)^2}\right) \,, \tag{5.5.2}$$

where $0 < \eta \ll 1$ is the transverse coordinate of the mass M source, $R^2(t) = X^i(t)X^i(t)$, and ${l_W}^2$ is the worldvolume angular momentum. We note that although there is a sign difference between the two equations, the potential is attractive in both the worldvolume and transverse coordinates since for all $x > 0$, $\tanh x \in (0, 1)$, so the logarithmic terms are negative definite.

Method 1: Fixed Points

Our first approach is to find fixed points of the geodesic equation where $P(t)$ is constant. We can then infer the four-dimensional Newton constant by substituting this fixed point into (5.5.1). At first glance, however, we find that there are no fixed points for $P(t)$. In order to generate one, recall that our 5-dimensional system can be embedded in 10 dimensions, where the ρ coordinate is paired with the angular coordinate χ, forming an $\mathbb{R}^2$. So we may suppose that by restoring nontrivial χ dependence, there will be an additional angular momentum term in (5.5.2). More precisely, we have

$$P''(t) + \frac{g^2}{2}P(t) - \frac{{l_T}^2}{P(t)^3} = \frac{6g^3\hat{\kappa}^2 M}{\pi^3 R(t)}\frac{\log(\eta)}{P(t)} + \mathcal{O}\left(\frac{1}{R(t)^2}\right) , \tag{5.5.3}$$

where l_T is the transverse angular momentum. If we ignore the higher order corrections involving the radius $R(t)$ and take $P(t) = P$ to be constant, this equation simplifies to

$$\frac{g^2}{2}P^4 - \frac{6g^3\hat{\kappa}^2 M}{\pi^3 R(t)}\log(\eta)P^2 - {l_T}^2 = 0 . \tag{5.5.4}$$

The only positive solution for this is

$$P = 2^{\frac{1}{4}}\sqrt{\frac{l_T}{g}} + \frac{3g^{\frac{3}{2}}\hat{\kappa}^2 M\log(\eta)}{2^{1/4}\pi^3 R(t)\sqrt{l_T}} + \mathcal{O}\left(\frac{1}{R(t)^2}\right) . \tag{5.5.5}$$

Of course, we can find the leading order of this expression by simply suppressing the quadratic term in P in equation (5.5.4). This reflects the structure of the background: since there is an attractive potential, there is a stable circular orbit where

$$P = 2^{\frac{1}{4}}\sqrt{\frac{l_T}{g}} . \tag{5.5.6}$$

The additional attractive potential from the mass M source 'squeezes' this orbit, but at large world volume radius this squeezing fades out. If we suppose there is some minimum non-zero transverse angular momentum l_T, as in the Bohr–Sommerfeld quantisation condition, then we may suppose that P takes this value [83]. One may make a similar interpretation for the value of the mass M source transverse coordinate η.

Substituting (5.5.6) into (5.5.1), we then find that the $R(t)$ equation becomes

$$R''(t) - \frac{{l_W}^2}{R(t)^3} = -\frac{6g\hat{\kappa}^2 M}{\pi^3 R(t)^2}\left(\log\tanh\left(2^{\frac{1}{4}}\sqrt{\frac{l_T}{g}}\right)\right)^2 + \mathcal{O}\left(\frac{1}{R(t)^3}\right) , \tag{5.5.7}$$

$$\approx -\frac{6g\hat{\kappa}^2\log\left(\sqrt{2}\frac{g}{l_T}\right)^2 M}{4\pi^3 R(t)^2} . \tag{5.5.8}$$

If we compare this to the usual radial geodesic equation in four dimensions

$$r''(t) - \frac{l_W^2}{r(t)^3} = -\frac{\kappa^2 M}{4\pi r(t)^2} , \tag{5.5.9}$$

we find a value for the effective four-dimensional gravitational coupling

$$\kappa = \frac{\sqrt{6g}}{\pi}\left|\log\left(\frac{\sqrt{2}l_T}{g}\right)\right|\hat{\kappa} \; , \tag{5.5.10}$$

where we recall that $\hat{\kappa}$ is the five-dimensional gravitational coupling constant.

Method 2: Quantum Localization

We can go beyond the above semiclassical picture, if we want to consider that our geodesic equation becomes nonsingular due to quantum effects. Specifically we may ask what the instantaneous worldvolume radial force is on a purely quantum test particle, defined by a separable wavefunction

$$\Psi(x^i, \rho) = \psi(x^i)\phi(\rho) \; , \tag{5.5.11}$$

where we assume that the worldvolume wavefunction ψ is some Gaussian wave packet with a negligible width compared to the worldvolume radius r of its centroid. To apply a quantum mechanical analysis, we note that our geodesic equations can be obtained from the following Lagrangian (calculated to give the geodesic equations (5.5.1) and (5.5.2) as its equation of motion)

$$L = \frac{1}{2}\left(\frac{d}{dt}X^i(t)\right)^2 + \frac{1}{2g^2}\left(\frac{d}{dt}P(t)\right)^2 - \frac{1}{2}P(t)^2 - \frac{\mu}{R(t)}\log\tanh(P(t)) \; , \tag{5.5.12}$$

where $\mu = \frac{12g\hat{\kappa}^2 M}{\pi^3}\log\tanh(\eta)$. If we assume that the X^i are effectively constant, the associated Hamiltonian is

$$H = \frac{1}{2}\Pi(t)^2 + \frac{g^2}{2}P(t)^2 + \frac{\mu}{r}\log\tanh(P(t)) \; . \tag{5.5.13}$$

Therefore, we may study functions $\phi_E(\rho)$ that solve the associated time independent Schrödinger equation (TISE),

$$E\phi_E(\rho) = \left(-\frac{\hbar^2}{2}\frac{d^2}{d\rho^2} + \frac{g^2}{2}\rho^2 + \frac{\mu}{r}\log\tanh(\rho)\right)\phi_E(\rho) \; , \tag{5.5.14}$$

and we will focus on the ground state $\phi_0 = \phi$, as we are interested in small (low-energy) quantum excitations.

The TISE (5.5.14) was derived with the assumption that the worldvolume motions X^i are effectively constant. This is a good approximation when $r^2 = X^i X^i \gg 1$. Now, if we assume that ρ is finite, then along with the assumption $r^2 \gg 1$, the TISE asymptotes to the equation describing a quantum harmonic oscillator, with its well-known solutions [62]. The ground state, in particular, is [75]

$$\phi(\rho) = \sqrt{2}\left(\frac{g}{\pi\hbar}\right)^{\frac{1}{4}} \exp\left(-\frac{g\rho^2}{2\hbar}\right) + \mathcal{O}\left(\frac{\mu}{r}\right) \ . \tag{5.5.15}$$

For $\rho \ll 1$ on the other hand, the logarithmic term in the TISE is no longer negligible even in the large r approximation, and the equation for the ground state at fixed r approximates instead to

$$\left(-\frac{\hbar^2}{2}\frac{d^2}{d\rho^2} + \frac{\mu}{r}\log(\rho)\right)\phi(\rho) = 0 \ . \tag{5.5.16}$$

To our knowledge, the exact solution to this differential equation is unknown. However, if we make a WKB approximation [13] (noting that the $f_n(\rho)$ will in general be complex)

$$\phi(\rho) = \exp\left(\frac{1}{\hbar}f_{-1}(\rho) + f_0(\rho) + \hbar f_1(\rho) + \mathcal{O}(\hbar^2)\right) \ , \tag{5.5.17}$$

then, to leading order in $\hbar$, we find,

$$\left(\frac{d}{d\rho}f_{-1}(\rho)\right)^2 = \frac{2\mu}{r}\log(\rho) \ , \tag{5.5.18}$$

which has solutions

$$f_{-1}^{\pm} = k^{\pm} \pm i\sqrt{\frac{2\mu}{r}}\left(\rho\sqrt{-\log\rho} - \frac{\sqrt{\pi}}{2}\mathrm{Erf}\left(\sqrt{-\log\rho}\right)\right) \ , \tag{5.5.19}$$

where $k^{\pm}$ are integration constants, and Erf is the error function [2]. Therefore in the $\rho \ll 1$ regime, ϕ is given by a superposition

$$\begin{aligned}\phi(\rho) = A\exp&\left(\frac{i}{\hbar}\sqrt{\frac{2\mu}{r}}\left(\rho\sqrt{-\log\rho} - \frac{\sqrt{\pi}}{2}\mathrm{Erf}\left(\sqrt{-\log\rho}\right)\right)\right)\\ &+ B\exp\left(-\frac{i}{\hbar}\sqrt{\frac{2\mu}{r}}\left(\rho\sqrt{-\log\rho} - \frac{\sqrt{\pi}}{2}\mathrm{Erf}\left(\sqrt{-\log\rho}\right)\right)\right) \ .\end{aligned} \tag{5.5.20}$$

Since ϕ is a ground-state quantum wavefunction, we will require that it obey the special Neumann boundary condition at $\rho = 0$:

$$\partial_\rho\phi\big|_{\rho=0} = 0 \, . \tag{5.5.21}$$

This is the same condition as that obeyed by the quantum harmonic oscillator ground state (5.5.15), as is appropriate for an S-wave ground state when one recalls that Equation (5.5.14) is the radial part of a

[75] Since the domain is the positive real line, we are a factor of $\sqrt{2}$ different from the standard normalization of the ground state of the quantum harmonic oscillator.

transverse two-dimensional Schrödinger problem in (ρ, χ). Condition (5.5.21) relates the coefficients A and B:

$$A = B \exp\left(i\sqrt{\frac{2\pi\mu}{r}}\right) . \tag{5.5.22}$$

We can now determine the remaining coefficient B by matching the large r limit of (5.5.20) with the harmonic oscillator ground state (5.5.15) at $\rho = 0$, which gives

$$B = \sqrt{\frac{g}{\pi\hbar}} . \tag{5.5.23}$$

In particular, we can match (5.5.20) with (5.5.15).

That is that the solution to the unperturbed problem supports a solution with the same asymptotic behavior of the perturbed problem at order $\mathcal{O}(\mu/r)$. If the perturbed problem had qualitatively different asymptotic solutions (say $\log(\rho)$ versus ρ^2) and specifically had dominated the unperturbed problem, then we could, in principle, see corrections that dominate the leading order of our estimate of κ. However, in this we can, up to corrections of order $\mathcal{O}(\mu/r)$, compute expectation values using just the solutions to the harmonic oscillator.

The expectation value we are interested in is the transverse-space dependent part of the right-hand-side of (5.5.1). This allows us to deduce the four-dimensional effective Newton constant. Explicitly, we find

$$\kappa^2 = \frac{24g\hat{\kappa}^2}{\pi^2} \langle \log\tanh(\rho) \rangle \log\tanh(\eta) , \tag{5.5.24}$$

with the expectation value for an operator $f(\rho)$ defined as

$$\langle f(\rho) \rangle = \int_0^\infty 2\sqrt{\frac{g}{\pi\hbar}} \exp\left(-\frac{g\rho^2}{\hbar}\right) f(\rho) d\rho + \mathcal{O}\left(\frac{\mu}{r}\right) . \tag{5.5.25}$$

We may similarly choose to consider both the test particle and the source to be governed by the same transverse quantum Schrödinger problem. Given that, we find the effective four-dimensional gravitational coupling at large r distance

$$\kappa = -\sqrt{24g}\frac{\hat{\kappa}}{\pi} \langle \log\tanh(\rho) \rangle . \tag{5.5.26}$$

We are unable to compute such expectation values analytically. But, if we set $\hbar/g = 1$, we can give a numerical approximation:

$$\kappa = \sqrt{g}\,(1.73338\ldots)\,\hat{\kappa} . \tag{5.5.27}$$

Method 3: Smeared Transverse Expectation Values

Of course, calculating expectation values given some transverse profile function does not require a fully quantum treatment. We can instead imagine measuring the instantaneous acceleration of a particle whose transverse position is drawn from a smeared distribution of possible positions in the transverse direction.

We may suppose that the test particles have $P(0) = P$ and suppose that the probability $\mathcal{P}$ of P taking a given value between $0 < a < b < \infty$ is

$$\mathcal{P}(a < P < b) = \int_a^b f_P(\rho) d\rho \, , \tag{5.5.28}$$

where we define our random variable, P, by its probability density function f_P. The average instantaneous acceleration we measure for a test particle drawn from this distribution is

$$\langle R''(0) \rangle - \left\langle \frac{l_W^2}{R(0)^3} \right\rangle = - \left\langle \frac{6\hat{\kappa}^2 M}{\pi^3 g R(0)^2} \log\tanh(P(0)) \log\tanh(\eta) \right\rangle + \mathcal{O}\left(\frac{1}{R(0)^3} \right) \, . \tag{5.5.29}$$

Assuming $R(0)$ and P are independent variables and our probability density function is correctly normalized, that is $\langle 1 \rangle = 1$, then

$$R''(0) - \frac{l_W^2}{R(0)^3} = - \frac{6g\hat{\kappa}^2 M}{\pi^3 R(0)^2} \langle \log\tanh(P) \rangle \log\tanh(\eta) + \mathcal{O}\left(\frac{1}{R(0)^3} \right) \, . \tag{5.5.30}$$

Here

$$\langle \log\tanh(P) \rangle = \int_0^\infty f_P(\rho) \log\tanh(\rho) d\rho \, . \tag{5.5.31}$$

We might choose to study any number of random distributions, but, given the suggestive form of the right-hand-side of equation (5.5.31) we will take

$$f_P(\rho) = \mu(\rho) \xi_0(\rho)^2 = \frac{12}{\pi^2} \sinh(2\rho) \left(\log\tanh(\rho) \right)^2 \, . \tag{5.5.32}$$

Given this,

$$\langle \log\tanh(P) \rangle = \frac{9\zeta(3)}{\pi^2} \, , \tag{5.5.33}$$

with $\zeta(z)$ the Riemann zeta function [61]. This determines the four-dimensional κ to be

$$\kappa^2 = - \frac{216\zeta(3) g\hat{\kappa}^2}{\pi^4} \log\tanh(\eta) \, . \tag{5.5.34}$$

We can similarly average to get an expected value for $\log\tanh(\eta)$, to find

$$\kappa = \sqrt{6g}\frac{18\zeta(3)}{\pi^3}\hat{\kappa}\ . \tag{5.5.35}$$

We may compare this with the numerical value of κ given by the quantum treatment of the geodesic equation (5.5.27), finding here

$$\kappa = \sqrt{g}\,(1.70932\ldots)\,\hat{\kappa}\ , \tag{5.5.36}$$

and observe that these two approaches calculations agree to 3 parts in 100. Although the numerical result from the quantum treatment required setting $\hbar/g = 1$, the result will not change significantly if $\hbar/g$ is set to another finite constant. This is because the quantum expectation value is dominated by behaviour of $\exp(-g\rho^2/\hbar)\log\tanh\rho$ near the origin, which only deviates very slowly as a function of the ratio $\hbar/g$. The result of (5.5.35) also agrees precisely with the value found in Reference [32] for the four-dimensional graviton self-coupling κ, up to corrections arising from the compactification of higher transverse dimensions other than ρ.

The Force Lines of the Newtonian Potential

In order to help visualizing the effect of the source near the origin on a test particle at some distance r away, and specifically to show how the resulting near field evolves into the far field, we have made approximate illustrations for Type II and Type III potentials.

These images were created by taking the leading orders of the potential in the near ($R \ll 1$) and far ($r \sim 1$) field limits and interpolating. The change brought about by the source perturbation needs to be considered in comparison to the effect of the unperturbed SS–CGP background. The effect of the background is a uniform attraction to $\rho = 0$ proportional to ρ. At small values of ρ, or for relatively massive sources, this background effect may be neglected. There is one additional scale of relevance, which is the ratio of g, the SS–CGP background parameter, to η, the height above the $\rho = 0$ plane at which the source is placed. In our illustrations we have chosen $\frac{\eta}{g} = 0.1$. We did not take any obvious limits, such as $\frac{\eta}{g} \to 0$ or ∞, because the Type III solution becomes infinite or vanishes in those limits respectively.

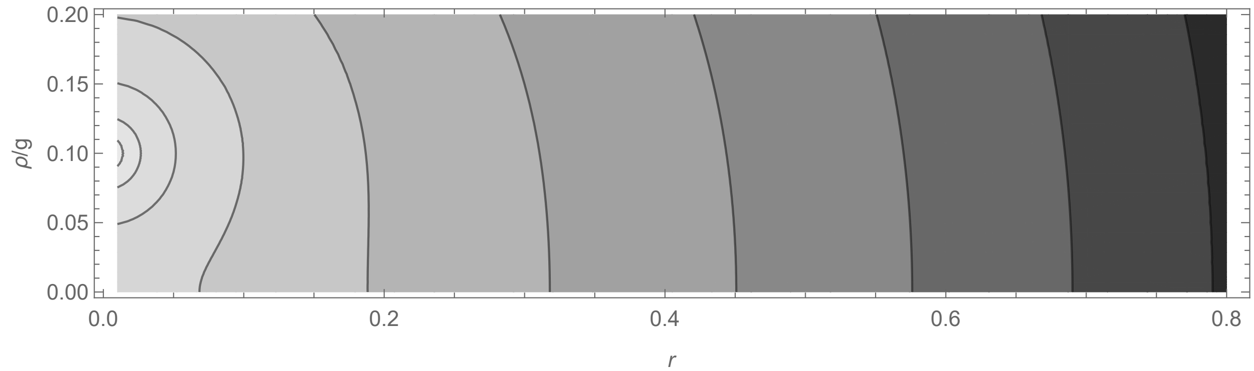

Figure 6: Equipotential surfaces of a Type II potential

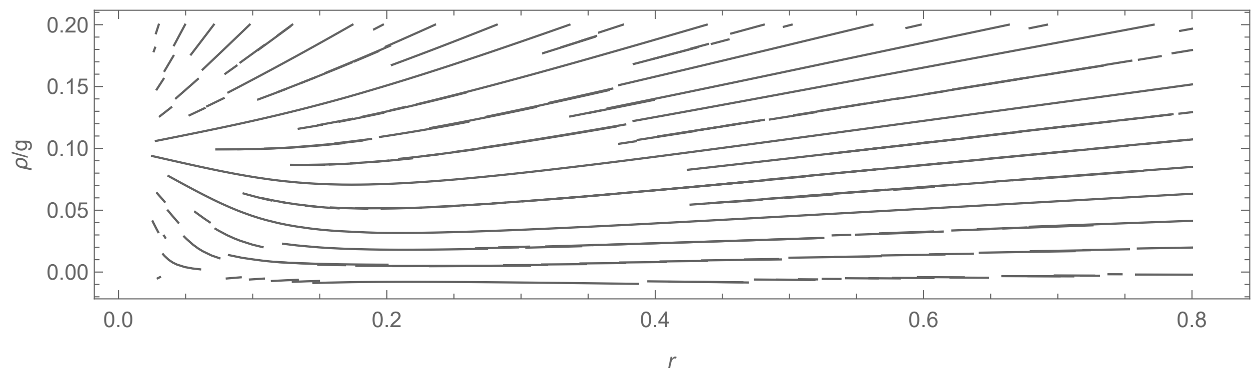

Figure 7: Force lines (gradient flows) of a Type II potential

We can see from Figures (6) and (7) that, near to the source, the Type II potential asymptotes to a spherically symmetric potential $\left(\frac{1}{R^3}\right)$. Note that the lines in the two figures are orthogonal to each other. Arbitrarily far away, the equipotential surface shapes asymptote to an oblate spheroid which has twice the radius in the ρ direction as in the r direction. It is not seen from the illustration that the Type II solution is exponentially decaying at large r. Overall, the particle is drawn towards the source with relative disregard (in comparison to Type III) for its ρ position.

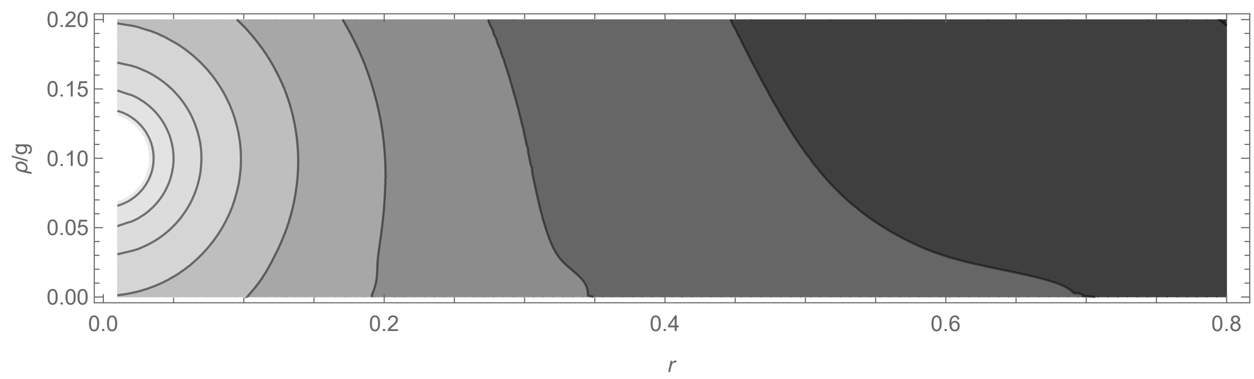

Figure 8: Equipotential surfaces of a Type III potential

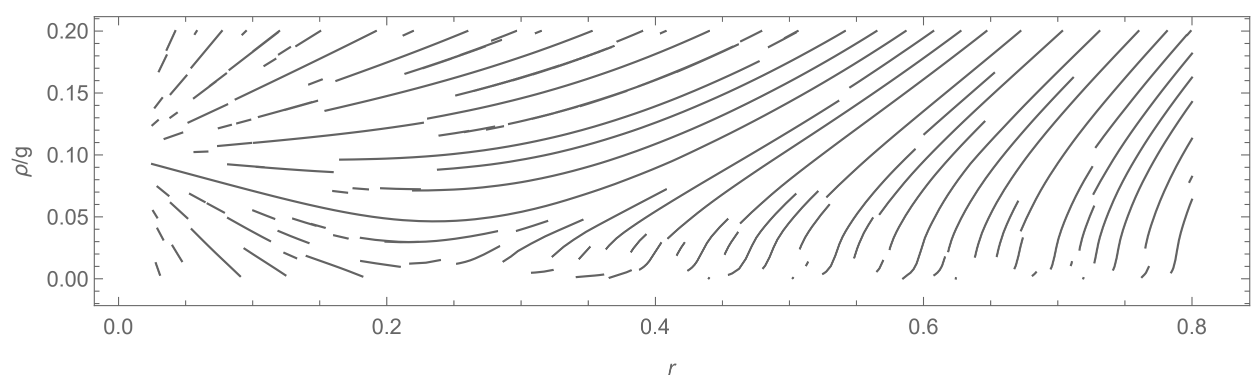

Figure 9: Force lines (gradient flows) of a Type III potential

Now contrast this with the Type III potential shown in figures (8) and (9). Near to the source on the SS–CGP background in the Type III situation, the potential behaves asymptotically in a similar fashion as in the Type II situation. The difference occurs for large r.

For the sake of clarity, we have regularized the $\xi_0 \propto \log\tanh(\rho)$ transverse wavefunction. The equation for the regularised $\tilde{\xi}_0$ is

$$\left(\partial_\rho^2 + 2\coth(2\rho)\right)\tilde{\xi}_0(\rho) = \frac{1}{\epsilon}\left(\tanh\left(\alpha\left(\epsilon - \rho\right)\right) - 1\right)\ . \tag{5.5.37}$$

We have chosen to regularise ξ_0 in this way so that all force lines in the illustration end on the perturbative source at the displace point $r = 0$, $\rho = \eta$. When $\alpha \gg 1$, the right-hand side of equation (5.5.37) approximates a step function, normalized so that it integrates to one over the half open integral. In our illustration we have chosen $\alpha = 100$ and $\epsilon = 0.02$.

One can see in Figures (8) and (9) that the Type III force lines concentrate as one approaches $\rho = 0$ or, alternately, that the equipotential surfaces spread out with increasing r along the $\rho = 0$ subsurface. The fact that the potential at large r is proportional to the ξ_0 transverse wavefunction is due to the Type III boundary condition (5.3.27), or, equivalently, to the presence of a boundary term placed at $\rho = 0$ in order to enforce the boundary condition. The $\xi_0 \to \tilde{\xi}_0$ regularization is equivalent to the smearing of that condition/source.

Due to the $\tilde{\xi}_0$ smearing/regularizing, the effect of the boundary term can be seen near to $\rho = 0$, as opposed to at $\rho = 0$ in our illustration. Specifically one sees the force lines on the far right travel downwards towards $\rho = 0$, in response to the presence of the boundary term. Close to $\rho = 0$, the force lines bend left as the r dependence of the boundary term draws them towards the source. Then as they approach the origin they then bend back upwards towards the source at $(r, \rho) = (0, \eta)$. If one removes the $\tilde{\xi}_0$ regularization, almost all force lines concentrate within the $\rho = 0$ subplane. That does not largely effect the long-range potential, but the regularized $\tilde{\xi}_0$ helps the visualization.

Due to the boundary condition/term at $\rho = 0$, the force in the Type III situation falls more slowly at large r than in Type II, that is it does not decay exponentially when $r \gg 1$. Instead, the potential has an $1/r$ falloff as we found in the Type III Green function (5.3.24). The total effect is similar to the RSII 'brane bending' as described by Giddings, Katz, and Randall [60].

6 Effective Field Theories of Scalars

What are the implications of dimensional reduction outside the standard Neumann-Neumann or periodic boundary conditions, that is outside of consistent truncations?

At the level of an effective field theory of only scalars, the effect of choosing different transverse spaces for our effective field theory is only relevant in how it affects the spectrum and overlap integrals, therefore it creates an ideal juxtaposition with standard Kaluza–Klein reduction (which we cover in section 7). In this section we will explore how inconsistent truncations can be 'integrated out' at the cost of nonlocal operators suppressed by the mass gap between the lightest particle, and the next lightest particle.

This section also lays out a procedure for accomplishing the dimensional reduction to maintain consistency between the higher- and lower-dimensional actions and equations of motion, which we dub the 'Dimensional Reduction Square'. We will describe the infinitude of possible higher-dimensional boundary terms which allow us to select any boundary condition, and attempt to argue for a standard form. We will then integrate out the heavy fields and note the structure of each of the possible corrections as we accomplish this, emphasizing when these are different to the corrections in a consistent truncation.

6.1 The Dimensional Reduction Square

In general, there are several technical problems with finding an effective field theory (EFT) with any non-trivial background or boundary conditions, especially at the level of the action. We will present this work in two steps, first we will find the free theory, and calculate the lower-dimensional degrees of freedom. Second we will calculate the interactions, and attempt to describe the theory in terms of its lightest 'effective' modes. We have discovered, during this procedure, we must ensure that we are preserving the following properties:

1. The extrema of our action must correspond to the solutions of the equations of motion given our boundary conditions. This is accomplished through specifying higher-dimensional boundary terms.

2. The higher-dimensional fields must be expressed in a basis which is complete, obeys our boundary conditions, and hopefully diagonalizes the transverse wave operator(s).

3. The higher-dimensional fields should be expressed in terms of lower-dimensional components which hopefully diagonalize the lower-dimensional degrees of freedom, and the lower-dimensional gauge transformations simultaneously.

Each of these seems an obvious requirement for the procedure of finding an EFT at the level of the action. However, the necessity of each of these steps can be illustrated with specific problems that arise in the lower-dimensional EFT when they are missed. The purpose of this section is to demonstrate deriving an EFT where the answer is already known, so that failures can be obviously diagnosed and the procedure can be derived and verified.

However, if one assumes that we did not have a concrete known answer, how does one identify when such a problem in the EFT has arisen? Generically we will find that the 'dimensional reduction square' will not commute. That is, when we take a higher-dimensional action, vary it to find higher-dimensional equations of motion, insert a Sturm–Liouville basis, and use linear independence to interpret the higher-dimensional equations of motion as lower-dimensional equations of motion, this should always produce the same physical system as when we take a higher-dimensional action, insert a Sturm–Liouville basis, integrate over the transverse dimension, and then vary the resulting lower-dimensional action to find lower-dimensional equations of motion.

$$\begin{tikzcd}
\mathcal{S}_{\text{higher}} \arrow[r, "\text{variation}"] \arrow[d, "\text{integration}", "\text{expansions}"'] \arrow[dr, phantom, "\circlearrowright"] & \text{EoM}_{\text{higher}} \arrow[d, "\text{expansions}"] \\
\mathcal{S}_{\text{lower}} \arrow[r, "\text{variation}"] & \text{EoM}_{\text{lower}}
\end{tikzcd} \tag{6.1.1}$$

We will spare the reader the somewhat laborious procedure of following the calculation of how the dimensional reduction square fails to commute when one of our enumerated points is missed, however, we will emphasize the relevant term in the lower-dimensional action that changes when the step is included correctly.

6.2 Boundary Terms in One Dimension

Let us first consider the mechanism for finding boundary terms in an arbitrary theory.

Consider Klein–Gordon theory [104] on a one-dimensional manifold with boundary

$$\mathcal{S}_{\text{bulk}} = \int_{\mathcal{M}^1} \frac{1}{2}\, \phi(t)\, {\partial_t}^2\, \phi(t)\, dt\ . \tag{6.2.1}$$

The variation of this action is

$$\delta\mathcal{S}_{\text{bulk}} = \int_{\mathcal{M}^1} \frac{1}{2}\, \delta\phi\, {\partial_t}^2\, \phi + \frac{1}{2}\, \phi {\partial_t}^2 \delta\phi\, dt\ . \tag{6.2.2}$$

To find the equations of motion we must add total derivatives

$$\delta\mathcal{S}_{\text{bulk}} = \int_{\mathcal{M}^1} \frac{1}{2}\,\delta\phi\,{\partial_t}^2\,\phi + \frac{1}{2}\phi\,{\partial_t}^2\,\delta\phi - \frac{1}{2}\partial_t\left(\phi\,\partial_t\,\delta\phi - \delta\phi\,\partial_t\,\phi\right)dt\ . \tag{6.2.3}$$

Applying Green's theorem ($\int_{\mathcal{M}^1}\left(\partial_t f(t)\right)dt = f(t)|_{\partial\mathcal{M}^1}$) we find

$$\delta\mathcal{S}_{\text{bulk}} = \int_{\mathcal{M}^1} \delta\phi\,{\partial_t}^2\,\phi\,dt + \frac{1}{2}\left(\phi\,\partial_t\,\delta\phi - \delta\phi\,\partial_t\,\phi\right)\Big|_{\partial\mathcal{M}^1}\ . \tag{6.2.4}$$

Requiring this to vanish ($\delta\mathcal{S} = 0$) for arbitrary variations in the bulk $\delta\phi(t)$, requires

$${\partial_t}^2\phi = 0\ . \tag{6.2.5}$$

We interpret these as the equations of motion of the system. Vanishing of the entire variation, however, requires that our boundary terms vanish either when the on-shell condition, or equations of motion, are applied or when the boundary conditions are applied.

6.3 Boundary Terms in One Dimension

The most generic boundary conditions which we may apply to a second-order set of equations of motion and still have non-trivial solutions are either one first-order or zeroth-order (in derivatives) condition on both boundaries or two first-order or zeroth-order conditions on one boundary. For this problem we will consider the former case. Assuming, without loss of generality, our lower boundary is $t = 0$ and our upper boundary is $t = 1$, our boundary conditions may always be written

$$a\,\partial_t\,\phi + b\,\phi\Big|_{t=0} = \phi_0\ , \qquad c\,\partial_t\,\phi + d\,\phi\Big|_{t=1} = \phi_1\ . \tag{6.3.1}$$

The lower (upper) boundary condition is generic when $\phi_0 \neq 0$ ($\phi_1 \neq 0$), special when $\phi_0 = 0$ ($\phi_1 = 0$), Dirichlet when $a = 0$ ($c = 0$), Neumann when $b = 0$ ($d = 0$), and mixed or Robin when $a \neq 0$ and $b \neq 0$ ($c \neq 0$ and $d \neq 0$).[76] Furthermore, we may use the conditions on our fields to derive conditions on our variations obey

$$a\,\partial_t\,\delta\phi + b\,\delta\phi\Big|_{z=0} = 0\ , \qquad c\,\partial_t\,\delta\phi + d\,\delta\phi\Big|_{z=1} = 0\ , \tag{6.3.2}$$

[76] Boundary conditions on the perturbation of a field are generically special.

since our variations take us between field configurations which obey our boundary conditions. Given this we may simplify our boundary term[77]

$$\begin{aligned}
&\frac{1}{2}\left(\phi\,\partial_t\,\delta\phi-\delta\phi\,\partial_t\,\phi\right)\Big|_{\partial\mathcal{M}^1}\\
=&\frac{1}{2}\left(\phi\,\partial_t\,\delta\phi-\delta\phi\,\partial_t\,\phi\right)\Big|_{t=1}-\frac{1}{2}\left(\phi\,\partial_t\,\delta\phi-\delta\phi\,\partial_t\,\phi\right)\Big|_{t=0}\\
=&\frac{1}{2}\left(\phi\left(-\frac{d}{c}\delta\phi\right)-\delta\phi\left(-\frac{d}{c}\phi+\frac{1}{c}\phi_1\right)\right)\Big|_{t=1}-\frac{1}{2}\left(\phi\left(-\frac{b}{a}\delta\phi\right)-\delta\phi\left(-\frac{b}{a}\phi+\frac{1}{a}\phi_0\right)\right)\Big|_{t=0}\\
=&-\frac{1}{2c}\phi_1\delta\phi\Big|_{t=1}+\frac{1}{2a}\phi_0\delta\phi_0\Big|_{t=0}\,.
\end{aligned}\tag{6.3.3}$$

Therefore if we add

$$\mathcal{S}_{\text{boundary}}=\frac{1}{2c}\;\phi\;\phi_1\phi\Big|_{t=1}-\frac{1}{2a}\;\phi\;\phi_0\Big|_{t=0}\;,\tag{6.3.4}$$

which obeys

$$\delta\mathcal{S}_{\text{boundary}}=-\frac{1}{2}\left(\phi\;\partial_t\;\delta\phi-\delta\phi\;\partial_t\;\phi\right)\Big|_{\partial\mathcal{M}^1}\;,\tag{6.3.5}$$

then the extrema of the action $\mathcal{S}=\mathcal{S}_{\text{bulk}}+\mathcal{S}_{\text{bound}}$ correspond to the solutions of the equations of motion (equation (6.2.5)) given our field obeys our boundary conditions (equation (6.3.1)).

This is not, however, the only boundary action we could add. Given

$$a\;\partial_t\;\phi+b\;\phi-\phi_0\Big|_{t=0}=0\;,\tag{6.3.6}$$

we may add

$$\mathcal{S}_{\text{lower}}=f_{\text{lower}}(a\partial_t\;\phi+b\phi-\phi_0)-f_{\text{lower}}(0)\Big|_{t=0}\;,\tag{6.3.7}$$

and a similar boundary term where f_{lower} is any smooth function. However, if we restrict our attention to second-degree polynomials which have 0 as a root, we find there is a four parameter family of possible actions which correspond to our equations of motion and boundary conditions.

Similarly we note we may add the bulk term

$$\int_{\mathcal{M}^1}\frac{1}{2a}\;\partial_t\;\phi\;\phi_0-\partial_t\;f_{\text{lower}}\;dt\;,\tag{6.3.8}$$

and absorb our lower boundary term into our bulk action and into our upper boundary term. Since adding such total-derivative term does not change the value of the action 'off-shell' (considering both equations of motion and boundary conditions) any two actions related by addition of such a boundary term are

[77] When a or c vanish we have generic Dirichlet conditions at the relevant boundary. In such a case the following calculation is invalid. However, given that this is the most studied example [80, 84], we allow it to be an exception to our analysis here.

equivalent actions (in classical field theory). This contrasts with the addition of f_{lower} on the boundary, since this changes the value of the action action away from its extrema.

Therefore, we want to define a 'canonical' form for an action. Since we may always write the second-order (in derivatives) bulk terms as 'derivative-field derivative-field' we say the canonical form of the action is that which has no second-order (in derivatives) bulk terms and has the lowest-order boundary polynomials (such as f_{lower}).[78]

For our current case the canonical action is

$$\mathcal{S} = -\frac{1}{2}\left(\int_{\mathcal{M}^1} \partial_t\,\phi\,\partial_t\,\phi\,dt\right) + \left(-\frac{d}{2c}\,\phi^2 + \frac{1}{c}\,\phi\,\phi_1\right)\Big|_{t=1} + \left(\frac{b}{2a}\,\phi^2 - \frac{1}{a}\,\phi\,\phi_0\right)\Big|_{t=0} . \tag{6.3.9}$$

While this appears less compact, it is exceedingly well behaved in the limits of either generic Dirichlet or special Neumann. In both limits $\mathcal{S}_{\text{bound}}$ vanishes.[79]

Most introductions to the calculus of variations either ignore considerations of boundary terms or restrict themselves to only the simplest cases of either Dirichlet or special Neumann [80, 84]. We see that this treatment is justified, so long as the action presented is of canonical form. In this case the equations of motion do not require careful treatment by addition of total derivatives and cancellation of boundary terms, but instead may simply be stated as the Euler–Lagrange equations

$$\frac{\partial \mathcal{S}}{\partial \phi} - \partial_t \frac{\partial \mathcal{S}}{\partial\left(\partial_t \phi\right)} = 0 . \tag{6.3.10}$$

To conclude, we will summarize our technique for finding boundary terms. We take the canonical bulk action, add a generic boundary term with undetermined coefficients, vary, apply our boundary conditions and equations of motion, solve the coefficient equation for each linearly independent boundary term, and reduce to the canonical boundary term.

6.4 The Quadratic Theory

We want to study the Klein–Gordon equation on a higher-dimensional product space ($\mathcal{M}^h = \mathcal{M}_l \times \mathcal{D}$) [117]

$$\left(\nabla^2 - M^2\right)\Phi(x,\,z) = \left(\Box + \Delta - M^2\right)\Phi(x,\,z) = 0 . \tag{6.4.1}$$

[78]We only claim this is a well defined quantity in the context of a flat boundary.

[79]This requires special treatment of denominators the limit $c \to 0$, for instance, however, in this limit, $\mathcal{S}_{\text{bound}}$ still vanishes.

Here $\Box$ is the d'Alembertian on the (Lorenzian) manifold $\mathcal{M}_l$ and Δ is the Laplacian on $\mathcal{D}$. If we suppose that Φ obeys some special boundary conditions at the upper and lower boundary[80] of our $\mathcal{D}$

$$(a\partial_z + b)\,\Phi(x,\,z)\Big|_{z\to l^+} = 0\,, \qquad (c\partial_z + d)\,\Phi(x,\,z)\Big|_{z\to u^-} = 0\,, \tag{6.4.2}$$

this Klein–Gordon equation corresponds to the following action

$$\begin{aligned}\mathcal{S}_{\text{free}} = &\int_{\mathcal{M}_l}\sqrt{-g_l(x)}\int_{\mathcal{D}}\mu(z)\left(-\frac{1}{2}\left(\nabla_\mu\Phi(x,\,z)\right)^2 - \frac{1}{2}\left(\partial_z\Phi(x,\,z)\right)^2 - \frac{1}{2}M^2\Phi(x,\,z)^2\right)d^lx dz\\ &-\int_{\mathcal{M}_l}\sqrt{-g_l(x)}\mu(z)\frac{1}{2}\frac{a}{b}\left(\partial_z\Phi(x,\,z)\right)^2\Big|_{z\to l^+}d^lx + \int_{\mathcal{M}_l}\sqrt{-g_l(x)}\mu(z)\frac{1}{2}\frac{c}{d}\left(\partial_z\Phi(x,\,z)\right)^2\Big|_{z\to u^-}d^lx\,.\end{aligned} \tag{6.4.3}$$

Furthermore, we may expand Φ in terms of the relevant transverse basis

$$\Phi(x,\,z) = \phi^\omega(x)f_\omega(z)\,. \tag{6.4.4}$$

Since the explicit boundary terms we added to $\mathcal{S}_{\text{free}}$ summed with the boundary terms arising from integration by parts of the transverse derivatives annihilate f_ω, we have

$$\begin{aligned}-\int_{\mathcal{D}}\mu(z)(\partial_z f_\omega(z))(\partial_z f_\sigma(z))dz + \left(\mu(z)\frac{b}{a}f_\omega(z)f_\sigma(z)\right)\Big|_{z\to l^+} - \left(\mu(z)\frac{d}{c}f_\omega(z)f_\sigma(z)\right)\Big|_{z\to u^-}\\ = \int_{\mathcal{D}}\mu(z)f_\omega(z)\Delta f_\sigma(z)dz\,.\end{aligned} \tag{6.4.5}$$

Using the orthonormality of our basis we find

$$\mathcal{S}_{\text{free}} = \int_{\mathcal{M}_l}\sqrt{-g_l(x)}\left(-\frac{1}{2}\left(\nabla_\mu\phi^\omega(x)\right)^2 - \frac{1}{2}\left(M^2+\omega^2\right)\phi^\omega(x)^2\right)d^lx\,. \tag{6.4.6}$$

From this we note several facts. First, neither negative higher-dimensional mass nor imaginary higher-dimensional eigenvalue indicate a tachyonic mode [21] (negative mass mode) in the lower-dimensional effective field theory, so long as the sum of these terms is overall positive $M^2+\omega^2>0$.

