Prog. Theor. Exp. Phys. 2022 013B04(28 pages)
DOI: 10.1093/ptep/ptab152

Majorana neutrino masses by D-brane instanton effects in magnetized orbifold models

Kouki Hoshiya[1], Shota Kikuchi[1], Tatsuo Kobayashi[1], Kaito Nasu[1], Hikaru Uchida[1,⋆], and Shohei Uemura[2]

[1] *Department of Physics, Hokkaido University, Sapporo 060-0810, Japan*
[2] *CORE of STEM, Nara Women's University, Nara 630-8506, Japan*
⋆E-mail: h-uchida@particle.sci.hokudai.ac.jp

Received October 11, 2021; Revised November 12, 2021; Accepted November 26, 2021; Published November 30, 2021

We study Majorana neutrino masses induced by D-brane instanton effects in magnetized orbifold models. We classify the possible cases where neutrino masses can be induced. Three and four generations are favored in order to generate neutrino masses by D-brane instantons. Explicit mass matrices have specific features. Their diagonalizing matrices correspond to the bimaximal mixing matrix in the case with even magnetic fluxes, independently of the modulus value τ. On the other hand, for odd magnetic fluxes, diagonalizing matrices correspond nearly to the tri-bimaximal mixing matrix near $\tau = i$, while they become the bimaximal mixing matrix for larger Im τ. For even fluxes, neutrino masses are modular forms of weight 1 on $T^2/\mathbb{Z}_2$, and they have symmetries such as S_4' and $\Delta'(96) \times \mathbb{Z}_3$.

Subject Index B27, B29, B33, B41, B54

1. Introduction

Superstring theory is a promising candidate for the unified theory of all the interactions including gravity, matter such as quarks and leptons, and the Higgs particle. It predicts six dimensions (6D) in addition to our four-dimensional (4D) spacetime. The 6D space must be compact. Geometrical aspects of the 6D compact space as well as gauge background fields determine phenomenological properties of particle physics such as generation numbers, flavor structure of quarks and leptons, and gauge and Yukawa coupling strengths.

Compactification with magnetic flux background is quite interesting in higher-dimensional theory and superstring theory. The magnetic flux background can lead to a 4D chiral theory even on the torus compactification [1–4], although one cannot realize a 4D chiral theory on the torus without magnetic fluxes. That is, chiral zero-modes appear and their number depends on the size of the magnetic flux. Their wavefunctions are non-trivially quasi-localized. Yukawa couplings as well as higher-order couplings are written by overlap integrals of wavefunctions [5,6]. They can be of $\mathcal{O}(1)$ or exponentially suppressed depending on the distances of quasi-localizing positions among quarks, leptons, and the Higgs mode.

In addition to the simple torus compactification, the orbifold compactification with magnetic flux is quite interesting. This can project out adjoint matter, i.e. open string moduli, and lead to numbers of chiral zero-modes different from the torus compactifictaion [7–9].[1] One can construct various models by properly choosing orbifold parities and Scherk–Schwarz (SS) phases

[1]For the numbers of zero-modes, see also Refs. [10–12].

as well as discrete Wilson lines.[2] The three-generation models in magnetized orbifold models have been classified [14–16]. Furthermore, realistic quark masses and mixing angles as well as the CP phase have been studied [17–20], as has realization of charged lepton masses. Recently, their flavor structure was studied from the viewpoint of modular symmetry [21–28].

Small neutrino masses can be realized by the see-saw mechanism as well as the Weinberg operators. Here, we concentrate on the see-saw mechanism, where right-handed neutrinos and their Majorana masses are introduced. Such Majorana masses can be induced by non-perturbative effects, i.e. D-brane instanton effects [29,30], although there is no mass scale perturbatively below the compactification scale in superstring theory and super Yang–Mills theory, which is the low-energy effective field theory of superstring theory. (For explicit models, see also Refs. [31–34].) It is very important to study explicitly which patterns of Majorana masses are induced by D-brane instanton effects. Thus, our purpose in this paper is to systematically study patterns of Majorana mass matrices induced by D-brane effects in magnetized orbifold models, in particular $T^2/\mathbb{Z}_2$ orbifold models.

When a D-brane instanton appears, new zero-modes, β_i and γ_j, also appear which correspond to open strings between the D-brane instanton and the D-branes for the right-handed neutrinos. We integrate the new zero-modes so as to obtain non-perturbative correction terms. The Majorana mass terms can be induced only when we have a certain number of the new zero-modes, β_i and γ_j. In magnetized orbifold models, the number of zero-modes is determined by the size of magnetic fluxes, $\mathbb{Z}_2$-parity, and SS phases. In this paper we systematically study D-brane configurations leading to Majorana neutrino masses and compute the patterns of neutrino mass matrices. One needs two universal zero-modes, which correspond to the Grassmann coordinates θ, in order to generate Majorana neutrino masses in 4D $N = 1$ supersymmetric theory [29,30]. We assume that such universal zero-modes remain through orientifold projection, and the other neutral zero-modes are projected out. In other words, we assume that the D-brane instanton is the so-called O(1) instanton. D-brane instatnons can have additional neutral zero-modes corresponding to the deformations of the wrapped cycles. These additional zero-modes can spoil the calculations, and they must be eliminated by orbifold projections or become massive. Furthermore, full stringy consistent models should satisfy the tadpole condition. However, such conditions on full string models depend on gauge groups of the visible sector, the quark sector, the charged lepton sector, and the hidden sector. Assuming these stringy consistency conditions, we focus on the neutrino sector and study possible patterns of Majorana neutrino mass matrices. Such D-brane configurations can actually be found in six-dimensional toroidal orbifolds, and the Majorana mass term has been calculated in the full stringy type-IIA orientifold model [33]. On the other hand, our approach is to study low-energy effective field theory in order to systematically study the possible patterns of neutrino mass matrices. We study supersymmetric Yang–Mills theory on the orbifold compactification with magnetic fluxes. We classify the possible configurations of magnetic fluxes and brane-instanton[3] configurations leading to Majorana neutrino mass matrices. We also study their symmetries.

This paper is organized as follows. In Sect. 2, we give a brief review of the number of zero-modes, their wavefunctions, and three-point couplings. In Sect. 3, we study systematically D-brane configurations, where Majorana neutrino masses can be induced by D-brane instanton

[2]For shifted orbifold models, see Ref. [13].

[3]The brane instanton would be a classical solution in low-energy effective supergravity theory.

effects. In Sect. 4, we explicitly compute the patterns of Majorana neutrino mass matrices. Section 5 presents our conclusions. In Appendix A, we briefly review the modular symmetry in magnetized torus and orbifold models. In Appendix B, we review wavefunctions on the $T^2/\mathbb{Z}_4$ orbifold.

2. Orbifold compactification with magnetic fluxes

Here we review torus and orbifold compactfications with magnetic fluxes [5,7–9]. We study magnetic flux compactification within the framework of higher-dimensional super Yang–Mills theory, which is the low-energy effective field theory of superstring theory. We concentrate on the extra two dimensions, T^2 and $T^2/\mathbb{Z}_2$. Our models can correspond to the D5-brane system as well as D7- and D9-brane systems, assuming that other extra dimensions are irrelevant to the mass ratios and mixing angles of the Majorana neutrino mass matrix, but relevant only to the overall factor of the mass matrix. Also, we assume that these D-brane systems preserve 4D $N = 1$ supersymmetry.

2.1 *Torus compactification with magnetic fluxes*

In this subsection we review zero-mode wavefunctions on a two-dimensional torus T^2 with $U(N)$ magnetic flux. First, T^2 is obtained by the identification $z \sim z + 1 \sim z + \tau$, where we use the complex coordinate $z \equiv x + \tau y$, and τ ($\tau \in \mathbb{C}$, $\mathrm{Im}\,\tau > 0$) is the complex structure modulus of T^2. Let us consider the following $U(N)$ magnetic flux on T^2:

$$F = \frac{\pi i}{\mathrm{Im}\,\tau}(dz \wedge d\bar{z})\begin{pmatrix} M_1 \mathbb{I}_{N_1\times N_1} & & \\ & \ddots & \\ & & M_n \mathbb{I}_{N_n\times N_n}\end{pmatrix}, \tag{1}$$

where $\mathbb{I}_{N_a\times N_a}$ ($a = 1, \ldots, n$) denotes the ($N_a \times N_a$) unit matrix with $\sum_{a=1}^{n} N_a = N$, and M_a ($a = 1, \ldots, n$) must be an integer. This magnetic flux is induced by the background gauge field

$$A(z) = \frac{\pi}{\mathrm{Im}\,\tau}\mathrm{Im}\,(\bar{z}dz)\begin{pmatrix} M_1 \mathbb{I}_{N_1\times N_1} & & \\ & \ddots & \\ & & M_n \mathbb{I}_{N_n\times N_n}\end{pmatrix}, \tag{2}$$

where we can always set Wilson lines vanishing by introducing SS phases instead, as shown in Ref. [8]. From this gauge background, $U(N)$ gauge symmetry is broken to $\Pi_{a=1}^{n} U(N_a)$.

Here, we consider a two-dimensional spinor on T^2 under the breaking $U(N) \to U(N_1) \times U(N_2)$ by the above background magnetic flux, i.e.

$$\psi(z) = \begin{pmatrix}\psi_+(z)\\ \psi_-(z)\end{pmatrix}, \qquad \psi_\pm(z) = \begin{pmatrix}\psi_\pm^{(11)}(z) & \psi_\pm^{(12)}(z)\\ \psi_\pm^{(21)}(z) & \psi_\pm^{(22)}(z)\end{pmatrix}, \tag{3}$$

where $\psi_\pm^{(11)}$ and $\psi_\pm^{(22)}$ correspond to $U(N_1)$ and $U(N_2)$ gauginos, respectively, and $\psi_\pm^{(12)}$ and $\psi_\pm^{(21)}$ correspond to $(N_1, \bar{N}_2)$ and $(\bar{N}_1, N_2)$ representations under $U(N_1) \times U(N_2)$. The above wavefunctions satisfy the boundary conditions

$$\psi_\pm^{(ab)}(z + 1) = e^{2\pi i \alpha_1^{(ab)}} e^{\pi i M_{ab}\frac{\mathrm{Im}\,z}{\mathrm{Im}\,\tau}} \psi_\pm^{(ab)}(z), \tag{4}$$

$$\psi_\pm^{(ab)}(z + \tau) = e^{2\pi i \alpha_\tau^{(ab)}} e^{\pi i M_{ab}\frac{\mathrm{Im}\,\bar{\tau}z}{\mathrm{Im}\,\tau}} \psi_\pm^{(ab)}(z), \tag{5}$$

where $M_{ab} \equiv M_a - M_b$ $(a, b = 1, 2)$ and $\alpha_i^{(ab)} \equiv \alpha_i^a - \alpha_i^b$ $(i = 1, \tau, a, b = 1, 2)$ denote the SS phases. When we solve the zero-mode Dirac equation,

$$i \not{D} \psi(z) = 0, \tag{6}$$

under the above boundary conditions with $M_{12} > 0$, only $\psi_+^{(12)}$ (as well as the anti-particle $\psi_-^{(21)}$) has an M_{12}-number of degenerate zero-modes. Then, a chiral theory is realized. The explicit form of the wavefunction with the magnetic flux M_{12} and the SS phases $(\alpha_1^{(12)}, \alpha_\tau^{(12)})$ is written by

$$\psi_{T^2}^{\left(j+\alpha_1^{(12)}, \alpha_\tau^{(12)}\right), |M_{12}|}(z) = \left(\frac{|M_{12}|}{\mathcal{A}^2}\right)^{1/4} e^{2\pi i \frac{(j+\alpha_1^{(12)})\alpha_\tau^{(12)}}{|M_{12}|}} e^{\pi i |M_{12}| z \frac{\mathrm{Im}\, z}{\mathrm{Im}\, \tau}} \vartheta \begin{bmatrix} \frac{j+\alpha_1^{(12)}}{|M_{12}|} \\ -\alpha_\tau^{(12)} \end{bmatrix} (|M_{12}| z, |M_{12}| \tau), \tag{7}$$

where $j \in \mathbb{Z}_{|M_{12}|}$, $\mathcal{A}$ denotes the area of T^2, and ϑ denotes the Jacobi theta function defined as

$$\vartheta \begin{bmatrix} a \\ b \end{bmatrix} (\nu, \tau) = \sum_{l \in \mathbb{Z}} e^{\pi i (a+l)^2 \tau} e^{2\pi i (a+l)(\nu+b)}. \tag{8}$$

The wavefunction satisfies the following normalization condition:

$$\int_{T^2} dz d\bar{z} \left(\psi_{T^2}^{\left(j+\alpha_1^{(12)}, \alpha_\tau^{(12)}\right), |M_{12}|}(z) \right)^* \psi_{T^2}^{\left(k+\alpha_1^{(12)}, \alpha_\tau^{(12)}\right), |M_{12}|}(z) = (2 \mathrm{Im}\, \tau)^{-1/2} \delta_{j,k}. \tag{9}$$

We note that the definition of the wavefunction in Eq. (7) is different from one in Ref. [8] by the phase factor $e^{2\pi i (j+\alpha_1^{(12)})\alpha_\tau^{(12)}/|M_{12}|}$, which does not affect the boundary conditions or the equation of motion. Similarly, we can consider the breaking case such as $U(N) \to \Pi_{a=1}^n U(N_a)$.

The three-point coupling, d^{ijk}, of the wavefunctions with the magnetic flux M_a and the SS phases $(\alpha_1^{(a)}, \alpha_\tau^{(a)})$ $(a = i, j, k)$ can be calculated as

$$\begin{aligned} d^{ijk} &= \int_{T^2} dz d\bar{z} \psi_{T^2}^{\left(i+\alpha_1^{(i)}, \alpha_\tau^{(i)}\right), |M_i|}(z) \cdot \psi_{T^2}^{\left(j+\alpha_1^{(j)}, \alpha_\tau^{(j)}\right), |M_j|}(z) \cdot \left(\psi_{T^2}^{\left(k+\alpha_1^{(k)}, \alpha_\tau^{(k)}\right), |M_k|}(z) \right)^* \\ &= c_{(M_i - M_j - M_k)} \exp \left\{ 2\pi i \left(\frac{\left(i+\alpha_1^{(i)}\right)\alpha_\tau^{(i)}}{M_i} + \frac{\left(j+\alpha_1^{(j)}\right)\alpha_\tau^{(j)}}{M_j} - \frac{\left(k+\alpha_1^{(k)}\right)\alpha_\tau^{(k)}}{M_k} \right) \right\} \\ &\quad \times \sum_{m=0}^{|M_k|-1} \vartheta \begin{bmatrix} \frac{|M_j|\left(i+\alpha_1^{(i)}\right) - |M_i|\left(j+\alpha_1^{(j)}\right) + |M_i M_j| m}{|M_i M_j M_k|} \\ 0 \end{bmatrix} \left(M_i \alpha_\tau^{(j)} - M_j \alpha_\tau^{(i)}, |M_i M_j M_k| \tau \right) \\ &\quad \times \delta_{\left(i+\alpha_1^{(i)}\right) + \left(j+\alpha_1^{(j)}\right) - \left(k+\alpha_1^{(k)}\right), |M_k| \ell - |M_i| m}, \end{aligned} \tag{10}$$

provided $M_i + M_j = M_k$, $\alpha_1^{(i)} + \alpha_1^{(j)} = \alpha_1^{(k)}$, and $\alpha_\tau^{(i)} + \alpha_\tau^{(j)} = \alpha_\tau^{(k)}$ are satisfied, where $\ell \in \mathbb{Z}$. Here, the coefficient $c_{(M_i - M_j - M_k)}$ is defined as

$$c_{(M_i - M_j - M_k)} = (2 \mathrm{Im}\, \tau)^{-1/2} \mathcal{A}^{-1/2} \left| \frac{M_i M_j}{M_k} \right|^{1/4}. \tag{11}$$

Table 1. The numbers of zero-modes with different SS phases.

