

# Minimal Area Surfaces, Euclidean Wilson loops and Riemann Theta Functions

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## Abstract.

The AdS/CFT correspondence relates Wilson loops in supersymmetric N=4 gauge theory to minimal area surfaces in anti-de Sitter space. In this paper we consider the case of Euclidean flat Wilson loops which are related to minimal area surfaces in Euclidean AdS<sub>3</sub> space. Using known mathematical results for such minimal area surfaces we describe an infinite parameter family of analytic solutions for closed Wilson loops. Further, we also consider examples of surfaces with two boundaries dual to correlators of two Wilson loops.

## 1. Introduction

The computation of Wilson loops, which are dual to minimal area surfaces, plays a central role in understanding the AdS/CFT correspondence [1, 2]. Few explicit examples are known of minimal area surfaces in *AdS* space. For closed, Euclidean Wilson loops (with constant scalar) the most studied one is the circular Wilson loop [3] which is dual to a half-sphere. Another interesting one is the lens shaped discussed in [4]. For infinite Wilson loops, parallel lines [2] and the cusp are known [5]. Concerning multiple contours, the known one consists of two concentric circles [6]. For Minkowski signature, light-like lines play an important role starting with the light-like cusp [7] and culminating with a large recent activity [8] in relation to scattering amplitudes following [9, 10].

Here we show that, in the case of flat Euclidean Wilson loops which are dual to minimal area surfaces in Euclidean *AdS*<sub>3</sub>, known results from the mathematical literature [11] allow us to find an infinite parameter family of solutions in terms of Riemann theta functions. The construction is based on [12] and follows previous results in [13, 14].

The presentation is organized as follows. We start by writing the equations of motion arriving at the cosh-Gordon equation. Next we solve the equations of motion in terms of theta functions and compute the regularized area. Finally we give examples of single curves and multiple curves together with the minimal area surfaces ending on them. More details can be found in the work [15].

## 2. Equations of motion

In this section we write the equations of motion for a minimal area surface in Euclidean *AdS*<sub>3</sub>. We use embedding coordinates  $X_{\mu=0\dots 3}$  parameterizing a space  $R^{3,1}$  and subjected to the constraint

$$X_0^2 - X_1^2 - X_2^2 - X_3^2 = 1, \quad (1)$$

Other useful coordinates are Poincare coordinates  $(X, Y, Z)$  given by:

$$X + iY = \frac{X_1 + iX_2}{X_0 - X_3}, \quad Z = \frac{1}{X_0 - X_3}. \quad (2)$$

The boundary is an  $R^2$  space and located at  $Z = 0$ . A string is parameterized by complex coordinates  $z = \sigma + i\tau$ ,  $\bar{z} = \sigma - i\tau$ . The action in conformal gauge is given by

$$S = \frac{1}{2} \int (\partial X_\mu \bar{\partial} X^\mu + \Lambda(X_\mu X^\mu - 1)) \, d\sigma \, d\tau \quad (3)$$

$$= \frac{1}{2} \int \frac{1}{Z^2} (\partial_a X \partial^a X + \partial_a Y \partial^a Y + \partial_a Z \partial^a Z) \, d\sigma \, d\tau, \quad (4)$$

where  $\Lambda$  is a Lagrange multiplier and the  $\mu$  indices are raised and lowered with the  $R^{3,1}$  metric. An Euclidean classical string is given by functions  $X_\mu(z, \bar{z})$  obeying the equations of motion:

$$\partial \bar{\partial} X_\mu = \Lambda X_\mu, \quad (5)$$

where  $\Lambda$ , the Lagrange multiplier is given by

$$\Lambda = -\partial X_\mu \bar{\partial} X^\mu. \quad (6)$$

These equations should be supplemented by the Virasoro constraints which read

$$\partial X_\mu \partial X^\mu = 0 = \bar{\partial} X_\mu \bar{\partial} X^\mu. \quad (7)$$