Second, at this level any further detail of the transverse problem beyond the spectrum of Δ given our boundary conditions is not relevant. If two transverse spaces have the same spectrum, then the lower-dimensional effective Klein–Gordon theories will agree.

Third, for an arbitrarily massive field in the higher dimension (e.g. when $M^2 \gg {\omega_0}^2 - {\omega_1}^2$ for the

[80] In the case where the transverse space is not one-dimensional we consider an S-wave expansion and $\mathcal{D}$ is the manifold of the radial coordinate only.

first two eigenvalues of a discrete spectrum) our lower-dimensional fields will have an effective symmetry mixing between these fields, similar to the broken chiral symmetry of the fermionic sector of quantum chromodynamics with only the three lightest quarks, up, down, and strange [16]. Experiments on such particles in such a universe would have additional effective symmetries.

At this level, any discussion of truncation is trivial. This is always true for a free theory. To see the novel implications of an unusual Sturm–Liouville basis we must introduce interactions, and explore either the unusual relationship between these couplings, for instance, between the cubic interaction and quartic interaction, or the effect of integrating out the heavy fields at cubic order.

6.5 Nonlinear Corrections to the Couplings

The first nontrivial interaction term in orders of fields (at zeroth-order in derivatives) are

$$\mathcal{S}_{\text{cubic}} + \mathcal{S}_{\text{quartic}} = \int_{\mathcal{M}_l} \sqrt{-g_l(x)} \int_{\mathcal{D}} \mu(z) \left(-\frac{\Lambda}{3} \Phi(x,\, z)^3 - \frac{K}{4} \Phi(x,\, z)^4 \right) dz d^l x \; . \tag{6.5.1}$$

Fortunately, as these are non-derivative terms, they require no additional boundary terms. In our effective field theory the cubic and quartic terms become, respectively

$$\mathcal{S}_{\text{cubic}} = \int_{\mathcal{M}_l} \sqrt{-g_l(x)} \left(-\frac{\Lambda}{3} \left(\int_{\mathcal{D}} \mu f_\omega f_\sigma f_\tau dz \right) \phi^\omega(x) \phi^\sigma(x) \phi^\tau(x) \right) d^l x \; , \tag{6.5.2}$$

$$\mathcal{S}_{\text{quartic}} = \int_{\mathcal{M}_l} \sqrt{-g_l(x)} \left(-\frac{K}{4} \left(\int_{\mathcal{D}} \mu f_\omega f_\sigma f_\tau f_\upsilon dz \right) \phi^\omega(x) \phi^\sigma(x) \phi^\tau(x) \phi^\upsilon(x) \right) d^l x \; . \tag{6.5.3}$$

We now swap to our 'I' notation for overlap integrals defined in equation (3.3.1). Note this term in principle indicates an interaction between any triplet of scalars, even when they correspond to different eigenvalues.

We are, of course, specifically interested in the case where our spectrum begins with a discrete state or a zero mode, which we label as the lightest mode, with eigenvalue l (note l can be, but is not necessarily, zero). This is opposed to the heavy modes, which we say have eigenvalues $\overline{\omega} > l$. In this case we note that this mode has some coupling to itself with effective couplings in the lower dimension,

$$\lambda = \Lambda I_{lll} \; , \qquad \kappa = K I_{llll} \; . \tag{6.5.4}$$

The relationship between the effective couplings is now highly dependent on which space we consider. However, in the case of a consistent tructation, for example, when f_l is a constant, we have

$$\left(I_{lll}\right)^2 = I_{llll} \; . \tag{6.5.5}$$

Therefore, for consistent truncations, any relationship between Λ and K is preserved at the level of the effective field theory; such as

$$\Lambda^2 = K \qquad \Rightarrow \qquad \lambda^2 = \kappa^2 \; . \tag{6.5.6}$$

However, if we consider any reduction where f_l is not a constant, which includes any inconsistent truncation where $l = 0$ by our argument in equation (3.3.5), then we have some nonlinear correction to our higher-dimensional relationship. For instance, for a flat interval (where our zero mode is as given in section 2.6.1) we have

$$I_{0000} - \left(I_{000}\right)^2 = 1/135b^2(45 - 96b^2 + 80b^4) \; , \tag{6.5.7}$$

where $b = 0$ is the limit where our Sturm–Liouville basis contains a constant zero mode.

This is especially relevant in the context of gauge theories, where the squared relationship of the cubic and quartic couplings is required for gauge invariance [38]. This is the fact that originally motivated us to the primary study of section 8. This is a generic property of the nonlinear (in order of fields) corrections to the couplings, which is equivalent to any inconsistent truncation with a non-constant zero mode.

6.6 Integrating Out Heavy Fields with Cubic Interactions

Expanding our cubic interaction term (equation (6.5.2)) in orders of heavy fields (that is $\mathcal{O}(1)$, $\mathcal{O}(\phi^{\overline{\omega}})$, $\mathcal{O}(\phi^{\overline{\omega}}\phi^{\overline{\sigma}})$, etc) followed we have

$$\mathcal{S}_{\text{cubic}} = \int_{\mathcal{M}_l} \sqrt{-g_l} \left(-\frac{\lambda}{3} {\phi^l}^3 - \Lambda I_{ll\overline{\omega}} {\phi^l}^2 \phi^{\overline{\omega}} - \Lambda I_{l\overline{\omega}\overline{\sigma}} \phi^l \phi^{\overline{\omega}} \phi^{\overline{\sigma}} - \frac{\Lambda}{3} I_{\overline{\omega}\overline{\sigma}\overline{\tau}} \phi^{\overline{\omega}} \phi^{\overline{\sigma}} \phi^{\overline{\tau}} \right) d^l x \; . \tag{6.6.1}$$

The associated equations of motion for our heavy fields of the action to this order are

$$\left(\Box - \left(M^2 + \overline{\omega}^2\right)\right) \phi^{\overline{\omega}} - \Lambda I_{ll\overline{\omega}} {\phi^l}^2 - 2\Lambda I_{l\overline{\omega}\overline{\sigma}} \phi^l \phi^{\overline{\sigma}} - \Lambda I_{\overline{\omega}\overline{\sigma}\overline{\tau}} \phi^{\overline{\sigma}} \phi^{\overline{\tau}} = 0 \; . \tag{6.6.2}$$

Solving this expression formally we have

$$\phi^{\overline{\omega}} = \left(\Box - (M^2 + \overline{\omega}^2)\right)^{-1} \left(\Lambda I_{ll\overline{\omega}} {\phi^l}^2 + 2\Lambda I_{l\overline{\omega}\overline{\sigma}} \phi^l \left(\Box - (M^2 + \overline{\sigma}^2)\right)^{-1} \left(\Lambda I_{ll\overline{\sigma}} {\phi^l}^2 + \ldots \right) + \ldots \right) \; . \tag{6.6.3}$$

Here the right hand side is found by inverting the propagator

$$\left(\Box - (M^2 + \overline{\omega}^2)\right)^{-1} = -\frac{1}{M^2 + \overline{\omega}^2} - \frac{\Box}{\left(M^2 + \overline{\omega}^2\right)^2} - \frac{\Box^2}{\left(M^2 + \overline{\omega}^2\right)^3} - \ldots = -\frac{1}{M^2 + \omega^2} \sum_{i=0}^{\infty} \left(\frac{\Box}{M^2 + \overline{\omega}^2} \right)^i \; , \tag{6.6.4}$$

then recursively substituting the expression into itself for the value of $\phi^{\overline{\omega}}$ or $\phi^{\overline{\sigma}}$, and so on. These estimates assume first that the heavy fields are unexcited in the absence of the light fields, upon relaxing this assumption we may define exact higher-dimensional sources, and that the momentum of the light modes is much smaller than the mass of the heavy modes

$$(M^2+\overline{\omega}^2)\phi^l \gg \Box\phi^l \ . \tag{6.6.5}$$

We note that the right hand side of equation (6.6.3), therefore, may be given explicitly in orders of $\frac{1}{M^2+\omega^2}$ or both ϕ^l and $\Box$, in practice, we will use both. For the sake of completeness we explicitly have

$$\phi^{\overline{\omega}} = -\frac{\Lambda}{M^2+\overline{\omega}^2} I_{ll\overline{\omega}} {\phi^l}^2 - \frac{\Lambda}{\left(M^2+\overline{\omega}^2\right)^2} I_{ll\overline{\omega}} \Box {\phi^l}^2 - \frac{2\Lambda}{\left(M^2+\overline{\omega}^2\right)\left(M^2+\overline{\sigma}^2\right)} I_{l\overline{\omega}\sigma} I_{ll\overline{\sigma}} {\phi^l}^3 + \mathcal{O}\left({\phi^l}^4, \partial^4\right) \ . \tag{6.6.6}$$

However, we will leave our expression in terms of the formal inverse for the moment and further condense our notation as $\mathcal{O}_\omega = \Box - \left(M^2+\omega^2\right)$. Substituting our expression for $\phi^{\overline{\omega}}$ into our quadratic and cubic action we have ($\mathcal{S}_{\text{free}} + \mathcal{S}_{\text{cubic}} =$)

$$\begin{aligned} \int_{\mathcal{M}_l} \sqrt{-g_l(x)} \Bigg(& \frac{1}{2}\phi^l \mathcal{O}_l \phi^l + \frac{1}{2} {\mathcal{O}_{\overline{\omega}}}^{-1} \left(\Lambda I_{ll\overline{\omega}} {\phi^l}^2 + \ldots\right) \mathcal{O}_{\overline{\omega}} {\mathcal{O}_{\overline{\omega}}}^{-1} \left(\Lambda I_{ll\overline{\omega}} {\phi^l}^2 + \ldots\right) \\ & - \frac{\lambda}{3} {\phi^l}^3 - \Lambda I_{ll\overline{\omega}} {\phi^l}^2 \mathcal{O}_{\overline{\omega}} {\mathcal{O}_{\overline{\omega}}}^{-1} \left(\Lambda I_{ll\overline{\omega}} {\phi^l}^2 + \ldots\right) + \mathcal{O}\left({\phi^l}^5, \partial^2\right) \Bigg) d^l x \ . \end{aligned} \tag{6.6.7}$$

We notice that we may simplify $\mathcal{O}_{\overline{\omega}} {\mathcal{O}_{\overline{\omega}}}^{-1} = 1$, and further notice that the effective interactions from setting the heavy fields onshell in the heavy fields' kinetic terms and the leading interactions including heavy fields are repeated terms with a relative $-\frac{1}{2}$. Simplifying we find

$$\mathcal{S}_{\text{free}} + \mathcal{S}_{\text{cubic}} = \int_{\mathcal{M}_l} \sqrt{-g_l(x)} \left(\frac{1}{2}\phi^l \mathcal{O}_l \phi^l - \frac{\lambda}{3} {\phi^l}^3 - \frac{1}{2} \frac{{I_{ll\overline{\omega}}}^2}{M^2+\overline{\omega}^2} \Lambda^2 {\phi^l}^4 + \mathcal{O}\left({\phi^l}^5, \partial^2\right) \right) d^l x \ . \tag{6.6.8}$$

The effective coupling at quartic order, $Y = -\frac{{I_{ll\overline{\omega}}}^2}{M^2+\overline{\omega}^2}$, is of the form noted in equation (3.3.7), which is given by integrals of the Green function against the interaction term.

Of course, the majority of study is focused on the case where these corrections identically vanish, or when $I_{ll\overline{\omega}} = 0$ which implies $X = 0$ [36, 45, 68, 86, 94]. However, these cases are mathematically rare, as we have argued in section 4, they only occur when the transverse space has finite volume, and for most choices of transverse space with finite volume, most choices of boundary condition lead to inconsistent truncations. Therefore we are interested in the case where the effective theory we have derived here is nontrivial and an accurate description of the low energy physics.

6.7 Explicit Corrections for a Flat Interval

Overall corrections to the effective theory of the lightest mode are suppressed by both the mass of the particles in the higher dimension and the effective mass induced by momentum in the transverse space of the heavy modes ($\overline{\omega}^2$). Of special interest are the cases when $M^2 + l^2 \ll M^2 + \overline{\omega}^2$. Of course we have the additional requirement that we study particles whose world-volume momentum is much smaller than the mass of the lightest heavy particles, where the corrections are largest.

If we consider a theory where the native mass term in the higher dimension is large, $M^2 \gg l^2$, then we are also obliged to consider only scattering of light particles that are relatively stationary. However, in the case where $M^2 = 0$ (or is simply relatively small) we have a 'window' of effective energies where we may probe the non-derivative corrections to our theory from the higher dimension before we measure higher-derivative corrections.

The simplest case in which to study this is the flat interval. We explicitly calculated the relevant sums of overlap integrals in section 3.3.3. For the full action $\mathcal{S} = \mathcal{S}_{\text{free}} + \mathcal{S}_{\text{cubic}} + \mathcal{S}_{\text{quartic}}$ we have, for $M = 0$, $l = 0$, $\phi = \phi^l = \phi^0$, given the 'pure ζ_0' (Neumann-Neumann) boundary conditions,

$$\mathcal{S} = \int_{\mathcal{M}_l} \sqrt{-\det g_l} \left(-\frac{1}{2}\partial_\mu\phi\partial^\mu\phi - \left(\frac{1}{\sqrt{2}}\right)\frac{\Lambda}{3}\phi^3 - \left(\left(\frac{1}{2}\right)\frac{K}{4} + (0)\frac{\Lambda^2}{2}\right)\phi^4 + \mathcal{O}\left(\phi^5, \partial^2\right)\right) d^l x \, . \tag{6.7.1}$$

By contrast, given the 'pure ξ_0' boundary conditions we have

$$\mathcal{S} = \int_{\mathcal{M}_l} \sqrt{-\det g_l} \left(-\frac{1}{2}\partial_\mu\phi\partial^\mu\phi - \left(\sqrt{\frac{27}{32}}\right)\frac{\Lambda}{3}\phi^3 - \left(\left(\frac{9}{10}\right)\frac{K}{4} + \left(\frac{57}{244}\right)\frac{\Lambda^2}{2}\right)\phi^4 + \mathcal{O}\left(\phi^5, \partial^2\right)\right) d^l x \, . \tag{6.7.2}$$

Therefore we see the higher-order interactions are enhanced, both without corrections from heavy fields, and then corrections to the heavy fields enhance these terms further.

7 Deriving the Kaluza–Klein Ansatz Perturbatively

We now turn our attention to the construction of effective field theories given our boundary conditions and our backgrounds. The first useful example of such a construction is the Kaluza–Klein (KK) ansatz [108]. The KK ansatz was developed in two stages. First Kaluza noticed that the equations of motion of a higher-dimensional metric, when a single transverse dimension is 'read separately,' reconstruct lower-dimensional equations of motion for a lower-dimensional metric and other lower-dimensional fields, given that the metric is independent of the separated dimension [77, 96]. Klein introduced the innovation of interpreting the separated dimension as a compact dimension, thus introducing a length scale to the theory which allows the interpretation as a lower-dimensional EFT [81].

In our language we would call Kaluza's work a Type-I reduction and Klein's addition a Type-III* reduction.[81] Most authors now focus their discussions of the KK EFT and its generalizations on dimensional reduction, that is whether it is consistent to truncate the massive degrees of freedom, etc. However, we are presently interested in an EFT which would both lead to an inconsistent truncation, and where gauge transformations and the lower-dimensional degrees of freedom are much harder to find. In solving the issues we encountered in our study, we developed an algorithmic method which can be generically applied to 'integrating out' extra dimensions into a lower-dimensional EFT. This section will present this algorithm in a well-known case so that the reader may both freely apply this algorithm to their problems, and to provide a reference for later sections.

7.1 The Gibbons–Hawking–York Boundary Term, Exactly

The biggest drawback of illustrating our procedure for finding a lower-dimensional EFT in the content of Kaluza–Klein theory is that many of the steps of this process result in no change when the transverse space is an r-cycle, that is a closed manifold without boundary. For this reason we will instead consider the theory of gravity on $(d+1)$-dimensional Minkowski with two parallel time-like boundaries[82] with Neumann-Neumann boundary conditions.[83] That is, our ($(d+1)$-dimensional) manifold is

$$\mathcal{M} = \mathbb{R}^{1,d-1} \times \mathcal{D} \, , \tag{7.1.1}$$

[81] Arguably the full KK ansatz is not a Type-III* reduction because the periodicity is not a boundary condition. However, the massless sector of the KK ansatz is given by a reduction with Neumann-Neumann conditions. We will remark more upon this when we discuss the spectrum of massive particles the theory.

[82] These are sometimes referred to as orbifolds.

[83] The massless theory is the same, however.

with coordinates (and indices) X^M on $\mathcal{M}$, x^μ on $\mathbb{R}^{1,d-1}$, and z (coordinate z) on $\mathcal{D}$. Without loss of generality we say our orbifolds lie at $z = \pm 1$, or that $\mathcal{D} = (-1, 1)$.

Our bulk action is the Einstein–Hilbert action [48, 71],

$$\mathcal{S}_{\text{EH}} = \int_{\mathcal{M}} \sqrt{-g}\ \frac{1}{2\hat{\kappa}^2} R(g)\ d^{d+1}X\ . \tag{7.1.2}$$

Here g_{MN} is our metric, $\hat{\kappa}$ is our higher-dimensional Einstein constant, $\sqrt{-g}$ is the square root of the determinant of our metric, and R is the Ricci-scalar associated with the Levi–Civita connection of the metric, ∇.

Famously, the extrema of Einstein–Hilbert action do not correspond to the solutions of the Einstein equation on a manifold with boundary when the metric obeys generic Dirichlet conditions. That is, we require the Gibbons–Hawking–York (GHY) boundary term [59, 131]

$$\mathcal{S}_{\text{GHY}} = -\int_{\partial\mathcal{M}} \sqrt{-\gamma} \frac{1}{\hat{\kappa}^2} K(\gamma) d^{d-1}Y\ , \tag{7.1.3}$$

Here γ_{mn} is the induced metric on the boundary, $K(\gamma)$ is the trace of the extrinsic curvature on the boundary and Y^m are some boundary coordinates. We can view this term as a geometrically elegant version of the canonical boundary term (as in section 6.2). However a derivation of the GHY term may be accomplished by fairly simple considerations of what the extrema of the action are.

When written in terms of partial derivatives of the metric, the Einstein–Hilbert action reads[84]

$$\begin{aligned}\mathcal{S}_{\text{EH}} = \int_{\mathcal{M}} \sqrt{-g} \bigg(&+ g^{MN} g^{PQ} \left(-\partial_M \partial_N g_{PQ} + \partial_M \partial_P g_{NQ}\right) \\ &+ g^{MN} g^{PQ} g^{RS} \bigg(+ \frac{3}{4} \partial_M g_{PR} \partial_N g_{QS} - \frac{1}{4} \partial_M g_{PQ} \partial_N g_{RS} + \partial_M g_{PQ} \partial_R g_{NS} \\ &\qquad - \frac{1}{2} \partial_M g_{PR} \partial_Q g_{NS} - \partial_M g_{NP} \partial_R g_{QS} \bigg) \bigg) d^{d+1}X\ .\end{aligned} \tag{7.1.4}$$

After integration by parts the bulk Lagrangian density takes the following canonical form[85]

$$\mathcal{L}_{ADM} = g^{MN} g^{PQ} g^{RS} \left(-\frac{1}{4} \partial_M g_{PR} \partial_N g_{QS} + \frac{1}{4} \partial_M g_{PQ} \partial_N g_{RS} - \frac{1}{2} \partial_M g_{NP} \partial_Q g_{RS} + \frac{1}{2} \partial_M g_{PR} \partial_Q g_{SN} \right)\ . \tag{7.1.5}$$

[84] We have momentarily set $\frac{1}{2\hat{\kappa}^2} = 1$ for the sake of margin space.

[85] This 'derivative-field derivative-field' form of gravity can serve as a first step to writing a fully nonlinear Hamiltonian for gravity. Therefore we attribute this form to the Arnowitt, Deser, and Misner [9].

The total derivative we add to convert between the two is

$$\int_{\mathcal{M}} \left(\partial^M \sqrt{-g}\left(-g^{PQ}\partial_M g_{PQ} + g^{PQ}\partial_P g_{MQ}\right)\right) d^d X \tag{7.1.6}$$

$$= \int_{\partial\mathcal{M}} \sqrt{-g}\left(-g^{PQ}\partial_M g_{PQ} + g^{PQ}\partial_P g_{MQ}\right) n^M d^d Y \;. \tag{7.1.7}$$

Here Y are our boundary coordinates and n^M is the outward facing unit normal to the boundary.

Upon variation the bulk action $\mathcal{S} = \int_{\mathcal{M}} \sqrt{-g}\, \mathcal{L}_{CG}\, d^{d+1}X$, where $\mathcal{L}_{CG}$ is as in equation (7.1.5), generates boundary terms, all of which are proportional to δg. Since, even for general Dirichlet conditions on g, δg obeys special Dirichlet conditions, the variation of this action vanishes when g obeys the equations of motion and obeys a general Dirichlet condition. These extrema agree exactly with the extrema of the Einstein–Hilbert action (equation (7.1.2)) plus the Gibbons–Hawking–York boundary term (equation (7.1.3)). Since both our boundary term and the GHY boundary term are first order in derivatives, this implies that our boundary term is the GHY boundary term up to some total derivative. However, since the boundary of a boundary vanishes, adding total derivative terms to the GHY boundary term leaves it invariant. Therefore, while equation (7.1.7) may, in principle, only agree up to total derivatives with equation (7.1.3), we have

$$\mathcal{S}_{EH} + \mathcal{S}_{GHY} = \int_{\mathcal{M}} \sqrt{-g}\, \mathcal{L}_{CG}\, d^d X \;. \tag{7.1.8}$$

7.2 The Gibbons–Hawking–York Boundary Term, Perturbatively

Traditionally we derive the massless Fierz–Pauli action by requiring a specific normalization of the kinetic term [53], and that the action be symmetric under the 'inhomogeneous part' of the graviton's transformation.

That is, assuming a flat background, we write

$$\begin{aligned}\mathcal{S} = \int_{\mathcal{M}} \big(&-\frac{1}{2}\partial^M \mathcal{H}^{PQ}\partial_M \mathcal{H}_{PQ} + a\, \partial^M \mathcal{H}^N{}_N \partial_M \mathcal{H}_P{}^P + b\, \partial^M \mathcal{H}_{MN}\partial^N \mathcal{H}_P{}^P \\ &+ c\, \partial^M \mathcal{H}_{MN}\partial_P \mathcal{H}^{NP} + d\, \partial^M \mathcal{H}^{PQ}\partial_P \mathcal{H}_{QM} \big)\, d^{d+1}X \;.\end{aligned} \tag{7.2.1}$$

Where $\mathcal{H}$ is a symmetric 2-tensor, $\mathcal{H}_{MN} = \mathcal{H}_{NM}$, $-\frac{1}{2}\partial^M \mathcal{H}^{PQ}\partial_M \mathcal{H}_{PQ}$ is our normalized kinetic term and $a, \ldots, d$ are undetermined coefficients multiplying all contractions of $\partial_M \mathcal{H}_{PQ}\partial_N \mathcal{H}_{RS}$. We define the transformation $\mathcal{H}'_{MN} = \mathcal{H}_{MN} + \partial_M \chi_N + \partial_N \chi_M$, where χ^M is any vector field and collect terms at $\mathcal{O}(\chi)$. The relevant portion of the integrand is

$$\begin{aligned}&(-2+2d)\partial^M \mathcal{H}^{PQ}\partial_M \partial_N \chi_Q + (4a+b)\partial^M \mathcal{H}^N{}_N \partial_M \partial_Q \chi^Q + b\partial^M \mathcal{H}^N{}_N \partial_Q \partial^Q \chi_M \\ &+(2b+2c)\partial^M \mathcal{H}_{MN}\partial^N \partial_Q \chi^Q + 2c\partial^M \mathcal{H}_{MN}\partial_Q \partial^Q \chi^N + 2d\partial_P \mathcal{H}_{MN}\partial^M \partial^N \chi^Q \;.\end{aligned} \tag{7.2.2}$$

There are no values for $a, \ldots, d$ for which this vanishes. However, if we integrate by parts off of $\mathcal{H}_{MN}$, our integrand becomes

$$(2-2d-2c)\mathcal{H}^{MN}\partial_Q\partial^Q\partial_M\chi_N - (4a+2b)\mathcal{H}_Q{}^Q\partial_P\partial^P\partial_M\chi^M - (2b+2c+2d)\mathcal{H}_{MN}\partial^M\partial^N\partial_Q\chi^Q \ . \quad (7.2.3)$$

This vanishes when $a=\frac{1}{2}$, $b=-1$, and $c=-1-d$. The degeneracy of this solution is again due to the degeneracy of boundary actions, since the terms multiplying c and d are related by integration by parts. We choose $c=0$ and $d=-1$.[86] To wit, we write the Fierz–Pauli action as

$$\mathcal{S}_{FP} = \int_{\mathcal{M}} \left(-\frac{1}{2}\partial^M\mathcal{H}^{PQ}\partial_M\mathcal{H}_{PQ} + \frac{1}{2}\partial^M\mathcal{H}^N{}_N\partial_M\mathcal{H}_P{}^P - \partial^M\mathcal{H}_{MN}\partial^N\mathcal{H}^P{}_P + \partial_M\mathcal{H}_{NP}\partial^N\mathcal{H}^{PM} \right) d^{d+1}X \ . \quad (7.2.4)$$

Further, we note that there is no action of a symmetric 2-tensor which is invariant under the transformation $\mathcal{H}'_{MN} = \mathcal{H}_{MN} + \partial_M\chi_N + \partial_N\chi_M$ with no boundary conditions on $\mathcal{H}$ or χ.

To further explain our point about the fact that the action is only symmetric up to boundary conditions on $\mathcal{H}$ or χ: if we consider a theory composed of objects which are invariant under our gauge transformation, such as the field-strength tensor in Maxwell or the kinetic term of a charged scalar in Scalar Electrodynamics, then we do not require integration by parts to eliminate our gauge transformation. In perturbation gravity, however, the transformation of different terms in the action must be cancelled against each-other via integration by parts. If we write our action in "derivative-field derivative-field" form, for instance, we can integrate by parts to express the transformation of our action int terms of only derivatives on the gauge transformation. This incurs a "field derivative-derivative-gauge parameter" boundary term, which can be zero because of our boundary conditions on either our perturbation or our gauge parameter. Since the action is not built out of terms that can be grouped into terms that are gauge-invariant without integration by parts, to understand gauge invariance of the action, we must consider our boundary terms and conditions.

Since the perturbed metric is a symmetric 2-tensor, and therefore the perturbation transforms inhomogeneously in proportion to the background metric, we expect the perturbation of the Einstein–Hilbert action to agree with the Fierz–Pauli action. However, if we define our metric to be $g_{MN} = \eta_{MN} + \mathcal{H}_{MN}$ and expand

[86] We will justify this choice shortly.

the Einstein–Hibert action in orders of $\mathcal{H}$, we find at second order[87]

$$\begin{aligned}\mathcal{S}_{EH} = \int_{\mathcal{M}} \Big(&\frac{3}{4}\partial_P \mathcal{H}_{MN}\partial^P\mathcal{H}^{MN} + \mathcal{H}^{MN}\partial^P\partial_P\mathcal{H}_{MN} - \frac{1}{4}\partial_P\mathcal{H}_M{}^M\partial^P\mathcal{H}_N{}^N - \frac{1}{2}\mathcal{H}_M{}^M\partial_P\partial^P\mathcal{H}_N{}^N \\ &+ \partial_M\mathcal{H}^{MN}\partial_N\mathcal{H}_P{}^P + \mathcal{H}^{MN}\partial_M\partial_N\mathcal{H}_P{}^P + \frac{1}{2}\mathcal{H}_P{}^P\partial_M\partial_N\mathcal{H}^{MN} + \mathcal{H}^{MN}\partial_M\partial_N\mathcal{H}_P{}^P \\ &- \frac{1}{2}\partial_P\mathcal{H}_{MN}\partial^M\mathcal{H}^{NP} - \partial^M\mathcal{H}_{MN}\partial_P\mathcal{H}^{NP} - 2\mathcal{H}_{MN}\partial_P\partial^M\mathcal{H}^{NP}\Big) d^{d+1}X \ .\end{aligned} \tag{7.2.5}$$

The boundary term (with unit boundary normal n^P) which converts between these two actions is[88]

$$\begin{aligned}\int_{\partial\mathcal{M}} (&\mathcal{H}^{MN}\partial_P\mathcal{H}_{MN} - \frac{1}{2}\mathcal{H}_M{}^M\partial_P\mathcal{H}_N{}^N + \frac{1}{2}\mathcal{H}_N{}^N\partial^M\mathcal{H}_{PM} \\ &+ \mathcal{H}_{PM}\partial^M\mathcal{H}_N{}^N - \mathcal{H}_{PM}\partial_N\mathcal{H}^{MN} - \mathcal{H}^{MN}\partial_N\mathcal{H}_{PM})n^P \ d^dY \ .\end{aligned} \tag{7.2.6}$$

Equation (7.2.6) agrees with the perturbation of the Gibbons–Hawking–York boundary term (equation (7.1.3)), which justifies our choice of $c = 0$ in defining the Fierz–Pauli action. Note, there is no ab initio way of finding the perturbation of the Einstein–Hilbert action (equation (7.2.5)) or the Gibbons–Hawking–York boundary term from the action of a symmetric 2-tensor, however we have to second order for $g_{MN} = \eta_{MN} + \mathcal{H}_{MN}$

$$\frac{1}{2}\mathcal{S}_{FP} = \mathcal{S}_{EH} + \mathcal{S}_{GHY} \ . \tag{7.2.7}$$

7.3 Diagonalizing the Lower-Dimensional Free Theory

For the remainder of the section we will consider the simplified case of $\mathcal{M}_l = \mathbb{R}^{1,d-1}$. If we take the (massless) Fierz–Pauli action (equation (7.2.4)) and separate one dimension (z) the bulk Lagrangian density becomes ($\mathcal{L}_{\text{bulk}} =$)

$$\begin{aligned}&-\frac{1}{2}\partial_\sigma\mathcal{H}_{\mu\nu}\partial^\sigma\mathcal{H}^{\mu\nu} - \frac{1}{2}\partial_z\mathcal{H}_{\mu\nu}\partial^z\mathcal{H}^{\mu\nu} + \frac{1}{2}\partial_\sigma\mathcal{H}_\mu{}^\mu\partial^\sigma\mathcal{H}_\nu{}^\nu + \frac{1}{2}\partial_z\mathcal{H}_\mu{}^\mu\partial^z\mathcal{H}_\nu{}^\nu - \partial^\mu\mathcal{H}_{\mu\nu}\partial^\nu\mathcal{H}_\sigma{}^\sigma + \partial_\sigma\mathcal{H}_{\mu\nu}\partial^\mu\mathcal{H}^{\nu\sigma} \\ &-\partial_\mu\mathcal{H}_{\nu z}\partial^\mu\mathcal{H}^{\nu z} + \partial_\nu\mathcal{H}_{\mu z}\partial^\mu\mathcal{H}^{\nu z} + 2\partial^z\mathcal{H}_{\mu\nu}\partial^\mu\mathcal{H}^{\nu z} - 2\partial_z\mathcal{H}_\nu{}^\nu\partial_\mu\mathcal{H}^{\mu z} - \partial^\mu\mathcal{H}_{\mu\nu}\partial^\nu\mathcal{H}_z{}^z + \partial_\sigma\mathcal{H}_\nu{}^\nu\partial^\sigma\mathcal{H}_z{}^z \ .\end{aligned} \tag{7.3.1}$$

If we attempt to interpret these as lower-dimensional equations of motion the leading terms $-\frac{1}{2}\partial_\sigma\mathcal{H}_{\mu\nu}\partial^\sigma\mathcal{H}^{\mu\nu}$ $-\frac{1}{2}\partial_z\mathcal{H}_{\mu\nu}\partial^z\mathcal{H}^{\mu\nu}$ could serve as the kinetic term and putative mass term for a lower-dimensional Fierz–Pauli action. Upon variation these two terms become

$$\left(\Box + \Delta\right)\mathcal{H}_{\mu\nu} \ , \tag{7.3.2}$$

[87] Since $\partial_M\eta_{PQ} = 0$ and $\nabla_M = \partial_M + \mathcal{O}(\mathcal{H})$, this is the first nonvanishing order.

[88] The Fierz–Pauli action is a factor of two larger than the second perturbation of Einstein–Hilbert action. We have left this factor in so that we agree with the standard normalization for both actions.

from which we recognize the scalar wave operator in the lower dimension, $\Box = \partial_\sigma \partial^\sigma$, and the scalar wave operator in the transverse dimension, $\Delta = {\partial_z}^2$. We want to study $\mathcal{H}_{MN}$ which obey Neumann-Neumann conditions therefore we define

$$\mathcal{H}_{MN}(x,\, z) = {H^n}_{MN}(x)\zeta_n(z)\ . \tag{7.3.3}$$

Here we sum over repeated Fourier indices (n) and ζ_n is as in section 2.6.1 when $b = 0$.

Now we have found a (perturbative) higher-dimensional action whose extrema agree with the solutions to Einstein's equations given a metric which obeys Neumann-Neumann boundary conditions at $z = \pm l$ (equation (7.3.1)), and we have expanded our higher-dimensional fields in a Sturm–Liouville basis consistent with this choice (equation (7.3.3)) we may integrate over the transverse space to find a lower-dimensional effective action. Applying orthonormality, our Lagrangian density becomes

$$\begin{aligned} -\frac{1}{2}\partial_\sigma {H^n}_{\mu\nu}\partial^\sigma H^{n\mu\nu} + \frac{1}{2}\partial_\sigma {H^n}_\mu{}^\mu \partial^\sigma {H^n}_\nu{}^\nu - \partial^\mu {H^n}_{\mu\nu}\partial^\nu {H^n}_\sigma{}^\sigma + \partial_\sigma {H^n}_{\mu\nu}\partial^\mu H^{n\nu\sigma} \\ -\frac{{\omega_n}^2}{2}\left({H^n}_{\mu\nu}H^{n\mu\nu} - {H^n}_\mu{}^\mu {H^n}_\nu{}^\nu\right) - \partial_\mu {H^n}_{\nu z}\partial^\mu H^{n\nu z} + \partial_\nu {H^n}_{\mu z}\partial^\mu H^{n\nu z} \\ -\partial^\mu {H^n}_{\mu\nu}\partial^\nu {H^n}_z{}^z + \partial_\sigma {H^n}_\nu{}^\nu \partial^\sigma {H^n}_z{}^z + 2\,{H^m}_{\mu\nu}\left(\partial^\mu H^{n\nu z} - \eta^{\mu\nu}\partial_\sigma H^{n\sigma z}\right)\int_{\mathcal{D}} \zeta_n \partial_z \zeta_m dz\ . \end{aligned} \tag{7.3.4}$$

This is almost lower-dimensional massless $(n = 0)$ and massive $(n \neq 0)$ Fierz–Pauli in terms of ${H^n}_{\mu\nu}$, with lower-dimensional Maxwell ${H^n}_{\mu z}$ terms, and lower-dimensional Klein–Gordon terms, except there are putative mixing terms between the world-volume only components of ${H^n}_{\mu\nu}$, the off-diagonal components ${H^n}_{\mu z}$, and the transverse only components ${H^n}_{zz}$.

To simplify we first note the off-diagonal mixing terms are given as $\int_{\mathcal{D}} \zeta_n \partial_z \zeta_m dz = I_{nm'}$, $\int_{\mathcal{D}} \zeta_n \partial_z \zeta_0 dz = 0$ and $\int_{\mathcal{D}} \zeta_0 \partial_z \zeta_m dz = -\sqrt{2}\sin\left(\frac{\pi}{2}n\right)$. Therefore we make the following field redefinition

$${H^n}_{\mu\nu} = {h^n}_{\mu\nu} + \alpha\eta_{\mu\nu}\phi^n\ , \tag{7.3.5}$$

$${H^n}_{\mu z} = {A^n}_\mu\ , \tag{7.3.6}$$

$${H^n}_{zz} = \phi^n\ , \tag{7.3.7}$$

where α is a dimension dependent normalization. Inserting this into our action we find

$$\begin{aligned} -\frac{1}{2}\left(\partial_\sigma {h^n}_{\mu\nu}\right)^2 + \frac{1}{2}\left(\partial_\sigma h^n\right)^2 - \partial^\mu {h^n}_{\mu\nu}\partial^\nu h^n + \partial_\sigma {h^n}_{\mu\nu}\partial^\mu h^{n\nu\sigma} - \alpha^2\left(\partial_\sigma \phi^n\right)^2 \\ -\frac{{\omega_n}^2}{2}\left(\eta^{\mu\rho}\eta^{\nu\sigma} - \eta^{\mu\nu}\eta^{\rho\sigma}\right)\left({h^n}_{\mu\nu} - I_{mn'}\partial_\mu {A^m}_\nu - I_{mn'}\partial_\nu {A^m}_\mu + 2\partial_\mu\partial_\nu\phi^n\right)^2\ . \end{aligned} \tag{7.3.8}$$

If we truncate our heavy fields ($h^{\overline{n}}{}_{\mu\nu}$, $A^{\overline{n}}{}_{\mu}$, and $\phi^{\overline{n}}$ where $\overline{n} > 0$) then we have, at the free level, a massless graviton, a massless scalar and

$$- \frac{1}{{\omega_n}^2} I_{m'0} I_{m'0} = - \sum_{n=1}^{\infty} 2 \frac{\sin^2\left(\frac{\pi}{2}n\right)}{\left(\frac{\pi}{2}n\right)^2} = -1 \; . \tag{7.3.9}$$

The consequence of this is

$$- \frac{{\omega_n}^2}{2} \left(\eta^{\mu\rho}\eta^{\nu\sigma} - \eta^{\mu\nu}\eta^{\rho\sigma}\right) \left(-I_{0n'}\partial_\mu A^0{}_\nu - I_{0n'}\partial_\nu A^0{}_\mu\right) \left(-I_{0n'}\partial_\rho A^0{}_\sigma - I_{0n'}\partial_\sigma A^0{}_\rho\right) = -F^0{}_{\mu\nu}F^{0\mu\nu} \; . \tag{7.3.10}$$

Here $F^n{}_{\mu\nu} = \partial_\mu A^n{}_\nu - \partial_\nu A^n{}_\mu$. So we can identify a massless vector as well.