$\|M\|$	1	2	3	4	5	6	7	8	9	10	11	12
SS phase $(\alpha_1, \alpha_\tau) = (0, 0)$												
$\mathbb{Z}_2$-even	1	2	2	3	3	4	4	5	5	6	6	7
$\mathbb{Z}_2$-odd	0	0	1	1	2	2	3	3	4	4	5	5
SS phase $(\alpha_1, \alpha_\tau) = (1/2, 0)$												
$\mathbb{Z}_2$-even	1	1	2	2	3	3	4	4	5	5	6	6
$\mathbb{Z}_2$-odd	0	1	1	2	2	3	3	4	4	5	5	6
SS phase $(\alpha_1, \alpha_\tau) = (0, 1/2)$												
$\mathbb{Z}_2$-even	1	1	2	2	3	3	4	4	5	5	6	6
$\mathbb{Z}_2$-odd	0	1	1	2	2	3	3	4	4	5	5	6
SS phase $(\alpha_1, \alpha_\tau) = (1/2, 1/2)$												
$\mathbb{Z}_2$-even	0	1	1	2	2	3	3	4	4	5	5	6
$\mathbb{Z}_2$-odd	1	1	2	2	3	3	4	4	5	5	6	6

To derive Eq. (10), we use the property

$$\vartheta\begin{bmatrix}\frac{r}{N_1}\\0\end{bmatrix}(\nu_1, N_1\tau)\times\vartheta\begin{bmatrix}\frac{s}{N_2}\\0\end{bmatrix}(\nu_2, N_2\tau)$$

$$=\sum_{m\in\mathbb{Z}_{N_1+N_2}}\vartheta\begin{bmatrix}\frac{r+s+N_1m}{N_1+N_2}\\0\end{bmatrix}(\nu_1+\nu_2, (N_1+N_2)\tau)$$

$$\times\vartheta\begin{bmatrix}\frac{N_2r-N_1s+N_1N_2m}{N_1N_2(N_1+N_2)}\\0\end{bmatrix}(\nu_1N_2-\nu_2N_1, N_1N_2(N_1+N_2)\tau). \tag{12}$$

2.2 *Orbifold compactification*

In this subsection we review zero-mode wavefunctions on the $T^2/\mathbb{Z}_2$ twisted orbifold with the magnetic flux in Eq. (1). The $T^2/\mathbb{Z}_2$ twisted orbifold is obtained further by identifying a $\mathbb{Z}_2$ twisted point $-z$ with z, i.e. $z\sim -z$. Then, the wavefunctions on the magnetized $T^2/\mathbb{Z}_2$ twisted orbifold with the magnetic flux M and the SS phases (α_1, α_τ) satisfy the boundary condition

$$\psi^{(j+\alpha_1,\alpha_\tau),|M|}_{T^2/\mathbb{Z}_2^m}(-z)=(-1)^m\psi^{(j+\alpha_1,\alpha_\tau),|M|}_{T^2/\mathbb{Z}_2^m}(z),\qquad m\in\mathbb{Z}_2 \tag{13}$$

in addition to Eqs. (4) and (5). These boundary conditions are simultaneously satisfied only in the case of $\mathbb{Z}_2$ SS phases,

$$(\alpha_1, \alpha_\tau)=(0,0), (1/2,0), (0,1/2), (1/2,1/2). \tag{14}$$

Hence, the wavefunctions on the magnetized $T^2/\mathbb{Z}_2$ twisted orbifold can be expressed by ones on magnetized T^2 as

$$\psi^{(j+\alpha_1,\alpha_\tau),|M|}_{T^2/\mathbb{Z}_2^m}(z)=\mathcal{N}\left(\psi^{(j+\alpha_1,\alpha_\tau),|M|}_{T^2}(z)+(-1)^m\psi^{(j+\alpha_1,\alpha_\tau),|M|}_{T^2}(-z)\right)$$

$$=\mathcal{N}\left(\psi^{(j+\alpha_1,\alpha_\tau),|M|}_{T^2}(z)+(-1)^{m-2\alpha_\tau}\psi^{(|M|-(j+\alpha_1),\alpha_\tau),|M|}_{T^2}(z)\right), \tag{15}$$

where

$$\mathcal{N}=\begin{cases}1/2 & (j+\alpha_1=0, |M|/2),\\ 1/\sqrt{2} & (\text{otherwise}).\end{cases} \tag{16}$$

Note that only when $j+\alpha_1=0$ is the factor $(-1)^{2\alpha_1}$ replaced by 1. Table 1 shows the numbers of the zero-modes. The normalization of the wavefunction and their three-point coupling are similarly obtained by replacing the wavefunctions on T^2 with ones on the $T^2/\mathbb{Z}_2$ twisted orbifold. Note that they are overall $\mathbb{Z}_2$ invariant.

3. Majorana neutrino masses by D-brane instanton effects

Here, we study Majorana neutrino masses induced by D-brane instanton effects in magnetized orbifold models.

3.1 *D-brane instanton effects*

We give a brief review of Majorana neutrino mass terms induced by D-brane instanton effects [29,30].

We consider two stacks of D-branes, D_{N1} and D_{N2}, with different magnetic fluxes. Zero-modes of open strings between these D-branes correspond to the neutrinos, N_a. We denote the difference of their magnetic fluxes M_N. The neutrino generation number is determined by M_N and boundary conditions such as SS phases and $\mathbb{Z}_2$ parity. We assume the D-brane instanton $D_{\rm inst}$ with magnetic flux, which has the zero-modes β_i (γ_j) between D_{N1} (D_{N2}) and $D_{\rm inst}$. The numbers of zero-modes β_i and γ_j are determined by magnetic fluxes in zero-mode equations, and the SS phases and $\mathbb{Z}_2$ parities of boundary conditions.

The Majorana mass terms of N_a due to D-brane instanton effects can be written by [29,30]

$$e^{-S_{\rm cl}(D_{\rm inst},M_{\rm inst})}\int d^2\beta d^2\gamma\, e^{-\sum_{ija} d_a^{ij}\beta_i\gamma_j N_a}, \tag{17}$$

where β_i and γ_j are Grassmannian, and d_a^{ij} denotes the three-point coupling among β_i, γ_j, and N_a. Here, $S_{\rm cl}(D_{\rm inst}, M_{\rm inst})$ denotes the classical action of the D-brane instanton written by the Dirac–Born–Infeld action, which depends on the D-brane instanton volume in the compact space and the magnetic flux. Mass terms can be induced only if each of β_i and γ_j has two zero-modes, i.e. β_1, β_2, γ_1, and γ_2. By the Grassmannian integral, we find

$$\int d^2\beta d^2\gamma\, e^{-\sum_{ija} d_a^{ij}\beta_i\gamma_j N_a} = N_a N_b\left(\varepsilon_{ij}\varepsilon_{k\ell}d_a^{ik}d_b^{j\ell}\right). \tag{18}$$

Thus, the mass matrix M_{ab} is obtained by

$$M_{ab} = e^{-S_{\rm cl}(D_{\rm inst},M_{\rm inst})}m_{ab}, \qquad m_{ab} = \left(\varepsilon_{ij}\varepsilon_{k\ell}d_a^{ik}d_b^{j\ell}\right). \tag{19}$$

Furthermore, we assume that there are just two neutral fermionic zero-modes, which are the zero-modes of the open string whose endpoints are attached to $D_{\rm inst}$. They are neutral under the gauge group on D_{N1} (D_{N2}), and correspond to the Grassmann coordinate of $N = 1$ SUSY. The number of neutral zero-modes also depends on the configuration of the D-brane instanton. In the tyep-IIA orientifold model, which is the T-dual of the magnetized torus, the D-brane instanton must wrap on an orientifold invariant cycle to generate non-perturbative superpotential [33]. This implies $M_{\rm inst} = 0$. Since the number of charged zero-modes (and neutrinos) depends only on the differences of magnetic fluxes, we can always set $M_{\rm inst} = 0$ without loss of generality.

3.2 *Classification of models with Majorana mass terms*

The number of zero-modes is required to be two for both β_i and γ_j in order to induce Majorana neutrino mass terms by D-brane instanton effects, as reviewed in the previous section. We consider the six-dimensional compact space which is a product of $T^2/\mathbb{Z}_2$ and four-dimensional compact space. Here, we focus on the two-dimensional $T^2/\mathbb{Z}_2$ orbifold, and we study the

Table 2. Configurations leading to two zero-modes for β_i and γ_j.

β_i, γ_j	$(\alpha_1, \alpha_\tau : M_{\beta,\gamma}, \mathbb{Z}_2$ parity)
i	(0, 0: 2, E)
ii	(0, 0: 3, E)
iii	(0, 0: 5, O)
iv	(0, 0: 6, O)
v	(1/2, 0: 3, E)
vi	(0, 1/2: 3, E)
vii	(1/2, 0: 4, E)
viii	(0, 1/2: 4, E)
ix	(1/2, 0: 4, O)
x	(0, 1/2: 4, O)
xi	(1/2, 0: 5, O)
xii	(0, 1/2: 5, O)
xiii	(1/2, 1/2: 4, E)
xiv	(1/2, 1/2: 5, E)
xv	(1/2, 1/2: 3, O)
xvi	(1/2, 1/2: 4, O)

models with two zero-modes, β_i and γ_j. When the instanton brane $\mathrm{D}_{\mathrm{inst}}$ wraps on the other four-dimensional compact space, each of β_i and γ_j must have only one zero-mode on the four-dimensional compact space such that the total zero-mode number is equal to two. Note that the total zero-mode number is a product of zero-mode numbers on $T^2/\mathbb{Z}_2$ and the other four-dimensional compact space.[4] Thus, the flavor structure, i.e. mass ratios and mixing angles, is determined by the configuration on $T^2/\mathbb{Z}_2$, although the other four-dimensional compact space contributes only to an overall factor of the mass matrix. Also, we assume that these D-brane systems preserve 4D $N = 1$ supersymmetry.

Two zero-modes can be realized by D-brane instantons with proper magnetic fluxes and SS phases, as reviewed in Sect. 2.2. These are shown in Table 2. In addition, the neutrinos must have non-vanishing three-point couplings d_a^{ij} with the zero-modes, β_i and γ_j. That leads to the following conditions:

$$M_\beta \pm M_\gamma = \pm M_N, \qquad (\alpha_1, \alpha_\tau)_\beta + (\alpha_1, \alpha_\tau)_\gamma = (\alpha_1, \alpha_\tau)_N, \tag{20}$$

where M_β, M_γ, and M_N are the magnetic fluxes in the zero-mode equations of β_i, γ_i, and neutrinos N_a, and $(\alpha_1, \alpha_\tau)_\beta$, $(\alpha_1, \alpha_\tau)_\gamma$, and $(\alpha_1, \alpha_\tau)_N$ are SS phases in the boundary conditions of β_i, γ_j, and neutrinos N_a, respectively. Note that the SS phases (α_1, α_τ) are defined modulo an integer. Furthermore, three-point couplings d_a^{ij} are allowed only if the product of $\mathbb{Z}_2$ parities of β_i, γ_j, and neutrinos N_a is $\mathbb{Z}_2$-even. Thus, when we fix magnetic fluxes, SS phases, and $\mathbb{Z}_2$ parities for β_j and γ_j, those are determined for the neutrinos. Then, we can find the generation number of the neutrinos, which can gain mass terms through the D-brane instanton effects. Such generation numbers are shown in Table 3 for $M_N = M_\beta + M_\gamma$. When $M_N = |M_\beta - M_\gamma|$, the neutrino generation number is 0 or 1.

[4]The number of zero-modes on six-dimensional toroidal orbifolds such as $T^6/(\mathbb{Z}_2 \times \mathbb{Z}_2)$ can be more subtle [57]. We concentrate here on D-brane configurations where the number of zero-modes arises on one two-dimensional torus for simplicity.

Table 3. The neutrino generation numbers for combinations of zero-modes β_i and γ_j.

$\beta_i \backslash \gamma_j$	i	ii	iii	iv	v	vi	vii	viii	ix	x	xi	xii	xiii	xiv	xv	xvi
i	3	3	3	3	3	3	3	3	3	3	3	3	3	3	3	3
ii	3	4	3	4	3	3	4	4	3	3	4	4	3	4	3	4
iii	3	3	6	6	4	4	4	4	5	5	5	5	5	5	4	4
iv	3	4	6	7	4	4	5	5	5	5	6	6	5	6	4	5
v	3	3	4	4	4	3	4	3	3	4	3	4	4	4	3	3
vi	3	3	4	4	3	4	3	4	4	3	4	3	4	4	3	3
vii	3	4	4	5	4	3	5	4	3	4	4	5	4	5	3	4
viii	3	4	4	5	3	4	4	5	4	3	5	4	4	5	3	4
ix	3	3	5	5	3	4	3	4	5	4	5	4	4	4	4	4
x	3	3	5	5	4	3	4	3	4	5	4	5	4	4	4	4
xi	3	4	5	6	3	4	4	5	5	4	6	5	4	5	4	5
xii	3	4	5	6	4	3	5	4	4	5	5	6	4	5	4	5
xiii	3	3	5	5	4	4	4	4	4	4	4	4	5	5	3	3
xiv	3	4	5	6	4	4	5	5	4	4	5	5	5	6	3	4
xv	3	3	4	4	3	3	3	3	4	4	4	4	3	3	4	4
xvi	3	4	4	5	3	3	4	4	4	4	5	5	3	4	4	5

Table 4. The combination numbers of zero-modes β_i and γ_j for the neutrino generation numbers. The number of all possible combinations β_i and γ_j is 256.

Neutrino generation number	3	4	5	6	7
Combination number of β and γ	81	108	54	12	1

Table 5. The combination numbers of zero-modes β_i and γ_j for the neutrino generation numbers such that the neutrino sector has vanishing SS phases. The number of all possible combinations β_i and γ_j is 64.

Neutrino generation number	3	4	5	6	7
Combination number of β and γ	27	18	12	6	1

Table 4 shows the combination numbers of zero-modes β_i and γ_j for the neutrino generation numbers. This implies that the three and four generations are statistically favored. Their probabilities are 32% and 42%.

Similarly, Table 5 shows the combination numbers of zero-modes β_i and γ_j for the neutrino generation numbers such that the neutrino sector has vanishing SS phases. This means the combinations of β_i and γ_j corresponding to the SS phases (0,0) and (0,0), (1/2, 1/2) and (1/2, 1/2), (1/2, 0) and (1/2, 0), and (0, 1/2) and (0, 1/2). Again, the three and four generations are statistically favored. Their probabilities are about 42% and 28%.

4. Majorana neutrino mass matrices

Here we show explicitly Majorana neutrino mass matrices induced by D-brane instanton effects. We restrict ourselves to the three-generation models, where neutrinos correspond to zero-modes with vanishing SS phases.

4.1 *Neutrino mass terms*

In the previous sections we have shown possible D-brane configurations, where Majorana neutrino mass terms can be induced by D-brane instanton effects. All of the possible effects can contribute to neutrino mass terms for a fixed configuration of D-branes D_{N1} and D_{N2}.

