Caio Vaz Pereira de Brito

On the linear stability and causality of transient fluid dynamics

Brasil

March, 2021

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Dissertation presented to the Departamento de Física of Universidade Federal Fluminense in partial fulfillment of the requirements for the degree of Master of Science.

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“There is no lemon so sour you can’t make something resembling lemonade.

(Jack Pearson on ‘This Is Us’)

Resumo

Neste trabalho, derivam-se as teorias relativísticas de Navier-Stokes e Israel-Stewart a partir da segunda lei da termodinâmica, considerando diferentes formulações para a quadricorrente de entropia. Para verificar o regime de aplicabilidade destas formulações na descrição de fluidos relativísticos, faz-se uma análise linear de estabilidade de ambas, na qual o sistema é perturbado ao redor de um estado global de equilíbrio. Ao passo que a teoria relativística de Navier-Stokes é linearmente acausal e instável, a causalidade e estabilidade lineares da teoria de Israel-Stewart são garantidas caso os coeficientes de transporte satisfaçam a um conjunto de condições. Em seguida, verifica-se novamente a causalidade e estabilidade desta teoria, porém considerando também dissipação por difusão de carga líquida, antes desprezada. Neste caso, há a ocorrência de termos de segunda ordem que acoplam as correntes dissipativas, os quais denominam-se *termos de acoplamento*. Analisa-se, então, como a presença destes termos afeta a estabilidade e causalidade lineares da teoria, e que novas condições estes implicam. Por último, analisa-se a estabilidade linear de uma derivação microscópica de terceira ordem para a viscosidade de cisalhamento, onde verifica-se ser sempre linearmente instável e acausal. Neste contexto, uma nova formulação é introduzida ao inserir novos coeficientes de transporte que possibilitam que a propriedade de estabilidade linear seja satisfeita. As condições que os novos coeficientes de transporte devem obedecer são então cuidadosamente derivadas.

Palavras-chave: Hidrodinâmica relativística dissipativa, análise linear de estabilidade, teoria de Israel-Stewart, hidrodinâmica de terceira ordem.

Abstract

In this work, we derive the relativistic Navier-Stokes theory and the Israel-Stewart theory from the second law of thermodynamics by considering different formulations for the entropy 4-current. In order to verify the applicability of these formulations in the description of relativistic fluids, we perform a linear stability analysis on both, in which the system is perturbed around a global equilibrium state. The relativistic Navier-Stokes theory is found to be linearly acausal and unstable, while the linear causality and stability of the Israel-Stewart theory are guaranteed if the transport coefficients satisfy a set of constraints. In the following, the linear causality and stability of Israel-Stewart theory is analyzed once again, only now considering also dissipation due to net-charge diffusion, neglected in the first analysis. In this case, there is the occurrence of second-order terms that couple the dissipative currents, which we called *coupling terms*. Then, we analyze how the presence of these terms affect the linear causality and stability of the theory and which conditions it further implies. Last, the linear stability of a third-order microscopic derivation for the shear-stress tensor is analyzed, and it is shown to be always linearly unstable and acausal. In this context, a novel formulation is presented as we introduce new transport coefficients that enable the fulfilling of linear stability. The conditions these coefficients must satisfy are then carefully derived.

Keywords: Relativistic dissipative fluid dynamics, linear stability analysis, Israel-Stewart theory, third-order fluid dynamics.

Contents

Introduction

Quantum chromodynamics (QCD) is the theory that describes the strong interactions, in terms of fundamental particles called *quarks* and *gluons*. The quarks have two distinct degrees of freedom intrinsic to QCD, namely flavor and color charge. So far, six different flavours have been observed: up, down, charm, strange, top and bottom, each of them containing specific mass and electric charge. The color charge is the QCD analogue of the electric charge [1] with the difference that three different types of color charge exist: red, green and blue. These three colors combined form a color neutral state. A color neutral particle formed by the combination of quarks is called a hadron, which can be divided in baryons, which are composed by three quarks, and mesons, which are composed by a quark and its corresponding anti-quark. Particles with a color charge interact with each other by the exchange of gluons, the force carriers of QCD. Unlike in quantum electrodynamics (QED), where the force carriers (photons) have zero electric charge, the gluons have color degrees of freedom. Therefore, this means that not only the quarks interact with other quarks, but also gluons can interact with other gluons. The interaction between the force carriers of the theory is portrayed by the presence of non-linear terms in the lagrangean of QCD (which are absent in QED), making any calculations from first principles an extremely complicated task.

Apart from the aforementioned remarks, QCD still displays two particular properties that have no analogue in QED. At low energies, quarks are strongly interacting, and this interaction becomes weaker as the energy is increased, a property called *asymptotic freedom* [2, 3, 4]. Another important property to be mentioned is *color confinement* [5], that prohibits the existence of free color charged particles, which have not been observed in nature to this day. In particular, if we try to split interacting quarks (whether in a baryon or in a meson), at some point, the amount of energy used to break this bound state is sufficient to, e.g., create other pairs of quark and anti-quark from vacuum (with opposite colors, e.g., red and anti-red), which will combine to form hadrons.

The property of asymptotic freedom further implies that at extremely high temperatures quarks and gluons are weakly interacting and cannot form bound states, thus leading to a *deconfined phase* of QCD matter. In this scenario, nuclear matter exists as a soup of elementary particles, the quark-gluon plasma (QGP) [6], which is believed to have occurred in nature in the early stages of our universe, right after the Big Bang, and also to compose the interior of compact stars. Due to confinement, at lower temperatures nuclear matter must exist as a collection of baryons and mesons and, thus, the existence of the QGP suggests the occurrence of a phase transition: from a phase in which the quarks are confined to another where they are deconfined. The properties of such a phase transition and the QGP itself are not well understood and correspond to a topic of research in high energy nuclear physics. These properties cannot be completely studied from first principles at this point and therefore experimental guidance is of the utmost importance.

In the past decades, ultra-relativistic heavy-ion collision have been widely performed in the experiments launched at the Relativistic Heavy Ion Collider (RHIC), in Brookhaven National Lab (Upton, USA), and at the Large Hadron Collider (LHC), at European Organization for Nuclear Research (CERN), at Geneva. Colliding heavy nuclei, such as gold and lead, at the highest energy densities achieved in laboratory, provide the most reliable approach to emulate the physical conditions of the early universe creating the theoretically predicted QGP in a controlled environment. This enables the possibility of analyzing new phases of nuclear matter, searching for phase transitions and understanding the phase diagram of QCD.

However, studying the hot and dense QCD matter produced in ultra-relativistic heavy-ion collisions is by no means a trivial task. This happens because the QGP is only formed in these collisions for a very small time and cannot be measured directly. The only accessible data are the momenta of the hadrons formed at the late stages of the collisions. In order to extract the properties of the QGP, one must model the whole collision, from the formation and evolution of the QGP to the final production of hadrons. The current understanding of heavy-ion collisions breaks it in the following different stages: the initial stage, in which the interaction between the Lorentz-contracted nuclei are modeled using either Glauber model [7, 8], IP-Glasma [9] or Color Glass Condensate (CGC) [10]; the pre-equilibrium stage, in which takes place the formation and thermalization of the QGP; the hydrodynamic stage, which describes the expansion of the QGP as a relativistic fluid; last, the freeze-out phase, where the non-equilibrium dynamics of hadrons is described by kinetic theory, in particular the relativistic Boltzmann equation.

The application of relativistic dissipative fluid dynamics to describe the evolution of the hot and dense QCD matter produced in heavy-ion collisions has a long history [11, 12, 13, 14, 15, 16, 17, 18]. In the early days, models using ideal fluid dynamics were employed to describe the experimental data measured at RHIC with relative success [19, 20]. However, an optimal agreement is only found when computing hydrodynamic simulations that use small but finite values of shear viscosity over entropy density ratio, η/s [21]. As a matter of fact, the values of viscosity extracted from data suggest that the QGP is the fluid with the lowest kinetic viscosity observed in nature. In particular, calculations using the AdS/CFT correspondence in quantum field theory lead to a conjecture of a lower bound to the shear viscosity coefficient over entropy density for a strongly coupled relativistic fluids, $\eta/s \geq 1/4\pi$, known as the KSS bound [22, 23, 24]. The proximity between the estimations of the shear viscosity to entropy density ratio of the QGP and the KSS bound further corroborates the statement that the QGP is an almost perfect fluid.

In this work, our goal is not to study heavy-ion collisions themselves. In fact, we are interested in exploring fluid dynamics as a theoretical framework. Throughout this dissertation, the essential ideas behind several formulations for relativistic dissipative fluid dynamics are presented. The most intuitive approach to obtain a relativistic fluid-dynamical formulation is to extend the non-relativistic Navier-Stokes theory, which has been successfully used to describe a wide range of fluids. However, relativistic generalizations of Navier-Stokes theory, derived by Eckart [25] and later by Landau and Lifshitz, independently, [26], are known to be ill-defined, containing intrinsic instabilities already in the linear regime when perturbed around an arbitrary global equilibrium state [27, 28, 29, 30]. In Refs. [31, 29, 30], such linear instabilities were shown to be related to the acausal nature of these theories, which allows perturbations to propagate with an infinite speed. These fundamental problems prohibit the application of traditional Navier-Stokes theory to describe any practical fluid-dynamical problem, may it be in the description of neutron star mergers or in the description of the QGP produced in heavy ion collisions. This problem will be addressed in Chapter 2.

The acausal nature of the relativistic Navier-Stokes theory is a severe problem for a relativistic theory, which further motivated the development of alternative formulations that fulfill such fundamental property. In this context, *linearly* stable and causal theories of relativistic fluid dynamics were later derived by Israel and Stewart, following the procedure initially developed by Grad [32] for non-relativistic systems. Israel and Stewart performed this task in two distinct ways: the first being a phenomenological derivation, based on the second law of thermodynamics [33], and the second being a microscopic derivation starting from the relativistic Boltzmann equation [34]. Similar theories have been widely developed in the past decades [35, 36, 37, 38, 39, 40, 41, 42, 43, 44, 45], but all carry the same fundamental aspects: in contrast to Navier-Stokes theory, such linearly causal theories of fluid dynamics include in their description the transient dynamics of the non-conserved dissipative currents. For this reason, they were initially named by Israel and Stewart as transient fluid dynamics (nowadays, they are often referred to as *second-order theories*). Here, we also note that novel causal extensions of *first-order theories* have been recently presented. In this case, the fluid-dynamical equations can be rendered causal by the inclusion of first-order time-like gradients (and not only space-like gradients, as traditionally done) [46, 47] in the constitutive relations satisfied by the dissipative currents. The causality of such novel formulations of fluid dynamics is not guaranteed and was shown to depend on the matching conditions that define the local equilibrium state [46].

At this point, it is important to remark that the theory formulated by Israel and Stewart is not guaranteed to be linearly causal and stable. As was first shown by Hiscock and Lindblom and, later, by Olson, such transient theories of fluid dynamics are only linearly causal and stable if their transport coefficients satisfy certain conditions [31, 48]. Such conclusions were obtained by analyzing the properties of the theory in the linear regime and by imposing that the perturbations around a global equilibrium state are stable and propagate subluminally. More recent analyses were developed in Ref. [29], including only the effects of bulk viscosity, and, later, in Ref. [30], which included the effects of both shear and bulk viscosity. In both of these papers constraints for the shear relaxation time were explicitly derived (these results will be reproduced in Chapter 2 of this dissertation in the absence of bulk viscosity). Nowadays, causality analyses have been performed also in the non-linear regime [49, 50] (in this case, including the effects of shear and bulk viscosity), where more general inequalities required to ensure the causal propagation of the theory were derived. In the latter case, the inequalities constrain not only the transport coefficients, but also the values of the dissipative currents (in the linear regime, the inequalities derived in Ref. [50] reduce to those derived in Refs. [29, 30]). Such constraints are relevant, e.g. for fluid-dynamical applications in heavy ion collisions, since the transport coefficients of QCD matter are not precisely known (often, they are completely unknown) and constraints on transport coefficients (and the values of the dissipative currents) can be extremely useful.

The main goal of this dissertation is to revisit the results described above and go beyond them by deriving constraints that are essential to the linear stability and causality of relativistic dissipative fluid dynamics. This will be done in two separate stages: the first being in the presence of net-charge and the second considering higher-order extensions of Israel-Stewart theory. We shall briefly elaborate on these topics below.

Recently, several programs to experimentally study QCD matter at finite net-baryon density have been put in motion at RHIC and at the Nuclotron-based Ion Collider fAcility (NICA), in the Joint Institute for Nuclear Research (Dubna, Russia), and, will be starting soon at the Facility for Antiproton and Ion Research (FAIR), in GSI Helmholtzzentrum für Schwerionenforschung (Darmstadt, Germany). Nevertheless, the more recent investigations on the stability and causality of fluid-dynamical descriptions [29, 30, 49, 50], have not yet considered the *complete* set of the Israel-Stewart equations, usually neglecting any dissipation by net-baryon diffusion and, also, possible diffusion-viscous coupling terms [1]. In this dissertation, we actually perform a linear stability analysis around global equilibrium of Israel-Stewart theory, including the effects of the shear-stress tensor and net-baryon diffusion 4-current (all effects of bulk viscous pressure are neglected). We find all the relevant modes of this theory and derive the conditions that these modes must satisfy in order to be stable and subluminal. With this result, we obtain new conditions that the shear and diffusion relaxation times must satisfy so that Israel-Stewart theory remains linearly causal and stable. We further find constraints for the transport coefficients that couple the shear-stress tensor and the net-baryon diffusion current (diffusion-viscous coupling).

[1] The work by Olson [48] considered the complete Israel-Stewart equations, including all sources of fluctuations. Nevertheless, they did not consider the limit of vanishing background net charge and did not explicitly evaluate the dispersion relations for the hydrodynamic modes.

In other words, we show that the inclusion of diffusion-viscous coupling in the equations of motion drives the theory unstable, if these transport coefficients do not satisfy certain bounds. Such novel constraints may be useful when such transport coefficients are included in the current fluid-dynamical simulations of the quark-gluon plasma.

In the last decade, the Israel-Stewart theory has been widely used to describe the dynamics of the QGP generated in heavy-ion collisions [51, 52, 53]. However, one may expect a better agreement with the experimental results with the inclusion of terms of higher order in gradients in the expressions for the dissipative currents, e.g., using third-order equations. In this sense, a derivation of third-order dissipative fluid dynamics from the second law of thermodynamics was developed in Ref. [54]. Furthermore, in Ref. [55], it was shown that the inclusion of higher-order terms in the dissipative currents may improve the agreement with numerical solutions of Boltzmann equation in comparison to the solutions given by the Israel-Stewart theory. Another relativistic third-order fluid-dynamical approach following kinetic theory calculations was performed in Ref. [56] in the relaxation time approximation using the Chapman-Enskog method for the Boltzmann equation. Once again, in order to verify if this is a suitable framework to describe relativistic fluids, we resort to a linear stability analysis. However, the third-order theory such as proposed in Ref. [56] leads to a parabolic equation of motion [57, 58] – see Appendix A – for the shear-stress tensor, being linearly acausal and unstable. In this scenario, we introduce a hyperbolic third-order fluid-dynamical formulation and study the conditions that must be satisfied in order to guarantee its linear stability. This will be carefully discussed in Chapter 4.

This dissertation is organized as follows: in Chapter 1 we discuss the foundations of relativistic fluid dynamics. First, we discuss the simplest case of ideal fluid dynamics, in which there are no dissipative currents. In the following, we derive the relativistic Navier-Stokes and Israel-Stewart theories from the second law of thermodynamics. Next, in Chapter 2, we reproduce a linear stability and causality analysis of both formulations considering only dissipation via shear-stress. We show the relativistic Navier-Stokes theory is indeed linearly acausal and unstable and obtain the constraints that ensure the linear causality and stability of Israel-Stewart theory. This is performed by decomposing the transverse and longitudinal degrees of freedom of both theories, in a procedure first presented in Ref. [59]. This process is carefully presented and used in every stability analysis developed in this dissertation. Furthermore, in Chapter 3 we analyze the linear stability and causality of Israel-Stewart considering dissipation not only via shear-stress, but also including net-charge diffusion effects. In particular, we explore how the inclusion of the diffusion-viscous couplings – transport coefficients, which are of second-order contributions, that couple the dissipative currents in the Israel-Stewart equations of motion – affects the linear stability and causality conditions the theory must fulfill. Moreover, in Chapter 4, we discuss the linear stability of the third-order fluid-dynamical formulation for the shear-stress tensor proposed in Ref. [56]. In this context, the parabolicity of this formulation leads to linear instabilities. In the following, we presented a novel third-order equation for the shear-stress tensor by introducing new transport coefficients that are essential to ensure the hyperbolicity of the new theory. We then finally analyze the linear stability of this novel formulation, deriving the constraints that must be satisfied in order to obtain a linearly stable theory. Last, all conclusions are summarized in Chapter 5, where we further discuss future perspectives regarding the results presented here.

Note that throughout this work we employ natural units, $c = \hbar = k_B = 1$, and adopt the mostly-minus convention for the Minkowski metric tensor, $g^{\mu\nu} = \text{diag}(+, -, -, -)$.

1 Relativistic Fluid Dynamics

Fluid dynamics is an effective theory that describes the small frequency and long wavelength dynamics of a many-body system. In order to be able to apply a fluid-dynamical description, there must be a separation between the macroscopic and microscopic scales of the system. In this limit, macroscopic variables, such as densities of conserved quantities, are expected to be sufficient to describe the dynamics of the corresponding system. In this chapter, we explore the fundamental concepts to construct relativistic ideal and dissipative fluid dynamics.

1.1 Thermodynamics

In this section, we briefly revisit some thermodynamic concepts that will be used later in further derivations. The first law of thermodynamics states that the energy, E, of an arbitrary system is always conserved, which is translated by the empirical expression

$$dE = TdS - PdV + \mu dN, , \tag{1.1}$$

where T is the temperature, S is the entropy, P is the thermodynamic pressure, V is the volume, μ is the chemical potential and N is the number of particles. This leads to the following Maxwell identities

$$\begin{aligned} \left(\frac{\partial E}{\partial S}\right)_{V,N} &= T, \\ \left(\frac{\partial E}{\partial V}\right)_{S,N} &= -P, \\ \left(\frac{\partial E}{\partial N}\right)_{S,V} &= \mu. \end{aligned} \tag{1.2}$$

Furthermore, the energy of a system is an *extensive quantity*, being directly proportional to the macroscopic variables of the system. Therefore, if we increase (decrease) its entropy, volume and number of particles by a certain amount, its internal energy will increase (decrease) proportionally. This statement can be mathematically expressed as

$$E(\lambda S, \lambda V, \lambda N) = \lambda E(S, V, N). \tag{1.3}$$

Then, using the chain rule, one can demonstrate that

$$\lambda E = S\left(\frac{\partial E}{\partial(\lambda S)}\right)_{\lambda V,\lambda N} + V\left(\frac{\partial E}{\partial(\lambda V)}\right)_{\lambda S,\lambda N} + N\left(\frac{\partial E}{\partial(\lambda N)}\right)_{\lambda S,\lambda V}. \tag{1.4}$$

If we take $\lambda = 1$ and use the partial derivatives of the energy, displayed in Eq. (1.2), we obtain the Euler equation

$$E = TS - PV + \mu N, \tag{1.5}$$

which combined with the first law of thermodynamics leads to the Gibbs-Duhem relation,

$$SdT - VdP + Nd\mu = 0. \tag{1.6}$$

Now, it is convenient to express all the relations derived above in terms of densities of the thermodynamic variables. First, it is possible to express the Euler equation and Gibbs-Duhem relation in terms of densities in the following way

$$\varepsilon + P = Ts + \mu n, \tag{1.7}$$

$$sdT - dP + nd\mu = 0, \tag{1.8}$$

where we defined the energy density, the entropy density and the particle density respectively as $\varepsilon \equiv E/V$, $s \equiv S/V$ and $n \equiv N/V$. Combining these two equations, we are able to express the first law of thermodynamics in terms of densities

$$d\varepsilon = Tds + \mu dn. \tag{1.9}$$

The relations derived in this section will be further used in the derivation of the fluid-dynamical equations of motion.

1.2 Ideal Relativistic Fluid Dynamics

The first case to be studied in this dissertation is ideal fluid dynamics, in which the fluid elements are assumed to be always in thermodynamic equilibrium. In this limit, we shall demonstrate that the system can be completely described just in terms of conservation laws and an equation of state. The conserved quantities required to describe the fluids considered in this work are the net-charge, energy and momentum. Naturally, the conservation of such quantities can be simply expressed in the form of continuity equations,

$$\partial_\mu N^\mu = 0, \qquad \partial_\mu T^{\mu\nu} = 0, \tag{1.10}$$

where N^μ is the net-charge 4-current and $T^{\mu\nu}$ is the energy-momentum tensor.

With the assumption of local thermodynamic equilibrium, it is possible to determine the explicit form of N^μ and $T^{\mu\nu}$. In practice, this is done by performing a Lorentz-boost to their *local rest frame* (LRF), in which the fluid elements are at rest. In this case, there is no energy flux, and thus $T^{i0}_{\rm LRF} = 0$. Furthermore, the momentum density is zero, and therefore the components $T^{0i}_{\rm LRF}$ also vanish. Finally, due to the assumption of thermodynamic equilibrium, the force per surface element between adjacent fluid elements is isotropic and equal to the thermodynamic pressure P, therefore $T^{ij}_{\mathrm{LRF}} = \delta^{ij} P$. Hence, the energy-momentum tensor has the following form in this frame

$$T^{\mu\nu}_{\mathrm{LRF}} = \begin{pmatrix} \epsilon & 0 & 0 & 0 \\ 0 & P & 0 & 0 \\ 0 & 0 & P & 0 \\ 0 & 0 & 0 & P \end{pmatrix}. \tag{1.11}$$

The thermodynamic pressure is determined by the net-charge density and energy density via an equation of state, $P = P(n, \varepsilon)$.

Furthermore, in the local rest frame there is no net-charge flux, since the fluid is at rest, and thus the spatial components N^{i}_{LRF} vanish. Therefore, the net-charge 4-current in the local rest frame can be written simply as

$$N^{\mu}_{\mathrm{LRF}} = \begin{pmatrix} n \\ 0 \\ 0 \\ 0 \end{pmatrix}, \tag{1.12}$$

where here n stands for the net-charge density. Finally, the entropy 4-current in the local rest frame can be written analogously as

$$S^{\mu}_{\mathrm{LRF}} = \begin{pmatrix} s \\ 0 \\ 0 \\ 0 \end{pmatrix}, \tag{1.13}$$

So far, the energy-momentum tensor, the net-charge 4-current and the entropy 4-current have been obtained in the fluid local rest frame. In order to obtain the general form of the conserved currents, a Lorentz-boost is performed with the 4-velocity $u^{\mu} = \gamma(1, \mathbf{V})$, with $\mathbf{V}$ being the fluid velocity, and $\gamma \equiv 1/\sqrt{1 - V^2}$ being the Lorentz factor. We remark that the 4-velocity is a normalized 4-vector, i.e., $u_{\mu} u^{\mu} = 1$. Then, the following expressions are obtained

$$T^{\mu\nu} = T^{\mu\nu}_{\mathrm{eq}} = \varepsilon u^{\mu} u^{\nu} - \Delta^{\mu\nu} P, \tag{1.14}$$

$$N^{\mu} = N^{\mu}_{\mathrm{eq}} = n u^{\mu}, \tag{1.15}$$

$$S^{\mu} = S^{\mu}_{\mathrm{eq}} = s u^{\mu}, \tag{1.16}$$

where $\Delta^{\mu\nu} \equiv g^{\mu\nu} - u^{\mu} u^{\nu}$ is the operator that projects onto the 3-space orthogonal to the 4-velocity u^{μ}. The expressions obtained above are associated to a fluid in thermodynamic equilibrium and thus the underscript "eq".

Since the thermodynamic pressure is given by an equation of state and the 4-velocity is normalized, N^{μ} and $T^{\mu\nu}$ contain a total of five independent degrees of freedom: n, ε, and

V. Therefore, the conservation of net-charge, energy and momentum, which constitute five equations, are sufficient to completely describe the dynamics of an ideal fluid. We now explicitly derive these equations of motion.

First, it is convenient to decompose the conservation of energy and momentum in two components, one parallel to the 4-velocity u^μ and another orthogonal to it. Thus, from Eqs. (1.10), the conservation of net-charge number, energy and momentum in an ideal fluid can be decomposed in the following way

$$u_\nu \partial_\mu T^{\mu\nu}_{\text{eq}} = 0, \tag{1.17}$$
$$\Delta^\lambda_\nu \partial_\mu T^{\mu\nu}_{\text{eq}} = 0, \tag{1.18}$$
$$\partial_\mu N^\mu_{\text{eq}} = 0, \tag{1.19}$$

which can be straightforwardly proven to respectively lead to the following equations of motion for the hydrodynamic variables

$$\dot{\varepsilon} + (\varepsilon + P)\theta = 0, \tag{1.20}$$
$$(\varepsilon + P)\dot{u}^\lambda - \nabla^\lambda P = 0, \tag{1.21}$$
$$\dot{n} + n\theta = 0, \tag{1.22}$$

where the dot denotes the comoving derivative, $\dot{A} = dA/d\tau = u^\mu \partial_\mu A$, while $\theta \equiv \partial_\mu u^\mu$ is the expansion rate and $\nabla^\mu \equiv \Delta^{\mu\nu}\partial_\nu$ is the orthogonal projection of the covariant derivative.

Moreover, an equation of motion for the entropy density can be obtained from Eqs. (1.20) and (1.22) using the thermodynamic relations given by Eq. (1.7) and (1.9). Then, one obtains

$$\dot{s} + s\theta = \partial_\mu (s u^\mu) = \partial_\mu S^\mu_{\text{eq}} = 0. \tag{1.23}$$

Clearly, the occurrence of a continuity equation suggests the conservation of entropy, since su^μ is the exact expression for the entropy 4-current of an ideal fluid. The reader may be asking how the addition of dissipative terms affects the conservation of entropy. In order to answer this and further questions, first it is essential to understand how the energy-momentum tensor and the net-charge 4-currents are affected by the presence of dissipative effects. This will be the topic of the next sections.

1.3 Dissipative Relativistic Fluid Dynamics

In the last section, the expressions for the energy-momentum tensor, net-charge and entropy 4-current were derived for an ideal fluid. Their domain of applicability is extremely narrow, due to the assumption of local thermodynamic equilibrium, and they must be revised by including non-equilibrium corrections.

1.3.1 Dissipative currents

In this section, we derive the fluid-dynamical equations taking into account dissipative effects. In this regime, one considers heat transfer between fluid elements, and the system can no longer be assumed to be always in local thermodynamic equilibrium. This implies that the fluid cannot be trivially described in terms of its equilibrium variables.

Nevertheless, we assume that the fluid is in a state that is *close* to a local equilibrium state. In this case, the energy-momentum tensor and the net-charge 4-current are written as

$$T^{\mu\nu} = T^{\mu\nu}_{\text{eq}} + \tau^{\mu\nu} = \varepsilon u^{\mu} u^{\nu} - \Delta^{\mu\nu} P + \tau^{\mu\nu}, \tag{1.24}$$

$$N^{\mu} = N^{\mu}_{\text{eq}} + n^{\mu} = n u^{\mu} + n^{\mu}, \tag{1.25}$$

where $\tau^{\mu\nu}$ and n^{μ} are the dissipative corrections associated to the energy-momentum tensor and the net-charge 4-current, respectively, with the latter being denominated net-charge diffusion 4-current. In order to fulfill angular momentum conservation, $\tau^{\mu\nu}$ is defined as a symmetric tensor, $\tau^{\mu\nu} = \tau^{\nu\mu}$. The next step is to further analyze the explicit form of this term.

1.3.2 Matching conditions

Before proceeding to the derivation of the explicit form of the dissipative currents and their respective contributions to the non-equilibrium equations of motion, it is fundamental to state the physical meaning of the equilibrium variables introduced in Eqs. (1.24) and (1.25). It is essential to redefine the hydrodynamic variables ε, n and u^{μ}, whose definitions were previously supported by the hypothesis of a frame in which fluid elements are in thermodynamic equilibrium.

In order to keep ε and n as the energy density and net-charge density of the fluid in the local rest frame, respectively, it is necessary to impose that

$$\varepsilon \equiv T^{00}_{\text{LRF}} = u_{\mu} u_{\nu} T^{\mu\nu}, \tag{1.26}$$

$$n \equiv N^{0}_{\text{LRF}} = u_{\mu} N^{\mu}, \tag{1.27}$$

while we further impose that $P = P(\varepsilon, n)$, the thermodynamic pressure of the fluid, remains being defined by an equation of state. These definitions have an immediate effect on the dissipative currents, rendering them orthogonal to u^{μ},

$$u_{\mu} n^{\mu} = u_{\mu} u_{\nu} \tau^{\mu\nu} = 0. \tag{1.28}$$

The definition of the fluid 4-velocity itself is more intricate. For dissipative fluids, it is no longer possible to define u^{μ} by the existence of a rest frame in which the fluid is in thermodynamic equilibrium. In particular, unlike in the case of ideal fluids, there is no frame where the energy and net-charge flow simultaneously vanish. Therefore, the 4-velocity must be defined in another way. There are two main definitions widely used in the study of relativistic hydrodynamics, known as the Eckart picutre and the Landau-Lifshitz picture.

In the Eckart picture [25], the 4-velocity is defined by following the net-charge flux,

$$N^\mu \equiv n u^\mu. \tag{1.29}$$

In the Laudau-Lifshitz picture [26], also known as the *energy frame*, the 4-velocity is defined by following the energy flux, being therefore, an eigenvector of the enery-momentum tensor

$$u_\nu T^{\mu\nu} \equiv \varepsilon u^\mu. \tag{1.30}$$

Throughout all the calculations present in this dissertation, we shall only employ the Landau-Lifshitz picture.

1.4 Equations of motion

The inclusion of the dissipative corrections to the energy-momentum tensor and net-charge 4-current, respectively defined as $\tau^{\mu\nu}$ and n^μ, was not explicitly explored and these currents still remain arbitrary. First, it is convenient to separate $\tau^{\mu\nu}$ in terms of a traceless part, defined as $\pi^{\mu\nu}$, and its trace, defined as -3Π. In this case, one obtains

$$\tau^{\mu\nu} = \pi^{\mu\nu} - \Delta^{\mu\nu}\Pi, \tag{1.31}$$

with $\Pi \equiv -\frac{1}{3}\Delta^{\alpha\beta}\tau_{\alpha\beta}$ being the bulk viscous pressure and $\pi^{\mu\nu} \equiv \Delta^{\mu\nu}_{\alpha\beta}\tau^{\alpha\beta}$ being the shear-stress tensor. Here, we further defined

$$\Delta^{\mu\nu}_{\alpha\beta} \equiv \frac{1}{2}\left(\Delta^\mu_\alpha \Delta^\nu_\beta + \Delta^\mu_\beta \Delta^\nu_\alpha\right) - \frac{1}{3}\Delta^{\mu\nu}\Delta_{\alpha\beta} \tag{1.32}$$

as the double traceless symmetric projection operator onto the 3-space orthogonal to the fluid 4-velocity u^μ.

Then, it is possible to rewrite the expression for the energy-momentum tensor, Eq. (1.24) – while the equation for the net-charge 4-current remains unchanged. These equations then read

$$T^{\mu\nu} = \varepsilon u^\mu u^\nu - \Delta^{\mu\nu}(P + \Pi) + \pi^{\mu\nu}, \tag{1.33}$$

$$N^\mu = n u^\mu + n^\mu. \tag{1.34}$$

Here, the bulk viscous pressure Π can immediately be understood as a viscous correction to the thermodynamic pressure P.

Naturally, the conservation of energy, momentum and net-charge number remain valid in the presence of dissipation. The dissipative terms that feature Eqs. (1.33) and (1.34) will just lead to the occurrence of new terms, in addition to the ones that already appeared in the equations of motion for an ideal fluid, Eqs. (1.20), (1.21) and (1.22). Thus, the general equations of motion for a dissipative relativistic fluid then read

$$\dot{\varepsilon} + (\varepsilon + P + \Pi)\theta - u_\nu \partial_\mu \pi^{\mu\nu} = 0, \tag{1.35}$$

$$(\varepsilon + P + \Pi)\dot{u}^\lambda - \nabla^\lambda (P + \Pi) + \Delta^\lambda_\nu \partial_\mu \pi^{\mu\nu} = 0, \tag{1.36}$$

$$\dot{n} + n\theta + \partial_\mu n^\mu = 0. \tag{1.37}$$

One can straightforwardly recover the equations of motion for the ideal case by simply taking the dissipative currents Π, n^μ, and $\pi^{\mu\nu}$ to zero.

Although the equations of motion which are product of the conservation of net-charge number, energy and momentum are now generalized to contemplate the presence of dissipative currents in a fluid, its dynamics still cannot be completely described. The conservation laws associated with an equation of state are sufficient to describe an ideal fluid. However, for dissipative fluids, the conservation of energy and momentum and net-charge provide five independent equations that feature fourteen independent fields, $n, \varepsilon, \Pi, u^\mu, n^\mu$ and $\pi^{\mu\nu}$. Hence, nine additional relations are required to close this set of equations. Such equations are the expressions for the three dissipative currents. In the following sections, we revisit two widely explored approaches to this problem: the relativistic Navier-Stokes theory and the Israel-Stewart theory.

1.5 Navier-Stokes theory

It was shown in Sec. 1.2 that the entropy 4-current of an ideal fluid satisfies a continuity equation, which implies that for an ideal fluid, the entropy itself is always conserved, as displayed in Eq. (1.23). This occurs due to the fact that in an ideal fluid, all fluid elements are always in thermodynamic equilibrium, and the second law of thermodynamics further implies that, in such case, the variation of entropy is zero. However, for a dissipative fluid, this statement is no longer valid. Instead, the occurrence of irreversible thermodynamic processes dictates the non-conservation of entropy. Therefore, for dissipative fluids, the entropy 4-current no longer satisfies a continuity equation.

The first efforts to derive an expression for the non-equilibrium entropy 4-current made addressed independently by Eckart [25] and Landau and Lifshitz [26]. They used the equations of motion from the conservation laws, Eqs. (1.35), (1.36) and (1.37) to obtain the following expression

$$\partial_\mu \left(s u^\mu - \frac{\mu}{T} n^\mu \right) = \frac{1}{T} \pi^{\mu\nu} \sigma_{\mu\nu} - \frac{1}{T} \Pi\theta - n^\mu \nabla_\mu \frac{\mu}{T}, \tag{1.38}$$

with $\sigma_{\mu\nu} \equiv \Delta_{\mu\nu}^{\alpha\beta}\partial_\alpha u_\beta$ being the shear tensor. They further identified the left-hand side of the equation as the divergence of the non-equilibrium entropy 4-current, defined as $S^\mu \equiv su^\mu - \frac{\mu}{T}n^\mu$. Naturally, the right-hand side is identified as the entropy production due to dissipative effects. If the dissipative currents are set to zero, one can easily recover the conservation of entropy in an ideal fluid, given by Eq. (1.23). Furthermore, the second law of thermodynamics states that the variation of entropy must be non-negative, leading to the following inequality

$$\frac{1}{T}\pi^{\mu\nu}\sigma_{\mu\nu} - \frac{1}{T}\Pi\theta - n^\mu\nabla_\mu\frac{\mu}{T} \geq 0, \tag{1.39}$$

Then, the simplest form to assure the entropy production is positive definite is by assuming each term is positive definite separately. Thus, the following *ansatz* is made, leading to the Navier-Stokes equations

$$\Pi \equiv -\zeta\theta, \tag{1.40}$$

$$n^\mu \equiv \kappa_n \nabla^\mu\left(\frac{\mu}{T}\right), \tag{1.41}$$

$$\pi^{\mu\nu} \equiv 2\eta\sigma^{\mu\nu}, \tag{1.42}$$

Here, the bulk viscosity, diffusion coefficient and shear viscosity were introduced as ζ, κ_n and η, respectively, and are positive quantities. These transport coefficients dictate the dissipative properties of the fluid, and must be obtained from a microscopic theory, such as kinetic theory. In this case, the entropy production reads

$$\partial_\mu S^\mu = \frac{1}{2\eta T}\pi_{\mu\nu}\pi^{\mu\nu} + \frac{1}{\zeta T}\Pi^2 - \frac{1}{\kappa_n T}n_\mu n^\mu, \tag{1.43}$$

since $n^\mu n_\mu \leq 0$ and $\pi^{\mu\nu}\pi_{\mu\nu} \geq 0$, and thus ensuring every term is quadratic and, therefore, positive definite, satisfying the second law of thermodynamics.