Later on we will be interested in finding the solutions in Poincare coordinates  $(X, Y, Z)$  but for the moment it is convenient to study the problem in embedding coordinates  $X_\mu$ . We can rewrite the equations using the matrix

$$\mathbb{X} = \begin{pmatrix} X_0 + X_3 & X_1 - iX_2 \\ X_1 + iX_2 & X_0 - X_3 \end{pmatrix} = X_0 + X_i \sigma^i, \quad (8)$$

where  $\sigma^i$  denote the Pauli matrices. Notice also that Poincare coordinates are simply given by

$$Z = \frac{1}{\mathbb{X}_{22}}, \quad X + iY = \frac{\mathbb{X}_{21}}{\mathbb{X}_{22}}. \quad (9)$$

The matrix  $\mathbb{X}$  satisfies

$$\mathbb{X}^\dagger = \mathbb{X}, \quad \det \mathbb{X} = 1, \quad \partial \bar{\partial} \mathbb{X} = \Lambda \mathbb{X}, \quad \det(\partial \mathbb{X}) = 0 = \det(\bar{\partial} \mathbb{X}), \quad (10)$$

as follows from the definition of  $\mathbb{X}$ , the constraint (1), the equations of motion (5) and the Virasoro constraints (7). These are non-linear equations which in principle are difficult to solve. However, known procedures (see [15]) reduce the equations to the cosh-Gordon equation which has known solutions in terms or Riemann theta functions. Therefore the main equation that has to be solved is:

$$\frac{\partial^2}{\partial z \partial \bar{z}} \alpha(z, \bar{z}) = 2 \cosh 2\alpha, \quad (11)$$

where  $\alpha(z, \bar{z})$  is a real function defined on the world-sheet. In fact,  $\alpha$  satisfies a more general equation that contains an arbitrary holomorphic function  $f(z)$  that we have set to one. This can always be done locally but, doing it globally as we do here implies a restriction in the type of solution we consider. Once a solution for  $\alpha(z, \bar{z})$  is found, the full solution  $\mathbb{X}(z, \bar{z})$  can be reconstructed by a procedure discussed also in [15]. Instead of explaining this procedure in detail we give the solution in the next section. It is straight-forward to check that it satisfies the equations of motion.

### 3. Solutions

The solutions are written in terms of theta functions [11, 16]. We need two theta functions, one has zero characteristics and the other fixed characteristics that we call  $\Delta_{1,2}$ . The latter we denote as:

$$\hat{\theta}(\zeta) = \theta \left[ \begin{matrix} \Delta_1 \\ \Delta_2 \end{matrix} \right] (\zeta) \tag{12}$$

In this paper, for definiteness, we use the convention of [16] as regards theta functions. The theta functions are associated to an auxiliary (genus  $g$ ) hyperelliptic Riemann surface determined as a subspace of  $\mathbb{C}^2$  by an equation

$$\mu^2 = \lambda \prod_{j=1}^{2g} (\lambda - \lambda_j) \tag{13}$$

where  $(\mu, \lambda)$  parameterize  $\mathbb{C}^2$ . If  $\hat{\theta}$  has an odd characteristic then

$$e^{2\alpha} = \frac{\theta^2(\zeta)}{\hat{\theta}^2(\zeta)}, \tag{14}$$

satisfies the cosh-Gordon equation provided we take

$$\zeta = 2\omega(p_1)\bar{z} + 2\omega(p_3)z. \tag{15}$$

where  $\omega$  are the normalized holomorphic differentials and  $p_{1,3}$  are two branch points of the Riemann surface such that

$$2 \int_{p_1}^{p_3} \omega = \Delta_2 + \Pi\Delta_1, \tag{16}$$

As discussed, associated to the solution of the cosh-Gordon equation one finds a solution for the minimal area surface which, in Poincare coordinates, is

$$Z = \left| \frac{\hat{\theta}(2 \int_{p_1}^{p_4})}{\hat{\theta}(\int_{p_1}^{p_4})\theta(\int_{p_1}^{p_4})} \right| \frac{|\theta(0)\theta(\zeta)\hat{\theta}(\zeta)| |e^{\mu z + \nu \bar{z}}|^2}{|\hat{\theta}(\zeta - \int_{p_1}^{p_4})|^2 + |\theta(\zeta - \int_{p_1}^{p_4})|^2}, \tag{17}$$