7.4 Diagonalizing the Gauge Transformations

If we consider a truncation of all of the heavy fields ($h^{\overline{n}}{}_{\mu\nu} = 0$, $A^{\overline{n}}{}_{\mu} = 0$, and $\phi^{\overline{n}} = 0$) then we have, as the first-order perturbation of the metric in the higher dimension, (We now introduce perturbation parameters for ease of reading ϵ denotes the order of the perturbation.)

$$\hat{g}_{MN} = \eta_{MN} + \epsilon\, \mathcal{H}_{MN} \; , \tag{7.4.1}$$

$$\hat{g}_{\mu\nu} = \eta_{\mu\nu} + \epsilon\, h_{\mu\nu} + \epsilon\, \alpha\, \eta_{\mu\nu}\phi \; , \tag{7.4.2}$$

$$\hat{g}_{\mu z} = \epsilon\, A_\mu \; , \tag{7.4.3}$$

$$\hat{g}_{zz} = \epsilon\, \phi \; . \tag{7.4.4}$$

Under the diffeomorphism defined by some higher-dimensional vector $\mathcal{X}^M$ the higher-dimensional metric transforms as (We will use γ to track orders of our diffeomorphism.)

$$\gamma\delta\hat{g}_{MN} + \mathcal{O}\left(\gamma^2\right) = \gamma\left(\mathcal{L}_{\mathcal{X}}\hat{g}\right)_{MN} = \gamma\left(\mathcal{L}_{\mathcal{X}}\eta\right)_{MN} + \gamma\left(\mathcal{L}_{\mathcal{X}}\delta\mathcal{H}\right)_{MN} \; . \tag{7.4.5}$$

Here $(\mathcal{L}_V Y)$ is the Lie derivative of Y with respect to V [126]. We have

$$\gamma\delta\hat{g}_{MN} = \gamma\partial_M\mathcal{X}_N + \gamma\partial_N\mathcal{X}_M + \gamma\epsilon\mathcal{X}^P\partial_P\mathcal{H}_{MN} + \gamma\epsilon\mathcal{H}_{PN}\partial_M\mathcal{X}^P + \gamma\epsilon\mathcal{H}_{MP}\partial_N\mathcal{X}^P \; . \tag{7.4.6}$$

Formally, we could expand $\mathcal{X}^M$ in terms of the entire transverse Sturm–Liouville basis, however, for the purpose of simplicity we will only consider $\mathcal{X}^M$ which is constant in z. Further we separate indices as

$$\mathcal{X}^\mu = X^\mu \ , \tag{7.4.7}$$

$$\mathcal{X}^z = \lambda \ . \tag{7.4.8}$$

From this we have, at zeroth-order perturbatively,

$$\delta\hat{g}_{\mu\nu} = \gamma\partial_\mu X_\nu + \gamma\partial_\nu X_\mu \ = \gamma\partial_\mu X_\nu + \gamma\partial_\nu X_\mu \ , \tag{7.4.9}$$

$$\delta\hat{g}_{\mu z} = \gamma\partial_z X_\mu + \gamma\partial_\mu\lambda \quad = \gamma\partial_\mu\lambda \ , \tag{7.4.10}$$

$$\delta\hat{g}_z = 2\gamma\partial_z\lambda \qquad\qquad = 0 \ . \tag{7.4.11}$$

From this we may identify the zeroth-order transformation in ϵ of our perturbed fields

$$\delta\hat{g}_{MN} = \delta\eta_{MN} + \epsilon\delta\mathcal{H}_{MN} \ . \tag{7.4.12}$$

In general, we could choose to absorb part of this transformation into the background (in this case, $\delta\eta_{MN}$). We choose not to, and matching these terms we have

$$\epsilon\delta h_{\mu\nu} = \gamma\partial_\mu X_\nu + \gamma\partial_\nu X_\mu \ , \tag{7.4.13}$$

$$\epsilon\delta A_\mu = \gamma\partial_\mu\lambda \ , \tag{7.4.14}$$

$$\epsilon\delta\phi = 0 \ . \tag{7.4.15}$$

This is a requirement for our transformations to be consistently defined in the higher dimension given our definition of $\mathcal{H}_{MN}$ in terms of $h_{\mu\nu}$, A_μ and ϕ. We take as a further condition that our gauge transformation is small relative to our perturbation $\epsilon >> \gamma$. Scrutinizing equations (7.4.13), (7.4.14), and (7.4.15) we notice that these transformations define the linear order of expected diffeomorphism and gauge transformations in the lower dimension. The standard definition of these variable's covariant transformations is

$$\delta h_{\mu\nu} = \frac{\gamma}{\epsilon}\left(\mathcal{L}_X\eta\right)_{\mu\nu} + \gamma\left(\mathcal{L}_X h\right)_{\mu\nu} \ , \tag{7.4.16}$$

$$\delta A_\mu = \frac{\gamma}{\epsilon}\partial_\mu\lambda + \gamma\left(\mathcal{L}_X A\right)_\mu \ , \tag{7.4.17}$$

$$\delta\phi = 0 + \gamma\left(\mathcal{L}_X\phi\right) \ . \tag{7.4.18}$$

We note that the terms we have written at linear-order in perturbations in equations (7.4.16), (7.4.17), and (7.4.18) are a choice, we could choose simply to set the next order of their transformation equal to whatever it was natively in the higher dimension. As a consequence of this choice, however, the next order of the transformations in the higher dimension no longer agrees:

$$\gamma \left(\mathcal{L}_{\mathcal{X}}\eta\right)_{\mu\nu} + \gamma\epsilon \left(\mathcal{L}_{\mathcal{X}}\mathcal{H}\right)_{\mu\nu} = \gamma\partial_\mu X_\nu + \gamma\partial_\nu X_\mu + \gamma\epsilon\alpha X^\sigma \eta_{\mu\nu}\partial_\sigma\phi \tag{7.4.19}$$

$$+ \gamma\epsilon X^\sigma \partial_\sigma h_{\mu\nu} + \gamma\epsilon h_{\sigma\nu}\partial_\mu X^\sigma + \gamma\epsilon h_{\mu\sigma}\partial_\nu X^\sigma$$

$$+ \gamma\epsilon\alpha\phi\eta_{\sigma\nu}\partial_\mu X^\sigma + \gamma\epsilon\alpha\phi\eta_{\mu\sigma}\partial_\nu X^\sigma + \gamma\epsilon A_\nu\partial_\nu\lambda + \gamma\epsilon A_\mu\partial_\mu\lambda \tag{7.4.20}$$

$$\neq \delta\hat{g}_{\mu\nu} = \gamma\partial_\mu X_\nu + \gamma\partial_\nu X_\mu + \gamma\epsilon\alpha X^\sigma \eta_{\mu\nu}\partial_\sigma\phi$$

$$+ \gamma\epsilon X^\sigma \partial_\sigma h_{\mu\nu} + \gamma\epsilon h_{\sigma\nu}\partial_\mu X^\sigma + \gamma\epsilon h_{\mu\sigma}\partial_\nu X^\sigma + \gamma\epsilon\alpha\eta_{\mu\nu}X^\sigma\partial_\sigma\phi \, . \tag{7.4.21}$$

There are several possible methods of addressing this inconsistency, and we will study them in general in the next section 7.5. However, it is simplest (and most consistent with the literature [108]) in this case to correct $\mathcal{H}_{MN}$ at second-order in perturbations as

$$\epsilon\mathcal{H}_{\mu\nu} = \epsilon \left(h_{\mu\nu} + \alpha\eta_{\mu\nu}\phi\right) + \epsilon^2 \left(\alpha\phi h_{\mu\nu} + A_\mu A_\nu\right) \, , \tag{7.4.22}$$

$$\epsilon\mathcal{H}_{\mu z} = \epsilon \left(A_\mu\right) + \epsilon^2 \left(\phi A_\mu\right) \, , \tag{7.4.23}$$

$$\epsilon\mathcal{H}_{zz} = \epsilon\phi \, . \tag{7.4.24}$$

Here the quadratic terms (in perturbations) of $\mathcal{H}_{\mu z}$ and $\mathcal{H}_{zz}$ are found through the same consistency requirement through which we found $\mathcal{H}_{\mu\nu}$ at second order.

If we repeat this process we will find cubic, quartic, and higher terms. However, the primary purpose of this procedure is to identify possible exact ansätze. Here we may recognize the Taylor expansion of the exponential function and guess

$$\hat{g}_{MN}dX^M dX^N = \exp(\epsilon\alpha\phi) \left(\eta_{\mu\nu} + \epsilon h_{\mu\nu} + \epsilon A_\mu A_\nu\right) dx^\mu dx^\nu + \epsilon \exp(\epsilon\alpha\phi) A_\mu dz dx^\mu + \epsilon\phi dz^2 \, , \tag{7.4.25}$$

This ansatz solves the relationship between the different orders of terms exactly and agrees with our diagonalization of the lower-dimensional action.

7.5 The Generic Recursion Equation

The technique we applied in the last section is actually applicable to an arbitrary dimensional reduction. That is, in the last section we showed how to derive the quadratic order (in lower-dimensional fields) of the higher-dimensional field definitions given the goal of diagonalizing a canonical transformation in terms of the lower-dimensional fields. This generates, at arbitrary order, the Kaluza–Klein ansatz, when we repeat the analysis of the previous section to arbitrary order. Furthermore, it is applicable to more possible lower-dimensional transformations. Therefore in this section we will define the generic recursion equation.

To generalize our system, take our usual background with our product manifold $\mathcal{M}_l \times_W \mathcal{M}_z$ and transverse spectrum f_0 and f_{ϖ} and their putative derivative basis counterparts ζ, f_0', and f_{ϖ}'. We specify a decomposition of our higher-dimensional perturbative (of order ϵ) fields Φ, in terms of our lower-dimensional perturbative (of order ϵ) fields ϕ, and our higher-dimensional transformation specification (of order γ) Λ in terms of lower-dimensional transformation specification (of order γ) λ order by order.

$$\epsilon\Phi = \epsilon\Phi_1 + \epsilon^2\Phi_2 + \epsilon^3\Phi_3 + \ldots = \epsilon(\phi f_0 + \phi^{\overline{\omega}} f_{\varpi} + \ldots) + \epsilon^2(a_2\phi^2 f_b(z) + \phi\phi^{\overline{\omega}} f_{b,\varpi} + \ldots) + \ldots \ , \tag{7.5.1}$$

$$\gamma\Lambda = \gamma\Lambda_0 + \gamma\epsilon\Lambda_1 + \ldots \qquad = \gamma(\lambda f_0 + \lambda^{\overline{\omega}} f_{\varpi} + \ldots) + \gamma\epsilon(d_1\lambda\phi f_d(z) + \ldots) + \ldots \ . \tag{7.5.2}$$

Here f_a, $\ldots$, f_d are other functions which we may expand in our bases $f_a = a_0 f_0 + a_1 f_1 + \ldots$. Also note that we have defined our transformation Λ in the higher dimension so that it explicitly allows for some dependence on the fields at nonlinear order (e.g. $d_1\lambda\phi$ in equation (7.5.2)).

The recursion equation requires the transformation of the lower-dimensional fields, which we specify in terms of the lower-dimensional quantities $\overline{\delta}\phi \sim \lambda + \ldots$ (we denote δ for our higher-dimensional transformations and $\overline{\delta}$ for our lower-dimensional transformations), to equal the transformation of the higher-dimensional fields. In the case where we only go to linear order in γ (that is, linear order in Λ) we may expand out our higher-dimensional transformation (given in terms of some function F)

$$\delta\Phi = F(\gamma\Lambda, \epsilon\Phi) - F(0, \epsilon\Phi) \tag{7.5.3}$$

$$\begin{aligned} =& \cancel{F^{0,0}(0,0)} + \gamma\Lambda F^{(1,0)}(0,0) + \epsilon\Phi \cancel{F^{(0,1)}(0,0)} \\ &+ \gamma^2\frac{1}{2}\Lambda^2 \cancel{F^{(2,0)}(0,0)} + \gamma\epsilon\Lambda\Phi F^{(1,1)}(0,0) + \epsilon^2\frac{1}{2}\Phi^2 \cancel{F^{(0,2)}(0,0)} \\ &+ \gamma^3\frac{1}{6}\Lambda^3 \cancel{F^{(3,0)}(0,0)} + \gamma^2\epsilon\frac{1}{2}\Lambda^2\Phi \cancel{F^{(2,1)}(0,0)} + \gamma\epsilon^2\frac{1}{2}\Lambda\Phi^2 F^{(1,2)}(0,0) + \epsilon^3\frac{1}{6}\Phi^3 \cancel{F^{(0,3)}(0,0)} + \mathcal{O}(\Phi^3) \ . \end{aligned} \tag{7.5.4}$$

We can expand this further in terms of the internal order ϵ for Λ and Φ. When we take the transformation of some term that will add one order of γ, however, it will not necessarily decrease the order of ϵ, since we may have terms such as $\delta\phi = \mathcal{L}_X\phi$ as in the previous section. Therefore we will define operators $delta^n$ which act on our higher-dimensional fields and select the ϵ^n order term. We may have an ϵ^0 term if we suppose that $\epsilon >> \gamma$

$$\epsilon\delta\Phi = \epsilon^0\delta^0\Phi + \epsilon^1\delta^1\Phi + \epsilon^2\delta^2\Phi + \mathcal{O}(\phi^3) \ , \tag{7.5.5}$$

$$\epsilon^0\delta^0\Phi = \gamma\epsilon^0\Lambda^0 F^{(1,0)}(0,0) \ , \tag{7.5.6}$$

$$\epsilon\delta^1\Phi = \gamma\epsilon\left(\Lambda_1 F^{(1,0)}(0,0) + \Lambda_0\Phi_1 F^{(1,1)}(0,0)\right) \ , \tag{7.5.7}$$

$$\epsilon^2\delta^2\Phi = \gamma\epsilon^2\left(\Lambda_2 F^{(1,0)}(0,0) + \Lambda_1\Phi_1 F^{(1,1)}(0,0) + \frac{1}{2}\Lambda_0\left(\Phi_1\right)^2 F^{(1,2)}(0,0)\right) \ . \tag{7.5.8}$$

Similarly we define a lower-dimensional transformation operator $\overline{\delta}$. Further, we define operators $\overline{\delta}^n$ which selects the ϵ^n term

$$\epsilon\overline{\delta}\phi = \epsilon^0\overline{\delta}^0\phi + \epsilon\overline{\delta}^1\phi + \epsilon^2\overline{\delta}^2\phi + \mathcal{O}(\phi^3) \ . \tag{7.5.9}$$

$\overline{\delta}$ is an operator which obeys the Leibniz rule [123]

$$\epsilon^2\left(\overline{\delta}\phi\psi\right) = \epsilon^2\left(\overline{\delta}\phi\right)\psi + \epsilon^2\phi\left(\overline{\delta}\psi\right) \ . \tag{7.5.10}$$

So we may expand $\overline{\delta}\Phi$ order by order as well, by expanding both $\Phi = \Phi_1 + \Phi_2 + \ldots$ and $\overline{\delta} = \overline{\delta}^1 + \overline{\delta}^2 + \ldots$.

$$\epsilon\overline{\delta}\Phi = \epsilon^0\left(\overline{\delta}^0\Phi_1\right) + \epsilon\left(\overline{\delta}^1\Phi_1 + \overline{\delta}^0\Phi_2\right) + \epsilon^2\left(\overline{\delta}^2\Phi_1 + \overline{\delta}^1\Phi_2 + \overline{\delta}^0\Phi_3\right) \ . \tag{7.5.11}$$

The recursion equation is

$$\delta\Phi = \overline{\delta}\Phi \ . \tag{7.5.12}$$

Specifying this order by order, it is

$$\gamma\epsilon^0\Lambda^0 F^{(1,0)}(0,0) = \epsilon^0\overline{\delta}^0\Phi_1 \ , \tag{7.5.13}$$

$$\gamma\epsilon\Lambda^1 F^{(1,0)}(0,0) + \gamma\epsilon\Lambda^0\Phi_1 F^{(1,1)}(0,0) = \epsilon\overline{\delta}^1\Phi_1 + \epsilon\delta^0\Phi_2 \ , \tag{7.5.14}$$

$$\gamma\epsilon^2\Lambda^2 F^{(1,0)}(0,0) + \gamma\epsilon^2\Lambda^1\Phi_1 F^{(1,1)}(0,0) + \gamma\epsilon^2\frac{1}{2}\Lambda^0\left(\Phi_1\right)^2 F^{(1,2)}(0,0) = \epsilon^2\overline{\delta}^2\Phi_1 + \epsilon^2\overline{\delta}^1\Phi_2 + \epsilon^2\overline{\delta}^0\Phi_3 \ . \tag{7.5.15}$$

If we suppose that some part of our definitions are unspecified, then either the field or transformation parameter decomposition ($\Phi^1, \ \Phi^2, \ \ldots$ or $\Lambda^0, \ \Lambda^1, \ \ldots$), or the definition of the lower-dimensional transformations ($\overline{\delta}^0\phi, \ \overline{\delta}^1\phi, \ \ldots$), or potentially some combination, is underspecified. This may allow a non-trivial field redefinition from one set of lower-dimensional ϕ that transform in one way to an alternative ϕ' with a different transformation, perhaps making some fields invariant under the transformation at the cost of a noncanonical transformation of a different field. This is relevant in the context of a non-linear realization, because we can prove that there does not exist any perturbative field redefinition that linearizes said realization. We will explore this possibility in the next section (8.6). In the end there is only one lower-dimensional physical theory, but using this technique we can express it in several different ways, and hopefully more easily connect to the literature.

8 Covert Symmetry Breaking in Scalar Electrodynamics

We have covered all prerequisite aspects of dimensional reduction in the context of inconsistent truncations to handle the case where we reduce a gauge theory with nonlinear interaction terms and boundary conditions other than Dirichlet or special Neumann. Specifically we understand what boundary terms we require in the higher-dimensional action (Section 7), and what corrections to the effective field theory we can expect (section 6). Additionally, from Section 7, we understand how our higher-dimensional gauge transformations become our lower-dimensional gauge transformations corresponding to massless gauge fields and our spontaneously broken lower-dimensional gauge transformations corresponding to massive vectors, gravitons. This understanding, however, was in the context of constant transverse zero modes and consistent truncations. The purpose of this section is to study what happens when we combine non-constant transverse zero modes and higher-dimensional gauge transformations.

We will accomplish this by studying free higher-dimensional Maxwell theory, then higher-dimensional Maxwell theory coupled to a complex scalar. We will learn that, similar to the case Maxwell theory with Neumann-Neumann conditions, we have one massless lower-dimensional vector degree of freedom and one massless scalar degree of freedom.

8.1 The Quadratic Theory

The higher-dimensional Maxwell equations on a totally flat manifold ($\mathcal{M}_l = \mathbb{R}^{1,d-1} \times \mathcal{D}$ as in equation (7.1.1)) are [106]

$$\partial_N \partial^N \mathcal{A}_M - \partial_M \partial^N \mathcal{A}_N = 0 \ . \tag{8.1.1}$$

If we separate out our indices we have

$$\left(\Box + \Delta\right) \mathcal{A}_\mu - \partial_\mu \partial^\nu \mathcal{A}_\nu - \partial_\mu \partial^z \mathcal{A}_z = 0 \ , \tag{8.1.2}$$

for our world-volume ($M = \mu$) equation and

$$\Box \mathcal{A}_z - \partial_z \partial^\mu \mathcal{A}_\mu = 0 \ , \tag{8.1.3}$$

for our transverse ($M = z$) equation.

We choose $\mathcal{D} = (-1, 1)$ for ease of demonstration. Our transverse boundary conditions are

$$(f_0(-1)\partial_z - b)\mathcal{A}_\mu(x,\, z)\Big|_{z\to -1^+} = 0 \;, \qquad (f_0(1)\partial_z - b)\mathcal{A}_\mu(x,\, z)\Big|_{z\to 1^-} = 0 \;. \tag{8.1.4}$$

Here f_0 is as in section (2.6.1). Note, these are only for the world-volume components. If we focus only on terms that include only world-volume components and transverse derivatives in the bulk action we have

$$\mathcal{S} = \ldots + \int \int_{-1}^{1} -\frac{1}{2} \left(\partial_z \mathcal{A}_\mu\right)^2 dz d^d x + \ldots \;. \tag{8.1.5}$$

As in equation (6.4.3), for the extrema of this action to correspond to the solutions of our equations of motion given the boundary conditions that allow for f_0 we require an additional boundary term

$$\mathcal{S} = \ldots + \int \frac{1}{2} \frac{f_0(z)}{b} \left(\partial_z \mathcal{A}_\mu\right)^2 \Big|_{z\to -1^+}^{z\to 1^-} d^d x \; + \ldots \;. \tag{8.1.6}$$

Of course, the total action must be invariant under gauge transformations. Explicitly, invariant under

$$\mathcal{A}'_\mu(x,\, z) = \mathcal{A}_\mu(x,\, z) + \partial_\mu \Lambda(x,\, z) \;, \tag{8.1.7}$$

$$\mathcal{A}'_z(x,\, z) = \mathcal{A}_z(x,\, z) + \partial_z \Lambda(x,\, z) \;, \tag{8.1.8}$$

where $\Lambda(x,\, z)$ is an arbitrary higher-dimensional function. The terms in equations (8.1.5) and (8.1.6) are paired with the appropriate $\partial_\mu \mathcal{A}_z$ terms to combine to gauge invariant quantities. The total action reads

$$\mathcal{S} = \int \int_{-1}^{1} \left\{ -\frac{1}{4} \left(\partial_\mu \mathcal{A}_\nu - \partial_\nu \mathcal{A}_\mu\right)^2 - \frac{1}{2} \left(\partial_\mu \mathcal{A}_z - \partial_z \mathcal{A}_\mu\right)^2 \right\} dz d^d x + \mathcal{S}_{\text{Maxwell boundary}} \;. \tag{8.1.9}$$

Here our boundary term is

$$\mathcal{S}_{\text{Maxwell boundary}} = \int \frac{1}{2} \frac{f_0(z)}{b} \left(\partial_\mu \mathcal{A}_z - \partial_z \mathcal{A}_\mu\right) \Big|_{z\to -1^+}^{z\to 1^-} d^d x \;. \tag{8.1.10}$$

This action is symmetric under arbitrary higher-dimensional gauge transformation. However, if Λ obeys different boundary conditions at $z \to \pm 1^{\mp}$ than $\mathcal{A}_\mu$, then our transformed fields, $\mathcal{A}'_\mu$ will not obey our boundary conditions. Therefore this system is only invariant under Λ which obey

$$(f_0(-1)\partial_z - b)\Lambda(x,\, z)\Big|_{z\to -1^+} = 0 \;, \qquad (f_0(1)\partial_z - b)\Lambda(x,\, z)\Big|_{z\to 1^-} = 0 \;. \tag{8.1.11}$$

At the free level this system is also invariant under transformations of $\mathcal{A}_z$ alone. It is

$$\mathcal{A}_z'(x,\, z) = \mathcal{A}_z(x,\, z) + \partial_z \Gamma(x,\, z) \ , \tag{8.1.12}$$

where Γ lies within the kernel of $\Box$ and Δ independently, or

$$\Box\Gamma = \Delta\Gamma = 0 \ . \tag{8.1.13}$$

Given that $\mathcal{A}_\mu$ obeys boundary conditions that allow for f_0, we should expand $\mathcal{A}_\mu$ in the $\{f_{\omega_i}\}$ basis. We will call this the $\{f_i\}$ basis for the sake of margin space. Further, we see that $\mathcal{A}_z$'s kinetic terms, those are $(\partial_\mu \mathcal{A}_z)^2$, are under not the standard $L_2(-1,1)$ inner product, but the augmented inner product associated with $\mathcal{A}_z$, described in section 2.5 due to the boundary terms described in equation (8.1.10). Together we have

$$\mathcal{A}_\mu(x,\, z) = A^i{}_\mu(x) f_i(z) \ , \tag{8.1.14}$$

$$\mathcal{A}_z(x,\, z) = B^i(x) f_i'(z) + C(x)\zeta(z) \ . \tag{8.1.15}$$

Note, we adopt the shorthand that $\phi^0 = \phi$ for any lower-dimensional field ϕ^i, or, if we omit the Sturm–Liouville index, we are referring to the mode corresponding to the smallest (in this case zero) eigenvalue.

With this we integrate over the transverse dimension in our higher-dimensional action and find ($Z = (\xi_0\, \zeta)$ where $(\cdot,\, \cdot)$ is as in section 2.5)

$$\mathcal{S} = \int \left\{ -\frac{1}{4} \left(\partial_\mu A^i{}_\nu - \partial_\nu A^i{}_\mu \right)^2 - \frac{1}{2} {\omega_i}^2 \left(A^i{}_\mu - \partial_\mu B^i \right)^2 + Z \partial^\mu C \left(A^0{}_\mu - \partial_\mu B^0 \right) \right\} d^d x \ . \tag{8.1.16}$$

This corresponds to a Stueckelberged [1] (Lorenz) gauge-fixed [19] lower-dimensional Maxwell field $A^0{}_\mu$ (B^0 is a Stueckelberg [122] and C is a Lagrange multiplier [117]) and a series of Stueckelbered Proca fields $A^i{}_\mu$ (B^i are Stueckelbergs).

To verify this we also decompose our higher-dimensional gauge parameter Λ as well as Γ,

$$\Lambda(x,\, z) = \lambda^i(x) f_i(z) \ , \qquad \Gamma(x,\, z) = \gamma(x) f_0(z) \ , \tag{8.1.17}$$

then our lower-dimensional fields transform as

$$A^{i\prime}{}_\mu(x) = \partial_\mu \lambda^i(x) \ , \tag{8.1.18}$$

$$B^{i\prime}(x) = \lambda^i(x) + \delta_{i0}\gamma(x) \ . \tag{8.1.19}$$

Combining each of these symmetries we deduce that the lower-dimensional theory comprises one lower-dimensional massless vector degree of freedom and a tower of lower-dimensional massive degrees of freedom with masses $m^2 = {\omega_i}^2$. However, a definitive calculation of the degrees of freedom requires either an analysis of our lower-dimensional equations in Fourier space or a Hamiltonian analysis.

8.2 A Fourier Space Analysis of the Degrees of Freedom

We define our lower-dimensional Fourier transforms as [106]

$$A^i{}_\mu(x) = \int \exp(-ik\cdot x)\left(a^i{}_\mu(k) + k_\mu \epsilon^i\right) d^dk \ , \tag{8.2.1}$$

$$B^i(x) = \int \exp(-ik\cdot x) b^i(k) d^dk \ , \qquad C(x) = \int \exp(-ik\cdot x) c(k) d^dk \ . \tag{8.2.2}$$

Here, without loss of generality, $a^i{}_\mu$ is linearly independent of k_μ. Our lower-dimensional equations of motion (from the variation of equation (8.1.16)) read

$$\Box A^i{}_\mu - \partial_\mu\partial^\nu A^i{}_\nu - {\omega_i}^2\left(A^i{}_\mu - \partial_\mu B^i\right) + Z\delta_{i0}\partial_\mu C = 0 \ , \tag{8.2.3}$$

$${\omega_i}^2\left(\Box B^i - \partial^\mu A^i{}_\mu\right) + Z\delta_{i0}\Box C = 0 \ , \tag{8.2.4}$$

$$Z\left(\Box B^0 - \partial^\mu A^0{}_\mu\right) = 0 \ , \tag{8.2.5}$$

for A^i, B^i and C, respectively. We note, first, that B^0 appears only in C's equation. Furthermore we note that B^i's equation is simply the divergence $A^i{}_\mu$'s equation for all i, and therefore they are redundant. In Fourier space our system's equations are (excluding the redundant equation (8.2.4))

$$k^2 a^i{}_\mu - k_\mu k^\nu a^i{}_\nu - {\omega_i}^2\left(a^i{}_\mu + k_\mu\epsilon^i + ik_\mu b^i\right) + iZ\delta_{i0}k_\mu c = 0 \ , \tag{8.2.6}$$

$$Z\left(k^2 b^0 + ik^\mu a^0{}_\mu + ik^2\epsilon^0\right) = 0 \ . \tag{8.2.7}$$

We are mainly interested in the zero mode's equation, which is

$$k^2 a_\mu - k_\mu k^\nu a_\nu + iZk_\mu c = 0 \ . \tag{8.2.8}$$

Since $a^i{}_\mu$ and k_μ are linearly independent, equation (8.2.6) implies

$$\left(k^2 - {\omega_i}^2\right) a^i{}_\mu = 0 \ , \tag{8.2.9}$$

$$k_\mu \left(-k^\nu a^i{}_\nu - {\omega_i}^2 \epsilon^i - i\omega_i b^i + iZ\delta_{i0} c\right) = 0 \ . \tag{8.2.10}$$

From equation (8.2.9) we learn that $a^i{}_\mu$ may only have support onshell. Restated $a^i{}_\mu = 0$ when $k^2 \neq {\omega_i}^2$.

Similarly, when $k^2 = 0$, b^0 and ϵ^0 do not appear in c's equation (8.2.8). Therefore $k^\nu a^0{}_\nu = 0$ onshell as well. Applying this to equation (8.2.10) for $i = 0$ we learn

$$\omega_0{}^2 \left(\epsilon^0 + ib^0\right) - iZc = 0 \ , \tag{8.2.11}$$

everywhere. Since $\omega_0{}^2 = 0$, this implies $c = 0$ everywhere. Combining equations (8.2.8) and (8.2.10) we have

$$k^\mu a^i{}_\mu + k^2 \epsilon^i + ik^2 b^i = 0 \ , \tag{8.2.12}$$

for all i. This entire quantity is gauge invariant, however ϵ^i and b^i both transform proportionally to the gauge parameter, that is, under a gauge transformation ($\lambda^i = \int \exp(-ik\cdot x) l^i(k) d^d k$)

$$\epsilon^{i\prime} = \epsilon^i - il^i \ , \qquad b^{i\prime} = b^i + l^i \ . \tag{8.2.13}$$

For the sake of clarity let us set $l^i = -i\epsilon^i$, or $\epsilon^{i\prime} = 0$ for all i. Since the difference of ϵ^i and ib^i must have the same support as $a^i{}_\mu$, this implies $b^i = 0$ offshell.

At the free level, we still have one final transformation for this system, which is, for $\gamma = \int \exp(-ik \cdot x) g(k) d^k$,

$$b^{0\prime} = b^0 + g \ . \tag{8.2.14}$$

Selecting $g^0 = -b^0$ we eliminate b^0 as well.

In the final count for $\bar{i} > 0$ we have the $d-2$ degrees of freedom of $a^{\bar{i}}{}_\mu$ which have mass $m^2 = \omega_{\bar{i}}^2$ with one more degree of freedom, $b^{\bar{i}}$, with the same mass. Together these compose a massive vector. A^0 comprises $d-2$ degrees of freedom. Further, at the free level we have one scalar degree of freedom that can be removed by our harmonic symmetry. Removing this degree of freedom, however, may not be well-defined at interacting order for our system.

8.3 Hamiltonian Analysis of the Quadratic Theory

To ease the calculation of the Hamiltonian of this theory we will gauge fix the lower-dimensional effective field theory, in the $B = 0$ gauge, and consider only the lightest fields.

$$\mathcal{L} = -\frac{1}{4}\left(\partial_\mu A_\nu - \partial_\nu A_\mu\right)^2 + Z\left(\partial^\mu C\right) A_\mu \ . \tag{8.3.1}$$

In this gauge we notice that our Lagrangian density is a limiting form of $R\xi$ gauge fixed Lagrangian density which generically reads [106]

$$\mathcal{L} = -\frac{1}{4}\left(\partial_\mu A_\nu - \partial_\nu A_\mu\right)^2 + C\partial^\mu A_\mu - \frac{\xi}{2}C^2 \ . \tag{8.3.2}$$

Here ξ is not a Sturm–Liouville eigenvalue but an arbitrary numeric constant named ξ by convention. Note we have rescaled C to eliminate Z. Given this Lagrangian we calculate the canonical momenta (which we name E_i and π)[89]

$$E_i = \frac{\partial L}{\partial \partial_0 A_i} = \partial_0 A_i - \partial_i A_0 \ , \qquad \pi = \frac{\partial L}{\partial \partial_0 A_0} = C \ . \tag{8.3.3}$$

We then note the equivalence of our lower-dimensional effective action in Hamiltonian and Lagrangian form to define our Hamiltonian density

$$\mathcal{H} = E_i \partial_0 A_i + \pi \partial_0 A_0 - \mathcal{L} \ , \tag{8.3.4}$$

$$\mathcal{H} = \frac{1}{2}{E_i}^2 + \frac{1}{4}{F_{ij}}^2 + \frac{\xi}{2}\pi^2 + A_0\left(\partial_i E_i\right) + \pi\left(\partial_i A_i\right) \ . \tag{8.3.5}$$

8.4 Higher-Dimensional Scalar Electrodynamics

In section 6 we analyzed a higher-dimensional interacting theory with a non-constant transverse zero mode and observed nonlinear (as a function of the number of fields) corrections to the couplings. How does this square with a gauge symmetry, which normally constrains the relationship between couplings at different orders?

Consider a higher-dimensional complex scalar Φ which is gauge covariant; its gauge transformation is [106]

$$\Phi'(x,\, z) = \exp(ie\Lambda(x,\, z))\Phi(x,\, z) \ , \tag{8.4.1}$$

[89] We choose 0 to be the index of our time coordinate.

(under higher-dimensional gauge transformations with parameter Λ). The (gauge covariant) equations of motion of higher-dimensional scalar electrodynamics are

$$\partial_N \partial^N \mathcal{A}_M - \partial_M \partial^N \mathcal{A}_N - ie\overline{\Phi}\left(\partial_M - ie\mathcal{A}_M\right)\Phi + ie\Phi\left(\partial_M + ie\mathcal{A}_M\right)\overline{\Phi} = 0\ , \tag{8.4.2}$$

$$\partial_N \partial^N \Phi - ie\left(\left(\partial_N \mathcal{A}^N\right)\Phi + 2\mathcal{A}_N \partial^N \Phi\right) - e^2 \mathcal{A}_N \mathcal{A}^N \Phi = 0\ . \tag{8.4.3}$$

The first challenge in defining our higher-dimensional field theory is in ensuring the covariance of our boundary conditions. For instance, consider a gauge-covariant complex scalar Φ and a linear special Neumann boundary condition at $z \to 1^+$

$$\partial_z \Phi(x,\, z)\Big|_{z\to 1^-} = 0\ . \tag{8.4.4}$$

If we consider a Φ and Φ' which both obey our boundary conditions and are related by a gauge transformation, then

$$\begin{aligned}\partial_z \Phi'(x,\, z)\Big|_{z\to 1^-} &= \exp(ie\Lambda(x,\, z)\left(ie\left(\partial_z \Lambda(x,\, z)\right)\Phi(x,\, z) + \partial_z \Phi(x,\, z)\right)\Big|_{z\to 1^-} \\ &= ie\left(\partial_z \Lambda(x,\, z)\right)\exp\left(ie\Lambda(x,\, z)\right)\Phi(x,\, z)\Big|_{z\to 1^-} = 0\ .\end{aligned} \tag{8.4.5}$$

That is, our boundary condition on Φ further restricts our possible gauge transformations. Note that this restriction agrees with the restriction we already observed from $\mathcal{A}_\mu$'s boundary condition (equation (8.1.11)) only when $\mathcal{A}_\mu$ obeys special Neumann boundary conditions.

To resolve this issue we must make our boundary condition gauge covariant itself, by appealing to the covariant derivative. For the case of special Robin boundary conditions we generalize our boundary conditions (as given in equation (6.4.2)) to

$$\left(a\left(\partial_z - ie\mathcal{A}_z(x,\, z)\right) + b\right)\Phi(x,\, z)\Big|_{z\to -1^-} = 0\ , \qquad \left(c\left(\partial_z - ie\mathcal{A}_z(x,\, z)\right) + d\right)\Phi(x,\, z)\Big|_{z\to 1^-} = 0\ , \tag{8.4.6}$$

and generalize our boundary terms (which are given for a real scalar field in equation (6.4.3) to

$$\mathcal{S}_{\text{scalar boundary}} = -\int \frac{a}{b}\left(\left|\partial_z - ie\mathcal{A}_z\right)\Phi\right|^2\Big|_{z\to -1^+} d^d x + \int \frac{c}{d}\left|\left(\partial_z - ie\mathcal{A}_z\right)\Phi\right|^2\Big|_{z\to 1^-} d^d x\ . \tag{8.4.7}$$

Our bulk higher-dimensional action now reads ($\mathcal{S}_{\text{SED bulk}} =$)

$$\int\int_{-1}^{1}\left\{-\frac{1}{4}\left(\partial_\mu \mathcal{A}_\nu - \partial_\nu \mathcal{A}_\mu\right)^2 - \frac{1}{2}\left(\partial_\mu \mathcal{A}_z - \partial_z \mathcal{A}_\mu\right)^2 - \left|\left(\partial_\mu - ie\mathcal{A}_\mu\right)\Phi\right|^2 - \left|\left(\partial_z - ie\mathcal{A}_z\right)\Phi\right|^2\right\} dz d^d x\ , \tag{8.4.8}$$

where $\mathcal{S}_{\text{Maxwell boundary}}$ is as given in equation (8.1.10). Our total action is

$$\mathcal{S} = \mathcal{S}_{\text{SED bulk}} + \mathcal{S}_{\text{Maxwell boundary}} + \mathcal{S}_{\text{scalar boundary}} \ . \tag{8.4.9}$$

Our boundary terms are constructed so that whenever we vary either $\mathcal{A}_\mu$ or Φ the terms found by integration by parts which collect on the boundary cancel the variation of the terms already on the boundary. For $\mathcal{A}_z$ however, since there are no terms in the bulk which include transverse derivatives of $\mathcal{A}_z$, the total variation on the boundary terms is, at the upper boundary for instance,

$$\int \left\{ \frac{1}{2} X \delta \mathcal{A}^z \partial^\mu \left(\partial_z \mathcal{A}_\mu - \partial_\mu \mathcal{A}_z\right) + \frac{c}{d} \delta \mathcal{A}^z \left(ie\overline{\Phi} \left(\partial_z - ie\mathcal{A}_z\right) \Phi - ie\Phi \left(\partial_z + ie\mathcal{A}_z\right) \overline{\Phi} \right) \right\} \Bigg|_{z\to 1^-} d^d x \ . \tag{8.4.10}$$

If we apply Φ's boundary condition the terms involving covariant derivatives of Φ cancel. Similarly if we consider the z component of our higher-dimensional vector's equation of motion

$$\Box \mathcal{A}_z - \partial_z \partial^\mu \mathcal{A}_\mu - ie\overline{\Phi} \left(\partial_z - ie\mathcal{A}_z\right) \Phi + ie\Phi \left(\partial_z + ie\mathcal{A}_z\right) \overline{\Phi} = 0 \ , \tag{8.4.11}$$

and impose Φ's boundary condition at $z \to 1^+$, the terms involving covariant derivatives of Φ cancel. Therefore, as in the case of free Maxwell, the terms from the variation on the boundary are proportional to the bulk equations of motion.

8.5 Nonlinear Boundary Conditions and Perturbative Field Redefinitions

Since we have linear boundary conditions for $\mathcal{A}_\mu$ and $\mathcal{A}_z$ we choose to expand them in the bases given in (8.1.14) and (8.1.15), respectively. However Φ obeys nonlinear boundary conditions, which require a more subtle treatment.

In principle for two different points on the world volume $x_a,\, x_b \in \mathbb{R}^{1,d-1}$, where the transverse components of our higher-dimensional vector do not agree at the boundary $\mathcal{A}_z(x_a,\, 1) \neq \mathcal{A}_z(x_b,\, 1)$, we have two separate boundary conditions

$$\left(c\partial_z + d - cie\mathcal{A}(x_{a/b},\, 1)\right) \Phi(x,\, z) \Big|_{z\to 1^-} = 0 \ . \tag{8.5.1}$$

There are several possible resolutions to this problem. First, we may consider only special Dirichlet conditions for Φ, that is $\Phi(x, \pm 1) = 0$. Since this is covariant without the inclusion of $\mathcal{A}_z$, this avoids any of the following discussion.

Second, we note we have a basis expansion for $\mathcal{A}_z$ which permits the transverse antiderivative, that is

$$\partial_z{}^{-1}\mathcal{A}_z(x,\, z) = B^i(x) f_i(z) + C(z) Z(z) \; . \tag{8.5.2}$$

Here $\partial_z Z(z) = \zeta(z)$.[90] If we consider the gauge transformation of our lower-dimensional fields, $\delta B^i(x) = \lambda^i(x)$ and $\delta C(x) = 0$, then there is a separate gauge-inert quantity we can define in the higher dimension

$$\Psi(x,\, z) = \exp\left(-ie\partial_z{}^{-1}\mathcal{A}_z(x,\, z)\right)\Phi(x,\, z) = \exp\left(-ieB^i(x)\xi_i(z) - ieC(x)Z(z)\right)\Phi(x,\, z) \; . \tag{8.5.3}$$

This Ψ obeys a linearized version of the boundary conditions Φ obeys. We therefore expand it in corresponding Sturm–Liouville basis, which we name g_n with eigenvalues $-\sigma_n{}^2$, $\left(a\partial_z + b\right) g_n(z)\Big|_{z\to -1^+ = 0}$, etc.

$$\Psi(x,\, z) = \psi^n(x) g_n(z) \; . \tag{8.5.4}$$

Third, if we expand Φ within the same basis as Ψ, that is

$$\Phi(x,\, z) = \phi^n(x) g_n(z) \; , \tag{8.5.5}$$

and Taylor expand the exponential in equation (8.5.3) about $\mathcal{A}_z = 0$, then

$$\psi^n(x) g_n(z) = \phi^n(x) g_n(z) - ieB^i(x)\phi^m f_i(z) g_m(z) - ieC(x)\phi^m(x) Z(z) g_m(z) + \mathcal{O}\left(\epsilon^3\right) \; . \tag{8.5.6}$$

Here ϵ represents the total order in lower-dimensional fields for an expression. If we suppose that the total functional dependence on the right hand side may be expanded in the g_n basis using the resolution of the identity

$$\delta(z - s) = g_n(z) g_n(s) \; , \tag{8.5.7}$$

then we have

$$\begin{aligned} \psi^n(x) g_n(z) = \phi^n(x) g_n(z) &- ieB^i(x)\phi^m(x) g_n(z) \int_{-1}^{1} f_i(s) g_m(s) g_n(s) ds \\ &- ieC(x)\phi^m(x) g_n(z) \int_{-1}^{1} Z(z) g_m(s) g_n(s) ds + \mathcal{O}\left(\epsilon^3\right) \; . \end{aligned} \tag{8.5.8}$$

[90] We momentarily ignore the x dependent constant (in z) function.