When the neutrinos appear as three $\mathbb{Z}_2$ even zero-modes corresponding to the magnetic flux $M_N = 4$ and 5, D-brane instanton effects to generate Majorana neutrino masses are unique, as shown in Sects. 4.2 and 4.4. On the other hand, when neutrinos appear as three $\mathbb{Z}_2$ odd zero-modes corresponding to the magnetic flux $M_N = 7$ and 8, the total mass terms are written by linear combinations with the factor $e^{-S_{\rm cl}(D_{\rm inst}, M_{\rm inst})}$. When the neutrinos correspond to the magnetic flux $M_N = 8$, their mass matrix is written by

$$M_{ab} = e^{-S_{\rm cl}\left(D_{\rm inst}, M_{\rm inst}^{(2\text{-}6\text{-}8)}\right)} m_{ab}^{(2\text{-}6\text{-}8)} + \sum_{(\alpha_1,\alpha_\tau)} e^{-S_{\rm cl}\left(D_{\rm inst}, M_{\rm inst}^{(3\text{-}5\text{-}8)}\right)} m_{ab}^{(3\text{-}5\text{-}8)(\alpha_1,\alpha_\tau)}$$
$$+ \sum_{(\alpha_1,\alpha_\tau)} e^{-S_{\rm cl}\left(D_{\rm inst}, M_{\rm inst}^{(4\text{-}4\text{-}8)}\right)} m_{ab}^{(4\text{-}4\text{-}8)(\alpha_1,\alpha_\tau)}. \tag{21}$$

Here, $M_{\rm inst}^{(2\text{-}6\text{-}8)}$ denotes the magnetic flux on the D-brane instanton to realize the case with $(M_\beta, M_\gamma, M_N) = (2, 6, 8)$, which is denoted in short by 2-6-8. Other notations such as $M_{\rm inst}^{(3\text{-}5\text{-}8)}$ and $M_{\rm inst}^{(4\text{-}4\text{-}8)}$ have similar meaning. Note that we can shift the magnetic fluxes on D_{N1}-, D_{N2}-, and $D_{\rm inst}$-branes by a constant to make their differences invariant and realize the 2-6-8 case. For example, when we choose magnetic fluxes on D_{N1}- and D_{N2}-branes such that $M_{\rm inst}^{(2\text{-}6\text{-}8)} = 0$ and $M_{\rm inst}^{(3\text{-}5\text{-}8)}$, $M_{\rm inst}^{(4\text{-}4\text{-}8)} \neq 0$, the first term is dominant, and other terms are exponentially suppressed:

$$M_{ab} \approx e^{-S_{\rm cl}(D_{\rm inst},0)} m_{ab}^{(2\text{-}6\text{-}8)}. \tag{22}$$

Similarly, when $M_N = 7$, the mass matrix is written by

$$M_{ab} = e^{-S_{\rm cl}\left(D_{\rm inst}, M_{\rm inst}^{(2\text{-}5\text{-}7)}\right)} m_{ab}^{(2\text{-}5\text{-}7)} + \sum_{(\alpha_1,\alpha_\tau)} e^{-S_{\rm cl}\left(D_{\rm inst}, M_{\rm inst}^{(3\text{-}4\text{-}7)}\right)}. \tag{23}$$

Under the choice $M_{\rm inst}^{(2\text{-}5\text{-}7)} = 0$ and $M_{\rm inst}^{(3\text{-}4\text{-}7)} \neq 0$, we have the approximation

$$M_{ab} \approx e^{-S_{\rm cl}(D_{\rm inst},0)} m_{ab}^{(2\text{-}5\text{-}7)}. \tag{24}$$

4.2 *Neutrino sector with $M_N = 4$*

Here, we study the mass matrix of the neutrino sector with $M_N = 4$. The only possibility is the 2-2-4 case which is constructed as follows. Two independent zero-modes for both β_i and γ_j are obtained by taking even wavefunctions under the magnetic flux, $M_\beta = M_\gamma = 2$. For the neutrino sector, three independent zero-modes are obtained by taking even wavefunctions under the magnetic flux, $M_N = 4$. Wavefunctions for β_i and γ_j are given by

$$\psi_{T^2/\mathbb{Z}_2^+}^{(0,0),2}(z) = \psi_{T^2}^{(0,0),2}(z), \qquad \psi_{T^2/\mathbb{Z}_2^+}^{(1,0),2}(z) = \psi_{T^2}^{(1,0),2}(z). \tag{25}$$

Wavefunctions for the neutrino sector are given by

$$\psi_{T^2/\mathbb{Z}_2^+}^{(0,0),4}(z) = \psi_{T^2}^{(0,0),4}(z), \qquad \psi_{T^2/\mathbb{Z}_2^+}^{(1,0),4}(z) = \frac{1}{\sqrt{2}}\left(\psi_{T^2}^{(1,0),4}(z) + \psi_{T^2}^{(3,0),4}(z)\right),$$
$$\psi_{T^2/\mathbb{Z}_2^+}^{(2,0),4}(z) = \psi_{T^2}^{(2,0),4}(z). \tag{26}$$

From the above wavefunctions, the d matrices can be computed as

$$d_1 = c_{(2\text{-}2\text{-}4)}\begin{pmatrix} \eta_0^{(16)} + \eta_8^{(16)} & 0 \\ 0 & \eta_4^{(16)} + \eta_{12}^{(16)} \end{pmatrix},$$
$$d_2 = c_{(2\text{-}2\text{-}4)}\sqrt{2}\begin{pmatrix} 0 & \eta_2^{(16)} + \eta_{10}^{(16)} \\ \eta_2^{(16)} + \eta_{10}^{(16)} & 0 \end{pmatrix},$$
$$d_3 = c_{(2\text{-}2\text{-}4)}\begin{pmatrix} \eta_4^{(16)} + \eta_{12}^{(16)} & 0 \\ 0 & \eta_0^{(16)} + \eta_8^{(16)} \end{pmatrix}, \tag{27}$$

where we have defined

$$\eta_N^{(n)} = \vartheta\begin{bmatrix} \frac{N}{n} \\ 0 \end{bmatrix}(0, n\tau). \tag{28}$$

The explicit form of the overall constant factor $c_{(2\text{-}2\,-\,4)}$ is shown by Eq. (11). The mass matrix is then given by

$$\boldsymbol{m}^{(2\text{-}2\text{-}4)} = c_{(2\text{-}2\text{-}4)}^2\begin{pmatrix} X_3 & 0 & X_1 \\ 0 & -\sqrt{2}X_2 & 0 \\ X_1 & 0 & X_3 \end{pmatrix}, \tag{29}$$

where the X_i (i=1, 2, 3) are defined as

$$X_1 = \left(\eta_0^{(16)} + \eta_8^{(16)}\right)^2 + \left(\eta_4^{(16)} + \eta_{12}^{(16)}\right)^2,$$
$$X_2 = \frac{1}{\sqrt{2}}\left(\left(\eta_2^{(16)} + \eta_{10}^{(16)}\right) + \left(\eta_6^{(16)} + \eta_{14}^{(16)}\right)\right)^2,$$
$$X_3 = 2\left(\eta_0^{(16)} + \eta_8^{(16)}\right)\left(\eta_4^{(16)} + \eta_{12}^{(16)}\right). \tag{30}$$

Next, we investigate the modular transformation behavior of $\boldsymbol{m}^{(2\text{-}2\text{-}4)}$. Under the S-transformation, $\tau \to -\frac{1}{\tau}$, X_i $(i = 1, 2, 3)$ are transformed as

$$\begin{pmatrix} X_1 \\ X_2 \\ X_3 \end{pmatrix} \xrightarrow{S} (-\tau)\frac{i}{2}\begin{pmatrix} 1 & \sqrt{2} & 1 \\ \sqrt{2} & 0 & -\sqrt{2} \\ 1 & -\sqrt{2} & 1 \end{pmatrix}\begin{pmatrix} X_1 \\ X_2 \\ X_3 \end{pmatrix}. \tag{31}$$

Under the T-transformation, $\tau \to \tau + 1$,

$$\begin{pmatrix} X_1 \\ X_2 \\ X_3 \end{pmatrix} \xrightarrow{T} \begin{pmatrix} 1 & 0 & 0 \\ 0 & i & 0 \\ 0 & 0 & -1 \end{pmatrix}\begin{pmatrix} X_1 \\ X_2 \\ X_3 \end{pmatrix}. \tag{32}$$

From the above results, it can be seen that X_1, X_2, and X_3 are modular forms of weight 1 and behave as a triplet of the group S_4'.[5]

Eigenvalues of $\boldsymbol{m}^{(2\text{-}2\text{-}4)}$ are given by

$$\lambda_1 = c_{(2\text{-}2\text{-}4)}^2(X_1 + X_3) = c_{(2\text{-}2\text{-}4)}^2\left(\left(\eta_0^{(16)} + \eta_8^{(16)}\right) + \left(\eta_4^{(16)} + \eta_{12}^{(16)}\right)\right)^2,$$
$$\lambda_2 = -c_{(2\text{-}2\text{-}4)}^2\sqrt{2}X_2 = -c_{(2\text{-}2\text{-}4)}^2\left(\left(\eta_2^{(16)} + \eta_{10}^{(16)}\right) + \left(\eta_6^{(16)} + \eta_{14}^{(16)}\right)\right)^2,$$
$$\lambda_3 = c_{(2\text{-}2\text{-}4)}^2(X_3 - X_1) = -c_{(2\text{-}2\text{-}4)}^2\left(\left(\eta_0^{(16)} + \eta_8^{(16)}\right) - \left(\eta_4^{(16)} + \eta_{12}^{(16)}\right)\right)^2. \tag{33}$$

[5] $\Gamma_4' \simeq S_4' \simeq \Delta'(24)$ is the double covering group of $\Gamma_4 \simeq S_4 \simeq \Delta(24)$.

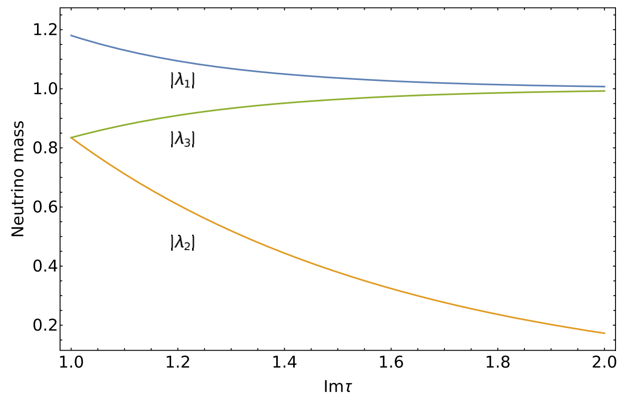

Fig. 1. $\mathrm{Im}\,\tau$ dependence ($1 \leq \mathrm{Im}\,\tau \leq 2$) of the absolute values of the mass eigenvalues in the 2-2-4 case at $\mathrm{Re}\,\tau = 0$.

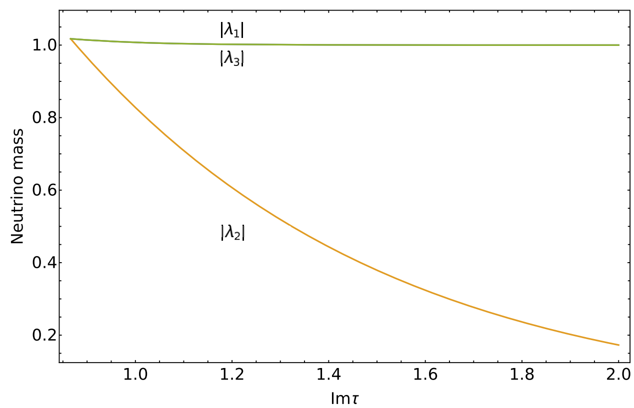

Fig. 2. $\mathrm{Im}\,\tau$ dependence ($\sqrt{3}/2 \leq \mathrm{Im}\,\tau \leq 2$) of the absolute values of the mass eigenvalues in the 2-2-4 case at $\mathrm{Re}\,\tau = 1/2$.

They can be approximated at large $\mathrm{Im}\,\tau$ as

$$\begin{aligned}
\lambda_1 &\approx c^2_{(2\text{-}2\text{-}4)} \left(\eta_0^{(16)}\right)^2 \approx c^2_{(2\text{-}2\text{-}4)}(1 + \cdots), \\
\lambda_2 &\approx -c^2_{(2\text{-}2\text{-}4)} 4 \left(\eta_2^{(16)}\right)^2 \approx -4c^2_{(2\text{-}2\text{-}4)}(e^{\frac{\pi i \tau}{2}} + \cdots), \\
\lambda_3 &\approx -c^2_{(2\text{-}2\text{-}4)} \left(\eta_0^{(16)}\right)^2 \approx -c^2_{(2\text{-}2\text{-}4)}(1 + \cdots).
\end{aligned} \tag{34}$$

The diagonalizing matrix P for all τ satisfying $P^{\mathrm{T}} \boldsymbol{m}^{(2\text{-}2\text{-}4)} P = \mathrm{diag}(\lambda_1, \lambda_2, \lambda_3)$ is

$$P = \begin{pmatrix} \frac{1}{\sqrt{2}} & 0 & \frac{-1}{\sqrt{2}} \\ 0 & 1 & 0 \\ \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \end{pmatrix}, \tag{35}$$

showing that the mixing angle is 45°. Note that the mass eigenstates become $\mathbb{Z}_2$-twisted and $\mathbb{Z}_2$-shifted eigenstates.[6] Figures 1 and 2 show the $\mathrm{Im}\,\tau$ dependence $\left(\sqrt{1 - (\mathrm{Re}\,\tau)^2} \leq \mathrm{Im}\,\tau \leq 2\right)$ of the absolute values of the mass eigenvalues λ_i ($i = 1, 2, 3$) in Eq. (33) at $\mathrm{Re}\,\tau = 0$ and $\mathrm{Re}\,\tau = 1/2$, respectively. Here, we set $c_{(2\text{-}2\text{-}4)} = 1$ for simplicity.

There are four interesting features in Figs. 1 and 2. First, we can find $|\lambda_2| = |\lambda_3|$ at $\tau = i$. This can be explained by considering that the point $\tau = i$ is invariant under S-transformation.

[6]They can be defined in $M \in 4\mathbb{Z}$. For details, see Refs. [25,26].

In other words, each mass eigenvalue λ_k ($k = 1, 2, 3$) at $\tau = i$ must be invariant under S-transformation. However, from Eqs. (31) and (35), the λ_k ($k = 1, 2, 3$) at $\tau = i$ are transformed under S-transformation as

$$\begin{pmatrix}\lambda_1\\ \lambda_2\\ \lambda_3\end{pmatrix} \xrightarrow{S} \begin{pmatrix}1 & 0 & 0\\ 0 & 0 & 1\\ 0 & 1 & 0\end{pmatrix}\begin{pmatrix}\lambda_1\\ \lambda_2\\ \lambda_3\end{pmatrix}. \tag{36}$$

Thus, in order for λ_k ($k = 1, 2, 3$) at $\tau = i$ to be invariant under S-transformation, it should be required that $\lambda_2 = \lambda_3$ at $\tau = i$. Actually, by using the relation

$$\eta_N^{(n)}(-1/\tau) = \sqrt{\frac{-i\tau}{n}} \sum_{N'=0}^{n-1} e^{2\pi i \frac{NN'}{n}} \eta_{N'}^{(n)}(\tau), \tag{37}$$

we can check that

$$\left(\eta_0^{(16)} + \eta_8^{(16)}\right) - \left(\eta_4^{(16)} + \eta_{12}^{(16)}\right) = \left(\eta_2^{(16)} + \eta_{10}^{(16)}\right) + \left(\eta_6^{(16)} + \eta_{14}^{(16)}\right) \qquad (\tau = i), \tag{38}$$

and then $\lambda_2 = \lambda_3$ is satisfied at $\tau = i$.

Second, we can find $|\lambda_1| \simeq |\lambda_3|$ and $|\lambda_2| \to 0$ at $\tau \to i\infty$. This can be explained by considering that the limit $\tau = i\infty$ is invariant under T-transformation. In other words, each mass eigenvalue λ_k ($k = 1, 2, 3$) at $\tau = i\infty$ must be invariant under T-transformation. However, from Eqs. (32) and (35), the λ_k ($k = 1, 2, 3$) are transformed under T-transformation as

$$\begin{pmatrix}\lambda_1\\ \lambda_2\\ \lambda_3\end{pmatrix} \xrightarrow{T} \begin{pmatrix}0 & 0 & -1\\ 0 & i & 0\\ -1 & 0 & 0\end{pmatrix}\begin{pmatrix}\lambda_1\\ \lambda_2\\ \lambda_3\end{pmatrix}. \tag{39}$$

Thus, in order for λ_k ($k = 1, 2, 3$) at $\tau = i\infty$ to be invariant under T-transformation, it should be required that $\lambda_1 = -\lambda_3$ and $\lambda_2 = i\lambda_2 \Leftrightarrow \lambda_2 = 0$ at $\tau = i\infty$. Actually, the λ_k ($k = 1, 2, 3$) in Eq. (34) at $\tau \to i\infty$ are estimated as $\lambda_1 \simeq -\lambda_3 \to 1$ and $\lambda_2 \to 0$.

Third, we can find $|\lambda_1| = |\lambda_3|$ at $\tau = \frac{1}{2} + i\mathrm{Im}\,\tau$. This can be explained by considering that $\tau = \frac{1}{2} + i\mathrm{Im}\,\tau$ is invariant under T-transformation and CP-transformation, where $CP : \tau \to -\bar{\tau}$. In other words, each mass eigenvalue λ_k ($k = 1, 2, 3$) at $\tau = \frac{1}{2} + i\mathrm{Im}\,\tau$ must be invariant under the $T \cdot CP$-transformation. Here, we can easily check that

$$CP : \lambda_k(\tau) \to \lambda_k(-\bar{\tau}) = \lambda_k^*(\tau) \qquad (k = 1, 2, 3). \tag{40}$$

Then, the λ_k ($k = 1, 2, 3$) are transformed under the $T \cdot CP$-transformation as

$$\begin{pmatrix}\lambda_1\\ \lambda_2\\ \lambda_3\end{pmatrix} \xrightarrow{T\cdot CP} \begin{pmatrix}0 & 0 & -1\\ 0 & i & 0\\ -1 & 0 & 0\end{pmatrix}\begin{pmatrix}\lambda_1^*\\ \lambda_2^*\\ \lambda_3^*\end{pmatrix}. \tag{41}$$

Thus, in order for the λ_k ($k = 1, 2, 3$) at $\tau = \frac{1}{2} + i\mathrm{Im}\,\tau$ to be invariant under the $T \cdot CP$-transformation, it should be required that $\lambda_1 = -\lambda_3^*$ and $\lambda_2 = i\lambda_2^*$ at $\tau = \frac{1}{2} + i\mathrm{Im}\,\tau$. Actually, the λ_k ($k = 1, 2, 3$) in Eq. (33) at $\tau = \frac{1}{2} + i\mathrm{Im}\,\tau$ are expressed as

$$\begin{aligned}
\lambda_1 &= \left(\left(|\eta_0^{(16)}| + |\eta_8^{(16)}|\right) + i\left(|\eta_4^{(16)}| + |\eta_{12}^{(16)}|\right)\right)^2,\\
\lambda_2 &= -4e^{\frac{\pi i}{4}}\left(|\eta_2^{(16)}| - |\eta_{10}^{(16)}|\right)^2,\\
\lambda_3 &= -\left(\left(|\eta_0^{(16)}| + |\eta_8^{(16)}|\right) - i\left(|\eta_4^{(16)}| + |\eta_{12}^{(16)}|\right)\right)^2,
\end{aligned} \tag{42}$$

and then they satisfy $\lambda_1 = -\lambda_3^*$ and $\lambda_2 = i\lambda_2^*$.