$$X + iY = e^{2\bar{\mu}\bar{z} + 2\nu z} \frac{\theta(\zeta - \int_{p_1}^{p_4})\overline{\theta(\zeta + \int_{p_1}^{p_4})} - \hat{\theta}(\zeta - \int_{p_1}^{p_4})\overline{\hat{\theta}(\zeta + \int_{p_1}^{p_4})}}{|\hat{\theta}(\zeta - \int_{p_1}^{p_4})|^2 + |\theta(\zeta - \int_{p_1}^{p_4})|^2}, \tag{18}$$

Here  $p_4$  denotes a point on the Riemann surface on which the solution depends. The notation  $\int_{p_1}^{p_4}$  indicates

$$\int_{p_1}^{p_4} = \int_{p_1}^{p_4} \omega \tag{19}$$

where  $\omega$  are the normalized holomorphic differentials. Finally  $\zeta$  is given by eq.(15). This gives the shape of the surface. Now we need to compute the area.

### 4. Computation of the area

The expectation value of the Wilson loop is determined by the (regularized) area of the minimal surface we described. The area can be computed as

$$A = 4 \int e^{2\alpha} d\sigma d\tau, \tag{20}$$

This integral actually diverges. To regulate the divergence we cut the surface at a value  $Z = \epsilon > 0$ . In the limit  $\epsilon \rightarrow 0^+$  the area behaves as

$$A = \frac{L}{\epsilon} + A_f, \tag{21}$$

where  $L$  should be the length of the Wilson loop and  $A_f$  is the finite part which is identified with the expectation value of the Wilson loop through:

$$\langle W \rangle = e^{-\frac{\sqrt{\lambda}}{2\pi} A_f}. \tag{22}$$

This finite piece can be computed from the solution as

$$A_f = -8D_{p_1 p_3} \ln \theta(0) \oint (\sigma d\tau - \tau d\sigma) + 2 \oint \frac{D_{p_1 p_3} \hat{\theta}(\zeta)}{|D_{p_1} \hat{\theta}(\zeta)|} d\ell. \tag{23}$$

where  $d\ell$  is the differential of arc in the world-sheet boundary.

This concludes our presentation of the analytical results obtained. Now we go into some examples to understand how these solutions look.

### 5. Example: single contour

We now illustrate the shape of the Wilson loops and their dual surfaces. Consider the function

$$\mu(\lambda) = i\sqrt{-i(\lambda + 1 - i)}\sqrt{-i(\lambda + 1 + i)}\sqrt{-i(\lambda - \frac{1+i}{2})}\sqrt{-i(\lambda - \frac{1-i}{2})}\sqrt{2-\lambda}\sqrt{\lambda}\sqrt{\lambda + \frac{1}{2}}, \tag{24}$$

where the square root is taken to have a cut on the negative real axis. The cuts in the complex plane are illustrated in fig.1. The period matrix is

$$\Pi = \begin{pmatrix} 0.5 + 0.64972i & 0.14972i & -0.5 \\ 0.14972i & -0.5 + 0.64972i & 0.5 \\ -0.5 & 0.5 & 0.639631i \end{pmatrix}. \tag{25}$$

To write the solution we choose the points  $p_1 = 0$  and  $p_3 = \infty$  which works well since a path between 0 and  $\infty$  defines an odd characteristic that can be taken to be

$$\Delta_1 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \quad \Delta_2 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}. \tag{26}$$

We therefore have

$$\hat{\theta}(\zeta) = \theta \begin{bmatrix} 0 & 0 & 1 \\ 1 & 1 & 1 \end{bmatrix} (\zeta). \tag{27}$$

We can now write

$$\zeta = 2(\omega(\infty)z + \omega(0)\bar{z}) = \begin{bmatrix} -0.4903\sigma - 0.19069\tau \\ -0.4903\sigma + 0.19069\tau \\ 0.59321\tau \end{bmatrix}, \tag{28}$$

where we used  $z = \sigma + i\tau$ . The shape of the Wilson loop and its dual surface are illustrated in figures 2 and 3 for two different choices of the point  $p_4$ .