We define the notation

$$I_{ij\ldots,mn\ldots} = \int_{-1}^{1} \left(f_i(z) f_j(z) \ldots\right) \left(g_m(z) g_n(z) \ldots\right) dz \ , \tag{8.5.9}$$

$$I_{ij'\ldots,m'n\ldots} = \int_{-1}^{1} \left(f_i(z) \left(\partial_z f_j(z)\right) \ldots\right) \left(\left(\partial_z g_m(z)\right) g_n(z) \ldots\right) dz \ , \tag{8.5.10}$$

$$I_{iZz\ldots,mn\ldots} = \int_{-1}^{1} \left(f_i(z) Z(z) \zeta(z) \ldots\right) \left(g_m(z) g_n(z) \ldots\right) dz \ . \tag{8.5.11}$$

Here, $I_{ij\ldots,mn\ldots}$ represents an overlap integral of arbitrarily many (i and j) elements of the $\{f_i(z)\}$ basis, arbitrarily many (z) $\zeta(z)$ and (Z) $Z(z)$, as well as (m and n) elements of $\{g_m(z)\}$. If any given index is followed by a prime, n' for example, that function appears under a transverse derivative.

Integrating equation (8.5.8) against an arbitrary g_n we find

$$\psi^n = \phi^n - ieI_{i,mn} B^i \phi^m - ieI_{Z,mn} C \phi^m + \mathcal{O}\left(\epsilon^3\right) \ . \tag{8.5.12}$$

This can be perturbatively redefined order by order to give

$$\phi^n = \psi^n + ieI_{i,mn} B^i \psi^m + ieI_{Z,mn} C \psi^m + \mathcal{O}\left(\epsilon^3\right) \ . \tag{8.5.13}$$

Therefore the corrections to our Sturm–Liouville basis due to our nonlinear boundary conditions can be understood as a perturbative series away from the g_n in orders of the transverse components of $\mathcal{A}_z$.

However, the lower-dimensional systems are the same between these two procedures. That is the dimensional reduction square commutes (equation (6.1.1)) when we expand Φ in our $\{g_n\}$ basis then make the redefinition in the lower dimension (equation (8.5.13)) or if we make the redefinition in the higher dimension (equation (8.5.8)) then expand Ψ in our $\{g_n\}$ basis. Therefore these two procedures define the same physical system.

8.6 Lower-Dimensional Scalar Electrodynamics, Original Variables

Expanding $\mathcal{A}_\mu$, $\mathcal{A}_z$, and Φ as in equations (8.1.14), (8.1.15), and (8.5.5) then integrate over the transverse dimension our lower-dimensional Lagrangian density $\mathcal{S} = \int \mathcal{L}_{\text{SED EFT}}\, d^dx$ becomes ($\mathcal{L}_{\text{SED EFT}} =$)

$$\begin{aligned}&-\frac{1}{4}\left(\partial_\mu {A^i}_\nu - \partial_\nu {A^i}_\mu\right)^2 - \frac{1}{2}{\omega_i}^2\left({A^i}_\mu - \partial_\mu B^i\right)^2 + Z\partial^\mu C\left({A^0}_\mu - \partial_\mu B^0\right) - |\partial_\mu \phi^n| - {\sigma_n}^2 |\phi^n|^2 \\ &-ieI_{i,mn}{A^i}_\mu\left(\overline{\phi}^m\partial^\mu\phi^n - \phi^m\partial^\mu\overline{\phi}^n\right) - ie\left(I_{i',mn'} - I_{i',m'n}\right)B^i\overline{\phi}^m\phi^n - ie\left(I_{z,mn'} - I_{z,m'n}\right)C\overline{\phi}^m\phi^n \\ &-e^2 I_{ij,mn}{A^i}_\mu {A^j}^\mu\overline{\phi}^m\phi^n - e^2 I_{i'j',mn}B^iB^j\overline{\phi}^m\phi^n - 2e^2 I_{i'z,mn}B^iC\overline{\phi}^m\phi^n - e^2 I_{zz,mn}C^2\overline{\phi}^m\phi^n \;.\end{aligned} \tag{8.6.1}$$

If we focus on the gauge charge of the lightest scalar field ϕ^0 with respect to the massless vector ${A^0}_\mu$, then we may define an effective coupling

$$e_{\text{eff}} = eI_{00,0} \;. \tag{8.6.2}$$

Usually gauge symmetry restricts the coupling of the theory to all orders given only the cubic coupling [38]. That is the quartic coupling must be the square of the cubic coupling, or $e_4 = (e_{\text{eff}})^2$. In this case, however as with section 6.5 the effective coupling at quartic order is

$$e_4 = e^2 I_{00,00} \;. \tag{8.6.3}$$

Since $I_{00,00} \neq (I_{00,0})^2$, this relationship is broken.

This unusual behavior is not due to the Stuckelbergs or the spontaneously created gauge fixing term in the vector's quadratic theory. Instead, it is due to a nonlinear realization of the gauge symmetry involving all of the heavy scalar fields. Expanding equation (8.4.1) with Λ in the $\{f_i\}$ basis we find, to first order in gauge parameter

$$\phi^{m\prime} = \phi^m + ieI_{i,nm}\lambda^i\phi^n \;. \tag{8.6.4}$$

If we gauge transform equation (8.6.1) and collect only terms containing one each of ${A^0}_\mu$, λ^0, ϕ^0, and $\overline{\phi}^0$, we find

$$\begin{aligned}&-ieI_{0,0n}{A^0}_\mu\left(\overline{\phi}^0\partial^\mu\left(ieI_{0,0n}\lambda^0\phi^0\right) + \left(-ieI_{0,0n}\lambda^0\overline{\phi}\right)\partial^\mu\phi^0 - \phi^0\partial^\mu\left(-ieI_{0,0n}\overline{\phi}^0\right)\right) \\ &\qquad -2e^2 I_{00,00}{A^0}_\mu\left(\partial^\mu\lambda^0\right)\overline{\phi}^0\phi^0 \;.\end{aligned} \tag{8.6.5}$$

Expanding this we find that these terms cancel. That is

$$I_{0,0n}I_{0,0n} = \int_{-1}^{1} f_0 g_0(z)g_n(z)dz \int_{-1}^{1} f_0(s)g_0(s)g_n(s)ds = \int_{-1}^{1} f_0(z)^2 g_0(z)^2 dz = I_{00,00} \;. \tag{8.6.6}$$

Why do the Stueckelberg appear in the interactions at all? Usually the mass terms are gauge invariant by themselves, since ϕ^n and $\overline{\phi}^n$ transform, at infinitesimal order in the gauge parameter, in opposite directions. However, if we consider the transformation of the mass terms here

$$-\sigma_n{}^2\left(\left(-ieI_{i,nm}\lambda^i\overline{\phi}^m\right)\phi^n+\overline{\phi}^n\left(ieI_{i,nm}\lambda^i\phi^m\right)\right)\ . \tag{8.6.7}$$

These two terms generically do not cancel. However, if we combine this with the transformation of the Stueckelberg at cubic order

$$-ie\left(I_{i',mn'}-I_{i',m'n}\right)\lambda^i\overline{\phi}^m\phi^n\ , \tag{8.6.8}$$

these terms cancel, since

$$\begin{aligned}-I_{i,nm}\sigma_n{}^2+I_{i,nm}\sigma_m{}^2&=\int_{-1}^{1}f_i(s)g_n(s)g_m(s)ds\int_{-1}^{1}g_n(z)\Delta g_n(z)dz-(n\leftrightarrow m)\\=\int_{-1}^{1}f_i(z)g_m(z)\Delta g_n(z)dz-(n\leftrightarrow m)&=\int_{-1}^{1}f_i'(z)g_m'(z)g_n(z)dz-(n\leftrightarrow m)=I_{i',m'n}-I_{i',mn'}\ .\end{aligned} \tag{8.6.9}$$

Notably, the antisymmetry of these terms implies that, if we consider a system of only the lightest scalar and massless vector, the scalar does not appear at cubic order.

In summation, the terms which include the gauge fields ${A^0}_\mu$ excepting the quadratic gauge fixing term, are invariant among themselves, taking the entire tower of gauge fields into account. The presence of the Stueckelberg field (B^0) is necessary because the off-diagonal transformation of mass terms corresponding to the scalars.

It is also worth noting that C has been promoted from a Lagrange multiplier to an auxiliary field, since there now are terms at quartic order involving C^2. C's equation of motion now reads,

$$Z\Box B^0-Z\partial^\mu {A^0}_\mu-ie\left(I_{z,mn'}-I_{z,nm'}\right)\overline{\phi}^m\phi^n-2e^2I_{i'z,mn}B^i\overline{\phi}^m\phi^n-2e^2I_{zz,mn}C\overline{\phi}^m\phi^n=0\ . \tag{8.6.10}$$

8.7 The Gauge Inert Basis and the Interaction Basis

Substituting the redefinition given in equation (8.5.3) into the higher-dimensional action we find the higher-dimensional bulk Lagrangian density ($\mathcal{S}_{\text{SED bulk}}=\int\int_{-1}^{1}\mathcal{L}_{\text{SED bulk}}dzd^dx$) becomes

$$\mathcal{L}_{\text{SED bulk}}=-\frac{1}{4}\left(\partial_\mu\mathcal{A}_\nu-\partial_\nu\mathcal{A}_\mu\right)^2-\frac{1}{2}\left(\partial_\mu\mathcal{A}_z-\partial_z\mathcal{A}_\mu\right)^2-\left|\left(\partial_\mu-ie\mathcal{A}_\mu+ie\partial_\mu{\partial_z}^{-1}\mathcal{A}_z\right)\Psi\right|^2-|\partial_z\Psi|^2\ . \tag{8.7.1}$$

Expanding $\mathcal{A}_\mu$, $\mathcal{A}_z$, and Ψ as in equations (8.1.14), (8.1.15), and (8.5.4) then integrating over the transverse dimension our lower-dimensional Lagrangian density $\mathcal{S} = \int \mathcal{L}_{\text{SED EFT}}\, d^dx$ becomes ($\mathcal{L}_{\text{SED EFT}} =$)

$$
\begin{gathered}
-\frac{1}{4}\left(\partial_\mu {A^i}_\nu - \partial_\nu {A^i}_\mu\right)^2 - \frac{1}{2}{\omega_i}^2\left({A^i}_\mu - \partial_\mu B^i\right)^2 + Z\partial^\mu C\left({A^0}_\mu - \partial_\mu B^0\right) - \left|\partial_\mu \psi^n\right| - {\sigma_n}^2\left|\psi^n\right|^2 \\
-ieI_{i,mn}\left({A^i}_\mu - \left(\partial_\mu B^i\right)\right)\left(\overline{\psi}^m \partial^\mu \psi^n - \psi^m \partial^\mu \overline{\psi}^n\right) + ieI_{Z,mn}\left(\partial_\mu C\right)\left(\overline{\psi}^m \partial^\mu \psi^n - \psi^m \partial^\mu \overline{\psi}^n\right) \\
-e^2 I_{ij,mn}\left({A^i}_\mu - \left(\partial_\mu B^i\right)\right)\left(A^{j\mu} - \left(\partial^\mu B^j\right)\right)\overline{\psi}^m \psi^n \\
-2e^2 I_{iZ,mn}\left({A^i}_\mu - \left(\partial_\mu B^i\right)\right)\left(\partial^\mu C\right)\overline{\psi}^m \psi^n - e^2 I_{ZZ,mn}\left(\partial_\mu C\right)\left(\partial^\mu C\right)\overline{\phi}^m \phi^n \ .
\end{gathered}
\tag{8.7.2}
$$

The gauge invariance of this action is obvious, since, with the exception of ${A^i}_\mu$'s kinetic term, which are inherently gauge invariant, every appearance of every vector is gauge inert, i.e. ${A^i}_\mu - \partial_\mu B^i$.

An important relationship between the action presented here (8.7.2) and the action presented in (8.6.1) is that they transform into each other given our perturbative field redefinition (equation (8.5.13) in the lower-dimensional effective field theory. For example, if we inspect the perturbative redefintion of the kinetic terms we find

$$
\begin{aligned}
\left|\partial_\mu \phi^n\right|^2 &= \left(\partial_\mu \overline{\psi}^n - ieI_{i,mn}\left(\partial_\mu B^i \overline{\psi}^m\right) - ieI_{Z,mn}\left(\partial_\mu C \overline{\psi}^m\right) + \mathcal{O}\left(\epsilon^3\right)\right) \\
&\quad \times \left(\partial^\mu \psi^n + ieI_{i,pn}\left(\partial^\mu B^i \psi^p\right) + ieI_{Z,pn}\left(\partial^\mu C \psi^p\right) + \mathcal{O}\left(\epsilon^3\right)\right) \\
&= \left|\partial_\mu \psi^n\right|^2 - ie\left(I_{i,mn}\left(\partial_\mu B^i\right) + I_{Z,mn}\left(\partial_\mu C\right)\right)\left(\overline{\psi}^m \partial^\mu \psi^n - \psi^m \partial^\mu \overline{\psi}^n\right) + \mathcal{O}\left(\epsilon^4\right) \ .
\end{aligned}
\tag{8.7.3}
$$

Here we have cancelled terms where derivatives do not act on either B^i or C.

One final question we may ask about field redefinitions is: can any field redefinition correct the unusual quartic coefficient. The answer is, first, no perturbative field redefinition can. This can be argued as a fact about gauge invariance which we will demonstrate in the context of the theory in the original variables after integrating the heavy fields out. However, if we expand the accepted space of field redefinitions, then we may define the most general redefinition at linear order in fields, defined by some operator K_{nm}

$$
\varphi^n = K_{nm}\phi^m \ , \tag{8.7.4}
$$

and consider its gauge transformation

$$
\varphi^{n\prime} = K_{nm}\phi^{m\prime} = K_{nm}\phi^m + ieK_{nm}I_{i,mp}\lambda^i\phi^p = \varphi^n + ieK_{nm}I_{i,mp}K^{pq}\lambda^i\varphi^q \ . \tag{8.7.5}
$$

Here we have invoked K_{pq}'s inverse, K^{pq}, defined so that $K^{np}K_{pm} = \delta^n{}_m$. We cannot simultaneously diagonalize $I_{i,mn}$ for all i, however, if we suppose that K_{mn} diagonalizes $I_{0,mn}$, that is

$$K_{mn}I_{i,np}K^{pq} = J\delta^{i0}\delta_m{}^q + J_{i,m}{}^q \ , \tag{8.7.6}$$

where J is some constant and $J_{0,m}{}^q = 0$, then we have

$$\varphi^{n\prime} = \varphi^n + ieJ\lambda^0\varphi^n + \mathcal{O}\left(\lambda^{\bar{i}}\right) + \mathcal{O}\left(\lambda^2\right) \ . \tag{8.7.7}$$

If we (inconsistently) truncate all heavy vectors and insert this field redefinition our Lagrangian density becomes

$$\begin{gathered} -\frac{1}{4}\left(\partial_\mu A_\nu - \partial_\nu A_\mu\right)^2 + Z\partial^\mu C\left(A_\mu - \partial_\mu B\right) \\ -\left|\left(\partial_\mu - ieJA_\mu\right)\varphi^n\right| \\ -\sigma_n{}^2\left|K^{nm}\varphi^m + ieL_i{}^{nm}B^i\varphi^m + ieL_z{}^{nm}C\varphi^m\right|^2 \ . \end{gathered} \tag{8.7.8}$$

That is, our covariant derivatives become diagonalized, but our mass terms become undiagonalized.

8.8 The Effective Field Theory of the Light Fields

Selecting the original basis, that is ϕ^n as in the Lagrangian density given in equation (8.6.1), we know that the theory is best approximated by $A^0{}_\mu$, B^0, C, and ϕ^0 with all of the massive vectors ($A^{\bar{i}}{}_\mu$ and $\overline{\phi}^{\overline{n}}$ with $\bar{i}$, $\overline{n} > 0$) and heavy scalars put onshell. We start by making the gauge choice that $B^{\bar{i}} = 0$. In this gauge the massive vectors' equations of motion read

$$\left(\Box - \omega_{\bar{i}}{}^2\right)A^{\bar{i}}{}_\mu - \partial_\mu\partial^\nu A^{\bar{i}}{}_\nu = ieI_{\bar{i},mn}\left(\overline{\phi}^m\partial_\mu\phi^n - \phi^m\partial_\mu\overline{\phi}^n\right) + 2e^2I_{\bar{i}j,mn}A^j{}_\mu\overline{\phi}^m\phi^n \ . \tag{8.8.1}$$

Also, our heavy scalars' equations of motion read

$$\begin{aligned} &\left(\Box - \sigma_{\overline{n}}{}^2\right)\phi^{\overline{n}} = \\ &\quad ieI_{i,\overline{n}m}\left(\left(\partial^\mu A^i{}_\mu\right)\phi^m + 2A^i{}_\mu\partial^\mu\phi^m\right) + ie\left(I_{0',\overline{n}m'} - I_{0',\overline{n}'m}\right)B^0\phi^m + ie\left(I_{z,\overline{n}m'} - I_{z,\overline{n}'m}\right)C\phi^m \\ &\quad + e^2I_{ij,\overline{n}m}A^i{}_\mu A^{j\mu}\phi^m + 2e^2I_{0'z,\overline{n}m}B^0C\phi^m + e^2I_{0'0',\overline{n}m}\left(B^0\right)^2\phi^m \ . \end{aligned} \tag{8.8.2}$$

Therefore to leading order in light fields (ϵ^n), and derivatives (∂^n), we have

$$A^{\bar{i}}{}_\mu = -ie\frac{I_{\bar{i},00}}{\omega_{\bar{i}}{}^2}\left(\overline{\phi}\partial_\mu\phi - \phi\partial_\mu\overline{\phi}\right) + \mathcal{O}\left(\epsilon^3\right) + \mathcal{O}\left(\partial^3\right) \ , \tag{8.8.3}$$

$$
\begin{aligned}
\phi^{\overline{n}} = &- ie\frac{I_{0,\overline{n}0}}{\sigma_{\overline{n}}{}^2}\left(\left(\partial^\mu A_\mu\right)\phi + 2A_\mu\partial^\mu\phi\right) - ie\frac{I_{0',\overline{n}'0} - I_{0',0'\overline{n}}}{\sigma_{\overline{n}}{}^2}B\phi - ie\frac{I_{z,\overline{n}'0} - I_{z,0'\overline{n}}}{\sigma_{\overline{n}}{}^2}C\phi \\
&- ie\frac{I_{0',\overline{n}0'} - I_{0',\overline{n}'0}}{\sigma_{\overline{n}}{}^4}\left(\Box B\phi\right) - ie\frac{I_{z,\overline{n}'0} - I_{z,0'\overline{n}}}{\sigma_{\overline{n}}{}^4}\left(\Box C\phi\right) + \mathcal{O}\left(\epsilon^3\right) + \mathcal{O}\left(\partial^3\right) \ .
\end{aligned}
\tag{8.8.4}
$$

Substituting these into our Lagrangian density (equation (8.6.1)) we find ($\mathcal{L}_{\text{SED EFT}} =$)

$$
\begin{gathered}
-\frac{1}{4}\left(\partial_\mu A_\nu - \partial_\nu A_\mu\right)^2 + Z\partial^\mu C\left(A_\mu - \partial_\mu B\right) - \left|\partial_\mu\phi\right|^2 - \sigma^2\left|\phi\right|^2 \\
-ieI_{0,00}A_\mu\left(\overline{\phi}\partial^\mu\phi - \phi\partial^\mu\overline{\phi}\right) - e^2 I_{00,00}A_\mu A^\mu\left|\phi\right|^2 \\
-e^2 I_{0'0',00}B^2\left|\phi\right|^2 - 2e^2 I_{0'z,00}BC\left|\phi\right|^2 - e^2 I_{zz,00}C^2\left|\phi\right|^2 \\
+\frac{1}{2}e^2\frac{I_{\overline{i},00}I_{\overline{i},00}}{\omega_{\overline{i}}{}^2}\left(\overline{\phi}\partial_\mu\phi - \phi\partial_\mu\overline{\phi}\right)^2 + e^2\frac{I_{0,0\overline{n}}I_{0,0\overline{n}}}{\sigma_{\overline{n}}{}^2}\left|\left(\partial^\mu A_\mu\right)\phi + 2A_\mu\partial^\mu\phi\right|^2 \\
+e^2\frac{\left(I_{0',\overline{n}'0} - I_{0',0'\overline{n}}\right)^2}{\sigma_{\overline{n}}{}^2}B^2\left|\phi\right|^2 + e^2\frac{\left(I_{z,\overline{n}'0} - I_{z,0'\overline{n}}\right)^2}{\sigma_{\overline{n}}{}^2}C^2\left|\phi\right|^2 \\
+\frac{1}{2}e^2\frac{I_{0,0\overline{n}}\left(I_{0',\overline{n}'0} - I_{0',0'\overline{n}}\right)}{\sigma_{\overline{n}}{}^2}\left(\left(\left(\partial^\mu A_\mu\right)\overline{\phi} + 2A_\mu\partial^\mu\overline{\phi}\right)B\phi + B\overline{\phi}\left(\left(\partial^\mu A_\mu\right)\phi + 2A_\mu\partial^\mu\phi\right)\right) \\
+\frac{1}{2}e^2\frac{I_{0,0\overline{n}}\left(I_{z,\overline{n}'0} - I_{z,0'\overline{n}}\right)}{\sigma_{\overline{n}}{}^2}\left(\left(\left(\partial^\mu A_\mu\right)\overline{\phi} + 2A_\mu\partial^\mu\overline{\phi}\right)C\phi + C\overline{\phi}\left(\left(\partial^\mu A_\mu\right)\phi + 2A_\mu\partial^\mu\phi\right)\right) \\
+2e^2\frac{\left(I_{0',\overline{n}'0} - I_{0',0'\overline{n}}\right)\left(I_{z,\overline{n}'0} - I_{z,0'\overline{n}}\right)}{\sigma_{\overline{n}}{}^2}BC\left|\phi\right|^2 + \frac{1}{2}e^2\frac{\left(I_{0',\overline{n}'0} - I_{0',0'\overline{n}}\right)^2}{\sigma_{\overline{n}}{}^4}\left(B\overline{\phi}\left(\Box B\phi\right) + \left(\Box B\overline{\phi}\right)B\phi\right) \\
+\frac{1}{2}e^2\frac{\left(I_{0',\overline{n}'0} - I_{0',0'\overline{n}}\right)\left(I_{z,\overline{n}'0} - I_{z,0'\overline{n}}\right)^2}{\sigma_{\overline{n}}{}^4}\left(B\overline{\phi}\left(\Box C\phi\right) + C\overline{\phi}\left(\Box B\phi\right) + \left(\Box B\overline{\phi}\right)C\phi + \left(\Box C\overline{\phi}\right)B\phi\right) \\
+\frac{1}{2}e^2\frac{\left(I_{z,\overline{n}'0} - I_{z,0'\overline{n}}\right)^2}{\sigma_{\overline{n}}{}^4}\left(C\overline{\phi}\left(\Box C\phi\right) + \left(\Box C\overline{\phi}\right)C\phi\right) + \mathcal{O}\left(\epsilon^5\right) + \mathcal{O}\left(\partial^3\right) \ .
\end{gathered}
\tag{8.8.5}
$$

Here, the gauge transformation of our lightest scalar field is also changed by setting $\phi^{\overline{n}}$ onshell. This is since

$$
\begin{aligned}
\phi' &= \phi + ieI_{0,00}\lambda\phi + ieI_{0,0\overline{n}}\lambda\phi^{\overline{n}} + \mathcal{O}\left(\lambda^2\right) \\
&= \phi + ieI_{0,00}\lambda\phi + e^2\frac{I_{0,0\overline{n}}{}^2}{\sigma_{\overline{n}}{}^2}\lambda\left(\left(\partial^\mu A_\mu\right)\phi + 2A_\mu\partial^\mu\phi\right) \\
&\quad + e^2\frac{I_{0,0\overline{n}}\left(I_{0',\overline{n}'0} - I_{0',0'\overline{n}}\right)}{\sigma_{\overline{n}}{}^2}\lambda B\phi + e^2\frac{I_{0,0\overline{n}}\left(I_{z,\overline{n}'0} - I_{z,0'\overline{n}}\right)}{\sigma_{\overline{n}}{}^2}\lambda C\phi + \mathcal{O}\left(\lambda^2\right) + \mathcal{O}\left(\partial^2\right) + \mathcal{O}\left(\epsilon^3\right) \ .
\end{aligned}
\tag{8.8.6}
$$

This causes ϕ's mass term in equation (8.8.5) to produce a term which cancels against the transformation of quartic order terms. From the mass term, if we collect only terms involving λ, A_μ, $\overline{\phi}$, ϕ, and a single derivative ∂_μ, we have

$$
-\sigma^2\overline{\phi}\left(e^2\frac{I_{0,0\overline{n}}{}^2}{\sigma_{\overline{n}}{}^2}\lambda\left(\left(\partial^\mu A_\mu\right)\phi + 2A_\mu\partial^\mu\phi\right)\right) - \sigma^2\left(\frac{I_{0,0\overline{n}}{}^2}{\sigma_{\overline{n}}{}^2}\lambda\left(\left(\partial^\mu A_\mu\right)\overline{\phi} + 2A_\mu\partial^\mu\overline{\phi}\right)\right)\phi \ . \tag{8.8.7}
$$

We note that this is identical in structure to the terms mixing A_μ, B, $\overline{\phi}$, ϕ, and a single derivative ∂_μ in our action (equation (8.8.5)). If we add total derivatives to write these terms so that no derivatives act on A_μ

these terms simplify to

$$2e^2\sigma^2 \frac{{I_{0,0\overline{n}}}^2}{{\sigma_{\overline{n}}}^2} A_\mu \partial^\mu \lambda \left|\phi\right|^2 \ . \tag{8.8.8}$$

That is, the higher-order transformation of the mass term, as well as the induced terms involving A_μ, B, $\overline{\phi}$, ϕ, and a single derivative ∂_μ, all cancel given

$$I_{00,00} - {I_{0,00}}^2 - \frac{I_{0,0\overline{n}} \left(I_{0',\overline{n}'0} - I_{0',0'\overline{n}}\right)}{{\sigma_{\overline{n}}}^2} - 2\sigma^2 \frac{{I_{0,0\overline{n}}}^2}{{\sigma_{\overline{n}}}^2} = 0 \ . \tag{8.8.9}$$

Of vital importance, there exist cases when $\frac{I_{0,0\overline{n}}\left(I_{0',\overline{n}'0} - I_{0',0'\overline{n}}\right)}{{\sigma_{\overline{n}}}^2} = 0$ and when $2\sigma^2 \frac{{I_{0,0\overline{n}}}^2}{{\sigma_{\overline{n}}}^2} = 0$ (especially when $\sigma^2 = 0$), but not simultaneously. Therefore a massive field could potentially have a perturbative redefinition to a field with a linearly realized gauge symmetry, but a massless field (such as a self-interacting gauge field) never can.

9 Covert Symmetry Breaking in Yang–Mills

This section will present a preliminary analysis of covert symmetry breaking in the context of a self-interacting gauge-theory, Yang–Mills [130].

9.1 Yang–Mills Boundary Terms and Conditions

At linear order, our study of the behavior of Yang–Mills is essentially a restatement of our study of Maxwell. First, we invoke our bulk higher-dimensional theory

$$\mathcal{S}_{\text{bulk}} = \int_{\mathbb{R}^{1,d-1}} \int_{-1}^{1} -\frac{1}{4}\,\text{tr}\left\{\left(\partial_M \mathcal{A}_N - \partial_N \mathcal{A}_M + g\left[\mathcal{A}_M,\, \mathcal{A}_N\right]\right)^2\right\} dz d^d x\ . \tag{9.1.1}$$

Here we have used the same coordinate split as in section 7. We of course also have the pithier expression $\mathcal{L}_{\text{bulk}} = -\frac{1}{4}\,\text{tr}\left\{{\mathcal{F}_{MN}}^2\right\}$ for our gluon field strength tensor $\mathcal{F}_{MN} = \partial_M \mathcal{A}_N - \partial_N \mathcal{A}_M + g\left[\mathcal{A}_M,\, \mathcal{A}_M\right]$.

Next we impose the following boundary conditions on the world-volume components of our field

$$\left(\partial_z - \frac{b}{a}\right) \mathcal{A}_\mu(x,\, z)\big|_{z=-1} = 0\ , \qquad \left(\partial_z - \frac{b}{a+2b}\right) \mathcal{A}_\mu(x,\, z)\big|_{z=1} = 0\ . \tag{9.1.2}$$

Next we vary our action and collect boundary terms at $z = \pm 1$.

$$\delta\mathcal{S}_{\text{bulk}} = \int_{\mathbb{R}^{1,d-1}} \left\{ \int_{-1}^{1} \text{tr}\left\{\delta\mathcal{A}^M\left(\partial^N \mathcal{F}_{NM} + g\left[\mathcal{A}^N,\, \mathcal{F}_{NM}\right]\right)\right\} dz + \text{tr}\left\{\delta\mathcal{A}^\mu \mathcal{F}_{\mu z}\right\}\Big|_{z=-1}^{z=1} \right\} d^d x\ . \tag{9.1.3}$$

Requiring this to vanish we learn the bulk equations of motion

$$\partial^N \mathcal{F}_{NM} + g\left[\mathcal{A}^N,\, \mathcal{F}_{NM}\right] = 0\ . \tag{9.1.4}$$

However, the boundary term which usually vanishes, does not. Normally it vanishes either due to the world-volume components of our field's variation obeying a special Dirichlet condition ($\delta\mathcal{A}^\mu\big|_{z=\pm 1} = 0$) or our world-volume components of our field obeying a special Neumann condition ($\partial_z \mathcal{A}_\mu\big|_{z=\pm 1} = 0$) and the transverse component of field obeying a special Dirichlet condition $\mathcal{A}_z\big|_{z=\pm 1} = 0$, or some mixture.

We wish to study Robin conditions, however, so we augment our action $\mathcal{S} = \mathcal{S}_{\text{bulk}} + \mathcal{S}_{z=\pm 1}$.

$$\mathcal{S}_{z=+1} = +\int_{\mathbb{R}^{1,d-1}} \frac{(a+b)+b}{b}\frac{1}{2}\,\mathrm{tr}\left\{{\mathcal{F}_{\mu z}}^2\right\}\Big|_{z=+1} d^dx \ , \tag{9.1.5}$$

$$\mathcal{S}_{z=-1} = -\int_{\mathbb{R}^{1,d-1}} \frac{(a+b)-b}{b}\frac{1}{2}\,\mathrm{tr}\left\{{\mathcal{F}_{\mu z}}^2\right\}\Big|_{z=-1} d^dx \ , \tag{9.1.6}$$

Given these terms the variation (at quadratic order) at the upper boundary, becomes

$$\int_{\mathbb{R}^{1,d-1}} \mathrm{tr}\left\{-\left(\left(\frac{a+2b}{b}\partial^z - 1\right)\delta\mathcal{A}^\mu\right)\mathcal{F}_{\mu z} - \delta\mathcal{A}^z\partial^\mu\mathcal{F}_{\mu z}\right\}\Bigg|_{z=1} d^dx \ . \tag{9.1.7}$$

The terms including $\delta\mathcal{A}_\mu$ vanish given that our variation takes us from a field configuration which obeys our boundary condition, to a field configuration which obeys our boundary condition. That is

$$\begin{gathered}\left(\partial_z - \frac{b}{a+2b}\right)\mathcal{A}'_\mu(x,\,z)\big|_{z=1} = 0 \qquad \text{and} \qquad \left(\partial_z - \frac{b}{a+2b}\right)\mathcal{A}_\mu(x,\,z)\big|_{z=1} = 0 \\ \Rightarrow \qquad \left(\partial_z - \frac{b}{a+2b}\right)\delta\mathcal{A}_\mu(x,\,z)\big|_{z=1} = 0 \ .\end{gathered} \tag{9.1.8}$$

Here we have used $\mathcal{A}'_\mu(x,\,z) = \mathcal{A}_\mu(x,\,z) + \delta\mathcal{A}_\mu(x,\,z)$. Furthermore, the boundary terms which include $\delta\mathcal{A}_z$ vanish given the (linear) equation of motion derived from varying $\delta\mathcal{A}_z$ in the bulk. Since the variation of $\delta\mathcal{A}_z$ is not constrained by any boundary condition on $\mathcal{A}_z$, we have our equation of motion independently at every point in our higher dimensional space, therefore the term on the boundary vanishes.

Of course, we only considered the quadratic variation/linear equations of motion. To covariantize the boundary operator that which acts on the variation of $\mathcal{A}_\mu$ we have

$$\left(\frac{a+2b}{b}\partial^z - 1\right)\delta\mathcal{A}^\mu\Bigg|_{z=+1} = 0 \qquad \rightarrow \qquad \left(\frac{a+2b}{b}D^z - 1\right)\delta\mathcal{A}^\mu\Bigg|_{z=+1} = 0 \ , \tag{9.1.9}$$

$$\left(\frac{a}{b}\partial^z - 1\right)\delta\mathcal{A}^\mu\Bigg|_{z=-1} = 0 \qquad \rightarrow \qquad \left(\frac{a}{b}D^z - 1\right)\delta\mathcal{A}^\mu\Bigg|_{z=-1} = 0 \ . \tag{9.1.10}$$

Here our expression for the covariant derivative, $D_z\delta\mathcal{A}_\mu = \partial_z\delta\mathcal{A}_\mu + g\left[\mathcal{A}_z,\,\delta\mathcal{A}_\mu\right]$, is sensible as the difference between two gauge fields is a covariant object, even if the gauge-field itself is not. This leaves us with a conundrum; we state boundary conditions on fields, not variations. However, we require a covariant nonlinear boundary condition on our variation, which would have to be derived from some non-covariant nonlinear boundary condition on our field. To find what the boundary condition on $\mathcal{A}_\mu$ is requires the recursion equation (7.5.12).

9.2 The Quadratic Lower-Dimensional Theory

First, however, allow us to determine what the physical degrees of freedom of our lower-dimensional theory are. There is a coordinate invariance for picking which Sturm–Liouville basis we desire to use, however, using a Sturm–Liouville basis which is incompatible with our boundary conditions will require careful collection of boundary terms in the lower dimension. Alternatively, using a Sturm–Liouville basis that is compatible with our boundary conditions leads to a straight-forward projection of our higher dimensional equations of motion into our lower dimension.

Given equation (9.1.2) the Sturm–Liouville basis for $\mathcal{A}_\mu$ which will diagonalize the quadratic lower dimensional action is our f basis. That is

$$\mathcal{A}_\mu(x,\, z) = A^i{}_\mu(x) f_{\omega_i}(z)\ . \tag{9.2.1}$$

Of course, we are particularly interested in our possibly massless zero mode $A^0{}_\mu$, which we shall also call A_μ. Similarly, we will diagonalize our action if we select the derivative basis for our transverse modes, as the relevant inner product (which is $(\cdot,\, \cdot)$ from equation (2.5.1)). That is

$$\mathcal{A}_z(x,\, z) = C(x)\zeta(z) + B^i(x) f'_{\omega_i}(z)\ . \tag{9.2.2}$$

Again, we may call B^0 as B. Inserting these terms into our action with boundary terms and collapsing the relevant delta-distributions (equations (2.4.2) and (2.5.13)) against the transverse integral and boundary terms we find, at quadratic order ($\mathcal{S} =$)

$$\int \operatorname{tr}\left\{-\frac{1}{4}\left(\partial_\mu A^i{}_\nu - \partial_\nu A^i{}_\mu\right)^2 - \frac{{\omega_i}^2}{2}\left(A^i{}_\mu - \partial_\mu B^i\right)^2 + \partial^\mu C\left(A_\mu - \partial_\mu B\right)\right\} d^d x + \mathcal{O}\left(\phi^3\right)\ . \tag{9.2.3}$$

Here ϕ^3 is a combination of fields at cubic order. We recognize a Stueckelberged Landau gauge-fixed action for our zero modes and Stueckelberged Proca action for our massive modes [122].

9.3 The Recursion Equation

From equation (9.1.9) we know precisely what nonlinear condition we require on our variation, and we know precisely what linear boundary condition we want on our fields. We know the gauge invariance of the action in the higher dimension and the inherited transformation of our lower-dimensional fields at linear order. To summarize these last two points, for some gauge group valued higher-dimensional function $\mathcal{Y}(x, z)$, our higher-dimensional fields transform as

$$\mathcal{A}'_M = \mathcal{Y}^{-1}\mathcal{A}_M\mathcal{Y} + \frac{1}{g}\,\mathcal{Y}^{-1}\partial_M\mathcal{Y}\ . \tag{9.3.1}$$

This is the full nonlinear transformation, however, to specify how it acts at linear order we require an infinitesimal expansion. To wit, we specify an adjoint valued higher-dimensional function $\upsilon(x,z) = \log(\mathcal{Y}(x,z))$, and state

$$\mathcal{A}'_M = \mathcal{A}_M + \frac{1}{g}\partial_M\upsilon + [\upsilon,\,\mathcal{A}_M] + \mathcal{O}\left(\upsilon^2\right)\ . \tag{9.3.2}$$

At linear order we can see that whatever boundary conditions $\mathcal{A}_\mu$ obeys, υ must also obey. If υ does not obey the boundary conditions of $\mathcal{A}_\mu$ then our gauge transformation will take us from a field configuration obeying our of boundary conditions to a field configuration not obeying those boundary conditions. Therefore we suppose that υ (to linear order) may be expanded using our f basis

$$\upsilon(x,\,z) = y^i(x)f_{\omega_i}(z)\ . \tag{9.3.3}$$

From this we find

$$A^{i\prime}{}_\mu = \left\langle f^i,\,\mathcal{A}'_\mu\right\rangle = A^i{}_\mu + \frac{1}{g}\partial_\mu y^i + \mathcal{O}(y\phi)\ , \tag{9.3.4}$$

$$B^{i\prime} = \left(\left(\partial_z f_i\right),\,\mathcal{A}'_z\right) = B^i + \frac{1}{g}y^i + \mathcal{O}(y\phi)\ , \tag{9.3.5}$$

$$C' = \left(\zeta,\,\mathcal{A}'_z\right) = C + \mathcal{O}(y\phi)\ . \tag{9.3.6}$$

This confirms that these fields indeed do transform at linear order as gauge fields (for $A^i{}_\mu$) and Stueckelberg fields (for B^i).

The next order of the transformations, however, will mix the transformation of all of our fields. The next order of our transformations is

$$A^{i\prime}{}_\mu = \ldots + \left[y^j,\,A^k{}_\mu\right]\langle f_i,\,f_jf_k\rangle + \mathcal{O}(y^2)\ , \tag{9.3.7}$$

$$B^{i\prime} = \ldots + \left[y^j,\,B^k\right](f'_i,\,f_jf'_k) + \left[y^j,\,C\right](f'_i,\,f_j\zeta) + \mathcal{O}(y^2)\ , \tag{9.3.8}$$

$$C' = \ldots + \left[y^j,\,B^k\right](\zeta,\,f_jf'_k) + \left[y^j,\,C\right](\zeta,\,f_j\zeta) + \mathcal{O}(y^2)\ . \tag{9.3.9}$$

We see even our auxiliary field (C) will transform. Despite these nonlinearities, we still find the action is gauge invariant by applying these transformations and our resolutions of the identity.