Fourth, we can find $|\lambda_1| = |\lambda_2| = |\lambda_3|$ at $\tau = \frac{1}{2} + \frac{\sqrt{3}}{2}i$. This can be explained by considering that the point $\tau = \frac{1}{2} + \frac{\sqrt{3}}{2}i$ is invariant under ST^{-1}-transformation. In other words, each mass eigenvalue λ_k ($k = 1, 2, 3$) at $\tau = \frac{1}{2} + \frac{\sqrt{3}}{2}i$ must be invariant under ST^{-1}-transformation. However, the λ_k ($k = 1, 2, 3$) at $\tau = \frac{1}{2} + \frac{\sqrt{3}}{2}i$ are transformed under ST^{-1}-transformation as

$$\begin{pmatrix} \lambda_1 \\ \lambda_2 \\ \lambda_3 \end{pmatrix} \xrightarrow{ST^{-1}} \begin{pmatrix} 0 & 0 & e^{-\frac{5\pi i}{6}} \\ e^{-\frac{5\pi i}{6}} & 0 & 0 \\ 0 & e^{-\frac{\pi i}{3}} & 0 \end{pmatrix} \begin{pmatrix} \lambda_1 \\ \lambda_2 \\ \lambda_3 \end{pmatrix}. \tag{43}$$

Thus, in order for λ_k ($k = 1, 2, 3$) at $\tau = \frac{1}{2} + \frac{\sqrt{3}}{2}i$ to be invariant under ST^{-1}-transformation, it should be required that $\lambda_1 = e^{\frac{5\pi i}{6}}\lambda_2 = e^{-\frac{5\pi i}{6}}\lambda_3$ at $\tau = \frac{1}{2} + \frac{\sqrt{3}}{2}i$. Actually, this can also be checked explicitly by considering that $\tau = \frac{1}{2} + \frac{\sqrt{3}}{2}i$ is invariant under ST^{-1}-transformation. Note that $\lambda_1 = -\lambda_3^*$ and $\lambda_2 = i\lambda_2^*$ should also be satisfied, and actually they are consistent with $\lambda_1 = e^{\frac{5\pi i}{6}}\lambda_2 = e^{-\frac{5\pi i}{6}}\lambda_3$.

4.3 *Neutrino sector with $M_N = 8$*

Here we study the mass matrix of the neutrino sector with $M_N = 8$. For this neutrino sector, the mass terms can be induced by several cases: 2-6-8, 3-5-8, and 4-4-8.

4.3.1 *2-6-8 case.* In the 2-6-8 case we take even wavefunctions under $M_\beta = 2$ and odd wavefunctions under $M_\gamma = 6$ for β_i and γ_j respectively. For the neutrino sector, we take odd wavefunctions under $M_N = 8$. The d matrices are given by

$$d_1 = c_{(2\text{-}6\text{-}8)} \begin{pmatrix} A_1 & 0 \\ 0 & A_3 \end{pmatrix}, \qquad d_2 = c_{(2\text{-}6\text{-}8)} \begin{pmatrix} 0 & A_2 \\ A_2 & 0 \end{pmatrix}, \qquad d_3 = c_{(2\text{-}6\text{-}8)} \begin{pmatrix} A_3 & 0 \\ 0 & A_1 \end{pmatrix}. \tag{44}$$

Here, A_i ($i = 1, 2, 3$) are defined as

$$\begin{aligned} A_1 &= \left(\eta_2^{(96)} - \eta_{14}^{(96)}\right) - \left(\eta_{34}^{(96)} - \eta_{46}^{(96)}\right), \\ A_2 &= \left(\eta_4^{(96)} - \eta_{28}^{(96)}\right) - \left(\eta_{20}^{(96)} - \eta_{44}^{(96)}\right), \\ A_3 &= \left(\eta_{10}^{(96)} - \eta_{22}^{(96)}\right) - \left(\eta_{26}^{(96)} - \eta_{38}^{(96)}\right). \end{aligned} \tag{45}$$

The mass matrix is then given by

$$\boldsymbol{m}^{(2\text{-}6\text{-}8)} = c_{(2\text{-}6\text{-}8)}^2 \begin{pmatrix} X_3 & 0 & X_1 \\ 0 & -\sqrt{2}X_2 & 0 \\ X_1 & 0 & X_3 \end{pmatrix}, \tag{46}$$

where X_i ($i = 1, 2, 3$) are defined as

$$X_1 = (A_1)^2 + (A_3)^2, \qquad X_2 = \sqrt{2}(A_2)^2, \qquad X_3 = 2A_1A_3. \tag{47}$$

Next, let us investigate the modular transformation behavior of the mass matrix, $\boldsymbol{m}^{(2\text{-}6\text{-}8)}$. Under the S-transformation, $\tau \to -\frac{1}{\tau}$, X_i ($i = 1, 2, 3$) are transformed as

$$\begin{pmatrix} X_1 \\ X_2 \\ X_3 \end{pmatrix} \xrightarrow{S} (-\tau)\frac{i}{2} \begin{pmatrix} 1 & \sqrt{2} & 1 \\ \sqrt{2} & 0 & -\sqrt{2} \\ 1 & -\sqrt{2} & 1 \end{pmatrix} \begin{pmatrix} X_1 \\ X_2 \\ X_3 \end{pmatrix}. \tag{48}$$

Under the T-transformation, $\tau \to \tau + 1$,

$$\begin{pmatrix} X_1 \\ X_2 \\ X_3 \end{pmatrix} \xrightarrow{T} e^{\frac{i\pi}{12}} \begin{pmatrix} 1 & 0 & 0 \\ 0 & e^{\frac{i\pi}{4}} & 0 \\ 0 & 0 & -1 \end{pmatrix} \begin{pmatrix} X_1 \\ X_2 \\ X_3 \end{pmatrix}. \tag{49}$$

From the above results, it can be seen that X_1, X_2, and X_3 are modular forms of weight 1 and behave as a triplet of the group $\Delta'(96) \times \mathbb{Z}_3$.[7]

4.3.2 *3-5-8 case.* In the 3-5-8 case there are four possible variations with different SS phases: (0,0), (1/2, 0), (0, 1/2), and (1/2, 1/2). The sum of four mass matrices[8] in equal ratio is given by

$$\begin{aligned} \boldsymbol{m}^{(3\text{-}5\text{-}8)} &= \boldsymbol{m}^{(3\text{-}5\text{-}8)(0,0)} + \boldsymbol{m}^{(3\text{-}5\text{-}8)(1/2,0)} + \boldsymbol{m}^{(3\text{-}5\text{-}8)(0,1/2)} + \boldsymbol{m}^{(3\text{-}5\text{-}8)(1/2,1/2)} \\ &= c^2_{(3\text{-}5\text{-}8)} \begin{pmatrix} Y_3 & 0 & Y_1 \\ 0 & -\sqrt{2}Y_2 & 0 \\ Y_1 & 0 & Y_3 \end{pmatrix}. \end{aligned} \tag{50}$$

Here, Y_1, Y_2, and Y_3 are defined as

$$\begin{aligned} Y_1 &= 2\sqrt{2}(D_9\tilde{D}_{37} + D_{33}\tilde{D}_{-11} + D_{-3}\tilde{D}_1 + D_{21}\tilde{D}_{-47}), \\ Y_2 &= 4(E_{-6}\tilde{E}_{-2} - E_{42}\tilde{E}_{34}), \\ Y_3 &= 2\sqrt{2}(D_9\tilde{D}_{-47} + D_{33}\tilde{D}_1 + D_{21}\tilde{D}_{37} + D_{-3}\tilde{D}_{-11}), \end{aligned} \tag{51}$$

where D_N, E_N, $\tilde{D}_N$, and $\tilde{E}_N$ are given by

$$\begin{aligned} D_N &= \eta^{(120)}_N - \eta^{(120)}_{N+30}, \quad & E_N &= \eta^{(120)}_N - \eta^{(120)}_{N+60}, \\ \tilde{D}_N &= D_N + D_{N+40}, \quad & \tilde{E}_N &= E_N + E_{N+40}. \end{aligned} \tag{52}$$

Under the modular transformation, Y_1, Y_2, and Y_3 are modular forms of weight 1 and behave as a triplet of the group $\Delta'(96) \times \mathbb{Z}_3$.

4.3.3 *4-4-8 case.* In the 4-4-8 case, there are three possible variations with different SS phases: (1/2, 0), (0, 1/2), and (1/2, 1/2). The sum of the three mass matrices in equal ratio is given by

$$\begin{aligned} \boldsymbol{m}^{(4\text{-}4\text{-}8)} &= \boldsymbol{m}^{(4\text{-}4\text{-}8)(1/2,0)} + \boldsymbol{m}^{(4\text{-}4\text{-}8)(0,1/2)} + \boldsymbol{m}^{(4\text{-}4\text{-}8)(1/2,1/2)} \\ &= c^2_{(4\text{-}4\text{-}8)} \begin{pmatrix} \sqrt{2}Z_6 - Z_3 & 0 & Z_1 + \sqrt{2}(Z_2 + Z_4) \\ 0 & -2(Z_1 + Z_5) & 0 \\ Z_1 + \sqrt{2}(Z_2 + Z_4) & 0 & \sqrt{2}Z_6 - Z_3 \end{pmatrix}. \end{aligned} \tag{53}$$

Here, the Z_i ($i = 1, 2, 3, 4, 5, 6$) are defined as

$$\begin{aligned} Z_1 &= (B_1)^2 + (B_3)^2, \quad & Z_2 &= \tfrac{1}{2\sqrt{2}}(B_0 - B_4)^2, \quad & Z_3 &= -2B_1B_3, \\ Z_4 &= \tfrac{1}{2\sqrt{2}}(B_0 + B_4)^2, \quad & Z_5 &= B_2(B_0 + B_4), \quad & Z_6 &= \sqrt{2}(B_2)^2, \end{aligned} \tag{54}$$

where B_N is given by

$$B_N = \eta^{(128)}_{4N} + \eta^{(128)}_{4N+32} + \eta^{(128)}_{4N+64} + \eta^{(128)}_{4N+96}. \tag{55}$$

[7] $\Delta'(96) \simeq (\mathbb{Z}_4 \times \mathbb{Z}'_4) \rtimes \mathbb{Z}_3 \rtimes \mathbb{Z}_4$, which is the double covering group of $\Delta(96) \subset \Gamma_8$ and is a subgroup of Γ'_8 [27].

[8] In particular, $m^{(3\text{-}5\text{-}8)}_{12}$ and $m^{(3\text{-}5\text{-}8)}_{23}$ (as well as $m^{(3\text{-}5\text{-}8)}_{21}$ and $m^{(3\text{-}5\text{-}8)}_{32}$) cancel each other by those components of $\boldsymbol{m}^{(3\text{-}5\text{-}8)(\alpha_1,0)}$ and $\boldsymbol{m}^{(3\text{-}5\text{-}8)(\alpha_1,1/2)}$ with $\alpha_1 = 0, 1/2$.

Under the modular transformation, the Z_i $(i = 1, 2, 3, 4, 5, 6)$ are modular forms of weight 1. It can be verified that Z_i $(i = 1, 2, 3)$ behave as a triplet of the group $\Delta'(96)$. On the other hand, Z_i $(i = 4, 5, 6)$ behave as a triplet of the group $S_4' \subset \Delta'(96)$.

4.3.4 *Total mass matrix and its eigenvalues.* Recall that the total mass matrix $\boldsymbol{M}^{(M_N=8)}$ is given by the sum of $\boldsymbol{m}^{(2\text{-}6\text{-}8)}$, $\boldsymbol{m}^{(4\text{-}4\text{-}8)}$, and $\boldsymbol{m}^{(3\text{-}5\text{-}8)}$, which are weighted by the factor $e^{-S_{\rm cl}(D_{\rm inst}, M_{\rm inst})}$ as shown in Eq. (21). In fact, $\boldsymbol{M}^{(M_N=8)}$ is diagonalized by the bimaximal mixing matrix,

$$P = \begin{pmatrix} \frac{1}{\sqrt{2}} & 0 & \frac{-1}{\sqrt{2}} \\ 0 & 1 & 0 \\ \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \end{pmatrix}, \tag{56}$$

meaning that we have $P^{\rm T}\boldsymbol{M}^{(M_N=8)}P = {\rm diag}(\lambda_1^{(M_N=8)}, \lambda_2^{(M_N=8)}, \lambda_3^{(M_N=8)})$ independent of the factor $e^{-S_{\rm cl}(D_{\rm inst}, M_{\rm inst})}$ as well as the modulus τ. This follows from the fact that Eqs. (46), (50), and (53) are all separately diagonalized by Eq. (56) for all τ. Note that the mass eigenstates become $\mathbb{Z}_2$-twisted and $\mathbb{Z}_2$-shifted eigenstates. Let us denote the eigenvalues of $\boldsymbol{m}^{(2\text{-}6\text{-}8)}$ by $\lambda_i^{(2\text{-}6\text{-}8)}$ $(i = 1, 2, 3)$. We understand $\lambda_i^{(3\text{-}5\text{-}8)}$ and $\lambda_i^{(4\text{-}4\text{-}8)}$ in the same way. Then, the eigenvalues $\lambda_i^{(M_N=8)}$ are given by

$$\begin{aligned}\lambda_i^{(M_N=8)} &= e^{-S_{\rm cl}(D_{\rm inst}, M_{\rm inst}^{(2\text{-}6\text{-}8)})}\lambda_i^{(2\text{-}6\text{-}8)} + e^{-S_{\rm cl}(D_{\rm inst}, M_{\rm inst}^{(3\text{-}5\text{-}8)})}\lambda_i^{(3\text{-}5\text{-}8)} \\ &\quad + e^{-S_{\rm cl}(D_{\rm inst}, M_{\rm inst}^{(4\text{-}4\text{-}8)})}\lambda_i^{(4\text{-}4\text{-}8)},\end{aligned} \tag{57}$$

where

$$\begin{aligned}\lambda_1^{(2\text{-}6\text{-}8)} &= c_{(2\text{-}6\text{-}8)}^2(X_1 + X_3) = c_{(2\text{-}6\text{-}8)}^2(A_1 + A_3)^2, \\ \lambda_2^{(2\text{-}6\text{-}8)} &= -c_{(2\text{-}6\text{-}8)}^2\sqrt{2}X_2 = -c_{(2\text{-}6\text{-}8)}^2 2(A_2)^2, \\ \lambda_3^{(2\text{-}6\text{-}8)} &= c_{(2\text{-}6\text{-}8)}^2(X_3 - X_1) = -c_{(2\text{-}6\text{-}8)}^2(A_1 - A_3)^2,\end{aligned} \tag{58}$$

$$\begin{aligned}\lambda_1^{(3\text{-}5\text{-}8)} &= c_{(3\text{-}5\text{-}8)}^2(Y_1 + Y_3) \\ &= c_{(3\text{-}5\text{-}8)}^2 2\sqrt{2}\left((D_9 + D_{21})(\tilde{D}_{37} + \tilde{D}_{-47}) + (D_{33} + D_{-3})(\tilde{D}_{-11} + \tilde{D}_1)\right), \\ \lambda_2^{(3\text{-}5\text{-}8)} &= -c_{(3\text{-}5\text{-}8)}^2\sqrt{2}Y_2 \\ &= -c_{(3\text{-}5\text{-}8)}^2 4\sqrt{2}(E_{-6}\tilde{E}_{-2} - E_{42}\tilde{E}_{34}), \\ \lambda_3^{(3\text{-}5\text{-}8)} &= c_{(3\text{-}5\text{-}8)}^2(Y_3 - Y_1) \\ &= -c_{(3\text{-}5\text{-}8)}^2 2\sqrt{2}\left((D_9 - D_{21})(\tilde{D}_{37} - \tilde{D}_{-47}) + (D_{33} - D_{-3})(\tilde{D}_{-11} - \tilde{D}_1)\right),\end{aligned} \tag{59}$$

$$\begin{aligned}\lambda_1^{(4\text{-}4\text{-}8)} &= c_{(4\text{-}4\text{-}8)}^2\left(\sqrt{2}(Z_2 + Z_4 + Z_6) + (Z_1 - Z_3)\right) \\ &= c_{(4\text{-}4\text{-}8)}^2\left((B_0^2 + B_4^2 + 2B_2^2) + (B_1 + B_3)^2\right), \\ \lambda_2^{(4\text{-}4\text{-}8)} &= -c_{(4\text{-}4\text{-}8)}^2 2(Z_1 + Z_5) \\ &= -c_{(4\text{-}4\text{-}8)}^2 2\left((B_1)^2 + (B_3)^2 + B_2(B_0 + B_4)\right), \\ \lambda_3^{(4\text{-}4\text{-}8)} &= c_{(4\text{-}4\text{-}8)}^2\left(\sqrt{2}(Z_6 - Z_2 - Z_4) - (Z_1 + Z_3)\right) \\ &= -c_{(4\text{-}4\text{-}8)}^2\left((B_0^2 + B_4^2 - 2B_2^2) + (B_1 - B_3)^2\right).\end{aligned} \tag{60}$$