With the purpose of being the simplest form to ensure the second law of thermodynamics from the Landau-Lifshitz non-equilibrium entropy 4-current, this *ansatz* takes into account only first-order terms in the dissipative currents. Hence, the Navier-Stokes theory is commonly referred to as a *first-order theory*. Although their addition render the solutions rather complicated, other approaches considering higher-order terms were also investigated. In the next section, we explore the Israel-Stewar theory, a *second-order theory*.

Equations of motion

Naturally, considering the dissipative currents given by the Navier-Stokes equations, the Eqs. (1.35), (1.36) and (1.37) can be explicitly written in terms of gradients of the fluid 4-velocity and chemical potential. These equations then read,

$$\dot{\varepsilon} + (\varepsilon + P - \zeta\theta)\theta - 2\eta\sigma_{\mu\nu}\sigma^{\mu\nu} = 0, \tag{1.44}$$

$$(\varepsilon + P - \zeta\theta)\dot{u}^{\lambda} - \nabla^{\lambda}(P - \zeta\theta) + 2\Delta^{\lambda}_{\nu}\partial_{\mu}(\eta\sigma^{\mu\nu}) = 0, \tag{1.45}$$

$$\dot{n} + n\theta + \partial_{\mu}\left(\kappa_n \nabla^{\mu}\frac{\mu}{T}\right) = 0, \tag{1.46}$$

Although these equations are obtained by the conservation of energy and momentum and net-charge number, which are always valid, they do not carry the same success of applicability of its non-relativistic version, a formulation widely employed to the study the dynamics of non-relativistic fluids, since it is an acausal and unstable framework. This will be discussed in the next chapter.

Acausality and instability

The Navier-Stokes equations for the dissipative currents arise from the second law of thermodynamics applied to the non-equilibrium entropy 4-current considering up to first-order terms in the dissipative currents, see Eq. (1.38). Although this formulation is suitable to mathematically satisfy the condition of a non-negative entropy production, it contains several non-physical features that forbid its application to describe relativistic fluids.

As it can be seen from the Navier-Stokes equations, Eqs. (1.40), (1.41) and (1.42), arbitrarily small gradients of the fluid 4-velocity and chemical potential instantaneously yields dissipative currents. This immediate response from the system manifests as the propagation of signals with infinite speed, since even the smallest gradients instantaneously lead to the occurrence of dissipative currents, which was shown to be directly associated with instabilities of the system [27, 28, 29, 30, 31]. This will be further analysed in detail in the next chapter. Therefore, an alternative formulation is still required to properly understand the dynamics of relativistic fluids without the occurrence of acausal and/or unstable behaviors.

1.5.1 A causal framework

In order to solve the problem of the superluminal velocity of signals propagation in a fluid, Cattaneo showed in his work for the non-relativistic heat transfer equation [60] that such parabolic equation must be converted into a hyperbolic equation by including a relaxation term in order to satisfy causality. In the context of the relativistic Navier-Stokes equations, this further implies that the dissipative currents must satisfy hyperbolic equations and not parabolic equations [57, 58] – since they are essentially acausal even in the non-relativistic regime –, which is performed by introducing relaxation time scales as new transport coefficients. In this case, the dissipative currents no longer satisfy constitutive relations, but instead fulfill equations of motion. Furthermore, applying the Maxwell-Cattaneo's approach [60, 61] to the equations for the dissipative currents, we then obtain

$$\tau_\Pi \dot{\Pi} + \Pi = -\zeta\theta, \tag{1.47}$$

$$\tau_n \dot{n}^\mu + n^\mu = \kappa_n \nabla^\mu \left(\frac{\mu}{T}\right), \tag{1.48}$$

$$\tau_\pi \dot{\pi}^{\mu\nu} + \pi^{\mu\nu} = 2\eta\sigma^{\mu\nu}, \tag{1.49}$$

where τ_Π, τ_n and τ_π are defined respectively as the bulk, diffusion and shear relaxation times. The inclusion of relaxation time scales is an essential feature to render the theory causal [62], leading to hyperbolic equations in the linear regime.

Hence, with the inclusion of the relaxation times, gradients of the fluid 4-velocity and chemical potential do not yield instantly to the occurrence of dissipative currents. Instead now it takes a finite time interval, given by the corresponding relaxation time scale, for each dissipative current to be generated from these gradients. Taking the relaxation time to zero, one straightforwardly recovers Navier-Stokes equations – a regime also known as the Navier-Stokes limit.

However the acausality problem was primarily addressed by the addition of relaxation time scales, their inclusion are essentially *ad hoc*, and does not carry a physical meaning *a priori*, besides fixing the superluminal signal propagation. Wherefore, a derivation based on more solid grounds is still required. With that in view, Israel and Stewart [33, 34] formulated a new relativistic fluid-dynamical theory, which will be further explored in the next section.

1.6 Israel-Stewart theory

The relativistic Navier-Stokes theory is built based on the conservation of net-charge, energy and momentum, and on the second law of thermodynamics, which are known to be always valid. Therefore, it is not clear why a theory formulated from such solid physical principles displays fundamental non-physical properties. Naturally, the physical problems that arise from such theory cannot be product of these properties, and must come from other sources.

The first successful efforts to fix the acausality and instability problems were due to Israel and Stewart's works based on the procedure introduced by Grad for non-relativistic systems [32]. In their works, Israel and Stewart derived fluid dynamics from the second law of thermodynamics [33] and from the relativistic Boltzmann equation [34] using Grad's method of moments generalized for relativistic fluids. Unlike the propositions by Eckart [25] and Landau and Lifshitz [26], in which only up to first-order terms in the dissipative currents were included in the construction of the entropy 4-current, Israel and Stewart further considered the possibility of contributions from second-order terms in the entropy, and therefore this formulation is called a *second-order theory*.

The second-order terms in the dissipative currents included in the analyses of Israel and Stewart play an essential role in enabling causal signal propagation in a theory, and can be further shown to be an essential feature to its stability and therefore cannot be neglected [30]. This section is dedicated to the derivation of Israel-Stewart theory from the second law of thermodynamics, a phenomenological approach.

1.6.1 Out-of-equilibrium entropy

One of the fundamental assumptions of ideal fluid dynamics is that the system must always be in local thermodynamic equilibrium. Naturally, in the presence of dissipation, this assumption no longer holds. Nevertheless, in order to keep using the same variables that are used to describe an ideal fluid (equilibrium variables), we resort to the assumption that a dissipative fluid is in a state that is *close* to local equilibrium. This considerably simplifies the understanding of its dynamics and further simplifies the following calculations.

We then assume the entropy 4-current S^μ depends only on the conserved currents N^μ and $T^{\mu\nu}$, an assumption previously made when deriving the relativistic Navier-Stokes theory. This is analogous to assume that S^μ is completely described by the hydrodynamic variables n, ϵ, Π, u^μ, n^μ and $\pi^{\mu\nu}$, and thus it is a function of the type $S^\mu = S^\mu(n, \varepsilon, \Pi, u^\mu, n^\mu, \pi^{\mu\nu})$. The entropy 4-current is then expanded around an equilibrium state in powers of the dissipative currents. Then, Israel and Stewart's approach was based on truncating this series in second order in the dissipative currents. In this case, the entropy 4-current can be written in the following general form

$$S^\mu = su^\mu - \frac{\mu}{T}n^\mu + Q^\mu + \mathcal{O}(3), \tag{1.50}$$

with Q^μ being the 4-vector that accounts all possible linear combinations of second order in the dissipative currents. Note that this expansion is constructed so the truncation in first order in the dissipative currents leads to the Landau-Lifshitz entropy, see Eq. (1.38), which is the starting point for the relativistic Navier-Stokes theory. Third-order (or higher) terms in the dissipative currents shall be neglected. The explicit form of Q^μ is written as

$$Q^\mu \equiv -\frac{1}{2}u^\mu \left(\delta_0\Pi^2 - \delta_1 n_\alpha n^\alpha + \delta_2 \pi_{\lambda\rho}\pi^{\lambda\rho}\right) - \gamma_0 \Pi n^\mu - \gamma_1 \pi^\mu_\nu n^\nu, \tag{1.51}$$

where δ_1, δ_2, δ_3, γ_0 e γ_1 are scalar functions of the chemical potential and temperature. These functions are transport coefficients and determine the dissipative properties of the fluid, and must be calculated within a microscopic framework, such as kinetic theory.

Naturally, the entropy 4-current introduced by Israel and Stewart leads to a different entropy production than the one derived in the context of the relativistic Navier-Stokes theory, with several additional terms, including second-order contributions. In this case, the second law of thermodynamics applied to the entropy 4-current considering contributions of second order in the dissipative currents, Eq. (1.50), leads to

$$\partial_\mu S^\mu = \frac{1}{T}\pi^{\mu\nu}\sigma_{\mu\nu} - \frac{1}{T}\Pi\theta - n^\mu \nabla_\mu \frac{\mu}{T} + \partial_\mu Q^\mu \geq 0. \tag{1.52}$$

Hence, the entropy production can be explicitly calculated as

$$\begin{aligned}\partial_\mu S^\mu = \frac{1}{T}\pi^{\mu\nu}&\left(\sigma_{\mu\nu} - \frac{1}{2}T\delta_2\Pi\pi_{\mu\nu}\theta - \gamma_1 T\Delta^{\alpha\beta}_{\mu\nu}\nabla_\alpha n_\beta - \frac{1}{2}Tn_\alpha\Delta^{\alpha\beta}_{\mu\nu}\nabla_\beta\gamma_1\right.\\ &\left.-\frac{1}{2}T\dot{\delta}_2\pi_{\mu\nu} - T\delta_2\Delta^{\alpha\beta}_{\mu\nu}\dot{\pi}_{\alpha\beta}\right) + \frac{\Pi}{T}\left(-\theta - \frac{1}{2}T\delta_0\Pi\theta - T\gamma_0\partial_\mu n^\mu\right.\\ &\left.-\frac{1}{2}Tn^\mu\nabla_\mu\gamma_0 - \frac{1}{2}T\dot{\delta}_0\Pi - T\delta_0\dot{\Pi}\right) + n^\mu\left(\nabla_\mu\alpha + \frac{1}{2}\delta_1\theta n_\mu + \delta_1\Delta^\lambda_\mu\dot{n}_\lambda\right.\\ &\left.-\gamma_0\nabla_\mu\Pi - \frac{1}{2}\Pi\nabla_\mu\gamma_0 - \gamma_1\Delta^\lambda_\mu\partial_\nu\pi^\nu_\lambda - \frac{1}{2}\pi^\nu_\mu\nabla_\nu\gamma_1 + \frac{1}{2}\dot{\delta}_1 n_\mu\right).\end{aligned} \tag{1.53}$$

Once again, the simplest form to ensure the entropy production is always positive is by taking each term to be positive by itself. Once again, a convenient *ansatz* to be taken in order to achieve such purpose is imposing that each term is quadratic in the dissipative currents, leading to Eq. (1.43). Then, from Eq. (1.53), the Israel-Stewart equations are obtained as

$$\begin{aligned}\tau_\pi\Delta^{\alpha\beta}_{\mu\nu}\dot{\pi}_{\alpha\beta} + \pi_{\mu\nu} &= 2\eta\sigma_{\mu\nu} - \frac{\tau_\pi}{2}\theta\pi_{\mu\nu} - \frac{1}{2}\pi_{\mu\nu}\eta T\frac{d}{d\tau}\left(\frac{\tau_\pi}{\eta T}\right) - 2\eta T\gamma_1\Delta^{\alpha\beta}_{\mu\nu}\nabla_\beta n_\alpha\\ &+ \eta T\Delta^{\alpha\beta}_{\mu\nu}\nabla_\alpha n_\beta, \end{aligned}\tag{1.54}$$

$$\begin{aligned}\tau_\Pi\dot{\Pi} + \Pi &= -\zeta\theta - \frac{1}{2}\tau_\Pi\Pi\theta - \frac{1}{2}\zeta T\Pi\frac{d}{d\tau}\left(\frac{\tau_\Pi}{\zeta T}\right) - \zeta T\gamma_0\partial_\mu n^\mu\\ &- \frac{1}{2}\zeta Tn^\mu\nabla_\mu\gamma_0, \end{aligned}\tag{1.55}$$

$$\begin{aligned}\tau_n\Delta^\lambda_\mu\dot{n}_\lambda + n_\mu &= \kappa_n\nabla_\mu\alpha + \frac{1}{2}\tau_n n_\mu\theta + \frac{1}{2}n_\mu\kappa_n\frac{d}{d\tau}\left(\frac{\tau_n}{\kappa_n}\right)\\ &- \frac{1}{2}\kappa_n\pi^\nu_\mu\nabla_\nu\gamma_1 + \kappa_n\gamma_1\Delta^\lambda_\mu\partial_\nu\pi^\nu_\lambda - \frac{1}{2}\kappa_n\Pi\nabla_\mu\gamma_0 - \kappa_n\gamma_0\nabla_\mu\Pi, \end{aligned}\tag{1.56}$$

with the relaxation times being defined as

$$\tau_\pi \equiv 2T\delta_2\eta, \quad \tau_\Pi \equiv T\delta_0\zeta, \quad \tau_n \equiv \kappa_n\delta_1. \tag{1.57}$$

Note the natural occurrence of relaxation time scales in the equations for the dissipative currents in the Israel-Stewart theory by considering second-order terms in the entropy 4-current, as it was first observed in the *ad-hoc* formulation following Maxwell-Cattaneo's equation for the dissipative currents in Sec. 1.5.1. Nevertheless, although this was achieved in the previous case by forcing the presence of time scales to enable the causality of the equations, here it was achieved using basic physical principles. Moreover, it is also important to note that the addition of second-order terms in the dissipative currents to the entropy 4-current leads to equations of motion, as it can be seen in the Israel-Stewart equations, and no longer constitutive relations, as it was the case in the relativistic Navier-Stokes theory. While in the relativistic Navier-Stokes theory the dissipative currents are dynamically dependent variables which are defined in terms of the equilibrium variables, in Israel-Stewart theory they are treated as independent variables and thus their dynamics must be accounted separately.

From these equations, it is possible to understand the physical meaning carried by the different transport coefficients. First, η, ζ, κ_n are related to the fluid viscosity. Then, $\tau_\pi, \tau_n, \tau_\kappa$ are the relaxation times, which are essential to preserve the theory's causality. Therefore, taking the relaxation times to zero, one must recover Navier-Stokes theory. Last, γ_0, γ_1, γ_2 are coupling parameters, which couple two different dissipative currents.

2 Linear Stability Analysis

In the previous chapter, a first and second-order fluid-dynamical formulations were explicitly derived. In this context, it was stated several times that the relativistic Navier-Stokes theory is a problematic formulation, with instabilities arising from its acausal behavior. In this sense, the relaxation times were introduced as an essential feature to guarantee subluminal signal propagation in a system, and will be proved fundamental to ensure stability as well. However, we still did not comment on how can one explicitly conclude if a given framework is suitable to describe relativistic fluids.

First, we shall note that relativistic fluids are known to exist in nature, and thus they must be stable systems. In particular, since we are dealing with relativistic formulations of fluid dynamics, causality is a property that must always be satisfied. In this sense, information in a fluid cannot travel with a speed that exceeds the speed of light. Furthermore, causality and stability are extremely related properties, and the violation of causality usually leads to instabilities in the theory and vice-verse, see Refs. [29, 30, 59], as we will also demonstrate in this dissertation. For a stable theory, it is expected for the system to return to its initial (equilibrium) state after perturbations are performed on it. In this case, we expect the occurrence of modes with amplitudes that are damped with time and thus decay towards the initial equilibrium state. On the other hand, in an unstable theory, even the smallest perturbations oscillate with time-increasing amplitudes.

It is then possible to analyze the properties of causality and stability of a fluid-dynamical theory by performing a *linear stability analysis*. In this case, the system is assumed to be initially in an equilibrium state, and then small perturbation in the hydrodynamic variables are performed. We shall assume a relativistic fluid initially in equilibrium, with energy density ε_0, net-baryon number n_0 and fluid 4-velocity u_0^μ, which will be often referred to as background fluid 4-velocity. Then, this system is slightly perturbed around an arbitrary equilibrium state, such as

$$\begin{aligned} \varepsilon = \varepsilon_0 + \delta\varepsilon,\ n = n_0 + \delta n,\ u^\mu = u_0^\mu + \delta u^\mu, \\ \Pi = \delta\Pi,\ n^\mu = \delta n^\mu,\ \pi^{\mu\nu} = \delta\pi^{\mu\nu}. \end{aligned} \tag{2.1}$$

Since the perturbations are taken around the equilibrium state of the system, the dissipative currents are the perturbations themselves. Furthermore, the perturbations can be taken to be as small as we desire. Therefore, we shall assume sufficiently small perturbations, and then quadratic (or higher order) terms in the perturbations can be neglected. This process is referred to as *linearization* and leads to considerably simpler equations. Throughout this chapter, we analyze the two fluid-dynamical formulations presented so far in their respective linear regimes: the relativistic Navier-Stokes and Israel-Stewart theories.

2.1 Navier-Stokes theory

Although its non-relativistic analogue is successfully employed to this day, the relativistic Navier-Stokes theory was shown to lead to intrinsic instabilities when perturbed around an equilibrium state, considering both the Landau-Lifshitz and Eckart pictures [27, 28, 31]. In particular, in Refs. [29, 30], the linearly unstable behavior of the relativistic Navier-Stokes theory was shown to be related to the acausal nature of the formulations by Eckart [25] and Landau and Lifshitz [26]. Hence, these non-physical features lead to the conclusion that this formulation is not a suitable framework to describe relativistic fluids.

In this section, we shall carefully investigate the instabilities that are a product of the acausal nature of the relativistic Navier-Stokes theory in the linear regime. The first step to achieve this goal is to write the linearized equations of motion retaining only up to first order terms in the perturbations. In this case, they read

$$u_0^\mu \partial_\mu \delta\varepsilon + (\varepsilon_0 + P_0)\partial_\mu \delta u^\mu = 0, \tag{2.2}$$

$$u_0^\mu \partial_\mu \delta n + n_0 \partial_\mu \delta u^\mu + \partial_\mu \delta n^\mu = 0, \tag{2.3}$$

$$(\epsilon_0 + P_0) u_0^\mu \partial_\mu \delta u^\lambda - \Delta_0^{\lambda\nu} \partial_\nu (\delta P + \delta\Pi) + \partial_\nu \delta\pi^{\lambda\nu} = 0, \tag{2.4}$$

with the projection operator onto the 3-space orthogonal to the fluid background 4-velocity defined as

$$\Delta_0^{\mu\nu} \equiv g^{\mu\nu} - u_0^\mu u_0^\nu. \tag{2.5}$$

Furthermore, since the 4-velocity of the fluid is normalized, one can straightforwardly demonstrate that, up to first order terms in the perturbations, the background 4-velocity and the fluctuations on the fluid 4-velocity are orthogonal. Moreover, since the dissipative currents are orthogonal to the fluid 4-velocity by construction, see Eq. (1.28), fluctuations of the dissipative currents are also orthogonal to the background 4-velocity up to first order in the perturbations. Thus,

$$u_0^\mu \delta u_\mu = \mathcal{O}(2) \approx 0, \tag{2.6}$$

$$u_0^\mu \delta n_\mu = \mathcal{O}(2) \approx 0, \tag{2.7}$$

$$u_0^\mu \delta \pi_{\mu\nu} = \mathcal{O}(2) \approx 0. \tag{2.8}$$

It is practical to express the hydrodynamic currents in Fourier space. In this dissertation we shall adopt the following convention for the Fourier transform of an arbitrary field

$$\tilde{M}(k^\mu) = \int d^4x \exp(-i x_\mu k^\mu)\, M(x^\mu), \tag{2.9}$$

$$M(x^\mu) = \int \frac{d^4k}{(2\pi)^4} \exp(i x_\mu k^\mu)\, \tilde{M}(k^\mu), \tag{2.10}$$

where we defined $k^\mu = (\omega, \mathbf{k})$, with ω being the frequency and $\mathbf{k}$ being the wave-vector. Furthermore, it is convenient to define the following additional covariant variables Ω and

κ^μ, with the first being the frequency and the latter being the wave 4-vector in the local rest frame of the unperturbed system,

$$\Omega \equiv u_0^\mu k_\mu, \tag{2.11}$$

$$\kappa^\mu \equiv \Delta_0^{\mu\nu} k_\nu. \tag{2.12}$$

We further define the covariant wavenumber in the local rest frame of the unperturbed system, κ, as

$$\kappa^\mu \kappa_\mu = -\kappa^2. \tag{2.13}$$

We remark that in this convention, a positive (negative) imaginary part of the modes leads to the decreasing (increasing) of the perturbations with time, thus leading to linear stability (instability). Therefore, we are particular interested in obtaining the constraints the transport coefficients must satisfy in order to render only modes with positive imaginary parts.

The linearized equations of motion in Fourier space read

$$\Omega\delta\tilde{\epsilon} + (\epsilon_0 + P_0)\kappa_\mu \delta\tilde{u}^\mu = 0, \tag{2.14}$$

$$\Omega\delta\tilde{n} + n_0\kappa_\mu\delta\tilde{u}^\mu + \kappa_\mu\delta\tilde{n}^\mu = 0, \tag{2.15}$$

$$(\epsilon_0 + P_0)\Omega\delta\tilde{u}^\mu - \kappa^\mu(\delta\tilde{P} + \delta\tilde{\Pi}) + \kappa_\nu\delta\tilde{\pi}^{\mu\nu} = 0. \tag{2.16}$$

It is also convenient to express the perturbations into longitudinal degrees of freedom (parallel to κ^μ) and transverse degrees of freedom (orthogonal to κ^μ). This procedure was first developed in Ref. [59].

In order to perform this task, it is possible to extend the tensor decomposition introduced in Appendix B to Fourier space. In this case, we define a projection operator that when applied on other tensors leads to components orthogonal do the wave 4-vector in the local rest of the unperturbed system. For this purpose, we define the following projection operator,

$$\Delta_\kappa^{\mu\nu} \equiv g^{\mu\nu} + \frac{\kappa^\mu\kappa^\nu}{\kappa^2}. \tag{2.17}$$

Therefore, a 4-vector can be decomposed in its transverse and longitudinal components with respect to the wave 4-vector in the local rest frame of the background system as

$$A^\mu = A_\parallel \kappa^\mu + A_\perp^\mu, \tag{2.18}$$

with the longitudinal component being defined as $A_\parallel = -\kappa_\mu A^\mu/\kappa$ while the transverse component is $A_\perp^\mu = \Delta^{\mu\nu}A_\nu$. A similar approach can be performed to decompose a traceless rank two tensor. In this case, it is first essential to introduce the double, symmetric, and traceless projection operator in Fourier space as

$$\Delta_{\alpha\beta}^{\mu\nu} = \frac{1}{2}\left(\Delta_{\alpha,\kappa}^\mu \Delta_{\beta,\kappa}^\nu + \Delta_{\beta,\kappa}^\mu \Delta_{\alpha,\kappa}^\nu\right) - \frac{1}{3}\Delta_\kappa^{\mu\nu}\Delta_{\alpha\beta,\kappa}. \tag{2.19}$$

Wherefore, the decomposition of an arbitrary traceless rank two tensor $A^{\mu\nu}$ can be performed as follows

$$A^{\mu\nu} = A_{\parallel}\frac{\kappa^{\mu}\kappa^{\nu}}{\kappa^2} + \frac{1}{3}A_{\parallel}\Delta_{\kappa}^{\mu\nu} + A_{\perp}^{\mu}\frac{\kappa^{\nu}}{\kappa} + A_{\perp}^{\nu}\frac{\kappa^{\mu}}{\kappa} + A_{\perp}^{\mu\nu}, \tag{2.20}$$

with its respective projections being defined as $A_{\parallel} \equiv \kappa_{\mu}\kappa_{\nu}A^{\mu\nu}/\kappa^2$, $A_{\perp}^{\mu} \equiv -\kappa^{\lambda}\Delta_{\kappa}^{\mu\nu}A_{\lambda\nu}/\kappa$, and $A_{\perp}^{\mu\nu} \equiv \Delta_{\kappa}^{\mu\nu\alpha\beta}A_{\alpha\beta}$. Thereby, the linearized equations for the conservation laws in Fourier space, Eqs. (2.14), (2.15) and (2.16), are split in two different components that decouple and can be solved independently, as it will be shown in the next sections. It is then possible to obtain the dispersion relations associated to the transverse and longitudinal modes.

Transverse modes

We first analyze the transverse projections of the equations of motion. These are obtained by projecting Eq. (2.16) onto the 3-space orthogonal to the background wave 4-vector κ^{μ}, see Eq. (2.18), thus leading to the following relation

$$(\epsilon_0 + P_0)\Omega\delta\tilde{u}_{\perp}^{\mu} - \kappa\delta\tilde{\pi}_{\perp}^{\mu} = 0. \tag{2.21}$$

Now one must provide a constitutive or dynamical equation satisfied by the partially transverse component of the shear-stress tensor. For the linearized Navier-Stokes theory, one obtains

$$\delta\tilde{\pi}_{\perp}^{\mu} = i\eta\kappa\delta\tilde{u}_{\perp}^{\mu}. \tag{2.22}$$

Then, inserting Eq. (2.22) into Eq. (2.21), we obtain

$$\left(\Omega - i\frac{\eta}{\varepsilon_0 + P_0}\kappa^2\right)\delta\tilde{u}_{\perp}^{\mu} = 0, \tag{2.23}$$

leading to the following dispersion relation for the transverse modes of Navier-Stokes theory

$$\Omega = i\tau_{\eta}\kappa^2, \tag{2.24}$$

with the intrinsic fluid-dynamical microscopic time-scale being defined as

$$\tau_{\eta} \equiv \frac{\eta}{\varepsilon_0 + P_0}. \tag{2.25}$$

The next step is to evaluate the behavior of the modes of Navier-Stokes theory considering both a zero and a non-zero value for the background fluid velocity. For the sake of convenience, we shall begin with the case of a static background fluid. In this case, the 4-velocity of the unperturbed system is simply $u_0^{\mu} = (1, 0, 0, 0)$, hence leading to,

$$\Omega = \omega \tag{2.26}$$

$$\kappa^2 = k^2. \tag{2.27}$$

Then, the dispersion relation associated to the transverse modes of the Navier-Stokes theory, given by Eq. (2.24), can be simply written as

$$\omega = i\tau_\eta k^2, \tag{2.28}$$

which corresponds to a single hydrodynamic mode, i.e., ω becomes zero in the vanishing wavenumber limit. We note that the frequency is purely imaginary and shows a strong resemblance with the modes obtained from the diffusion equation. This is the first hint of a non-physical behavior of the relativistic Navier-Stokes theory, since it is well known that the diffusion equation is acausal [57]. Furthermore, this dispersion relation leads to an exponential attenuation of the wave amplitude with a time-scale of the order of τ_η. It is then possible to conclude that when the unperturbed system is at rest, the theory is linearly stable – which is not an unexpected result, since this regime corresponds essentially to the non-relativistic limit of the theory (small velocities regime), in which the Navier-Stokes theory does not display any non-physical behavior.

Naturally, a more interesting case to be explored in this analysis resides on considering perturbations on top of a non-static background fluid. Perturbations around a moving fluid unveil intrinsically relativistic phenomena, since the perturbation themselves may be small, while the fluid velocity is not, and therefore might unfold more information on the linear stability of the theory. For the sake of convenience, throughout this dissertation, when analyzing perturbations on top of a moving fluid, we always take the background fluid velocity and the wave-vector to be in the same direction, e.g., the x axis, and thus we assume $u_0^\mu = \gamma(1, V, 0, 0)$ and $k^\mu = (\omega, k, 0, 0)$, unless stated otherwise. In this case, the frequency and wave-vector in the local rest frame of the unperturbed system for this case are written as

$$\Omega = \gamma(\omega - Vk), \tag{2.29}$$

$$\kappa^2 = \gamma^2(\omega V - k)^2, \tag{2.30}$$

Thus, in this case the dispersion relation for the transverse modes, Eq. (2.24), reads

$$\omega^2 V^2 + \left(\frac{i}{\gamma\tau_\eta} - 2Vk\right)\omega - \frac{i}{\gamma\tau_\eta}Vk + k^2 = 0. \tag{2.31}$$

In this work, for the sake of convenience, we shall adopt dimensionless variables. In order to obtain these variables, we re-scale all variables in terms of the hydrodynamic time scale, $\hat{A} = A[\tau_\eta]$. In this case, the re-scaled dispersion relation reads

$$\hat{\omega}^2 V^2 + \left(\frac{i}{\gamma} - 2V\hat{k}\right)\hat{\omega} - \frac{i}{\gamma}V\hat{k} + \hat{k}^2 = 0. \tag{2.32}$$

Note that for perturbations on top of a moving background there is the occurrence of a new mode in the dispersion relation. This is a surprising result, since a change in the reference frame – the case of a moving background can be understood as a Lorentz transformation from a static background – should not interfere in the number of modes of the theory. Thus, for a moving background, the modes are written as

$$\hat{\omega}(\hat{k}) = \frac{1 + 2i\gamma V\hat{k} \pm \sqrt{1 + \frac{4iV\hat{k}}{\gamma}}}{2i\gamma V^2}. \tag{2.33}$$

Naturally, in the small velocities regime one recovers the dispersion relation for a static background, a case in which the theory was shown to be linearly stable, while the other goes to infinity in such regime. This suggests a discontinuity in the modes as a function of the background velocity, which is a non-physical behavior that must be corrected. Furthermore, the new mode increases exponentially with time and therefore corresponds to the first mathematical evidence of the linear instability of the relativistic Navier-Stokes theory. The linearly unstable behavior can be easily visualized in the vanishing wavenumber limit. In this case, the dominant terms can be written as

$$\hat{\omega}(\hat{k} \to 0) \approx \begin{cases} -\frac{i}{\gamma V^2}, \\ V\hat{k} + i\frac{\hat{k}^2}{\gamma^3}. \end{cases} \tag{2.34}$$

One can immediately notice that perturbations on a moving background fluid lead to the occurrence of a new non-hydrodynamic mode, i.e., ω does not vanish for $k = 0$. Considering the convention adopted for the Fourier transform in this work, this mode has a negative imaginary part and thus is linearly unstable, even for $k = 0$, which corresponds to the homogeneous limit. Therefore, perturbations on a fluid initially in equilibrium leads to exponentially increasing oscillations, a sign of *linear* instability, a non-physical behavior. On the other hand, the bottom solution has both a propagating and an oscillating part and it is linearly stable. As it was previously mentioned, one can straightforwardly recover the dispersion relation for perturbations on a static background fluid, see Eq. (2.28), by simply taking $V \to 0$.

We display the solutions of Eq. (2.32), i.e., the transverse modes of the relativistic Navier-Stokes theory for perturbations on top of a moving background in Fig. 1. Here, we explicitly see the additional unstable non-hydrodynamic mode. Furthermore, note that the instability of such mode occurs not only in the vanishing wavenumber limit, but also for arbitrary values of k.

Longitudinal modes

The occurrence of an additional mode when considering perturbations on top of a moving fluid, the discontinuity of such modes as a function of the background velocity, and the linear instability even for $k = 0$ are enough evidence to discard the Navier-Stokes theory as a suitable formulation for the description of relativistic fluids. Still, for the sake of completeness, we shall carry on the present stability analysis for the longitudinal

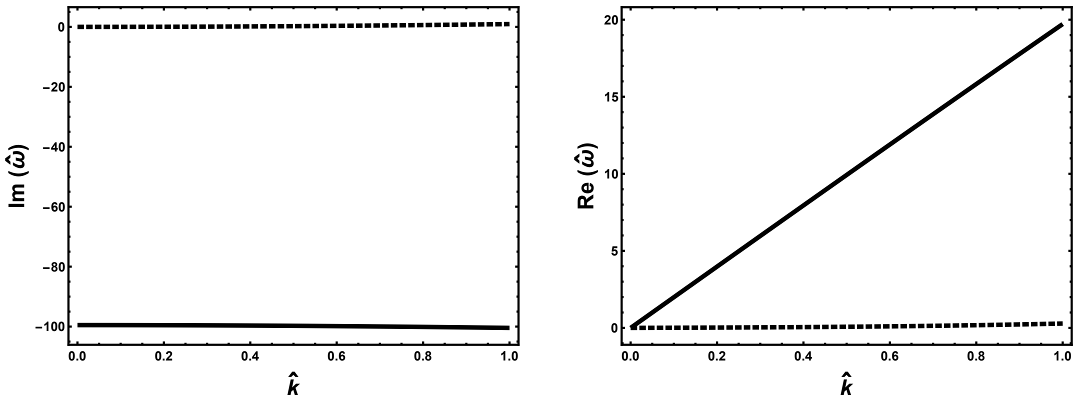

Figure 1 – Imaginary and real parts of the transverse modes of the relativistic Navier-Stokes theory for perturbations on moving background fluid considering $V = 0.1$.

modes of the theory. Here, the analysis is simplified by neglecting any fluctuations on the net-charge and net-charge diffusion, $\delta\tilde{n} = 0$ and $\delta\tilde{n}^\mu = 0$.

In order to obtain the longitudinal modes of the relativistic Navier-Stokes theory, we shall project Eq. (2.16) with κ_μ, following the same procedure presented earlier in this chapter. Equation (2.14) is already written in terms of the longitudinal variables and requires no manipulation. Since net-charge and net-charge diffusion are being neglected, Eq. (2.15) is simply disregarded. Thus, the equations for the longitudinal degrees of freedom can be expressed as

$$\Omega\delta\tilde{\epsilon} - \kappa(\epsilon_0 + P_0)\delta\tilde{u}_\parallel = 0, \tag{2.35}$$

$$(\epsilon_0 + P_0)\Omega\delta\tilde{u}_\parallel - \kappa(\delta\tilde{P} + \delta\tilde{\Pi} + \delta\tilde{\pi}_\parallel) = 0. \tag{2.36}$$

These equations are valid for an arbitrary fluid-dynamical theory, since the linearized dissipative currents have not been specified yet. In Navier-Stokes theory, the longitudinal components of the linearized dissipative currents in Fourier space are given by,

$$\delta\tilde{\Pi} = i\zeta\kappa\delta\tilde{u}_\parallel, \tag{2.37}$$

$$\delta\tilde{\pi}_\parallel = \frac{4}{3}i\eta\kappa\delta\tilde{u}_\parallel. \tag{2.38}$$

Therefore, the equations can be written as

$$\Omega\delta\tilde{\epsilon} - \kappa(\epsilon_0 + P_0)\delta\tilde{u}_\parallel = 0, \tag{2.39}$$

$$(\epsilon_0 + P_0)\Omega\delta\tilde{u}_\parallel - \kappa\delta\tilde{P} - i\kappa^2\left(\zeta + \frac{4}{3}\eta\right)\delta\tilde{u}_\parallel = 0. \tag{2.40}$$

Furthermore, it is possible to express them in the following matrix form

$$\begin{pmatrix} \Omega & -\kappa \\ -c_s^2\kappa & \Omega - i\tau_{\text{eff}}\kappa^2 \end{pmatrix} \begin{pmatrix} \frac{\delta\tilde{\epsilon}}{\epsilon_0+P_0} \\ \delta\tilde{u} \end{pmatrix} = 0, \tag{2.41}$$

with the speed of sound in the fluid being identified as,

$$c_\mathrm{s}^2 = \left.\frac{\partial P_0}{\partial \epsilon_0}\right|_{s_0,n_0}, \tag{2.42}$$

and a new time scale, associated to the longitudinal modes, being defined as

$$\tau_\mathrm{eff} \equiv \frac{\zeta + \frac{4}{3}\eta}{\epsilon_0 + P_0}. \tag{2.43}$$

Naturally, non-trivial solutions are obtained by requiring the determinant of the matrix on the left-hand side of Eq. (2.41) to vanish, leading to the dispersion relation for the longitudinal modes of Navier-Stokes theory

$$\Omega^2 - i\tau_\mathrm{eff}\Omega\kappa^2 - c_\mathrm{s}^2\kappa^2 = 0. \tag{2.44}$$

Analogous to the procedure adopted for the transverse modes, we re-scale the dispersion relation in order to work exclusively with dimensionless variables. However, in the following, we re-scale the variables using τ_eff, $\bar{A} = A[\tau_\mathrm{eff}]$. Then, the dispersion relation associated to the longitudinal modes of the relativistic Navier-Stokes theory reads

$$\bar{\Omega}^2 - i\bar{\Omega}\bar{\kappa}^2 - c_\mathrm{s}^2\bar{\kappa}^2 = 0. \tag{2.45}$$

As before, we begin the linear stability analysis of the longitudinal modes considering the case where the unperturbed system is at rest. In this scenario, the dispersion relation becomes

$$\bar{\omega}^2 - i\bar{\omega}\bar{k}^2 - c_\mathrm{s}^2\bar{k}^2 = 0, \tag{2.46}$$

and its solutions are given by

$$\bar{\omega}(\bar{k}) = i\frac{\bar{k}^2}{2} \pm \bar{k}\sqrt{c_s^2 - \frac{\bar{k}^2}{4}}. \tag{2.47}$$

It can be seen that by taking $\tau_\mathrm{eff} = 0$, the dispersion relation for ideal fluids – the equation of a plane wave – $\omega(k) = \pm c_\mathrm{s} k$, is immediately recovered.