This is tantamount to declaring whatever our true boundary conditions are, they are such that equation (9.1.9) is obeyed. Given these conditions, we still have the coordinate invariance of choosing whichever Sturm–Liouville basis to expand $\mathcal{A}_M$ we desire. We choose our $\{f_i\}$ basis, despite our fields not obeying the boundary condition that our fs obey, and the cost of this is that our lower-dimensional action has a nonlinear mixing of our fields.

To restate the last point colloquially, we may suppose our fields as currently given by the linear order, since any Sturm–Liouville basis is a basis for all $L_2(\mathcal{D})$ functions, are the fields to arbitrary order. This is the same as studying standard Kaluza–Klein reduction in terms of the undiagonalized higher-dimensional fields (equation (7.4.1)).

Alternately we may declare that equations (9.3.4), (9.3.5), and (9.3.6), are exact[91] and attempt to find a nonlinear field redefintion which solves the recursion equation. That is we suppose

$$\mathcal{A}_\mu(x,z) = A^i{}_\mu(x) f_i(z) + \mathcal{O}(\phi^2) \ , \tag{9.3.10}$$

as well as some similar expansion for $\mathcal{A}_z$, and attempt to solve the recursion equation. This, it eventuates, is impossible. One might suspect that the presence of our lower-dimensional Stueckelberg field would make finding such a transformation trivial. This is because almost all terms required at some order ($y\phi^{n-1}$) may be cancelled by the transformation of the Stueckelberg at the same order, heuristically $-B\phi^{n-1}$. The one field that this cannot work for, however, is the Stueckelberg itself. It is

$$\mathcal{A}'_z = [y^i, B^j] f_i f'_j + [y^i, C] f_i \zeta \ . \tag{9.3.11}$$

While the term including C may be cancelled easily by $[B^i, C] f_{\omega_i} \zeta$, there is no possible redefinition which can cancel the $[y^i, B^j]$ term, instead, we must consider a redefinition of both $\mathcal{A}_z$ and $\mathcal{Y}$. We will spare the reader the most general $\mathcal{A}$ redefinition which obeys the necessary diffeomorphism covariant [126] structure[92] and instead consider

$$\mathcal{A}_\mu = A^i{}_\mu f_i + [B^i,\, A^j{}_\mu] g_{ij} + [B^i,\, \partial_\mu B^j] h_{ij} + \mathcal{O}(\phi^3) \ , \tag{9.3.12}$$

$$\mathcal{A}_z = C\zeta + B^i f'_i + [B^i,\, C] k_i + [B^i,\, B^j] l_{ij} + \mathcal{O}(\phi^3) \ , \tag{9.3.13}$$

$$\mathcal{Y} = y_i h_i + [y^i, B^j] p_{ij} + \mathcal{O}(\phi^2) \ . \tag{9.3.14}$$

[91] We will handle the desired nonlinear transformation of the gauge group shortly.
[92] Although considering the full redefinition is easy using computer algebra.

Alternatively we say

$$\mathcal{A}^1{}_\mu = A^i{}_\mu f_i \ , \qquad \mathcal{A}^2{}_\mu = [B^i,\, A^j{}_\mu]g_{ij} + [B^i,\, \partial_\mu B^j]h_{ij} \ , \tag{9.3.15}$$

$$\mathcal{A}^1{}_z = C\zeta + B^i f_i' \ , \qquad \mathcal{A}^2{}_z = [B^i,\, C]k_i + [B^i,\, B^j]l_{ij} \ , \tag{9.3.16}$$

$$\mathcal{Y}^0 = y^i f_i \ , \qquad \mathcal{Y}^1 = [y^i, B^j]p_{ij} \ . \tag{9.3.17}$$

Here $g_{ij}, \dots, p_{ij}$ are as yet unknown transverse functions. The first order of our recursion equation is already solved, and the following order is

$$\hat{\delta}^2\mathcal{A}_M = \frac{1}{g}\partial_M\mathcal{Y}^1 + [\mathcal{Y}^0,\, \mathcal{A}^1{}_M] = \delta^2\mathcal{A}_M = \delta^0\mathcal{A}^2{}_M + \delta^1\mathcal{A}^1{}_M = \delta^2\mathcal{A}_M \ . \tag{9.3.18}$$

Given equations (9.3.4), (9.3.5), and (9.3.6) are exact, then $\delta^1\Phi = 0$, that is, the fields only transform non-homogeneously at linear order. We find

$$\begin{aligned} &\frac{1}{g}\partial_\mu\left([y^i,\, B^j]p_{ij}\right) + [y^i f_i,\, A^j{}_\mu f_j] \\ &= \frac{1}{g}[y^i,\, A^j{}_\mu]g_{ij} + \frac{1}{g}[y^i,\, \partial_\mu B^j]h_{ij} + \frac{1}{g}[B^i,\, \partial_\mu y^j](g_{ij} + h_{ij}) \ , \end{aligned} \tag{9.3.19}$$

$$\begin{aligned} &\frac{1}{g}\partial_z\left([y^i,\, B^j]p_{ij}\right) + [y^i f_i, C\zeta + B^j f_j'] \\ &= \frac{1}{g}[y^i,\, C]k_i + \frac{1}{g}[y^i,\, B^j]\left(l_{ij} - l_{ji}\right) \ . \end{aligned} \tag{9.3.20}$$

Matching unique terms in $\mathcal{A}_\mu$'s recursion equation we learn

$$g[y^i,\, A^j{}_\mu]f_i f_j = [y^i,\, A^j{}_\mu]g_{ij} \ . \tag{9.3.21}$$

A solution to this equation is

$$g_{ij} = g\, f_i f_j \ . \tag{9.3.22}$$

Similarly we learn

$$[\partial_\mu y^i,\, B^j]p_{ij} = [B^i,\, \partial_\mu y^j]\left(g_{ij} + h_{ij}\right) \ , \qquad [y_i,\, \partial_\mu B^j]p_{ij} = [y^i,\, \partial_\mu B^i]h_{ij} \ . \tag{9.3.23}$$

A solution is

$$h_{ij} = p_{ij} = -\frac{1}{2}g_{ij} = -\frac{1}{2}g\, f_i f_j \ . \tag{9.3.24}$$

Substituting this into $\mathcal{A}_z$'s recursion equation we learn

$$k_i = g\, f_i \zeta \ , \qquad l_{ij} = \frac{1}{2} g\, f_i f_j' \ . \tag{9.3.25}$$

Restating our redefinition given this we have

$$\begin{aligned}
\mathcal{A}^2{}_\mu &= g\,[B^i,\, A^j{}_\mu] f_i f_j - \frac{1}{2} g\,[B^i,\, \partial_\mu B^j] f_i f_j \ , && (9.3.26)\\
\mathcal{A}^2{}_z &= g\,[B^i,\, C] f_i \zeta - \frac{1}{2} g\,[B^i,\, B^j] f_i f_j' \ , && (9.3.27)\\
\mathcal{Y}^1 &= -\frac{1}{2} g\,[y^i, B^j] f_i f_j \ . && (9.3.28)
\end{aligned}$$

We can then repeat this process at arbitrary order in our recursion equation. However, we can identify that $\mathcal{Y}$ is obeying the Baker–Campbell–Hausdorff identity at this order [8], and hypothesize that it will obey this identity to arbitrary order. That is, if we define

$$\mathcal{N} = \exp\left(g\, B^i f_i\right) \ , \tag{9.3.29}$$

we have

$$\mathcal{N}' = \exp\left(g\, B^i f_i + g\, y^i f_i\right) = \mathcal{N} \exp\left(\log\left(\exp\left(-g\, B^i f_i\right) \exp\left(g\, B^i f_i + g y^i f_i\right)\right)\right) = \mathcal{N}\mathcal{Y} \ . \tag{9.3.30}$$

Given this we can define

$$\begin{aligned}
\mathcal{A}_\mu &= \mathcal{N}^{-1}\left(A^i{}_\mu - \partial_\mu B^i\right) f_i\, \mathcal{N} + \frac{1}{g} \mathcal{N}^{-1} \partial_\mu \mathcal{N} \ , && (9.3.31)\\
\mathcal{A}_z &= \mathcal{N}^{-1} C \zeta\, \mathcal{N} + \frac{1}{g} \mathcal{N}^{-1} \partial_z \mathcal{N} \ . && (9.3.32)
\end{aligned}$$

This field redefinition exactly obeys equation (9.3.1), for example,

$$\begin{aligned}
\mathcal{A}'_\mu &= \left(\mathcal{N}'\right)^{-1}\left(A^{i\prime}{}_\mu - \partial_\mu B^{i\prime}\right) f_i\, \mathcal{N}' + \frac{1}{g}\left(\mathcal{N}'\right)^{-1} \partial_\mu \mathcal{N}' \\
&= \mathcal{Y}^{-1}\mathcal{N}^{-1}\left(A^i{}_\mu + \partial_\mu y^i - \partial_\mu B^i - \partial_\mu y^i\right) f_i\, \mathcal{N}\mathcal{Y} + \frac{1}{g}\mathcal{Y}^{-1}\mathcal{N}^{-1}\partial_\mu \mathcal{N}\mathcal{Y} + \frac{1}{g}\mathcal{Y}^{-1}\mathcal{N}^{-1}\mathcal{N}\partial_\mu \mathcal{Y} \qquad (9.3.33)\\
&= \mathcal{Y}^{-1}\mathcal{A}_\mu \mathcal{Y} + \frac{1}{g}\mathcal{Y}^{-1}\partial_\mu \mathcal{Y} \ .
\end{aligned}$$

However, these redefinitions are only possible when we either restrict what gauge groups we study to those covered by the exponential map [125], or we restrict our study to perturbations about the identity small enough that the Baker–Campbell–Hausdorff identity converges [17].

9.4 The Higher-Order Effective Field Theory

We notice that our field redefinition appears to be a gauge transformation away from $\mathcal{N} = 1$, $B^i = 0$. Therefore to discover the lower-dimensional field theory to general order we need set this gauge transformation in the higher dimension, reduce the theory, then gauge transform back in the lower dimension.

Our higher-dimensional bulk theory is

$$\begin{aligned}\mathcal{S}_{\text{bulk}} = \int_{\mathbb{R}^{1,d-1}} \int_{-1}^{1} & -\frac{1}{4}\,\text{tr}\left\{\left(\partial_\mu A^i{}_\nu f_i - \partial_\nu A^i{}_\mu f_i + g\left[A^i{}_\mu,\ A^i{}_\nu\right] f_i f_j\right)^2\right\} \\ & -\frac{1}{2}\,\text{tr}\left\{\left(\partial_\mu C\zeta - A^i{}_\mu f_i' + g[A^i{}_\mu,\, C] f_i\zeta\right)^2\right\} dz d^d x\ .\end{aligned} \tag{9.4.1}$$

Our gauge-fixed higher-dimensional boundary theory is

$$\mathcal{S}_{z=\pm1} = \pm \int_{\mathbb{R}^{1,d-1}} \frac{(a+b)\pm b}{b}\frac{1}{2}\,\text{tr}\left\{\left(\partial_\mu C\zeta - A^i{}_\mu f_i' + g[A^i{}_\mu,\, C] f_i\zeta\right)^2\right\}\Big|_{z=\pm1} d^d x\ , \tag{9.4.2}$$

Therefore our gauge-fixed lower-dimensional Lagrangian density is ($\mathcal{L} =$)

$$\begin{aligned}\text{tr}\Bigg\{ & -\frac{1}{4}\left(\partial_\mu A^i{}_\nu + \partial_\nu A^i{}_\mu\right)^2 - {\omega_i}^2\left(A^i{}_\mu\right)^2 + \left(\partial^\mu C\right) A_\mu \\ & - g\left(\partial_\mu A^i{}_\nu\right)[A^{j\mu},\, A^k{}_\nu]\,\langle f_i f_j f_k\rangle - \frac{g^2}{4}[A^i{}_\mu,\, A^j{}_\nu][A^{k\mu},\, A^{l\nu}]\,\langle f_i f_j f_k f_l\rangle) \\ & - g\left(\partial_\mu C\right)[A^{i\mu},\, C]\left(\zeta^2 f_i\right) - gA^i{}_\mu[A^{j\mu},\, C]\,(\zeta f_i' f_j) - \frac{g^2}{2}[A^{i\mu},\, C][A^{j\mu},\, C]\left(\zeta^2 f_i f_j\right)\Bigg\}\ .\end{aligned} \tag{9.4.3}$$

We may find the gauge invariant version of this Lagrangian by replacing $A^i{}_\mu \to A^i{}_\mu - \partial_\mu B^i$. Under this replacement every combination of $A^i{}_\mu$ and B^i is manifestly gauge inert and, like the case for Maxwell, B^i's equation of motion is simply the transverse derivative of $A^i{}_\mu$'s equation of motion.

If we follow our program from the previous section and integrate our massive vectors out we generically find that corrections to the theory start at quartic order in A_μ, B, and C and second order in derivatives. This is since the cubic couplings (in the action, quadratic in the equations of motion) between A and $A^{\bar{i}}$ are derivative couplings. Therefore the gauge transformation of our vector only closes at higher order (both cubic and quartic) due to the presence of the Steuckelberg field. We therefore ask, is there any field redefinition that will remove this scalar?

We define the most generic perturbative second-order field redefinition involving only A_μ and up to first derivatives of B. That is

$$a_\mu = A_\mu + X[B,\, A_\mu] + Y[B,\, \partial_\mu B] + \mathcal{O}\left(\phi^3\right) \ , \tag{9.4.4}$$

$$A_\mu = a_\mu - X[B,\, a_\mu] - Y[B,\, \partial_\mu B] + \mathcal{O}\left(\partial_\mu,\, \phi^3\right) \ . \tag{9.4.5}$$

Here X and Y are arbitrary constants. Similarly we define the most generic perturbative second-order gauge transformation redefinition involving only B. That is

$$y = u + Z[B,\, u] + \mathcal{O}\left(\phi^2\right) \ , \qquad\qquad u = y - Z[B,\, y] + \mathcal{O}\left(\phi^2\right) \ . \tag{9.4.6}$$

Here Z is another arbitrary constant. The gauge transformation of a_μ may be found by substituting the gauge transformation of A_μ into the definition of a_μ and then substituting the solution of a_μ's equation in terms of A_μ. That is

$$\begin{aligned}
a'_\mu =& A'_\mu + X[B',\, A'_\mu] + Y[B',\, \partial_\mu B'] + \mathcal{O}\left(\partial_\mu,\, \phi^3\right) \\
=& A_\mu + g\partial_\mu y + X[B,\, A_\mu] + X[y,\, A_\mu] + X[B,\, a_\mu] \\
& \qquad + Y[B,\, \partial_\mu B] + Y[y,\, \partial_\mu B] + Y[B,\, \partial_\mu y] + \mathcal{O}\left(\partial_\mu,\, \phi^2, y\right) \\
=& a_\mu - X[B,\, a_\mu] - Y[B,\, \partial_\mu B] + \partial_\mu u - Z[\partial_\mu B,\, u] - Z[B,\, \partial_\mu u] \\
& \qquad + X[B,\, a_\mu] + X[u,\, a_\mu] + X[u,\, \partial_\mu B] \\
& \qquad + Y[B,\, \partial_\mu B] + Y[u,\, \partial_\mu B] + Y[B, \partial_\mu B] + \mathcal{O}\left(\partial_\mu,\, \phi^2, y\right) \ .
\end{aligned} \tag{9.4.7}$$

Therefore if we select $Y = -\frac{1}{2}X$, and $Z = \frac{1}{2}X$ we have

$$a'_\mu = a_\mu + \partial_\mu u + X[u,\, a_\mu] + \mathcal{O}\left(\partial_\mu,\, \phi^2, y\right) \ . \tag{9.4.8}$$

That is, we can undo our higher-dimensional gauge-fixing in the lower-dimension. Furthermore if we select $X = gI_{000}$ we find a lower-dimensional Lagrangian density ($\mathcal{L} =$)

$$\begin{aligned}
\mathrm{tr}\Bigg\{ & -\frac{1}{4}\left(\partial_\mu a_\nu - \partial_\nu a_\mu\right)^2 + \left(\partial^\mu C\right)\left(\overline{a}_\mu\right) - gI_{000}\left(\partial_\mu a_\nu\right)[a^\mu,\, a^\nu] \\
& - \frac{g^2}{4}\left(I_{000}\right)^2 [a_\mu,\, a_\nu][a^\mu,\, a^\nu] - \frac{g^2}{4} X[\overline{a}_\mu,\, \overline{a}_\nu][\overline{a}^\mu,\, \overline{a}^\nu] \\
& - \frac{1}{2} gI_{000}\left(\partial^m uC\right)[B,\, \overline{a}_\mu] + g^2\left(I_{000}\right)^2 \partial^\mu C\left(\mathcal{O}\left(B^2,\, a_\mu\right) + \mathcal{O}\left(\partial_\mu, B^3\right)\right) \\
& - g\left(\zeta^2 f_0\right)\left(\partial_\mu C\right)[\overline{a}^\mu,\, C] - g\left(\zeta f'_0 f_0\right)\overline{a}_\mu[\overline{a}^\mu,\, C] - \frac{g^2}{2}\left(\zeta^2 f_0 f_0\right)[\overline{a}^\mu,\, C][\overline{a}^\mu,\, C]\Bigg\} \ .
\end{aligned} \tag{9.4.9}$$

Here $X = I_{0000} - (I_{000})^2$ is a novel seagull coefficient and $\overline{a}_\mu = a_\mu - \partial_\mu B$ is the gauge inert (at linear order) combination of a_μ and B in the lower dimension. We have not explicitly calculated the cubic redefinition of A_μ and its inverse in terms of a_μ, however, it is sufficient to state the action, and therefore the equations of motion, at linear order in B. We can use this to find B's equation exactly since it is given by the derivative of A_μ's equation from equation (9.4.3).

In short, we see that, like for the case of scalar electrodynamics (in section 8.6) we have a promotion of our Lagrange multiplier C to an auxiliary field. Further analysis of this action will have to wait for later authors.

10 Summary

In this text, we first, in section 2, preemptively studied the conditions under which a one-dimensional Laplacian will have a zero mode, the conditions that give that zero mode, what the remaining spectrum of the problem will be, and a related augmented inner product. In section 3 we showed what we could compute from our Sturm–Liouville bases, that was we showed how to calculate Green functions, their augmented and shifted relatives, complicated sums of overlap integrals, and further Green functions on product spaces.

In section 4 we studied the perturbation problem about maximally symmetric warped product spaces, discussed how it related to the exact problem, and catalogued how different effective behavior could be quickly diagnosed from different boundary conditions. We then supplemented our catalogue, in section 5, by studying a specific problem and relating its solutions to each other.

We then shifted again and, in section 6, studied the effects of selecting boundary conditions (in the context of Klein–Gordon theory) other than Type I, Type II, and Type III * in a dimensional reduction. We contrasted such problems, in 7, to Type III * reductions with tensorial components, and described how to diagonalize effective degrees of freedom and gauge transformations.

Finally, in section 8, we studied what happens when you combine these two complexities, and studied a Type III † reduction of an interacting higher-dimensional gauge theory in detail. Then, in section 9, we laid the groundwork for studying Type III † self-interacting higher-dimensional gauge theory, and showed how the relevant recursion equations can be solved using the exponential map.

From this we can see our two main foci were then:

- Identifying when the perturbation problem in a higher dimension can be effectively lower-dimensional.
- Building a toolkit for understanding the effective field theories about such backgrounds.

Towards our first focus, to study theories that cannot be consistently truncated, we must generalize from the standard Fourier basis expansion of our higher-dimensional field to a Sturm–Liouville basis. In studying the Sturm–Liouville theory, focusing on zero modes (so that we may have an effectively lower-dimensional solution once we apply Green's formula for a product space) given by Laplacians we discover that we can easily construct the space of zero modes, one is always a constant and the other is always given by quadrature. We then discover that we can easily specify what boundary conditions correspond to what zero modes, and that the zero modes will appear (or not appear) in the basis in one of four ways:

- We may have a one-dimensional space of normalizable zero modes.[93]

[93] Some of these bases correspond contain negative eigenvalue (and therefore negative mass) modes.

- We may have a space where only the constant zero mode is normalizable.
- We may have a space where only the non-constant zero mode is normalizable.
- We may have a space with no normalizable zero modes.

We then find evidence of the adage 'all perturbative physics is contained in the zero mode'. Specifically, we show how the presence or absence of a zero mode determines the leading behavior of Green functions for Laplacians on product spaces.

Finally we study the perturbation problem about a warped product background. We find that, for gravitons, that product space Laplacian is the relevant quantity. We then explore how the graviton's equation can inform black hole solutions. We find that we can find some solutions exactly, i.e. Type I solutions and sometimes Type II solutions, but that, in the context of an infinite volume transverse space, you may never have lower-dimensional effective physics linked to a localized higher-dimensional source. We then explore how to compact the space to find an approximate solution to a consistent truncation, which we call Type III *, or how sometimes the transverse space will allow for a nonconstant zero mode and how these can persist even when the transverse space has infinite volume, which we call Type III †.

We then give an example of a Type III † reduction, show that the solution to the product space Laplacian is a valid approximation of the leading order of the theory, and describe several techniques for extracting the lower-dimensional physical constants from this theory.

Towards our second focus, our study of Sturm–Liouville theory further informs further investigations of Green functions. We find that manipulations of the zero mode[94] can give us the sum of the overlap integrals of heavy modes. Further we discover that theories with non-constant zero modes, which necessarily correspond to the context of inconsistent truncations, have many novel corrections that are simple to identify, but pose possible issues for gauge invariance. Furthermore we understand that it does not require any explicit knowledge of the remainder of the spectrum to study the corrections to the theory. This is a unique feature of our context of inconsistent truncations and can possibly be leveraged to understand exact solutions in self-interacting theories in a new way.

[94] As well as its closely related cousin the functions in the kernel of the square of the Laplacian.

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A Shape Invariant Schrödinger Problems

One of the most powerful tools for solving one-dimensional Schrödinger problems is the application of shape invariance [75]. Shape invariance is built up in stages, from initial theoretical observations of factorization, to the relations of two differential operators in so-called supersymmetric quantum mechanics or SUSYQM [31], to the relations of infinitely many differential operators all linked by factorization.

Shape invariance is not a problem with infinitely large reach. Its reach is limited by its power, as it allows you to solve entire classes of problems, given that they are described by highly restrictive requirements. Finding all possible problems described by those restrictions leads one to discover that there are precisely six possibilities.

These six problems have important applications in many physically important problems such as spherical harmonics, meson fields, and the Keppler problems [75]. The underlying reason why is revealed with a slight generalization of shape invariance, from a univariate shape invariant problem to a bivariate problem, will allow us to define additional operators that will relate these problems to several different Lie algebras, revealing the underlying symmetry of the problems in which the potentials are frequently found.

From a mathematical perspective shape invariance informs us about how we might use a series of differential operators to relate a family of special functions to itself through change of parameters. This is possibly the most efficient method of finding recurrence relations, and often of calculating these special functions [61].

Lastly, these problems can serve as toy models for many more intricate problems in quantum field theory, etc. However, the application of these tools are frequently superseded by others such as the theory of special functions, and the manipulation of Green functions in Sturm–Liouville theory, as they are for this work. However, for the sake of completeness, and in the hope that the study of these tools might reach an audience which might apply them, they are included here.

The organization of this appendix is as follows.

1. We define Darboux transformations, and find the superpotential for an arbitrary Schrödinger operator.

2. We then define SUSYQM, and introduce the superspace formulation of such.

3. We repeat Darboux transformations [5] to relate an entire (shape invariant) family of problems [75].

4. We use the definition of shape invariance to find all possible shape invariant families.

5. We generalize shape invariance to a bivariate family and show how they recreate a Lie algebra.

A.1 Darboux Transformations

In the shape invariance literature Schrödinger problems are usually defined in terms of a Hamiltonian H [75],

$$-\partial_x{}^2 + V(x) = H \ , \tag{A.1.1}$$

and its eigenvalue problem

$$H\psi_l(x) = E_l\psi_l(x) \ . \tag{A.1.2}$$

We note that we may universally transform the eigenvalue problem of a Laplacian into this form using the technique demonstrated in equation (2.2.22).[95]

Given such a problem we might attempt to find a factorization, that is operators A and its adjoint under the standard L_2 inner product $A^\dagger$ so that

$$H = AA^\dagger = (\partial_x + W(x))(-\partial_x + W(x)) \ . \tag{A.1.3}$$

Here $A = \partial_x + W(x)$ and its adjoint is $A^\dagger = -\partial_x + W(x)$. Simplifying the right hand side of equation (A.1.3), we have

$$-\partial_x{}^2 + W'(x) + W(x)^2 = -\partial_x{}^2 + V(x) \ . \tag{A.1.4}$$

Taken as a nonlinear ordinary differential equation for $W(x)$ we recognize a Riccati equation and propose the variable redefinition $W(x) = \frac{y'(x)}{y(x)}$ [101]. Given this we find

$$\frac{y''(x)}{y(x)} = V(x) \ , \tag{A.1.5}$$

which we may restate

$$Hy(x) = 0 \ . \tag{A.1.6}$$

That is, finding a factorization of a Schrödinger problem is equivalent to finding a vacuum solution to the same. This $W(x)$ is referred to as the 'superpotential' of the Hamiltonian H. However, there is slightly more flexibility in defining $W(x)$ then there generically is in finding eigenstates of the Hamiltonian, since any $y(x)$ which lies within the kernel of H, whether it obeys the boundary conditions of the Schrödinger problem or not, defines a unique superpotential [75]. Therefore there exists a one parameter family of superpotentials.

[95] To complete the dictionary explicitly we have $E_l = -\omega^2$ and $l_\omega = \psi_l$.

For example, for the free Hamiltonian $V(x) = 0$ we have $y = \cos^2(\theta) + \sin^2(\theta)x$, for any θ, and therefore[96]

$$W(x) = \frac{1}{\cot^2(\theta) + x} \ . \tag{A.1.7}$$

Given a factorized Hamiltonian (that is, given explicit A and $A^\dagger$), we may define a one-parameter family of Hamiltonia H_θ defined via Darboux transformation as

$$H_\theta = A^\dagger A = -\partial_x{}^2 + \left(W(x)^2 - W'(x)\right) \ . \tag{A.1.8}$$

For example, the family of H_θ defined via Darboux transformation of the free Hamiltonian is

$$H_\theta = \partial_x{}^2 + \frac{2}{\left(\cot^2(\theta) + x\right)^2} \ . \tag{A.1.9}$$

Given any eigenvalue to the original Hamiltonian H, $\psi_l(x)$, we may apply our factor $A^\dagger$ to find a eigenvalue to H_θ. That is

$$H\psi_l = AA^\dagger\psi_l = E_l\psi_l \qquad \Rightarrow \qquad H_\theta A^\dagger\psi_l = A^\dagger AA^\dagger\psi_l = A^\dagger H\psi_l = E_l A^\dagger\psi_l \ . \tag{A.1.10}$$

Similarly, given a solution to H_θ we may find a solution to H.

One notable exception to this technique is any function, $\phi(x)$, which lies within the kernel of $A^\dagger$. These obey

$$A^\dagger\phi(x) = -\phi'(x) + \frac{y'(x)}{y(x)}\phi = 0 \qquad \Rightarrow \qquad \frac{\phi'(x)}{\phi(x)} = \frac{y'(x)}{y(x)} \ . \tag{A.1.11}$$

If we integrate the left and right hand sides of this final expression we find $\log(\phi(x)) = \log(y(x)) + c$. Therefore $\phi(x) = \exp(c)y(x)$. That is, the mode we used to define our superpotential $W(x)$ is not related to a solution of H_θ by Darboux transformation.

A.2 SUSYQM

If we consider two Hamiltonia related by Darboux transformation $H_- = AA^\dagger$ and $H_+ = A^\dagger A$ then we may define an overall Hamiltonian $\mathcal{H}$ defined as[97]

$$\mathcal{H} = \frac{1}{2}\begin{pmatrix} H_- & 0 \\ 0 & H_+ \end{pmatrix} \ . \tag{A.2.1}$$

[96] Since non-trivial $y(x)$ constitute a two parameter family, we may be tempted to surmise there is a two parameter family of W. However, the overall scale of any solution cancels in the variable redefinition, leaving us with only one parameter.

[97] We have introduced a factor of $\frac{1}{2}$ to standardize the later conversion to the Lagrangian formalism.

Further we may define supercharges

$$\mathcal{Q} = \begin{pmatrix} 0 & 0 \\ A & 0 \end{pmatrix} , \qquad \mathcal{Q}^\dagger = \begin{pmatrix} 0 & A^\dagger \\ 0 & 0 \end{pmatrix} . \tag{A.2.2}$$

These obey the anticommuntation relations

$$2\mathcal{H} = \{\mathcal{Q},\, \mathcal{Q}^\dagger\} . \tag{A.2.3}$$

We may rewrite this as

$$\mathcal{H} = \frac{1}{2}\left(p^2 + W(x)^2\right)\hat{1} - [\sigma_+,\, \sigma_-]W'(x) . \tag{A.2.4}$$

Here p is the standard momentum operator. This Hamiltonian (with its associated time-dependent Schrödinger problem) has the associated Lagrangian

$$\mathcal{L} = \frac{1}{2}\dot{x}(t)^2 + i\sigma_-\partial_t\sigma_+ - \frac{1}{2}W(x)^2 + \frac{1}{2}[\sigma_+,\, \sigma_-]W'(x) . \tag{A.2.5}$$

Which is used as a toy problem to study supersymmetry in the context of a field one-dimensional field theory.

A.3 Shape Invariance

Of course, there is no reason that we must stop at two related Schrödinger problems. We may repeatedly select a specific factorization and Darboux transform our problems for an infinite family of H_n given by

$$H_n = {A_{n-1}}^\dagger A_{n-1} + L_{n-1} = A_n {A_n}^\dagger + L_n . \tag{A.3.1}$$

Here we have chosen to shift the eigenvalues of the n^{th} problem by L_n, which leaves the eigenfunctions invariant.

Each of these Hamiltonians has its own associated eigenfunctions $\psi_l^n(x)$ which obey

$$H_n\psi_l^n(x) = E_l^n\psi_l^n(x) . \tag{A.3.2}$$

Even with shifting our eivenvalues during our Darboux transformation, our kinetic terms, i.e. $-{\partial_x}^2$, are invariant. Therefore we only need track what occurs to the potentials

$$V_n(x) = W_{n-1}(x)^2 - W'_{n-1}(x) + L_{n-1} = W'_n(x) + W_n(x)^2 + L_n . \tag{A.3.3}$$

Subtracting the n^{th} and $n-1^{\text{st}}$ factorizations of H_n we find

$$W_n(x)^2 + W_n'(x) + L_n - W_{n-1}(x)^2 + W_{n-1}'(x) - L_{n-1} = 0 \ . \tag{A.3.4}$$

We, of course, can always find solutions to these equations for any series of Hamiltonians related by sequential Darboux transformations, which we have argued always exist. Shape invariance is the additional requirement that W_n and L_n may be expanded in terms of finite power series of n. This restriction allows us to, given either knowledge of all of the modes in H_0's spectrum, or knowledge of each of our superpotentials, W_n, give the full spectrum for each of our operators. Furthermore, the restriction to finite power series in n allows us to explicitly find all families of shape invariant potentials. We would first note that there is a seemingly trivial solution with

$$W_n(x) = f_n \ , \qquad L_n = {f_n}^2 \ . \tag{A.3.5}$$

However, this form of relation, of the free Schrödinger problem to itself, even with an arbitrary shift in energy, is simply a restatement of the quantum harmonic oscillator [75], which is a special case of shape invariance, yes, but is also best studied in a more symbolically tractable case. We will attempt to investigate solutions that relate distinct $V_n(x)$.

We now assume

$$W_n(x) = W(x) + nX(x) \ , \tag{A.3.6}$$

and substitute to find

$$2\left(W'(x) + W(x)X(x)\right) + (2n-1)\left(X(x)^2 + X'(x)\right) = L_{n-1} - L_n \ , \tag{A.3.7}$$

for all n. We may rewrite the left hand side

$$\left(n^2\left(X(x)^2 + X'(x)\right) + 2n\left(W'(x) + W(x)X(x)\right)\right) \tag{A.3.8}$$

$$-\left((n-1)^2\left(X(x)^2 + X'(x)\right) + 2(n-1)\left(W'(x) + W(x)X(x)\right)\right) \ . \tag{A.3.9}$$

If we assume L_n may be written as a power series of n, we have

$$L_n = -n^2\left(X(x)^2 + X'(x)\right) - 2n\left(W'(x) + W(x)X(x)\right) + f(x) \ , \tag{A.3.10}$$

where $f(x)$ is an arbitrary function of x. If we assume $L_0 = 0$, we have $f(x) = 0$. In such a case we have

$$X(x)^2 + X'(x) = -c^2 \ , \qquad W'(x) + W(x)X(x) = -ck \ . \tag{A.3.11}$$

Here c and k are arbitrary constants. Solving these equations sequentially we have

$$X(x) = -c\tan(cx + ca) \ , \qquad W(x) = b\sec(c(a - x)) + k\tan(c(a - x)) \ . \tag{A.3.12}$$

Here a and b are two further integration constants.

We may further consider $X(x)^2 + X'(x) = 0$ and $W'(x) + W(x)X(x) = -l$, for a simpler definition of L_n, however, all such combinations can be considered as limiting forms of the explicit case when $c \to 0^+$ and $ck \to l$, or some similar combination of limits. More relevant is the question of how our assumption of the form of $W_n(x)$ might be relaxed.

If we assume $W_n(x) = \ldots + n^2 Y(x)$ is a quadratic polynomial in n, then we discover

$$4n^3 Y(x)^2 + \mathcal{O}(n^2) = L_{n-1} - L_n \ . \tag{A.3.13}$$

If we expand the left and right in series in n, each coefficient function of a given order in n on the left must be x independent, therefore $\partial_x Y(x)^2 = 0$. Furthermore the addition of an an^2 term in $W_n(x)$ causes a similarly non-differential requirement for the n term in $W_n(x)$. If one follows this chain of implication we find $W_n(x) = f(n)$ after including any term beyond linear order in n.

However, if we generalize from Taylor series to Laurent series, assuming

$$W_n(x) = \frac{1}{n}U(x) + W(x) + nX(x) \ , \tag{A.3.14}$$

and repeat this calculation we will discover a new solution with

$$X(x) = -c\tan(cx + ca) \ , W(x) = 0 \ , U(x) = b \ . \tag{A.3.15}$$

With similar limiting cases.

Assembling all of this we find six relevant sublimits of these problems, labeled cases A through F in [75].

B Asymptotic Approximations of Solutions with a Transverse Pöschl–Teller Potential

B.1 Orthonormalised Transverse Wavefunctions

To eke out any higher precision than the expression given in equation (5.3.20), first note that our separated solutions

$$\left(\partial_r^2 + \frac{2}{r}\partial_r + g^2\left(\partial_\rho^2 + 2\coth(2\rho)\partial_\rho\right)\right) f^\omega(r)\zeta_\omega(\rho) = 0 \ , \tag{B.1.1}$$

have transverse factors $\zeta_\omega(\rho)$, which, after changing variables to $y = \cosh(2\rho)$ and $\zeta_\omega(\rho) = \psi_\omega(\cosh(2\rho)) = \psi_\omega(y)$, solve

$$\left(4\partial_y\left(y^2-1\right)\partial_y + \omega^2\right)\psi_\omega(y) = 0 \ . \tag{B.1.2}$$

This is a known version of Legendre's differential equation with the general solution given by Legendre functions (since the order is in general complex) [61].

$$\psi_\omega(y) = a_\omega \mathcal{P}_{-\frac{1}{2}+\frac{\sqrt{1-\omega^2}}{2}}(y) + b_\omega \mathcal{Q}_{-\frac{1}{2}+\frac{\sqrt{1-\omega^2}}{2}}(y) \ . \tag{B.1.3}$$

The Legendre functions of the second type ($\mathcal{Q}$) have a logarithmic divergence as $y \to 1$. For the moment we want to consider only solutions that are regular when $r \neq 0$ and $\rho \to 0$ ($y \to 1$), so we consider solutions involving only the Legendre function of the first type ($\mathcal{P}$).

Returning to the ρ variables, we now investigate orthonormality. We require

$$\int_0^\infty \sinh(2\rho)\zeta_\omega(\rho)\zeta_\upsilon(\rho)d\rho = \delta(\omega-\upsilon) \ . \tag{B.1.4}$$

Applying our transverse operator and integrating by parts, we find that this integral may be given purely in terms of contact terms at infinity. We recall the identity[98]

$$\lim_{R\to\infty}\frac{1}{\omega^2-\upsilon^2}\left(\omega\sin(\omega\rho)\cos(\upsilon\rho) - \upsilon\cos(\omega\rho)\sin(\upsilon\rho)\right)\bigg|_{\rho=R} \propto \delta(\omega-\upsilon) \ . \tag{B.1.5}$$

Our solutions do not asymptote to sinusoidal functions with frequency ω. Instead, they asymptote with frequency $\sigma = \sqrt{\omega^2-1}$ as can be seen both from the asymptotic form of Equation (B.1.2), and via the

[98] We ignore momentarily the numerical factors of the form $\sqrt{2}$, $\sqrt{\pi}$, etc.

properties of Legendre functions. Specifically, the large y asymptote is

$$\mathcal{P}_\nu(y) \sim B\left(\nu+\frac{1}{2},\frac{1}{2}\right)^{-1}(2y)^{-\nu-1} \ , \tag{B.1.6}$$

where B is the Euler beta function [61]. There are actually two asymptotic regimes that we need to consider: when $Re(\nu) > -\frac{1}{2}$ and when $Re(\nu) < -\frac{1}{2}$ (although they actually agree in the present case). Furthermore, we will need the connection formula for Legendre functions, the definition of the Euler beta function and the reflection formula for Euler gamma functions:

$$\mathcal{P}_\nu(y) = \mathcal{P}_{-1-\nu}(y) \ , \qquad B(x,y) = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)} \ , \qquad \Gamma(z)\Gamma(1-z) = \pi\csc(\pi z) \ . \tag{B.1.7}$$

These allow us to derive the necessary normalization a_ω so that the amplitude of our solutions as $\rho\to\infty$ is ω independent. That is, given

$$\zeta_\omega(\rho) \propto \sqrt{\pi\sigma\tanh\left(\frac{\pi\sigma}{2}\right)}\mathcal{P}_{-\frac{1}{2}+\frac{i\sigma}{2}}(\cosh(2\rho)) \ , \tag{B.1.8}$$

we have $\sqrt{\sinh(2\rho)}\zeta_\omega(\rho) \sim 2\sin(\sigma\rho+\delta)$. The shift δ is irrelevant for orthonormalization. These almost satisfy the equation that we require. We require one additional normalization, since the asymptotic frequency is given by a function of the separation constant, rather than the separation constant we naïvely expected. That is, since $\sigma = \sqrt{\omega^2-1}$, we have

$$\begin{aligned}\int_0^\infty \sinh(2\rho)\left(\sqrt{\pi\sigma\tanh\left(\frac{\pi\sigma}{2}\right)}\mathcal{P}_{-\frac{1}{2}+\frac{i\sigma}{2}}(\cosh(2\rho))\right)\left(\sqrt{\pi\tau\tanh\left(\frac{\pi\tau}{2}\right)}\mathcal{P}_{-\frac{1}{2}+\frac{i\tau}{2}}(\cosh(2\rho))\right)d\rho \\ = \delta\left(\sigma-\tau\right) \ .\end{aligned} \tag{B.1.9}$$

To build our Green functions we require this integral to generate a delta function distribution with respect to ω, not σ. We use the well-known following property of delta function distributions,

$$\delta(f(\omega)-f(\tau)) = \frac{\delta(\omega-\tau)}{f'(\omega)} \ , \tag{B.1.10}$$

then divide by the derivative of the function of the asymptotic frequency with respect to the separation constant, to find the correctly normalized transverse wavefunctions. At the end, they are

$$\zeta_\omega(\rho) = \sqrt{\frac{\pi\left(\sigma^2+1\right)}{\sigma}\tanh\left(\frac{\pi\sigma}{2}\right)}\mathcal{P}_{-\frac{1}{2}+i\frac{\sigma}{2}}\left(\cosh(2\rho)\right) \ , \tag{B.1.11}$$

written in terms of σ and

$$\zeta_\omega(\rho) = \sqrt{\frac{\pi\omega^2}{\sqrt{\omega^2-1}} \tanh\left(\frac{\pi}{2}\sqrt{\omega^2-1}\right)} \mathcal{P}_{-\frac{1}{2}+\frac{\sqrt{1-\omega^2}}{2}} \left(\cosh(2\rho)\right) \ , \tag{B.1.12}$$

when written in terms of ω. These now, by construction, obey the identity

$$\int_1^\infty \zeta_\omega(\rho)\zeta_\omega(\eta) d\omega = \frac{\delta(\rho-\eta)}{\sinh(2\rho)} \ . \tag{B.1.13}$$

We set $\eta = 0$ for ease since $\mathcal{P}_\nu(0) = 1$ for all ν.

