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Quantum corrections to newtonian potential and generalized uncertainty principle

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Abstract. We use the leading quantum corrections to the newtonian potential to compute the deformation parameter of the generalized uncertainty principle. By assuming just only General Relativity as theory of Gravitation, and the thermal nature of the GUP corrections to the Hawking spectrum, our calculation gives, to first order, a specific numerical result. We briefly discuss the physical meaning of this value, and compare it with the previously obtained bounds on the generalized uncertainty principle deformation parameter.

1. Introduction

Gravitation plays a pivotal role in the generalization of Heisenberg uncertainty principle (HUP), from the early attempts [1], to the more modern proposals, as string theory, loop quantum gravity, deformed special relativity, and studies of black hole physics [2, 3, 4, 5, 6, 7].

The dimensionless deforming parameter of the generalized uncertainty principle (GUP), usually denoted by β, is not (in principle) fixed by the theory, although it is generally assumed to be of order one (this happens, in particular, in some models of string theory, see for instance Ref. [2]). Therefore, many studies try to set bounds on β. For instance, in Refs. [8], [9], [10] a specific (in general, non linear) representation of the operators in the deformed fundamental commutator is utilized [1]

$$\left[\hat{X}, \hat{P}\right] = i\,\hbar \left(1 + \beta\, \frac{\hat{P}^2}{m_{\mathrm{p}}^2}\right) \tag{1}$$

in order to compute corrections to quantum mechanical quantities, such as energy shifts in the spectrum of the hydrogen atom, or to the Lamb shift, the Landau levels, Scanning Tunneling

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[1] We shall work with $c = k_B = 1$, but explicitly show the Newton constant G_{N} and Planck constant $\hbar$. We also recall that the Planck length is defined as $\ell_{\mathrm{p}}^2 = G_{\mathrm{N}}\,\hbar/c^3$, the Planck energy as $\mathcal{E}_{\mathrm{p}}\,\ell_{\mathrm{p}} = \hbar\,c/2$, and the Planck mass as $m_{\mathrm{p}} = \mathcal{E}_{\mathrm{p}}/c^2$, so that $G_{\mathrm{N}} = \ell_{\mathrm{p}}/2\,m_{\mathrm{p}}$ and $\hbar = 2\,\ell_{\mathrm{p}}\,m_{\mathrm{p}}$.

Microscope, charmonium levels, etc. The bounds so obtained on β are quite stringent, ranging from $\beta < 10^{21}$ to $\beta < 10^{50}$.

In Refs. [11] and [12], a different path is proposed in order to find bounds on β. A deformation of classical newtonian mechanics is introduced by modifying the standard Poisson brackets in a way that resembles the quantum commutator

$$[\hat{x}, \hat{p}] = i\,\hbar\left(1 + \beta_0\,\hat{p}^2\right) \Rightarrow \{X, P\} = \left(1 + \beta_0\,P^2\right) \tag{2}$$

where $\beta_0 = \beta/m_{\mathrm{p}}^2$. However, in the limit $\beta \to 0$, Ref. [11] recovers *only* the newtonian mechanics but *not* General Relativity, and GR corrections must be added as an extra structure. The physical relevance of this approach (and the bound they get for β) remains therefore questionable.

Finally, in Refs. [13], [14], [15], authors consider the gravitational interaction when evaluating bounds on β. In the first two papers, a covariant formalism is used, which firstly is defined in Minkowski space, with the metric $\eta_{\mu\nu} = \mathrm{diag}(1, -1, -1, -1)$, and then easily generalized to curved space-times via the standard procedure $\eta_{\mu\nu} \to g_{\mu\nu}$. However, these two papers (as the previous ones) start from a deformation of classical Poisson brackets, although posited in covariant form. They obtain interesting consequences, like a β-deformed geodesic equation, which leads to a violation of the Equivalence Principle. This formalism remains anyway covariant when $\beta \to 0$ and it reproduces the standard GR results in the limit $\beta \to 0$ (unlike papers as Ref. [11]). In the third paper, Ref. [15], on the contrary, Poisson brackets and classical newtonian mechanics remain untouched. Therefore, the Equivalence Principle is preserved, and the equation of motion of a test particle is still given by the standard geodesic equation. Additionally, also here GR and standard quantum mechanics are recovered, when $\beta \to 0$. The bounds on β proposed by papers which take into account gravity range from $\beta < 10^{19}$, for those papers admitting a violation of Equivalence Principle, to $\beta < 10^{69}$ for the papers preserving the aforesaid principle.