It is further possible to extract information about the stability of the theory by analyzing the behavior of the modes in the limits of small and large wavenumber, respectively. In these regimes, the modes become

$$\bar{\omega}(\bar{k} \to 0) = \pm c_s\bar{k} + i\frac{\bar{k}^2}{2} + O(\bar{k}^3) \tag{2.48}$$

$$\bar{\omega}(\bar{k} \to \infty) = i\frac{\bar{k}^2}{2}(1 \pm 1). \tag{2.49}$$

Therefore, for small wavenumbers, the real part of the dispersion relation is a sound wave. Moreover, the imaginary part leads to the exponential damping of the perturbations and is linearly stable. The time scale τ_eff then dictates how fast these modes are damped. For large wavenumbers, the mode either vanishes or has a positive imaginary part, once again leading to a stable solution. Once again we shall point out that this result is expected, since perturbations on a static background corresponds to a non-relativistic regime, where Navier-Stokes theory is linearly stable.

As it was observed for the transverse modes, a more interesting case arises when perturbations on a non-static background fluid are considered. Once again, we assume the background velocity is in the same direction of the wave-vector. Then, the dispersion relation associated to the longitudinal modes of the relativistic Navier-Stokes theory for perturbations on top of a moving fluid reads

$$(\bar{\omega} - V\bar{k})^2 - i\gamma(\bar{\omega} - V\bar{k})(\bar{\omega}V - \bar{k})^2 - c_s^2(\bar{\omega}V - \bar{k})^2 = 0. \tag{2.50}$$

Analogous to what was also observed for the transverse modes, the dispersion relation for the longitudinal modes for a non-static background is a polynomial of one degree higher in comparison to the case of perturbations on a static background fluid. As it will be shown, the new mode is non-hydrodynamic and linearly unstable.

For the sake of simplicity, we shall analyze the linear stability of these modes for a vanishing wavenumber. In this limit, the modes are written as

$$\bar{\omega}(\bar{k} = 0) = \begin{cases} -i\frac{1-c_s^2V^2}{\gamma V^2} \\ \pm 0. \end{cases} \tag{2.51}$$

Clearly, the new mode is non-hydrodynamic since it does not vanish for $k = 0$. Since it is always negative and purely imaginary, it corresponds to a linearly unstable mode, leading to perturbations that increase exponentially. We display the longitudinal modes of the relativistic Navier-Stokes theory for perturbations on top of a moving background fluid in Fig. 2.

As it was first observed for the transverse modes, there is the occurrence of an additional non-hydrodynamic mode, which is linearly unstable. Furthermore, we observe a discontinuity of the modes with the background fluid velocity, analogous to what was seen for the transverse modes: the modes are linearly stable for a vanishing background velocity but become unstable for non-zero values of V. Note that the linear instability of the non-hydrodynamic longitudinal mode is not restricted to the vanishing wavenumber case, but is also observed when considering arbitrary values of k.

Therefore, after analyzing how small perturbations behave in the framework of relativistic Navier-Stokes theory and explicitly calculating the transverse and longitudinal modes of this theory, it is then possible to conclude this formulation is indeed linearly unstable and shall be discarded as a suitable description of relativistic fluids.

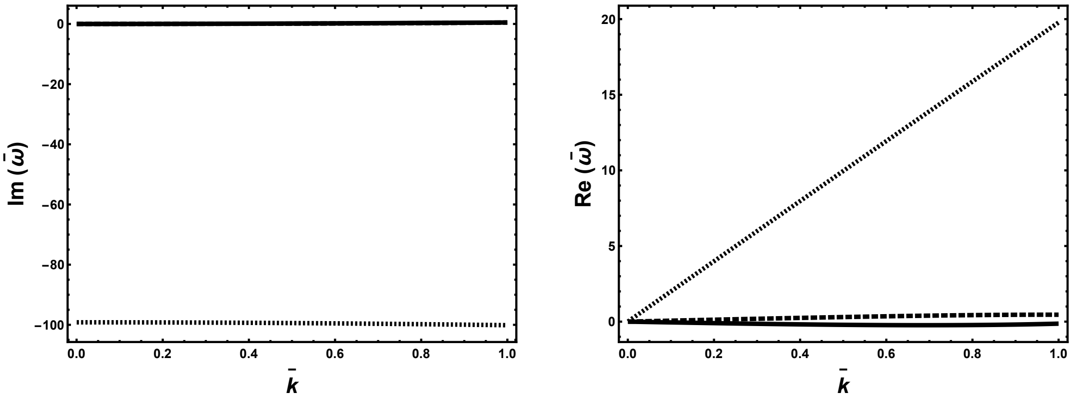

Figure 2 – Imaginary and real parts of the longitudinal modes of the relativistic Navier-Stokes theory for perturbations on moving background fluid considering $c_s^2 = 1/3$ and $V = 0.1$.

2.1.1 Causality and instability of Navier-Stokes theory

For the sake of convenience, so far in this work we only considered fluids with a background velocity that is in the same as the wave-vector, which leads to Eqs. (2.29) and (2.30). These relations can be understood as a Lorentz boost of a 4-vector with components ω and k. Thus, it is possible to obtain the dispersion relation satisfied by perturbations on a non-static background fluid by performing a Lorentz-boost on the dispersion relation satisfied by perturbations performed on a static background fluid. The Lorentz boost of a 4-vector with time component ω and spatial component (in the same direction of the boost) k reads

$$\begin{pmatrix} \omega \\ k \end{pmatrix} = \begin{pmatrix} \gamma & -\gamma V \\ -\gamma V & \gamma \end{pmatrix} \begin{pmatrix} \omega' \\ k' \end{pmatrix} = \begin{pmatrix} \gamma\omega' - \gamma V k \\ -\gamma V \omega' + \gamma k' \end{pmatrix}. \tag{2.52}$$

It is then possible to understand the linear instability of Navier-Stokes theory in the following way. Both transverse and longitudinal modes show a similar behavior in the large wavenumber limit, considering perturbations on a static background fluid, see Eqs. (2.28) and (2.49). These expressions can be summarized by the following diffusion-like dispersion relation

$$\omega = iDk^2, \tag{2.53}$$

with D being the diffusion coefficient. Hence, when considering a non-static background, this equation is transformed as

$$\omega' - Vk' = i\gamma D(k' - V\omega')^2, \tag{2.54}$$

where the occurrence of an additional mode, in comparison to the static case, is observed, analogous to what has been derived previously in this section for both transverse and longitudinal modes. Naturally, the hydrodynamic mode is once again stable, while the non-hydrodynamic mode for $k = 0$ reads

$$\omega' = -\frac{i}{\gamma D V^2}. \tag{2.55}$$

Therefore, a diffusive dispersion relation, when expressed in a moving frame, necessarily leads to the occurrence of a new linearly unstable non-hydrodynamic mode, that was absent in the rest frame. Then, it is possible to conclude that the diffusion-like behavior of the modes of the relativistic Navier-Stokes theory at large wavenumbers is the physical origin of the linear instabilities displayed by the theory.

Finally, since the relativistic Navier-Stokes theory is linearly unstable, the next step is to investigate if the inclusion of relaxation times in the equations for the dissipative currents is sufficient to guarantee the linear stability of the theory. In particular, in the next section, we shall analyze if the Israel-Stewart theory is a suitable description for relativistic fluids.

2.2 Israel-Stewart theory

The re-formulation of the entropy 4-current including up to second-order terms in the dissipative currents leads to hyperbolic equations for the dissipative currents in the linear regime, a necessary condition to ensure causality. However, the causality and stability of the Israel-Stewart theory have not yet been proved. In order to address this problem, a linear stability analysis of the Israel-Stewart theory is presented in this section, with the goal of verifying whether the inclusion of relaxation times is sufficient to render a linearly causal and stable fluid-dynamical formulation.

In this case, analogous to what has been performed in the linear stability analysis of Navier-Stokes theory, the system is assumed to be initially in a global equilibrium state, and then perturbations around such state are performed, such as in Eq. (2.1). In this chapter, we neglect coupling between dissipative currents and restrict ourselves to the analysis of the simplest version of Israel-Stewart theory. An analysis taking into account diffusion-viscous will be developed in detail in Chapter 3.

In this case, the linearized Israel-Stewart equations read

$$D_0 \delta\Pi + \frac{\delta\Pi}{\tau_\Pi} + \frac{\zeta}{\tau_\Pi}\partial_\mu \delta u^\mu = 0, \tag{2.56}$$

$$D_0 \delta n_\mu + \frac{\delta n_\mu}{\tau_n} - \frac{\kappa_n}{\tau_n}\nabla^0_\mu \delta\alpha = 0, \tag{2.57}$$

$$D_0 \delta\pi_{\mu\nu} + \frac{\delta\pi_{\mu\nu}}{\tau_\pi} - 2\frac{\eta}{\tau_\pi}\Delta^{\alpha\beta,0}_{\mu\nu}\partial_\alpha \delta u_\beta = 0, \tag{2.58}$$

with $D_0 \equiv u^\mu_0 \partial_\mu$ being the comoving derivative with respect to the background fluid velocity, $\nabla^\mu_0 \equiv \Delta^{\mu\nu}_0 \partial_\nu$ being defined as the projected derivative in the direction orthogonal to u_0^μ and $\Delta_0^{\mu\nu\alpha\beta} \equiv \frac{1}{2}\Delta_0^{\mu\alpha}\Delta_0^{\nu\beta} + \frac{1}{2}\Delta_0^{\mu\beta}\Delta_0^{\nu\alpha} - \frac{1}{3}\Delta_0^{\mu\nu}\Delta_0^{\alpha\beta}$. The stability of Israel-Stewart theory has been investigated in the linear regime considering dissipation via bulk [29], shear-stress [30], net-charge diffusion [63], and also when coupling between shear-stress and net-charge diffusion is taken into account in Refs. [31, 48, 59]. The present section is dedicated to the investigation of the linear stability of Israel-Stewart theory in a simple case, neglecting any dissipation by bulk viscosity and net-charge diffusion, i.e., $\delta\tilde{n} = \delta\tilde{n}^\mu = \delta\tilde{\Pi} = 0$, thus only considering dissipation by shear-stress – the only non-trivial equation of motion from the Israel-Stewart theory is therefore Eq. (2.58).

In order to analyze the linear stability of Israel-Stewart theory we shall compute the linearized equation for the shear-stress tensor in Fourier space

$$(i\Omega\tau_\pi + 1)\,\delta\tilde{\pi}^{\mu\nu} = i\eta\left(\kappa^\mu\delta\tilde{u}^\nu + \kappa^\nu\delta\tilde{u}^\mu - \frac{2}{3}\Delta_0^{\mu\nu}\kappa_\lambda\delta\tilde{u}^\lambda\right), \tag{2.59}$$

and plug it into the conservation laws. Similarly to what was done for Navier-Stokes theory, this analysis will be divided between transverse and longitudinal degrees of freedom, since once again they decouple and can be solved independently.

Transverse modes

Naturally, since Eq. (2.16) is the only tensorial equation among the equations of motion that arise from the conservation laws, it is the only one that has transverse components, which were already calculated in Eq. (2.21). The partially transverse projection of Eq. (2.59) is given by,

$$i\tau_\pi\Omega\delta\tilde{\pi}_\perp^\mu + \delta\tilde{\pi}_\perp^\mu = i\eta\kappa\delta\tilde{u}_\perp^\mu. \tag{2.60}$$

Substituting Eq. (2.60) into Eq. (2.21) we obtain the following equation for the transverse component of the velocity field,

$$\left(\frac{\tau_\pi}{\tau_\eta}\Omega^2 - i\frac{\Omega}{\tau_\eta} - \kappa^2\right)\delta\tilde{u}_\perp^\mu = 0. \tag{2.61}$$

For the sake of convenience, we work with dimensionless variables and re-scale all variables by the time scale τ_η, $\hat{A} = A[\tau_\eta]$. Hence, the re-scaled dispersion relation for the transverse modes of Israel-Stewart theory is written in the following form

$$\hat{\tau}_\pi\hat{\Omega}^2 - i\hat{\Omega} - \hat{\kappa}^2 = 0. \tag{2.62}$$

As before, the first case to be considered is the one in which the velocity of the background fluid is zero, $V = 0$. In this case, $\Omega = \omega$ and $\kappa = k$, and the dispersion relation becomes simply

$$\hat{\omega}\left(i\hat{\tau}_\pi\hat{\omega} + 1\right) - i\hat{k}^2 = 0. \tag{2.63}$$

The solutions of this equation are

$$\hat{\omega}_{T,\pm}^{\text{shear}} = \frac{i \pm \sqrt{4\hat{\tau}_\pi k^2 - 1}}{2\hat{\tau}_\pi}. \tag{2.64}$$

Note that even in the case in which the background fluid is at rest, we have the occurrence of two modes, with one of them being non-hydrodynamic. This is different to what was observed in Navier-Stokes theory, where there was only one hydrodynamic mode in such case. We note that the non-hydrodynamic mode found above is stable as long as the relaxation time is positive. This is in agreement with microscopic calculations of this transport coefficient evaluated in kinetic theory [64, 65]. Moreover, there is a critical value of the wavenumber, $k_c = 1/2\sqrt{\hat{\tau}_\pi}$, below which the modes are purely imaginary and, therefore, non-propagating. The transverse modes of Israel-Stewart theory for perturbations on a static background fluid are displayed in Fig. 3 considering a shear relaxation time calculated from the Boltzmann equation, via the 14-moment approximation, in the ultra-relativistic limit, $\hat{\tau}_\pi = 5$ [64]. We note that these modes do not change qualitatively when different values of $\hat{\tau}_\pi$ are employed.

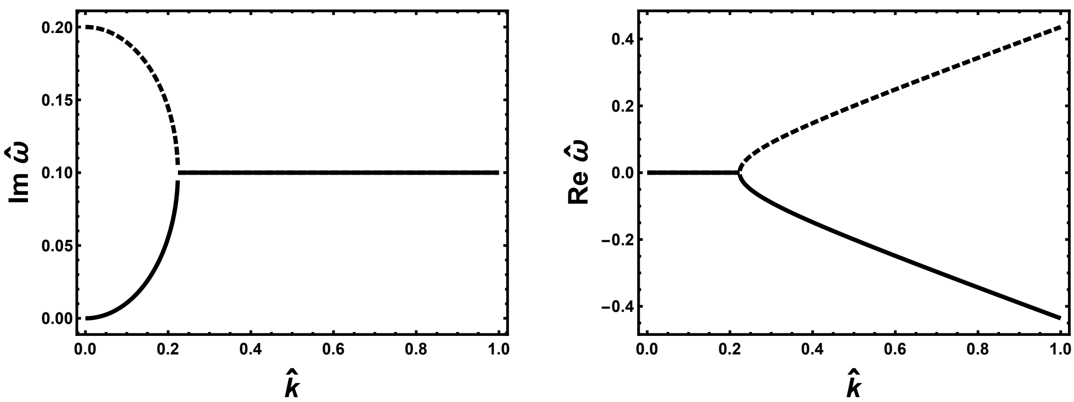

Figure 3 – Imaginary and real parts of the transverse modes of the Israel-Stewart theory for perturbations on a static fluid considering $\hat{\tau}_\pi = 5$.

Nevertheless, when analyzing such modes in the Navier-Stokes limit, i.e., when the shear relaxation time is set to zero, it is possible to observe that the non-hydrodynamic mode diverges, while the hydrodynamic mode goes to the usual diffusion-like solution, a remarkable feature of Navier-Stokes theory, $\hat{\omega}^{\text{shear}}_{T,-} = ik^2$. One can also write the solution above in the small wavenumber limit,

$$\hat{\omega}^{\text{shear}}_{T,+} = \frac{i}{\hat{\tau}_\pi} - i\hat{k}^2 + \mathcal{O}\left(\hat{k}^4\right), \tag{2.65}$$

$$\hat{\omega}^{\text{shear}}_{T,-} = i\hat{k}^2 + i\hat{\tau}_\pi \hat{k}^4 + \mathcal{O}\left(\hat{k}^6\right), \tag{2.66}$$

where we can see that the hydrodynamic mode resembles a solution from the diffusion equation in this regime. On the other hand, in the large wavenumber limit, these modes behave as

$$\hat{\omega}^{\text{shear}}_{T,\pm} = \frac{i}{2\hat{\tau}_\pi} \pm \left(\frac{\hat{k}}{\sqrt{\hat{\tau}_\pi}} - \frac{1}{8\hat{\tau}_\pi^{3/2}\hat{k}}\right) + \mathcal{O}\left(\frac{1}{\hat{k}^4}\right). \tag{2.67}$$

Moreover, in this limit the modes have also a propagating part, and they are not purely imaginary anymore, hence leading to the occurrence of an asymptotic group velocity. In this case, the modes are linearly causal if this velocity is subluminal [30, 66], which is guaranteed by the following condition

$$\lim_{\hat{k}\to\infty}\left|\frac{\partial \mathrm{Re}(\hat{\omega})}{\partial \hat{k}}\right| \leq 1 \Longrightarrow \hat{\tau}_\pi \geq 1 \Longrightarrow \tau_\pi \geq \frac{\eta}{\varepsilon_0 + P_0} \tag{2.68}$$

A more interesting case to be analyzed is, once again, the case of perturbations on top of a moving background fluid. As it was done in the stability analysis of Navier-Stokes theory, we shall assume that the velocity of the unperturbed system and the wave-vector are parallel. Therefore, Ω and κ satisfy Eqs. (2.29) and (2.30), respectively, and thus the dispersion relation, Eq. (2.62), is written as

$$\hat{\tau}_\pi \gamma(\hat{\omega} - V\hat{k})^2 - i(\hat{\omega} - V\hat{k}) - \gamma(\hat{\omega}V - \hat{k})^2 = 0. \tag{2.69}$$

The first point to be noted here is the fact that the number of modes does not increase (nor decrease) for perturbations on top of a moving fluid in comparison to the static case, which was a non-physical behavior we observed in the relativistic Navier-Stokes theory. However, for perturbations on a moving background, the imaginary part of the non-hydrodynamic mode is not always positive definite. Instead, there are conditions the transport coefficients – in particular, the relaxation times – must fulfill in order to render linearly stable modes. This can be easily seen when analyzing the vanishing wavenumber limit of Eq. (2.69), in which the non-hydrodynamic mode becomes

$$\hat{\omega}^{\mathrm{shear}}_{T,+}(\hat{k} = 0) = \frac{i}{\gamma(\hat{\tau}_\pi - V^2)}. \tag{2.70}$$

Naturally, the theory must be stable for any value of k. As a matter of fact, it is particularly important that the modes remain stable when $k = 0$, since this corresponds to the case of homogeneous perturbations. Furthermore, the theory should be stable for any value of the background fluid velocity V. The aforementioned conditions are guaranteed to hold as long as the relaxation time satisfies

$$\hat{\tau}_\pi > 1 \Longrightarrow \tau_\pi > \tau_\eta, \tag{2.71}$$

which is the linear stability condition originally derived in Ref. [30]. In particular, this condition is identical to the causality condition obtained for perturbations on a static background fluid, see Eq. (2.68).

The non-hydrodynamic transverse mode $\hat{\omega}^{\mathrm{shear}}_{T,+}$ is displayed in Fig. 4 for three different values of background velocity V as a function of the shear relaxation time $\hat{\tau}_\pi$ in the vanishing wavenumber limit.

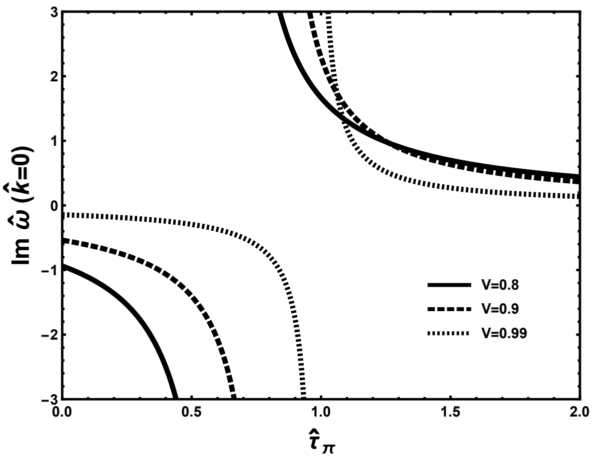

Figure 4 – Imaginary part of the non-hydrodynamic transverse mode $\hat{\omega}^{\text{shear}}_{T,+}$ for $\hat{k} = 0$, considering three different values of the background velocity, $V = 0.8$, $V = 0.9$ and $V = 0.99$ and $\hat{\tau}_\pi = 5$.

Longitudinal modes

Naturally, after deriving the stability conditions associated to the transverse degrees of freedom, the next step is to analyze the constraints that must be imposed in order to obtain linearly causal and stable longitudinal modes. To obtain the projected equations for the longitudinal degrees of freedom we reproduce the same steps of the analysis of the relativistic Navier-Stokes theory, following the procedure presented previously in this chapter. In the absence of bulk viscous pressure and, also, setting the net-charge to be zero, the longitudinal components of the conservation laws become

$$\Omega \delta\tilde{\epsilon} - \kappa(\epsilon_0 + P_0)\delta\tilde{u}_\parallel = 0, \tag{2.72}$$

$$(\epsilon_0 + P_0)\Omega\delta\tilde{u}_\parallel - \kappa(\delta\tilde{P} + \delta\tilde{\pi}_\parallel) = 0. \tag{2.73}$$

Then, to analyze the linear stability of Israel-Stewart theory, the longitudinal projection of the shear-stress tensor must be inserted in the last equation. This projection can be calculated as

$$i\tau_\pi \Omega \delta\tilde{\pi}_\parallel + \delta\tilde{\pi}_\parallel - \frac{4\eta}{3} i\kappa\delta\tilde{u}_\parallel = 0 \Longrightarrow \delta\tilde{\pi}_\parallel = \frac{4\eta}{3(i\tau_\pi\Omega + 1)} i\kappa\delta\tilde{u}_\parallel. \tag{2.74}$$

These equations can be written in the following matrix form

$$\begin{pmatrix} \Omega & -\kappa \\ -c_s^2\kappa(i\Omega\tau_\pi + 1) & \Omega(i\Omega\tau_\pi + 1) - i\frac{4}{3}\tau_\eta\kappa^2 \end{pmatrix} \begin{pmatrix} \frac{\delta\tilde{\epsilon}}{\epsilon_0 + P_0} \\ \delta\tilde{u}_\parallel \end{pmatrix} = 0, \tag{2.75}$$

Non-trivial solutions of this equation are obtained when the determinant is zero. Therefore, the dispersion relation related to the longitudinal modes is obtained as

$$i\tau_\pi\Omega^3 + \Omega^2 - i\frac{4}{3}\tau_\eta\kappa^2\Omega - i\tau_\pi c_s^2\kappa^2\Omega - c_s^2\kappa^2 = 0. \tag{2.76}$$

Once again, it is further convenient to introduce dimensionless variables by employing the same re-scaling process used in the analysis of the transverse modes, expressing the dimensionful variables in terms of the hydrodynamic time-scale, such as $\hat{A} = A[\tau_\eta]$. In this notation, Eq. (2.76) can be written as

$$i\hat{\tau}_\pi\hat{\Omega}^3 + \hat{\Omega}^2 - \frac{4}{3}i\hat{\kappa}^2\hat{\Omega} - i\hat{\tau}_\pi c_s^2\hat{\kappa}^2\hat{\Omega} - c_s^2\hat{\kappa}^2 = 0. \tag{2.77}$$

As it was done so far, we begin the linear stability analysis of the longitudinal modes considering perturbations around a static background fluid. In this case, the dispersion relation simply becomes

$$i\hat{\tau}_\pi\hat{\omega}^3 + \hat{\omega}^2 - \frac{4}{3}i\hat{k}^2\hat{\omega} - i\hat{\tau}_\pi c_s^2\hat{k}^2\hat{\omega} - c_s^2\hat{k}^2 = 0. \tag{2.78}$$

The solutions of Eq. (2.78) are displayed in Fig. 5, once again considering the shear relaxation time calculated from kinetic theory using the 14-moment approximation, in the ultra-relativistic limit, $\hat{\tau}_\pi = 5$ [64]. We also used that $c_s^2 = 1/3$. These modes were first obtained in Ref. [30] and further reproduced in Ref. [59]. Note that the dashed lines represent the hydrodynamic modes, which are propagating and have a degenerate imaginary part, while the solid line corresponds to the non-hydrodynamic (and non-propagating) mode. In contrast to what has been observed in the stability analysis of Navier-Stokes theory, performed in the previous section, a non-hydrodynamic mode appears in both transverse and longitudinal degrees of freedom of Israel-Stewart theory.

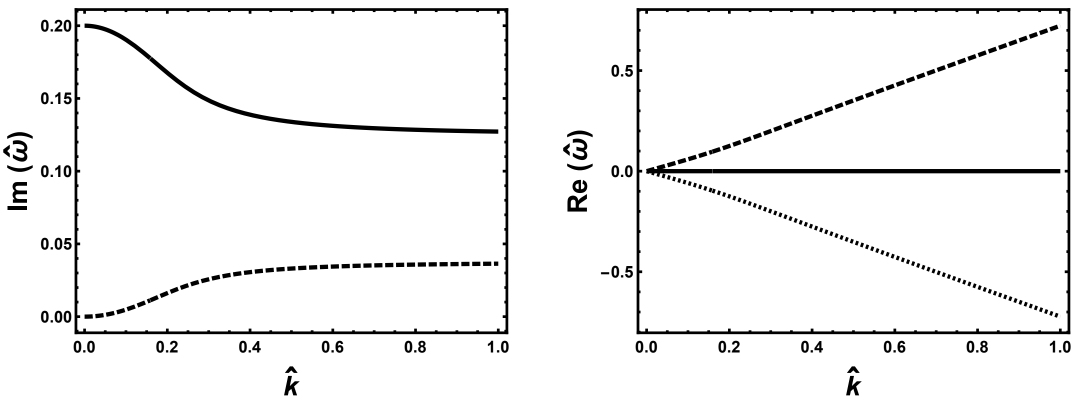

Figure 5 – Imaginary and real parts of the longitudinal modes of the Israel-Stewart theory for perturbations on a static fluid, $V = 0$, considering $\hat{\tau}_\pi = 5$ and $c_s^2 = 1/3$

The general solutions of Eq. (2.78) are rather complicated and will not be written in their explicit form. Nevertheless, it is still useful to write the asymptotic limit of these solutions in the small and large wavenumber limits. In the first case, they read,

$$\omega_{\pm}^{\text{sound}} = \pm\frac{1}{\sqrt{3}}\hat{k} + \frac{2}{3}i\hat{k}^2 + \mathcal{O}\left(\hat{k}^3\right), \tag{2.79}$$

$$\omega_{L}^{\text{shear}} = \frac{i}{\hat{\tau}_\pi} - \frac{4}{3}i\hat{k}^2 + \mathcal{O}\left(\hat{k}^4\right). \tag{2.80}$$

In this limit, the hydrodynamic modes reduce to the solutions found in Navier-Stokes. The non-hydrodynamic mode has no analogue in Navier-Stokes theory. Moreover, in the large wavenumber limit the longitudinal modes can be written as

$$\omega_{\pm}^{\text{sound}} = \pm\sqrt{\frac{4+\hat{\tau}_\pi}{3\hat{\tau}_\pi}}\hat{k} + \frac{2i}{\hat{\tau}_\pi\left(4+\hat{\tau}_\pi\right)} + \mathcal{O}\left(\frac{1}{\hat{k}}\right), \tag{2.81}$$

$$\omega_{L}^{\text{shear}} = \frac{i}{4+\hat{\tau}_\pi} + \mathcal{O}\left(\frac{1}{\hat{k}}\right). \tag{2.82}$$

In this limit, the hydrodynamic mode remains with a propagating part. In order to ensure the causality of such modes, we must impose that the asymptotic group velocity is subluminal [66], leading to the following constrain for the relaxation time [30]

$$\lim_{k\to\infty}\left|\frac{\partial \text{Re}(\omega)}{\partial k}\right| \leq 1 \Longrightarrow \hat{\tau}_\pi \geq 2 \Longrightarrow \tau_\pi \geq \frac{2\eta}{\varepsilon_0 + P_0}. \tag{2.83}$$

However, instabilities in the Israel-Stewart theory may arise in the non-hydrodynamic modes when performing perturbations on top of a moving fluid. Once again, we assume that the background fluid velocity is in the same direction of the wave-vector. In this case, the dispersion relation is written as

$$i\hat{\tau}_\pi\gamma(\hat{\omega}-\hat{k}V)^3+(\hat{\omega}-\hat{k}V)^2-\frac{4}{3}\gamma i(\hat{\omega}V-\hat{k})^2(\hat{\omega}-\hat{k}V)-i\hat{\tau}_\pi c_{\text{s}}^2\gamma(\hat{\omega}V-\hat{k})^2(\hat{\omega}-\hat{k}V)-c_{\text{s}}^2(\hat{\omega}V-\hat{k})^2 = 0. \tag{2.84}$$

As it was first observed for the transverse modes, the number of longitudinal modes of the Israel-Stewart theory does not increase for perturbations on top of a moving fluid, unlike in the relativistic Navier-Stokes theory. Clearly, a stability analysis for arbitrary values of wavenumber can be extremely complicated. Therefore, we look at the linear stability of the modes in the homogeneous limit, $k = 0$, hence deriving necessary conditions the theory must satisfy in order to yield stable modes. In this case, considering $c_{\text{s}}^2 = 1/3$, the hydrodynamic modes are found to be stable, while the non-hydrodynamic mode reads

$$\hat{\omega}_{L}^{\text{shear}} = \frac{i\left(3-V^2\right)}{\gamma\left[3\hat{\tau}_\pi - \left(\hat{\tau}_\pi+4\right)V^2\right]} \tag{2.85}$$

This mode must be stable for any value of the background fluid velocity. This can be guaranteed as long as the transport coefficients – in particular, the shear relaxation time – satisfy certain constraints. In this case, the linear stability condition associated to the longitudinal modes is

$$\hat{\tau}_\pi > 2 \Longrightarrow \tau_\pi \geq \frac{2\eta}{\varepsilon_0 + P_0}. \tag{2.86}$$

The linear stability condition for the longitudinal modes, given by Eq. (2.86), is stronger than the condition obtained for the transverse modes, Eq. (2.71). Furthermore, this condition is identical to the linear causality condition, Eq. (2.83). The non-hydrodynamic mode obtained for perturbations on top of a moving fluid in the vanishing wavenumber limit, Eq. (2.85), is displayed for several values of background velocity as a function of the shear relaxation time in Fig. 6.

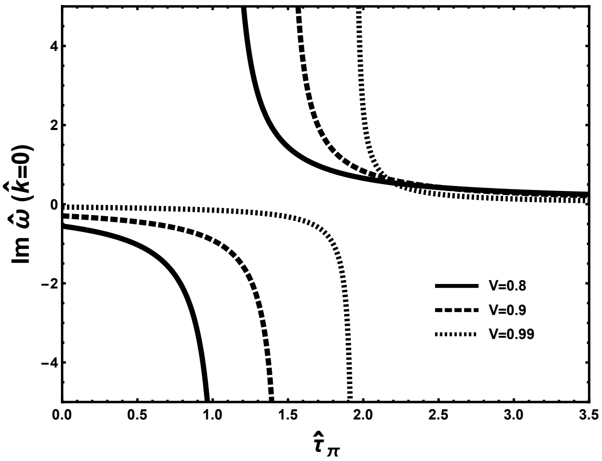

Figure 6 – Imaginary part of the non-hydrodynamic longitudinal mode considering three different values of the background velocity, $V = 0.8$, $V = 0.9$ and $V = 0.99$, for $c_s^2 = 1/3$.

In this chapter we analyzed the stability of Navier-Stokes and Israel-Stewart theories in the linear regime. The first was shown to be linearly unstable, a direct consequence of the theory's parabolicity [57, 58]. We showed that perturbations on a static fluid yields exclusively stable hydrodynamic modes. On the other hand, when considering perturbations on a moving fluid new unstable non-hydrodynamic modes appear. This discontinuity in the number of modes with respect with the background fluid velocity is, by itself, non-physical and must be corrected.

The linear stability of Israel-Stewart was studied considering dissipation only via shear-stress. However, unlike in the case of Navier-Stokes theory, the theory's hyperbolicity guarantees the number of modes does not change with the background fluid velocity. In particular, non-hydrodynamic modes now occur even for perturbations on a static fluid. We showed that such a version of Israel-Stewart theory can be made linearly causal and stable as long as certain conditions are satisfied by the shear relaxation time. In particular, we note that the linear causality and stability conditions obtained analyzing the longitudinal modes are stronger than the ones obtained for the transverse modes. Finally, we concluded that Israel-Stewart theory, considering only the dynamics of the shear-stress tensor, is linearly causal and stable as long as $\tau_\pi > \frac{2\eta}{\varepsilon_0 + P_0}$.

3 Linear stability of Israel-Stewart theory with net-charge

The last chapter was entirely dedicated to reproduce the results obtained from linear stability analyses of a first-order fluid-dynamical formulation – Navier-Stokes theory – and a second-order formulation – Israel-Stewart theory considering only dissipation by shear-stress. The first does not carry the same success as its non-relativistic analogue, as the parabolicity [57, 58] of the equations for the dissipative currents leads to superluminal sign propagation and linear instabilities – an issue attributed to the first-order character of the out-of-equilibrium entropy 4-current.

On the other hand, although the reformulation of the entropy 4-current, different than the one introduced as the baseline of the relativistic Navier-Stokes theory, considering not only first-order terms in the dissipative currents, but also second-order terms, yields the occurrence of relaxation times that are essential to the theory's causality, the linear stability of the Israel-Stewart theory is not automatically guaranteed. As it was demonstrated in the last chapter, the transport coefficients – in particular, the relaxation times of the theory – must satisfy certain constraints in order to the theory to be linearly causal and stable. The linear stability analysis presented in Chapter 2 takes into account only dissipation via shear-stress, but a derivation including also bulk yet neglecting any coupling between dissipative currents can be straightforwardly extended [30].

Heavy-ion collisions at extremely high energies produce a huge number of baryons and their corresponding anti-baryons, which corresponds to a near-vanishing baryon chemical potential, $\mu \approx 0$. Furthermore, since on average the net-baryon number is zero, the fluid-dynamical evolution of the QGP in such conditions is performed neglecting the dynamics of net-baryon diffusion. With the purpose of achieving a better understanding of QCD in this regime, heavy-ion collisions at lower energies are being performed in RHIC and LHC. Naturally, this type of collision produces a smaller number of baryons and anti-baryons from vacuum. In this case, the difference between those quantities becomes more appreciable and one can properly analyze the regime of a non-vanishing baryon chemical potential. In particular, when modelling heavy-ion collisions at lower energies, the net-baryon diffusion current can no longer be neglected and it necessarily has to be taken into account in the fluid-dynamical evolution of the system.