As for the worldvolume factors $f^\omega(\eta)$, we know the fundamental solution to the corresponding worldvolume differential equation,

$$\left(\partial_r{}^2 + \frac{2}{r}\partial_r - g\omega^2\right) \frac{\exp\left(-g^2\omega r\right)}{4\pi r} = \frac{\delta(r)}{4\pi r^2} \ . \tag{B.1.14}$$

We may then write the fundamental solution to the total Laplacian

$$G(r,\rho) = \int_1^\infty \frac{\exp\left(-g\omega r\right)}{4\pi r} \left(\frac{\pi\omega^2}{\sqrt{\omega^2-1}} \tanh\left(\frac{\pi}{2}\sqrt{\omega^2-1}\right)\right) \mathcal{P}_{-\frac{1}{2}+\frac{\sqrt{1-\omega^2}}{2}} \left(\cosh(2\rho)\right) d\omega \ . \tag{B.1.15}$$

Alternately, we may state the integral in terms of σ

$$\int_0^\infty \frac{\exp\left(-g\sqrt{\sigma^2+1}r\right)}{4\pi r} \pi\sqrt{\sigma^2+1} \tanh\left(\frac{\pi\sigma}{2}\right) \mathcal{P}_{-\frac{1}{2}+i\frac{\sigma}{2}}(\cosh(2\rho)) d\sigma \ . \tag{B.1.16}$$

B.2 The Ray Trick

No general form of the integral (B.1.16) is known to our knowledge, as it involves an integral with respect to the order of a Legendre function. However, we can find some limits of this integral. Let us introduce the ray trick. If we want to consider the limit of some integral, say

$$I(r,\rho) = \int_0^\infty \frac{\exp(-\omega r)}{4\pi r} \cos(\omega\rho) d\omega \ , \tag{B.2.1}$$

we can take explicit ratios of $r = xt$ and $\rho = t$ as $t \to 0^+$. Then our integral becomes

$$I(x,t) = \int_0^\infty \frac{\exp(-\omega xt)}{4\pi xt} \cos(\omega t) d\omega \ . \tag{B.2.2}$$

Multiplying this integral by xt^2, and taking the limit $t \to 0^+$, we define

$$J(x) = \lim_{t\to 0^+} xt^2 \int_0^\infty \frac{\exp(-\omega xt)}{4\pi xt} \cos(\omega t) d\omega \,. \tag{B.2.3}$$

This can be rewritten as

$$J(x) = \lim_{t\to 0^+} \int_0^\infty t f_x(\omega t) d\omega \,, \tag{B.2.4}$$

which, after a variable redefinition $y = \omega t$, becomes

$$J(x) \lim_{t\to 0^+} \int_0^\infty f_x(y) t\, t^{-1} dy = \lim_{t\to 0^+} \int_0^\infty f_x(y) dy \,, \tag{B.2.5}$$

where, crucially, the integrand is t *independent*. Thus,

$$J(x) = \lim_{t\to 0^+} xt^2 \int_0^\omega \frac{\exp(-\omega xt)}{4\pi xt} \cos(\omega t) d\omega = \frac{1}{4\pi} \int_0^\infty \exp(-xy)\cos(y) dy = \frac{1}{4\pi} \frac{x}{1+x^2} \,. \tag{B.2.6}$$

We can now divide by the factor that we used to get the equation into the t independent form and we find

$$I(x,t) = \int_0^\infty \frac{\exp(-\omega xt)}{4\pi xt} \cos(\omega t) d\omega \sim \frac{1}{4\pi} \frac{1}{t^2 x} \frac{x}{1+x^2} = \frac{1}{4\pi} \frac{1}{(1+x^2)t^2} = \frac{1}{4\pi} \frac{1}{r^2+\rho^2} \,. \tag{B.2.7}$$

This gives us the expected value of the integral in the $r \sim \rho \sim 0$ region.

B.3 The $R \ll 1$ Expansion

Let us verify that the solution in equation (B.1.15) is the same (or at least proportional to) the solution in equation (5.3.11). We begin by multiplying the total function by the SS–CGP parameter g, redefining $\tilde{r} = gr$, then dropping the tilde. That is, by rescaling r and G^N by g we may find the solution in terms of the integral when $g = 1$.

$$\begin{aligned} G^N(r,\rho) &= g \int_1^\infty \frac{\exp(-\omega \tilde{r})}{2\pi \tilde{r}} \left(\frac{\pi \omega^2}{\sqrt{\omega^2 - 1}} \tanh\left(\frac{\pi}{2} \sqrt{\omega^2-1} \right) \right) \mathcal{P}_{-\frac{1}{2} + \frac{\sqrt{1-\omega^2}}{2}} (\cosh(2\rho))\, d\omega \\ &= g\tilde{G}(gr,\rho) \,, \end{aligned} \tag{B.3.1}$$

where $\tilde{G} = G^N\big|_{g=1}$. We then break the dual space into low frequency and high frequency contributions separated at a value Λ, writing the integrand as $E(\omega, r, \rho)$

$$\tilde{G}(r,\rho) = \int_1^\Lambda E(\omega,r,\rho) d\omega + \int_\Lambda^\infty E(\omega,r,\rho) d\omega \,. \tag{B.3.2}$$

Let us now focus on the large frequency integral. When $\omega \gg 1$ most of the terms of the integral simplify. We use the following asymptotic forms for square root, hyperbolic tangent, and Legendre functions [2]

$$
\begin{aligned}
\sqrt{\omega^2-1} &\sim \omega - \frac{1}{2\omega} + \mathcal{O}(\omega^{-3}) \,, \\
\tanh(X) &\sim 1 - 2\exp(-2X) + \mathcal{O}\left(\exp(-4X)\right) \,, \\
\mathcal{P}_{-\frac{1}{2}+i\frac{\omega}{2}}(\cosh(2\rho))\sqrt{\sinh 2\rho} &\sim \sqrt{2\rho}J_0(\omega\rho) + \text{subleading} \,.
\end{aligned}
\tag{B.3.3}
$$

Given these expansions and a sufficiently large cutoff, we may now write the high frequency integral in terms of a new simpler integrand $\mathcal{E}$, plus subleading corrections

$$
\begin{aligned}
\int_\Lambda^\infty E(\omega,r,\rho)d\omega &= \pi\int_\Lambda^\infty \frac{\exp(-\omega r)}{4\pi r}\omega J_0(\omega\rho)\frac{\sqrt{2\rho}}{\sqrt{\sinh(2\rho)}} + \mathcal{O}\left(\frac{1}{\omega}\right)d\omega \\
&= \int_\Lambda^\infty \mathcal{E}(\omega,r,\rho) + \mathcal{O}\left(\frac{1}{\omega}\right)d\omega \,.
\end{aligned}
\tag{B.3.4}
$$

This integral is still unknown. However, using the fundamental theorem of calculus, we may approximate it in the small Λ limit:

$$
\begin{aligned}
&\int_\Lambda^\infty \mathcal{E}(\omega,r,\rho)d\omega = \int_0^\infty \mathcal{E}(\omega,r,\rho)d\omega - \int_0^\Lambda \mathcal{E}(\omega,r,\rho)d\omega \\
&= \frac{\sqrt{2\rho}}{4\sqrt{\sinh(2\rho)}}\left(\frac{1}{R^3} + \frac{\Lambda^2}{2r} - \frac{\Lambda^3}{3} + \frac{\left(2r^2-\rho^2\right)\Lambda^4}{r} + \mathcal{O}\left(\Lambda^5\right)\right) \,,
\end{aligned}
\tag{B.3.5}
$$

where we recall that $R^2 = g^2r^2 + \rho^2$. We will address the validity of the small Λ limit momentarily. The low frequency contribution may be done using different approximations of these functions. First we shift $\omega = \tilde{\omega} + 1$ so that our integral is from $\tilde{\omega} = 0$ to $\tilde{\Lambda} = \Lambda - 1$. Our integrand becomes

$$
\begin{aligned}
\int_1^\Lambda E(\omega,r,\rho)d\omega = \exp(-r)\int_0^{\tilde{\Lambda}} &\frac{\exp(-\tilde{\omega}r)}{4\pi r}\left(\frac{\pi^2}{2} + \left(\pi^2 - \frac{\pi^4}{12}\right)\tilde{\omega} + \mathcal{O}\left(\tilde{\omega}^2\right)\right) \\
&\times\left(1 + \frac{1}{4}\rho^2\left(-\tilde{\omega}^2 - 2\tilde{\omega} - 1\right) + \mathcal{O}\left(\rho^4\right)\right)d\tilde{\omega} \,.
\end{aligned}
\tag{B.3.6}
$$

We can expand this in the small $\tilde{\Lambda}$ and small ρ limit to find

$$
\int_1^\Lambda E(\omega,r,\rho)d\omega = \frac{\pi\tilde{\Lambda}e^{-r}}{8r} - \frac{\pi\tilde{\Lambda}\rho^2e^{-r}}{32r} + \mathcal{O}\left(\tilde{\lambda}^2\right) + \mathcal{O}\left(\rho^4\right) \,.
\tag{B.3.7}
$$

Therefore as $R \to 0$, $G \to \frac{1}{R^3}$. We confirm that this is (proportional to) the solution given above. All terms that contain factors of the cutoff accurately represent the forms of corrections. However, since the cutoff is arbitrary the exact function will, of course, be independent of the cutoff, but the actual coefficients of these corrections remain unknown.

B.4 The $\rho \ll 1$ Expansion

First we set $\rho = 0$ when $\mathcal{P}_\nu(1) = 1$. We shift our integrand as before to find

$$\tilde{G}(r,0) = \exp(-r)\int_0^\infty \frac{\exp(-\omega r)}{4\pi r}\left(\frac{\pi\left(\omega+1\right)^2}{\sqrt{(\omega+1)^2-1}}\tanh\left(\frac{\pi}{2}\sqrt{(\omega+1)^2-1}\right)\right)d\omega\ . \tag{B.4.1}$$

This integral still escapes the domain of known integrals giving named functions, but we can expand the integrand excluding the $\exp(-\omega r)$ term in the small ω limit. This is valid when r becomes large as all large ω terms become exponentially suppressed. This gives us the following series

$$\tilde{G}(r,0) = \frac{\exp(-r)}{4\pi}\left(\frac{\pi^2}{r^2} - \frac{2\pi^2\left(\pi^2-3\right)}{3r^3} + \frac{2\pi^2\left(15-25\pi^2+8\pi^4\right)}{15r^4} + \mathcal{O}\left(\frac{1}{r^6}\right)\right)\ . \tag{B.4.2}$$

If we expand our transverse functions at small ρ, we find the following series

$$\mathcal{P}_{-\frac{1}{2}+\frac{\sqrt{1-(\omega+1)^2}}{2}}\left(\cosh(2\rho)\right) = 1 + \left(-\frac{1}{4} - \frac{\omega}{2} - \frac{\omega^2}{4}\right)\rho^2 + \left(\frac{11}{192} + \frac{7\omega}{48} + \frac{13\omega^2}{96} + \frac{\omega^3}{16} + \frac{\omega^4}{64}\right)\rho^4 + \mathcal{O}\left(\rho^6\right)\ . \tag{B.4.3}$$

Using these two series we can find $\exp(r)\tilde{G}(r,\rho)$ to arbitrary order in $\frac{1}{r}$ and ρ. Furthermore, we may find the exact coefficient of the leading term in the expansion in $\frac{\exp(-r)}{r}$ by first substituting $\omega = 0$ into our transverse wavefunction. We find

$$\mathcal{P}_{-\frac{1}{2}}\left(\cosh(2\rho)\right) = \frac{2}{\pi}K\left(-\sinh^2(\rho)\right)\ , \tag{B.4.4}$$

where K is the complete elliptic integral of the first kind. The best estimate we have for $\tilde{G}$ is therefore

$$\begin{aligned}\tilde{G}(r,\rho) =& \frac{\exp(-r)}{4r^2}K\left(-\sinh^2(\rho)\right)\\ &+\frac{\exp(-r)}{4\pi r^3}\left(-\left(\frac{\pi^4}{12}+\pi^2\right)+\left(\frac{\pi^4}{48}-\frac{\pi^2}{2}\right)\rho^2+\left(\frac{25\pi^2}{192}-\frac{11\pi^4}{2304}\right)\rho^4\right)\ ,\end{aligned} \tag{B.4.5}$$

up to corrections of order $\mathcal{O}\left(\frac{\exp(-r)}{r^4}\right)$ or $\mathcal{O}\left(\rho^5\right)$. We find a similar solution when we assume $\tilde{G}$ is given in an expansion in $\exp(-r)r^{-n}f_n(\rho)$ with minimum $n = 2$ and solve equation (B.1.1) in the large r limit order by order, using the same technique as for finding the large R expansion. Unfortunately the first sourced order (f_3, the coefficient of $\exp(-r)r^{-3}$) cannot be solved analytically except for the case when $\rho \ll 1$.

B.5 The $\rho \gg 1$ Expansion

When $\rho \gg 1$ we may approximate our differential equation as

$$\left(\partial_r^2 + \frac{2}{r}\partial_r + \partial_\rho^2 + 2\partial_\rho\right)\tilde{G}(r,\rho) = 0 \ . \tag{B.5.1}$$

To simplify, we change variables to

$$\tilde{G}(r,\rho) = \frac{\exp\left(-r-\rho\right)}{r}U(r,\rho) \ . \tag{B.5.2}$$

We can further simplify our differential equation by multiplying by $r\exp(r+\rho)$. We find f satisfies

$$\left(\partial_r^2 + \partial_\rho^2 - 1\right)U(r,\rho) = 0 \ . \tag{B.5.3}$$

Unfortunately we cannot translate our boundary conditions onto any condition on this U, other than that it must not grow exponentially fast as $\rho \to \infty$ of $r \to \infty$. We may, however, suppose the ansatz that it has a Laurent series in r starting with $\frac{1}{r}$. Given that choice and using the same technique as for small R we find

$$U(r,\rho) = \frac{a_1\rho + b_1}{r} + \frac{-\frac{a_1}{3}\rho^3 - b_1\rho^2 + a_2\rho + b_2}{r^2} + \mathcal{O}\left(\frac{1}{r^3}\right) \ . \tag{B.5.4}$$

Returning to the actual solution given in integral form we may approximate the Legendre function when $\rho \gg 1$ as above:

$$\mathcal{P}_\nu\left(\cosh(2\rho)\right) \sim \frac{1}{2}\left(B\left(\nu+\frac{1}{2},\frac{1}{2}\right)^{-1}\exp(2\rho)^{-1-\nu} + B\left(\mu+\frac{1}{2},\frac{1}{2}\right)^{-1}\exp(2\rho)^{-1-\mu}\right) \ , \tag{B.5.5}$$

where ν and μ are complex conjugates given that the real part of $\nu = -\frac{1}{2}$. Using the mirror symmetry of gamma functions ($\Gamma(z^*) = \Gamma(z)^*$) we may identify $B(\nu+\frac{1}{2},\frac{1}{2})^{-1} = m\exp(i\delta)$ for some real variables m and δ. Given our $\nu = -\frac{1}{2} + i\frac{\sigma}{2}$, we have

$$\mathcal{P}_\nu\left(\cosh(2\rho)\right) \sim \exp\left(-\rho\right)m\frac{1}{2}\left(\exp(i\delta)\exp(-i\sigma\rho) + \exp(-i\delta)\exp(i\sigma\rho)\right) \ . \tag{B.5.6}$$

Simplifying, we find

$$\mathcal{P}_{-\frac{1}{2}+i\frac{\sigma}{2}}\left(\cosh(2\rho)\right) \sim \exp\left(-\rho\right)m\cos\left(\rho\sigma - \delta\left(\sigma\right)\right) \ . \tag{B.5.7}$$

Since we require an expansion of this quantity in ω we may no longer ignore the frequency shift, δ. The formulae for m and δ are

$$m = \sqrt{\frac{\Gamma\left(\frac{1}{2}\right)^2 \Gamma\left(-\frac{i\sigma}{2}\right)\Gamma\left(\frac{i\sigma}{2}\right)}{\Gamma\left(\frac{1}{2}-\frac{i\sigma}{2}\right)\Gamma\left(\frac{i\sigma}{2}+\frac{1}{2}\right)}}\,, \qquad \delta = \arctan\left(\frac{\mathrm{Im}\left\{B(-i\frac{\sigma}{2},\frac{1}{2})\right\}}{\mathrm{Re}\left\{B(-i\frac{\sigma}{2},\frac{1}{2})\right\}}\right)\,, \tag{B.5.8}$$

where we may use the reflection formula for gamma functions to find an exact value for the first and a Taylor expansion for the second:

$$m = \sqrt{\frac{2\pi}{\sigma\tanh\left(\frac{\pi}{2}\sigma\right)}}\,, \qquad \delta = \frac{\pi}{2} - \frac{1}{2}\left(\psi_0(1) - \psi_0\left(\frac{1}{2}\right)\right)\sigma + \frac{1}{48}\left(\psi_2(1) - \psi_2\left(\frac{1}{2}\right)\right) + \mathcal{O}(\sigma^3)\,. \tag{B.5.9}$$

Inserting (B.5.9) into (B.5.7) then into (B.1.15), we must then change coordinates from $\sigma = \sqrt{\omega^2-1}$ to ω, then $\omega = \tilde{\omega} - 1$ to $\tilde{\omega}$. We may then expand the integrand (save $\exp(-\tilde{\omega}r)$) as a Taylor series in $\tilde{\omega}$. This becomes

$$\begin{aligned}\tilde{G}(r,\rho) = \frac{\exp(-r-\rho)}{4\pi r}\int_0^\infty \exp(-\tilde{\omega}r)\Bigg(&\frac{\pi^2\left(4\rho+\log(16)\right)}{4\sqrt{2}} \\ &-\frac{\pi^2\left((\rho+\log(2))\left(4\left(\rho(\rho+\log(4))-6+\log^2(2)\right)+\pi^2\right)+6\zeta(3)\right)}{12\sqrt{2}}\tilde{\omega} + \mathcal{O}\left(\tilde{\omega}^2\right)\Bigg)d\tilde{\omega}\,.\end{aligned} \tag{B.5.10}$$

From this we approximate

$$\begin{aligned}\tilde{G}(r,\rho) = \frac{\exp(-r-\rho)}{4\pi r}\Bigg(&\frac{\pi^2(\rho+\log(2))}{\sqrt{2}r} \\ &-\frac{\pi^2\left((\rho+\log(2))\left(4\left(\rho(\rho+\log(4))-6+\log^2(2)\right)+\pi^2\right)+6\zeta(3)\right)}{12\sqrt{2}r^2} + \mathcal{O}\left(\frac{1}{r^3}\right)\Bigg)\,,\end{aligned} \tag{B.5.11}$$

which we see obeys the expansion that we derived previously as the most general solution.

C Fermi-Normal Coordinates and Self-Asymptotic Problems

C.1 A Five-Dimensional Cognate Sigma Model

We note that the geodesic equations ((5.5.1) and (5.5.2)) are associated with a sigma model in five dimensions

$$\mathcal{S} = -\frac{1}{2}\int d^5X\sqrt{g}g^{\mu\nu}\partial_\mu F\partial_\nu F \ . \tag{C.1.1}$$

Here, our background line element takes the form

$$ds^2 = \sinh^\beta(2\rho)\left(dx^2 + dy^2 + dz^2 + d\rho^2 + \sinh^{2-2\beta}(2\rho)d\chi\right) \ . \tag{C.1.2}$$

One obvious geodesic that includes the origin ($x = y = z = \rho = 0$) is given by

$$X_l^\mu = (v_x t,\ v_y t,\ v_z t,\ v\rho t,\ \chi_0)^\mu \ . \tag{C.1.3}$$

When $\beta = 0$ and $\rho \gg 1$ this becomes approximately

$$ds^2 = dx^2 + dy^2 + dz^2 + d\rho^2 + \exp(4\rho)d\chi^2 \ . \tag{C.1.4}$$

Where we have rescaled χ by a factor of two. The associated geodesic equations are

$$x''(t) = 0\ , \quad y''(t) = 0\ , \quad z''(t) = 0\ , \quad \rho''(t) - 2\exp\left(4\rho(t)\right)\left(\chi'(t)\right)^2 = 0\ , \quad \chi''(t) + 4\rho'(t)\chi'(t) = 0\ . \tag{C.1.5}$$

The most general solution with non trivial time dependence of χ to the geodesic equation is

$$X_t^\mu = \left(v_x t + x_0,\ v_y t + y_0,\ v_z t + z_0,\ \log(2) - \frac{1}{4}\log\left(c^2\text{sech}^2\left(\frac{c}{2}t + k\right)\right),\ \chi_0 + \frac{c}{8}\tanh\left(\frac{c}{2}t + k\right)\right)^\mu \ . \tag{C.1.6}$$

This geodesic has the property that $\rho'(t) = 0$ when $k = 0$, which means at any arbitrary point in our manifold, the metric that passes through that point and the origin can be given by some X_l^μ. Then, when $\rho \gg 1$, the space of orthogonal geodesics is spanned by some collection of geodesics described by X_t^μ, generically with $\rho = \rho_0$ and $\chi = \chi_0$, or with $v_x = v_y = v_z = 0$ and $k = 0$.

C.2 Asymptotic Fermi Normal Coordinates

Before diving into what the Fermi normal coordinates [88] are for the full system we will derive them in a less specialized case. We will first derive the asymptotic case of the sigma model currently under consideration, then for the point of clarity, show how some solutions to the asymptotic problem are solutions to the original problem. Then, for the sake of completeness, we will derive the Fermi normal coordinates to show how the correspondence was discovered.

The asymptotic line element when $\beta = 0$ (and rescaling $\chi \to \frac{1}{2}\chi$) is

$$ds_A{}^2 = ds_3{}^2 + d\rho^2 + \exp(4\rho)d\chi^2 \ . \tag{C.2.1}$$

The associated Laplacian is

$$\Delta_A = \Delta_3 + \partial_\rho{}^2 + 2\ \partial_\rho + \exp(-4\rho)\partial_\chi{}^2 \ . \tag{C.2.2}$$

We note the known χ independent fundamental solutions by first considering

$$\exp(\rho)\Delta_A\left[\exp(-\rho)f(r,\rho)\right] = \left(\Delta_3 + \partial_\rho{}^2 - 1\right)f(r,\rho) \ . \tag{C.2.3}$$

That is, when we multiply our solution by an exponential we find the operator on our remaining functional dependence is the Helmholtz operator in four dimensions with $k = 1$ [61]. The fundamental solution to the Helmholtz operator in $n > 3$ dimensions is

$$\left(\Delta_n - 1\right)f(R) = \frac{\delta(R)}{V_{n-1}R^{n-1}} \quad \Rightarrow \quad f(R) = -\frac{1}{(2\pi)^{\frac{n}{2}}R^{\frac{n}{2}-1}}K_{\frac{n}{2}-1}\left(R\right) \ . \tag{C.2.4}$$

Where V_{n-1} is the volume of the unit $n-1$ sphere and K_α is the modified Bessel function of the second kind with order, also known as the MacDonald function. So our problem has the χ independent fundamental solution

$$\Delta_A F(r,\rho) = \frac{\delta(r)\delta(\rho)}{V_2 r^2 \exp 2\rho} \quad \Rightarrow \quad F(r,\rho) = -\frac{\exp(-\rho)}{4\pi^2\sqrt{r^2+\rho^2}}K_1\left(\sqrt{r^2+\rho^2}\right) \ . \tag{C.2.5}$$

However, if we consider a specific mixture of χ dependence, we make the following observation:

$$\Delta_A F(r, \frac{1}{2}\exp(2\rho)\chi) = \left(\Delta_3 + 4\partial_A\left(1 + A^2\right)\partial_A\right)F(r,A) \ . \tag{C.2.6}$$

Here we defined $A = \frac{1}{2}\exp(2\rho)\chi$. Alternately we can consider an analytic completion of a variable transformation in the original problem

$$\left(\Delta_3 + {\partial_\rho}^2 + 2\coth(2\rho)\partial_\rho\right) F(r, i\cosh(2\rho)) = \left(\Delta_3 + 4\partial_B \left(1+B^2\right)\partial_B\right) F(r,B) \ . \tag{C.2.7}$$

Here we defined $B = i\cosh(2\rho)$. So we see that some solutions to the asymptotic problem constitute solutions to the original problem.

To see why this happens, we will now finally derive the Fermi normal coordinates. The procedure for deriving Fermi normal coordinates is a five step procedure. First, find the first set of 'trunk' geodesics and value of the vielbeins along it. Second, find a family of linearly independent intersecting 'arm' geodesics (name chosen for similarity to saguaro anatomy). Third, orthogonalize and solve the 'arm' parameters so that they intersect the 'trunks.' Fourth, define the Fermi normal coordinates and solve for the parameters of 'arm' geodesics. Finally, substitute the Fermi normal coordinates into the expressions for the geodesics, to define the coordinate transformation.

Let us see this in action. We will begin by focusing only on finding Fermi normal coordinates for the ρ, χ subspace. We begin by noting lines of constant χ always form geodesics, so we choose our 'trunk' geodesics to be

$$\tau(T)^\mu = (0,0,0,T,\chi_0)^\mu \ . \tag{C.2.8}$$

Fortunately our vielbeins are parallelly transported trivially along these geodesics, so the first step is complete.

Second, we must solve the geodesic equation in our background with non-zero derivative in the χ direction. Considering the following ansatz

$$\alpha(\sigma)^\mu = (0,0,0,f(\sigma),g(\sigma))^\mu \ , \tag{C.2.9}$$

the geodesic equation is

$$\frac{d^2\alpha^\mu}{d\sigma^2} + \Gamma^\mu_{\theta\phi}\frac{d\alpha^\theta}{d\sigma}\frac{d\alpha^\phi}{d\sigma} = \Big(0,\ 0,\ 0,\ f''(\sigma) - 2\exp\left(4f(\sigma)\right)\left(g'(\sigma)\right)^2,\ g''(\sigma) + 4f'(\sigma)g'(\sigma)\Big)^\mu = 0\ . \tag{C.2.10}$$

We note the following solution for $f'(\sigma)$ given the χ component of the geodesic equation

$$f'(\sigma) = -\frac{1}{4}\frac{g''(\sigma)}{g'(\sigma)} \quad \Rightarrow \quad f(\sigma) = -\frac{1}{4}\log\left(g'(\sigma)\right) + C\ . \tag{C.2.11}$$

Actually, we have discovered a general property of metrics of the form

$$ds^2 = {ds_W}^2 + d\rho^2 + A(\rho)d\chi^2 \ . \tag{C.2.12}$$

The ρ (f), and χ (g) components of the geodesics (other than those of constant χ) are always related as

$$f(\sigma) = -A^{-1}\left(g'(\sigma)\right) + C \ . \tag{C.2.13}$$

We can leverage this later to find geodesics using the exact metric. For simplicity we choose f as given above with $C = 0$ and the remaining component of our geodesic equation becomes

$$-\frac{1}{4}\frac{g'''(\sigma)}{g'(\sigma)} + \frac{1}{4}\left(\frac{g''(\sigma)}{g'(\sigma)}\right)^2 - 2g'(\sigma) = 0 \ . \tag{C.2.14}$$

This third-order ODE has the solutions

$$g(\sigma) = \frac{1}{4}c\tanh(c\sigma + k) + l \ . \tag{C.2.15}$$

This concludes the second step.

The orthogonality condition

$$g_{\mu\nu}\frac{d\alpha(\sigma)^\mu}{d\sigma}\frac{d\tau(T)^\mu}{dT} = 0 \ , \tag{C.2.16}$$

is satisfied when $\frac{d}{d\sigma}f(\sigma)|_{\sigma=0} = 0$. This requires

$$f'(\sigma) = \frac{1}{2}c\tanh(c\sigma + k)\bigg|_{\sigma=0} = 0 \ . \tag{C.2.17}$$

This may be satisfied if either $c = 0$ or $\tanh(k) = 0$. The former corresponds to σ independent solutions, and is therefore not the case of interest at the moment. Changing the value k given that $\tanh(k) = 0$ is equivalent to changing the value of C, so we consider $k = 0$ for convenience. Next we require these geodesics intersect at some point specified by the first parameter T when $\sigma = 0$ therefore we require

$$f(0) = -\frac{1}{2}\log\left(\frac{1}{2}c\,\mathrm{sech}(c\cdot 0)\right) = T \quad \Rightarrow \quad c = 2\exp(-2T) \ . \tag{C.2.18}$$

We therefore have

$$\alpha^\mu(\sigma) = \left(0,\ 0,\ 0,\ T - \frac{1}{2}\log\left(\mathrm{sech}(2\exp(-2T)\sigma)\right),\ \frac{1}{2}\exp(-2T)\tanh\left(2\exp(-2T)\sigma\right) + \chi_0\right)^\mu \ . \tag{C.2.19}$$

Which concludes the third step.

Fermi normal coordinates are T, which is the affine parameter of the distance you must travel along a 'trunk' geodesic, and X, which is linearly related to the affine parameter of the distance you must travel along the 'arm' geodesic. Consider the following image, which represents the coordinates in our spacetime:

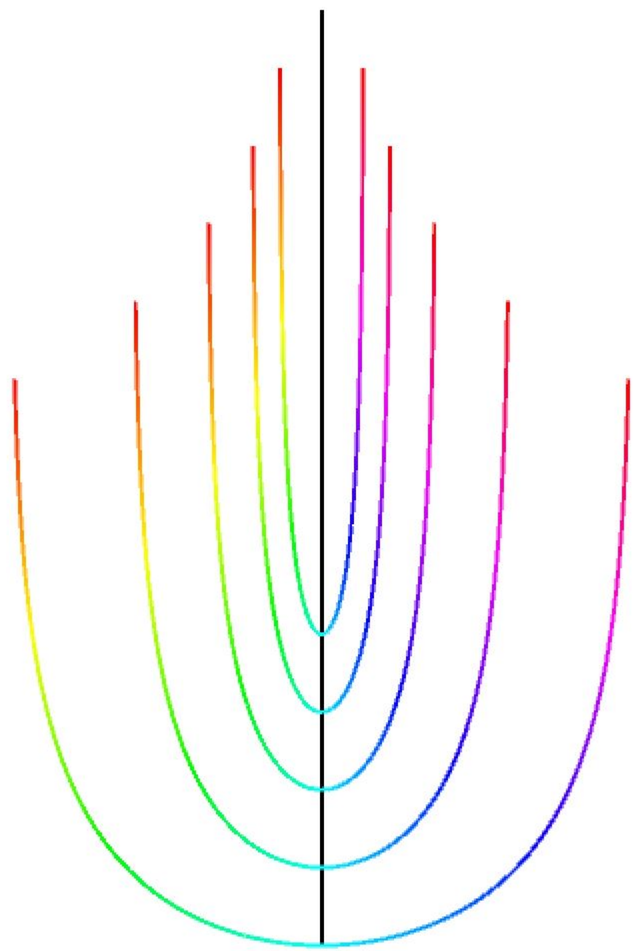


Figure 10: An illustration of the geodesics of the SS–CGP spacetime

The 'trunk' is drawn in black, distance along that trunk selects the colored lines drawn with equal spacing in the T direction. The color along that line represents the X coordinate. To get to any point you travel T up the 'trunk' until you find the correct 'arm' then travel out the 'arm' X until you arrive at your destination. We have already given T by definition in our 'trunk' geodesic. The equation for X is

$$X^i = \sigma \left(\frac{d\alpha(\sigma)^\mu}{d\sigma} e_\mu{}^i(\alpha(\sigma)) \right) \Bigg|_{\sigma=0} . \tag{C.2.20}$$

This allows for, in principle, many such geodesics indexed by an additional parameter i. In our case we have only one 'arm' parameter which gives us

$$X = \sigma \frac{c^2}{4} \exp(2T) \quad \Rightarrow \quad \sigma = 2\exp(2T)\, X \ . \tag{C.2.21}$$

This concludes the fourth step.

With all of our ingredients assembled we write the value of ρ and χ in terms of the arm geodesics with their parameters written in terms of the Fermi normal coordinates. That is

$$\rho = f(\sigma) = T - \frac{1}{2}\log\left(\operatorname{sech}(2X)\right) \ , \tag{C.2.22}$$

$$\chi = g(\sigma) = -\frac{1}{2}\exp(-2T)\tanh(2X) \ . \tag{C.2.23}$$

For record keeping we also include the inverse transformations

$$T = \rho - \frac{1}{4}\log\left(4\chi^2\exp(4\rho) + 1\right) \ , \tag{C.2.24}$$

$$X = \frac{1}{2}\operatorname{arcsinh}\left(-2\chi\exp(2\rho)\right) \ . \tag{C.2.25}$$

We can now substitute these into our asymptotic metric to find

$$ds_A{}^2 = ds_3{}^2 + \frac{1}{2}\left(1 + \cosh(4X)\right) dT^2 + dX^2 \ . \tag{C.2.26}$$

This has the associated Laplacian

$$\Delta_A = \left(\Delta_3 + \operatorname{sech}^2(2X)\partial_T{}^2 + \partial_X{}^2 + 2\tanh(2X)\partial_X\right) \ . \tag{C.2.27}$$

This becomes, after a shift into complex X and T

$$\Delta_A F(r, iT, X + \frac{i\pi}{4}) = \left(\Delta_3 + \operatorname{csch}^2(2Y)\partial_S{}^2 + \partial_Y{}^2 + 2\coth(2Y)\partial_Y\right) F(r, S, Y) \ . \tag{C.2.28}$$

Here we have chosen $Y = X + \frac{i\pi}{4}$ and $S = iT$. Therefore we discover that χ independent solutions to the original problem are T independent solutions to the asymptotic problem.

C.3 Exact Fermi Normal Coordinates

Now that we have seen the process once, let us repeat this procedure for the full metric. Given a non-constant $\rho = f(\sigma)$ and $\chi = g(\sigma)$ dependence our χ geodesic equation becomes

$$2f'(\sigma)\tanh(f(\sigma))g'(\sigma) + 2f'(\sigma)\coth(f(\sigma))g'(\sigma) + g''(\sigma) = 0 \,. \tag{C.3.1}$$

We notice all terms including f are paired with a factor of g', therefore we consider the ansatz

$$f(\sigma) \to \eta(g'(\sigma)) \,. \tag{C.3.2}$$

Our χ geodesic equation becomes

$$4g'(\sigma)g''(\sigma)\eta'\left(g'(\sigma)\right)\coth\left(2\eta\left(g'(\sigma)\right)\right) + \eta''(\sigma) = 4x\eta'(x)\coth(2\eta(x))g''(t) + g''(t) = 0 \,, \tag{C.3.3}$$

where we have made the substitution $g'(\sigma) = x$. Dividing by the common factor of g' we find

$$\eta(x) = \frac{1}{2}\text{arcsinh}\left(\frac{A}{\sqrt{x}}\right) \quad \Rightarrow \quad f(\sigma) = \frac{1}{2}\text{arcsinh}\left(\frac{A}{\sqrt{g'(\sigma)}}\right) \,. \tag{C.3.4}$$

Given this substitution the ρ geodesic equation becomes

$$\frac{A\left(16A^4g'(\sigma)^3 - 2A^2g''(\sigma)^2 + 32A^2g'(\sigma)^4 + 2g^{(3)}(\sigma)g'(\sigma)\left(A^2 + g'(\sigma)\right) + 16g'(\sigma)^5 - 3g'(\sigma)g''(\sigma)^2\right)}{\sqrt{g'(\sigma)}\sqrt{\frac{A^2}{g'(\sigma)} + 1}\left(A^2 + g'(\sigma)\right)} = 0 \,. \tag{C.3.5}$$

One solution is

$$g(\sigma) = D - \frac{1}{2}\arctan\left(\frac{-16A^3B + i\left(16A^2 + B^2\right)^2 e^{AB(C+\sigma)} + AB^3}{8A^2B^2}\right) \,. \tag{C.3.6}$$

We now build our solutions by imposing $f(0) = T$, $f'(0) = 0$, $g(0) = 0$, $\sigma g'(0)e^5_\chi = X$ and setting all remaining constants (there is one more that corresponds to rescalings of T) to unity and find our geodesics are given in terms of the Fermi normal coordinates as

$$\rho(T,\ X) = \frac{1}{2}\text{arcsinh}\left(\frac{1}{\sqrt{2}}\sqrt{\cosh(4X)\cosh^2(2T) + \sinh^2(2T) - 1}\right) \,, \tag{C.3.7}$$

$$\chi(T,\ X) = \frac{1}{2}\left(\text{gd}(2T) - \text{arccot}\left(\frac{2\sinh(2T)}{e^{-4X}\cosh^2(2T) + \sinh^2(2T) - 1}\right)\right) \,. \tag{C.3.8}$$

Here gd is the Gudermannian function. under this substitution our metric becomes

$$ds^2 = ds_3{}^2 + \frac{1}{2}\left(1 + \cosh(4X)\right) dT^2 + dX^2\ . \tag{C.3.9}$$

So we see the exact correspondence between the problem and its asymptote through Fermi normal coordinates.