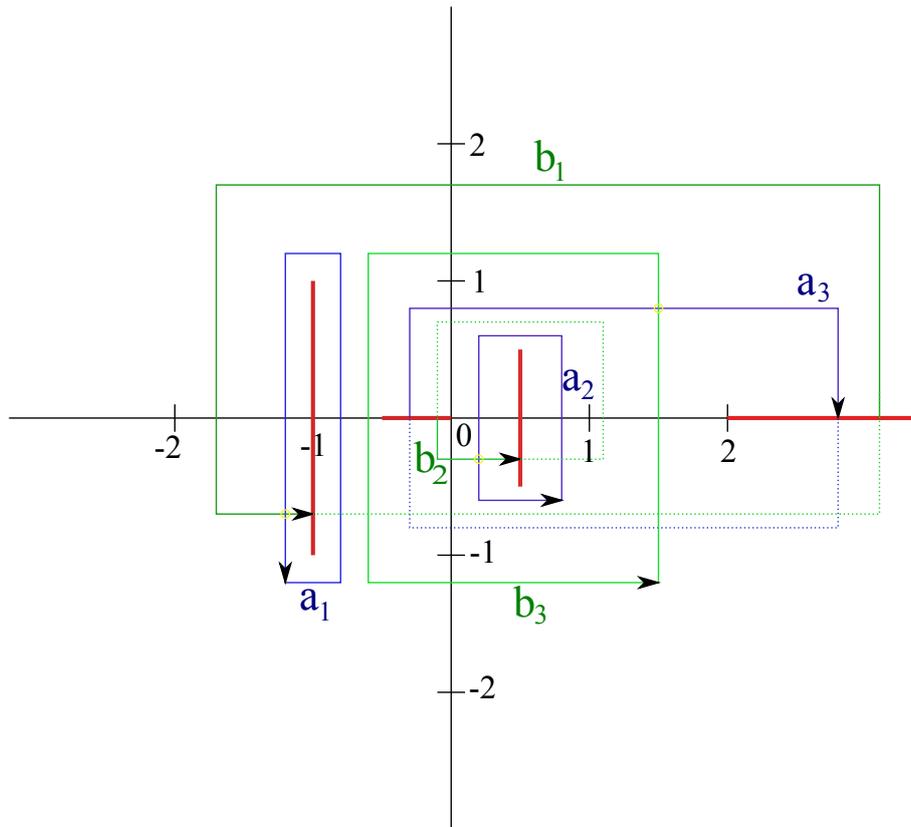


Figure 1: Genus three Riemann surface. A basis of fundamental cycles is depicted with solid paths being in the upper sheet and dotted lines in the lower sheet.

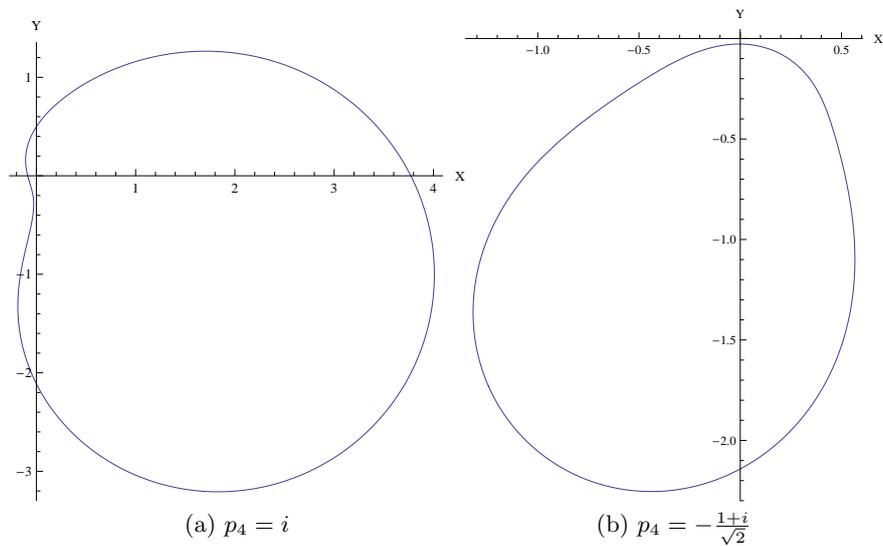


Figure 2: Shape of the Wilson loops for two different choices of the point  $p_4$ .