Furthermore, one may ask whether the linear stability conditions of the Israel-Stewart theory remain unchanged once coupling between dissipative currents is taken into account. The first developments on such task have been performed by Olson in Ref. [48] for the complete set of equations of motion of the theory considering the Landau-Lifshitz picture. However, the linear stability conditions derived in this work are rather convoluted and not explicitly written in terms of the transport coefficients, e.g., coupling terms (also referred to as *diffusion-viscous coupling*) and diffusion relaxation time. Furthermore, this analysis cannot be directly extended to the case in which one considers perturbations around a vanishing net-charge, the case studied in this chapter. A preliminary stability analysis including net-charge diffusion was developed in Ref. [63], neglecting dissipation via shear-stress. In Ref. [59], we developed a more general linear stability of Israel-Stewart, with only dissipation by bulk viscous pressure being neglected. In our work, that shall be presented in this chapter and corresponds to the main results of this dissertation, we calculated the dispersion relations of the theory and obtained linear causality and stability conditions in terms of the transport coefficients of the theory. This chapter is dedicated to the understanding and detailed reproduction of these results.

3.1 Equations of motion

First, since the net-baryon diffusion current is being taken into account in this analysis, the conservation of charge is no longer trivial, and its evolution must be coupled to the conservation of energy and momentum, to determine the fluid-dynamical evolution of the system. Furthermore, the conservation laws constitute five equations, while they are composed by thirteen independent fields (since bulk is being neglected in this analysis, $\Pi = 0$). Therefore, as it was discussed previously, these conservation laws must be supplemented by equations for the dissipative currents, which are given by the Israel-Stewart equations. In the present analysis, we will consider the Israel-Stewart equations obtained by kinetic theory calculations [41, 64, 65],

$$\begin{aligned}\tau_n \dot{n}^{\langle\mu\rangle} + n^\mu &= \kappa_n \nabla^\mu \alpha_B - n_\nu \omega^{\nu\mu} - \delta_{nn} n^\mu \theta + \ell_{n\pi} \Delta^{\mu\nu} \nabla_\lambda \pi^\lambda_\nu \\ &\quad - \tau_{n\pi} \pi^{\mu\nu} \nabla_\nu P - \lambda_{nn} n_\nu \sigma^{\mu\nu} - \lambda_{n\pi} \pi^{\mu\nu} \nabla_\nu \alpha_B, \end{aligned} \tag{3.1}$$

$$\begin{aligned}\tau_\pi \dot{\pi}^{\langle\mu\nu\rangle} + \pi^{\mu\nu} &= 2\eta \sigma^{\mu\nu} + 2\tau_\pi \pi_\lambda^{\langle\mu} \omega^{\nu\rangle\lambda} - \delta_{\pi\pi} \pi^{\mu\nu} \theta - \tau_{\pi\pi} \pi^{\lambda\langle\mu} \sigma^{\nu\rangle}_\lambda \\ &\quad - \tau_{\pi n} n^{\langle\mu} \nabla^{\nu\rangle} P + \ell_{\pi n} \nabla^{\langle\mu} n^{\nu\rangle} + \lambda_{\pi n} n^{\langle\mu} \nabla^{\nu\rangle} \alpha_B, \end{aligned} \tag{3.2}$$

where $\alpha_B \equiv \mu_B/T$, with μ_B being the baryon chemical potential and T the temperature. The vorticity tensor is defined as $\omega^{\mu\nu} \equiv \left(\nabla^\mu u^\nu - \nabla^\nu u^\mu\right)/2$, and the diffusion-viscous couplings are introduced as $\ell_{n\pi}$ and $\ell_{\pi n}$. We further employ the notation, $A^{\langle\mu\rangle} \equiv \Delta^\mu_\nu A^\nu$, and $A^{\langle\mu\nu\rangle} \equiv \Delta^{\mu\nu}_{\alpha\beta} A^{\alpha\beta}$.

Naturally, in the equations of motion for the dissipative currents presented above, there is the occurrence of several transport coefficients. As it can be concluded from the analysis performed in the previous chapter, in order to obtain a linearly causal and stable formulation, they cannot assume arbitrary values. Moreover, the effects of the transport coefficients can be studied in the *linear* regime, which has been performed in

Refs. [29, 30, 31, 48], where the relaxation times were found to be necessary in order to contemplate causality and lead to linearly stable modes. In particular, this analysis is focused on the understanding of the role the diffusion-viscous couplings $\ell_{n\pi}$ and $\ell_{\pi n}$ play on the linear stability and causality of Israel-Stewart theory. We then obtain constraints for linear causality and stability in terms of the fundamental hydrodynamic variables and display the modes of the theory, a task that had not been done so far.

Here, analogously to the linear stability of Navier-Stokes and Israel-Stewart theories presented in Chapter 2, we consider perturbations around a global equilibrium state with energy density ε_0, vanishing net-baryon number density $n_{B,0} = 0$ and fluid 4-velocity u_0^μ. Naturally, since the system is assumed to be initially in an equilibrium state, the equilibrium values of the dissipative currents are zero, $n_0^\mu = 0$, and $\pi_0^{\mu\nu} = 0$. Wherefore, since bulk viscosity is being neglected in this analysis, the perturbations can be expressed in the form of

$$\varepsilon = \varepsilon_0 + \delta\varepsilon,\ n_B = \delta n_B,\ u^\mu = u_0^\mu + \delta u^\mu,\ n^\mu = \delta n^\mu,\ \pi^{\mu\nu} = \delta\pi^{\mu\nu}. \tag{3.3}$$

The next step is to linearize the fluid-dynamical equations and further compute the modes of the theory. Then we will be able to determine the linear stability and causality conditions of the Israel-Stewart with diffusion-viscous coupling. In this chapter, we employ a slightly different notation when compared to Chapter 2. First, we re-express the linearized conservation laws, Eqs. (2.2), (2.3) and (2.4), in the following form,

$$D_0\left(\frac{\delta\varepsilon}{w_0}\right) + \nabla_0^\mu \delta u_\mu = \mathcal{O}(2) \approx 0, \tag{3.4}$$

$$D_0 \delta u^\mu - \nabla_0^\mu \left(\frac{\delta P}{w_0}\right) + \nabla_0^\nu \delta\chi_\nu^\mu = \mathcal{O}(2) \approx 0, \tag{3.5}$$

$$D_0\left(\frac{\delta n_B}{n_0}\right) + \nabla_0^\mu \delta\xi_\mu = \mathcal{O}(2) \approx 0, \tag{3.6}$$

where we remind the reader that $D_0 \equiv u_0^\mu \partial_\mu$ is the comoving derivative with respect to the background fluid velocity and $\nabla_0^\mu \equiv \Delta_0^{\mu\nu}\partial_\nu$ is the linearized projected derivative. Moreover, $w_0 \equiv \varepsilon_0 + P_0$ is the enthalpy and $n_0 \equiv (\varepsilon_0 + P_0)/4T$ is the particle number density at vanishing chemical potential. The hydrodynamic currents $\delta\chi^{\mu\nu} \equiv \delta\pi^{\mu\nu}/(\varepsilon_0 + P_0)$ and $\delta\xi^\mu \equiv \delta n^\mu/n_0$ were further introduced as the re-scaled dissipative currents of shear-stress tensor and net-baryon diffusion, respectively, in order to work only with dimensionless variables. Furthermore, the equations of motion of Israel-Stewart theory obtained from kinetic theory for the net-charge diffusion, Eq. (3.1), and for the shear-stress tensor, Eq. (3.2), must also be linearized. Then, they are written as

$$\tau_n D_0 \delta\xi^\mu + \delta\xi^\mu = \frac{\bar{n}_B}{n_0}\tau_\kappa \nabla_0^\mu \delta\alpha_B + \mathcal{L}_{n\pi}\nabla_0^\nu \delta\chi_\nu^\mu, \tag{3.7}$$

$$\tau_\pi D_0 \delta\chi^{\mu\nu} + \delta\chi^{\mu\nu} = 2\tau_\eta \Delta_0^{\mu\nu\alpha\beta}\partial_\alpha \delta u_\beta + \mathcal{L}_{\pi n}\Delta_0^{\mu\nu\alpha\beta}\partial_\alpha \delta\xi_\beta, \tag{3.8}$$

where the following transport coefficients were introduced

$$\tau_\kappa \equiv \frac{\kappa_n}{\bar{n}_B}, \;\; \mathcal{L}_{\pi n} \equiv \frac{\ell_{\pi n}}{4T}, \;\; \mathcal{L}_{n\pi} \equiv 4T\ell_{n\pi}, \tag{3.9}$$

with $\bar{n}_B$ being the baryon number density, not to be confused with the net-baryon number density. The last two coefficients, $\mathcal{L}_{\pi n}$ and $\mathcal{L}_{n\pi}$, are second-order terms, and therefore are a feature of Israel-Stewart theory, and they correspond to coupling parameters between the dissipative currents. Since the stability of Israel-Stewart theory has been widely investigated in the linear regime in the absence of coupling terms [28, 29, 30, 31, 48], as already explained in the previous chapter, it is interesting to see whether such formulation remains linearly stable when diffusion-viscous coupling is included.

Analogous to what has been performed to study the linear stability of Navier-Stokes theory and Israel-Stewart theory in the absence of diffusion-viscous coupling, it is convenient to express the linearized fluid-dynamical equations in Fourier space, following the procedure developed in Chapter 2. Naturally, the Fourier transform of the equations of motion were already derived in the last chapter, and thus remain such as defined in Eqs. (2.14), (2.15) and (2.16) – however, note that here we take the background net-baryon number density to zero, which further simplifies Eq. (2.15). We shall rewrite these equations as

$$\Omega\frac{\delta\tilde{\varepsilon}}{w_0} + \kappa^\mu \delta\tilde{u}_\mu = 0, \tag{3.10}$$

$$\Omega\delta\tilde{u}^\mu - \kappa^\mu\frac{\delta\tilde{P}}{w_0} + \kappa^\nu\delta\tilde{\chi}^\mu_\nu = 0, \tag{3.11}$$

$$\Omega\frac{\delta\tilde{n}_B}{n_0} + \kappa^\mu\delta\tilde{\xi}_\mu = 0. \tag{3.12}$$

Furthermore, the equations of motion for the dissipative currents in Fourier space are

$$(i\tau_n\Omega + 1)\delta\tilde{\xi}^\mu = i\frac{\bar{n}_B}{n_0}\tau_\kappa\kappa^\mu\delta\tilde{\alpha}_B + i\mathcal{L}_{n\pi}\kappa_\nu\delta\tilde{\chi}^{\mu\nu}, \tag{3.13}$$

$$(i\tau_\pi\Omega + 1)\delta\tilde{\chi}^{\mu\nu} = 2i\tau_\eta\left[\kappa^{(\mu}\delta\tilde{u}^{\nu)} - \frac{1}{3}\Delta_0^{\mu\nu}\kappa_\lambda\delta\tilde{u}^\lambda\right] + i\mathcal{L}_{\pi n}\left[\kappa^{(\mu}\delta\tilde{\xi}^{\nu)} - \frac{1}{3}\Delta^{\mu\nu}\kappa_\lambda\delta\tilde{\xi}^\lambda\right] \tag{3.14}$$

Once again, as it was done so far, this analysis will also be divided into transverse and longitudinal degrees of freedom by decomposing the perturbations into components in the orthogonal and parallel direction with respect to κ^μ, respectively, following the procedure introduced in Ref. [59]. Here, even in the presence of diffusion-viscous coupling these components can once more be solved independently, which considerably simplifies the solutions. The equations related to the longitudinal modes can be obtained by simply contracting Eqs. (3.11) and (3.13) with the tensor κ^μ/κ, and Eq. (3.14) with $\kappa^\mu\kappa^\nu/\kappa^2$. The equations (3.10) and (3.12), on the other hand, are already expressed in terms of their respective longitudinal components. Thus, the equations related o the longitudinal modes can be summarized as

$$\begin{aligned}
\Omega\frac{\delta\tilde{\varepsilon}}{w_0} - \kappa\delta\tilde{u}_{\parallel} &= 0, &(3.15)\\
\Omega\delta\tilde{u}_{\parallel} - \kappa\frac{\delta\tilde{\varepsilon}}{3w_0} - \kappa\delta\tilde{\chi}_{\parallel} &= 0, &(3.16)\\
\Omega\frac{\delta\tilde{n}_B}{n_0} - \kappa\delta\tilde{\xi}_{\parallel} &= 0, &(3.17)\\
\left(i\hat{\tau}_n\hat{\Omega}+1\right)\delta\tilde{\xi}_{\parallel} + i\hat{\mathcal{L}}_{n\pi}\hat{\kappa}\delta\tilde{\chi}_{\parallel} &= i\hat{\tau}_\kappa\hat{\kappa}\frac{\delta\tilde{n}_B}{n_0}, &(3.18)\\
\left(i\hat{\tau}_\pi\hat{\Omega}+1\right)\delta\tilde{\chi}_{\parallel} - \frac{2}{3}i\hat{\mathcal{L}}_{\pi n}\hat{\kappa}\delta\tilde{\xi}_{\parallel} &= \frac{4}{3}i\hat{\kappa}\delta\tilde{u}_{\parallel}. &(3.19)
\end{aligned}$$

In deriving the above equations, we have already made assumptions regarding the equation of state. We assume an equation of state of a gas composed solely of a massless particle and its corresponding antiparticle. In this case, the perturbation of pressure and chemical potential can be expressed as

$$\begin{aligned}
\delta\tilde{P} &= \frac{1}{3}\delta\tilde{\varepsilon}, &(3.20)\\
\delta\tilde{\alpha}_B &= \frac{\delta\tilde{n}_B}{\bar{n}_B}. &(3.21)
\end{aligned}$$

Overall, the equations related to the longitudinal modes can be written in the following matrix form

$$\begin{pmatrix}
\Omega & 0 & 0 & -\kappa & 0\\
0 & \Omega & -\kappa & 0 & 0\\
0 & -\frac{\kappa}{3} & \Omega & 0 & -\kappa\\
-i\hat{\tau}_\kappa\hat{\kappa} & 0 & 0 & i\hat{\tau}_n\hat{\Omega}+1 & i\hat{\mathcal{L}}_{n\pi}\hat{\kappa}\\
0 & 0 & -\frac{4}{3}i\hat{\kappa} & -\frac{2}{3}i\hat{\mathcal{L}}_{\pi n}\hat{\kappa} & i\hat{\tau}_\pi\hat{\Omega}+1
\end{pmatrix}
\begin{pmatrix}
\delta\tilde{n}_B/n_0\\
\delta\tilde{\varepsilon}/w_0\\
\delta\tilde{u}_{\parallel}\\
\delta\tilde{\xi}_{\parallel}\\
\delta\tilde{\chi}_{\parallel}
\end{pmatrix} = 0. \quad (3.22)$$

Then, the dispersion relation associated to the longitudinal modes is further obtained when the determinant of the matrix on the left-hand side of this equation vanishes, leading to the following equation

$$\left[\left(\hat{\Omega}^2-\frac{1}{3}\hat{\kappa}^2\right)(i\hat{\tau}_\pi\hat{\Omega}+1)-\frac{4}{3}i\hat{\kappa}^2\hat{\Omega}\right]\left[\hat{\Omega}(i\hat{\tau}_n\hat{\Omega}+1)-i\hat{\tau}_\kappa\hat{\kappa}^2\right]-\frac{2}{3}\hat{\mathcal{L}}_{\pi n}\hat{\mathcal{L}}_{n\pi}\left(\hat{\Omega}^2-\frac{1}{3}\hat{\kappa}^2\right)\hat{\Omega}\hat{\kappa}^2 = 0. \quad (3.23)$$

Therefore, the next step is to calculate the equations related to the transverse modes of the theory, which are obtained projecting Eqs. (3.11) and (3.13) with $\Delta^{\kappa}_{\mu\lambda}$ and Eq. (3.14) with $\Delta^{\kappa}_{\mu\lambda}\kappa_\nu$. As it was previously mentioned, Eqs. (3.10) and (3.12) are already expressed in terms of the longitudinal components and hence their transverse components are zero. In this case, we obtain the following equations

$$\begin{aligned}
\hat{\Omega}\delta\tilde{u}^{\lambda}_{\perp} - \hat{\kappa}\delta\tilde{\chi}^{\lambda}_{\perp} &= 0, &(3.24)\\
(i\hat{\tau}_n\hat{\Omega}+1)\delta\tilde{\xi}^{\lambda}_{\perp} + i\hat{\mathcal{L}}_{n\pi}\hat{\kappa}\delta\tilde{\chi}^{\lambda}_{\perp} &= 0, &(3.25)\\
(i\hat{\tau}_\pi\hat{\Omega}+1)\delta\tilde{\chi}^{\lambda}_{\perp} - i\hat{\kappa}\delta\tilde{u}^{\lambda}_{\perp} - i\frac{\hat{\mathcal{L}}_{\pi n}}{2}\hat{\kappa}\delta\tilde{\xi}^{\lambda}_{\perp} &= 0. &(3.26)
\end{aligned}$$

As it was done for the equations related to the longitudinal modes, these equations can also be expressed in the following matrix form

$$\begin{pmatrix} \hat{\Omega} & -\hat{\kappa} & 0 \\ 0 & i\hat{\mathcal{L}}_{n\pi}\hat{\kappa} & i\hat{\Omega}\hat{\tau}_n + 1 \\ -i\hat{\kappa} & i\hat{\tau}_\pi\hat{\Omega} + 1 & -i\frac{\hat{\mathcal{L}}_{\pi n}}{2}\hat{\kappa} \end{pmatrix} \begin{pmatrix} \delta\tilde{u}^\mu_\perp \\ \delta\tilde{\chi}^\mu_\perp \\ \delta\tilde{\xi}^\mu_\perp \end{pmatrix} = 0, \tag{3.27}$$

hence leading to the dispersion relation for the transverse modes of the theory

$$\left[\left(1 + i\hat{\tau}_\pi\hat{\Omega}\right)\hat{\Omega} - i\hat{\kappa}^2\right]\left(1 + i\hat{\tau}_n\hat{\Omega}\right) - \frac{1}{2}\hat{\mathcal{L}}_{\pi n}\hat{\mathcal{L}}_{n\pi}\hat{\Omega}\hat{\kappa}^2 = 0. \tag{3.28}$$

Here, the reader may ask why only the partially transverse component of the shear-stress tensor was considered in this derivation. This is due to the fact that the fully transverse component, given by $\delta\tilde{\chi}^{\mu\nu}_\perp$, decouples from the perturbations related to energy density and velocity fluctuations and its dynamics can be solved independently. Without loss of generality, this contribution is further neglected in this analysis.

Furthermore, the effect of the diffusion-viscous coupling terms can be clearly seen when analyzing the dispersion relation for the longitudinal and transverse modes, Eqs. (3.23) and (3.28), respectively. In both cases, the dispersion relations related to fluctuations of net-charge diffusion current and shear-stress tensor are *coupled*, and thus cannot be solved independently, which naturally is the case when the coupling terms are set to zero. Therefore, the addition of diffusion-viscous couplings in the equation for the dissipative currents naturally leads to more complicated solutions, which will be analyzed in detail throughout this chapter. Throughout this work, we shall assume, unless stated otherwise, that

$$\mathcal{L}_{n\pi}\mathcal{L}_{\pi n} < 0, \tag{3.29}$$

which is supported by kinetic theory calculations [34, 64, 65]. Furthermore, this constraint is obtained in the phenomenological derivation of Israel-Stewart theory from the second law of thermodynamics [34, 67].

This chapter will be divided in two parts: the first is dedicated to analyze the linear stability and causality of Israel-Stewart theory in the presence of net-charge diffusion current while neglecting diffusion-viscous coupling. We further recover linear stability conditions for the transport coefficients related to shear-stress derived previously, and further obtain a new set of constraints related to transport coefficients associated to net-charge diffusion. In the second part, the linear stability of the Israel-Stewart is analyzed taking into account diffusion-viscous coupling. Then, we find new constraints for the transport coefficients, in particular for the product of the coupling terms, which is the main result of this dissertation and has been published in Ref. [59].

3.2 Linear stability analysis in the absence of coupling terms

As it was done in the previous sections, we shall begin studying the perturbations on the transverse degrees of freedom of the Israel-Stewart theory, and then proceeding to the analysis of the perturbations on its longitudinal degrees of freedom.

Transverse modes

The dispersion relation associated to the transverse modes of Israel-Stewart in the absence of diffusion-viscous coupling can be straightforwardly obtained by simply taking $\hat{\mathcal{L}}_{n\pi} = \hat{\mathcal{L}}_{\pi n} = 0$ in Eq. (3.28). In this case, we have

$$\left[\left(1 + i\hat{\tau}_\pi\hat{\Omega}\right)\hat{\Omega} - i\hat{\kappa}^2\right]\left(1 + i\hat{\tau}_n\hat{\Omega}\right) = 0. \tag{3.30}$$

Here, unlike what is observed in the dispersion relation including the coupling terms, Eq. (3.28), the equations related to fluctuations on energy and momentum and net-charge diffusion are clearly decoupled, and can be solved separately, as it was previously mentioned. Thus, we have to solve the following independent equations

$$\left(1 + i\hat{\tau}_\pi\hat{\Omega}\right)\hat{\Omega} - i\hat{\kappa}^2 = 0, \tag{3.31}$$

$$1 + i\hat{\tau}_n\hat{\Omega} = 0. \tag{3.32}$$

This analysis shall begin with the case where perturbations are performed over a static fluid. In this scenario, the background fluid velocity is given by its local rest frame, $u_0^\mu = (1, 0, 0, 0)$, leading simply to $\Omega = \omega$ and $\kappa = k$. Thus, the solutions of these expression are written, respectively, as

$$\hat{\omega}^{\text{shear}}_{T,\pm} = i\frac{1 \pm \sqrt{1 - 4\hat{\tau}_\pi\hat{k}^2}}{2\hat{\tau}_\pi}, \tag{3.33}$$

$$\hat{\omega}^{\text{diff}}_{T} = \frac{i}{\hat{\tau}_n}. \tag{3.34}$$

Here one can immediately note the occurrence of two non-hydrodynamic modes, given by $\hat{\omega}^{\text{shear}}_{T,+}$ and $\hat{\omega}^{\text{diff}}_{T}$, and one hydrodynamic mode, defined as $\hat{\omega}^{\text{shear}}_{T,-}$. In fact, the modes $\hat{\omega}^{\text{shear}}_{T,\pm}$ were previously obtained in the linear stability analysis of Israel-Stewart theory considering only dissipation via shear-stress, see Eq. (2.64) in Sec. 2.2. Wherefore, the linear causality condition for such modes, previously introduced in Eq. (2.68) as $\hat{\tau}_\pi \geq 1$, as first derived in Ref. [30], remains valid also in the presence of net-charge diffusion but without taking into account diffusion-viscous coupling. Furthermore, since in the present analysis net-charge diffusion current is also accounted, there is the occurrence of another transient mode, given by $\hat{\omega}^{\text{diff}}_{T}$, related to net-charge diffusion fluctuations, which is linearly stable as long as the diffusion relaxation time is positive,

$$\tau_n \geq 0. \tag{3.35}$$

Naturally, since this mode carries no dependence on the wavenumber k, and the relaxation times are positive definite transport coefficients, this mode is linearly stable for perturbations on a static background fluid. On top of that, it is a purely imaginary mode, and hence has no propagating part, carrying no contribution for linear causality. Although the modes $\hat{\omega}_{T,\pm}^{\text{shear}}$ have already been displayed in Fig. 3, it is rather interesting to display them one more time, now simultaneously with the mode which is the product of fluctuations in the net-baryon current, defined as $\hat{\omega}_{T}^{\text{diff}}$, for the sake of comparison, in Fig. 7. Once again the values of the shear and diffusion relaxation times employed here were obtained from kinetic theory calculations using the 14-moment approximation in the ultra-relativistic limit, $\hat{\tau}_\pi = 5$ and $\hat{\tau}_n = 27/4$ [64].

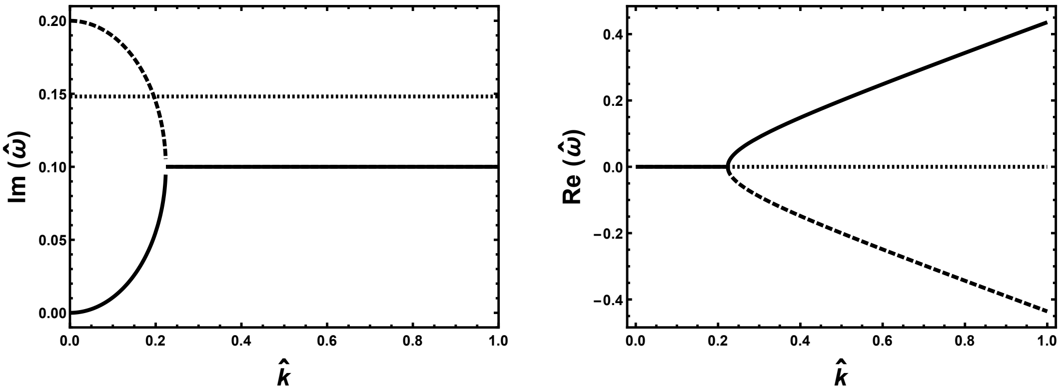

Figure 7 – Imaginary and real parts of the transverse modes of the Israel-Stewart theory in the absence of diffusion-viscous coupling for perturbations on a static background fluid considering $\hat{\tau}_\pi = 5$ and $\hat{\tau}_n = 27/4$.

As it was pointed out several times throughout this work, unstable modes usually occur for perturbations on top of a moving fluid. Once again, for the sake of convenience, we shall assume that the background fluid velocity is parallel to the wave-vector. In this scenario, the linear causality and stability of the modes $\hat{\omega}_{T,\pm}^{\text{shear}}$, the solutions of Eq. (3.31), was already investigated in Sec. 2.2, leading to the linear stability condition given by Eq. (2.71) as $\hat{\tau}_\pi \geq 1$, which is identical to the linear stability causality for perturbations on a static background fluid. On the other hand, Eq. (3.32) was not present in the previous analysis and shall be further investigated in detail. In this case, it then reads

$$1 + i\gamma\hat{\tau}_n(\hat{\omega} - V\hat{k}) = 0, \tag{3.36}$$

which has the following solution

$$\hat{\omega}_{T}^{\text{diff}} = V\hat{k} + \frac{i}{\gamma\hat{\tau}_n}. \tag{3.37}$$

This mode has a transient part, that relaxes to equilibrium with the Lorentz-dilated diffusion relaxation time $\gamma\hat{\tau}_n$, while it propagates with a group velocity that is equal to the background fluid velocity V. Naturally, it is linearly causal for subluminal values of the background fluid velocity and linearly stable. Overall, the transverse mode related to net-charge fluctuations does not have any contribution to the theory's linear causality and stability. On the other hand, the linear causality and stability conditions for the modes related to fluctuations of the energy-momentum tensor remain unchanged when including net-charge diffusion in the absence of diffusion-viscous coupling, due to the fact that both of these dispersion relations decouple in such case. These modes are compared in Fig. 8, where the solutions of Eq. (3.30) are displayed for three different values of background velocity, $V = 0.1$, $V = 0.4$, and $V = 0.9$, and using the same values for the relaxation times as before, $\hat{\tau}_\pi = 5$ and $\hat{\tau}_n = 27/4$.

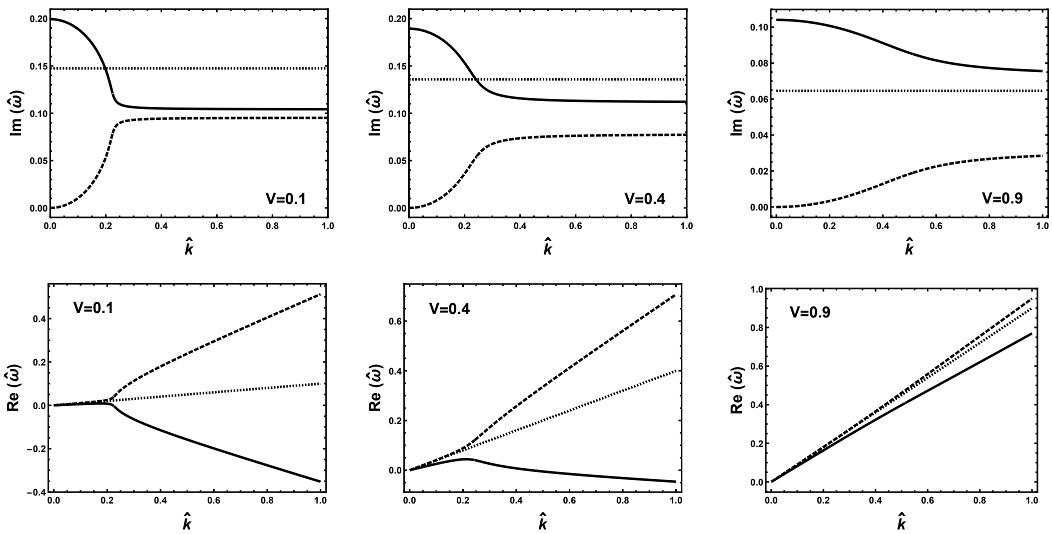

Figure 8 – Imaginary and real parts of the transverse modes of Israel-Stewart theory in the absence of diffusion-viscous coupling for perturbations around a moving fluid, considering $V = 0.1$, $V = 0.4$, and $V = 0.9$, in the absence of coupling terms, considering $\hat{\tau}_\pi = 5$ and $\hat{\tau}_n = 27/4$.

Furthermore, one example of an unstable fluid configuration is shown in Fig. 9, where the mode $\hat{\omega}_+^{\text{shear}}$ is displayed considering $\hat{\tau}_\pi = 0.5$, a value that does not satisfy the linear causality and stability conditions derived in Section 2.2 for the transverse modes.

Longitudinal modes

In the absence of coupling between the dissipative currents, the dispersion relation associated to the longitudinal modes of Israel-Stewart theory, Eq. (3.23), takes the form

$$\left[\left(\hat{\Omega}^2 - \frac{1}{3}\hat{\kappa}^2\right)(i\hat{\tau}_\pi\hat{\Omega} + 1) - \frac{4}{3}i\hat{\kappa}^2\hat{\Omega}\right]\left[\hat{\Omega}(i\hat{\tau}_n\hat{\Omega} + 1) - i\hat{\tau}_\kappa\hat{\kappa}^2\right] = 0. \tag{3.38}$$

As it was first observed for the transverse modes, the dispersion relations associated to energy-momentum and net-baryon current fluctuations decouple, since there is no

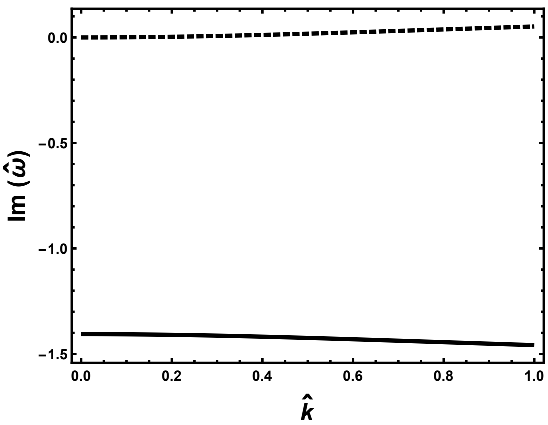

Figure 9 – The imaginary part of the unstable shear mode for $\hat{\tau}_\pi = 0.5$ for a background velocity fluid $V = 0.9$, in the absence of coupling terms, considering $\hat{\tau}_\pi = 5$.

diffusion-viscous coupling. In this case, these equation are solved separately as

$$\hat{\Omega}(i\hat{\tau}_n\hat{\Omega} + 1) - i\hat{\tau}_\kappa\hat{\kappa}^2 = 0, \tag{3.39}$$

$$\left(\hat{\Omega}^2 - \frac{1}{3}\hat{\kappa}^2\right)(i\hat{\tau}_\pi\hat{\Omega} + 1) - \frac{4}{3}i\hat{\kappa}^2\hat{\Omega} = 0. \tag{3.40}$$

The dispersion relation associated to energy-momentum tensor fluctuations, Eq. (3.40), was previously obtained in Eq. (2.76), when the Israel-Stewart theory considering only dissipation via shear-stress tensor was analyzed, in Sec. 2.2. In that case, the linear causality and stability of the modes was carefully investigated, which further lead to the condition given by $\hat{\tau}_\pi \geq 2$, see Eqs. (2.83) and (2.86), respectively. Therefore, since the inclusion of net-charge diffusion current in the dissipative currents without the presence of diffusion-viscous couplings does not change the modes associated to energy and momentum fluctuations, this derivation will not be revisited here, and we shall focus on the modes that are product of net-baryon current fluctuations and the linear stability and causality condition related to them.

The analysis once again starts with the case where we consider perturbations on a static background fluid. For the sake of illustration, the imaginary and real parts of the longitudinal modes are displayed in Fig. 10, taking $\hat{\tau}_\pi = 5$, $\hat{\tau}_n = 27/4$ and $\hat{\tau}_\kappa = 9/16$ [64]. In this scenario, the solutions of Eq. (3.39) are written simply as

$$\omega^{\mathrm{B}}_{L,\pm} = i\frac{1 \pm \sqrt{1 - 4\hat{\tau}_n\hat{\tau}_\kappa\hat{k}^2}}{2\hat{\tau}_n}. \tag{3.41}$$

Clearly, there is the occurrence of a hydrodynamic mode and a non-hydrodynamic mode, $\omega^{\mathrm{B}}_{L,-}$ and $\omega^{\mathrm{B}}_{L,+}$, respectively. In the small wavenumber limit, these modes can be written such as

$$\omega^{\mathrm{B}}_{L,-} = i\hat{\tau}_\kappa\hat{k}^2 + i\hat{\tau}_n\hat{\tau}_\kappa^2\hat{k}^4 + \mathcal{O}\left(\hat{k}^6\right), \tag{3.42}$$

$$\omega^{\mathrm{B}}_{L,+} = \frac{i}{\hat{\tau}_n} - i\hat{\tau}_\kappa\hat{k}^2 + \mathcal{O}\left(\hat{k}^4\right). \tag{3.43}$$

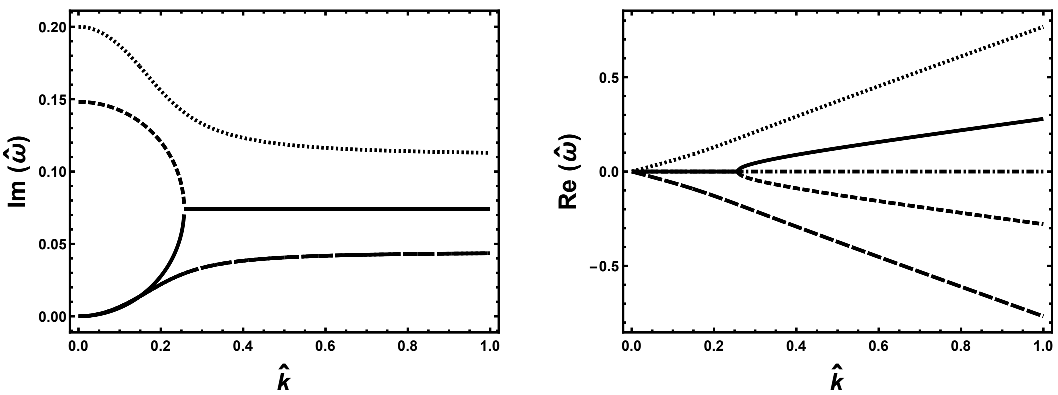

Figure 10 – Imaginary and real parts of the longitudinal modes of the Israel-Stewart theory in the absence of diffusion-viscous coupling for perturbations on a static background fluid considering $\hat{\tau}_\pi = 5$, $\hat{\tau}_n = 27/4$ and $\hat{\tau}_\kappa = 9/16$.