D Overlap Integrals of Multiple Legendre Functions

D.1 Outline

In the following Appendix the solution to the integral (over both $(-1,1)$ and $(1,\infty)$) of n–many Legendre functions is derived. The general form of the Integral of n many Legendre functions over $(-1,1)$ is given recursively by

$$\int_{-1}^{1} P_{\mu_1}(x)\cdots P_{\mu_m}(x)Q_{\nu_1}(x)\cdots Q_{\nu_n}(x)dx = I^{\mu_1\cdots\mu_m}_{\nu_1\cdots\nu_n} \ , \tag{D.1.1}$$

(with certain conditions on μ_i and ν_j given below), where $I^{\mu_1\cdots\mu_m}_{\nu_1\cdots\nu_n}$ may be defined as

$$I^{\mu_1\cdots\mu_m}_{\nu_1\cdots\nu_n} = \sum_{\substack{i_1,\cdots,i_m=0\\ j_1,\cdots,j_m=0}}^{\infty} \frac{2i_1+1}{2} I^{i_1\mu_1}\cdots\frac{2i_m+1}{2}I^{i_m\mu_m}\frac{2j_1+1}{2}I^{j_1}_{\nu_1}\cdots\frac{2j_n+1}{2}I^{j_n}_{\nu_n}I^{i_1\cdots i_m j_1\cdots j_n} \ , \tag{D.1.2}$$

and $I^{ijk_1\cdots k_n}$ with integral indices may be reduced by

$$I^{ijk_1\cdots k_n} = \sum_{k=|i-j|}^{i+j} \begin{pmatrix} i & j & k \\ 0 & 0 & 0 \end{pmatrix}^2 I^{kk_1\cdots k_2} \ . \tag{D.1.3}$$

$I^{\mu_1\mu_2}$, I^{μ}_{ν}, and $I_{\nu_1\nu_2}$ with complex arguments are given by

$$I^{\mu_1\mu_2} = \frac{2\pi\sin(\pi(\mu_1-\mu_2)) - 4\sin(\pi\mu_1)\sin(\pi\mu_2)\left(\mathcal{H}_{\mu_1}-\mathcal{H}_{\mu_2}\right)}{\pi^2(\mu_1-\mu_2)(\mu_1+\mu_2+1)} \ , \tag{D.1.4}$$

$$I^{\mu}_{\nu} = \frac{1-\cos(\pi(\mu-\nu)) - 2\pi^{-1}\sin(\pi\mu)\cos(\pi\nu)\left(\mathcal{H}_{\mu}-\mathcal{H}_{\nu}\right)}{(\mu-\nu)(\mu+\nu+1)} \ , \tag{D.1.5}$$

$$I_{\nu_1\nu_2} = \frac{\frac{\pi}{2}\sin(\pi(\nu_1-\nu_2)) - (1+\cos(\pi\nu_1)\cos(\pi\nu_2))\left(\mathcal{H}_{\nu_1}-\mathcal{H}_{\nu_2}\right)}{(\nu_1-\nu_2)(\nu_1+\nu_2+1)} \ , \tag{D.1.6}$$

where $\mathcal{H}_\nu$ is the harmonic number $\mathcal{H}_{nu} = \psi(\nu+1)+\gamma$ where $\gamma = \psi(1)$ is the Euler–Mascheroni constant and $\begin{pmatrix} i & j & k \\ m & n & p \end{pmatrix}$ are the Clebsch–Gordan coefficients and $I^{m_1m_2}$, I^m_n, and $I_{n_1n_2}$ for integral arguments are

$$I^{m_1m_2} = \frac{2}{2m_1+1}\delta_{m_1m_2} \ , \tag{D.1.7}$$

$$I^m_n = \frac{1-(-1)^{m+n}}{(m-n)(m+n+1)} \ , \tag{D.1.8}$$

$$I_{n_1n_2} = \frac{(-1+(-1)^{n_1+n_2})(\mathcal{H}_{n_1}-\mathcal{H}_{n_2})}{(n_1-n_2)(n_1+n_2+1)} \ . \tag{D.1.9}$$

The general form of the integral of n many Legendre functions over $(1, \infty)$ is given recursively by

$$\int_1^\infty \mathcal{P}_{\mu_1}(z) \cdots \mathcal{P}_{\mu_m}(z) \mathcal{Q}_{\nu_1}(z) \cdots \mathcal{Q}_{\nu_n}(z) dz = J^{\mu_1 \cdots \mu_m}_{\nu_1 \cdots \nu_n} \ , \tag{D.1.10}$$

(with certain conditions on μ_i and ν_j given below) where $J^{\mu_1 \cdots \mu_m}_{\nu_1 \cdots \nu_n}$ satisfies the equation

$$\begin{gathered}
J^{\mu_1 \cdots \mu_m}_{\nu_1 \cdots \nu_n} \\
+ I^{\mu_1 \cdots \mu_m}_{\nu_1 \cdots \nu_n} + \left(-\frac{i\pi}{2}\right) \sum_{i=1}^{n} I^{\nu_i \mu_1 \cdots \mu_m}_{\nu_1 \cdots \cancel{\nu_i} \cdots \nu_n} \\
+ \left(-\frac{i\pi}{2}\right)^2 \sum_{i=1}^{n-1} \sum_{j=i+1}^{n} I^{\nu_i \nu_j \mu_1 \cdots \mu_m}_{\nu_1 \cdots \cancel{\nu_i} \cdots \cancel{\nu_j} \cdots \nu_n} + \cdots + \left(-\frac{i\pi}{2}\right)^n I^{\nu_1 \cdots \nu_n \mu_1 \cdots \mu_m} \\
+ (-1)^n e^{i\pi(\mu_1 + \cdots + \mu_m - \nu_1 - \cdots - \nu_n)} J^{\mu_1 \cdots \mu_m}_{\nu_1 \cdots \nu_n} \\
+ (-1)^n \left(-\frac{2}{\pi}\right) \sum_{i=1}^{m} e^{i\pi(\mu_1 + \cdots + \cancel{\mu_i} + \cdots + \mu_m - \nu_1 - \cdots - \nu_n)} \sin(\pi\mu_i) J^{\mu_1 \cdots \cancel{\mu_i} \cdots \mu_n}_{\mu_i \nu_i \cdots \nu_n} \\
+ (-1)^n \left(-\frac{2}{\pi}\right)^2 \sum_{i=1}^{m-1} \sum_{j=i+1}^{m} e^{i\pi(\mu_1 + \cdots + \cancel{\mu_i} + \cdots + \cancel{\mu_j} + \cdots + \mu_m - \nu_1 - \cdots - \nu_n)} \times \\
\sin(\pi\mu_i) \sin(\pi\mu_j) J^{\mu_1 \cdots \cancel{\mu_i} \cdots \cancel{\mu_j} \cdots \mu_n}_{\mu_i \mu_j \nu_i \cdots \nu_n} \\
+ \cdots + (-1)^n \left(-\frac{2}{\pi}\right)^m e^{i\pi(-\nu_1 - \cdots - \nu_n} \prod_{i=1}^{m} \sin(\pi\mu_i) J_{\mu_1 \cdots \mu_m \nu_1 \cdots \nu_n} \\
= 0 \ .
\end{gathered} \tag{D.1.11}$$

This implies that a solution to such an integral over $(1, \infty)$ involving m Legendre functions of first type and n Legendre functions of second type requires the solution to m integrals over $(1, \infty)$ involving $m-1$ Legendre functions of first type and $n+1$ Legendre functions of second type, and $\binom{m}{2}$ integrals over $(1, \infty)$ of $m-2$ Legendre functions of first type and $n+2$ Legendre fuctions of second type, and so on. The m integrals over $(1, \infty)$ involving $m-1$ Legendre functions of first type, in principle involve $m-1$ integrals over $(1, \infty)$ involving $m-2$ Legendre functions of first type, however, each of these terms is duplicated in the previous expansion. Additionaly thre are n^2 integrals over $(-1, 1)$ which need be done to solve any integral involving n Legendre functions of second type, however, these are already given in terms of a series. This problem does terminate and the answer can be given in terms of roughly $m^2 n^2$ series.

Calculating integrals of $I^{\mu_1 \mu_2}$, I^{μ}_{ν}, or $I_{\nu_1 \nu_2}$ type is accomplished through the use of the self-adjoint properties of the Legendre differential equation. Further integrals of $I^{\mu \cdots}_{\nu \cdots}$ type are done through use of the orthogonal system which Legendre polynomials themselves provide, combined with utilization of Clebsch–Gordon coefficients, relating these solutions back to solutions involving two Legendre polynomials. Finally, calculating integrals of the $J^{\mu\cdots}_{\nu\cdots}$ type is done via using opportunistic contour integration, relating different segments of the contour integral to the original integral and previously known integrals using the analytic completion of Legendre functions.

D.2 Introduction to Legendre Functions, Analytic Continuation

The Legendre differential equation [61]

$$\frac{d}{dz}\left[\left(1-z^2\right)\frac{d}{dz}\left[y\left(z\right)\right]\right]+\nu\left(\nu+1\right)y\left(z\right)=0\ , \tag{D.2.1}$$

is a second–order differential equation with regular singular points at -1, 1, and ∞, (i.e. of Fuchsian type). Its solutions may be written in terms of the Riemann P–symbol, or hypergeometric functions, which will converge about ± 1 or ∞. In general the singular points of the above differential equation are branch points of the solutions. Depending on whether a solution on $(-1,1)\ni x$ or $(1,\infty)\ni z$ is desired, Legendre functions of the first and second type are denoted

$$P_\nu(x) \text{ and } Q_\nu(x)\ , \tag{D.2.2}$$

$$\text{or} \tag{D.2.3}$$

$$\mathcal{P}_\nu(z) \text{ and } \mathcal{Q}_\nu(z)\ , \tag{D.2.4}$$

which are linearly independent within each pair and related via analytic continuation between each pair. Further, for real $\nu(\nu+1)$ where ν is complex, Q_ν and $Q_{\overline{\nu}}$ are also linearly independent. To calculate the above integrals, only their asymptotic behavior and the relation between different branches is necessary. These identities are readily available in many sources which differ, unfortunately, in convention.

Here we use the following conventions. First, relating Legendre functions which have principle branch $(1,\infty)$ to themselves, the value of Legendre functions after following a contour which has winding number $\frac{1}{2}$ about 1 and -1 ($\mathrm{Arg}\left\{z\right\}<\pi$ and $|z|>1$)

$$\mathcal{P}_\nu\left(ze^{i\pi}\right) = e^{i\pi\nu}\mathcal{P}_\nu(z)-\frac{2}{\pi}\sin(\pi\nu)\mathcal{Q}_\nu(z)\ , \tag{D.2.5}$$

$$\mathcal{Q}_\nu\left(ze^{i\pi}\right) = -\,e^{-i\pi\nu}\mathcal{Q}_\nu(z)\ . \tag{D.2.6}$$

Next, relating Legendre functions with principle branch $(1, \infty)$ to Legendre functions on $(-1, 1)$, the value of Legendre functions after following a contour which has winding number $\frac{1}{2}$ about 1 and 0 about -1

$$\mathcal{P}_\nu(x) = P_\nu(x) \ , \tag{D.2.7}$$

$$\mathcal{Q}_\nu(x) = Q_\nu(x) - \frac{i\pi}{2} P_\nu(x) \ . \tag{D.2.8}$$

Calculating these integrals also requires knowledge of the asymptotic behavior of these functions, specifically as $z \to 1$

$$\mathcal{P}_\nu(z) \sim 1 \ , \tag{D.2.9}$$

$$\mathcal{Q}_\nu(z) \sim \frac{1}{2} \log\left(\frac{z+1}{z-1}\right) \ , \tag{D.2.10}$$

for $\nu \notin \{-1, -2, \cdots\}$. Similarly, as $z \to \infty$

$$\mathcal{P}_\nu(z) \sim \frac{\Gamma(\nu + 1/2)}{\pi^{\frac{1}{2}}\Gamma(\nu + 1)}(2z)^\nu \ , \tag{D.2.11}$$

$$\mathcal{P}_\nu(z) \sim \frac{\Gamma(-\nu - 1/2)}{\pi^{\frac{1}{2}}\Gamma(-\nu)(2z)^{\nu+1}} \ , \tag{D.2.12}$$

$$\mathcal{Q}_\nu(z) \sim \frac{\pi^{\frac{1}{2}}\Gamma(\nu + 1)}{\Gamma(\nu + \frac{3}{2})(2z)^{\nu+1}} \ , \tag{D.2.13}$$

for $\Re\{\nu\} > -\frac{1}{2}$ or $\Re\{\nu\} < -\frac{1}{2}$, respectively for the first and second formulae for Legendre functions of the first type and for $\nu \notin \{-1, -\frac{3}{2}, -2, -\frac{5}{2}, \cdots\}$ for Legendre functions of second type. Finally, for $x \to 1^-$

$$P_\nu(x) \sim 1 \ , \tag{D.2.14}$$

$$Q_\nu(x) \sim \frac{1}{2} \log\left(\frac{1+x}{1-x}\right) - \mathcal{H}_\nu \ , \tag{D.2.15}$$

for $\nu \notin \{-1, -2, \cdots\}$, and for $x \to -1^+$ the formulae

$$P_\nu(-x) = -\frac{2}{\pi} \sin(\pi\nu) Q_\nu(x) + \cos(\pi\nu) P_\nu(x) \ , \tag{D.2.16}$$

$$Q_\nu(-x) = -\frac{\pi}{2} \sin(\pi\nu) P_\nu(x) - \cos(\pi\nu) Q_\nu(x) \ , \tag{D.2.17}$$

may be used. That is as $\xi \to -1^+$, $\xi = -x$, $x \to 1^-$

$$P_\nu(\xi) = P_\nu(-x) \sim -\frac{2}{\pi}\sin(\pi\nu)\left(\frac{1}{2}\log\left(\frac{1+x}{1-x}\right) - \mathcal{H}_\nu\right) + \cos(\pi\nu)\ , \tag{D.2.18}$$

$$Q_\nu(\xi) = Q_\nu(-x) \sim -\frac{\pi}{2}\sin(\pi\nu) - \cos(\pi\nu)\left(\frac{1}{2}\log\left(\frac{1+x}{1-x}\right) - \mathcal{H}_\nu\right)\ . \tag{D.2.19}$$

Finally, in the following analysis one specific recurrence relation will be essential

$$(1-x^2)\frac{d}{dx}\left[R_\nu(x)\right] = \frac{\nu(\nu+1)}{2\nu+1}\left(R_{\nu-1}(x) - R_{\nu+1}(x)\right)\ , \tag{D.2.20}$$

where R_ν is either solution to the Legendre differential equation.

D.3 $I^{\mu_1\mu_2}$, I^{μ}_{ν}, and $I_{\nu_1\nu_2}$

The integral of two solutions to an operator of Sturm–Liouville type, such as the Legendre differential operator, is a well studied problem. A simple example given here is, since

$$-\nu(\nu+1)R_\nu(x) = \frac{d}{dx}\left[(1-x^2)\frac{d}{dx}\left[R_\nu(x)\right]\right]\ , \tag{D.3.1}$$

where R_ν is a Legendre function of first or second type, the integral

$$\int_{\mathcal{D}} R_\mu(x)R_\nu(x)dx = \frac{-1}{\mu(\mu+1)}\int_{\mathcal{D}}\frac{d}{dx}\left[(1-x^2)\frac{d}{dx}\left[R_\mu(x)\right]\right]R_\nu dx \tag{D.3.2}$$

$$= \frac{-1}{\nu(\nu+1)}\int_{\mathcal{D}} R_\mu\frac{d}{dx}\left[(1-x^2)\frac{d}{dx}\left[R_\nu(x)\right]\right]dx\ , \tag{D.3.3}$$

therefore for $\mu \neq \nu$ the integral may be rewritten as

$$\left(\mu(\mu+1) - \nu(\nu+1)\right)\int_{\mathcal{D}} R_\mu(x)R_\nu(x)dx =$$
$$\int_D\left\{R_\mu(x)\frac{d}{dx}\left[(1-x^2)\frac{d}{dx}\left[R_\nu(x)\right]\right] - \frac{d}{dx}\left[(1-x^2)\frac{d}{dx}\left[R_\mu(x)\right]\right]R_\nu(x)\right\}dx\ , \tag{D.3.4}$$

which may then be partially evaluated using integration by parts

$$
\begin{aligned}
&(\mu(\mu+1)-\nu(\nu+1))\int_{\mathcal{D}} R_\mu(x)R_\nu(x)dx = \\
&\left\{(1-x^2)\left(R_\mu(x)\frac{d}{dx}\left[R_\nu(x)\right]-\frac{d}{dx}\left[R_\mu(x)\right]R_\nu(x)\right)\right\}\Bigg|_{\partial\mathcal{D}} \qquad \text{(D.3.5)}\\
&-\int_{\mathcal{D}}\left\{\frac{d}{dx}\left[R_\mu(x)\right](1-x^2)\frac{d}{dx}\left[R_\nu(x)\right]-(1-x^2)\frac{d}{dx}\left[R_\mu(x)\right]\frac{d}{dx}\left[R_\nu(x)\right]\right\}dx\ , \qquad \text{(D.3.6)}
\end{aligned}
$$

each term in the final integrand above cancels and the formula may be simplified

$$\int_{\mathcal{D}} R_\mu(x)R_\nu(x)dx = \frac{\left\{(1-x^2)\left(R_\mu(x)\frac{d}{dx}\left[R_\nu(x)\right]-\frac{d}{dx}\left[R_\mu(x)\right]R_\nu(x)\right)\right\}\Big|_{\partial\mathcal{D}}}{\mu(\mu+1)-\nu(\nu+1)}\ . \qquad \text{(D.3.7)}$$

If this limit can, in general, be evaluated, then the integral of $\int_{\mathcal{D}} R_\mu(x)R_\mu(x)dx$ may be identifiable with the limit as $\mu\to\nu$. Often, if $\mathcal{D}$ is non-compact, the solution may be identified with a Dirac delta–function. Therefore $I^{\mu_1\mu_2}$, I^μ_ν, and $I_{\nu_1\nu_2}$ are given by

$$I^{\mu_1\mu_2} = \frac{\left\{(1-x^2)\left(P_{\mu_1}(x)\frac{d}{dx}\left[P_{\mu_2}(x)\right]-\frac{d}{dx}\left[P_{\mu_1}(x)\right]P_{\mu_2}(x)\right)\right\}\Big|_{-1}^{1}}{\mu_1(\mu_1+1)-\mu_2(\mu_2+1)}\ , \qquad \text{(D.3.8)}$$

$$I^\mu_\nu = \frac{\left\{(1-x^2)\left(P_\mu(x)\frac{d}{dx}\left[Q_\nu(x)\right]-\frac{d}{dx}\left[P_\mu(x)\right]Q_\nu(x)\right)\right\}\Big|_{-1}^{1}}{\mu(\mu+1)-\nu(\nu+1)}\ , \qquad \text{(D.3.9)}$$

$$I_{\nu_1\nu_2} = \frac{\left\{(1-x^2)\left(Q_{\nu_1}(x)\frac{d}{dx}\left[Q_{\nu_2}(x)\right]-\frac{d}{dx}\left[Q_{\nu_1}(x)\right]Q_{\nu_2}(x)\right)\right\}\Big|_{-1}^{1}}{\nu_1(\nu_1+1)-\nu_2(\nu_2+1)}\ . \qquad \text{(D.3.10)}$$

Each of these boundary terms must be done in the limit, and the value of this integral is given in [61]

$$I^{\mu_1\mu_2} = \frac{2\pi\sin(\pi(\mu_1-\mu_2))-4\sin(\pi\mu_1)\sin(\pi\mu_2)\left(\mathcal{H}_{\mu_1}-\mathcal{H}_{\mu_2}\right)}{\pi^2(\mu_1-\mu_2)(\mu_2+\mu_1+1)}\ . \qquad \text{(D.3.11)}$$

Similarly,

$$I^\mu_\nu = \frac{1-\cos(\pi(\mu-\nu))-2\pi^{-1}\sin(\pi\mu)\cos(\pi\nu)\left(\mathcal{H}_\mu-\mathcal{H}_\nu\right)}{(\mu-\nu)(\mu+\nu+1)}\ , \qquad \text{(D.3.12)}$$

$$I_{\nu_1\nu_2} = \frac{\frac{\pi}{2}\sin(\pi(\nu_1-\nu_2))-(1+\cos(\pi\nu_1)\cos(\pi\nu_2))\left(\mathcal{H}_{\nu_1}-\mathcal{H}_{\nu_2}\right)}{(\nu_1-\nu_2)(\nu_2+\nu_1+1)}\ , \qquad \text{(D.3.13)}$$

D.4 $I^{\mu\cdots}_{\nu\cdots}$

Simplifying more complicated integrals of multiple Legendre functions of both types may be done by first considering $I^{\mu_1\mu_2\mu_3}$ where $\mu_i \in \mathbb{Z}^+$

$$I^{\mu_1\mu_2\mu_3} = \int_{-1}^{1} P_{\mu_1}(x)P_{\mu_2}(x)P_{\mu_3}(x)dx = 2\begin{pmatrix}\mu_1 & \mu_2 & \mu_3\\ 0 & 0 & 0\end{pmatrix}^2 . \tag{D.4.1}$$

To see this first use the expansion of Legendre polynomials in terms of Wigner $3j$ symbols [2]. This may be done pairwise on any two of the integral degree Legendre polynomials. Since the resultant sum is finite, and integration is linear, the sum and integral commute.

$$\begin{aligned} I^{\mu_1\mu_2\mu_3} &= \int_{-1}^{1} P_{\mu_1}(x)P_{\mu_2}(x)P_{\mu_3}(x)dx \\ &= \int_{-1}^{1} \sum_{j=|\mu_1-\mu_2}^{\mu 1+\mu 2} 2(2j+1)\begin{pmatrix}\mu_1 & \mu_2 & j\\ 0 & 0 & 0\end{pmatrix}^2 P_j(x)P_{\mu_3}(x)dx \\ &= \sum_{j=|\mu_1-\mu_2}^{\mu 1+\mu 2} 2(2j+1)\begin{pmatrix}\mu_1 & \mu_2 & \mu_3\\ 0 & 0 & 0\end{pmatrix}^2 \int_{-1}^{1} P_j(x)P_{\mu_3}(x)dx \, , \end{aligned} \tag{D.4.2}$$

next, integrate and simplify the sum using of the Kronecker delta.

$$I^{\mu_1\mu_2\mu_3} = \sum_{j=|\mu_1-\mu_2|}^{\mu 1+\mu 2} 2(2j+1)\begin{pmatrix}\mu_1 & \mu_2 & j\\ 0 & 0 & 0\end{pmatrix}^2 \frac{\delta_{j\mu_3}}{2\mu_3+1} = 2\begin{pmatrix}\mu_1 & \mu_2 & \mu_3\\ 0 & 0 & 0\end{pmatrix}^2 . \tag{D.4.3}$$

The above technique generalizes to m many integral degree Legendre polynomials

$$I^{\mu_1\mu_2\cdots k_m} = \sum_{\mu=|\mu_1-\mu_2|}^{\mu_1+\mu_2} 2(2\mu+1)\begin{pmatrix}\mu_1 & \mu_2 & \mu\\ 0 & 0 & 0\end{pmatrix}^2 I^{\mu\mu_3\cdots\mu_m} . \tag{D.4.4}$$

However, integrals of more than four Legendre polynomials simplify to finite sums.

$$I^{\mu_1\mu_2\mu_3\mu_4} = \sum_{\mu=|\mu_1-\mu_2|}^{\mu_1+\mu_2} 2(2\mu+1)\begin{pmatrix}\mu_1 & \mu_2 & \mu\\ 0 & 0 & 0\end{pmatrix}^2 2\begin{pmatrix}\mu_3 & \mu_4 & \mu\\ 0 & 0 & 0\end{pmatrix}^2 . \tag{D.4.5}$$

This does allow for direct calculation of some of these integrals. For instance, when $|\mu_1 - \mu_2| > (\mu_3 + \mu_4)$ $I^{\mu_1\mu_2\mu_3\mu_4} = 0$.

Next, to evaluate integrals of Legendre functions of non-integral degree, the nature of the complete orthogonal system consisting of Legendre polynomials may be employed. That is, for any piecewise continuous square integrable function $f(x)$ where $x \in (-1, 1)$,

$$f(x) = \sum_{i=0}^{\infty} a_i P_i(x) \ , \tag{D.4.6}$$

where

$$a_i = \frac{2i+1}{2} \int_{-1}^{1} f(x) P_i(x) dx \ . \tag{D.4.7}$$

Note the correction to the integral since Legendre polynomials are not an complete orthonormal system. In practice this implies if the above integral can be done, then the integral of f with m many Legendre polynomials may also be done. Therefore since $I^{\mu_1 \mu_2}$ and I^{μ}_{ν} are known the general integral may be rewritten

$$\begin{aligned}
I^{\mu_1 \cdots \mu_m}_{\nu_1 \cdots \nu_n} &= \int_{-1}^{1} P_{\mu_1}(x) \cdots P_{\mu_m}(x) Q_{\nu_1}(x) \cdots Q_{\nu_n}(x) dx \\
= \int_{-1}^{1} \sum_{i_1=0}^{\infty} & \frac{2i_1+1}{2} I^{i_1 \mu_1} P_{i_1}(x) \cdots \sum_{i_m=0}^{\infty} \frac{2i_m+1}{2} I^{i_m \mu_m} P_{i_m}(x) \times \\
& \sum_{j_1=0}^{\infty} \frac{2j_1+1}{2} I^{j_1}_{\nu_1} P_{j_1}(x) \cdots \sum_{j_n=0}^{\infty} \frac{2j_n+1}{2} I^{j_n}_{\nu_n} P_{j_n}(x) dx \ .
\end{aligned} \tag{D.4.8}$$

Then, assuming the sums and integrals commute, the integral may be calculated

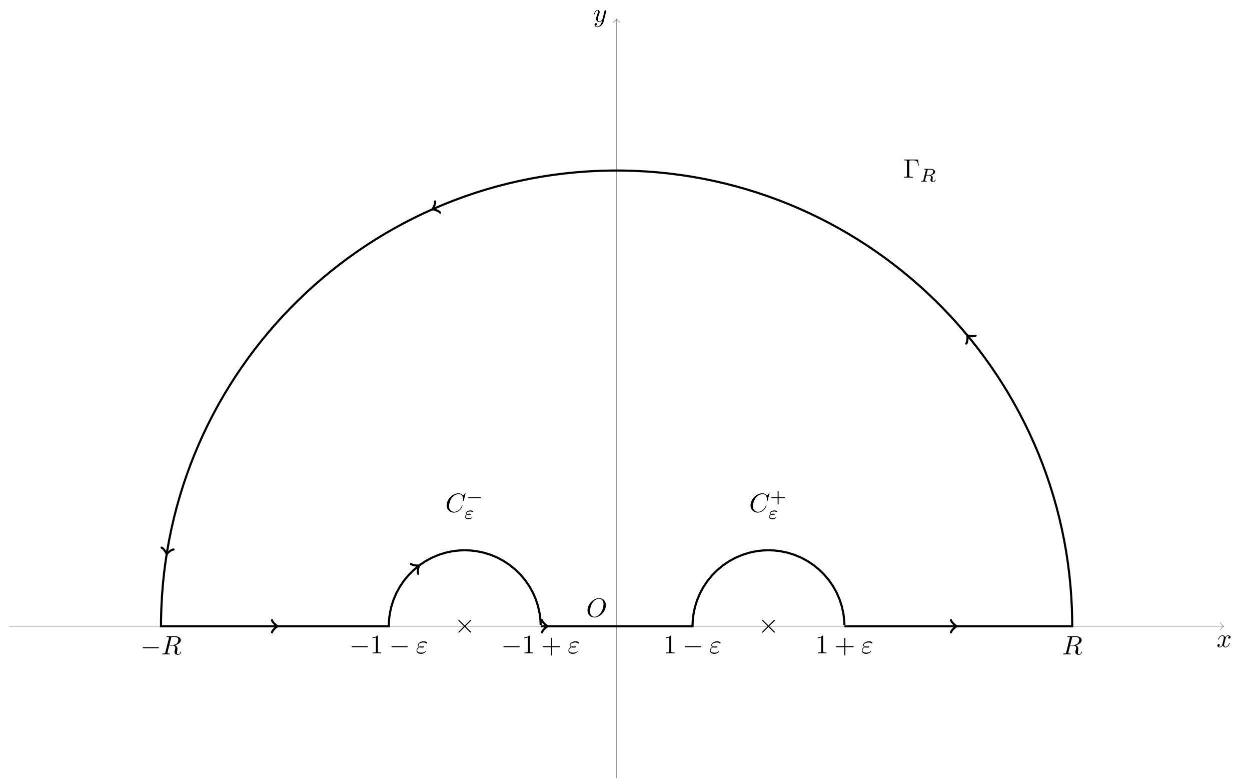
$$\begin{aligned}
& I^{\mu_1 \cdots \mu_m}_{\nu_1 \cdots \nu_n} \\
&= \sum_{\substack{i_1, \cdots, i_m = 0 \\ j_1, \cdots, j_m = 0}}^{\infty} \frac{2i_1+1}{2} I^{i_1 \mu_1} \cdots \frac{2i_m+1}{2} I^{i_m \mu_m} \frac{2j_1+1}{2} I^{j_1}_{\nu_1} \cdots \frac{2j_n+1}{2} I^{j_n}_{\nu_n} \times \\
& \qquad \int_{-1}^{1} P_{i_1}(x) \cdots P_{i_m}(x) P_{j_1}(x) \cdots P_{j_n}(x) dx \\
&= \sum_{\substack{i_1, \cdots, i_m = 0 \\ j_1, \cdots, j_m = 0}}^{\infty} \frac{2i_1+1}{2} I^{i_1 \mu_1} \cdots \frac{2i_m+1}{2} I^{i_m \mu_m} \frac{2j_1+1}{2} I^{j_1}_{\nu_1} \cdots \frac{2j_n+1}{2} I^{j_n}_{\nu_n} I^{i_1 \cdots i_m j_1 \cdots j_n} \ .
\end{aligned} \tag{D.4.9}$$

That the integrals and sum commute due to Fubini's theorem, when the integral is guaranteed to be finite. This point requires stressing. If there is some argument to show that for a given combination of $\mu \cdots$ and $\nu \cdots$ then the above power series is equivalent to the integral.

D.5 $J^{\mu\cdots}_{\nu\cdots}$

The technique of expressing Legendre functions as a series of Legendre polynomials does not work for $\mathcal{P}_\mu(z)$ or $\mathcal{Q}_\nu(z)$ since the Legendre polynomials do not form an orthonormal basis on $z \in (1, \infty)$, however contour integration does allow for relating the J type integrals to the I type integrals with certain requirements on the arguments.

Before looking directly at J type integrals, consider the following contour



This contour contains none of the singularities of Legendre functions, i.e. the integral of a product of Legendre functions over this entire contour should equal zero.

The integral over either C_ε^+ or C_ε^- tends to zero for any combination of Legendre functions. This is because

$$\mathcal{P}_\nu(z) \sim 1 \,, \tag{D.5.1}$$

$$\mathcal{Q}_\nu(z) \sim \frac{1}{2}\log\left(\frac{z+1}{z-1}\right) \,, \tag{D.5.2}$$

as stated above, therefore

$$\int_{C_\varepsilon^+} \mathcal{P}_\mu(z)\cdots\mathcal{Q}_\nu(z)\cdots dz = \int_0^\pi i\varepsilon e^{-i\varepsilon\theta}\mathcal{P}_\mu(-e^{-i\varepsilon\theta})\cdots\mathcal{Q}_\nu(-e^{-i\varepsilon\theta})\cdots d\theta \,. \tag{D.5.3}$$

Note the substitution $z = -e^{-I\varepsilon\theta}$, $dz = i\varepsilon e^{-i\varepsilon\theta}d\theta$. Applying the asymptotic form of these functions, that is

$$\mathcal{Q}_\nu(1 - i\varepsilon\theta + \cdots) \sim \frac{1}{2}(\log(2) - \log(-i\varepsilon\theta)) \sim -\frac{1}{2}\log(-i\varepsilon\theta) \; , \tag{D.5.4}$$

therefore the integrand is asymptotic to $A\,\varepsilon \log(\varepsilon)^n$ where A is constant. Furthermore substituting $x \to e^t$

$$\lim_{x\to 0^+} x\log(x)^n = \lim_{t\to-\infty} e^t \log(e^t)^n = \lim_{t\to-\infty} e^t t^n = 0 \; . \tag{D.5.5}$$

Therefore

$$\lim_{\varepsilon\to 0^+} i\varepsilon e^{-i\varepsilon\theta}\mathcal{P}_\mu(-e^{-i\varepsilon\theta})\cdots\mathcal{Q}_\nu(-e^{-i\varepsilon\theta})\cdots = \lim_{\varepsilon\to 0^+} i\varepsilon e^{-i\varepsilon\theta}1\cdots\frac{1}{2}(\log(2) - \log(-i\epsilon\theta))\cdots = 0 \; . \tag{D.5.6}$$

In fact, this limit converges uniformly, that is,

$$\begin{aligned}
&\forall\, \theta \in (0,\pi) \text{ and } \delta \in \mathbb{R}^+ \\
&\exists\, \varepsilon \in \mathbb{R}^+ \text{ so that} \\
&|i\varepsilon e^{-i\varepsilon\theta}\mathcal{P}_\mu(-e^{-i\varepsilon\theta})\cdots\mathcal{Q}_\nu(-e^{-i\varepsilon\theta})\cdots| < \delta \; .
\end{aligned} \tag{D.5.7}$$

Therefore, since the interval is finite, the limit and integral commute and the integral vanishes. The integral over C_ε^- may also be related, using the analytic continuation formulae given above, to finite product of logarithms of an infinitesimal with an infinitesimal, which also vanishes.

Finally, consider the integral over Γ_R, using a similar substitution, $z = Re^{i\theta}$, $dz = iRe^{i\theta}d\theta$, we find

$$\int_{\Gamma_R} \mathcal{P}_{\mu_1}(z)\cdots\mathcal{Q}_{\nu_1}(z)\cdots dz = \int_0^\pi iRe^{i\theta}\mathcal{P}_{\mu_1}(Re^{i\theta})\cdots\mathcal{Q}_{\nu_1}(Re^{i\theta})\cdots d\theta \; . \tag{D.5.8}$$

Now the precise nature of the degrees of these polynomials must be considered. Specifically given $l \le m$ where $\Re\{\mu_i\} > -\frac{1}{2}$ and $\Re\{\mu_j\} < -\frac{1}{2}$ for $i \le l$ and $j > l$, respectively, the asymptotic form of the above integrand is

$$\begin{gathered}
iRe^{i\theta}\mathcal{P}_{\mu_1}(Re^{i\theta})\cdots\mathcal{Q}_{\nu_1}(Re^{i\theta})\cdots \sim \\
iRe^{i\theta}\frac{\Gamma(\mu_1+1/2)}{\pi^{\frac{1}{2}}\Gamma(\mu_1+1)}(2Re^{i\theta})^{\mu_1}\cdots\frac{\Gamma(-\mu_{l+1}-1/2)}{\pi^{\frac{1}{2}}\Gamma(-\mu_{l+1})(2Re^{i\theta})^{\mu_{l+1}+1}}\cdots\frac{\pi^{\frac{1}{2}}\Gamma(\nu_1+1)}{\Gamma(\nu_1+\frac{3}{2})(2Re^{i\theta})^{\nu_1+1}}\cdots \sim \\
A\; e^{i\theta(1+\mu_1+\cdots-\mu_{l+1}-1-\cdots-\nu_1-1\cdots)}\; R^{1+\mu_1+\cdots-\mu_{l+1}-1-\cdots-\nu_1-1\cdots} \; ,
\end{gathered} \tag{D.5.9}$$

where A is a constant. This converges uniformly to zero if $\Re\{1 + \mu_1 + \cdots - \mu_{l+1} - 1 - \cdots - \nu_1 - 1 \cdots\} = \Re\{\mu_1 + \cdots - \mu_{l+1} \cdots - \nu_1 \cdots - (m - l + n) + 1\} < 0$, and may converge to a representation of a delta function for specific values of μs and νs. Therefore given the above condition on the degrees, this integral is also equal to zero.

Therefore integrals on $(-R, -1-\varepsilon)$, $(-1+\varepsilon, 1-\varepsilon)$, and $(1+\varepsilon, R)$ may be related to Js and Is. Specifically, apply the formula giving the asymptotic completions above

$$
\begin{aligned}
&\int_{-\infty}^{-1} \mathcal{P}_{\mu_1}(z) \cdots \mathcal{Q}_{\nu_1}(z) \cdots dz \\
&= - \int_{\infty}^{1} \mathcal{P}_{\mu_1}(-s) \cdots \mathcal{Q}_{\nu_1}(-s) \cdots ds \\
&= \int_{1}^{\infty} \left(e^{i\pi\mu_1} \mathcal{P}_{\mu_1}(s) - \frac{2}{\pi} \sin \pi\mu_1 \mathcal{Q}_{\mu_1}(s) \right) \cdots \left(-e^{-i\pi\nu_1} \mathcal{Q}_{\nu_1}(s) \right) \cdots ds \\
&= (-1)^n e^{i\pi(\mu_1 + \cdots - \nu_1 - \cdots)} J^{\mu_1 \cdots}_{\nu_1 \cdots} + (-1)^n e^{i\pi(\mu_1 + \cdots + \mu_m - \nu_1 - \cdots - \nu_n)} J^{\mu_1 \cdots \mu_m}_{\nu_1 \cdots \nu_n} \\
&+ (-1)^n \left(-\frac{2}{\pi} \right) \sum_{i=1}^{m} e^{i\pi(\mu_1 + \cdots + \cancel{\mu_i} + \cdots + \mu_m - \nu_1 - \cdots - \nu_n)} \sin(\pi\mu_i) J^{\mu_1 \cdots \cancel{\mu_i} \cdots \mu_n}_{\mu_i \nu_i \cdots \nu_n} \\
&+ (-1)^n \left(-\frac{2}{\pi} \right)^2 \sum_{i=1}^{m-1} \sum_{j=i+1}^{m} e^{i\pi(\mu_1 + \cdots + \cancel{\mu_i} + \cdots + \cancel{\mu_j} + \cdots + \mu_m - \nu_1 - \cdots - \nu_n)} \times \\
&\qquad \sin(\pi\mu_i) \sin(\pi\mu_j) J^{\mu_1 \cdots \cancel{\mu_i} \cdots \cancel{\mu_j} \cdots \mu_n}_{\mu_i \mu_j \nu_i \cdots \nu_n} \\
&+ \cdots + (-1)^n \left(-\frac{2}{\pi} \right)^m e^{i\pi(-\nu_1 - \cdots - \nu_n} \prod_{i=1}^{m} \sin(\pi\mu_i) J_{\mu_1 \cdots \mu_m \nu_1 \cdots \nu_n} \,.
\end{aligned}
\tag{D.5.10}
$$

Here, the first equality is given via the substitution $s = -z$, the second is via the analytic continuation formulae given in Eqs. (D.2.5) and (D.2.6), and the third is given by distributing the integrand, commuting the finite summation and integral, and applying the definition of $J^{\mu \cdots}_{\nu \cdots}$. This requires, for the evaluation of an integral with m Legendre functions of the first type in the integrand, the evaluation of roughly m^2 new integrals, however, since each new J type integral involves at least one fewer Legendre function of the first type int the integrand, the recursion terminates.

Similarly, for the integral over $(-1,1)$

$$\begin{aligned}&\int_{-1}^{1}\mathcal{P}_{\mu_1}(z)\cdots\mathcal{Q}_{\nu_1}(z)\cdots dz\\&=\int_{-1}^{1}P_{\mu_1}(z)\cdots\left(Q_{\nu_1}(z)-\frac{i\pi}{2}P_{\nu_1}(z)\right)\cdots dz\\&=I^{\mu_1\cdots\mu_m}_{\nu_1\cdots\nu_n}+\left(-\frac{i\pi}{2}\right)\sum_{i=1}^{n}I^{\nu_i\mu_1\cdots\mu_m}_{\nu_1\cdots\cancel{\nu_i}\cdots\nu_n}\\&+\left(-\frac{i\pi}{2}\right)^2\sum_{i=1}^{n-1}\sum_{j=i+1}^{n}I^{\nu_i\nu_j\mu_1\cdots\mu_m}_{\nu_1\cdots\cancel{\nu_i}\cdots\cancel{\nu_j}\cdots\nu_n}+\cdots+\left(-\frac{i\pi}{2}\right)^n I^{\nu_1\cdots\nu_n\mu_1\cdots\mu_m}\,.\end{aligned}\tag{D.5.11}$$

This equality is given using the same method as above for the second and third equalities.

Since the semicircular contours contribute zero in the limit, and all of the poles of Legendre functions lie outside the integrand, the sum of these three integrals is zero

$$\int_{-\infty}^{-1}\mathcal{P}_{\mu_1}(z)\cdots\mathcal{Q}_{\nu_1}(z)\cdots dz+\int_{-1}^{1}\mathcal{P}_{\mu_1}(z)\cdots\mathcal{Q}_{\nu_1}(z)\cdots dz+\int_{1}^{\infty}\mathcal{P}_{\mu_1}(z)\cdots\mathcal{Q}_{\nu_1}(z)\cdots dz=0\,.\tag{D.5.12}$$

Therefore $J^{\mu_1\cdots\mu_m}_{\nu_1\cdots\nu_n}$ may be given as

$$\begin{aligned}&J^{\mu_1\cdots\mu_m}_{\nu_1\cdots\nu_n}\\&=\left(-(-1)^n e^{i\pi(\mu_1+\cdots+\mu_m-\nu_1-\cdots-\nu_n)}-1\right)^{-1}\left(I^{\mu_1\cdots\mu_m}_{\nu_1\cdots\nu_n}+\left(-\frac{i\pi}{2}\right)\sum_{i=1}^{n}I^{\nu_i\mu_1\cdots\mu_m}_{\nu_1\cdots\cancel{\nu_i}\cdots\nu_n}\right.\\&+\left(-\frac{i\pi}{2}\right)^2\sum_{i=1}^{n-1}\sum_{j=i+1}^{n}I^{\nu_i\nu_j\mu_1\cdots\mu_m}_{\nu_1\cdots\cancel{\nu_i}\cdots\cancel{\nu_j}\cdots\nu_n}+\cdots+\left(-\frac{i\pi}{2}\right)^n I^{\nu_1\cdots\nu_n\mu_1\cdots\mu_m}\\&+(-1)^n\left(-\frac{2}{\pi}\right)\sum_{i=1}^{m}e^{i\pi(\mu_1+\cdots+\cancel{\mu_i}+\cdots+\mu_m-\nu_1-\cdots-\nu_n)}\sin(\pi\mu_i)J^{\mu_1\cdots\cancel{\mu_i}\cdots\mu_n}_{\mu_i\nu_i\cdots\nu_n}\\&+(-1)^n\left(-\frac{2}{\pi}\right)^2\sum_{i=1}^{m-1}\sum_{j=i+1}^{m}e^{i\pi(\mu_1+\cdots+\cancel{\mu_i}+\cdots+\cancel{\mu_j}+\cdots+\mu_m-\nu_1-\cdots-\nu_n)}\times\\&\sin(\pi\mu_i)\sin(\pi\mu_j)J^{\mu_1\cdots\cancel{\mu_i}\cdots\cancel{\mu_j}\cdots\mu_n}_{\mu_i\mu_j\nu_i\cdots\nu_n}\\&\left.+\cdots+(-1)^n\left(-\frac{2}{\pi}\right)^m e^{i\pi(-\nu_1-\cdots-\nu_n}\prod_{i=1}^{m}\sin(\pi\mu_i)J_{\mu_1\cdots\mu_m\nu_1\cdots\nu_n}\right)\,.\end{aligned}\tag{D.5.13}$$

D.6 Examples

D.6.1 $J_{0\cdots 0n}$

First consider $J_{0\cdots 0n}$, the integral of n Legendre functions of second-type of order 0. The recursive formula given above is invalid when applied to this integral for n odd, however one of the values can be changed from 0 to $\epsilon \in \mathbb{R}^+$, and a limit can be taken. Alternately $\mathcal{Q}_0(z)$ may be simplified to $\operatorname{arccoth}(z)$ and the substitution $z = \coth(t)$ may be used

$$J_{0\cdots 0n} = \int_1^\infty \mathcal{Q}_0(z)^n dz = \int_\infty^0 \operatorname{arccoth}(\coth(t))(-\operatorname{csch}^2(t))dt \tag{D.6.1}$$

$$= \frac{n!}{2^{n+1}} \frac{2^{n+1}}{\Gamma(n+1)} \int_0^\infty t \operatorname{csch}^2(t)dt = \frac{n!}{2^{n+1}}\zeta(n)\ . \tag{D.6.2}$$

D.6.2 $J_{0\nu}$

Another integral locatable in an integral table, and a valuable acid test of the above formula is the integral of a $\mathcal{Q}_0$ and $\mathcal{Q}_\nu$. Integral tables give this value as

$$J_{0\nu} = \frac{\mathcal{H}_\nu - \mathcal{H}_0}{\nu(\nu+1)}\ . \tag{D.6.3}$$

The recursive formula gives the integral as

$$J_{0\nu} = (-(-1)^2 e^{-i\pi\nu} - 1)^{-1} \left(I_{0\nu} - \frac{i\pi}{2}(I_0^\nu + I_\nu^0) - \frac{\pi^2}{4} I^{0\nu} \right) = \frac{\mathcal{H}_\nu}{\nu(\nu+1)}\ . \tag{D.6.4}$$

Given, $\mathcal{H}_0 = 0$, the formulae agree.