In the present paper, following our main work in Ref. [16], we arrive directly to compute the value of β by comparing two different low energy (first order in $\hbar$) corrections for the expression of the Hawking temperature. The first is due to the GUP, and therefore involves β. The second correction, instead, is obtained by including the deformation of the metric due to quantum corrections to the newtonian potential. We require the two corrections to match (at the first order), and therefore we get a specific numerical value for β. It results to be of order of unity, in agreement with the general belief and with some particular models of string theory.

2. GUP-deformed Hawking temperature

One of the most common forms of deformation of the HUP (as well as the form of GUP that we are going to study in this paper) is

$$\Delta x\,\Delta p \geq \frac{\hbar}{2}\left(1 \;+\; \beta\,\frac{4\,\ell_{\mathrm{p}}^2}{\hbar^2}\,\Delta p^{\,2}\right) = \frac{\hbar}{2}\left[1 + \beta\left(\frac{\Delta p}{m_{\mathrm{p}}}\right)^2\right] \tag{3}$$

which, for mirror-symmetric states (with $\langle\hat{p}\rangle^2 = 0$), can be equivalently written in terms of commutators as

$$[\hat{x}, \hat{p}] = i\hbar\left[1 + \beta\left(\frac{\hat{p}}{m_{\mathrm{p}}}\right)^2\right] \tag{4}$$

since $\Delta x\,\Delta p \geq (1/2)|\langle[\hat{x}, \hat{p}]\rangle|$.

As is well known from the argument of the Heisenberg microscope [17], the size δx of the smallest detail of an object, theoretically detectable with a beam of photons of energy E, is roughly given by

$$\delta x \simeq \frac{\hbar}{2\,E} \tag{5}$$

since larger and larger energies are required to explore smaller and smaller details. From the uncertainty relation (3), we see that the GUP version of the standard Heisenberg formula (5) is

$$\delta x \simeq \frac{\hbar}{2\,E} + 2\,\beta\,\ell_{\mathrm{p}}^2\,\frac{E}{\hbar} \tag{6}$$

which relates the (average) wavelength of a photon to its energy E [2]. Conversely, using relation (6), one can compute the energy E of a photon with a given (average) wavelength $\lambda \simeq \delta x$. To compute the thermal GUP corrections to the Hawking spectrum, we follow the arguments of Refs. [18, 19, 20, 21, 22, 23, 24], and we arrive to the expression

$$T = \frac{\hbar}{8\pi GM}\left(1 + \frac{\beta\,m_{\mathrm{p}}^2}{4\pi^2\,M^2} + \cdots\right)\ . \tag{7}$$

where it is evident that to zero order in β, we recover the usual well known Hawking formula. We stress that we are here assuming that the correction induced by the GUP has a thermal character, and, therefore, it can be cast in the form of a shift of the Hawking temperature. Of course, there are also different approaches, where the corrections do not respect the exact thermality of the spectrum, and thus need not be reducible to a simple shift of the temperature. An example is the corpuscular model of a black hole of Ref. [25]. In this model, the emission is expected to gain a correction of order $1/N$, where $N \sim (M/m_{\mathrm{p}})^2$ is the number of constituents, and it becomes important when the mass M approaches the Planck mass.

3. Leading quantum correction to the newtonian potential
After early results by Duff [26], the leading quantum correction to the newtonian potential has been computed by Donoghue, by assuming General Relativity as fundamental theory of Gravity. In a series of beautiful papers (see for instance Ref. [27]) he reformulated General Relativity as an effective field theory, and, in particular, he considered two heavy bodies close to rest. The leading quantum correction derived from this model shows a long-distance *quantum* effect. More recently, Donoghue and other authors found that the gravitational interaction between the two objects can be described by the potential energy [28]

$$U(r) = -\frac{GMm}{r}\left(1 + \frac{3G(M+m)}{rc^2} + \frac{41}{10\pi}\frac{\ell_P^2}{r^2}\right). \tag{8}$$

The first correction term does not contain any power of $\hbar$, so it is a classical effect, due to the non-linear nature of General Relativity. However, the second correction term, i.e., the last term of (8), is a true quantum effect, linear in $\hbar$. The potential generated by the mass M reads

$$V(r) = -\frac{GM}{r}\left(1 + \frac{3GM}{r}(1 + \frac{m}{M}) + \frac{41}{10\pi}\frac{\ell_P^2}{r^2}\right). \tag{9}$$

4. Metric mimicking the quantum corrected newtonian potential
The effective newtonian potential [3] produced by a metric of the very general class

$$ds^2 = F(r)dt^2 - g_{ik}(x_1, x_2, x_3)dx^i dx^k \tag{10}$$

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[2] Here, the standard dispersion relation $E = p\,c$ is assumed.
[3] The effective newtonian potential is produced by the metric given in (10) for a point particle which moves slowly, in a stationary and weak gravitational field, i.e., quasi-Minkowskian far from the source, $r \to \infty$.