It is straightforwardly possible to conclude that both modes are linearly stable for perturbations on a static background fluid, since the leading terms are positive definite. Note that the leading term of the hydrodynamic mode $\omega^{\mathrm{B}}_{L,-}$ is the dispersion related usually obtained in Navier-Stoke theory, while the non-hydrodynamic mode $\omega^{\mathrm{B}}_{L,+}$ does not exist in this formulation. Furthermore, no information regarding the linear causality of these modes can be inferred from this expansion. In order to be able to extract any information on their linear causality, we must analyze the asymptotic group velocity by looking at their behavior at the large wavenumber limit. In this case,

$$\omega^{\mathrm{B}}_{L,\pm} = \frac{i}{2\hat{\tau}_n} \pm \sqrt{\frac{\hat{\tau}_\kappa}{\hat{\tau}_n}} \left(\hat{k} - \frac{1}{8\hat{\tau}_n \hat{\tau}_\kappa \hat{k}} \right) + \mathcal{O}\left(\frac{1}{\hat{k}^3} \right). \tag{3.44}$$

The modes $\omega^{\mathrm{diff}}_{L,\pm}$ become propagating when the wavenumber k is greater than a particular value, given by $\hat{k} > 1/\left(2\sqrt{\hat{\tau}_n \hat{\tau}_\kappa}\right)$, which leads to the occurrence of a real part on both of these modes. Furthermore, causality dictates that the asymptotic group velocity must be subluminal [66], leading to the following constraint

$$\lim_{k\to\infty} \left| \frac{\partial \mathrm{Re}(\omega)}{\partial k} \right| \leq 1 \Longrightarrow \hat{\tau}_n \geq \hat{\tau}_\kappa. \tag{3.45}$$

We then look at these modes when considering perturbations on top of a moving fluid, in order to obtain the linear stability conditions that such modes must satisfy. In this case, Eq. (3.39) reads

$$\left(\gamma\hat{\omega} - \gamma V \hat{k}\right)\left[i\hat{\tau}_n \left(\gamma\hat{\omega} - \gamma V \hat{k}\right) + 1\right] - i\hat{\tau}_\kappa \left(\hat{\omega}\gamma V - \gamma\hat{k}\right)^2 = 0. \tag{3.46}$$

An analysis of these modes for arbitrary values of wavenumber k can be extremely complicated, and therefore shall not be explored in this work. Instead, for the sake of simplicity, as it was performed for the previous cases, we look at the modes in the vanishing wavenumber limit, which further supplies necessary, even though not always sufficient, linear stability conditions. Thus, since $\omega^{\mathrm{B}}_{L,-}$ is a hydrodynamic mode, it vanishes in such limit, while the non-hydrodynamic mode $\omega^{\mathrm{B}}_{L,+}$ does not. In this scenario, these modes read

$$\omega^{\mathrm{B}}_{L,-} = 0, \;\; \omega^{\mathrm{B}}_{L,+} = \frac{i}{\gamma\left(\hat{\tau}_n - \hat{\tau}_\kappa V^2\right)}. \tag{3.47}$$

Naturally, these modes must be linearly stable for any value of the background velocity V, and the stronger constraint arises when it assumes the maximal value allowed by causality. The mode $\omega^{\mathrm{B}}_{L,-}$ is trivial and hence always stable, while the mode $\omega^{\mathrm{B}}_{L,+}$ must satisfy the following linear stability condition in order to satisfy linear stability

$$\hat{\tau}_n \geq \hat{\tau}_\kappa. \tag{3.48}$$

This stability condition is equivalent to the linear causality condition, Eq. (3.45), which is in agreement with the connection between the causality and stability of fluid dynamics, first developed by Pu *et al.* in Ref. [30].

Therefore, whereas the linear causality and stability conditions obtained for the modes from energy and momentum fluctuations are unchanged for vanishing diffusion-viscous coupling, the constraints obtained for the diffusion relaxation time, the linear causality and stability conditions related to the modes associated to net-charge fluctuations, given by Eqs. (3.45) and (3.48), respectively, are novel constraints. We published these results for the first time in Ref. [59].

The modes that are obtained as the solutions of Eq. (3.38) for a moving background fluid are displayed in Fig. 11, considering $\hat{\tau}_\pi = 5$, $\hat{\tau}_n = 27/4$, and $\hat{\tau}_\kappa = 9/16$, for three values of the background velocity, $V = 0.1$, $V = 0.4$, and $V = 0.9$.

Further, in Fig. 12, two examples of unstable fluid configurations, i.e., fluids that do not satisfy the linear stability conditions derived in Eqs. (2.86) and (3.48) are displayed. On the left panel we analyze an unstable case in which $\hat{\tau}_n < \hat{\tau}_\kappa$. Here, we considered $\hat{\tau}_n = 3/16$ and $\hat{\tau}_\kappa = 9/16$ for an unperturbed system with velocity $V = 0.9$. In this scenario, the longitudinal non-hydrodynamic mode related to net-baryon current fluctuations is linearly unstable. On the right panel, we analyze the case where $\hat{\tau}_\pi = 0.9$ for an unperturbed system with velocity $V = 0.9$. Again, there is the occurrence of a linearly unstable non-hydrodynamic mode, related to fluctuations of the shear-stress tensor.

Throughout this chapter, the linear stability and causality conditions for the shear relaxation time were recovered and found to be unchanged, while novel constraints for the diffusion relaxation time were first derived. In particular, we showed that the causality conditions obtained for perturbations on a static background fluid are identical to the stability conditions obtained for perturbations on top of a moving background fluid. Nevertheless, these conditions are valid in the regime where there is no coupling between these two dissipative currents. The next step is to investigate the values such transport

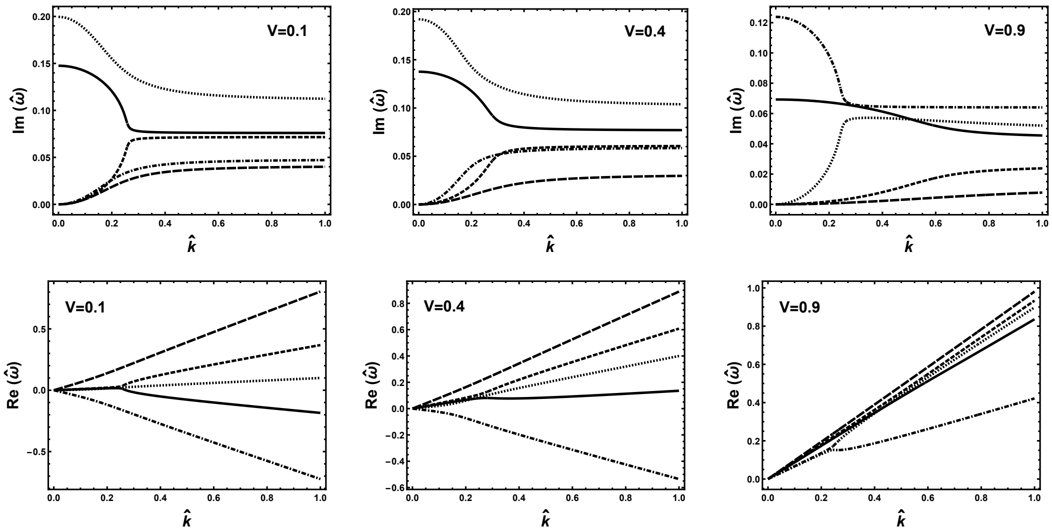

Figure 11 – Imaginary and real parts of the longitudinal modes for perturbations around a moving fluid, considering $V = 0.1$, $V = 0.4$, and $V = 0.9$, in the absence of coupling terms, considering $\hat{\tau}_\pi = 5$, $\hat{\tau}_n = 27/4$, and $\hat{\tau}_\kappa = 9/16$.

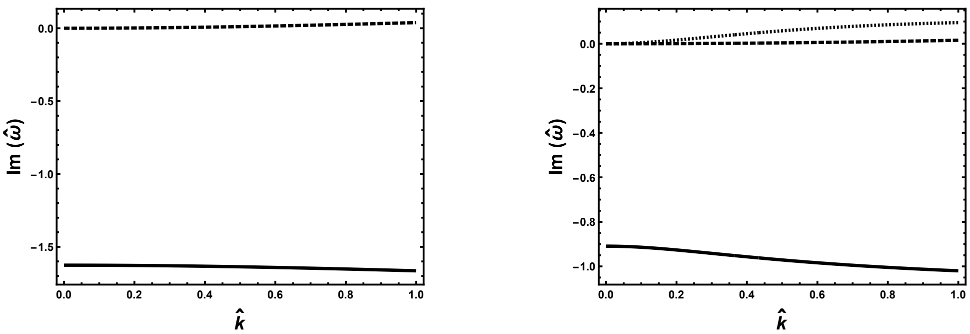

Figure 12 – Imaginary part of the unstable longitudinal mode related to baryon-number fluctuations for $\hat{\tau}_\kappa = 9/16$ and $\hat{\tau}_n = 3/16$ (left panel) and imaginary part of the unstable shear mode for $\hat{\tau}_\pi = 0.9$ (right panel). In both cases we consider $V = 0.9$.

coefficients can assume in order to have linearly stable modes when taking into account diffusion-viscous coupling.

3.3 Linear stability analysis in the presence of coupling terms

The linear stability and causality of the modes that are the solutions of the dispersion relation related to energy-momentum tensor fluctuations were suppressed in the last section, since they were carefully analyzed back in Sec. 2.2 and remain unchanged in the absence of diffusion-viscous coupling. However, when diffusion-viscous coupling is turned on, the dispersion relation associated to energy-momentum tensor and net-baryon number fluctuations can no longer be factorized. Instead, they are actually connected by the so called coupling terms. This further leads to novel constraints for these transport coefficients, which will be analyzed in detail in this section. We then analyze the possible values the product of the coupling terms can assume in order to render linearly causal and stable modes.

Transverse modes

The dispersion relation for the transverse modes of the Israel-Stewart theory in the presence of coupling, shown in Eq. (3.28), can be conveniently expressed in the following form

$$-\mathcal{A}\hat{\Omega}^3 + i\mathcal{B}\hat{\Omega}^2 + \left(1 + \mathcal{C}\hat{\kappa}^2\right)\hat{\Omega} - i\hat{\kappa}^2 = 0, \tag{3.49}$$

where we defined the quantities

$$\mathcal{A} \equiv \hat{\tau}_\pi \hat{\tau}_n, \tag{3.50}$$

$$\mathcal{B} \equiv \hat{\tau}_n + \hat{\tau}_\pi, \tag{3.51}$$

$$\mathcal{C} \equiv \hat{\tau}_n - \frac{1}{2}\hat{\mathcal{L}}_{\pi n}\hat{\mathcal{L}}_{n\pi}. \tag{3.52}$$

One can immediately note that once the coupling terms are included, the dispersion relation associated to energy-momentum tensor fluctuations and net-baryon current fluctuations no longer factorize, leading to new – and *coupled* – solutions. Even though such addition renders essentially more complicated solutions, it does not increase the number of modes of the theory. However, the modes now also carry a dependence on the product of the coupling terms, included within the definition of the variable $\mathcal{C}$. Moreover, an interesting case to be observed is the Navier-Stokes limit, where the shear and diffusion relaxation times are set to zero. In this case, one no longer recovers a simple Navier-Stokes dispersion relation. Instead, one obtains the solution

$$\hat{\Omega} = \frac{i\hat{\kappa}^2}{1 - \frac{1}{2}\hat{\mathcal{L}}_{\pi n}\hat{\mathcal{L}}_{n\pi}\hat{\kappa}^2}. \tag{3.53}$$

Naturally, the modes are always linearly stable if the product of the coupling terms is negative. In fact, this was an assumption we took initially. The modes in the Navier-Stokes limit are displayed in Fig. 13 considering a positive, a negative and a vanishing product of the coupling terms for perturbations on a static background fluid. It is possible to conclude that if the product of the coupling terms is zero, one recovers the dispersion relation associated to the transverse modes of the relativistic Navier-Stokes theory, which is linearly stable for perturbations on a static background fluid. Furthermore, one can straightforwardly note that for a positive value of the product of the coupling terms, there is the occurrence of a mode that is stable in the small wavenumber limit but at one point becomes unstable as the value of k is increased.

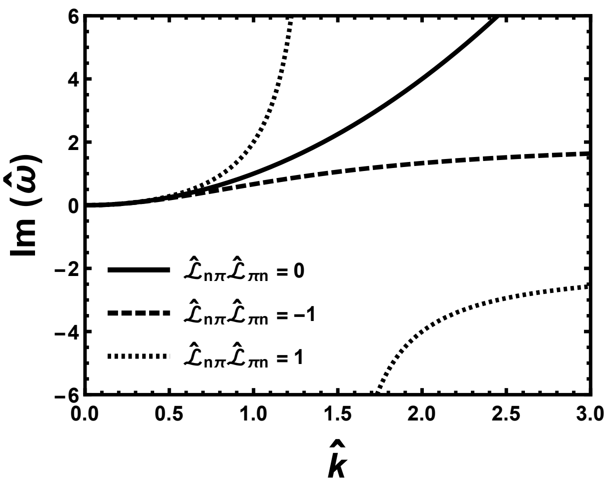

Figure 13 – Imaginary part of the transverse mode in the Navier-Stokes limit, i.e., considering $\hat{\tau}_\pi = \hat{\tau}_n = 0$ for three different values for the product of the coupling terms, $\hat{\mathcal{L}}_{\pi n}\hat{\mathcal{L}}_{n\pi} = -1$, $\hat{\mathcal{L}}_{\pi n}\hat{\mathcal{L}}_{n\pi} = 0$ and $\hat{\mathcal{L}}_{\pi n}\hat{\mathcal{L}}_{n\pi} = 1$.

In the absence of relaxation times, the linear stability of the theory's only mode is guaranteed by a negative value for the product of the coupling terms. However, the relaxation times are essential transport coefficients to render a causal fluid-dynamical formulation [15, 18]. In this sense, although the consistency of the assumptions aforementioned were checked to be valid in the Navier-Stokes limit, the linear causality and stability must be analyzed for non-zero values of the relaxation times. Following the same construction employed so far in this work, we begin looking at the modes for perturbations on a static background fluid. The dispersion relation associated to the transverse modes in this case reads

$$-\mathcal{A}\hat{\omega}^3 + i\mathcal{B}\hat{\omega}^2 + \left(1 + \mathcal{C}\hat{k}^2\right)\hat{\omega} - i\hat{k}^2 = 0. \tag{3.54}$$

We first study these solutions in two different limits: for small and large wavenumber. In the small wavenumber limit, the modes can be written as

$$\hat{\omega}^{\text{shear}}_{T,+} = \frac{i}{\hat{\tau}_\pi} + \frac{i}{2}\frac{2\left(\hat{\tau}_n - \hat{\tau}_\pi\right) - \hat{\mathcal{L}}_{\pi n}\hat{\mathcal{L}}_{n\pi}}{\hat{\tau}_\pi - \hat{\tau}_n}\hat{k}^2 + \mathcal{O}\left(\hat{k}^4\right), \tag{3.55}$$

$$\hat{\omega}^{\text{shear}}_{T,-} = i\hat{k}^2 + \mathcal{O}\left(\hat{k}^4\right), \tag{3.56}$$

$$\hat{\omega}^{\text{diff}}_{T} = \frac{i}{\hat{\tau}_n} + \frac{i\hat{\mathcal{L}}_{\pi n}\hat{\mathcal{L}}_{n\pi}}{2(\hat{\tau}_\pi - \hat{\tau}_n)}\hat{k}^2 + \mathcal{O}\left(\hat{k}^4\right). \tag{3.57}$$

Taking the product of the coupling terms to zero, one straightforwardly recovers Eqs. (2.65), (2.66) and (3.34), respectively. Therefore, the addition of the coupling terms, $\hat{\mathcal{L}}_{\pi n}\hat{\mathcal{L}}_{n\pi}$, can be understood to lead to contributions of higher order in the wavenumber k. Furthermore, in the large wavenumber limit, these modes can be expressed as

$$\hat{\omega}_{T,\pm}^{\text{shear}} = \pm\sqrt{\frac{\mathcal{C}}{\mathcal{A}}}\hat{k} + i\frac{\mathcal{BC}-\mathcal{A}}{2\mathcal{AC}} + O\left(\frac{1}{\hat{k}}\right), \quad (3.58)$$

$$\hat{\omega}_{T}^{\text{diff}} = \frac{i}{\mathcal{C}} + O\left(\frac{1}{\hat{k}}\right). \quad (3.59)$$

So far, analyzing the modes for perturbations on a static fluid at the large wavenumber limit have provided linear causality conditions, while the stability conditions arise from the analysis of perturbations on top of a moving fluid. However, in this case we are able to extract information not only on the linear causality of the modes, but also on their linear stability by looking at their behavior at large values of k. In this scenario, in order to have stable modes for perturbations on a static fluid, these conditions must be satisfied: $\mathcal{C}$ is real and positive and $\mathcal{BC}-\mathcal{A}>0$. Hence, we obtain the following linear stability conditions

$$\mathcal{C} > 0 \Longrightarrow \hat{\mathcal{L}}_{\pi n}\hat{\mathcal{L}}_{n\pi} < 2\hat{\tau}_n, \quad (3.60)$$

$$\mathcal{BC}-\mathcal{A} > 0 \Longrightarrow \hat{\mathcal{L}}_{\pi n}\hat{\mathcal{L}}_{n\pi} < 2\frac{\hat{\tau}_n^2}{\hat{\tau}_n+\hat{\tau}_\pi}. \quad (3.61)$$

These constraints are identical to the linear stability conditions obtained using the Routh-Hurwitz criterion [68, 69, 70], hence being valid for any value of wavenumber k, not only in the large wavenumber limit, as they were derived here. The condition given by Eq. (3.61) is the strongest between the two of them and imposes restrictions on the values that the coupling terms can assume. Furthermore, unlike the case without diffusion-viscous coupling, where the linear stability of Israel-Stewart theory for perturbations around a background at rest is always guaranteed, now there is an inequality the product of the coupling terms must satisfy in order to ensure the linear stability of the modes in such scenario. Moreover, a linear causality condition can be extracted from the expansion of the modes in the large wavenumber limit, leading to

$$\lim_{k\to\infty}\left|\frac{\partial \text{Re}(\omega)}{\partial k}\right| = \frac{\mathcal{C}}{\mathcal{A}} \leq 1 \Longrightarrow \hat{\mathcal{L}}_{\pi n}\hat{\mathcal{L}}_{n\pi} \geq -2\hat{\tau}_n\left(\hat{\tau}_\pi - 1\right). \quad (3.62)$$

For the sake of illustration, the modes $\hat{\omega}_{T,\pm}^{\text{shear}}$ and $\hat{\omega}_{T}^{\text{diff}}$ are displayed in Fig. 14 for negative values of the product of the coupling terms, $\hat{\mathcal{L}}_{\pi n}\hat{\mathcal{L}}_{n\pi} = -0.25, -1, -4$, and in Fig. 15 for positive values of it, $\hat{\mathcal{L}}_{\pi n}\hat{\mathcal{L}}_{n\pi} = 0.25, 2, 6$, both cases for a static background, $V=0$.

We note that it is possible to obtain stable modes even for positive values of $\hat{\mathcal{L}}_{\pi n}\hat{\mathcal{L}}_{n\pi}$, see Eq. (3.61). As the value of $\hat{\mathcal{L}}_{\pi n}\hat{\mathcal{L}}_{n\pi}$ becomes negative, the coupling terms affect the imaginary parts of the non-hydrodynamic modes to degenerate at larger values of wavenumber. On the other hand, if $\hat{\mathcal{L}}_{\pi n}\hat{\mathcal{L}}_{n\pi}$ is positive, the non-hydrodynamic mode related to the diffusion 4-current, $\hat{\omega}_{T}^{\text{diff}}$, becomes degenerate with the hydrodynamic mode $\hat{\omega}_{T,-}^{\text{shear}}$

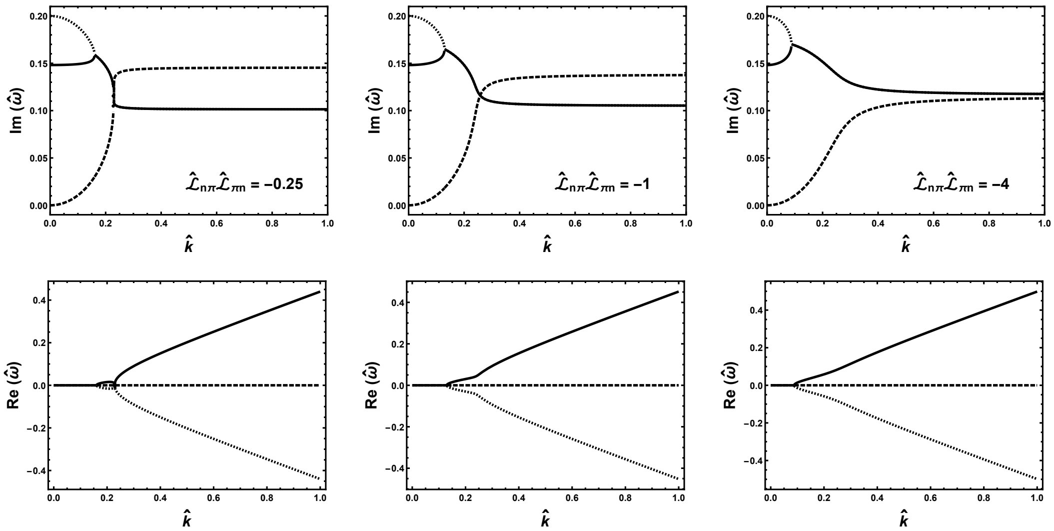

Figure 14 – Imaginary and real parts of the transverse modes considering $\hat{\tau}_\pi = 5$ and $\hat{\tau}_n = 27/4$ for three negative values of the product of the coupling terms, $\hat{\mathcal{L}}_{\pi n}\hat{\mathcal{L}}_{n\pi} = -0.25, -1, -4$, for a static background, $V = 0$.

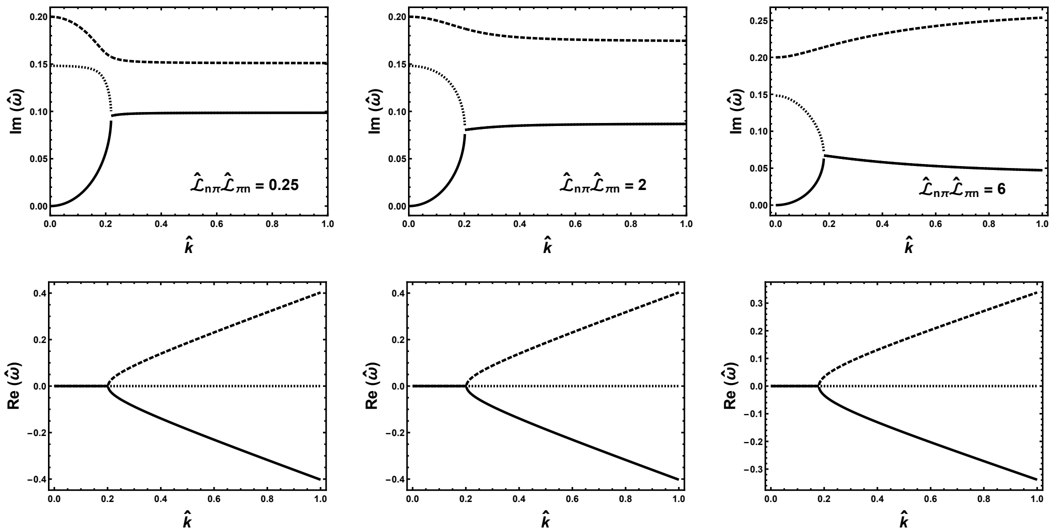

Figure 15 – Imaginary and real parts of the transverse modes considering $\hat{\tau}_\pi = 5$ and $\hat{\tau}_n = 27/4$ for three positive values of the product of the coupling terms, $\hat{\mathcal{L}}_{\pi n}\hat{\mathcal{L}}_{n\pi} = 0.25, 2, 6$, for a static background, $V = 0$.

instead, when the wavenumber increases. In Fig. 16, we show a case in which the modes are driven linearly unstable by a positive product of the coupling terms, $\hat{\mathcal{L}}_{\pi n}\hat{\mathcal{L}}_{n\pi} = 10$, even though the relaxation times employed are the ones used so far, calculated by Boltzmann equation [64] that fulfill the linear causality and stability conditions derived in this section.

Then, the next step is to look at the modes for perturbations on top of a moving background fluid. Once again, we take the background fluid velocity to be in the same direction of the wave-vector. The transverse modes are displayed in Fig. 17 for a negative

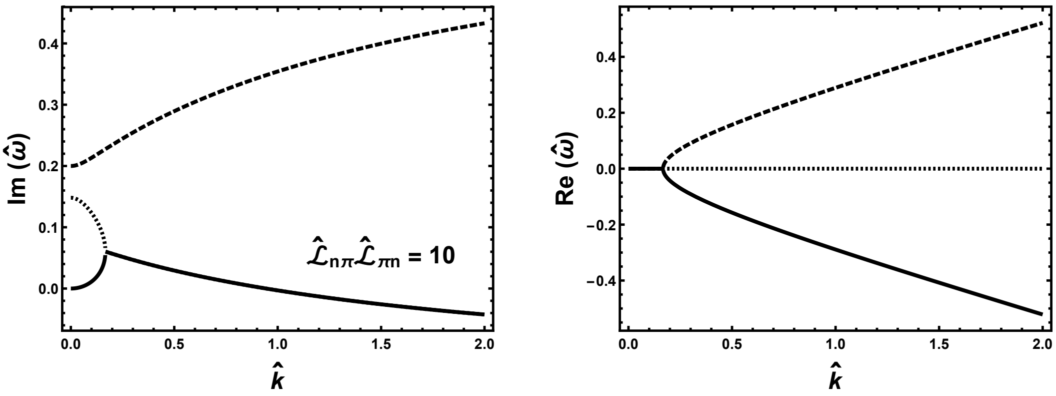

Figure 16 – Imaginary and real parts of the transverse modes considering $\hat{\tau}_\pi = 5$ and $\hat{\tau}_n = 27/4$ for an unstable value for the product of the coupling terms, $\hat{\mathcal{L}}_{\pi n}\hat{\mathcal{L}}_{n\pi} = 10$, for a static background, $V = 0$.

value of the product of the coupling terms, $\hat{\mathcal{L}}_{\pi n}\hat{\mathcal{L}}_{n\pi} = -1$, and in Fig. 18 for a positive value of it, $\hat{\mathcal{L}}_{\pi n}\hat{\mathcal{L}}_{n\pi} = 2$, in both cases for three different values of the background velocity, $V = 0.1$, $V = 0.4$ and $V = 0.9$.

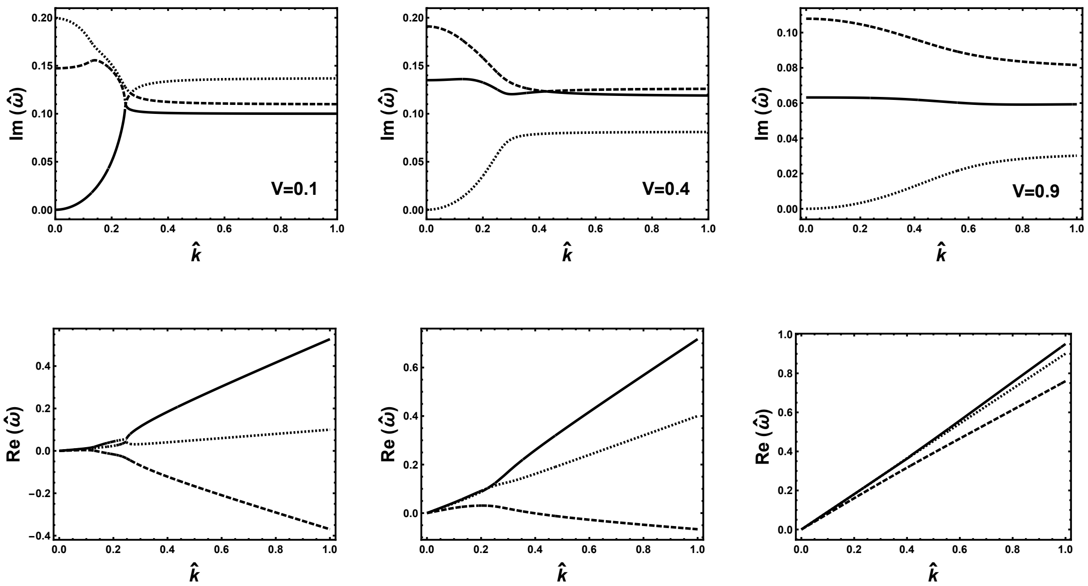

Figure 17 – Imaginary and real parts of the transverse modes considering a negative value for the product of the coupling terms, $\hat{\mathcal{L}}_{\pi n}\hat{\mathcal{L}}_{n\pi} = -1$, for three different values of the background velocity $V = 0.1$, $V = 0.4$, and $V = 0.9$.

The linear stability condition for the modes considering arbitrary values of wavenumber is extremely complicated, and not always can be expressed analytically and thus it will not be investigated here. Instead, once again we resort to the analysis of the modes in the vanishing wavenumber limit. This simplifies considerably the calculations and allows

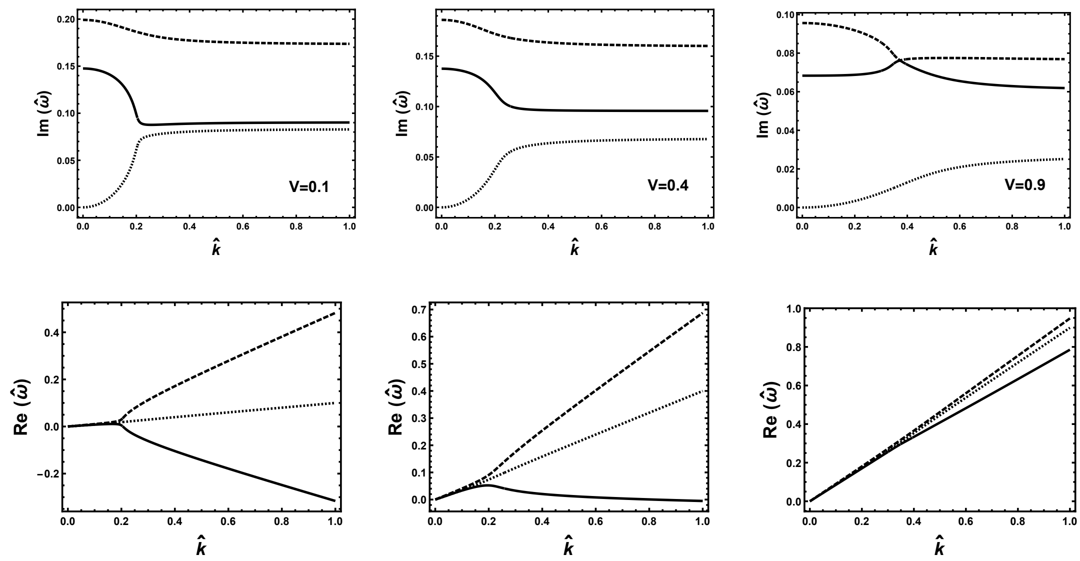

Figure 18 – Imaginary and real parts of the transverse modes considering a positive value for the product of the coupling terms, $\hat{\mathcal{L}}_{\pi n}\hat{\mathcal{L}}_{n\pi} = 2$, for three different values of the background velocity $V = 0.1$, $V = 0.4$, and $V = 0.9$.

us to provide basic necessary conditions for linear stability. In this case, the dispersion relation associated to the transverse modes reads

$$-\mathcal{A}\left(\gamma\hat{\omega}\right)^3 + i\mathcal{B}\left(\gamma\hat{\omega}\right)^2 + \left[1 + \mathcal{C}\left(\gamma\hat{\omega}V\right)^2\right]\left(\gamma\hat{\omega}\right) - i\left(\gamma\hat{\omega}V\right)^2 = 0, \tag{3.63}$$

where the solutions are simply

$$\hat{\omega} = 0, \gamma\hat{\omega}_{\pm} = i\frac{\left(\mathcal{B} - V^2\right) \pm \sqrt{\left(\mathcal{B} - V^2\right)^2 - 4\left(\mathcal{A} - \mathcal{C}V^2\right)}}{\mathcal{A} - \mathcal{C}V^2}. \tag{3.64}$$

Clearly, there is one hydrodynamic mode and two non-hydrodynamic modes, as it was expect, since it was the case for perturbations on a static fluid, see Eqs. (3.55), (3.56) and (3.57). Furthermore, if the background velocity is taken to zero, $V = 0$, one straightforwardly recovers the modes $\hat{\omega}^{\text{shear}}_{T,+} \sim i/\hat{\tau}_{\pi}$ and $\hat{\omega}^{\text{diff}}_{T} \sim i/\hat{\tau}_{n}$, while the hydrodynamic mode obtained above corresponds simply to $\hat{\omega}^{\text{shear}}_{T,-}$.

The stability of these modes requires that the numerator and denominator must have the same sign. Moreover, in order to avoid problematic discontinuities on the modes, which would lead to instabilities, we once again impose the numerator and denominator do not charge their signs for any causal value of the background velocity. For the sake of simplicity, we then look at them when the background velocity is zero with the purpose of analyzing their signs. Thus, since $\mathcal{A}$ is a positive definite quantity, the positive sign of the denominator is obtained with $\mathcal{A} - \mathcal{C}V^2$ for any causal value of the background fluid velocity V, which further implies

$$\mathcal{A} > \mathcal{C}. \tag{3.65}$$

Furthermore, the numerator must also be positive in order to the transverse modes to be linearly stable in the presence of diffusion-viscous coupling. Wherefore, it is essential that the term outside the square root to be bigger than the term inside it, ensuring the linear stability of both non-hydrodynamic modes. Since we have already shown that $\mathcal{A} > \mathcal{C}$ is a condition that must be necessarily satisfied, the term inside the square root is either positive and therefore smaller than the term outside or it is negative, leading to a real part of the modes, either way ensuring their linear stability. Nevertheless, one can show that for negative values of the product of the coupling terms, $\hat{\mathcal{L}}_{n\pi}\hat{\mathcal{L}}_{\pi n} \leq 0$, which is the case considered here, the term inside the square root is positive and smaller than $\mathcal{B} - V^2$, and thus does not affect the sign of the imaginary part of the modes. Finally, the positive sign of the numerator is guaranteed if the following condition is satisfied.

$$\mathcal{B} > 1. \tag{3.66}$$

It is convenient to express both of these linear stability conditions in terms of the transport coefficients. Hence,

$$\mathcal{B} > 1 \Longrightarrow \hat{\tau}_n + \hat{\tau}_\pi > 1, \tag{3.67}$$

$$\mathcal{A} > \mathcal{C} \Longrightarrow \hat{\mathcal{L}}_{\pi n}\hat{\mathcal{L}}_{n\pi} \geq -2\hat{\tau}_n \left(\hat{\tau}_\pi - 1\right). \tag{3.68}$$

On top of that, the linear stability conditions can be simplified if we take only negative values for the product of the coupling terms, an assumption made in the beginning of this chapter. In this case, we have the following constraints

$$|\hat{\mathcal{L}}_{\pi n}\hat{\mathcal{L}}_{n\pi}| \leq 2\hat{\tau}_n \left(\hat{\tau}_\pi - 1\right), \tag{3.69}$$

$$\hat{\tau}_\pi \geq 1, \tag{3.70}$$

$$\hat{\tau}_n \geq 0. \tag{3.71}$$

Therefore, in this case, the linear stability conditions for the shear and diffusion relation times are identical to the constraints they must satisfy in the absence of coupling terms, as it was derived in Eqs. (2.71) and (3.35). Although the product of the coupling terms was assumed to be negative, we note that is possible to obtain linearly stable theories considering positive values, as long as they satisfy the condition given by Eq. (3.69). Actually, such cases further allows the violation of the condition $\hat{\tau}_\pi \geq 1$, in which only positive values of the product of the coupling terms provide linear stability. However, this scenario shall not be further explored here.

Finally, it is convenient to express Eq. (3.69) in terms of the original transport coefficients that feature the Israel-Stewart equations, without the re-scaling factors. It then reads

$$|\ell_{\pi n}\ell_{n\pi}| \leq 2\tau_n \left(\tau_\pi - \tau_\eta\right). \tag{3.72}$$

A similar condition can be recovered from Olson's original work [48] by imposing that the transverse characteristic velocities are subluminal [Eq. (91) of the aforementioned paper].