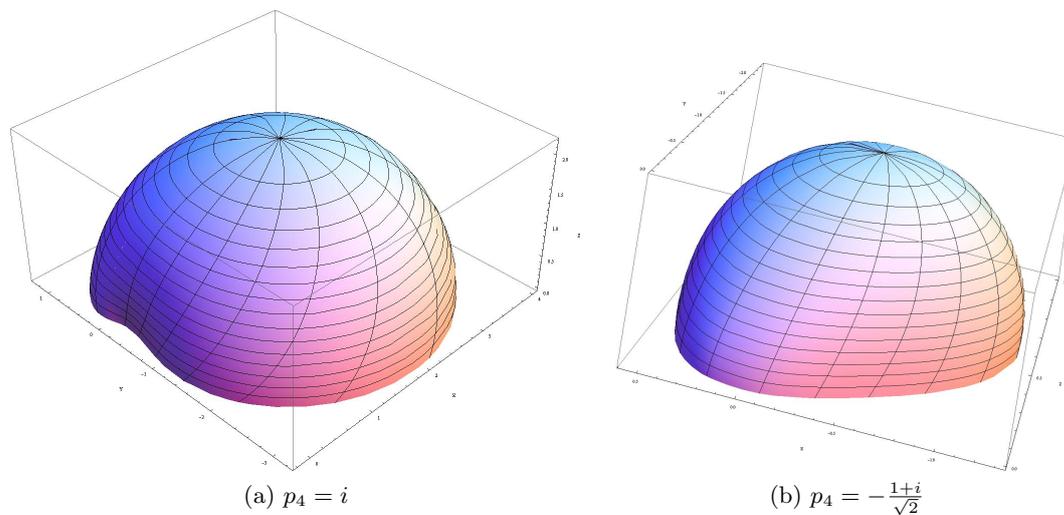


Figure 3: Minimal area surfaces ending on the contours illustrated in fig.2. We emphasize that the surfaces are known analytically.

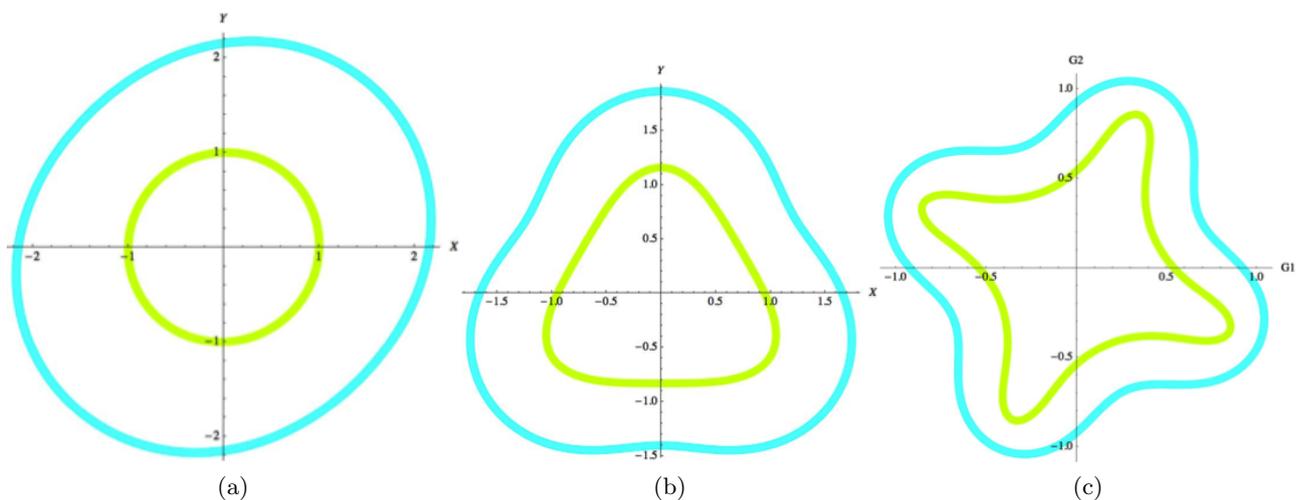


Figure 4: Shape of multiple contours. The minimal area surfaces connects the interior and exterior contours.