where $r = |\mathbf{x}| = (x_1^2 + x_2^2 + x_3^2)^{1/2}$, and x_1, x_2, x_3 are the standard Cartesian coordinates, can be obtained with well known procedures [29], and is of the form

$$V(r) \simeq \frac{1}{2}\left(F(r) - 1\right) \tag{11}$$

or, equivalently,

$$F(r) \simeq 1 + 2V(r) . \tag{12}$$

We can therefore write down the metric which is able to mimic the quantum corrected newtonian potential proposed by Donoghue. Recalling (9), we have

$$F(r) \simeq 1 + 2V(r) = 1 - \frac{2GM}{r} - \frac{6\,G^2M^2}{r^2}\left(1 + \frac{m}{M}\right) - \frac{41}{5\pi}\frac{G^3M^3}{r^3}\left(\frac{\ell_P}{GM}\right)^2 .$$

Let us now define

$$\epsilon(r) = -\frac{6\,G^2M^2}{r^2}\left(1 + \frac{m}{M}\right) - \frac{41}{5\pi}\frac{G^3M^3}{r^3}\left(\frac{\ell_P}{GM}\right)^2 . \tag{13}$$

Therefore, $F(r)$ will now be of the form

$$F(r) = 1 - \frac{2GM}{r} + \epsilon(r) \tag{14}$$

and it is evident that when r is large, then $|\epsilon(r)| \ll 2GM/r$.

5. Temperature from a deformed Schwarzschild metric

We can legitimately wonder what kind of deformed Hawking temperature can be inferred from a deformed Schwarzschild metric as in (14) [4] The deformation (14) makes sense when $|\epsilon(r)| \ll GM/r$. We can also introduce a regulatory small parameter ε and, thus, we can write $\epsilon(r) \equiv \varepsilon\phi(r)$. At the end of the calculation, ε can go to unity. Of course, we look for the lowest order correction in the dimensionless parameter ε. The horizon's equation, i.e., $F(r) = 0$, now reads

$$r - 2\,G\,M + \varepsilon\,r\,\phi(r) = 0 . \tag{15}$$

Such equations can be solved, in a first approximation in ε, as follows. First, we formulate (15) in a general form

$$x = a + \varepsilon f(x) . \tag{16}$$

It is obvious that if ε is set equal to zero, then the solution will be $x_0 = a$. If ε is slightly different from zero, then we can try a test solution of the form $x_0 = a + \eta(\varepsilon)$ where $\eta(\varepsilon) \to 0$ for $\varepsilon \to 0$. Substituting the aforesaid test solution in (16), we get $x_0 = a + \varepsilon f(x_0)$ which means $\eta = \varepsilon f(a + \eta)$. To first order in η, we have $\eta = \varepsilon[f(a) + f'(a)\eta]$ from which we obtain $\eta = \varepsilon f(a)/[1 - \varepsilon f'(a)]$. Therefore, to first order in ε, the general solution of (16) reads

$$x_0 = a + \frac{\varepsilon f(a)}{1 - \varepsilon f'(a)} .$$

Applying this formula to (15), we get the solution

$$r_H = a - \frac{\varepsilon\,a\,\phi(a)}{1 + \varepsilon\,[\phi(a) + a\,\phi'(a)]} \tag{17}$$

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[4] Recently, it was argued that in the special case in which $\epsilon(r) \sim 1/r^2$, the specific metric (14) could have some drawbacks in the context of GUP formalism [30]. However, none of those drawbacks appear here and, thus, there is no problem to employ (14) in our present study.

where $a = 2GM$.

The Hawking temperature is given by

$$T = \frac{\hbar}{4\pi} F'(r_H) \ . \tag{18}$$

Using Eq. (14) for $F(r)$, and Eq. (17) for r_H, and expanding in ε, one arrives to the deformed Hawking temperature as

$$T = \hbar \frac{F'(r_H)}{4\pi} = \frac{\hbar}{4\pi a} \left\{ 1 + \varepsilon \left[2\phi(a) + a\phi'(a) \right] + \varepsilon^2 \phi(a) \left[\phi(a) - 2a\phi'(a) - a^2 \phi''(a) \right] + \dots \right\} \ . \tag{19}$$