This condition is satisfied in calculations from the Boltzmann equation [64, 65]. So far, we are not aware of any microscopic calculations that does not satisfy this condition.

Longitudinal modes

The dispersion relation for the longitudinal modes was derived in the beginning of this chapter, in Eq. (3.23). We then re-write this equation in a more convenient form, such as

$$-\mathcal{A}\hat{\Omega}^5 + i\mathcal{B}\hat{\Omega}^4 + \left(1 + 2\mathcal{A}\mathcal{S}\hat{\kappa}^2\right)\hat{\Omega}^3 - i\frac{\mathcal{B}\mathcal{D}}{3}\hat{\kappa}^2\hat{\Omega}^2 - \frac{1}{3}\left(1 + \mathcal{E}\hat{\kappa}^2\right)\hat{\Omega}\hat{\kappa}^2 + i\frac{\hat{\tau}_\kappa}{3}\hat{\kappa}^4 = 0, \quad (3.73)$$

where we used the definition of the following variables

$$\mathcal{S} \equiv \frac{\mathcal{A} + 3\hat{\tau}_\pi\hat{\tau}_\kappa + 4\mathcal{C}}{6\mathcal{A}}, \quad (3.74)$$

$$\mathcal{D} \equiv \frac{\mathcal{B} + 3\hat{\tau}_\kappa + 4}{\mathcal{B}}, \quad (3.75)$$

$$\mathcal{E} \equiv 4\hat{\tau}_\kappa + \hat{\tau}_\pi\hat{\tau}_\kappa + \frac{4}{3}\left(\mathcal{C} - \hat{\tau}_n\right). \quad (3.76)$$

We further define other useful variables that will be employed later

$$\mathcal{M} \equiv \frac{\mathcal{E}}{3\mathcal{A}}, \quad (3.77)$$

$$\mathcal{R} \equiv \sqrt{\mathcal{S}^2 - \mathcal{M}}. \quad (3.78)$$

The stability analysis once again begins with the simplest case, in which perturbations are performed on a static background fluid. In this case, the dispersion relation associated to the longitudinal modes becomes simply

$$-\mathcal{A}\hat{\omega}^5 + i\mathcal{B}\hat{\omega}^4 + \left(1 + 2\mathcal{A}\mathcal{S}\hat{k}^2\right)\hat{\omega}^3 - i\mathcal{B}\mathcal{D}\hat{k}^2\hat{\omega}^2 - \frac{1}{3}\left(1 + \mathcal{E}\hat{k}^2\right)\hat{\omega}\hat{k}^2 + i\frac{\hat{\tau}_\kappa}{3}\hat{k}^4 = 0. \quad (3.79)$$

The solutions of Eq. (3.79) are displayed in Fig. 19 for a negative value of the coupling term $\hat{\mathcal{L}}_{\pi n}\hat{\mathcal{L}}_{n\pi} = -1$ (upper panels), and also for a positive value $\hat{\mathcal{L}}_{\pi n}\hat{\mathcal{L}}_{n\pi} = 2$ (bottom panels). The inclusion of the coupling produces a similar behavior when compared to the transverse modes, see Figs. 14 and 15, where the sign of the product of the coupling terms dictates which modes merge at large values of wavenumber. Similar to what was observed for the transverse modes, the imaginary part of the longitudinal modes becomes constant at large wavenumber. The real parts of the longitudinal modes do not show any qualitative variation at large wavenumber as we change the sign of the coupling term.

Furthermore, since this is a fifth-degree polynomial, its solutions are extremely complicated and are not explicitly derived here considering arbitrary values of wavenumber. Instead, in order to be able to extract any information regarding the linear stability of these modes, we resort once again to look at the modes in the small and large wavenumber

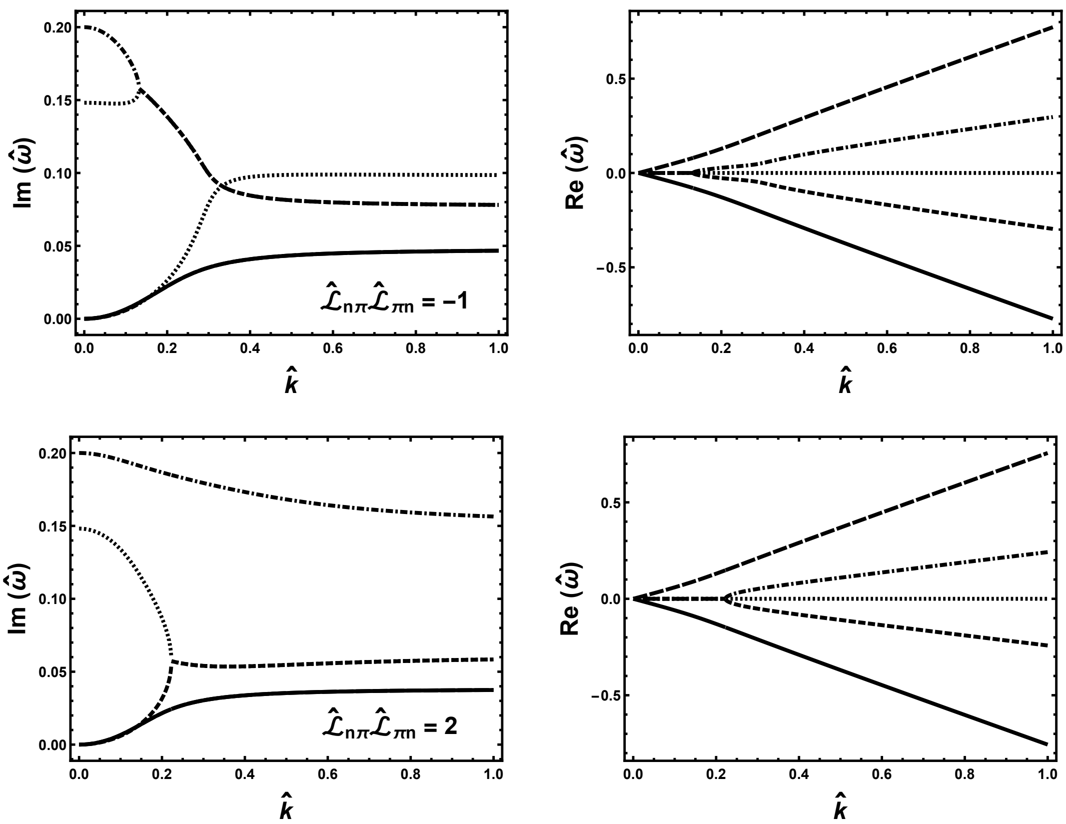

Figure 19 – Imaginary and real parts of the longitudinal modes considering a negative and a positive values for the product of the coupling terms, $\hat{\mathcal{L}}_{\pi n}\hat{\mathcal{L}}_{n\pi} = -1$ and $\hat{\mathcal{L}}_{\pi n}\hat{\mathcal{L}}_{n\pi} = 2$, considering a static background fluid, $V = 0$.

limits. In the first regime, they read

$$\omega_{L,+}^{\mathrm{B}} = \frac{i}{\hat{\tau}_n} + \mathcal{O}\left(\hat{k}^2\right), \tag{3.80}$$

$$\omega_{L}^{\mathrm{shear}} = \frac{i}{\hat{\tau}_\pi} + \mathcal{O}\left(\hat{k}^2\right), \tag{3.81}$$

$$\omega_{L,-}^{\mathrm{B}} = i\hat{\tau}_\kappa \hat{k}^2 + \mathcal{O}\left(\hat{k}^3\right), \tag{3.82}$$

$$\omega_{\pm}^{\mathrm{sound}} = \pm\frac{1}{\sqrt{3}}\hat{k} + \frac{2}{3}i\hat{k}^2 + \mathcal{O}\left(\hat{k}^3\right). \tag{3.83}$$

There are two non-hydrodynamic mode and three hydrodynamic modes. One can immediately see that the latter behave like the modes from Navier-Stokes theory in the small wavenumber limit. In particular, in this regime, the longitudinal modes of Israel-Stewart theory are linearly stable. Moreover, at the large wavenumber limit, the modes can be written as

$$\hat{\omega} = \pm\hat{k}\sqrt{\mathcal{S}\pm\mathcal{R}} + \frac{i}{\mathcal{A}}\frac{3\mathcal{B}\left(\mathcal{S}\pm\mathcal{R}\right)^2 - \mathcal{B}\mathcal{D}\left(\mathcal{S}\pm\mathcal{R}\right) + \hat{\tau}_\kappa}{15\left(\mathcal{S}\pm\mathcal{R}\right)^2 - 18\mathcal{S}\left(\mathcal{S}\pm\mathcal{R}\right) + 3\mathcal{M}} + \mathcal{O}\left(\frac{1}{\hat{k}}\right), \tag{3.84}$$

$$\hat{\omega} = i\frac{\hat{\tau}_\kappa}{\mathcal{E}} + \mathcal{O}\left(\frac{1}{\hat{k}}\right). \tag{3.85}$$

Similar to what has been first observed for the transverse modes, the inclusion of diffusion-viscous coupling leads to occurrence of not only novel linear causality conditions, but also linear stability conditions for perturbations on a static background fluid. Therefore, these modes are linearly stable if their imaginary parts are positive and thus we have the following necessary conditions

(i) $\hat{\tau}_\kappa/\mathcal{E} > 0$;

(ii) $\sqrt{\mathcal{S} \pm \mathcal{R}}$ is real;

(iii) $\left[3\mathcal{B}\left(\mathcal{S} \pm \mathcal{R}\right)^2 - \mathcal{B}\mathcal{D}\left(\mathcal{S} \pm \mathcal{R}\right) + \hat{\tau}_\kappa\right] / \left[15\left(\mathcal{S} \pm \mathcal{R}\right)^2 - 18\mathcal{S}\left(\mathcal{S} \pm \mathcal{R}\right) + 3\mathcal{M}\right]$ is positive.

while linear causality is ensured by a subluminal asymptotic group velocity, which is simply guaranteed by

$$\lim_{k\to\infty}\left|\frac{\partial \mathrm{Re}(\omega)}{\partial k}\right| = \sqrt{\mathcal{S} \pm \mathcal{R}} \leq 1. \tag{3.86}$$

In particular, there are several requirements for linear stability. The next step is to express the conditions derived above in terms of the transport coefficients. First, condition (i) is satisfied if $\mathcal{E} > 0$, leading to a new constraint for the product of the coupling terms, given by

$$\hat{\mathcal{L}}_{\pi n}\hat{\mathcal{L}}_{n\pi} < \frac{3}{2}\hat{\tau}_\kappa\left(\hat{\tau}_\pi + 4\right). \tag{3.87}$$

Note that this condition is automatically satisfied if the product of the coupling terms is not positive, $\hat{\mathcal{L}}_{\pi n}\hat{\mathcal{L}}_{n\pi} \leq 0$. Furthermore, condition (ii) is fulfilled if $\mathcal{S} > \mathcal{R}$, which is automatically satisfied for negative values of the product of the coupling terms. Moreover, it further implies the following inequality

$$\mathcal{S}^2 > \mathcal{M}. \tag{3.88}$$

We shall return to this condition later, and analyze it more carefully when considering perturbations on top of a moving background fluid. Last, condition (iii) further implies

$$3\mathcal{B}\left(\mathcal{S} + \mathcal{R}\right)^2 - \mathcal{B}\mathcal{D}\left(\mathcal{S} + \mathcal{R}\right) + \hat{\tau}_\kappa > 0, \tag{3.89}$$

$$3\mathcal{B}\left(\mathcal{S} - \mathcal{R}\right)^2 - \mathcal{B}\mathcal{D}\left(\mathcal{S} - \mathcal{R}\right) + \hat{\tau}_\kappa < 0. \tag{3.90}$$

All the constraints listed above are automatically fulfilled if the product of the coupling terms is negative, and hence do not provide any new information on the relations that dictate the linear stability of the theory. However, if one does not assume such statement and further consider the possibility of positive values for the product of the coupling terms, an essentially new set of constraints will appear, although we shall not explore this case thoroughly in this work, as we pointed out earlier in this chapter. Furthermore, these inequalities can be shown to be equivalent to the linear stability conditions obtained by using the Routh-Hurwitz criterion [68, 69, 70], and thus are valid not only in the regime of large values of k, but also for any wavenumber.

The final step of this analysis is to investigate the longitudinal modes of the Israel-Stewart theory in the presence of net-charge diffusion for perturbations on top of a moving background fluid. Since the solution of the dispersion relation, a fifth degree polynomial, would be even more complicated in this case, we analyze the modes in the vanishing wavenumber limit, as it was done in the previous cases. Analogously, we shall consider a background fluid velocity that is parallel to the wave-vector. Hence, the dispersion relation, Eq. (3.23), is written as

$$(\gamma\hat{\omega})^3\left[-3\mathcal{A}\left(1-2\mathcal{S}V^2+\mathcal{M}V^4\right)\gamma^2\hat{\omega}^2+i\left(\hat{\tau}_\kappa V^4+3\mathcal{B}-\mathcal{B}V^2\right)\gamma\hat{\omega}+3-V^2\right]=0. \quad (3.91)$$

There are three hydrodynamic and two non-hydrodynamic modes, as observed for perturbations on a static background fluid. The latter then read

$$\gamma\hat{\omega}_\pm = i\frac{\hat{\tau}_\kappa V^4+\mathcal{B}\left(3-\mathcal{D}V^2\right)\pm\sqrt{\left[\hat{\tau}_\kappa V^4+\mathcal{B}\left(3-\mathcal{D}V^2\right)\right]^2-12\mathcal{A}\left(3-V^2\right)\left(1-2\mathcal{S}V^2+\mathcal{M}V^4\right)}}{3\mathcal{A}\left(1-2\mathcal{S}V^2+\mathcal{M}V^4\right)}. \quad (3.92)$$

Naturally, as it was previously discussed several times, in order to have only linearly stable modes, the imaginary part of the modes must be positive for all possible values of V, which is guaranteed if both the numerator and denominator have the same sign for all possible values the background velocity can assume. Furthermore, with the purpose of avoiding non-physical discontinuities on the modes, both are assumed to maintain their signs for any causal value of the background velocity V. Thus, these signs can be straightforwardly verified in the limit where the background fluid is at rest, $V=0$. In this case, the denominator can be immediately shown to be positive, thus leading to

$$1-2\mathcal{S}V^2+\mathcal{M}V^4>0,\ \forall\ \ 0\leq V\leq 1. \quad (3.93)$$

Note that Eq. (3.93) is a polynomial function which is quadratic in V^2 and positive at $V=0$. The condition above is satisfied if the smallest root of this polynomial is larger than 1. This guarantees that the function is positive in the interval of $0\leq V^2\leq 1$, where the greatest value the background fluid velocity can assume is the speed of light, and further instabilities would only occur in the non-physical region in which the background velocity assumes acausal values. This is ensured by the following inequality

$$\mathcal{S}-\mathcal{R}\geq\mathcal{M}. \quad (3.94)$$

This relation is identical to the linear causality condition derived for perturbations on a static background fluid, see Eq. (3.86). This can be straightforwardly seen using the definition of the variable $\mathcal{M}=\mathcal{S}^2-\mathcal{R}^2$, which further leads to

$$\mathcal{S}-\mathcal{R}\geq(\mathcal{S}-\mathcal{R})(\mathcal{S}+\mathcal{R})\Longrightarrow\mathcal{S}+\mathcal{R}\leq 1\Rightarrow-\frac{3}{2}\hat{\tau}_\pi\left(\hat{\tau}_n-\hat{\tau}_\kappa\right)-\hat{\tau}_n\left(\hat{\tau}_\pi-2\right)\leq\hat{\mathcal{L}}_{\pi n}\hat{\mathcal{L}}_{n\pi}, \quad (3.95)$$

where we used the previously derived stability conditions $\mathcal{S} - \mathcal{R} > 0$ and $\mathcal{R} \geq 0$. This is another evidence that shows the strong connection between the linear stability conditions obtained for perturbations on top of a moving background and the causality conditions satisfied by perturbations on top of a static background. Furthermore, the inequalities given in Eq. (3.95) can be used to derive further constraints for the linear stability of the theory,

$$\mathcal{M} = (\mathcal{S}+\mathcal{R})(\mathcal{S}-\mathcal{R}) \leq 1 \Rightarrow \frac{3}{2}\hat{\tau}_\kappa \left(\hat{\tau}_\pi + 4\right) - \frac{9}{2}\mathcal{A} \leq \hat{\mathcal{L}}_{\pi n}\hat{\mathcal{L}}_{n\pi}, \tag{3.96}$$

$$\mathcal{S} \leq \frac{1+\mathcal{M}}{2} \Rightarrow -\frac{2}{3}\hat{\mathcal{L}}_{\pi n}\hat{\mathcal{L}}_{n\pi} \leq \left(\hat{\tau}_\pi - 2\right)\left(\hat{\tau}_n - \hat{\tau}_\kappa\right). \tag{3.97}$$

So far, after imposing that neither the numerator nor the denominator in Eq. (3.92) change their signs for any causal value for the background fluid velocity, the latter was shown to be positive and to lead to further inequalities that must be necessarily fulfilled. Therefore, the next step is to investigate the numerator, which thus must be positive as well, in order to render only linearly stable modes. In this case, there are two possible scenarios: in the first, the term inside the square root is negative, resulting in a real part of the modes that does not contribute to the linear stability of the modes; in the second, the term inside the square root is positive, yet smaller than the term outside it. One can show that the term inside the square root in Eq. (3.92) is positive definite as long as we assume that the product of the coupling terms is negative, and the linear stability of the non-hydrodynamic longitudinal modes is guaranteed if the following relation is satisfied

$$\frac{\hat{\tau}_\kappa}{3\mathcal{B}}V^4 - \frac{\mathcal{D}}{3}V^2 + 1 \geq 0, \ \forall \ \ 0 \leq V \leq 1. \tag{3.98}$$

The following analysis is analogous to what has been performed to the denominator of Eq. (3.92). The inequality above also corresponds to a quadratic polynomial function of V^2, and it is satisfied if its smallest root of is greater than 1, rendering the function positive in the physical interval in which the background velocity does not exceed the speed of light, further leading to a novel constraint for the relaxation times of the Israel-Stewart theory

$$\hat{\tau}_\pi + \hat{\tau}_n \geq \hat{\tau}_\kappa + 2. \tag{3.99}$$

This condition is satisfied if the values for the transport coefficients such as calculated by Boltzmann equation [33, 64, 65] are employed, such as used so far throughout this work. Finally, if the product of the coupling terms is negative, the linear stability conditions derived for the longitudinal modes can be summarized as

$$|\hat{\mathcal{L}}_{\pi n}\hat{\mathcal{L}}_{n\pi}| \leq \frac{3}{2}\left(\hat{\tau}_\pi - 2\right)\left(\hat{\tau}_n - \hat{\tau}_\kappa\right), \tag{3.100}$$

$$\hat{\tau}_\pi \geq 2, \tag{3.101}$$

$$\hat{\tau}_n \geq \hat{\tau}_\kappa. \tag{3.102}$$

It is interesting to note that, as it was observed for the Israel-Stewart theory in the absence of diffusion-viscous coupling, see Chapter 2, the linear stability conditions in the presence of coupling obtained looking at the longitudinal modes are stronger than the constraints obtained for the transverse modes. Furthermore, the linear stability condition for the shear relaxation time such as derived in the absence of diffusion-viscous coupling, Eq. (2.86), is still valid also when coupling is taken into account, Eq. (3.101). There is also the occurrence of a linear stability condition for the diffusion relaxation time, given by Eq. (3.102), which is also valid even in the absence of diffusion-viscous coupling, see Eq. (3.48). The linear stability conditions for the shear and diffusion relaxation times such as obtained in the absence of diffusion-viscous coupling can be recovered from Eq. (3.100) if the product of the coupling terms is set to zero.

For the sake of completeness, the solution of Eq. (3.23) for perturbations of top of a moving background are displayed in Figs. 20 and 21 considering a negative and a positive value for the product of the coupling terms, $\hat{\mathcal{L}}_{\pi n}\hat{\mathcal{L}}_{n\pi} = -1$ and $\hat{\mathcal{L}}_{\pi n}\hat{\mathcal{L}}_{n\pi} = 2$, respectively. We also show several examples of configurations that are driven unstable by the coupling terms in Fig. 22, taking the following values for the product of the coupling terms that violate Eq. (3.100), $\hat{\mathcal{L}}_{\pi n}\hat{\mathcal{L}}_{n\pi} = -40, -45, -50, -60$.

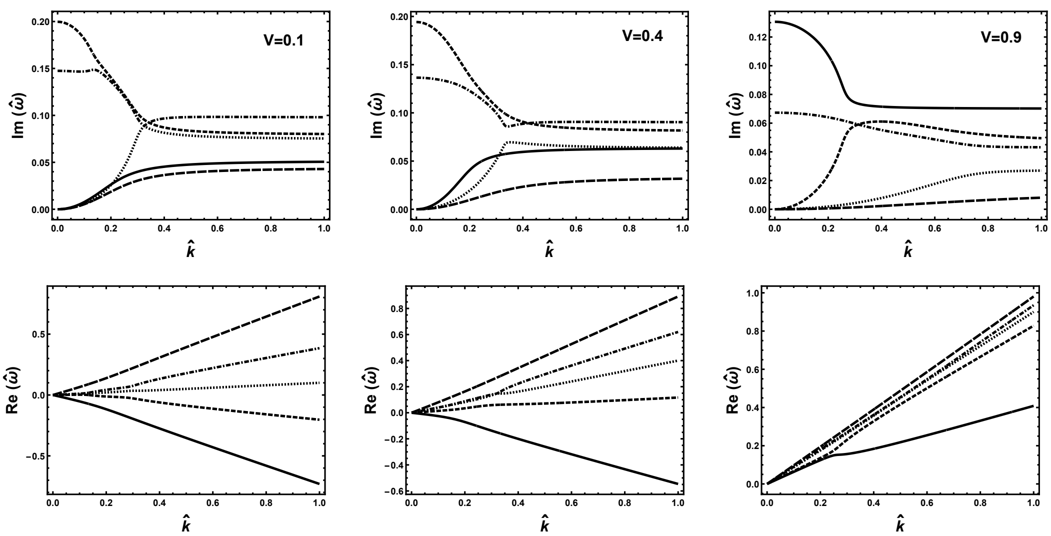

Figure 20 – Imaginary and real parts of the longitudinal modes considering a negative value for the product of the coupling terms, $\hat{\mathcal{L}}_{\pi n}\hat{\mathcal{L}}_{n\pi} = -1$, for three different values of background velocity, $V = 0.1$, $V = 0.4$, and $V = 0.9$.

As it was first unveiled in Ref. [30] considering dissipation via shear-stress and bulk viscous pressure, yet neglecting net-charge diffusion, in the Israel-Stewart theory, the linear causality conditions obtained for perturbations around a background fluid at rest are equivalent to linear stability conditions for perturbations on top of a moving fluid, which is also observed in the presence of diffusion-viscous coupling. Furthermore, we note

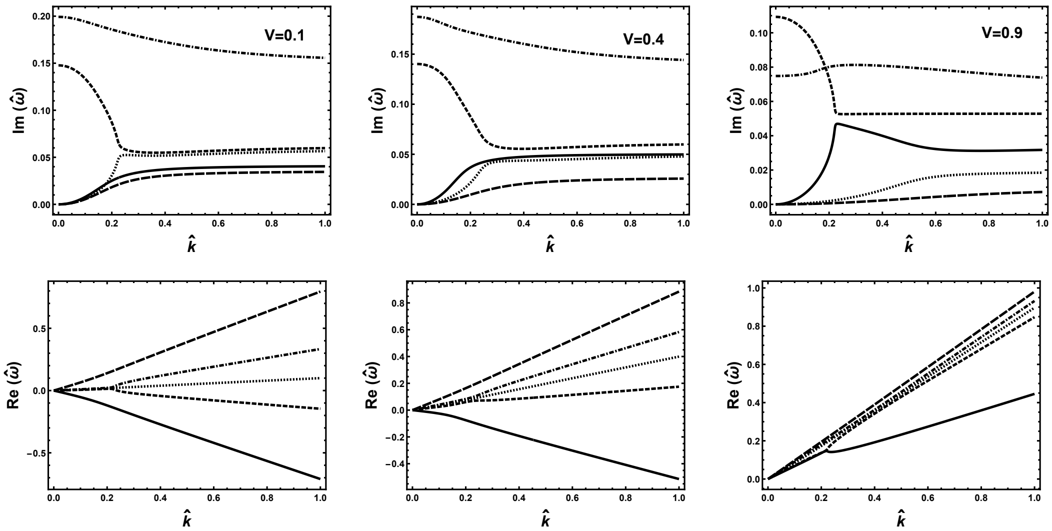

Figure 21 – Imaginary and real parts of the longitudinal modes considering a positive value for the product of the coupling term, $\hat{\mathcal{L}}_{\pi n}\hat{\mathcal{L}}_{n\pi} = 2$, for three different values of background velocity $V = 0.1$, $V = 0.4$, and $V = 0.9$.

that the stability conditions obtained by Olson in Ref. [48] also include the effects of diffusion-viscous coupling considering all three dissipative currents simultaneously, while bulk viscosity has been neglected in the original analysis of Ref. [59] and thus it was not further investigated here. However, in this case the constraints for the transport coefficients were written in a more convoluted form and constraints for the diffusion-viscous coupling terms were not explicitly derived. Furthermore, and more importantly, the linear stability conditions obtained by Olson cannot be trivially extended to the case of a vanishing background net-baryon number density and, thus, it is not possible to directly compared them to the results presented in this chapter. This happens because the perturbations defined by Olson diverge in the limit of vanishing net-charge, which is inconsistent with the assumptions made by Olson when deriving the linear stability conditions (in theorem A of Ref. [48], Olson assumes that the perturbations do not diverge when deriving linear stability conditions).

Nevertheless, some of the linear *causality* conditions derived by Olson are equivalent to those derived in this chapter and published in Ref. [59]. The linear causality condition for the transverse characteristic velocities calculated by Olson [Eq. (91) of Ref. [48]] is equivalent to the corresponding linear causality condition derived in this dissertation, see Eq. (3.62), in the limit of vanishing net-charge. The same does not occur when comparing the linear causality conditions derived for the longitudinal modes. In this case, the difficulty in the comparison lies in taking the limit of vanishing bulk viscosity and relaxation time, for which case the result diverges. In order to compare with Olson's result, we would need to include the effects of bulk viscosity from start as well.

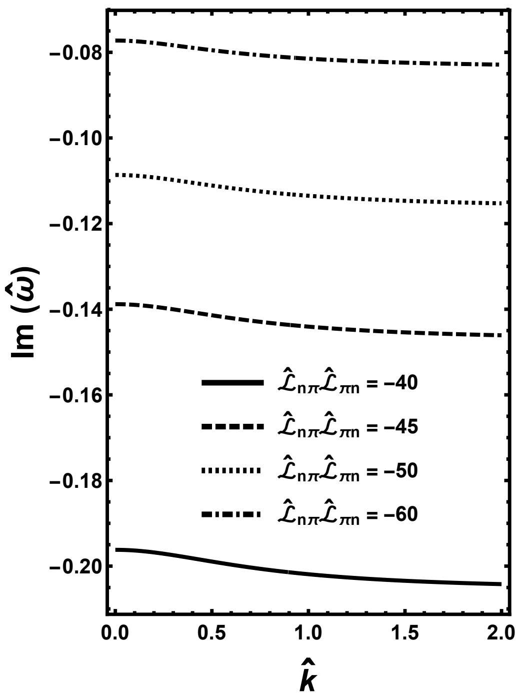

Figure 22 – Imaginary parts of the unstable shear modes for different negative values of the coupling term, $\hat{\mathcal{L}}_{\pi n}\hat{\mathcal{L}}_{n\pi} = -40, -45, -50, -60$, for $V = 0.99$.

4 Third-order fluid dynamics

In the first section of Chapter 2, we explored the linear causality and stability of the relativistic Navier-Stokes theory considering the presence of bulk viscous pressure and shear-stress, with net-charge diffusion being neglected in this analysis. As it was shown in this chapter, the modes of the relativistic Navier-Stokes theory are *always* linearly unstable for perturbations on top of a moving background fluid, a severe problem that is directly connected to the acausal character of the constitutive relations for the dissipative currents [29, 30] defined in this formulation. These equations come from the second law of thermodynamics applied to the first-order non-equilibrium entropy 4-current derived in Chapter 1.

In this scenario, a new formulation for the entropy 4-current was proposed by Israel and Stewart, in which they also took into account the presence of second-order terms in the entropy 4-current. These terms were shown to lead to the occurrence of relaxation times that are essential to ensure the causality of the theory, among other first and second-order terms. In the second section of Chapter 2, the linear stability of Israel-Stewart theory considering only dissipation via shear-stress was revisited. In particular, we showed that the relaxation times must satisfy certain inequalities in order to the theory to have only linearly causal and stable modes.

One may wonder what are the implications of considering higher-order terms in the equations of motion for the dissipative currents, see Ref. [55]: what is the regime of linear causality and stability of a higher-order fluid-dynamical formulation? Naturally, higher-order terms must affect the linear stability of relativistic fluid dynamics in the same way the inclusion of net-charge diffusion and diffusion-viscous couplings also lead to a new set of constraints in the Israel-Stewart theory.

In this chapter, we perform an analysis which is independent from the cases studied so far in this work. Here, we investigate the linear stability of a third-order fluid-dynamical formulation such as proposed in Ref. [56], showing it is an ill-defined theory with non-physical behaviors that resemble the ones observed in the relativistic Navier-Stokes theory. Furthermore, we connect the linear instability of this theory to its parabolic nature [57, 58] and introduce a novel formulation, which can be constructed to be linearly stable. We then derive the linear stability conditions of this formulation.

4.1 Parabolic third-order fluid dynamics

This section is dedicated to the linear stability analysis of the parabolic third-order formulation first derived in Ref. [56]. In this work, a third-order equation of motion for the shear-stress tensor is obtained from the relativistic Boltzmann equation using the Chapman-Enskog method (effects due to bulk viscous pressure and net-charge were neglected). This leads to the following dynamical equation for the shear-stress tensor

$$\dot{\pi}^{\langle\mu\nu\rangle} = 2\frac{\eta}{\tau_\pi}\sigma^{\mu\nu} - \frac{1}{\tau_\pi}\pi^{\mu\nu} + \frac{4}{35}\nabla^{\langle\mu}\left(\tau_\pi \nabla_\alpha \pi^{\nu\rangle\alpha}\right) - \frac{2}{7}\nabla_\alpha\left(\tau_\pi \nabla^{\langle\mu}\pi^{\nu\rangle\alpha}\right) - \frac{1}{7}\nabla_\alpha\left(\tau_\pi \nabla^\alpha \pi^{\langle\mu\nu\rangle}\right) + \cdots, \tag{4.1}$$

which corresponds to Eq. (16) of the aforementioned paper. Note that the dots represent terms that are non-linear and thus do not contribute to the linear stability analysis that follows and, hence, were omitted in the equality above. Naturally, a third-order equation of motion for the shear-stress tensor has several additional terms that are not present in the Israel-Stewart theory, see Eq. (3.2). In particular, the last three terms in the right-hand side of the equation above are novel linear contributions of third-order. The goal of this chapter is to analyze the effects of the inclusion of these terms in the linear stability of the theory. Throughout this chapter, only dissipation via shear-stress will be considered.

Once again, the linear stability analysis is performed considering perturbations on top of a fluid initially in a global equilibrium state,

$$\varepsilon = \varepsilon_0 + \delta\varepsilon, \;\; u^\mu = u_0^\mu + \delta u^\mu, \;\; \pi^{\mu\nu} = \delta\pi^{\mu\nu}. \tag{4.2}$$

The linearized equations of motion that come from the conservation of energy and momentum have been derived previously in Eqs. (2.2) and (2.4). In this chapter, we do not consider any kind of conserved charges, thus the remaining equation of motion, Eq. (2.3), is not required. Furthermore, the linearized third-order equation for the shear-stress tensor becomes

$$\begin{aligned} D_0\delta\pi^{\mu\nu} + \frac{1}{\tau_\pi}\delta\pi^{\mu\nu} &= \frac{\eta}{\tau_\pi}\left[2\nabla_0^{(\mu}\delta u^{\nu)} - \frac{2}{3}\Delta_0^{\mu\nu}\partial_\lambda \delta u^\lambda\right] - \frac{1}{7}\tau_\pi \nabla_\lambda^0 \nabla_0^\lambda \delta\pi^{\mu\nu} \\ &- \frac{6}{35}\tau_\pi \nabla_\lambda^0\left[2\nabla_0^{(\mu}\delta\pi^{\nu)\lambda} - \frac{1}{3}\Delta_0^{\mu\nu}\nabla_\beta^0 \delta\pi^{\beta\lambda}\right], \end{aligned} \tag{4.3}$$

where we remind the reader that $D_0 \equiv u_0^\mu \partial_\mu$ is the comoving derivative with respect to the background fluid velocity and $\nabla_0^\mu \equiv \Delta_0^{\mu\nu}\partial_\nu$ is the linearized projected derivative. The last two terms on the right-hand side of the equation above are corrections due to third-order contributions in the equation of motion for the shear-stress tensor. If these terms are set to zero, one recovers the linearized Israel-Stewart equation for the shear-stress tensor in the absence of diffusion-viscous coupling, which was investigated in Chapter 2. In this case, the transport coefficients – in particular, the relaxation times – must satisfy certain inequalities in order to guarantee that Israel-Stewart theory is linearly causal and stable. Therefore, one may ask whether the inclusion of third-order terms in the equation for the shear-stress tensor either maintains the properties of linear causality and stability observed in the Israel-Stewart or yields novel constraints for the transport coefficients in order to fulfill linear causality and stability.

Once again, we express the linearized fluid-dynamical equations in Fourier space. The Fourier transform of the continuity equation associated to energy and momentum were already derived, and are given in Eqs. (2.14) and (2.16). The linearized parabolic third-order equation of motion for the shear-stress tensor in Fourier space reads

$$\begin{aligned} i\Omega\delta\tilde{\pi}^{\mu\nu} + \frac{1}{\tau_\pi}\delta\tilde{\pi}^{\mu\nu} &= i\frac{\eta}{\tau_\pi}\left[2\kappa^{(\mu}\delta\tilde{u}^{\nu)} - \frac{2}{3}\Delta_0^{\mu\nu}\kappa_\lambda\delta\tilde{u}^\lambda\right] - \frac{1}{7}\tau_\pi\kappa^2\delta\tilde{\pi}^{\mu\nu} + \\ &+ \frac{6}{35}\kappa_\lambda\tau_\pi\left[\kappa^{(\mu}\delta\tilde{\pi}^{\nu)\lambda} - \frac{1}{3}\Delta_0^{\mu\nu}\kappa_\beta\delta\tilde{\pi}^{\beta\lambda}\right]. \end{aligned} \tag{4.4}$$

The next step is to decompose the linearized fluid-dynamical equations in Fourier space into transverse and longitudinal degrees of freedom, since they decouple and can be solved independently, as already shown in Chapter 2.

Transverse modes

The transverse component of the linearized third-order equation for the shear stress tensor, given by Eq. (4.4), is obtained by simply projecting this equation with $\kappa_\mu\Delta^\lambda_{\nu,\kappa}$, see the tensor decomposition in Fourier space developed in Chapter 2. Moreover, the transverse component of the equation of motion for the conservation of energy and momentum was already derived, see Eq. (2.21). Then, the equations that describe the transverse degrees of freedom of this theory can be summarized as follows

$$\hat{\Omega}\delta\tilde{u}^\mu_\perp - \hat{\kappa}\frac{\delta\tilde{\pi}^\mu_\perp}{\varepsilon_0 + P_0} = 0, \tag{4.5}$$

$$\left(i\hat{\tau}_\pi\hat{\Omega} + \frac{8}{35}\hat{\tau}_\pi^2\hat{\kappa}^2 + 1\right)\frac{\delta\tilde{\pi}^\mu_\perp}{\varepsilon_0 + P_0} - i\hat{\kappa}\delta\tilde{u}^\mu_\perp = 0. \tag{4.6}$$

For the sake of consistency, the variables here are re-scaled in terms of the hydrodynamic time scale τ_η in order to work only with dimensionless variables and further being able to properly compare the results derived in this chapter with the previous ones. Furthermore, these equations can be written in the following matrix form

$$\begin{pmatrix} i\hat{\tau}_\pi\hat{\Omega} + \frac{8}{35}\hat{\tau}_\pi^2\hat{\kappa}^2 + 1 & -i\hat{\kappa} \\ -\hat{\kappa} & \hat{\Omega} \end{pmatrix}\begin{pmatrix} \frac{\delta\tilde{\pi}^\mu_\perp}{\varepsilon_0+P_0} \\ \delta\hat{u}^\mu_\perp \end{pmatrix} = 0, \tag{4.7}$$

which leads to the following dispersion relation

$$\hat{\Omega}\left(i\hat{\tau}_\pi\hat{\Omega} + \frac{8}{35}\hat{\tau}_\pi^2\hat{\kappa}^2 + 1\right) - i\hat{\kappa}^2 = 0. \tag{4.8}$$

Note that if the quadratic term inside the parentheses – which corresponds to a third-order contribution – is set to zero, one recovers the dispersion relation satisfied by the transverse modes of Israel-Stewart theory considering only dissipation via shear-stress, see Eq. (2.62). The linear stability of Israel-Stewart theory in this regime is already known and has been carefully analysed in Chapter 2.