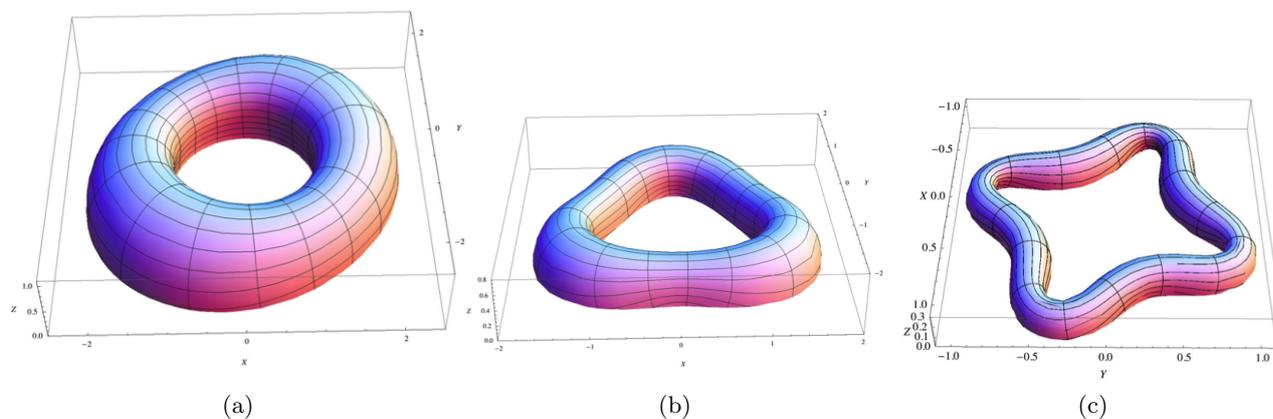


Figure 5: Shape of the minimal area surfaces ending on the contours in fig.4.

## 6. Example: Multiple contours

By changing the position of the cuts, one can find solutions ending on two contours. Since the construction is completely similar we just illustrate the results in figs. 4 and 5.