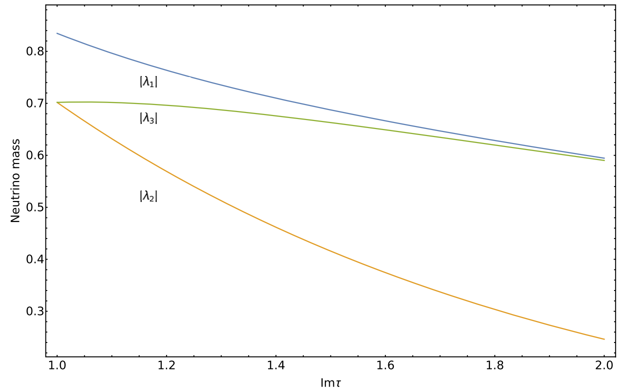

Fig. 3. The $\mathrm{Im}\,\tau$ dependence ($1 \leq \mathrm{Im}\,\tau \leq 2$) of the absolute values of the mass eigenvalues in the 2-6-8 case at $\mathrm{Re}\,\tau = 0$.

They can be approximated by

$$\lambda_1^{(2\text{-}6\text{-}8)} \approx c^2_{(2\text{-}6\text{-}8)} \left(\eta_2^{(96)}\right)^2 \approx c^2_{(2\text{-}6\text{-}8)}(e^{\frac{\pi i \tau}{12}} + \cdots),$$
$$\lambda_2^{(2\text{-}6\text{-}8)} \approx -c^2_{(2\text{-}6\text{-}8)} 2\left(\eta_4^{(96)}\right)^2 \approx -2c^2_{(2\text{-}6\text{-}8)}(e^{\frac{\pi i \tau}{3}} + \cdots),$$
$$\lambda_3^{(2\text{-}6\text{-}8)} \approx -c^2_{(2\text{-}6\text{-}8)} \left(\eta_2^{(96)}\right)^2 \approx -c^2_{(2\text{-}6\text{-}8)}(e^{\frac{\pi i \tau}{12}} + \cdots), \tag{61}$$

$$\lambda_1^{(3\text{-}5\text{-}8)} \approx c^2_{(3\text{-}5\text{-}8)} 2\sqrt{2}\eta_1^{(120)}\eta_3^{(120)} \approx 2\sqrt{2}c^2_{(3\text{-}5\text{-}8)}(e^{\frac{\pi i \tau}{12}} + \cdots),$$
$$\lambda_2^{(3\text{-}5\text{-}8)} \approx -c^2_{(3\text{-}5\text{-}8)} 4\sqrt{2}\eta_6^{(120)}\eta_2^{(120)} \approx -4\sqrt{2}c^2_{(3\text{-}5\text{-}8)}(e^{\frac{\pi i \tau}{3}} + \cdots),$$
$$\lambda_3^{(3\text{-}5\text{-}8)} \approx -c^2_{(3\text{-}5\text{-}8)} 2\sqrt{2}\eta_1^{(120)}\eta_3^{(120)} \approx -2\sqrt{2}c^2_{(3\text{-}5\text{-}8)}(e^{\frac{\pi i \tau}{12}} + \cdots), \tag{62}$$

$$\lambda_1^{(4\text{-}4\text{-}8)} \approx c^2_{(4\text{-}4\text{-}8)} \left(\left(\eta_0^{(128)}\right)^2 + \left(\eta_4^{(128)}\right)^2\right) \approx c^2_{(4\text{-}4\text{-}8)}(1 + e^{\frac{\pi i \tau}{4}} + \cdots),$$
$$\lambda_2^{(4\text{-}4\text{-}8)} \approx -c^2_{(4\text{-}4\text{-}8)} 2\left(\left(\eta_4^{(128)}\right)^2 + \eta_8^{(128)}\eta_0^{(128)}\right) \approx -2c^2_{(4\text{-}4\text{-}8)}(e^{\frac{\pi i \tau}{4}} + e^{\frac{\pi i \tau}{2}} + \cdots),$$
$$\lambda_3^{(4\text{-}4\text{-}8)} \approx -c^2_{(4\text{-}4\text{-}8)} \left(\left(\eta_0^{(128)}\right)^2 + \left(\eta_4^{(128)}\right)^2\right) \approx -c^2_{(4\text{-}4\text{-}8)}(1 + e^{\frac{\pi i \tau}{4}} + \cdots). \tag{63}$$

In order to numerically study the behavior of the mass eigenvalues $\lambda_i^{(M_N=8)}$ under the change in the modulus τ, let us use the relationship shown by Eq. (22). This allows us to write

$$\lambda_i^{(M_N=8)} \approx e^{-S_{\mathrm{cl}}(D_{\mathrm{inst}}, M_{\mathrm{inst}}^{(2\text{-}6\text{-}8)})}\lambda_i^{(2\text{-}6\text{-}8)}. \tag{64}$$

Thus, we are led to concentrate on the analysis of $\lambda_i^{(2\text{-}6\text{-}8)}$. Hereafter, we omit the overall factor $e^{-S_{\mathrm{cl}}(D_{\mathrm{inst}}, M_{\mathrm{inst}}^{(2\text{-}6\text{-}8)})}$. Figures 3 and 4 show the $\mathrm{Im}\,\tau$ dependence ($\sqrt{1-(\mathrm{Re}\,\tau)^2} \leq \mathrm{Im}\,\tau \leq 2$) of the absolute values of the mass eigenvalues $\lambda_i^{(2\text{-}6\text{-}8)}$ in Eq. (58) at $\mathrm{Re}\,\tau = 0$ and $\mathrm{Re}\,\tau = 1/2$, respectively. Here, we set $c_{(2\text{-}6\text{-}8)} = 1$ for simplicity.

There are also four interesting features in Figs. 3 and 4: $|\lambda_2| = |\lambda_3|$ at $\tau = i$, $|\lambda_1| \simeq |\lambda_3|$ and $|\lambda_2| \to 0$ at $\tau \to i\infty$, $|\lambda_1| = |\lambda_3|$ at $\tau = \frac{1}{2} + i\mathrm{Im}\,\tau$, and $|\lambda_1| = |\lambda_2| = |\lambda_3|$ at $\tau = \frac{1}{2} + \frac{\sqrt{3}}{2}i$. As with the 2-2-4 case, these can be explained by considering that the points $\tau = i$, $i\infty$, $\tau = \frac{1}{2} + i\mathrm{Im}\,\tau$, and $\frac{1}{2} + \frac{\sqrt{3}}{2}i$ are invariant under S-, T-, $T \cdot CP$-, and ST^{-1}-transformations, respectively. These features also appear in $\lambda_i^{(3\text{-}5\text{-}8)}$ and $\lambda_i^{(4\text{-}4\text{-}8)}$ as well as $\lambda_i^{(M_N=8)}$.

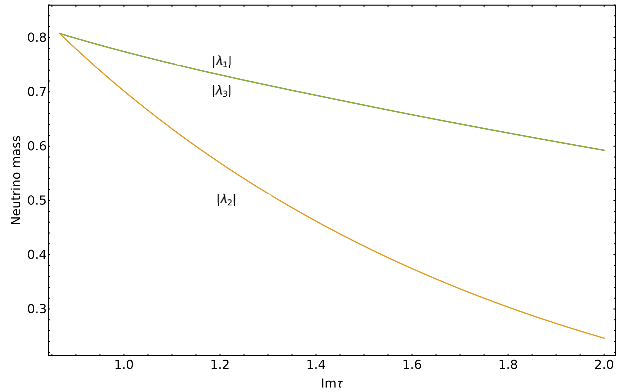

Fig. 4. The $\mathrm{Im}\,\tau$ dependence ($\sqrt{3}/2 \leq \mathrm{Im}\,\tau \leq 2$) of the absolute values of the mass eigenvalues in the 2-6-8 case at $\mathrm{Re}\,\tau = 1/2$.

4.4 *Neutrino sector with $M_N = 5$*

Here, we study the mass matrix of the neutrino sector with $M_N = 5$. The 2-3-5 case is the only possible D-brane configuration. We take even wavefunctions under $M_\beta = 2$ and $M_\gamma = 3$ for β_i and γ_j respectively. For the neutrino sector, we take even wavefunctions under $M_N = 5$. The d matrices are given by

$$d_1 = c_{(2\text{-}3\text{-}5)} \begin{pmatrix} \sqrt{2}\eta_{12}^{(30)} & \eta_2^{(30)} + \eta_8^{(30)} \\ \sqrt{2}\eta_3^{(30)} & \eta_7^{(30)} + \eta_{13}^{(30)} \end{pmatrix},$$

$$d_2 = c_{(2\text{-}3\text{-}5)} \begin{pmatrix} \eta_0^{(30)} & \sqrt{2}\eta_{10}^{(30)} \\ \eta_{15}^{(30)} & \sqrt{2}\eta_5^{(30)} \end{pmatrix},$$

$$d_3 = c_{(2\text{-}3\text{-}5)} \begin{pmatrix} \sqrt{2}\eta_6^{(30)} & \eta_4^{(30)} + \eta_{14}^{(30)} \\ \sqrt{2}\eta_9^{(30)} & \eta_1^{(30)} + \eta_{11}^{(30)} \end{pmatrix}. \tag{65}$$

The mass matrix is then given by

$$\boldsymbol{m}^{(2\text{-}3\text{-}5)} = c_{(2\text{-}3\text{-}5)}^2 \begin{pmatrix} X & U & V \\ U & Y & W \\ V & W & Z \end{pmatrix}, \tag{66}$$

where X, Y, Z, U, V, and W are defined as

$$\begin{aligned}
X &= 2\sqrt{2}\eta_{12}^{(30)} D_7 - 2\sqrt{2}\eta_3^{(30)} D_{-2} \approx -2\sqrt{2}\eta_3^{(30)}\eta_2^{(30)} \approx -2\sqrt{2}(e^{\frac{13\pi i\tau}{30}} + \cdots), \\
Y &= 2\sqrt{2}\eta_0^{(30)}\eta_5^{(30)} - 2\sqrt{2}\eta_{10}^{(30)}\eta_{15}^{(30)} \approx 2\sqrt{2}\eta_0^{(30)}\eta_5^{(30)} \approx 2\sqrt{2}(e^{\frac{25\pi i\tau}{30}} + \cdots), \\
Z &= 2\sqrt{2}\eta_6^{(30)} D_1 - 2\sqrt{2}\eta_9^{(30)} D_4 \approx 2\sqrt{2}\eta_6^{(30)}\eta_1^{(30)} \approx 2\sqrt{2}(e^{\frac{37\pi i\tau}{30}} + \cdots), \\
U &= -\eta_{15}^{(30)} D_{-2} + 2\eta_5^{(30)}\eta_{12}^{(30)} + \eta_0^{(30)} D_7 - 2\eta_3^{(30)}\eta_{10}^{(30)} \approx \eta_0^{(30)}\eta_7^{(30)} \approx e^{\frac{49\pi i\tau}{30}} + \cdots, \\
V &= -\sqrt{2}\eta_3^{(30)} D_4 + \sqrt{2}\eta_6^{(30)} D_7 + \sqrt{2}\eta_{12}^{(30)} D_1 - \sqrt{2}\eta_9^{(30)} D_{-2} \\
&\approx -\sqrt{2}\eta_3^{(30)}\eta_4^{(30)} \approx -\sqrt{2}(e^{\frac{25\pi i\tau}{30}} + \cdots), \\
W &= -\eta_{15}^{(30)} D_4 + 2\eta_5^{(30)}\eta_6^{(30)} + \eta_0^{(30)} D_1 - 2\eta_9^{(30)}\eta_{10}^{(30)} \approx \eta_0^{(30)}\eta_1^{(30)} \approx e^{\frac{\pi i\tau}{30}} + \cdots.
\end{aligned} \tag{67}$$

Here, we have defined

$$D_N = \eta_N^{(30)} + \eta_{N+10}^{(30)}. \tag{68}$$

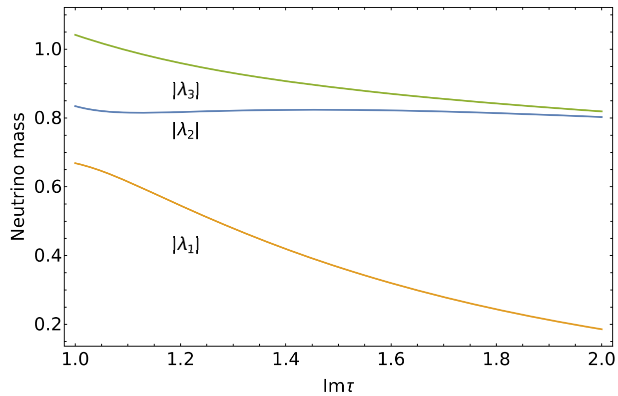

Fig. 5. The Im τ dependence ($1 \leq \mathrm{Im}\,\tau \leq 2$) of the absolute values of the mass eigenvalues in the 2-3-5 case at $\mathrm{Re}\,\tau = 0$.

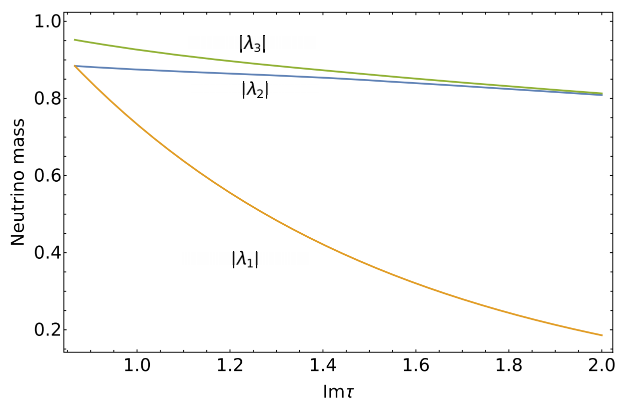

Fig. 6. The Im τ dependence ($\sqrt{3}/2 \leq \mathrm{Im}\,\tau \leq 2$) of the absolute values of the mass eigenvalues in the 2-3-5 case at $\mathrm{Re}\,\tau = 1/2$.

Let us now look at eigenvalues of $\boldsymbol{m}^{(2\text{-}3\text{-}5)}$. Unlike the $M_N = 4$ and 8 cases, $\boldsymbol{m}^{(2\text{-}3\text{-}5)}$ is not generally diagonalized by the bimaximal mixing matrix, but the diagonalization depends on the value of the modulus τ. This may come from the fact that $\mathbb{Z}_2$-shifts are not symmetric for the neutrino sector with $M_N = 5$ and the SS phase (0,0). That is, the SS phase (0,0) transforms to (0, 1/2) and (1/2, 0) under $z \to z + 1/2$ and $z \to z + \tau/2$, respectively.[9] Thus, we diagonalize it numerically. Figures 5 and 6 show the Im τ dependence ($\sqrt{1-(\mathrm{Re}\,\tau)^2} \leq \mathrm{Im}\,\tau \leq 2$) of the absolute values of the mass eigenvalues at $\mathrm{Re}\,\tau = 0$ and $\mathrm{Re}\,\tau = 1/2$, respectively. Here, we set $c_{(2\text{-}3\text{-}5)} = 1$ for simplicity.