The next step is to analyze whether the modes from the third-order formulation proposed in Ref. [56] are linearly causal and stable, and if the occurrence of additional terms in the equation of motion for the shear-stress tensor brings new constraints for this theory. Naturally, the first case that will be studied is when perturbations on a background fluid at rest are performed. In this case, the dispersion relation associated to the transverse mode becomes simply

$$\hat{\omega}\left(i\hat{\tau}_\pi\hat{\omega}+\frac{8}{35}\hat{\tau}_\pi^2\hat{k}^2+1\right)-i\hat{k}^2=0. \tag{4.9}$$

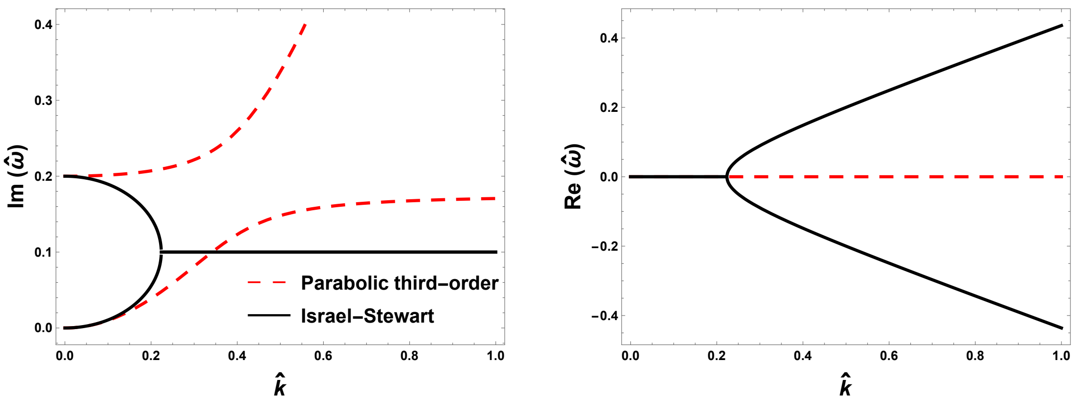

Figure 23 – Imaginary and real parts of the transverse modes of Israel-Stewart theory (solid black lines) and parabolic third-order fluid dynamics (red dashed lines) for perturbations on a static background fluid, considering $\hat{\tau}_\pi = 5$ [64].

The solutions of this equation are

$$\hat{\omega}_\pm = i\left[\frac{1+\frac{8}{35}\hat{\tau}_\pi^2\hat{k}^2 \pm \sqrt{\left(1+\frac{8}{35}\hat{\tau}_\pi^2\hat{k}^2\right)^2-4\hat{k}^2\hat{\tau}_\pi}}{2\hat{\tau}_\pi}\right], \tag{4.10}$$

and they are displayed in Fig. 23 (red dashed lines), where we also display the shear modes of the Israel-Stewart theory (solid black lines). As can be seen from this plot, there is the occurrence of one hydrodynamic and one non-hydrodynamic mode. However, unlike the shear modes of Israel-Stewart theory, the non-hydrodynamic mode of the parabolic third-order theory proposed in Ref. [56] does not saturate at large wavenumber, i.e., it keeps increasing as the wavenumber increases. Note that the non-hydrodynamic mode has the same behavior observed on the hydrodynamic mode of relativistic Navier-stokes theory, since it behaves as $\omega \sim ik^2$, as $k \to \infty$. For perturbations on a static fluid, we note that all modes of the parabolic third order theory are linearly stable.

In order to analyze a more interesting (intrinsic relativistic) case, we then consider perturbations on a moving fluid. For the sake of convenience we once again assume that the background fluid velocity is parallel to the wave-vector. In this case, the dispersion relation is obtained by inserting Eqs. (2.29) and (2.30) into Eq. (4.8). As before, we discuss the stability of the modes in the vanishing wavenumber limit, where necessary stability conditions can be derived. Then, the dispersion relation associated to the transverse modes reads

$$(\gamma\hat{\omega})\left[i\hat{\tau}_\pi(\gamma\hat{\omega}) + \frac{8}{35}\hat{\tau}_\pi^2(\gamma\hat{\omega}V)^2 + 1\right] - i(\gamma\hat{\omega}V)^2 = 0. \tag{4.11}$$

One can immediately note that perturbations on top of a moving fluid lead to the occurrence of an additional mode – in particular, another non-hydrodynamic mode. This is a remarkably problematic feature carried by parabolic formulations, first observed in the relativistic Navier-Stokes theory in Chapter 2. This is already a hint that the parabolic third-order formulation, such as the one proposed in Ref. [56], may be linearly unstable. The non-hydrodynamic modes in the vanishing wavenumber limit are

$$\hat{\omega}^{\pm}_{\text{shear}} = \frac{35i}{16\gamma\hat{\tau}_\pi^2 V^2}\left[(V^2 - \hat{\tau}_\pi) \pm \sqrt{(V^2 - \hat{\tau}_\pi)^2 + \frac{32}{35}V^2\hat{\tau}_\pi^2}\right]. \tag{4.12}$$

Since the term inside the square root in the above equation is positive definite and greater than the term outside it, the mode $\omega^{-}_{\text{shear}}$ is always linearly unstable. This is exactly the additional mode that is not present when perturbing a static fluid. On the other hand, $\omega^{+}_{\text{shear}}$ corresponds to a linearly stable non-hydrodynamic mode, already present in the static background case. For the sake of illustration, the solutions of the dispersion relation, Eq. (4.8), for perturbations on top of a moving are displayed in Fig. 24 as a function of the wavenumber k.

Before proceeding, we shall point out that even though this analysis was performed for $k = 0$, it is enough to discredit the theory, as its linear stability must be valid for any value of wavenumber. However, for the sake of completeness, we shall carry on the linear stability analysis and dedicate the next section to study the linear stability of the longitudinal modes as well.

The remaining, yet unnecessary, transverse equation can be obtained by projecting Eq. (4.4) with $\Delta_{\mu\nu\alpha\beta,\kappa}$, leading to the dispersion relation associated to the fully transverse component of the shear-stress tensor. In this case, we obtain

$$\left(i\hat{\tau}_\pi\hat{\Omega} + \frac{1}{7}\hat{\tau}_\pi^2\hat{\kappa}^2 + 1\right)\delta\tilde{\pi}^{\mu\nu}_{\perp} = 0. \tag{4.13}$$

This equation is independent of the remaining hydrodynamic perturbations and can be solved directly. It leads to the following dispersion relation,

$$\hat{\Omega} = i\left(\frac{1}{7}\hat{\tau}_\pi\hat{\kappa}^2 + \frac{1}{\hat{\tau}_\pi}\right). \tag{4.14}$$

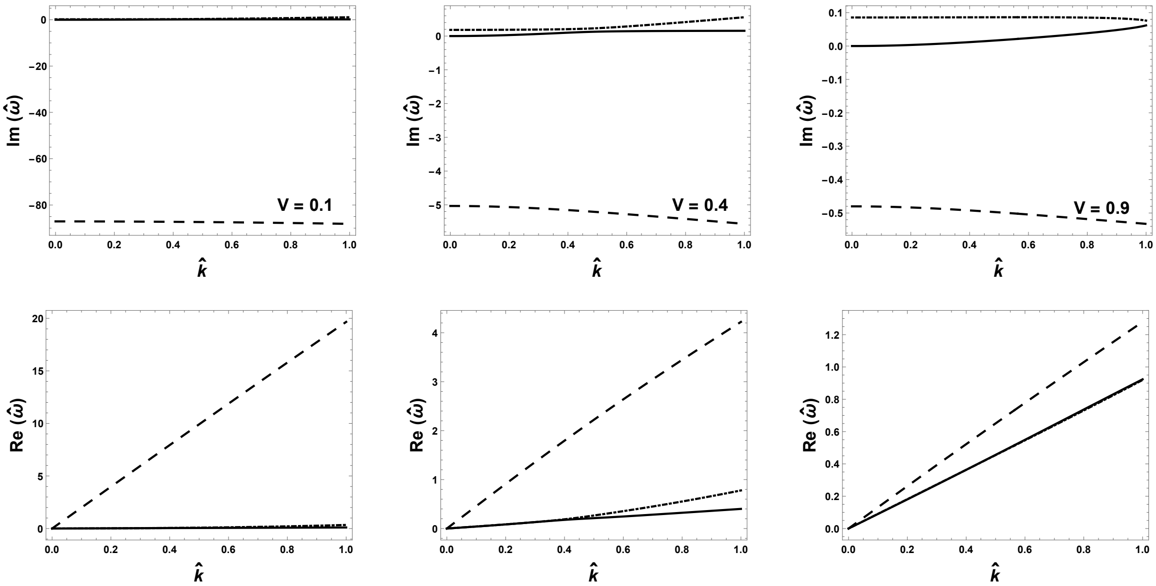

Figure 24 – Imaginary and real parts of the transverse modes for $\hat{\tau}_\pi = 5$, considering three different values for the background velocity $V = 0.1$, $V = 0.4$, and $V = 0.9$.

Clearly, if the term due to third-order contributions is neglected, we simply obtain a linearly causal [the linear causality condition remains identical to the shear modes derived in Ref. [30]] and stable transient mode that behaves such as $\Omega \sim i/\tau_\pi$. However, if such term is taken into account, the dispersion relation becomes a diffusion-like equation, similar to what is observed for the linear stability analysis of the relativistic Navier-Stokes theory in Chapter 2, with an additional correction due to the third order contributions. This is another strong evidence on the linear instability of the fluid-dynamical proposition of Ref. [56].

Longitudinal modes

The longitudinal components of the Fourier projections of the equations of motion from the conservation of energy and momentum were already calculated, see Eqs. (2.35) and (2.36). The longitudinal component of Eq. (4.4) is obtained by projecting this equation with $\kappa_\mu \kappa_\nu$, see Chapter 2. Hence, since bulk is being neglected in this analysis, the longitudinal equations are summarized as

$$\hat{\Omega}\frac{\delta\tilde{\varepsilon}}{\varepsilon_0 + P_0} - \hat{\kappa}\delta\tilde{u}_\parallel = 0, \tag{4.15}$$

$$\hat{\Omega}\delta\tilde{u}_\parallel - \hat{\kappa}c_s^2\frac{\delta\tilde{\varepsilon}}{\varepsilon_0 + P_0} - \hat{\kappa}\frac{\delta\tilde{\pi}_\parallel}{\varepsilon_0 + P_0} = 0, \tag{4.16}$$

$$\left(i\hat{\tau}_\pi\hat{\Omega} + \frac{9}{35}\hat{\tau}_\pi^2\hat{\kappa}^2 + 1\right)\frac{\delta\tilde{\pi}_\parallel}{\varepsilon_0 + P_0} - \frac{4i}{3}\hat{\kappa}\delta\tilde{u}_\parallel = 0. \tag{4.17}$$

It is possible to write the equation for the longitudinal modes in the following matrix form

$$\begin{pmatrix} i\hat{\tau}_\pi\hat{\Omega} + \frac{9}{35}\hat{\tau}_\pi^2\hat{\kappa}^2 + 1 & -i\frac{4}{3}\hat{\kappa} & 0 \\ -\hat{\kappa} & \hat{\Omega} & -c_s^2\hat{\kappa} \\ 0 & -\hat{\kappa} & \hat{\Omega} \end{pmatrix} \begin{pmatrix} \frac{\delta\tilde{\pi}_\parallel}{\varepsilon_0+P_0} \\ \delta\tilde{u}_\parallel \\ \frac{\delta\varepsilon}{\varepsilon_0+P_0} \end{pmatrix} = 0, \tag{4.18}$$

where non-trivial solutions are obtained when the determinant vanishes, leading to the dispersion relation,

$$\left(\hat{\Omega}^2 - c_s^2\hat{\kappa}^2\right)\left(i\hat{\tau}_\pi\hat{\Omega} + \frac{9}{35}\hat{\tau}_\pi^2\hat{\kappa}^2 + 1\right) - \frac{4i}{3}\hat{\Omega}\hat{\kappa}^2 = 0. \tag{4.19}$$

Note that if the term that contains third-order contributions is set to zero, one immediately recovers the dispersion relation of Israel-Stewart theory, see Eq. (2.77).

As it was done so far in this work, it is convenient to begin the linear stability analysis by looking at the modes for perturbations on a static background fluid. In this case, the dispersion relation associated to the longitudinal modes, Eq. (4.19), then reads

$$\left(\hat{\omega}^2 - c_s^2\hat{k}^2\right)\left(i\hat{\tau}_\pi\hat{\omega} + \frac{9}{35}\hat{\tau}_\pi^2\hat{k}^2 + 1\right) - \frac{4i}{3}\hat{\omega}\hat{k}^2 = 0. \tag{4.20}$$

The solutions of this equation are displayed in Fig. 25 (red dashed lines), where we also display the longitudinal modes of Israel-Stewart theory (black solid lines), for the sake of a quantitative comparison. There are two hydrodynamic modes, which have degenerated imaginary parts, and a single non-hydrodynamic mode, as it can be seen in the left panel. The non-hydrodynamic mode behaves as $\omega \sim ik^2$ in the large wavenumber limit, which is a feature usually observed in the relativistic Navier-Stokes theory. Furthermore, all three modes are linearly stable, since their imaginary parts are positive for any value of wavenumber. As it was explicitly shown in Fig. 23 and Fig. 25 both transverse and longitudinal modes of the parabolic third-order formulation presented here, respectively, are well behaved for perturbations on a static background fluid.

We now consider the behavior of the longitudinal modes for perturbations on a non-static background fluid. As it was done so far in this work, we consider a background fluid velocity that is parallel to the wave-vector, which further leads to Eqs. (2.29) and (2.30). Once again, we analyze analytically the modes at zero wavenumber, in order to obtain fundamental constraints that may be satisfied for this theory. In this regime, the dispersion relation associated to the longitudinal modes reads

$$\left[(\gamma\hat{\omega})^2 - c_s^2(\gamma\hat{\omega}V)^2\right]\left[i\hat{\tau}_\pi(\gamma\hat{\omega}) + \frac{9}{35}\hat{\tau}_\pi^2(\gamma\hat{\omega}V)^2 + 1\right] - \frac{4i}{3}(\gamma\hat{\omega})(\gamma\hat{\omega}V)^2 = 0. \tag{4.21}$$

As it was also observed for the transverse modes, perturbations on top of a moving fluid display an additional non-hydrodynamic mode. Thus, there are two hydrodynamic modes and two non-hydrodynamic ones, with the latter being given by

$$\hat{\omega}^\pm = \frac{35i}{54\gamma}\left[\frac{(3\hat{\tau}_\pi - 4V^2 - 3\hat{\tau}_\pi V^2 c_s^2) \pm \sqrt{(3\hat{\tau}_\pi - 4V^2 - 3\hat{\tau}_\pi V^2 c_s^2)^2 + \frac{324}{35}(1 - c_s^2 V^2)^2\hat{\tau}_\pi^2 V^2}}{\hat{\tau}_\pi^2 V^2 (c_s^2 V^2 - 1)}\right]. \tag{4.22}$$

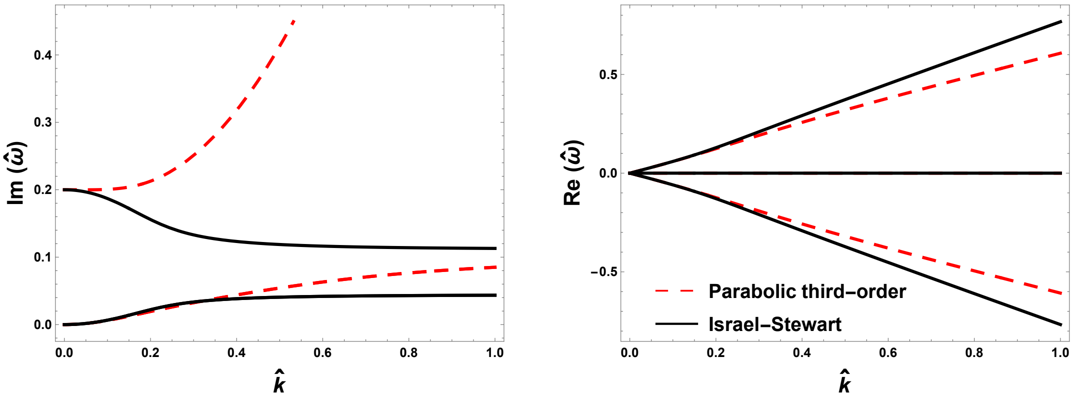

Figure 25 – Imaginary and real parts of the longitudinal modes of Israel-Stewart theory (solid black lines) and parabolic third-order fluid-dynamics (red dashed lines) for perturbations on a static background considering $\hat{\tau}_\pi = 5$ [64] and using the speed of sound in the ultra-relativistic regime, $c_s^2 = 1/3$.

In order to obtain exclusively linearly stable modes, their imaginary part must be positive for all possible values of the background velocity, which is guaranteed if both numerator and denominator have the same sign. As it was assumed in the previous analyses, it is required that neither the numerator nor the denominator change their signs for any value of the background velocity in the causal interval, $0 \leq V \leq 1$, otherwise leading to a problematic discontinuity in the modes. In this scenario, it is then straightforward to see the denominator is always negative. Therefore, a stable mode is achieved if the numerator is negative as well. However, since the term inside the square root is greater than the term outside it, it necessarily renders the mode $\hat{\omega}^+$ linearly unstable. This mode can be identified as the linearly unstable non-hydrodynamic mode that arises when perturbations on top of a moving fluid are performed. The solutions of the dispersion relation for the longitudinal modes, Eq. (4.19) are displayed in Fig. 26 for three different values of the background velocity in the ultra-relativistic limit as a function of wavenumber k.

As it is expected from the analysis performed for the transverse modes, the new non-hydrodynamic modes that appear when perturbations on a moving fluid are performed are linearly unstable not only in the vanishing wavenumber limit, but also for any value of k. On the other hand, the modes that are already present for perturbations on a static background fluid remain linearly stable for any value of wavenumber and background velocity.

As it was shown throughout this section, the theory proposed in Ref. [56] is not a suitable framework to describe the dynamics of relativistic fluids. It is a parabolic theory and was shown to be linearly unstable even in the vanishing wavenumber limit, a severe problem that must be corrected in order to obtain a theory that can be applied in the

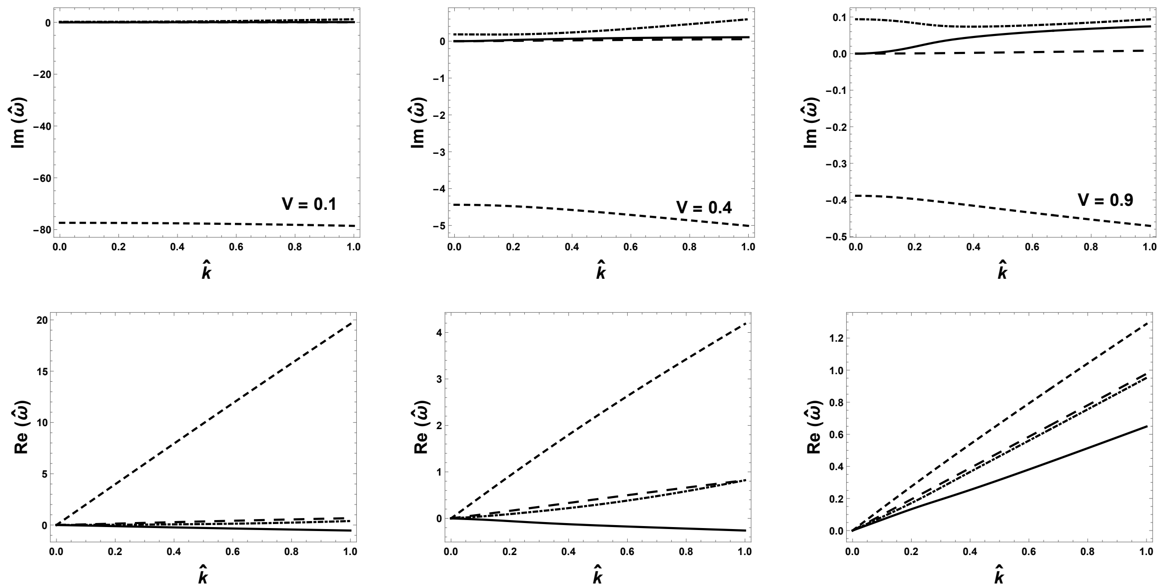

Figure 26 – Imaginary and real parts of the longitudinal modes for $\hat{\tau}_\pi = 5$, considering three different values for the background velocity $V = 0.1$, $V = 0.4$, and $V = 0.9$ in the ultra-relativistic regime $c_s^2 = 1/3$.

study of heavy-ion collisions. In particular, the theory was shown to be linearly unstable for *any* values of the background velocity and wavenumber. This issue cannot be corrected by changing the transport coefficients.

The aforementioned problem is similar to what is observed in relativistic Navier-Stokes theory and, thus, one may employ an approach analogous to the one proposed by Maxwell-Cattaneo [60, 61] in order to restore the hyperbolicity of the equation. In this scenario, one would obtain an alternative third-order hyperbolic formulation in which these problems should be no longer present. In the next section we analyze how is it possible to correct this theory and the implications for the linear stability of doing so.

4.2 Hyperbolic third-order fluid dynamics

In the last section, the third-order fluid-dynamical formulation proposed in Ref. [56] was shown to be linearly unstable. Furthermore, the occurrence of an additional linearly unstable non-hydrodynamic mode for perturbations on a moving fluid can be understood as an aftermath of the parabolicity of the equations. In this scenario, the following question naturally arises: how can one derive a linearly stable third-order fluid-dynamical formulation?

In this section we present a novel hyperbolic third-order formulation for the equation of motion for the shear-stress tensor and further perform a linear stability analysis of the proposed equations in order to verify its stability. First, note that the third-order equation of motion for shear-stress tensor was originally written as

$$\dot{\pi}^{\langle\mu\nu\rangle} = 2\frac{\eta}{\tau_\pi}\sigma^{\mu\nu} - \frac{1}{\tau_\pi}\pi^{\mu\nu} + \frac{4}{35}\nabla^{\langle\mu}\left(\tau_\pi\nabla_\alpha\pi^{\nu\rangle\alpha}\right) - \frac{2}{7}\nabla_\alpha\left(\tau_\pi\nabla^{\langle\mu}\pi^{\nu\rangle\alpha}\right) - \frac{1}{7}\nabla_\alpha\left(\tau_\pi\nabla^\alpha\pi^{\langle\mu\nu\rangle}\right) + \cdots, \tag{4.23}$$

with the dots denoting contribution of non-linear terms in the stability analysis that follows. One way to render this equation hyperbolic is by converting all gradients of the shear-stress tensor into an independent dynamical variable

$$\nabla^{\langle\alpha}\pi^{\mu\nu\rangle} \longrightarrow \rho^{\alpha\mu\nu}, \tag{4.24}$$

with the brackets denoting the contraction with a triple symmetric traceless projection operator onto the space orthogonal to the 4-velocity, $\nabla^{\langle\mu}\pi^{\nu\lambda\rangle} \equiv \Delta^{\mu\nu\lambda}_{\alpha\beta\gamma}\nabla^{\langle\alpha}\pi^{\beta\gamma\rangle}$. This projection operator is a sixth-rank tensor, defined as follows

$$\begin{aligned}
\Delta^{\mu\nu\lambda}_{\alpha\beta\rho} \equiv{}& \frac{1}{6}\left[\Delta^\mu_\alpha\left(\Delta^\nu_\beta\Delta^\lambda_\rho + \Delta^\nu_\rho\Delta^\lambda_\beta\right) + \Delta^\mu_\beta\left(\Delta^\nu_\alpha\Delta^\lambda_\rho + \Delta^\nu_\rho\Delta^\lambda_\alpha\right) + \Delta^\mu_\rho\left(\Delta^\nu_\alpha\Delta^\lambda_\beta + \Delta^\nu_\beta\Delta^\lambda_\alpha\right)\right] \\
-{}& \frac{1}{15}\left[\Delta^{\mu\nu}\left(\Delta^\lambda_\alpha\Delta_{\beta\rho} + \Delta^\lambda_\beta\Delta_{\alpha\rho} + \Delta^\lambda_\rho\Delta_{\alpha\beta}\right) + \Delta^{\mu\lambda}\left(\Delta^\nu_\alpha\Delta_{\beta\rho} + \Delta^\nu_\beta\Delta_{\alpha\rho} + \Delta^\nu_\rho\Delta_{\alpha\beta}\right)\right. \\
+{}& \left.\Delta^{\nu\lambda}\left(\Delta^\mu_\alpha\Delta_{\beta\rho} + \Delta^\mu_\beta\Delta_{\alpha\rho} + \Delta^\mu_\rho\Delta_{\alpha\beta}\right)\right].
\end{aligned} \tag{4.25}$$

In this case, Eq. (4.23) is re-expressed in the following form

$$\tau_\pi\dot{\pi}^{\langle\mu\nu\rangle} + \pi^{\langle\mu\nu\rangle} = 2\eta\sigma^{\mu\nu} - \tau_\pi\nabla_\alpha\rho^{\alpha\mu\nu} + \cdots, \tag{4.26}$$

where the dots once again denote all possible non-linear terms that do not contribute to a linear stability analysis.

The next step is to impose that $\rho^{\mu\nu\lambda}$ is not simply proportional to gradients of the shear-stress tensor, but instead relaxes to such quantities exponentially,

$$\tau_\rho\dot{\rho}^{\alpha\mu\nu} + \rho^{\alpha\mu\nu} = \frac{3}{7}\eta_\rho\nabla^{\langle\alpha}\pi^{\mu\nu\rangle} + \text{non-linear terms}, \tag{4.27}$$

with τ_ρ being introduced as a novel relaxation time and η_ρ as an effective viscosity coefficient associated to the new hydrodynamic variable $\rho^{\alpha\mu\nu}$. Then, Eqs. (4.26) and (4.27) are no longer explicitly parabolic. Note that if we take $\tau_\rho \to 0$ and $\eta_\rho \longrightarrow \tau_\pi$, we recover the original parabolic theory. Finally, we remark that an analogous equation of motion for the new hydrodynamic current $\rho^{\mu\nu\lambda}$ can be derived from the Boltzmann equation, see ref. [64].

Next, we extend the previous linear stability analysis to also consider perturbations in $\rho^{\mu\nu\lambda}$,

$$\varepsilon = \varepsilon_0 + \delta\varepsilon,\;\; u^\mu = u^\mu_0 + \delta u^\mu,\;\; \pi^{\mu\nu} = \delta\pi^{\mu\nu},\;\; \rho^{\mu\nu\lambda} = \delta\rho^{\mu\nu\lambda}. \tag{4.28}$$

In this case, the linearized Eqs. (4.26) and (4.27) become

$$\tau_\pi D_0\delta\pi^{\mu\nu} + \delta\pi^{\mu\nu} = \eta\left(\nabla^\mu_0\delta u^\nu + \nabla^\nu_0\delta u^\mu - \frac{2}{3}\Delta^{\mu\nu}_0\partial_\lambda\delta u^\lambda\right) - \tau_\pi\nabla^0_\alpha\delta\rho^{\alpha\mu\nu}, \tag{4.29}$$

$$\begin{aligned}
\tau_\rho D_0\delta\rho^{\mu\nu\lambda} + \delta\rho^{\mu\nu\lambda} ={}& \eta_\rho\left[\frac{1}{7}\left(\nabla^\lambda_0\delta\pi^{\mu\nu} + \nabla^\nu_0\delta\pi^{\mu\lambda} + \nabla^\mu_0\delta\pi^{\nu\lambda}\right) + \right. \\
&- \left.\frac{2}{35}\left(\Delta^{\mu\nu}_0\nabla^0_\alpha\delta\pi^{\lambda\alpha} + \Delta^{\mu\lambda}_0\nabla^0_\alpha\delta\pi^{\nu\alpha} + \Delta^{\nu\lambda}_0\nabla^0_\alpha\delta\pi^{\mu\alpha}\right)\right].
\end{aligned} \tag{4.30}$$

The next step is to calculate the Fourier transform of Eqs. (4.29) and (4.30), obtaining

$$\begin{aligned}\left(i\Omega\tau_{\pi}+1\right)\delta\tilde{\pi}^{\mu\nu} &= i\eta\left(\kappa^{\mu}\delta\tilde{u}^{\nu}+\kappa^{\nu}\delta\tilde{u}^{\mu}-\frac{2}{3}\Delta^{\mu\nu}\kappa_{\lambda}\delta\tilde{u}^{\lambda}\right)-i\tau_{\pi}\kappa_{\alpha}\delta\tilde{\rho}^{\alpha\mu\nu}, \quad (4.31)\\ \left(i\Omega\tau_{\rho}+1\right)\delta\tilde{\rho}^{\mu\nu\lambda} &= i\eta_{\rho}\left[\frac{1}{7}\left(\kappa^{\lambda}\delta\tilde{\pi}^{\mu\nu}+\kappa^{\nu}\delta\tilde{\pi}^{\mu\lambda}+\kappa^{\mu}\delta\tilde{\pi}^{\nu\lambda}\right)+\right.\\ &- \left.\frac{2}{35}\left(\Delta^{\mu\nu}\kappa_{\alpha}\delta\tilde{\pi}^{\lambda\alpha}+\Delta^{\mu\lambda}\kappa_{\alpha}\delta\tilde{\pi}^{\nu\alpha}+\Delta^{\nu\lambda}\kappa_{\alpha}\delta\tilde{\pi}^{\mu\alpha}\right)\right]. \quad (4.32)\end{aligned}$$

Note that, on the right-hand side of Eq. (4.31), only the projection $\kappa_{\alpha}\delta\tilde{\rho}^{\alpha\mu\nu}$ appears. Hence, we shall only consider the equation of motion for this projected variable in our analysis,

$$\begin{aligned}\left(i\Omega\tau_{\rho}+1\right)\kappa_{\lambda}\delta\tilde{\rho}^{\mu\nu\lambda} &= -\frac{i}{7}\eta_{\rho}\kappa^{2}\delta\tilde{\pi}^{\mu\nu}+\frac{3i}{35}\eta_{\rho}\left(\kappa_{\alpha}\kappa^{\mu}\delta\tilde{\pi}^{\nu\alpha}+\kappa_{\alpha}\kappa^{\nu}\delta\tilde{\pi}^{\mu\alpha}\right)\\ &- \frac{2i}{35}\eta_{\rho}\Delta^{\mu\nu}\kappa_{\alpha}\kappa_{\lambda}\delta\tilde{\pi}^{\alpha\lambda}. \quad (4.33)\end{aligned}$$

Similar to what has been performed so far, this analysis will be divided in terms of transverse and longitudinal modes.

Transverse modes

The next step is to study the transverse degrees of freedom of the hyperbolic third-order theory and analyze the linear stability of the corresponding modes. The transverse component of Eq. (4.33) is obtained by the following projection

$$\left(-\frac{\kappa_{\mu}}{\kappa}\Delta^{\alpha}_{\nu,\kappa}\right)\kappa_{\lambda}\delta\tilde{\rho}^{\mu\nu\lambda}=-\frac{8i}{35}\frac{\eta_{\rho}\kappa^{2}}{i\Omega\tau_{\rho}+1}\delta\tilde{\pi}^{\alpha}_{\perp}. \quad (4.34)$$

Therefore, inserting this equation in the partially transverse projection of Eq. (4.31), we obtain

$$\left(i\hat{\tau}_{\pi}\hat{\Omega}+\frac{8}{35}\frac{\hat{\eta}_{\rho}\hat{\tau}_{\pi}\hat{\kappa}^{2}}{i\hat{\Omega}\hat{\tau}_{\rho}+1}+1\right)\frac{\delta\tilde{\pi}^{\mu}_{\perp}}{\varepsilon_{0}+P_{0}}-i\hat{\kappa}\delta\tilde{u}^{\mu}_{\perp}=0. \quad (4.35)$$

Then, the equations that describe the transverse degrees of freedom of the novel third-order theory can be written in a matrix form as

$$\begin{pmatrix} i\hat{\tau}_{\pi}\hat{\Omega}+\frac{8}{35}\frac{\hat{\eta}_{\rho}\hat{\tau}_{\pi}\hat{\kappa}^{2}}{i\hat{\Omega}\hat{\tau}_{\rho}+1}+1 & -i\hat{\kappa}\\ -\hat{\kappa} & \hat{\Omega}\end{pmatrix}\begin{pmatrix}\frac{\delta\tilde{\pi}^{\mu}_{\perp}}{\varepsilon_{0}+P_{0}}\\ \delta\hat{u}^{\mu}_{\perp}\end{pmatrix}=0, \quad (4.36)$$

In this case, the dispersion relation associated to the transverse modes read

$$\hat{\Omega}\left(i\hat{\tau}_{\pi}\hat{\Omega}+\frac{8}{35}\frac{\hat{\tau}_{\pi}\hat{\eta}_{\rho}\hat{\kappa}^{2}}{i\hat{\Omega}\hat{\tau}_{\rho}+1}+1\right)-i\hat{\kappa}^{2}=0. \quad (4.37)$$

One can straightforwardly recover the dispersion relation for the parabolic third-order formulation by simply taking $\hat{\eta}_{\rho}=\hat{\tau}_{\pi}$ and $\hat{\tau}_{\rho}=0$, see Eq. (4.8). In the last section, we showed such formulation is linearly unstable.

Once again, we begin looking at the transverse modes for perturbations on a static background fluid, $V = 0$. In this case, the dispersion relation, Eq. (4.37), reads simply

$$\hat{\omega}\left(i\hat{\tau}_\pi\hat{\omega} + \frac{8}{35}\frac{\hat{\tau}_\pi\hat{\eta}_\rho\hat{k}^2}{i\hat{\omega}\hat{\tau}_\rho + 1} + 1\right) - i\hat{k}^2 = 0. \tag{4.38}$$

The solutions of this equation are displayed in Fig. 27 in comparison to the transverse modes of Israel-Stewart theory for perturbations on a static background fluid. The hyperbolic third-order formulation has three transverse modes, while both the parabolic third order formulation and Israel-Stewart theory have two transverse modes each. Considering the values for the transport coefficients employed here, the modes are found to be linearly stable not only in the vanishing wavenumber regime, but also for any value of k. We note that the additional mode observed in this case is non-hydrodynamic. Furthermore, while the transverse modes of the parabolic third-order theory have vanishing real parts for perturbations on a static background fluid, the modes of the hyperbolic formulation have non-zero real parts in such scenario.