D.6.3 $J_{00\nu}$

Consider

$$J_{00\nu} = \int_1^\infty \mathcal{Q}_0(z)^2 \mathcal{Q}_\nu(z) dz\ , \tag{D.6.5}$$

where $\Re\{\nu\} = -\frac{1}{2}$. The hypothetically problematic top contour converges to zero by the above argument since $\Re\{-0-0-\nu-3+1\} < 0 \Leftrightarrow \Re\{\nu\} > -2$. Therefore

$$\begin{aligned} &J_{00\nu} = \\ &\left(-(-1)^3 e^{i\pi(-0-0-\nu)} - 1\right)^{-1} \left(I_{003} + \left(-\frac{i\pi}{2}\right)\left(I_{0\nu}^0 + I_{0\nu}^0 + I_{00}^\nu\right) \right. \\ &\left. + \left(-\frac{i\pi}{2}\right)^2 \left(I_\nu^{00} + I_0^{0\nu} + I_0^{0\nu}\right) + \left(-\frac{i\pi}{2}\right)^3 I^{00\nu} \right)\ . \end{aligned} \tag{D.6.6}$$

This may be simplified since $P_0(z) = 1$, therefore $I^{0\mu\cdots}_{\nu\cdots} = I^{\mu\cdots}_{\nu\cdots}$, unless there are exactly two arguments. Therefore

$$\begin{aligned} &J_{00\nu} = \\ &-\frac{1}{e^{-i\pi\nu}+1}\left(I_{003} + \left(-\frac{i\pi}{2}\right)\left(I_{0\nu} + I_{0\nu} + I^{\nu}_{00}\right)\right. \\ &\left. + \left(-\frac{i\pi}{2}\right)^2 \left(I^0_\nu + I^\nu_0 + I^\nu_0\right) + \left(-\frac{i\pi}{2}\right)^3 I^{0\nu}\right). \end{aligned} \tag{D.6.7}$$

Each three–argument integral may be given in terms of series of two–argument integrals

$$\begin{aligned} &J_{00\nu} = \\ &-\frac{1}{e^{-i\pi\nu}+1}\left(\sum_{i,j=1}^{\infty}\sum_{k=|i-j|}^{i+j} \frac{2i+1}{2} I^i_0 \frac{2j+1}{2} I^j_0 \frac{2k+1}{2} I^k_\nu 2\begin{pmatrix} i & j & k \\ 0 & 0 & 0\end{pmatrix}^2\right. \\ &+ \left(-\frac{i\pi}{2}\right)\left(I_{0\nu} + I_{0\nu} + \sum_{i,j=1}^{\infty}\sum_{k=|i-j|}^{i+j} \frac{2i+1}{2} I^i_0 \frac{2j+1}{2} I^j_0 \frac{2k+1}{2} I^{k\nu} 2\begin{pmatrix} i & j & k \\ 0 & 0 & 0\end{pmatrix}^2\right) \\ &\left. + \left(-\frac{i\pi}{2}\right)^2 \left(I^0_\nu + I^\nu_0 + I^\nu_0\right) + \left(-\frac{i\pi}{2}\right)^3 I^{0\nu}\right). \end{aligned} \tag{D.6.8}$$

Each two–argument integral may be given explicitly

$$\begin{aligned} &J_{00\nu} = \\ &-\frac{1}{e^{-i\pi\nu}+1}\left(\sum_{i,j=1}^{\infty}\sum_{k=|i-j|}^{i+j} \frac{2i+1}{2}\frac{1-(-1)^i}{i(i+1)}\frac{2j+1}{2}\frac{1-(-1)^j}{j(j+1)}\frac{2k+1}{2}\frac{1-(-1)^k\cos(\pi\nu)}{(k-\nu)(k+\nu+1)} 2\begin{pmatrix} i & j & k \\ 0 & 0 & 0\end{pmatrix}^2\right. \\ &+ \left(-\frac{i\pi}{2}\right)\left(2\frac{\frac{\pi}{2}\sin(\pi\nu) - (1+\cos(\pi\nu))\mathcal{H}_\nu}{\nu(\nu+1)}\right. \\ &\left.+ \sum_{i,j=1}^{\infty}\sum_{k=|i-j|}^{i+j} \frac{2i+1}{2}\frac{1-(-1)^i}{i(i+1)}\frac{2j+1}{2}\frac{1-(-1)^j}{j(j+1)}\frac{2k+1}{2}\frac{-2(-1)^k\sin(\pi\nu)}{\pi(k-\nu)(k+\nu+1)}\begin{pmatrix} i & j & k \\ 0 & 0 & 0\end{pmatrix}^2\right) \\ &+ \left(-\frac{i\pi}{2}\right)^2\left(\frac{\cos(\pi\nu)-1}{\nu(\nu+1)} + 2\frac{\pi - \pi\cos(\pi\nu) - 2\sin(\pi\nu)\mathcal{H}_\nu}{\pi\nu(\nu+1)}\right) \\ &\left. + \left(-\frac{i\pi}{2}\right)^3 \frac{2\sin(\pi\nu)}{\pi\nu(\nu+1)}\right). \end{aligned} \tag{D.6.9}$$

E A Consistent Truncation about the SS–CGP Background

We will start with the bosonic sector of ten-dimensional type I supergravity,

$$\hat{\mathcal{L}}_{10} = \hat{R}\hat{*}1 - \frac{1}{2}d\hat{\phi}\wedge\hat{*}d\hat{\phi} - \frac{1}{2}e^{-\hat{\phi}}\hat{H}_{(3)}\wedge\hat{*}\hat{H}_{(3)}\,, \tag{E.0.1}$$

where $\hat{H}_{(3)} = d\hat{B}_{(2)}$. The solution of interest is the lifted Salam–Sezgin vacuum with an NS5-brane inclusion. This was derived in Reference [32], and we reproduce it here:

$$\begin{aligned} d\hat{s}_{10}^2 &= W(\rho)^{-\frac{1}{4}}\Big(\eta_{\mu\nu}dx^\mu dx^\nu + dy^2 + \frac{1}{4g^2}\Big(d\psi + \mathrm{sech}2\rho\big(d\chi + \cos\theta\, d\varphi\big)\Big)^2 + \frac{1}{g^2}W(\rho)\, ds_{EH}^2\Big)\,,\\ e^{2\hat{\phi}} &= W(\rho)\,, \quad \hat{B}_{(2)} = \frac{1}{4g^2}\big((1+k)d\chi + \mathrm{sech}2\rho\, d\psi\big)\wedge\big(d\chi + \cos\theta\, d\varphi\big)\,, \end{aligned} \tag{E.0.2}$$

where $W(\rho) = \mathrm{sech}2\rho - k\log\tanh\rho$, and ds_{EH}^2 is the Eguchi-Hanson metric

$$ds_{EH}^2 = \cosh 2\rho\Big(d\rho^2 + \frac{1}{4}(\tanh 2\rho)^2\big(d\chi + \cos\theta\, d\varphi\big)^2 + \frac{1}{4}\big(d\theta^2 + \sin^2\theta\, d\varphi^2\big)\Big)\,. \tag{E.0.3}$$

The coordinates ψ, y, and χ are S^1 coordinates, (θ, φ) parametrizes an S^2, $\rho \in [0, \infty)$ is the non-compact radius, and k is a positive constant. The six-dimensional worldvolume of the NS5-brane is parametrized by (x^μ, ψ, y). Our goal is to reduce the type I theory on $T^3 \ni (y, \psi, \chi)$ via the usual Kaluza–Klein methods, and S^2 on the background given in (E.0.2) to obtain a five-dimensional theory.

E.1 $10 \to 9$: Reduce on y

The background metric in (E.0.2) does not have a fibre over the circle parametrised by y. So, the appropriate Kaluza–Klein ansatz is

$$d\hat{s}_{10}^2 = e^{-\frac{1}{2\sqrt{7}}\phi_2}ds_9^2 + e^{\frac{\sqrt{7}}{2}\phi_2}dy^2\,, \quad \hat{B}_{(2)} = B_{(2)}\,, \quad \hat{\phi} = \phi_1\,, \tag{E.1.1}$$

where the un-hatted fields are eight-dimensional fields. The resulting equations of motion are encoded in the action

$$\mathcal{L}_9 = R{*}1 - \frac{1}{2}d\phi_i\wedge{*}d\phi_i - \frac{1}{2}e^{-\phi_1 + \frac{1}{\sqrt{7}}\phi_2}H_{(3)}\wedge{*}H_{(3)}\,, \tag{E.1.2}$$

where $H_{(3)} = dB_{(2)}$, and $i \in \{1, 2\}$. The background solution (E.0.2) reduces to

$$
\begin{aligned}
ds_9^2 &= W^{-\frac{2}{7}}\Big(\eta_{\mu\nu}dx^\mu dx^\nu + dy^2 + \frac{1}{4g^2}\Big(d\psi + \mathrm{sech}2\rho\big(d\chi + \cos\theta\, d\varphi\big)\Big)^2 + \frac{1}{g^2}W\, ds^2_{EH}\Big)\,,\\
e^{2\phi_1} &= e^{-2\sqrt{7}\phi_2} = W\,, \quad B_{(2)} = \frac{1}{4g^2}\big((1+k)d\chi + \mathrm{sech}2\rho\, d\psi\big)\wedge\big(d\chi + \cos\theta\, d\varphi\big)\,. \qquad \text{(E.1.3)}
\end{aligned}
$$

We observe that in the background solution, the combination $\phi_1 + \sqrt{7}\phi_2 = 0$. This suggests a field redefinition,

$$
\begin{pmatrix} \Phi_2 \\ \Phi_1 \end{pmatrix} = \frac{1}{2\sqrt{2}}\begin{pmatrix} 1 & \sqrt{7} \\ -\sqrt{7} & 1 \end{pmatrix}\begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix}\,. \qquad \text{(E.1.4)}
$$

Substituting this into (E.1.2), and noting that the transformation matrix (E.1.4) is orthogonal, we have

$$
\mathcal{L}_9 = R{*}1 - \frac{1}{2}d\Phi_i \wedge {*}d\Phi_i - \frac{1}{2}e^{2\sqrt{\frac{2}{7}}\Phi_1}H_{(3)} \wedge {*}H_{(3)}\,, \qquad \text{(E.1.5)}
$$

and the background solution is

$$
\Phi_2 = 0\,, \quad e^{-\sqrt{\frac{7}{2}}\Phi_1} = W\,. \qquad \text{(E.1.6)}
$$

From (E.1.5), we find that Φ_2 is decoupled, so there is a consistent truncation of the nine-dimensional theory given by $\Phi_2 = 0$, $\Phi_1 = \phi$. For completeness, the truncated theory is given by

$$
\hat{\mathcal{L}}_9 = \hat{R}\hat{*}1 - \frac{1}{2}d\hat{\phi} \wedge \hat{*}d\hat{\phi} - \frac{1}{2}e^{2\sqrt{\frac{2}{7}}\hat{\phi}}\hat{H}_{(3)} \wedge \hat{*}\hat{H}_{(3)}\,, \qquad \text{(E.1.7)}
$$

where we have reintroduced hats on all nine-dimensional fields. We will take (E.1.7) as the starting point for the next reduction step.

E.2 $9 \to 8$: Reduce on ψ

The background metric in (E.1.3) is fibred over the $\widetilde{\psi} = \psi/2g$ coordinate. This suggests that the appropriate Kaluza–Klein ansatz is

$$
d\hat{s}_9^2 = e^{-\frac{1}{\sqrt{21}}\phi_2}ds_8^2 + e^{\frac{6}{\sqrt{21}}\phi_2}\big(d\widetilde{\psi} + \mathcal{A}_{(1)}\big)^2\,, \quad \hat{B}_{(2)} = B_{(2)} + A_{(1)} \wedge d\widetilde{\psi}\,, \quad \hat{\phi} = \phi_1\,. \qquad \text{(E.2.1)}
$$

The resulting equations of motion are encoded in the action

$$\begin{aligned}\mathcal{L}_8 = R{*}1 - \frac{1}{2}d\phi_i \wedge {*}d\phi_i - \frac{1}{2}e^{\sqrt{\frac{7}{3}}\phi_2}\mathcal{F}_{(2)} \wedge {*}\mathcal{F}_{(2)} \\ - \frac{1}{2}e^{2\sqrt{\frac{2}{7}}\phi_1 - \frac{5}{\sqrt{21}}\phi_2}F_{(2)} \wedge {*}F_{(2)} - \frac{1}{2}e^{2\sqrt{\frac{2}{7}}\phi_1 + \frac{2}{\sqrt{21}}\phi_2}H_{(3)} \wedge {*}H_{(3)}\,,\end{aligned} \tag{E.2.2}$$

where $\mathcal{F}_{(2)} = d\mathcal{A}_{(1)}$, $F_{(2)} = dA_{(1)}$, $H_{(3)} = dB_{(2)} - dA_{(1)} \wedge \mathcal{A}_{(1)}$, and $i \in \{1,2\}$. The background solution in (E.1.3) reduces to

$$\begin{aligned}ds_8^2 = W^{-\frac{1}{3}}\Big(\eta_{\mu\nu}dx^\mu dx^\nu + \frac{W}{g^2}ds^2_{EH}\Big)\,, \quad B_{(2)} = \frac{1+k}{4g^2}\cos\theta\, d\chi \wedge d\varphi\,, \\ \mathcal{A}_{(1)} = -A_{(1)} = \frac{1}{2g}\text{sech}2\rho\,(d\chi + \cos\theta\, d\varphi)\,, \quad e^{-\sqrt{\frac{7}{2}}\phi_1} = e^{-\sqrt{21}\phi_2} = W\,.\end{aligned} \tag{E.2.3}$$

From this, we observe that $\phi_1 - \sqrt{6}\phi_2 = 0$. This suggests the field redefinition

$$\begin{pmatrix}\Phi_2 \\ \Phi_1\end{pmatrix} = \frac{1}{\sqrt{7}}\begin{pmatrix}1 & -\sqrt{6} \\ \sqrt{6} & 1\end{pmatrix}\begin{pmatrix}\phi_1 \\ \phi_2\end{pmatrix}. \tag{E.2.4}$$

For the background solution, we have

$$\Phi_2 = 0\,, \quad e^{-\sqrt{3}\Phi_1} = W\,. \tag{E.2.5}$$

Now, substituting the field redefinition into (E.2.2), and noting that the transformation matrix (E.2.4) is orthogonal, we obtain

$$\begin{aligned}\mathcal{L}_8 = R{*}1 - \frac{1}{2}d\Phi_i \wedge {*}d\Phi_i - \frac{1}{2}e^{-\sqrt{2}\Phi_2 + \frac{1}{\sqrt{3}}\Phi_1}\mathcal{F}_{(2)} \wedge {*}\mathcal{F}_{(2)} \\ - \frac{1}{2}e^{\sqrt{2}\Phi_2 + \frac{1}{\sqrt{3}}\Phi_1}F_{(2)} \wedge {*}F_{(2)} - \frac{1}{2}e^{\frac{2}{\sqrt{3}}\Phi_1}H_{(3)} \wedge {*}H_{(3)}\,.\end{aligned} \tag{E.2.6}$$

Let us now examine the equations of motion of $\mathcal{A}_{(1)}$, $A_{(1)}$, and Φ_2:

$$\begin{aligned}\mathcal{A}_{(1)}:&\quad d\big(e^{-\sqrt{2}\Phi_2 + \frac{1}{\sqrt{3}}\Phi_1}{*}d\mathcal{A}_{(1)}\big) - e^{\frac{2}{\sqrt{3}}\Phi_1}dA_{(1)} \wedge {*}H_{(3)} = 0\,, \\ A_{(1)}:&\quad d\big(e^{\sqrt{2}\Phi_2 + \frac{1}{\sqrt{3}}\Phi_1}{*}dA_{(1)}\big) - e^{\frac{2}{\sqrt{3}}\Phi_1}d\mathcal{A}_{(1)} \wedge {*}H_{(3)} = 0\,, \\ \Phi_2:&\quad d{*}d\Phi_2 = \frac{1}{2\sqrt{2}}e^{\frac{1}{\sqrt{3}}\Phi_1}\big(e^{\sqrt{2}\Phi_2}F_{(2)} \wedge {*}F_{(2)} - e^{-\sqrt{2}\Phi_2}\mathcal{F}_{(2)} \wedge {*}\mathcal{F}_{(2)}\big)\,.\end{aligned} \tag{E.2.7}$$

These equations admit the solution

$$\mathcal{A}_{(1)} = \pm A_{(1)}\,, \quad \Phi_2 = 0\,. \tag{E.2.8}$$

For our background solution, we have $\mathcal{A}_{(1)} = -A_{(1)}$. Using this simplifying ansatz, we find that the rest of the equations of motion are encoded in the action

$$\mathcal{L}_8 = R{*}1 - \frac{1}{2}d\phi\wedge{*}d\phi - e^{\frac{1}{\sqrt{3}}\phi}F_{(2)}\wedge{*}F_{(2)} - \frac{1}{2}e^{\frac{2}{\sqrt{3}}\phi}H_{(3)}\wedge{*}H_{(3)}\,, \tag{E.2.9}$$

where $F_{(2)} = dA_{(1)}$, $H_{(3)} = dB_{(2)} + dA_{(1)}\wedge A_{(1)}$, and we relabelled $\Phi_1 = \phi$. To put the action in canonical form, we have to rescale $A_{(1)}$ by a factor of $1/\sqrt{2}$. The final eight-dimensional theory is

$$\hat{\mathcal{L}}_8 = \hat{R}\hat{*}1 - \frac{1}{2}d\hat{\phi}\wedge\hat{*}d\hat{\phi} - \frac{1}{2}e^{\frac{1}{\sqrt{3}}\hat{\phi}}\hat{F}_{(2)}\wedge\hat{*}\hat{F}_{(2)} - \frac{1}{2}e^{\frac{2}{\sqrt{3}}\hat{\phi}}\hat{H}_{(3)}\wedge\hat{*}\hat{H}_{(3)}\,, \tag{E.2.10}$$

where $\hat{F}_{(2)} = d\hat{A}_{(1)}$, $\hat{H}_{(3)} = d\hat{B}_{(2)} + \frac{1}{2}d\hat{A}_{(1)}\wedge\hat{A}_{(1)}$, and we have restored the hats for all eight-dimensional fields.

E.3 $8 \to 7$: Reduce on χ

The background solution (E.2.3) is fibred over $\widetilde{\chi} = \chi/2g$. The appropriate Kaluza–Klein ansatz is then

$$\begin{aligned} d\hat{s}_8^2 &= e^{-\frac{1}{\sqrt{15}}\phi_2}ds_7^2 + e^{\sqrt{\frac{5}{3}}\phi_2}\big(d\widetilde{\chi} + \tilde{\mathcal{A}}_{(1)}\big)^2\,, \\ \hat{B}_{(2)} &= B_{(2)} + B_{(1)}\wedge d\widetilde{\chi}\,, \quad \hat{A}_{(1)} = A_{(1)} + \sigma\, d\widetilde{\chi}\,, \quad \hat{\phi} = \phi_1\,. \end{aligned} \tag{E.3.1}$$

The resulting equations of motion are encoded in the action

$$\begin{aligned} \mathcal{L}_7 = R{*}1 &- \frac{1}{2}d\phi_i\wedge{*}d\phi_i - \frac{1}{2}e^{\frac{1}{\sqrt{3}}\phi_1 - \sqrt{\frac{5}{3}}\phi_2}d\sigma\wedge{*}d\sigma - \frac{1}{2}e^{2\sqrt{\frac{3}{5}}\phi_2}\mathcal{F}_{(2)}\wedge{*}\mathcal{F}_{(2)} \\ &- \frac{1}{2}e^{\frac{1}{\sqrt{3}}\phi_1 + \frac{1}{\sqrt{15}}\phi_2}F_{(2)}\wedge{*}F_{(2)} - \frac{1}{2}e^{\frac{2}{\sqrt{3}}\phi_1 - \frac{4}{\sqrt{15}}\phi_2}H_{(2)}\wedge{*}H_{(2)} - \frac{1}{2}e^{\frac{2}{\sqrt{3}}\phi_1 + \frac{2}{\sqrt{15}}\phi_2}H_{(3)}\wedge{*}H_{(3)}\,, \end{aligned} \tag{E.3.2}$$

where $\mathcal{F}_{(2)} = d\tilde{\mathcal{A}}_{(1)}$, $F_{(2)} = dA_{(1)} - d\sigma\wedge\tilde{\mathcal{A}}_{(1)}$, $H_{(2)} = dB_{(1)} + \frac{1}{2}(\sigma dA_{(1)} - d\sigma\wedge A_{(1)})$, $H_{(3)} = dB_{(2)} + \frac{1}{2}dA_{(1)}\wedge A_{(1)} - H_{(2)}\wedge\tilde{\mathcal{A}}_{(1)}$, and $i\in\{1,2\}$. The background solution (E.2.3) reduces to

$$\begin{aligned} ds_7^2 &= \frac{(\sinh 2\rho)^{\frac{2}{5}}}{(W\cosh 2\rho)^{\frac{1}{5}}}\Big(\eta_{\mu\nu}dx^\mu dx^\nu + \frac{W\cosh 2\rho}{g^2}d\rho^2 + \frac{W\cosh 2\rho}{4g^2}\big(d\theta^2 + \sin^2\theta\, d\varphi^2\big)\Big)\,, \\ \sigma &= \sqrt{2}\mathrm{sech}2\rho\,, \quad \tilde{\mathcal{A}}_{(1)} = \frac{1}{2g}\cos\theta\, d\varphi\,, \quad A_{(1)} = \sigma\tilde{\mathcal{A}}_{(1)}\,, \quad B_{(1)} = -(1+k)\tilde{\mathcal{A}}_{(1)}\,, \\ e^{-\sqrt{3}\phi_1} &= W\,, \quad e^{\sqrt{\frac{5}{3}}\phi_2} = W^{\frac{2}{3}}(\sinh 2\rho)^2\mathrm{sech}2\rho\,, \quad B_{(2)} = 0\,. \end{aligned} \tag{E.3.3}$$

It is convenient to perform the field redefinition

$$\begin{pmatrix}\Phi_1\\ \Phi_2\end{pmatrix} = \frac{1}{\sqrt{6}}\begin{pmatrix}1 & -\sqrt{5}\\ \sqrt{5} & 1\end{pmatrix}\begin{pmatrix}\phi_1\\ \phi_2\end{pmatrix}. \tag{E.3.4}$$

The background solution is then

$$e^{-\sqrt{2}\Phi_1} = W(\sinh 2\rho)^2\mathrm{sech}2\rho\,, \quad e^{\sqrt{10}\Phi_2} = W^{-1}(\sinh 2\rho)^2\mathrm{sech}2\rho\,. \tag{E.3.5}$$

Substituting the field redefinition into (E.3.2), and noting that the transformation matrix (E.3.4) is orthogonal, we find that

$$\begin{aligned}\hat{\mathcal{L}}_7 = {} & \hat{R}\hat{*}1 - \frac{1}{2}d\hat{\Phi}_i\wedge\hat{*}d\hat{\Phi}_i - \frac{1}{2}e^{\sqrt{2}\hat{\Phi}_1}d\hat{\sigma}\wedge\hat{*}d\hat{\sigma} - \frac{1}{2}e^{-\sqrt{2}\hat{\Phi}_1+\sqrt{\frac{2}{5}}\hat{\Phi}_2}\hat{\mathcal{F}}_{(2)}\wedge\hat{*}\hat{\mathcal{F}}_{(2)}\\ & - \frac{1}{2}e^{\sqrt{\frac{2}{5}}\hat{\Phi}_2}\hat{F}_{(2)}\wedge\hat{*}\hat{F}_{(2)} - \frac{1}{2}e^{\sqrt{2}\hat{\Phi}_1+\sqrt{\frac{2}{5}}\hat{\Phi}_2}\hat{H}_{(2)}\wedge\hat{*}\hat{H}_{(2)} - \frac{1}{2}e^{2\sqrt{\frac{2}{5}}\hat{\Phi}_2}\hat{H}_{(3)}\wedge\hat{*}\hat{H}_{(3)}\,,\end{aligned} \tag{E.3.6}$$

where we have restored the hats to the seven-dimensional fields.

E.4 $7\to 5$: Reduce on S^2

The reduction ansatz that is consistent with the seven-dimensional background solution (E.3.3) is

$$\begin{aligned}& d\hat{s}_7^2 = e^{-\frac{2}{\sqrt{15}}\Phi_3}ds_5^2 + \frac{1}{4g^2}e^{\sqrt{\frac{3}{5}}\Phi_3}ds^2(S^2)\,, \quad \hat{\Phi}_{1,2} = \Phi_{1,2}\,, \quad \hat{\sigma} = \sigma\,,\\ & \hat{\mathcal{F}}_{(2)} = -\frac{1}{2g}\,\mathrm{vol}(S^2)\,, \quad \hat{F}_{(2)} = -\frac{\sigma}{2g}\,\mathrm{vol}(S^2)\,, \quad \hat{H}_{(2)} = -\frac{\sigma^2+m}{4g}\,\mathrm{vol}(S^2)\,, \quad \hat{H}_{(3)} = 0\,,\end{aligned} \tag{E.4.1}$$

where $ds^2(S^2)$ and $\mathrm{vol}(S^2)$ are the metric and volume form on the unit 2-sphere respectively, m is a constant, and all un-hatted fields are five-dimensional fields. The ansatz for the field strengths is consistent with the seven-dimensional Bianchi identities:

$$d\hat{\mathcal{F}}_{(2)} = 0\,, \quad d\hat{F}_{(2)} = d\hat{\sigma}\wedge\hat{\mathcal{F}}_{(2)}\,, \quad d\hat{H}_{(2)} = d\hat{\sigma}\wedge\hat{F}_{(2)}\,, \quad d\hat{H}_{(3)} = \frac{1}{2}\hat{F}_{(2)}\wedge\hat{F}_{(2)} - \hat{H}_{(2)}\wedge\hat{\mathcal{F}}_{(2)}\,. \tag{E.4.2}$$

Let us first look at the seven-dimensional gauge field equations of motion:

$$\hat{B}_{(2)}: \quad d\big(e^{2\sqrt{\frac{2}{5}}\hat{\Phi}_2}\hat{*}\hat{H}_{(3)}\big) = 0\,, \tag{E.4.3}$$

$$\hat{B}_{(1)}: \quad d\big(e^{\sqrt{2}\hat{\Phi}_1+\sqrt{\frac{2}{5}}\hat{\Phi}_2}\hat{*}\hat{H}_{(2)}\big) - e^{2\sqrt{\frac{2}{5}}\hat{\Phi}_2}\hat{\mathcal{F}}_{(2)}\wedge\hat{*}\hat{H}_{(3)} = 0\,, \tag{E.4.4}$$

$$\hat{A}_{(1)}: \quad d\big(e^{\sqrt{\frac{2}{5}}\hat{\Phi}_2}\hat{*}\hat{F}_{(2)}\big) + e^{\sqrt{2}\hat{\Phi}_1+\sqrt{\frac{2}{5}}\hat{\Phi}_2}d\hat{\sigma}\wedge\hat{*}\hat{H}_{(2)} + e^{2\sqrt{\frac{2}{5}}\hat{\Phi}_2}\hat{F}_{(2)}\wedge\hat{*}\hat{H}_{(3)} = 0\,, \tag{E.4.5}$$

$$\hat{\mathcal{A}}_{(1)}: \quad d\big(e^{-\sqrt{2}\hat{\Phi}_1+\sqrt{\frac{2}{5}}\hat{\Phi}_2}\hat{*}\hat{\mathcal{F}}_{(2)}\big) + e^{\sqrt{\frac{2}{5}}\hat{\Phi}_2}d\hat{\sigma}\wedge\hat{*}\hat{F}_{(2)} - e^{2\sqrt{\frac{2}{5}}\hat{\Phi}_2}\hat{H}_{(2)}\wedge\hat{*}\hat{H}_{(3)} = 0\,, \tag{E.4.6}$$

$$\hat{\sigma}: \quad d\big(e^{\sqrt{2}\hat{\Phi}_1}\hat{*}d\hat{\sigma}\big) - e^{\sqrt{\frac{2}{5}}\hat{\Phi}_2}\hat{\mathcal{F}}_{(2)}\wedge\hat{*}\hat{F}_{(2)} - e^{\sqrt{2}\hat{\Phi}_1+\sqrt{\frac{2}{5}}\hat{\Phi}_2}\hat{F}_{(2)}\wedge\hat{*}\hat{H}_{(2)} = 0\,. \tag{E.4.7}$$

We note that

$$\hat{*}\,\mathrm{vol}(S^2) = 4g^2 e^{-\frac{8}{\sqrt{15}}\Phi_3}\,\mathrm{vol}(M_5)\,, \quad \hat{*}d\hat{\sigma} = \frac{1}{4g^2}(*d\sigma)\wedge\mathrm{vol}(S^2)\,, \tag{E.4.8}$$

where $\mathrm{vol}(M_5)$ and $*$ are the volume form and Hodge star defined with respect to the five-dimensional metric ds_5^2 in (E.4.1) respectively. From this, we find that $\hat{*}\hat{\mathcal{F}}_{(2)}$, $\hat{*}\hat{F}_{(2)}$, and $\hat{*}\hat{H}_{(2)}$ are all proportional to $\mathrm{vol}(M_5)$, which is a top-form on M_5. This means that $d(e^S\hat{*}\hat{\mathcal{F}}_{(2)}) = d(e^S\hat{*}\hat{F}_{(2)}) = d(e^S\hat{*}\hat{H}_{(2)}) = 0$ for any field $S\in C^\infty(M_5)$, and $d\hat{\sigma}\wedge\hat{*}\hat{F}_{(2)} = d\hat{\sigma}\wedge\hat{*}\hat{H}_{(2)} = 0$. Therefore, the only non-trivial equation from the above is the $\hat{\sigma}$ equation. After some algebra, we find that the $\hat{\sigma}$ equation reads

$$d\big(e^{\sqrt{2}\Phi_1}*d\sigma\big) = 2g^2 e^{\sqrt{\frac{2}{5}}\Phi_2-\frac{8}{\sqrt{15}}\Phi_3}\big(2+e^{\sqrt{2}\Phi_1}(\sigma^2+m)\big)\sigma*1\,, \tag{E.4.9}$$

where we used the identity $*1 = \mathrm{vol}(M_5)$.

Next, we have the seven-dimensional dilaton equations,

$$d\hat{*}d\hat{\Phi}_1 + \frac{1}{\sqrt{2}}\big(e^{-\sqrt{2}\hat{\Phi}_1+\sqrt{\frac{2}{5}}\hat{\Phi}_2}\hat{\mathcal{F}}_{(2)}\wedge\hat{*}\hat{\mathcal{F}}_{(2)} - e^{\sqrt{2}\hat{\Phi}_1}d\hat{\sigma}\wedge\hat{*}d\hat{\sigma} - e^{\sqrt{2}\hat{\Phi}_1+\sqrt{\frac{2}{5}}\hat{\Phi}_2}\hat{H}_{(2)}\wedge\hat{*}\hat{H}_{(2)}\big) = 0\,, \tag{E.4.10}$$

$$d\hat{*}d\hat{\Phi}_2 - \frac{1}{\sqrt{10}}e^{\sqrt{\frac{2}{5}}\hat{\Phi}_2}\big(e^{-\sqrt{2}\hat{\Phi}_1}\hat{\mathcal{F}}_{(2)}\wedge\hat{*}\hat{\mathcal{F}}_{(2)} + \hat{F}_{(2)}\wedge\hat{*}\hat{F}_{(2)} + e^{\sqrt{2}\hat{\Phi}_1}\hat{H}_{(2)}\wedge\hat{*}\hat{H}_{(2)}\big) = 0\,, \tag{E.4.11}$$

where we have substituted the ansatz $\hat{H}_{(3)} = 0$. Using (E.4.8), we find that these equations become

$$d*d\Phi_1 = \frac{1}{\sqrt{2}}e^{\sqrt{2}\Phi_1}d\sigma\wedge*d\sigma + \frac{g^2}{\sqrt{2}}e^{\sqrt{\frac{2}{5}}\Phi_2-\frac{8}{\sqrt{15}}\Phi_3}\big(e^{\sqrt{2}\Phi_1}(\sigma^2+m)^2 - 4e^{-\sqrt{2}\Phi_1}\big)*1\,, \tag{E.4.12}$$

$$d*d\Phi_2 = \frac{1}{\sqrt{10}}g^2 e^{\sqrt{\frac{2}{5}}\Phi_2-\frac{8}{\sqrt{15}}\Phi_3}\big(e^{\sqrt{2}\Phi_1}(\sigma^2+m)^2 + 4e^{-\sqrt{2}\Phi_1} + 4\sigma^2\big)*1\,. \tag{E.4.13}$$

Finally, we have the seven-dimensional Einstein equation

$$\begin{aligned}\hat{R}_{MN} &= \frac{1}{2}\partial_M\hat{\Phi}_i\partial_N\hat{\Phi}_i + \frac{1}{2}e^{\sqrt{2}\hat{\Phi}_1}\partial_M\hat{\sigma}\partial_N\hat{\sigma} \\ &+ \frac{1}{2}e^{-\sqrt{2}\hat{\Phi}_1+\sqrt{\frac{2}{5}}\hat{\Phi}_2}\Big(\hat{\mathcal{F}}_{MP}\hat{\mathcal{F}}_N{}^P - \frac{1}{10}\big(\hat{\mathcal{F}}_{(2)}\big)^2\hat{g}_{MN}\Big) \\ &+ \frac{1}{2}e^{\sqrt{\frac{2}{5}}\hat{\Phi}_2}\Big(\hat{F}_{MP}\hat{F}_N{}^P - \frac{1}{10}\big(\hat{F}_{(2)}\big)^2\hat{g}_{MN}\Big) \\ &+ \frac{1}{2}e^{\sqrt{2}\hat{\Phi}_1+\sqrt{\frac{2}{5}}\hat{\Phi}_2}\Big(\hat{H}_{MP}\hat{H}_N{}^P - \frac{1}{10}\big(\hat{H}_{(2)}\big)^2\hat{g}_{MN}\Big)\,,\end{aligned} \tag{E.4.14}$$

where $i \in \{1,2\}$, and we have substituted the $\hat{H}_{(3)} = 0$ ansatz. We have to consider the equations where the indices M, N lie in the five-dimensional directions and the S^2 directions independently. Let $A, B, \ldots$ be the five-dimensional indices, and $m, n, \ldots$ be the S^2 indices. The $\hat{R}_{mn}$ equations give

$$d{*}d\Phi_3 = -\frac{4g^2}{\sqrt{15}}e^{\sqrt{\frac{2}{5}}\Phi_2-\frac{8}{\sqrt{15}}\Phi_3}\big(e^{\sqrt{2}\Phi_1}(\sigma^2+m)^2 + 4\sigma^2 + 4e^{-\sqrt{2}\Phi_1} - 10e^{-\sqrt{\frac{2}{5}}\Phi_2+\sqrt{\frac{3}{5}}\Phi_3}\big){*}1\,. \tag{E.4.15}$$

The $\hat{R}_{Am}$ equations give a $0 = 0$ identity, and the remaining $\hat{R}_{AB}$ equations read

$$\begin{aligned}R_{AB} &= \frac{1}{2}\partial_A\Phi_i\partial_B\Phi_i + \frac{1}{2}e^{\sqrt{2}\Phi_1}\partial_A\sigma\partial_B\sigma \\ &+ \frac{2g^2}{3}e^{\sqrt{\frac{2}{5}}\Phi_2-\frac{8}{\sqrt{15}}\Phi_3}\Big(e^{-\sqrt{2}\Phi_1} + \sigma^2 + \frac{1}{4}e^{\sqrt{2}\Phi_1}(\sigma^2+m)^2 - 4e^{-\sqrt{\frac{2}{5}}\Phi_2+\sqrt{\frac{3}{5}}\Phi_3}\Big)g_{AB}\,,\end{aligned} \tag{E.4.16}$$

where $i \in \{1,2,3\}$. The five-dimensional equations (E.4.9), (E.4.12), (E.4.13), (E.4.15), and (E.4.16) can be obtained from the action

$$\mathcal{L}_5 = R{*}1 - \frac{1}{2}d\Phi_i\wedge{*}d\Phi_i - \frac{1}{2}e^{\sqrt{2}\Phi_1}d\sigma\wedge{*}d\sigma - V{*}1\,, \tag{E.4.17}$$

where V is the scalar potential given by

$$V = 2g^2e^{\sqrt{\frac{2}{5}}\Phi_2-\frac{8}{\sqrt{15}}\Phi_3}\Big(e^{-\sqrt{2}\Phi_1} + \sigma^2 + \frac{1}{4}e^{\sqrt{2}\Phi_1}(\sigma^2+m)^2 - 4e^{-\sqrt{\frac{2}{5}}\Phi_2+\sqrt{\frac{3}{5}}\Phi_3}\Big)\,. \tag{E.4.18}$$

The five-dimensional Newton constant is related to the ten-dimensional one by

$$\hat{\kappa}^2 = \frac{g^4\hat{\kappa}_{10}^2}{2\pi^3 l_y}\,. \tag{E.4.19}$$

The seven-dimensional background solution (E.3.3) is now reduced to

$$ds_5^2 = (W\cosh 2\rho)^{\frac{1}{3}}(\sinh 2\rho)^{\frac{2}{3}}\Big(\eta_{\mu\nu}dx^\mu dx^\nu + \frac{W\cosh 2\rho}{g^2}d\rho^2\Big)\,, \quad e^{-\sqrt{2}\Phi_1} = W(\sinh 2\rho)^2 \mathrm{sech} 2\rho\,,$$
$$e^{\sqrt{10}\Phi_2} = W^{-1}(\sinh 2\rho)^2\mathrm{sech}2\rho\,, \quad e^{\sqrt{15}\Phi_3} = (W\cosh 2\rho)^4(\sinh 2\rho)^2\,, \quad \sigma = \sqrt{2}\mathrm{sech}2\rho\,. \tag{E.4.20}$$

The NS5-brane charge k is related to the parameter m by $k = -1 - m/2$. Since $k \geq 0$, we find that $m \leq -2$. For the purposes of the present work, we set $k = 0$, so $m = -2$.