There are also several interesting features. First, since the point $\tau = i$ is invariant under S-transformation, the mass eigenvalues at $\tau = i$ must be invariant under S-transformation. Then, the mass eigenstates must also be invariant under S-transformation. We note that S-transformation at $\tau = i$ induces a $\mathbb{Z}_4$-twist for the coordinate of T^2, i.e. $z \to iz$. Hence, the mass eigenstates must be $\mathbb{Z}_4$-twisted eigenstates. The diagonalizing matrix P at $\tau = i$ satisfying $P^{\mathrm{T}}\boldsymbol{m}^{(2\text{-}3\text{-}5)}P = \mathrm{diag}(\lambda_1, \lambda_2, \lambda_3)$ corresponds to the unitary transformation from the $\mathbb{Z}_2$-twisted

[9] From Eqs. (7) and (15), we can easily check that

$$\psi^{(j+0,0),5}_{T^2/\mathbb{Z}_2^+}\left(z+\frac{1}{2}\right) = (-1)^j e^{\frac{5}{2}\pi i \frac{\mathrm{Im}\,z}{\mathrm{Im}\,\tau}} \psi^{(j+0,\frac{1}{2}),5}_{T^2/\mathbb{Z}_2^+}(z), \qquad \psi^{(j+0,0),5}_{T^2/\mathbb{Z}_2^+}\left(z+\frac{\tau}{2}\right) = e^{\frac{5}{2}\pi i \frac{\mathrm{Im}\,\bar{\tau}z}{\mathrm{Im}\,\tau}} \psi^{(2-j+\frac{1}{2},0),5}_{T^2/\mathbb{Z}_2^+}(z).$$

basis to the $\mathbb{Z}_4$-twisted basis, i.e. $\psi_{T^2/\mathbb{Z}_4} = P^{\mathrm{T}}\psi_{T^2/\mathbb{Z}_2}$, given as

$$P = \begin{pmatrix} \sqrt{\frac{23-3\sqrt{5}}{30-2\sqrt{5}}} & -\frac{1}{\sqrt{5-\sqrt{5}}} & 0 \\ \frac{2}{\sqrt{30-2\sqrt{5}}} & \sqrt{\frac{3-\sqrt{5}}{5-\sqrt{5}}} & \sqrt{\frac{3+\sqrt{5}}{7+\sqrt{5}}} \\ -\sqrt{\frac{3+\sqrt{5}}{30-2\sqrt{5}}} & -\frac{1}{\sqrt{5-\sqrt{5}}} & \frac{2}{\sqrt{7+\sqrt{5}}} \end{pmatrix}. \tag{69}$$

This is nearly the tri-bimaximal mixing matrix, i.e.

$$|P_{ij}|^2 \approx \begin{pmatrix} 0.64 & 0.36 & 0 \\ 0.16 & 0.28 & 0.57 \\ 0.21 & 0.36 & 0.43 \end{pmatrix}. \tag{70}$$

Its derivation is shown in Appendix B. Actually, this is consistent with the numerical analysis.

Second, we can find that $|\lambda_2| \simeq |\lambda_3|$ and $|\lambda_1| \to 0$ at $\tau \to i\infty, \frac{1}{2} + i\infty$. Moreover, the mass matrices $\boldsymbol{m}^{(2\text{-}3\text{-}5)}$ are diagonalized by almost the bimaximal mixing matrix. These can be explained by considering T-invariance. Note that since neutrinos with $M_N = 5$ and SS phase (0,0) transform into ones with SS phase (0, 1/2) under T-transformation,[10] we cannot consider T-transformation for those neutrinos in general. However, in the limit $\mathrm{Im}\,\tau \to \infty$, the SS phases can be negligible.[11] In the limit $\mathrm{Im}\,\tau \to \infty$, an approximate T-transformation appears as well as $\mathbb{Z}_2$-shifts for the neutrinos. Then, the mass eigenvalues at $\tau \to i\infty, \frac{1}{2} + i\infty$ must be almost invariant under T-transformation as well as $\mathbb{Z}_2$-shifts. Thus, as with the $M_N = 4$ and 8 cases, the mass eigenstates at $\tau \to i\infty, \frac{1}{2} + i\infty$ become $\mathbb{Z}_2$-twisted and almost $\mathbb{Z}_2$-shifted eigenstates, which means the mass matrix is diagonalized by almost the bimaximal mixing matrix. Moreover, since the eigenstates of T-transformation are not the $\mathbb{Z}_2$-twisted and (almost) $\mathbb{Z}_2$-shifted basis but the $\mathbb{Z}_2$-twisted basis, two eigenvalues for two mass eigenstates transformed by almost the bimaximal mixing matrix become the same, i.e. $|\lambda_2| \simeq |\lambda_3|$. Actually, from Eq. (67), the mass matrix $\boldsymbol{m}^{(2\text{-}3\text{-}5)}$ at $\tau \to i\infty, \frac{1}{2} + i\infty$ is estimated as

$$\boldsymbol{m}^{(2\text{-}3\text{-}5)} \approx e^{\frac{\pi i\tau}{30}} \begin{pmatrix} -2\sqrt{2}e^{\frac{2\pi i\tau}{5}} & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}. \tag{71}$$

Then, it is diagonalized by the bimaximal mixing matrix and the eigenvalues satisfy $\lambda_2 \simeq -\lambda_3$ and $\lambda_1 \to 0$ at $\tau \to i\infty, \frac{1}{2} + i\infty$.

Third, we can find $|\lambda_1| = |\lambda_2|$ at $\tau = \frac{1}{2} + \frac{\sqrt{3}}{2}i$. This can be explained by considering that the point $\tau = \frac{1}{2} + \frac{\sqrt{3}}{2}i$ is invariant under ST^{-1}- and $T \cdot CP$-transformations. However, unlike the $M_N = 4$ and 8 cases, $|\lambda_3|$ is not the same as $|\lambda_2|$. In addition, the diagonalizing matrix departs from the bimaximal mixing matrix slightly, though it can still be approximated by the bimaximal mixing matrix. This may be because the difference of the SS phases by T-transformation as well as $\mathbb{Z}_2$-shifts cannot be ignored, though we can consider the simultaneous transformation of T-transformation and $\mathbb{Z}_2$-shifts. Then, the point $\tau = \frac{1}{2} + \frac{\sqrt{3}}{2}i$ is also invariant under $S \cdot CP$-transformation. Actually, on the mass eigenstates, $|\lambda_1|$ and $|\lambda_2|$ are exchanged under $S \cdot CP$-transformation, while $|\lambda_3|$ is invariant. This means that $|\lambda_1| = |\lambda_2|$ at $\tau = \frac{1}{2} + \frac{\sqrt{3}}{2}i$ is required.

[10] For details, see Ref. [27].

[11] In this limit, the shifts $z \to z + 1$ and $z \to z + 2$ cannot be distinguished, which means the difference of the SS phase α_1 can be negligible. Similarly, the shifts $z \to z + \frac{1}{2}$ and $z \to z + 1$ cannot be distinguished, which means the difference of the SS phase α_τ can be negligible.

4.5 *Neutrino sector with $M_N = 7$*

Here, we study the mass matrix for the neutrino sector with $M_N = 7$. For this neutrino sector the mass terms can be induced by the 2-5-7 and 3-4-7 cases.

4.5.1 *2-5-7 case.* In the 2-5-7 case, we take even wavefunctions under $M_\beta = 2$ and odd wavefunctions under $M_\gamma = 5$ for β_i and γ_j respectively. For the neutrino sector, we take odd wavefunctions under $M_N = 7$. The d matrices are given by

$$d_1 = c_{(2\text{-}5\text{-}7)} \begin{pmatrix} -E_{18} & E_4 \\ E_{-3} & -E_{11} \end{pmatrix},$$
$$d_2 = c_{(2\text{-}5\text{-}7)} \begin{pmatrix} D_2 & -D_{16} \\ -D_{23} & D_9 \end{pmatrix},$$
$$d_3 = c_{(2\text{-}5\text{-}7)} \begin{pmatrix} F_{-8} & -F_6 \\ -F_{13} & F_{-1} \end{pmatrix}. \tag{72}$$

Here, D_N, E_N, and F_N are defined as

$$D_N = \eta_N^{(70)} - \eta_{N+10}^{(70)}, \qquad E_N = \eta_N^{(70)} - \eta_{N+20}^{(70)}, \qquad F_N = \eta_N^{(70)} - \eta_{N+30}^{(70)}. \tag{73}$$

The mass matrix is then given by

$$\boldsymbol{m}^{(2\text{-}5\text{-}7)} = c_{(2\text{-}5\text{-}7)}^2 \begin{pmatrix} X & U & V \\ U & Y & W \\ V & W & Z \end{pmatrix}, \tag{74}$$

where X, Y, Z, U, V, and W are defined as

$$\begin{aligned}
X &= 2E_{18}E_{11} - 2E_{-3}E_4 \approx -2\eta_3^{(70)}\eta_4^{(70)} \approx -2(e^{\frac{5\pi i\tau}{14}} + \cdots), \\
Y &= -2D_{23}D_{16} + 2D_2D_9 \approx 2\eta_2^{(70)}\eta_9^{(70)} \approx 2(e^{\frac{17\pi i\tau}{14}} + \cdots), \\
Z &= 2F_{-8}F_{-1} - 2F_{13}F_6 \approx 2\eta_8^{(70)}\eta_1^{(70)} \approx 2(e^{\frac{13\pi i\tau}{14}} + \cdots), \\
U &= D_{23}E_4 - D_9E_{18} - D_2E_{11} + D_{16}E_{-3} \approx -\eta_2^{(70)}\eta_{11}^{(70)} \approx -(e^{\frac{25\pi i\tau}{14}} + \cdots), \\
V &= E_{-3}F_6 - E_{11}F_{-8} - E_{18}F_{-1} + E_4F_{13} \approx \eta_3^{(70)}\eta_6^{(70)} \approx e^{\frac{9\pi i\tau}{14}} + \cdots, \\
W &= -D_{23}F_6 + D_9F_{-8} + D_2F_{-1} - D_{16}F_{13} \approx \eta_2^{(70)}\eta_1^{(70)} \approx e^{\frac{\pi i\tau}{14}} + \cdots.
\end{aligned} \tag{75}$$

4.5.2 *3-4-7 case.* In 3-4-7 case, there are three possible variations with the different SS phases (1/2, 0), (0, 1/2), and (1/2, 1/2). The sum of the three mass matrices in equal ratio is given by

$$\begin{aligned}
\boldsymbol{m}^{(3\text{-}4\text{-}7)} &= \boldsymbol{m}^{(3\text{-}4\text{-}7)(1/2,0)} + \boldsymbol{m}^{(3\text{-}4\text{-}7)(0,1/2)} + \boldsymbol{m}^{(3\text{-}4\text{-}7)(1/2,1/2)} \\
&= c_{(3\text{-}4\text{-}7)}^2 \begin{pmatrix} X^{(3\text{-}4\text{-}7)} & U^{(3\text{-}4\text{-}7)} & V^{(3\text{-}4\text{-}7)} \\ U^{(3\text{-}4\text{-}7)} & Y^{(3\text{-}4\text{-}7)} & W^{(3\text{-}4\text{-}7)} \\ V^{(3\text{-}4\text{-}7)} & W^{(3\text{-}4\text{-}7)} & Z^{(3\text{-}4\text{-}7)} \end{pmatrix},
\end{aligned} \tag{76}$$

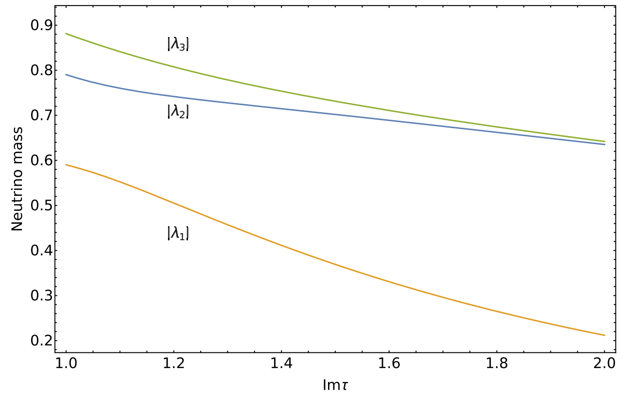

Fig. 7. The $\mathrm{Im}\,\tau$ dependence ($1 \le \mathrm{Im}\,\tau \le 2$) of the absolute values of the mass eigenvalues in the 2-5-7 case at $\mathrm{Re}\,\tau = 0$.

where

$$\begin{aligned}
X^{(3\text{-}4\text{-}7)} &= 2(S_{(-9,-61)} + R_{(30,44)}) \approx -2\sqrt{2}\eta_5^{(336)}\eta_9^{(336)} \approx -2\sqrt{2}e^{\frac{53\pi i\tau}{168}}, \\
Y^{(3\text{-}4\text{-}7)} &= 2(S_{(-57,-13)} + R_{(6,20)}) \approx 2\sqrt{2}\eta_{13}^{(336)}\eta_{15}^{(336)} \approx 2\sqrt{2}e^{\frac{197\pi i\tau}{168}}, \\
Z^{(3\text{-}4\text{-}7)} &= 2(S_{(-81,11)} + R_{(-18,-4)}) \approx 2\sqrt{2}\eta_3^{(336)}\eta_{17}^{(336)} \approx 2\sqrt{2}e^{\frac{149\pi i\tau}{168}}, \\
U^{(3\text{-}4\text{-}7)} &= S_{(-9,-13)} + S_{(15,-37)} + R_{(6,44)} + R_{(30,20)} \approx -\sqrt{2}\eta_{15}^{(336)}\eta_{19}^{(336)} \approx -\sqrt{2}e^{\frac{293\pi i\tau}{168}}, \\
V^{(3\text{-}4\text{-}7)} &= S_{(-33,11)} + S_{(39,-61)} + R_{(30,-4)} + R_{(-18,44)} \approx \sqrt{2}\eta_9^{(336)}\eta_{11}^{(336)} \approx \sqrt{2}e^{\frac{101\pi i\tau}{168}}, \\
W^{(3\text{-}4\text{-}7)} &= S_{(39,-13)} + S_{(15,11)} + R_{(6,-4)} + R_{(-18,20)} \approx \sqrt{2}\eta_3^{(336)}\eta_1^{(336)} \approx \sqrt{2}e^{\frac{5\pi i\tau}{168}}.
\end{aligned} \tag{77}$$

Here, we defined

$$\begin{aligned}
S_{(M,N)} &= \frac{1}{\sqrt{2}}(Q_{(M,N)} - P_{(M,N)}), \\
P_{(M,N)} &= B_M(B_N + B_{N+56}) - B_{14-N}(B_{14-M} + B_{(14-M)+56}), \\
Q_{(M,N)} &= E_M(D_N - D_{N+56}) + D_{14-N}(E_{14-M} - E_{(14-M)+56}), \\
R_{(M,N)} &= G_M F_N - A_{M-42}(F_{N-42} + F_{(N-42)+84}), \\
A_N &= \eta_N^{(336)} - \eta_{N+168}^{(336)}, \\
B_N &= \left(\eta_N^{(336)} + \eta_{N+168}^{(336)}\right) - \left(\eta_{N+42}^{(336)} + \eta_{(N+42)+168}^{(336)}\right), \\
D_N &= A_N - A_{N+42}, \\
E_N &= A_N + A_{N+42}, \\
F_N &= A_N - A_{N+56}, \\
G_N &= A_N - A_{N+84}.
\end{aligned} \tag{78}$$

4.5.3 *Full mass matrix and mass eigenvalues.* Finally, let us look at the mass eigenvalues of the full mass matrix $\boldsymbol{M}^{(M_N=7)}$. Unlike the $M_N = 8$ case, diagonalization of $\boldsymbol{M}^{(M_N=7)}$ depends on the factors $e^{-S_{\mathrm{cl}}(D_{\mathrm{inst}}, M_{\mathrm{inst}})}$ as well as the modulus τ in general. Here, we evaluate $\boldsymbol{M}^{(M_N=7)}$ with $\boldsymbol{m}^{(2\text{-}5\text{-}7)}$ as in Eq. (24). Then, we diagonalize $\boldsymbol{m}^{(2\text{-}5\text{-}7)}$ numerically, where we omit the overall factor $e^{-S_{\mathrm{cl}}(D_{\mathrm{inst}}, M_{\mathrm{inst}})}$. Figures 7 and 8 show the $\mathrm{Im}\,\tau$ dependence ($\sqrt{1-(\mathrm{Re}\,\tau)^2} \le \mathrm{Im}\,\tau \le 2$) of

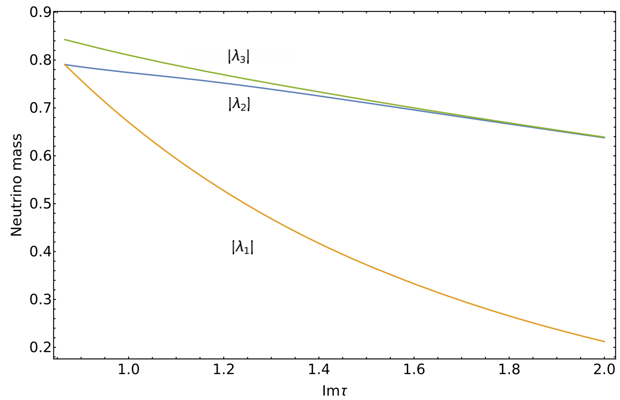

Fig. 8. The $\mathrm{Im}\,\tau$ dependence ($\sqrt{3}/2 \leq \mathrm{Im}\,\tau \leq 2$) of the absolute values of the mass eigenvalues in the 2-5-7 case at $\mathrm{Re}\,\tau = 1/2$.

the absolute values of the mass eigenvalues at $\mathrm{Re}\,\tau = 0$ and $\mathrm{Re}\,\tau = 1/2$, respectively. Here, we set $c_{(2\text{-}5\text{-}7)} = 1$ for simplicity.