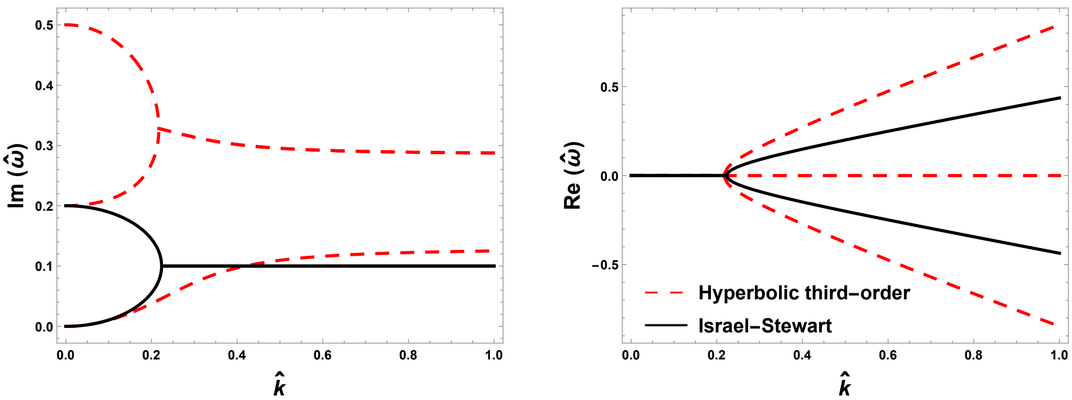

Figure 27 – Imaginary and real parts of the transverse modes of Israel-Stewart theory (solid black lines) and hyperbolic third-order fluid dynamics (red dashed lines) for perturbations on a static background fluid, considering $\hat{\tau}_\pi = \hat{\eta}_\rho = 5$ [64] and $\hat{\tau}_\rho = 2$.

We now consider the case of perturbations on a moving fluid. This case is usually more interesting since instabilities in fluid-dynamical theories are usually displayed by such perturbations. Once again, we assume that the background fluid velocity is parallel to the wave-vector. Furthermore, for the sake of simplicity, we analyze the stability of these modes in the vanishing wavenumber limit, $k = 0$, which corresponds to homogeneous perturbations. In this case, the dispersion relation for the transverse modes of the hyperbolic fluid-dynamical formulation, Eq. (4.37), simply reads

$$\gamma\hat{\omega}\left[i\hat{\tau}_\pi(\gamma\hat{\omega}) + \frac{8}{35}\frac{\hat{\tau}_\pi\hat{\eta}_\rho(\gamma\hat{\omega}V)^2}{i(\gamma\hat{\omega})\hat{\tau}_\rho + 1} + 1\right] - i(\gamma\hat{\omega}V)^2 = 0. \tag{4.39}$$

As it was also observed for perturbations on a static fluid, there is one hydrodynamic mode, and two non-hydrodynamic modes. The latter can be written as

$$\hat{\omega}^{\pm}_{\text{shear}} = \frac{i}{2\gamma}\left[\frac{\hat{\tau}_\pi + \hat{\tau}_\rho - V^2 \pm \sqrt{(\hat{\tau}_\pi + \hat{\tau}_\rho - V^2)^2 + \frac{32}{35}\hat{\tau}_\pi\hat{\eta}_\rho V^2 + 4\hat{\tau}_\rho(V^2 - \hat{\tau}_\pi)}}{\hat{\tau}_\rho(\hat{\tau}_\pi - V^2) - \frac{8}{35}\hat{\eta}_\rho\hat{\tau}_\pi V^2}\right]. \quad (4.40)$$

These modes are stable if their imaginary part is positive, which is guaranteed if both numerator and denominator have the same sign. Once again, we impose neither change their signs for any causal value of the background velocity V, otherwise resulting in a discontinuity in the modes. We note that both the denominator and the numerator are positive for $V = 0$. Thus, they must remain positive for all physical values of velocity.

In order for the denominator to be positive for $0 \leq V \leq 1$, the transport coefficients must satisfy,

$$(\hat{\tau}_\pi - 1)\hat{\tau}_\rho > \frac{8}{35}\hat{\tau}_\pi\hat{\eta}_\rho. \quad (4.41)$$

Furthermore, in order for the numerator to be positive for all physical values of velocity, the relaxation times must satisfy the condition

$$\hat{\tau}_\pi + \hat{\tau}_\rho > 1, \quad (4.42)$$

which guarantees that the term outside the square root in the numerator is always positive. We note that this constraint reduces to the linear causality and stability conditions obtained for Israel-Stewart theory, Eqs. (2.68) and (2.71), respectively, if $\tau_\rho = 0$. However, we further remark that the stability condition (4.41) forbids this limit – a linearly stable theory can only be obtained if τ_ρ is non-zero. Finally, note that conditions (4.41) and (4.42) combined guarantee that the square root in the numerator, if real, is always smaller than $\hat{\tau}_\pi + \hat{\tau}_\rho - V^2$, leading to a stable transverse mode. If the square root in the numerator is not real, then it does not contribute to the stability of the mode.

For the sake of illustration, the transverse modes are displayed as function of wavenumber in Fig. 28, considering three values for the background velocity and transport coefficients that satisfy the linear stability conditions for the transverse modes, Eqs. (4.41) and (4.42). Here, one can see that, for perturbations on a moving background fluid, the imaginary part of the non-hydrodynamic modes (upper panels) are no longer degenerated for large values of the wavenumber, a behavior also observed in Israel-Stewart theory, see Chapter 3.

In Fig. 29 we display the transverse modes of the theory considering values for the transport coefficients that violate the linear stability condition, Eq. (4.41), for a background velocity of $V = 0.9$. Here one can easily identify the occurrence of a linearly unstable (non-hydrodynamic) mode. Note that the unstable mode is also acausal, as it can be seen by looking at its real part. As it was mentioned several times throughout this work, causality and stability are usually related and must be satisfied simultaneously. However, in this analysis we shall not study the linear causality of the novel third-order theory.

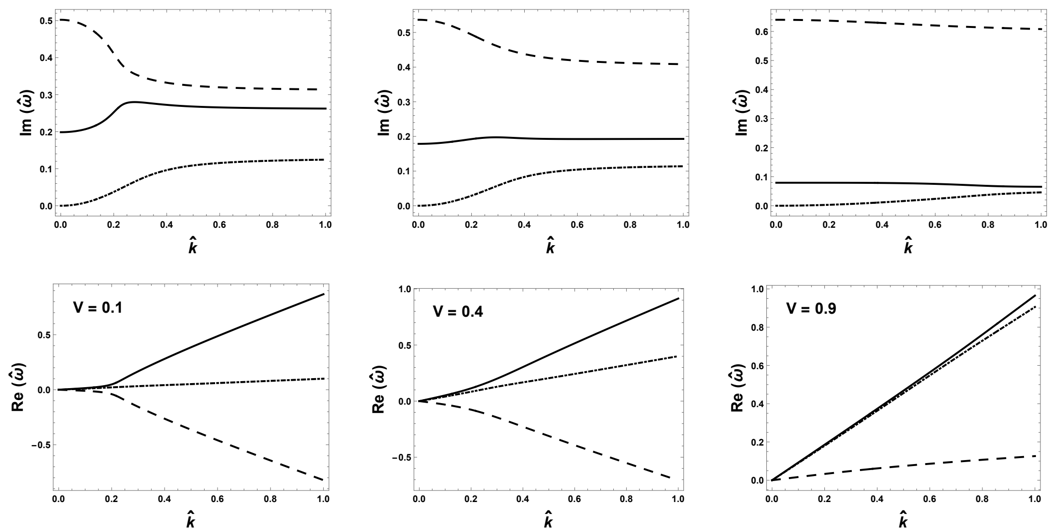

Figure 28 – Imaginary and real parts of the transverse modes for $\hat{\tau}_\pi = \hat{\eta}_\rho = 5$, $\hat{\tau}_\rho = 2$, considering three different values for the background velocity $V = 0.1$, $V = 0.4$, and $V = 0.9$, respectively

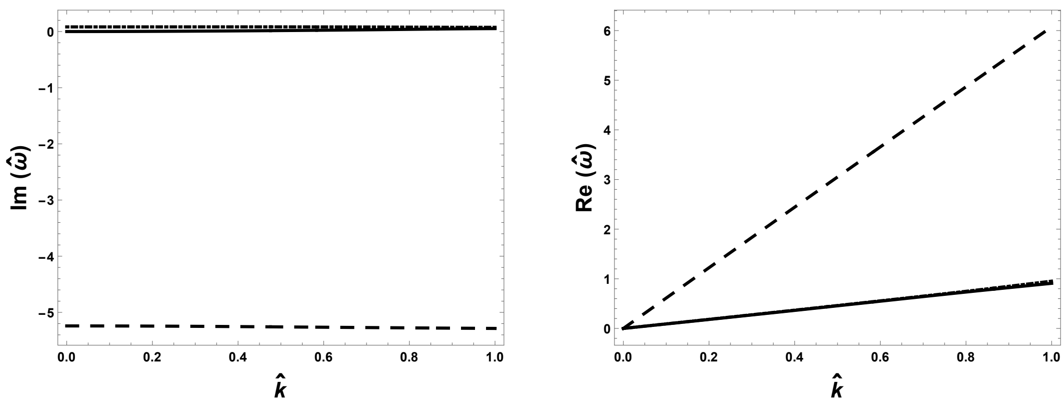

Figure 29 – Imaginary and real parts of the transverse modes for $\hat{\tau}_\pi = \hat{\eta}_\rho = 5$, $\hat{\tau}_\rho = 1$, considering the background velocity as $V = 0.9$.

The analysis developed in this section is restricted to the transverse modes of this theory. Therefore, the next step is to analyze the effects of the inclusion of the novel transport coefficients on the longitudinal modes of the theory and the constraints they must satisfy in order to be linearly stable.

Longitudinal modes

We now analyze the case of the longitudinal modes. The longitudinal component of Eq. (4.33) is obtained by contracting it with $\kappa_\mu \kappa_\nu$, thus leading to

$$(i\Omega\tau_\rho + 1)\left(\frac{\kappa_\mu\kappa_\nu}{\kappa^2}\right)\kappa_\lambda \delta\tilde{\rho}^{\mu\nu\lambda} = -\frac{9i}{35}\eta_\rho \kappa^2 \delta\tilde{\pi}_\parallel. \tag{4.43}$$

Inserting this result in the longitudinal projection of Eq. (4.31), we obtain

$$\left(i\hat{\tau}_\pi\hat{\Omega} + \frac{9}{35}\frac{\hat{\eta}_\rho\hat{\tau}_\pi\hat{\kappa}^2}{i\hat{\Omega}\hat{\tau}_\rho + 1} + 1\right)\frac{\delta\tilde{\pi}_\parallel}{\varepsilon_0 + P_0} - \frac{4i}{3}\hat{\kappa}\delta\tilde{u}_\parallel = 0. \tag{4.44}$$

It is possible to write the equation for the longitudinal modes in the following matrix form

$$\begin{pmatrix} i\hat{\tau}_\pi\hat{\Omega} + \frac{9}{35}\frac{\hat{\eta}_\rho\hat{\tau}_\pi\hat{\kappa}^2}{i\hat{\Omega}\hat{\tau}_\rho+1} + 1 & -i\frac{4}{3}\hat{\kappa} & 0 \\ -\hat{\kappa} & \hat{\Omega} & -c_s^2\hat{\kappa} \\ 0 & -\hat{\kappa} & \hat{\Omega} \end{pmatrix}\begin{pmatrix} \frac{\delta\tilde{\pi}_\parallel}{\varepsilon_0+P_0} \\ \delta\tilde{u}_\parallel \\ \frac{\delta\varepsilon}{\varepsilon_0+P_0} \end{pmatrix} = 0. \tag{4.45}$$

Therefore, the dispersion related to the longitudinal degrees of freedom of the hyperbolic third-order formulation introduced in this chapter reads

$$\left(\hat{\Omega}^2 - c_s^2\hat{\kappa}^2\right)\left(i\hat{\tau}_\pi\hat{\Omega} + \frac{9}{35}\frac{\hat{\tau}_\pi\hat{\eta}_\rho\hat{\kappa}^2}{i\hat{\Omega}\hat{\tau}_\rho + 1} + 1\right) - \frac{4i}{3}\hat{\Omega}\hat{\kappa}^2 = 0. \tag{4.46}$$

Again, one can straightforwardly recover the dispersion relation for the longitudinal modes using the formulation developed in Ref. [56], Eq. (4.19), by simply taking the novel relaxation time to zero, $\tau_\rho = 0$ and the viscosity coefficient to be $\eta_\rho = \tau_\pi$. In particular, the dispersion relation associated to the longitudinal modes of the Israel-Stewart in the absence of coupling, see Eq. (2.77), is immediately recovered if both of these transport coefficients are set to zero. Once again, the inclusion of a relaxation term in the equation for the hydrodynamic current $\rho^{\mu\nu\lambda}$, see Eq. (4.27), leads to the a dispersion relation one order higher than the one obtained for the parabolic theory. However, in the present case, the number of modes does not increase when considering perturbations on top of a moving fluid – which was previously observed for the transverse modes of the theory.

Once again, we first look at the longitudinal modes of the theory for perturbations on a static fluid. In this case, the dispersion relation associated to the longitudinal modes, Eq. (4.46), is written as

$$\left(\hat{\omega}^2 - c_s^2\hat{k}^2\right)\left(i\hat{\tau}_\pi\hat{\omega} + \frac{9}{35}\frac{\hat{\tau}_\pi\hat{\eta}_\rho\hat{k}^2}{i\hat{\omega}\hat{\tau}_\rho + 1} + 1\right) - \frac{4i}{3}\hat{\omega}\hat{k}^2 = 0. \tag{4.47}$$

The solutions of this equations are displayed in Fig. 30 in comparison with the corresponding longitudinal modes of Israel-Stewart theory, and are found to be linearly stable for the values of the transport coefficients employed here.

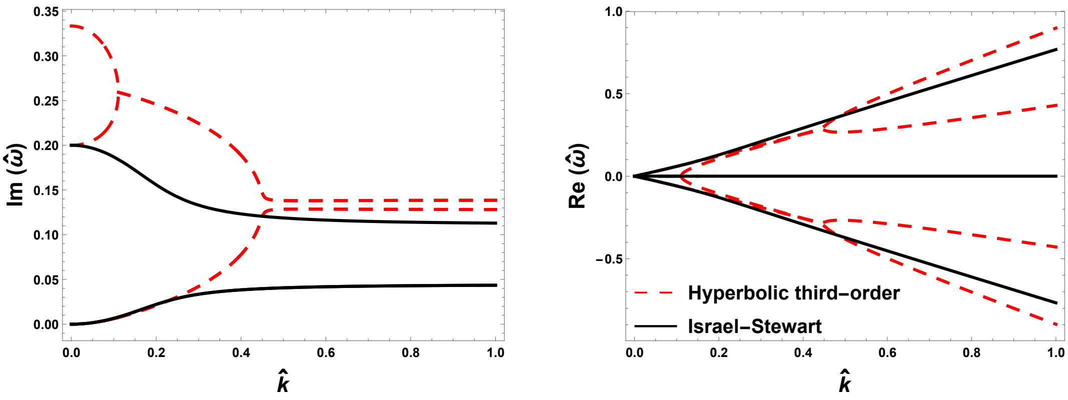

Figure 30 – Imaginary and real parts of the longitudinal modes of Israel-Stewart theory (solid black lines) and hyperbolic third-order fluid dynamics (red dashed line) for perturbations on a static background fluid, considering $\hat{\tau}_\pi = \hat{\eta}_\rho = 5$ [64] and $\hat{\tau}_\rho = 3$ in the ultra-relativistic regime, $c_s^2 = 1/3$

Next, we consider the longitudinal modes for perturbations on a moving fluid. In this case, we assume that the background velocity is parallel to the wave-vector, thus $\hat{\Omega}$ and $\hat{\kappa}$ are written in the form depicted in Eqs. (2.29) and (2.30). Moreover, for the sake of simplicity, we analyze the modes in the vanishing wavenumber limit, a case in which we can obtain *necessary* linear stability conditions the hyperbolic third-order formulation must satisfy. In this case, the dispersion relation associated to the longitudinal modes, Eq. (4.46), can be written as

$$\left(\mathcal{AB} + 4\hat{\tau}_\rho V^2\right)(\gamma\hat{\omega})^2 + i\left(3\mathcal{AC} - 4V^2\right)(\gamma\hat{\omega}) + 3\mathcal{A} = 0, \tag{4.48}$$

where the following variables were introduced

$$\mathcal{A} \equiv 1 - c_s^2 V^2, \tag{4.49}$$
$$\mathcal{B} \equiv 3\hat{\tau}_\pi\left(\frac{9}{35}\hat{\eta}_\rho V^2 - \hat{\tau}_\rho\right), \tag{4.50}$$
$$\mathcal{C} \equiv \hat{\tau}_\pi + \hat{\tau}_\rho. \tag{4.51}$$

There are two hydrodynamic longitudinal modes, which are linearly stable in this regime, and two non-hydrodynamic longitudinal modes. The stability of the latter modes is not guaranteed and must be analyzed. The explicit form of the non-hydrodynamic modes is

$$\hat{\omega}^{\pm} = -i\frac{3\mathcal{AC} - 4V^2 \pm \sqrt{(3\mathcal{AC} - 4V^2)^2 + 12\mathcal{A}\left(\mathcal{AB} + 4\hat{\tau}_\rho V^2\right)}}{2\gamma\left(\mathcal{AB} + 4\hat{\tau}_\rho V^2\right)}, \tag{4.52}$$

Therefore, these modes of the novel third-order theory are stable if both the numerator and denominator have opposite signs, leading to a positive imaginary part of the modes. Once again, we make the assumption that neither the numerator nor the denominator change their signs for any causal value of the background velocity V, otherwise leading to a non-physical discontinuity. Taking the background fluid velocity to be zero, $V = 0$, one can see that the numerator is positive definite, and thus the denominator must be negative. The latter is guaranteed as long as the following inequality is satisfied,

$$\hat{\tau}_\rho > \frac{27}{35}\hat{\eta}_\rho\hat{\tau}_\pi \frac{1 - c_{\rm s}^2}{3\hat{\tau}_\pi\left(1 - c_{\rm s}^2\right) - 4}. \tag{4.53}$$

In the ultra-relativistic limit, i.e., when the speed of sound is given by $c_{\rm s}^2 = 1/3$, this relation further simplifies to

$$(\hat{\tau}_\pi - 2)\hat{\tau}_\rho > \frac{9}{35}\hat{\tau}_\pi\hat{\eta}_\rho. \tag{4.54}$$

As it was also observed when analyzing Israel-Stewart theory, both in the absence and in the presence of diffusion-viscous coupling, the linear stability condition obtained for the novel relaxation time from the longitudinal modes, Eq. (4.54), is stronger than the constraint derived from the transverse modes, Eq. (4.41). The next step is to evaluate the condition that a positive numerator implies.

First, we note that, in order to obtain linearly stable modes, the term inside the square root in the numerator must be either: positive *and* smaller than the term outside, or negative. Both conditions are guaranteed by imposing $3\mathcal{AC} - 4V^2 \geq 0$. Naturally, this constraint must be valid for any physical value of the fluid velocity, V. In this case, the strongest condition possible is obtained considering the maximum value for the background velocity, $V = 1$. Then, we have

$$3(1 - {\rm c}_{\rm s}^2)(\hat{\tau}_\pi + \hat{\tau}_\rho) \geq 4, \tag{4.55}$$

which, in the ultra-relativistic limit, simply reduces to

$$\hat{\tau}_\pi + \hat{\tau}_\rho \geq 2. \tag{4.56}$$

Note that the stability conditions derived above reduce to the causality and stability conditions derived for Israel-Stewart theory, Eqs. (2.83) and (2.86), in the limit of vanishing τ_ρ and η_ρ. Nevertheless, we remark that simply taking $\tau_\rho = 0$ is forbidden by Eq. (4.54).

The solutions of Eq. (4.46) for perturbations on a moving fluid are displayed in Fig. 31. In these plots, we only considered values for the transport coefficients that satisfy the linear stability conditions derived in this section. In this scenario, we note that the modes are linearly stable beyond the vanishing wavenumber limit. Once again, as it was also observed for the transverse modes, the degeneracy between the imaginary part of the modes that occurs for large values of wavenumber disappears when considering perturbations on a moving fluid, as displayed in the upper panels.

For the sake of illustration and comparison, an example of linearly unstable longitudinal modes is shown in Fig. 32. Here, we use values that violate the constraints

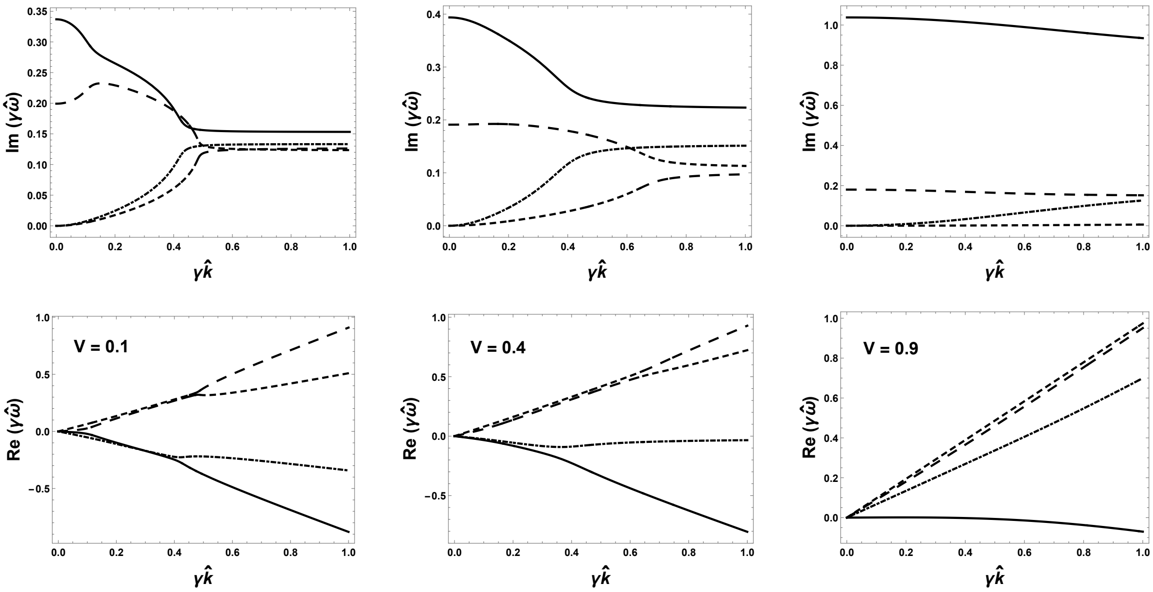

Figure 31 – Imaginary and real parts of the longitudinal modes for $\hat{\tau}_\pi = \hat{\eta}_\rho = 5$, $\hat{\tau}_\rho = 3$, considering three different values for the background velocity $V = 0.1$, $V = 0.4$, and $V = 0.9$, respectively, in the ultra-relativistic regime, $c_s^2 = 1/3$.

obtained in this section. As it was first observed for the transverse modes, the unstable longitudinal mode is also linearly acausal, as can be seen by looking at its real part in the right panel.

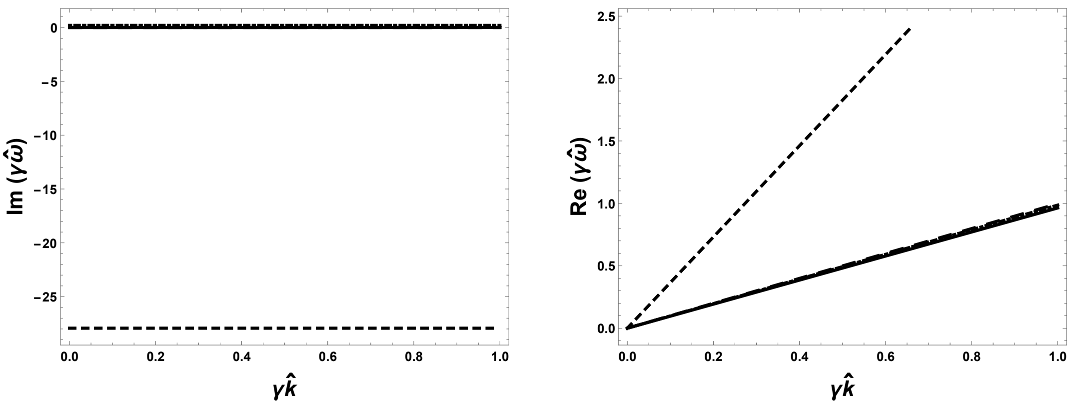

Figure 32 – Imaginary and real parts of the longitudinal modes for $\hat{\tau}_\pi = \hat{\eta}_\rho = 5$, $\hat{\tau}_\rho = 2$, considering the background velocity as $V = 0.99$.

After carefully analyzing both transverse and longitudinal modes of the novel third-order formulation, we concluded that the inclusion of the transport coefficients τ_ρ and η_ρ are essential to obtain a linearly stable theory. However, the novel transport coefficients cannot assume arbitrary values in order to produce only linearly stable modes, and must be somehow constrained. In this chapter, we obtained such conditions from the dispersion relation satisfied by perturbations on a moving fluid in the vanishing wavenumber limit. Once again, as it was first observed when analyzing the linear stability of Israel-Stewart theory, see Chapter 2, the stability conditions associated to the longitudinal modes are stronger than the ones associated to the transverse modes of the theory. In summary, the stability conditions are

$$\hat{\tau}_\rho > \frac{27}{35}\hat{\eta}_\rho\hat{\tau}_\pi \frac{1-c_{\rm s}^2}{3\hat{\tau}_\pi\left(1-c_{\rm s}^2\right)-4}, \tag{4.57}$$

$$\hat{\tau}_\pi + \hat{\tau}_\rho \geq \frac{4}{3(1-{\rm c}_{\rm s}^2)}. \tag{4.58}$$

5 Conclusions and Perspectives

In this dissertation, we investigated the linear causality and stability of several different relativistic fluid-dynamical formulations in the linear regime. We derived conditions that the transport coefficients must satisfy so that these theories yield only causal and stable modes. In the following, we briefly summarize each chapter and provide our conclusions and future perspectives.

In Chapter 1 we presented two fluid-dynamical formulations: relativistic Navier-Stokes theory and Israel-Stewart theory, both obtained from a phenomenological derivation using the second law of thermodynamics. The main difference between them lies on the definition of the entropy 4-current. While in the first, only up to first-order terms in the dissipative currents are taken into account in the expression for the entropy, see Eq. (1.38), in the latter up to second-order terms are accounted, see Eq. (1.53).

In Chapter 2, we performed a linear stability analysis of both theories around global equilibrium. We reproduced the well known result that relativistic Navier-Stokes theory is linearly acausal and unstable. We further rederived the linear causality and stability conditions for Israel-Stewart theory considering dissipation only via shear-stress. The linear stability condition for the shear relaxation time, τ_π, in this case is then given by

$$\tau_\pi \geq \frac{2\eta}{\varepsilon_0 + P_0}, \tag{5.1}$$

where η is the shear viscosity, ε_0 is the energy density and P_0 is the thermodynamic pressure.

In Chapter 3, we present the main results of this dissertation. Here, we extended the linear stability analysis of Israel-Stewart theory performed in Chapter 2 by further including the effects of net-charge diffusion. In particular, we investigate the effects that second order terms (that are linear in the dissipative currents) that couple one dissipative current with the other, the diffusion-viscous couplings, can have on the linear causality and stability of the theory.

We first considered the case where the coupling terms are zero. In this case, the modes related to fluctuations of energy, momentum and net-baryon number decouple. The dispersion relation for the modes related to fluctuations of energy and momentum obtained here are identical to the ones first derived in Ref. [30] and explicitly reproduced in Chapter 2. The linear stability condition for the shear relaxation time, τ_π, in this case remains being $\tau_\pi \geq \frac{2\eta}{\varepsilon_0+P_0}$. Furthermore, we also obtained a novel linear causality and stability condition for the net-baryon diffusion relaxation time, τ_n,

$$\tau_n \geq \frac{\kappa_n}{\bar{n}_B}, \tag{5.2}$$

where κ_n is the diffusion coefficient and $\bar{n}_B$ the baryon number density.

We then investigated how the introduction of the aforementioned coupling terms affects these stability conditions. In order to be consistent with kinetic theory calculations and the derivation of fluid dynamics from the second law of thermodynamics, the product of the coupling terms was assumed to be negative. We then showed that the linear stability conditions for the relaxation times are not modified by the inclusion of the viscous-diffusion coupling terms. Furthermore, we obtained a linear stability condition that must be satisfied by the coupling terms themselves, $\ell_{\pi n}$ and $\ell_{n\pi}$, published in Ref. [59], given by

$$|\ell_{\pi n}\ell_{n\pi}| \leq \frac{3}{2}\left(\tau_\pi - \frac{2\eta}{\varepsilon_0 + P_0}\right)\left(\tau_n - \frac{\kappa_n}{\bar{n}_B}\right). \tag{5.3}$$

Once more, it is crucial to emphasize that these conditions are obtained assuming $\ell_{\pi n}\ell_{n\pi} < 0$. However, we showed some example that the system can be stable for $\ell_{\pi n}\ell_{n\pi} > 0$, but these cases were not studied thoroughly.

Although these are novel results which constrain the values the transport coefficients can assume, in particular the diffusion-viscous couplings, they are obtained considering a vanishing bulk viscosity and can be generalized when such dissipative current is taken into account as in Ref. [48]. However, it is not possible to recover the results displayed here simply taking the bulk viscosity and relaxation time to zero in the analysis developed by Olson in the aforementioned paper, otherwise leading to divergent equations. Therefore, in order to obtain the linear causality and stability conditions explicitly in terms of the hydrodynamic variables such as derived in Ref. [59], it is essential to include bulk viscous pressure in the calculations from start. This task shall be performed in a future work.

In Chapter 4, we analyzed the linear stability of third-order fluid dynamics. The parabolic third-order fluid-dynamical formulation for the shear-stress tensor proposed in Ref. [56] was shown to be linearly unstable. In particular, there is the occurrence of an additional unstable non-hydrodynamic mode for perturbations on a moving background fluid for both transverse and longitudinal degrees of freedom of the theory. This problem is similar to what was observed in relativistic Navier-Stokes theory.

We then propose a novel third-order theory introducing a new dynamical variable $\rho^{\mu\nu\lambda}$ that couples to the shear-stress tensor. We then require that this new current satisfies a relaxation equation. We derived the linear stability conditions the new transport coefficients τ_ρ and η_ρ, the relaxation time and viscosity coefficient associated to this novel dissipative current, shall fulfill. In terms of the hydrodynamic variables, these conditions are given by

$$\left[3\tau_\pi\left(1-c_{\rm s}^2\right)-4\frac{\eta}{\varepsilon_0+P_0}\right]\tau_\rho \quad > \quad \frac{27}{35}\eta_\rho\tau_\pi\left(1-c_{\rm s}^2\right), \tag{5.4}$$

$$3(1-{\rm c}_{\rm s}^2)(\tau_\pi+\tau_\rho) \quad \geq \quad \frac{4\eta}{\varepsilon_0+P_0}. \tag{5.5}$$

A complete non-linear third-order fluid-dynamical theory was not explicitly derived here and will be investigated in an upcoming work. Furthermore, we are also interested in performing simulations using our hyperbolic formulation and further compare them with results from the parabolic third-order theory and Israel-Stewart theory. In order to check how the improvements to preserve causality and stability performed in this work shall enhance the agreement with exact solution of the relativistic Boltzmann equation, we shall investigate this theory within the framework of the highly-symmetrical boost-invariant Bjorken expansion [71].

APPENDIX A – Classification of partial differential equations

In this dissertation, we have used the terms *hyperbolic* and *parabolic* to refer to the linearized fluid-dynamical equations. However, we have not formally defined this terminology. This is the purpose of this appendix, where we follow the discussion developed in Refs. [57, 58].

Second-order partial differential equations can be divided in three different types with very characteristic properties and solutions, namely *elliptic*, *parabolic* and *hyperbolic*. Let us consider a linear differential operator of second order $\mathcal{L}$ whose action on an arbitrary function of two independent variables x and y, given by $u = u(x, y)$, results in the most general second-order partial differential equation

$$\mathcal{L}[u] = au_{xx} + 2bu_{xy} + cu_{yy} + du_x + eu_y + fu = g, \tag{A.1}$$

with a, b, c, d, e, f and g being not simultaneously vanishing functions of the variables x and y. The first three terms that contain second-order derivatives are called the *principal part* of the equation, as they essentially carry the properties of its solutions. It is possible to define another linear differential operator $\mathcal{L}_0$ that accounts only the principal part of the equation above. In this case,

$$\mathcal{L}_0[u] = au_{xx} + 2bu_{xy} + cu_{yy}. \tag{A.2}$$

The different classes of partial differential equations are carried by the relations between the coefficients a, b and c. In particular, a partial differential equation is

- **elliptic** if $b^2 - ac < 0$ (e.g. the Laplace equation);
- **parabolic** if $b^2 - ac = 0$ (e.g. the diffusion equation);
- **hyperbolic** if $b^2 - ac > 0$ (e.g. the wave equation).

Note that these relations were obtained based on considering two independent variables. However, a generalization for the case with N independent variables can be straightforwardly extended. In such case, it is necessary to analyze the elements of the matrix formed by the coefficients of the second-order derivatives. If the eigenvalues of such matrix are all negative or all positive, the equation is elliptic. If at least one is zero, it is a parabolic equation. If there is at least one positive (negative) and the rest is negative (positive), it is a hyperbolic equation.

In this dissertation, we actually determine the parabolicity of partial differential equations by comparing the highest order of the spatial and time derivatives in the equations. If there are spatial derivatives of higher order than the time derivatives, we say the equation is parabolic. This is consistent with the definitions presented above. If the highest order of the time derivative is larger than the highest order spatial derivatives, then we refer to the equation as hyperbolic. For the equations of motion discussed in this dissertation, this turns out to be true.

APPENDIX B – Tensor decomposition

It is always possible to decompose an arbitrary tensor in two parts: one parallel and the other orthogonal to a given normalized 4-vector, e.g., a^μ, which is normalized by definition $a_\mu a^\mu = \pm 1$. Such 4-vector may be, for instance, the 4-velocity of a fluid, the wave 4-vector, and so on. Thereby, the projection operator $\Delta^{\mu\nu}$ is defined as a mathematical object whose purpose is split parallel and orthogonal terms to the 4-vector a^μ. Therefore

$$\Delta^{\mu\nu} = g^{\mu\nu} - \frac{a^\mu a^\nu}{a_\lambda a^\lambda}. \tag{B.1}$$

This tensor acts projecting 4-vectors onto the orthogonal space to a^μ, and one can straightforwardly notice that $\Delta^{\mu\nu} a_\mu = \Delta^{\mu\nu} a_\nu = 0$. Therefore, an arbitrary tensor A^μ can be expressed in the following form

$$A^\mu = A_\parallel a^\mu + A^\mu_\perp, \tag{B.2}$$

with $A_\parallel \equiv a_\mu A^\mu$ and $A^\mu_\perp \equiv \Delta^{\mu\nu} A_\nu$ being the parallel and orthogonal parts of A^μ with respect to the 4-vector a^μ respectively.

An analogous analysis can be extended to the case of an arbitrary second-rank tensor $B^{\mu\nu}$. In this case, an extremely convenient decomposition has the following form

$$B^{\mu\nu} = B_\parallel a^\mu a^\nu + \alpha^\mu a^\nu + \alpha^\nu a^\mu - \beta \Delta^{\mu\nu} + \alpha^{\mu\nu}, \tag{B.3}$$

where the following definitions have been employed

$$B_\parallel \equiv \frac{1}{(a_\lambda a^\lambda)^2} a_\mu a_\nu B^{\mu\nu},\ \alpha^\mu \equiv \frac{1}{a_\lambda a^\lambda} \Delta^\mu_\sigma a_\rho B^{\sigma\rho},\ \alpha^{\mu\nu} \equiv \Delta^{\mu\nu}_{\alpha\beta} B^{\alpha\beta},\ \beta \equiv -\frac{1}{3} \Delta_{\alpha\beta} B^{\alpha\beta}, \tag{B.4}$$

with $\Delta^{\mu\nu}_{\alpha\beta} = \frac{1}{2}\Delta^\mu_\alpha \Delta^\nu_\beta + \frac{1}{2}\Delta^\mu_\beta \Delta^\nu_\alpha - \frac{1}{3}\Delta^{\mu\nu}\Delta_{\alpha\beta}$ being the traceless symmetric projection operator. Here, $B_\parallel$ is the component that is parallel to $B^{\mu\nu}$ on both indices, α^μ is a partially orthogonal 4-vector, $\alpha^{\mu\nu}$ is a symmetric traceless second-rank, while all the trace information is carried by the term β.

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