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## 7. References

- [1] J. Maldacena, “The large  $N$  limit of superconformal field theories and supergravity,” *Adv. Theor. Math. Phys.* **2**, 231 (1998) [*Int. J. Theor. Phys.* **38**, 1113 (1998)], [hep-th/9711200](#),  
S. S. Gubser, I. R. Klebanov and A. M. Polyakov, “Gauge theory correlators from non-critical string theory,” *Phys. Lett. B* **428**, 105 (1998) [[arXiv:hep-th/9802109](#)],  
E. Witten, “Anti-de Sitter space and holography,” *Adv. Theor. Math. Phys.* **2**, 253 (1998) [[arXiv:hep-th/9802150](#)].
- [2] J. M. Maldacena, “Wilson loops in large  $N$  field theories,” *Phys. Rev. Lett.* **80**, 4859 (1998) [[arXiv:hep-th/9803002](#)],  
S. J. Rey and J. T. Yee, “Macroscopic strings as heavy quarks in large  $N$  gauge theory and anti-de Sitter supergravity,” *Eur. Phys. J. C* **22**, 379 (2001) [[arXiv:hep-th/9803001](#)].
- [3] D. E. Berenstein, R. Corrado, W. Fischler and J. M. Maldacena, “The Operator product expansion for Wilson loops and surfaces in the large  $N$  limit,” *Phys. Rev. D* **59**, 105023 (1999) [[arXiv:hep-th/9809188](#)],  
D. J. Gross and H. Ooguri, “Aspects of large  $N$  gauge theory dynamics as seen by string theory,” *Phys. Rev. D* **58**, 106002 (1998) [[arXiv:hep-th/9805129](#)],  
J. K. Erickson, G. W. Semenoff and K. Zarembo, “Wilson loops in  $N = 4$  supersymmetric Yang-Mills theory,” *Nucl. Phys. B* **582**, 155 (2000) [[arXiv:hep-th/0003055](#)],  
N. Drukker and D. J. Gross, “An exact prediction of  $N = 4$  SUSYM theory for string theory,” *J. Math. Phys.* **42**, 2896 (2001) [[arXiv:hep-th/0010274](#)],  
V. Pestun, “Localization of gauge theory on a four-sphere and supersymmetric Wilson loops,” [arXiv:0712.2824 \[hep-th\]](#).
- [4] N. Drukker, S. Giombi, R. Ricci and D. Trancanelli, “Supersymmetric Wilson loops on  $S^{*3}$ ,” *JHEP* **0805**, 017 (2008) [[arXiv:0711.3226 \[hep-th\]](#)].
- [5] N. Drukker, D. J. Gross and H. Ooguri, “Wilson loops and minimal surfaces,” *Phys. Rev. D* **60**, 125006 (1999) [[arXiv:hep-th/9904191](#)].
- [6] N. Drukker and B. Fiol, “On the integrability of Wilson loops in  $AdS(5) \times S^{*5}$ : Some periodic ansatze,” *JHEP* **0601**, 056 (2006) [[arXiv:hep-th/0506058](#)].
- [7] M. Kruczenski, “A note on twist two operators in  $N = 4$  SYM and Wilson loops in Minkowski signature,” *JHEP* **0212**, 024 (2002) [[arXiv:hep-th/0210115](#)].
- [8] See e.g.  
J. Maldacena and A. Zhiboedov, “Form factors at strong coupling via a Y-system,” *JHEP* **1011**, 104 (2010) [[arXiv:1009.1139 \[hep-th\]](#)],  
L. F. Alday, B. Eden, G. P. Korchemsky, J. Maldacena and E. Sokatchev, “From correlation functions to Wilson loops,” [arXiv:1007.3243 \[hep-th\]](#),  
L. F. Alday, D. Gaiotto, J. Maldacena, A. Sever and P. Vieira, “An Operator Product Expansion for Polygonal null Wilson Loops,” [arXiv:1006.2788 \[hep-th\]](#).
- [9] L. F. Alday and J. M. Maldacena, “Gluon scattering amplitudes at strong coupling,” *JHEP* **0706**, 064 (2007) [[arXiv:0705.0303 \[hep-th\]](#)].
- [10] L. F. Alday and J. Maldacena, “Null polygonal Wilson loops and minimal surfaces in Anti-de-Sitter space,” *JHEP* **0911**, 082 (2009) [[arXiv:0904.0663 \[hep-th\]](#)].
- [11] M. Babich and A. Bobenko, “Willmore Tori with umbilic lines and minimal surfaces in hyperbolic space”, *Duke Mathematical Journal* **72**, No. 1, 151 (1993).
- [12] E. D. Belokolos, A. I. Bobenko, V. Z. Enol’skii, A. R. Its, V. B. Matveev, “Algebro-Geometric Approach to Nonlinear Integrable Equations,” Springer-Verlag series in Non-linear Dynamics, Springer-Verlag Berlin Heidelberg NewYork (1994).

- [13] N. Dorey and B. Vicedo, “On the dynamics of finite-gap solutions in classical string theory,” JHEP **0607**, 014 (2006) [arXiv:hep-th/0601194].
- [14] K. Sakai and Y. Satoh, “Constant mean curvature surfaces in  $AdS_3$ ,” JHEP **1003**, 077 (2010) [arXiv:1001.1553 [hep-th]].
- [15] R. Ishizeki, M. Kruczenski and S. Ziamas, “Notes on Euclidean Wilson loops and Riemann Theta functions,” Phys. Rev. D **85**, 106004 (2012) [arXiv:1104.3567 [hep-th]].
- [16] H. M. Farkas and I. Kra, “Riemann Surfaces”, Graduate Texts in Mathematics, Second Edition, Springer-Verlag, New, Berlin, Heidelberg (1991).