Again, there are several features that can be explained as with the $M_N = 5$ case. These features also appear in $\boldsymbol{m}^{(3\text{-}4\text{-}7)}$ as well as $\boldsymbol{M}^{(M_N=7)}$. In particular, it is interesting that in the large $\mathrm{Im}\,\tau$ limit, both matrices $\boldsymbol{m}^{(2\text{-}5\text{-}7)}$ and $\boldsymbol{m}^{(3\text{-}4\text{-}7)}$ can be approximated by

$$\boldsymbol{m}^{(2\text{-}5\text{-}7)} \approx a_{(2\text{-}5\text{-}7)}\boldsymbol{m}^{(7)}, \qquad \boldsymbol{m}^{(3\text{-}4\text{-}7)} \approx a_{(3\text{-}4\text{-}7)}\boldsymbol{m}^{(7)}, \tag{79}$$

where

$$\boldsymbol{m}^{(7)} = \begin{pmatrix} -2q_7 & -q_7^6 & q_7^2 \\ -q_7^6 & 2q_7^4 & 1 \\ q_7^2 & 1 & 2q_7^3 \end{pmatrix}, \qquad q_7 = e^{2\pi i\tau/7}. \tag{80}$$

It seems that the $\mathbb{Z}_{14}$ symmetry appears, originating from the T-transformation in the matrix $\boldsymbol{m}^{(7)}$. In addition, the overall factors $a_{(2\text{-}5\text{-}7)}$ and $a_{(3\text{-}4\text{-}7)}$ are written as

$$a_{(2\text{-}5\text{-}7)} = e^{\pi i\tau/14}, \qquad a_{(3\text{-}4\text{-}7)} = \sqrt{2}e^{5\pi i\tau/168}. \tag{81}$$

This leads to different phases under the T-transformation.[12]

5. Conclusion

We have studied Majorana neutrino masses induced by D-brane instanton effects in magnetized orbifold models. We have systematically studied the D-brane configurations, where neutrino masses can be induced. Three and four generations are favorable in order to generate Majorana neutrino masses by D-brane instanton effects. Also, we have computed explicit patterns of neutrino mass matrices. These matrices have specific features. Our basis is the $T^2/\mathbb{Z}_2$ orbifold basis. The diagonalizing matrices of neutrino mass matrices are the bimaximal mixing matrix in the case with even magnetic fluxes, independent of the modulus value τ. On the other hand, for odd magnetic fluxes, the diagonalizing matrices correspond nearly to the tri-bimaximal mixing matrix near $\tau = i$, while they become the bimaximal one for larger $\mathrm{Im}\,\tau$.[13]

For even magnetic fluxes, the neutrino masses are modular forms of weight 1 on $T^2/\mathbb{Z}_2$, and they have symmetries such as S_4' and $\Delta'(96) \times \mathbb{Z}_3$. These modular form structures of

[12] The anomaly would be relevant to these structures. Study of the modular symmetry anomaly is beyond our scope, and will be studied elsewhere.

[13] The point $\tau = i$ may be favorable from the viewpoint of modulus stabilization [35].

Majorana neutrino masses can provide us with the ultraviolet completion for the recent bottom-up approach to constructing modular flavor symmetric models [36–46]. We can extend our analysis on the $T^4/\mathbb{Z}_2$ orbifold with two moduli, τ_1 and τ_2. When we identify $\tau = \tau_1 = \tau_2$ as in Ref. [26], the Majorana neutrino masses would correspond to modular forms of weight 2 with symmetries Γ_N.

The patterns of Majorana neutrino mass matrices are quite interesting. For example, we can realize almost the tri-bimaximal mixing matrix as the diagonalizing matrix in our $T^2/\mathbb{Z}_2$ orbifold basis. However, this diagonalizing matrix is not physically observable. We have to examine the charged lepton mass matrix and Dirac neutrino mass matrix. Then, we can discuss the mixing angles in the lepton sector, i.e. the Pontecorvo–Maki–Nakagawa–Sakata matrix. Three-generation models have been studied in Refs. [14–16]. It is very interesting to combine those analyses with our results here to analyze light neutrino masses and the mixing angles in the lepton sector. We will study this elsewhere.

In the present paper we have assumed that stringy consistency conditions are satisfied without a precise discussion, and it is not clear whether or not our results can always be realized by full-fledged stringy models. For instance, if there are additional zero-modes between D-brane instantons and hidden D-branes, the Majorana mass term is not generated by the D-brane instanton and it vanishes. However, we should emphasize that such an effect does not change the form of the Majorana mass matrix, but removes the non-perturbative Majorana mass. Thus, a non-zero Majorana mass matrix generated by a D-brane instatnon on a toroidal orbifold must be included in our results. Hence, our results are general.

The Majorana mass terms may have corrections due to other non-perturbative effects, e.g. effects due to instanton branes localized at singular points of $T^2/\mathbb{Z}_2$, although $U(1)$ charge conservation would give a severe constraint. Note that in our case, $U(1)$ charges are conserved by the combination of neutrinos and zero-modes β_i and γ_j. To obtain two generations of zero-modes with such a localized instanton brane, we would need a proper zero-mode structure on the other four-dimensional compact space. At any rate, such corrections are beyond our scope.

It is important how to stabilize the modulus τ, because the mass matrix form depends on the value τ. One stabilization mechanism is due to three-form fluxes. For example, in Ref. [35] it was shown that certain values of τ are statistically favorable. One of the favorable values is $\tau = \frac{1}{2} + \frac{\sqrt{3}}{2}i$, where three mass eigenvalues are degenerate for $M_N = 4, 8$ while two mass eigenvalues are degenerate for $M_N = 5, 7$. Another favorable one is $\tau = i$, where two mass eigenvalues are degenerate for $M_N = 4, 8$, while the diagonalizing matrix is given by Eq. (69).

It is also important to extend our analysis to other $T^2/\mathbb{Z}_N$ orbifolds [8,9] as well as resolved orbifolds [47,48]. We will study this elsewhere.

Acknowledgments

T.K. was supported in part by MEXT KAKENHI Grant Number JP19H04605. H.U. was supported by Grant-in-Aid for JSPS Research Fellows No. JP20J20388.

Funding

Open Access funding: SCOAP^3.

Table A1. The modulus τ and the operators γ satisfying $\gamma(\tau) = \tau$.

τ	γ
i	S
$\frac{1}{2} + \frac{\sqrt{3}}{2}i\ (-\frac{1}{2} + \frac{\sqrt{3}}{2}i)$	$ST^{-1}, T \cdot CP, S \cdot CP\ (ST, T^{-1} \cdot CP, S \cdot CP)$
$\frac{1}{2} + i\mathrm{Im}\,\tau\ \left(-\frac{1}{2} + i\mathrm{Im}\,\tau\right)$	$T \cdot CP\ (T^{-1} \cdot CP)$
$(\mathrm{Re}\,\tau) + i\infty$	T

Appendix A. Modular symmetry

Here, we review the modular symmetry on T^2 and modular forms [49–52]. The modular transformation for the modulus τ as well as the coordinate z of T^2 are defined as

$$\gamma : \tau \to \gamma(\tau) = \frac{a\tau + b}{c\tau + d}, \quad \gamma : z \to \gamma(z) = \frac{z}{c\tau + d}, \quad \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2, \mathbb{Z}) \equiv \Gamma. \quad \text{(A1)}$$

They are generated by the following S and T transformations:

$$\begin{aligned} S &: \tau \to S(\tau) = -\tfrac{1}{\tau}, \quad S : z \to S(z) = -\tfrac{z}{\tau}, \quad S = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \in \Gamma, \\ T &: \tau \to T(\tau) = \tau + 1, \quad T : z \to T(z) = z, \quad T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \in \Gamma, \end{aligned} \quad \text{(A2)}$$

which satisfy $S^4 = (ST)^3 = \mathbb{I}$. In particular, the $S^2 = -\mathbb{I}$ transformation for the modulus τ is identified with the identity $\mathbb{I}$. In this sense, $\bar{\Gamma} \equiv \Gamma/\{\pm\mathbb{I}\}$ is called the (inhomogeneous) modular group. Note that the CP transformation for the modulus τ [53,54] as well as the coordinate z are also defined as

$$CP : \tau \to CP(\tau) = -\bar{\tau}, \qquad CP : z \to CP(z) = -\bar{z}. \quad \text{(A3)}$$

Then, the fundamental region, F, of the modulus τ becomes

$$F = \left\{ \tau \in \mathbb{C} \,\middle|\, -\frac{1}{2} \le \mathrm{Re}\,\tau \le \frac{1}{2},\ \sqrt{1 - (\mathrm{Re}\,\tau)^2} \le \mathrm{Im}\,\tau \right\}. \quad \text{(A4)}$$

Table A1 shows the specific points of modulus τ and the operators $\gamma \in \bar{\Gamma}$ satisfying $\gamma(\tau) = \tau$.

Now, let us review the modular forms. First, we introduce the principal congruence subgroup of level N defined as

$$\Gamma(N) \equiv \left\{ h = \begin{pmatrix} a' & b' \\ c' & d' \end{pmatrix} \in \Gamma \,\middle|\, \begin{pmatrix} a' & b' \\ c' & d' \end{pmatrix} \equiv \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} (\mathrm{mod}\ N) \right\}. \quad \text{(A5)}$$

The modular forms, $f(\tau)$, of the integral weight k for $\Gamma(N)$ are the holomorphic functions of τ transforming under the modular transformation as

$$\begin{aligned} f(\gamma(\tau)) &= J_k(\gamma, \tau)\rho(\gamma) f(\tau), \quad J_k(\gamma, \tau) = (c\tau + d)^k, \quad \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma, \\ f(h(\tau)) &= J_k(h, \tau) f(\tau), \quad J_k(h, \tau) = (c'\tau + d')^k, \quad h = \begin{pmatrix} a' & b' \\ c' & d' \end{pmatrix} \in \Gamma(N), \\ \rho(h) &= \mathbb{I}, \end{aligned} \quad \text{(A6)}$$

where $J_k(\gamma, \tau)$ is the automorphy factor and ρ is the unitary representation of the quotient group $\Gamma'_N \equiv \Gamma/\Gamma(N)$, satisfying the relations

$$\begin{aligned} &\rho(Z) = \rho(S)^2 = (-1)^k \mathbb{I}, \quad \rho(Z)^2 = \rho(S)^4 = [\rho(S)\rho(T)]^3 = \mathbb{I}, \\ &\rho(T)^N = \mathbb{I}, \quad \rho(Z)\rho(T) = \rho(T)\rho(Z). \end{aligned} \quad \text{(A7)}$$

Table B1. The number of zero-modes of $\mathbb{Z}_4$-twisted eigenmodes with SS phase (0,0).

$\|M\|$	$4n$	$4n+1$	$4n+2$	$4n+3$
$p=1$	$n+1$	$n+1$	$n+1$	$n+1$
$p=-1$	n	n	$n+1$	$n+1$
$p=i$	n	n	n	$n+1$
$p=-i$	$n-1$	n	n	n

In particular, in the case of an even weight k, ρ is also the unitary representation of the finite modular subgroup $\Gamma_N \equiv \bar{\Gamma}/\bar{\Gamma}(N)$, where $\bar{\Gamma}(N) \equiv \Gamma(N)/\{\pm \mathbb{I}\}$ for $N = 1, 2$ and $\bar{\Gamma}(N) \equiv \Gamma(N)$ for $N > 2$. Moreover, it is well known that $\Gamma_2 \simeq S_3$, $\Gamma_3 \simeq A_4$, $\Gamma_4 \simeq S_4$, and $\Gamma_5 \simeq A_5$ [55], and also that Γ'_N becomes the double covering group of Γ_N [56].

Appendix B. $\mathbb{Z}_4$-twisted basis

Here, we derive the basis transformation in Eq. (69) from the $\mathbb{Z}_2$-twisted basis into the $\mathbb{Z}_4$-twisted basis. First, when we consider the $T^2/\mathbb{Z}_4$-twisted orbifold, the modulus must be $\tau = i$. The $\mathbb{Z}_4$-twisted eigenstates with eigenvalues $p = \pm 1, \pm i$ and SS phase (0,0) [8–10] can be constructed by

$$\begin{aligned}
&\psi_{T^2/\mathbb{Z}_4^p}(z) \\
&= \mathcal{N}\left(\psi_{T^2}^{(j,0),|M|}(z) + p^{-1}\psi_{T^2}^{(j,0),|M|}(iz) + p^{-2}\psi_{T^2}^{(j,0),|M|}(-z) + p^{-3}\psi_{T^2}^{(j,0),|M|}(-iz)\right) \\
&= \sum_{k=0}^{|M|-1} \mathcal{N}\left(\left(\delta_{j,k} + p^{-2}\delta_{|M|-j,k}\right) + \frac{p^{-1}}{\sqrt{|M|}}\left(e^{\frac{2\pi i jk}{|M|}} + p^{-2}e^{-\frac{2\pi i jk}{|M|}}\right)\right)\psi_{T^2}^{(k,0),|M|}(z),
\end{aligned} \tag{B1}$$

where $\mathcal{N}$ denotes the normalization factor. Here, we use the S-transformation for wavefunctions on T^2,

$$\psi_{T^2}^{(j,0),|M|}\left(-\frac{z}{\tau}, -\frac{1}{\tau}\right) = (-\tau)^{1/2} \sum_{k=0}^{|M|-1} \frac{e^{\frac{\pi i}{4}}}{\sqrt{M}} e^{\frac{2\pi i jk}{|M|}} \psi_{T^2}^{(k,0),|M|}(z,\tau), \tag{B2}$$

at $\tau = i$ since the $\mathbb{Z}_4$-twist, $z \to iz$, can be induced by S-transformation at $\tau = i$. Then, the number of zero-modes is given in Table B1.