It is noteworthy that the only function $\phi(r)$ that annihilates the first-order in ε temperature correction term is the solution of the differential equation $2\phi(r) + r\phi'(r) = 0$, namely $\phi(r) = A/r^2$, where A is an arbitrary constant. In particular, for the function $\phi(r) = G^2 M^2/r^2$, the coefficient of ε in (19) is zero, and the coefficient of ε^2 is $-1/16$. It is also interesting to investigate what kind of function will eliminate the second-order in ε correction term. This function will be the solution of the differential equation $r^2 \phi''(r) + 2r\phi'(r) - \phi(r) = 0$ which is an Euler equation. Its characteristic equation is of the form $\lambda^2 + \lambda - 1 = 0$ with roots $\lambda_1 = (-1 - \sqrt{5})/2$ and $\lambda_2 = (-1 + \sqrt{5})/2$. So, the functions which remove the ε^2-correction term are $\phi_1(r) = r^{-|\lambda_1|}$ and $\phi_2(r) = r^{\lambda_2}$.

6. Computing β

We are now in the position to compute the temperature generated by the metric (14), by simply employing (19). Therefore, the metric-deformed Hawking temperature is of the form [5]

$$T = \frac{\hbar}{4\pi a} \left\{ 1 + \left[2\epsilon(a) + a\epsilon'(a) \right] + \dots \right\} \tag{20}$$

while the GUP-deformed Hawking temperature reads

$$T = \frac{\hbar}{8\pi GM} \left(1 + \frac{\beta\, m_{\mathrm{p}}^2}{4\pi^2\, M^2} + \dots \right) \ . \tag{21}$$

By comparing the two respective first-order correction terms in the two previous expansions, we obtain

$$\beta = \frac{4\pi^2 M^2}{m_{\mathrm{p}}^2} \left[2\epsilon(a) + a\epsilon'(a) \right] \ . \tag{22}$$

Using now expression (13) for $\epsilon(r)$, we get

$$2\epsilon(r) + r\epsilon'(r) = \frac{B}{r^3} \tag{23}$$

with $B = \frac{41}{5\pi} \frac{G^3 M^3}{} \left(\frac{\ell_P}{GM} \right)^2$. Therefore,

$$2\epsilon(a) + a\epsilon'(a) = \frac{B}{8G^3 M^3} \tag{24}$$

and using (22), the parameter β will get the value

$$\beta = \frac{4\pi^2 M^2}{m_{\mathrm{p}}^2} \frac{41}{40\pi} \left(\frac{\ell_P}{GM} \right)^2 = \frac{82\pi}{5} \ . \tag{25}$$

[5] Notice that from (13) we have $\epsilon(r) \sim 1$ for $r \sim a$, so this would seem to spoil the expansion (20) when $r \sim a$. On the contrary, we can always imagine to first expand the temperature $T(r) = \hbar F'(r)/4\pi$ for $r \gg a$, when $\epsilon(r)$ is small. Then, the term in $1/r^2$ disappears from the expansion of $T(r)$ because of the condition $2\phi(r) + r\phi'(r) = 0$. Finally, we take the limit $r \to a$, and this yields (20).

7. Conclusions

In this work we have computed the value of the deformation parameter β of the GUP. We obtain this result by computing in two different ways the Hawking temperature for a Schwarzschild black hole.

The first way consists in using the GUP (in place of the standard HUP) to compute the Hawking formula. We get an expression of the temperature containing a correction term depending on β, i.e., the GUP-deformed Hawking temperature (7).

The second way involves the consideration of the quantum correction to the newtonian potential, computed years ago by Donoghue and others. The corrections to the newtonian potential imply naturally a quantum correction to the Schwarzschild metric. Therefore, the Hawking temperature computed through this quantum corrected Schwarzschild metric result to get corrections in respect to the standard Hawking expression, i.e., the metric-deformed Hawking temperature (20).

The request that the first-order corrections of the two different expressions of Hawking temperature must coincide, fixes unambiguously the numerical value of β to be $82\pi/5$.

Finally, a couple of comments are in order here. First, this numerical value is of order one, as expected from several string theory models, and from versions of GUP derived through gedanken experiments. In particular, this is the first time, to our knowledge, that a specific value is obtained for β by starting from the minimal assumptions we made. Second, as we know, in the last years much research has focused on the experimental bounds of the *size* of β, and several experiments have been proposed to test GUPs in the laboratory. In fact, it has been shown that one does not need to reach the Planck energy scale to test GUP corrections. Among the more elaborated proposals, where conditions can be created in a lab, are those of the groups of Refs. [31, 32, 33]. However, it is also worth of note that the best bounds on β presented in the literature are still by far much larger than the value computed here. This could require, presumably, a big leap in the experimental designs and techniques in order to search this region for the parameter β.

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