Now, we derive the $\mathbb{Z}_4$-twisted basis explicitly from three $\mathbb{Z}_2$-even modes of $M = 5$. In this case, one of the eigenvalues is $p = -1$ and the other two eigenvalues are $p = 1$. The $\mathbb{Z}_4$-twisted eigenstates are obtained by

$$\psi_{T^2/\mathbb{Z}_4^p} = \mathcal{N}\left(\begin{pmatrix} 1 & & \\ & 1 & \\ & & 1 \end{pmatrix} + \frac{p^{-1}}{\sqrt{5}}\begin{pmatrix} 1 & \sqrt{2} & \sqrt{2} \\ \sqrt{2} & \frac{\sqrt{5}-1}{2} & -\frac{\sqrt{5}+1}{2} \\ \sqrt{2} & -\frac{\sqrt{5}+1}{2} & \frac{\sqrt{5}-1}{2} \end{pmatrix}\right)\begin{pmatrix} \psi_{T^2/\mathbb{Z}_2^+}^{(0,0),5} \\ \psi_{T^2/\mathbb{Z}_2^+}^{(1,0),5} \\ \psi_{T^2/\mathbb{Z}_2^+}^{(2,0),5} \end{pmatrix}, \tag{B3}$$

where

$$\psi_{T^2/\mathbb{Z}_2^+}^{(0,0),5} = \psi_{T^2}^{(0,0),5}, \quad \psi_{T^2/\mathbb{Z}_2^+}^{(1,0),5} = \frac{1}{\sqrt{2}}\left(\psi_{T^2}^{(1,0),5} + \psi_{T^2}^{(4,0),5}\right), \quad \psi_{T^2/\mathbb{Z}_2^+}^{(2,0),5} = \frac{1}{\sqrt{2}}\left(\psi_{T^2}^{(2,0),5} + \psi_{T^2}^{(3,0),5}\right). \tag{B4}$$

Then, the mode with $p = -1$ is obtained as

$$\psi^1_{T^2/\mathbb{Z}_4^{-1}} = \mathcal{N}'\left(\frac{\sqrt{5}-1}{\sqrt{2}}\psi^{(0,0),5}_{T^2/\mathbb{Z}_2^+} - \psi^{(1,0),5}_{T^2/\mathbb{Z}_2^+} - \psi^{(2,0),5}_{T^2/\mathbb{Z}_2^+}\right)$$
$$= \sqrt{\frac{3-\sqrt{5}}{5-\sqrt{5}}}\psi^{(0,0),5}_{T^2/\mathbb{Z}_2^+} - \frac{1}{\sqrt{5-\sqrt{5}}}\psi^{(1,0),5}_{T^2/\mathbb{Z}_2^+} - \frac{1}{\sqrt{5-\sqrt{5}}}\psi^{(2,0),5}_{T^2/\mathbb{Z}_2^+} \qquad \left(\mathcal{N}' = \frac{1}{\sqrt{5-\sqrt{5}}}\right), \tag{B5}$$

while one mode with $p = 1$ is obtained as

$$\psi^1_{T^2/\mathbb{Z}_4^{+1}} = \mathcal{N}'\left(\sqrt{2}\psi^{(0,0),5}_{T^2/\mathbb{Z}_2^+} + \frac{3\sqrt{5}-1}{2}\psi^{(1,0),5}_{T^2/\mathbb{Z}_2^+} - \frac{\sqrt{5}+1}{2}\psi^{(2,0),5}_{T^2/\mathbb{Z}_2^+}\right)$$
$$= \frac{2}{\sqrt{30-2\sqrt{5}}}\psi^{(0,0),5}_{T^2/\mathbb{Z}_2^+} + \sqrt{\frac{23-3\sqrt{5}}{30-2\sqrt{5}}}\psi^{(1,0),5}_{T^2/\mathbb{Z}_2^+}$$
$$- \sqrt{\frac{3+\sqrt{5}}{30-2\sqrt{5}}}\psi^{(2,0),5}_{T^2/\mathbb{Z}_2^+} \quad \left(\mathcal{N}' = \frac{1}{\sqrt{15-\sqrt{5}}}\right). \tag{B6}$$

The other mode with $p = 1$ is obtained by orthogonalizing

$$\psi_{T^2/\mathbb{Z}_4^{+1}} = \mathcal{N}'\left(\frac{\sqrt{5}+1}{\sqrt{2}}\psi^{(0,0),5}_{T^2/\mathbb{Z}_2^+} + \psi^{(1,0),5}_{T^2/\mathbb{Z}_2^+} + \psi^{(2,0),5}_{T^2/\mathbb{Z}_2^+}\right)$$

to the mode in Eq. (B6) through the Gram–Schmidt process as

$$\psi^2_{T^2/\mathbb{Z}_4^{+1}} = \sqrt{\frac{3+\sqrt{5}}{7+\sqrt{5}}}\psi^{(0,0),5}_{T^2/\mathbb{Z}_2^+} + \frac{2}{\sqrt{7+\sqrt{5}}}\psi^{(2,0),5}_{T^2/\mathbb{Z}_2^+}. \tag{B7}$$

Therefore, the $\mathbb{Z}_4$-twisted basis obtained from the $\mathbb{Z}_2$-twisted even modes of $M = 5$, $\psi_{T^2/\mathbb{Z}_4} = P^{\mathrm{T}}\psi_{T^2/\mathbb{Z}_2}$, is

$$\begin{pmatrix} \psi^1_{T^2/\mathbb{Z}_4^{+1}} \\ \psi^1_{T^2/\mathbb{Z}_4^{-1}} \\ \psi^2_{T^2/\mathbb{Z}_4^{+1}} \end{pmatrix} = \begin{pmatrix} \sqrt{\frac{23-3\sqrt{5}}{30-2\sqrt{5}}} & \frac{2}{\sqrt{30-2\sqrt{5}}} & -\sqrt{\frac{3+\sqrt{5}}{30-2\sqrt{5}}} \\ -\frac{1}{\sqrt{5-\sqrt{5}}} & \sqrt{\frac{3-\sqrt{5}}{5-\sqrt{5}}} & -\frac{1}{\sqrt{5-\sqrt{5}}} \\ 0 & \sqrt{\frac{3+\sqrt{5}}{7+\sqrt{5}}} & \frac{2}{\sqrt{7+\sqrt{5}}} \end{pmatrix} \begin{pmatrix} \psi^{(1,0),5}_{T^2/\mathbb{Z}_2^+} \\ \psi^{(0,0),5}_{T^2/\mathbb{Z}_2^+} \\ \psi^{(2,0),5}_{T^2/\mathbb{Z}_2^+} \end{pmatrix}. \tag{B8}$$

This gives P as in Eq. (69).

References

[1] C. Bachas, arXiv:hep-th/9503030 [Search inSPIRE].

[2] R. Blumenhagen, L. Goerlich, B. Kors, and D. Lust, J. High Energy Phys. **0010**, 006 (2000) [arXiv:hep-th/0007024] [Search inSPIRE].

[3] C. Angelantonj, I. Antoniadis, E. Dudas, and A. Sagnotti, Phys. Lett. B **489**, 223 (2000) [arXiv:hep-th/0007090] [Search inSPIRE].

[4] R. Blumenhagen, B. Kors, and D. Lust, J. High Energy Phys. **0102**, 030 (2001) [arXiv:hep-th/0012156] [Search inSPIRE].

[5] D. Cremades, L. E. Ibanez, and F. Marchesano, J. High Energy Phys. **05** (2004), 079 [arXiv:hep-th/0404229] [Search inSPIRE].

[6] H. Abe, K. S. Choi, T. Kobayashi, and H. Ohki, J. High Energy Phys. **06**, 080 (2009) [arXiv:0903.3800 [hep-th]] [Search inSPIRE].

[7] H. Abe, T. Kobayashi, and H. Ohki, J. High Energy Phys. **09** (2008), 043 [arXiv:0806.4748 [hep-th]] [Search inSPIRE].

[8] T. H. Abe, Y. Fujimoto, T. Kobayashi, T. Miura, K. Nishiwaki, and M. Sakamoto, J. High Energy Phys. **1401**, 065 (2014) [arXiv:1309.4925 [hep-th]] [Search inSPIRE].
[9] T. H. Abe, Y. Fujimoto, T. Kobayashi, T. Miura, K. Nishiwaki, and M. Sakamoto, Nucl. Phys. B **890**, 442 (2014) [arXiv:1409.5421 [hep-th]] [Search inSPIRE].
[10] T. Kobayashi and S. Nagamoto, Phys. Rev. D **96**, 096011 (2017) [arXiv:1709.09784 [hep-th]] [Search inSPIRE].
[11] M. Sakamoto, M. Takeuchi, and Y. Tatsuta, Phys. Rev. D **102**, 025008 (2020) [arXiv:2004.05570 [hep-th]] [Search inSPIRE].
[12] M. Sakamoto, M. Takeuchi, and Y. Tatsuta, arXiv:2010.14214 [hep-th] [Search inSPIRE].
[13] Y. Fujimoto, T. Kobayashi, T. Miura, K. Nishiwaki, and M. Sakamoto, Phys. Rev. D **87**, 086001 (2013) [arXiv:1302.5768 [hep-th]] [Search inSPIRE].
[14] H. Abe, K. S. Choi, T. Kobayashi, and H. Ohki, Nucl. Phys. B **814**, 265 (2009) [arXiv:0812.3534 [hep-th]] [Search inSPIRE].
[15] T. h. Abe, Y. Fujimoto, T. Kobayashi, T. Miura, K. Nishiwaki, M. Sakamoto, and Y. Tatsuta, Nucl. Phys. B **894**, 374 (2015) [arXiv:1501.02787 [hep-ph]] [Search inSPIRE].
[16] K. Hoshiya, S. Kikuchi, T. Kobayashi, Y. Ogawa, and H. Uchida, arXiv:2012.00751 [hep-th] [Search inSPIRE].
[17] H. Abe, T. Kobayashi, H. Ohki, A. Oikawa, and K. Sumita, Nucl. Phys. B **870**, 30 (2013) [arXiv:1211.4317 [hep-ph]] [Search inSPIRE].
[18] H. Abe, T. Kobayashi, K. Sumita, and Y. Tatsuta, Phys. Rev. D **90**, 105006 (2014) [arXiv:1405.5012 [hep-ph]] [Search inSPIRE].
[19] Y. Fujimoto, T. Kobayashi, K. Nishiwaki, M. Sakamoto, and Y. Tatsuta, Phys. Rev. D **94**, 035031 (2016) [arXiv:1605.00140 [hep-ph]] [Search inSPIRE].
[20] T. Kobayashi, K. Nishiwaki, and Y. Tatsuta, J. High Energy Phys. **04**, 080 (2017) [arXiv:1609.08608 [hep-th]] [Search inSPIRE].
[21] T. Kobayashi, S. Nagamoto, S. Takada, S. Tamba, and T. H. Tatsuishi, Phys. Rev. D **97**, 116002 (2018) [arXiv:1804.06644 [hep-th]] [Search inSPIRE].
[22] T. Kobayashi and S. Tamba, Phys. Rev. D **99**, 046001 (2019) [arXiv:1811.11384 [hep-th]] [Search inSPIRE].
[23] Y. Kariyazono, T. Kobayashi, S. Takada, S. Tamba, and H. Uchida, Phys. Rev. D **100**, 045014 (2019) [arXiv:1904.07546 [hep-th]] [Search inSPIRE].
[24] H. Ohki, S. Uemura, and R. Watanabe, Phys. Rev. D **102**, 085008 (2020) [arXiv:2003.04174 [hep-th]] [Search inSPIRE].
[25] S. Kikuchi, T. Kobayashi, S. Takada, T. H. Tatsuishi, and H. Uchida, Phys. Rev. D **102**, 105010 (2020) [arXiv:2005.12642 [hep-th]] [Search inSPIRE].
[26] S. Kikuchi, T. Kobayashi, H. Otsuka, S. Takada, and H. Uchida, J. High Energy Phys. **11**, 101 (2020) [arXiv:2007.06188 [hep-th]] [Search inSPIRE].
[27] S. Kikuchi, T. Kobayashi, and H. Uchida, arXiv:2101.00826 [hep-th] [Search inSPIRE].
[28] Y. Almumin, M. C. Chen, V. Knapp-Perez, S. Ramos-Sanchez, M. Ratz, and S. Shukla, arXiv:2102.11286 [hep-th] [Search inSPIRE].
[29] R. Blumenhagen, M. Cvetic, and T. Weigand, Nucl. Phys. B **771**, 113 (2007) [arXiv:hep-th/0609191] [Search inSPIRE].
[30] L. E. Ibanezand A. M. Uranga, J. High Energy Phys. **03**, 052 (2007) [arXiv:hep-th/0609213] [Search inSPIRE].
[31] L. E. Ibanez, A. N. Schellekens, and A. M. Uranga, J. High Energy Phys. **06**, 011 (2007) [arXiv:0704.1079 [hep-th]] [Search inSPIRE].
[32] S. Antusch, L. E. Ibanez, and T. Macri, J. High Energy Phys. **09**, 087 (2007) [arXiv:0706.2132 [hep-ph]] [Search inSPIRE].
[33] M. Cvetic, R. Richter, and T. Weigand, Phys. Rev. D **76**, 086002 (2007) [arXiv:hep-th/0703028] [Search inSPIRE].
[34] T. Kobayashi, Y. Tatsuta, and S. Uemura, Phys. Rev. D **93**, 065029 (2016) [arXiv:1511.09256 [hep-ph]] [Search inSPIRE].
[35] K. Ishiguro, T. Kobayashi, and H. Otsuka, J. High Energy Phys. **03**, 161 (2021) [arXiv:2011.09154 [hep-ph]] [Search inSPIRE].
[36] F. Feruglio, arXiv:1706.08749 [hep-ph] [Search inSPIRE].

[37] T. Kobayashi, K. Tanaka, and T. H. Tatsuishi, Phys. Rev. D **98**, 016004 (2018) [arXiv:1803.10391 [hep-ph]] [Search inSPIRE].
[38] J. T. Penedoand S. T. Petcov, Nucl. Phys. B **939**, 292 (2019) [arXiv:1806.11040 [hep-ph]] [Search in SPIRE].
[39] J. C. Criado and F. Feruglio, SciPost Phys. **5**, 042 (2018) [arXiv:1807.01125 [hep-ph]] [Search inSP IRE].
[40] T. Kobayashi, N. Omoto, Y. Shimizu, K. Takagi, M. Tanimoto, and T. H. Tatsuishi, J. High Energy Phys. **1811**, 196 (2018) [arXiv:1808.03012 [hep-ph]] [Search inSPIRE].
[41] P. P. Novichkov, J. T. Penedo, S. T. Petcov, and A. V. Titov, J. High Energy Phys. **1904**, 005 (2019) [arXiv:1811.04933 [hep-ph]] [Search inSPIRE].
[42] P. P. Novichkov, J. T. Penedo, S. T. Petcov, and A. V. Titov, J. High Energy Phys. **1904**, 174 (2019) [arXiv:1812.02158 [hep-ph]] [Search inSPIRE].
[43] F. J. deAnda, S. F. King, and E. Perdomo, arXiv:1812.05620 [hep-ph] [Search inSPIRE].
[44] H. Okada and M. Tanimoto, Phys. Lett. B **791**, 54 (2019) [arXiv:1812.09677 [hep-ph]] [Search in SPIRE].
[45] T. Kobayashi, Y. Shimizu, K. Takagi, M. Tanimoto, T. H. Tatsuishi, and H. Uchida, Phys. Lett. B **794**, 114 (2019) [arXiv:1812.11072 [hep-ph]] [Search inSPIRE].
[46] P. P. Novichkov, S. T. Petcov, and M. Tanimoto, Phys. Lett. B **793**, 247 (2019) [arXiv:1812.11289 [hep-ph]] [Search inSPIRE].
[47] T. Kobayashi, H. Otsuka, and H. Uchida, J. High Energy Phys. **08**, 046 (2019) [arXiv:1904.02867 [hep-th]] [Search inSPIRE].
[48] T. Kobayashi, H. Otsuka, and H. Uchida, J. High Energy Phys. **03**, 042 (2020) [arXiv:1911.01930 [hep-ph]] [Search inSPIRE].
[49] R. C. Gunning, Lectures on Modular Forms (Princeton University Press, Princeton, NJ, 1962).
[50] B. Schoeneberg, Elliptic Modular Functions (Springer, Berlin, 1974).
[51] N. Koblitz, Introduction to Elliptic Curves and Modular Forms (Springer, Berlin, 1984).
[52] J. H. Bruinier, G. V. D. Geer, G. Harder, and D. Zagier, The 1-2-3 of Modular Forms (Springer, Berlin, 2008).
[53] A. Baur, H. P. Nilles, A. Trautner, and P. K. S. Vaudrevange, Phys. Lett. B **795**, 7 (2019) [arXiv:1901.03251 [hep-th]] [Search inSPIRE].
[54] P. P. Novichkov, J. T. Penedo, S. T. Petcov, and A. V. Titov, J. High Energy Phys. **1907**, 165 (2019) [arXiv:1905.11970 [hep-ph]] [Search inSPIRE].
[55] R. de Adelhart Toorop, F. Feruglio, and C. Hagedorn, Nucl. Phys. B **858**, 437 (2012) [arXiv:1112.1340 [hep-ph]] [Search inSPIRE].
[56] X. G. Liuand G. J. Ding, J. High Energy Phys. **1908**, 134 (2019) [arXiv:1907.01488 [hep-ph]] [Sear ch inSPIRE].
[57] R. Blumenhagen, M. Cvetic, F. Marchesano, and G. Shiu, J. High Energy Phys. **03**, 050 (2005) [arXiv:hep-th/0502095] [Search inSPIRE].